AN

ARITHMETIC

COLLEGES AND SCHOOLS.

BY CLAUDIUS CROZET,

PRINCIPAL OF THE RICHMOND ACADEMY, LATE STATE ENGINEER OF VIRGINIA, AND ALSO OF LOUISIANA; PRESIDENT OF JEFFERSON COLLEGE, LOUISIANA, AND FORMERLY PROFESSOR OF ENGINEERING AT WEST POINT.

RICHMOND, VA.:
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TABLE OF PYTHAGORAS.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

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PREFACE.

There may be some presumption in publishing, with the pretension of having added some improvement, a work on a subject which has already been treated by so many authors. But the late advances in the mathematics would seem to call for some modifications in the manner of teaching their first branch, so as to make it a more direct introduction to higher subjects, than the Arithmetics which are usually put in the hands of students. I have, indeed, frequently wondered that a science depending altogether upon reasoning should so generally be attempted to be taught by practical rules, learned by rote, commonly forgotten by the pupil in a few days, and which, thus taught, can be correctly and readily applied only when, by long usage and maturity of mind, the arithmetician has acquired a kind of intuitive perception of the fundamental principles on which these rules depend, and of the train of reasoning by which they are obtained.

I have no hesitation in saying that if all the time and labor expended in learning these practical rules and acquiring readiness and skill in their application, were bestowed upon the study of the reasoned principles of the science, the same time and labor would suffice to obtain a competent knowledge not only of arithmetic itself, but likewise of the two other elementary branches of mathematics; Algebra and Geometry.

Let a young man begin arithmetic at the age when his reasoning powers are sufficiently matured; let him then be made to investigate and comprehend thoroughly the mathematical principles of each operation as he proceeds, and I venture to assert that a few months will suffice him to become skilled in every operation that can be performed by arithmetic: guided by the unbroken thread of reasoning, he never will forget the rules, and will perform his calculations with neatness and always by the simplest methods.

My experience has taught me that it is between the ages of twelve and fifteen, according to the degree of precocity of each individual, that the study of reasoned arithmetic should begin. Before that period, a child can only master the four plain rules, which he ought to learn as soon as practicable; they being for him then a mere mechanical process. But it is only by long habit that he can perform with readiness these four fundamental operations; and therefore he should begin early. But, beyond this training of the numerical faculties of the mind, I would not torment a child with the abstruseness of arithmetic before the above-named period.

In the following pages, I have endeavored to avoid both the obscurity and confusion of purely practical arithmetics, which furnish no thread to guide the youthful mind in the perplexing labyrinth in which he is frequently lost, and the other extreme of abstruseness, which some theoretical treatises are obnoxious to. My object has been to secure the knowledge of this science in a short, and, above all, in a permanent way, and to make the transition from it to algebra simple and natural. While I have endeavored to facilitate the progress of the student, I have not forgotten to make the work convenient for the teacher, who will find in it numerous carefully graduated examples.

In these examples, I have given the answers only when no part of the operation is supplied by them. Answers, in that case, are a great assistance to the teacher, while they merely warn the pupil of his mistake, without doing away the necessity of his going through the details of the exercises.

The work is divided into Lessons, some of which may be omitted with beginners; while others may be only rapidly reviewed with more advanced pupils, the work being intended for both classes of students.

This treatise will consist of two parts:

The first, containing only the elementary principles and fundamental rules of arithmetic.

The second, their combinations and applications to questions of practical utility, and particularly the manner of reducing these to numerical statements.

TABLE OF CONTENTS.

PART I.

CHAPTER I.

CHAPTER II.

CONTAINING THE FOUR ELEMENTARY RULES.

CHAPTER III.

CONTAINING DECIMAL FRACTIONS.

CHAPTER IV.

CONTAINING VULGAR FRACTIONS.

CHAPTER V.

CONTAINING RULES RELATIVE TO FACTORS AND DIVISORS.

PART II.

CHAPTER VI.

CONTAINING OPERATIONS IN VULGAR FRACTIONS AND PROPORTIONS.

CHAPTER VII.

CONTAINING DENOMINATE NUMBERS.

CHAPTER VIII.

CONTAINING PRACTICAL QUESTIONS DEPENDING ON PROPORTIONS.

APPENDIX.

To Teachers.—A few slight mistakes are purposely introduced in some of the answers, in order that the pupil may learn to be sure of himself, and not force his results.

PART I.

CHAPTER I.

CONTAINING DEFINITIONS AND NUMERATION.

LESSON I.

INTRODUCTION AND DEFINITIONS.

1. Mathematics constitute the science which furnishes rules and methods for measuring and calculating everything susceptible of augmentation or diminution.

2. Arithmetic is the first branch of mathematics, and confines itself altogether to numbers: *it is the science of numbers.*

3. A Number is the collection of several things or *Units*, classed under the same name.

4. A Unit is any individual thing: *a horse, an apple, one dollar*, are different units.

The *unit* of a number is one of the things it expresses: thus in *six cents, ten cents, one cent* is the *unit, six* and *ten* the *numbers;* in five apples, one apple is the unit, five is the number; in four hours, one hour is the unit, four the number, &c.

Units, however, are seldom absolutely such; most frequently they have only a *relative* character, and several units of a certain nature may be collected so as to form a different unit under another name.

Thus *twenty sheep* may make a unit under the name of *one score; three feet* become a single unit when called *one yard.*

Twenty-five units, called *cents*, make the single unit *one quarter.*

One hundred cents are the unit called *one dollar.*
Ten dollars are the unit, *one eagle.*
Twelve things make a unit, as *one dozen.*
Twelve dozen form a new unit, as *one gross*, &c.
Arithmetic considers only this numerical relation of different units.

5. The particular unit to which the value of others is referred, is called the *unit of comparison;* it must be carefully ascertained before proceeding in any calculation.

Thus if *gallons, quarts, pints,* or any quantity of a liquid be sold, the *unit of comparison* will be the gallon, if the liquid be sold by the *gallon;* it will be the quart, if it sells by the *quart*, &c.

In the measurement of distances, the *mile* is the *unit of comparison.*

In measuring timber, it is the *foot* for the length, and commonly the *inch* for the breadth, &c.

6. The unit of comparison is called simply *one*, or *unity*, when numbers are considered abstractly; that is, without designation of the nature of their units. Numbers so considered are called *abstract numbers;* as when we say, *One, two, ten, one hundred*, &c.

7. When the nature of the units of a number is designated, it used to be called *a concrete number;* but, of late, it has more properly received the name of *denominate number;* as *two dollars, ten apples, one hundred horses.*

8. The word *quantity* is also frequently used instead of number, to designate any amount in a more indefinite sense. Thus, we say *a certain quantity of things; this quantity* is greater than *that*, &c.

9. All the operations of arithmetic consist in discovering and setting down:

1st. The particular relative value of certain units.
2d. The quantity or number of these units.

Questions.—What are mathematics? What is arithmetic? What is a number? What is a single thing called? In ten cents, six apples, twenty miles, &c., what is the unit? What is the number? Are units absolute or relative? Give examples of relative units. What is the unit of comparison? Give examples. What is an abstract number? What is a denominate number? Give examples. What is meant by quantity? What do the operations of arithmetic consist in?

LESSON II.

NUMERATION AND NOTATION.

1. Numeration is the systematic expression of numbers.

The progressive formation of abstract numbers consists in joining one unit to another, then one more to this sum, and thus continuing to add *one* to each successive sum. In this way, any number, however large, may be formed.

Thé successive sums or numbers thus formed, must each have a particular name. All these names compose the *spoken numeration.*

2. As there can be no limit to the formation of numbers, it will be conceived that, not only it would be impossible to recollect so many independent names, but that no use could be made of them, if these names did not convey to the mind a ready and correct idea of their relation of value.

Hence the necessity of *a system of Numeration*, and the importance of a simple NOTATION; that is, of a *simple method of writing numbers*, consisting of a convenient *nomenclature* and *scale*, by which the relative value of numbers may be readily understood and applied.

3. The nomenclature adopted in modern times forms the *decimal system of numeration* (that is, of which *ten* is the base), chosen evidently on account of our ten fingers.

For, it was natural to count successively on the fingers One, two, three, four, five, six, seven, eight, nine, ten, and then to begin again and go through another *ten;* each successive *ten* being recorded by a mark.

Then all the marks standing for *tens*, or collection of ten units, must be counted in their turn, by means of the same primary names; so that there would be found successively,

One ten,	called - - -	*Ten.*
Two tens,	" - - -	*Twenty.*
Three tens,	" - - -	*Thirty.*
Four tens,	" - - -	*Forty.*
Five tens,	" - - -	*Fifty.*
Six tens,	" - - -	*Sixty.*
Seven tens,	" - - -	*Seventy.*
Eight tens,	" - - -	*Eighty.*
Nine tens,	" - - -	*Ninety.*

And, finally,

Ten tens,	" - - -	*One hundred.*

Another mark must now be made for a collection of *ten tens*, the new name, *one hundred*, being introduced for it; and then successive collections of ten tens may be counted in the same manner, under the names of

One hundred, *two hundreds*, &c., to *Ten hundreds*,

counting by hundreds, as had been done by tens and by simple units.

By an extension of the same process, *ten hundreds* receive the new name of *one thousand;* and every time another ten hundreds is counted, an additional thousand is recorded, and there is successively formed

One thousand, *two thousands*, &c., to *Ten thousands.*

4. In this way, each successive aggregate of ten is counted as a whole, and becomes a *new unit* of a different order; and a *scale* is formed, in which each new unit is ten times greater than the preceding one.

On this account,

Simple units are called		*Units of the 1st order.*
Tens,	"	*Units of the 2d order.*
Hundreds,	"	*Units of the 3d order.*
Thousands,	"	*Units of the 4th order.*
Ten thousands,	"	*Units of the 5th order.*
&c.		&c.

And so on. Beyond thousands, additional simplifications are introduced, which will be better understood by means of the NOTATION OR WRITTEN NUMERATION.

N. B.—When units are mentioned without designation of any particular order, simple units are always understood.

Questions.—What is numeration? Spoken numeration? Written numeration? Notation? Nomenclature? What is our system called? Why? From what probable cause? How many units form one ten? How many tens one hundred? How many hundreds one thousand? What are simple units called? Tens? Hundreds? Thousands? &c. How many units of the 1st order is a unit of the 2d order, of the 3d, of the 4th, 5th, equal to? When the word Unit is used alone, what is understood?

LESSON III.

NOTATION.

1. The numbers in arithmetic are all expressed by means of *ten Arabic characters*, called Figures, which were introduced into Europe by the Moors, about eight or nine hundred years ago. They are:

1,	-	called	-	one.
2,	-	"	-	two.
3,	-	"	-	three.
4,	-	"	-	four.
5,	-	"	-	five.
6,	-	"	-	six.
7,	-	"	-	seven.
8,	-	"	-	eight.
9,	-	"	-	nine.
0,	-	"	-	zero, cipher, or nought.

N. B.—*Zero* appears to me the preferable term. We say the thermometer stands at *zero; zero degree;* in algebra, a quantity is made equal to *zero*, and not *cipher*.

These figures were formerly all called *ciphers*. Hence the word *ciphering*, to express the operations of arithmetic. Cipher is now used only for 0.

2. The first nine figures are called *significant figures*, because each signifies a certain number of units.

The character 0 has of itself no value; *it denotes only the absence of a thing*. It, nevertheless, acts a most important part in the arithmetical expression of numbers, as will presently be seen.

3. Now, from what has been said above of spoken numeration, it will be easy to conceive how any number can be written with the nine significant figures; since it will be sufficient to set down successively the number of units, of tens, of hundreds, &c., it contains.

Suppose that a certain quantity of things, of eggs, for example, is counted, and there are found

Six *hundreds*, four *tens*, and seven *units* (eggs): with the figures we have adopted, it might be written,

6 *hundreds*, 4 *tens*, and 7 *units* (eggs),

or, simply, 647,

by dispensing with the words *hundreds, tens, units;* an omission which can produce no confusion, since these names are merely indicative of the order of the units of each figure, and this order can be as readily understood from the place each figure occupies:

The first figure to the right being	7	units of the first order.
The second is evidently	4	tens or units of the 2d order.
The third "	6	hundreds or units of the 3d order.

4. Thus we arrive at the fundamental law and simple arrangement of our Numeration: that

I. *In the Decimal system of Numeration, the value of each Unit is made ten times greater for each place it is removed to the left.*

II. *The value of a unit depends consequently on the place it occupies.*

5. But it may happen that the number to be written may contain no unit of a particular order. How, then, if we omit the names of the different orders of units, are we to fix them in the relative places which determine their values?

By putting the character 0 *in the place of the absent orders of units.*

Thus, in the above example, if there were no tens, the number being then 6 *hundreds* and 7 *units*,
would be written 607,
where the 6 occupies still its proper place, with the assistance of 0.

Again, if there were no units, the number would be only 6 *hundreds*,
which would be set down 600.
That is, 6 *hundreds, no tens,* and *no units.*

6. In general, therefore, *to write a number with figures, set down the significant figures as enunciated, and fix their orders, by filling up the place of each absent order of units by a zero.*

7. It may be remarked that, in a number, each significant figure has two kinds of values:

The first, *its absolute value,* which is *the intrinsic number of its units.*

The second, *its relative value,* which *depends on the place it occupies.*

Questions.—What characters are used to express numbers? How are they called? By whom and when were they introduced? What are the significant figures? Why are they so called? What is the name of the tenth character? What does it denote? How is a number to be expressed with the nine significant figures? What use is made of zero or cipher? What is the fundamental law of the decimal system? Why is it called decimal? What does the relative value of a figure depend on? How should a number be set down? What is the absolute value of a figure? What, its relative value?

LESSON IV.

1. According to what has been said in the preceding Lesson, of the use of the character 0,

One ten,	or	*Ten,*	is written,	10
Two tens,	called	*Twenty,*	"	20
Three tens,	"	*Thirty,*	"	30
Four tens,	"	*Forty,*	"	40
Five tens,	"	*Fifty,*	"	50
Six tens,	"	*Sixty,*	"	60
Seven tens,	"	*Seventy,*	"	70

Eight tens,	called	*Eighty*,	is written	80
Nine tens,	"	*Ninety*,	"	90

by placing 0 on the right of the significant figures 1, 2, 3, 4, 5, &c., which express respectively the number of tens.

In the same way will

One hundred, two hundreds, three hundreds, &c., be written respectively,

100 , 200 , 300 , &c.

One thousand, two thousands, three thousands, &c.

1000 , 2000 , 3000 , &c.

The *hundreds* with two ciphers, to show the absence of both tens and units.

The *thousands* with three, to show that there are no hundreds, tens, nor units.

And the same for any order, with a corresponding number of ciphers.

2. All intermediate numbers between 10 and 20, between 20 and 30, &c., up to One hundred, are expressed, as was said before (III. 3*), by setting down first the number of tens, and then on its right the number of units. Thus:

One ten and one,	called	*Eleven*,	is written	11
One ten and two,	"	*Twelve*,	"	12
One ten and three,	"	*Thirteen*,	"	13
One ten and four,	"	*Fourteen*,	"	14
One ten and five,	"	*Fifteen*,	"	15
One ten and six,	"	*Sixteen*,	"	16
One ten and seven,	"	*Seventeen*,	"	17
One ten and eight,	"	*Eighteen*,	"	18
One ten and nine,	"	*Nineteen*,	"	19
Then two tens,	"	*Twenty*,	"	20
Two tens and one,	"	*Twenty-one*,	"	21
Two tens and two,	"	*Twenty-two*,	"	22
&c.		&c.		&c.

Any intermediate number between 100 and 200, 200 and 300, &c., up to one thousand, will be composed of

* Numbers between brackets are references to other paragraphs, which the reader should not omit to turn to. The Roman figures indicate the Lesson, and the others the paragraph in it.

three figures, and written after the same manner as the number 647 (III. 3), by setting down successively from left to right the figure of *hundreds*, then *tens*, and lastly *units.*

3. We can, therefore, write any number:
From *one* to *ten*, with *one figure.*
From *ten* to *one hundred*, with *two.*
From *one hundred* to *one thousand*, with *three*, &c.

4. The largest number with one figure being 9
" " with two figures " 99
" " with three " " 999
&c. &c.

In general, *the largest number which can be written with any number of figures is composed of as many* 9, and one more simple unit added to each makes it respectively 10, 100, 1000, &c.; that is, raises it to a superior unit of the next order above that number.

Questions.—How is 1, 2, 3, &c., tens written? How 1, 2, 3, 4, &c., hundreds? 1, 2, 3, &c., thousands? How do you write intermediate numbers between 10 and 20, 20 and 30, &c.? How intermediate numbers between exact hundreds? How many figures will express numbers under 10? Between 10 and one hundred? How many thence to one thousand? Which is the largest in each case?

EXERCISES.

Write four tens. One hundred and twenty. Fifteen tens. 4 single units, 2 tens, and 5 hundreds. 6 units of the first order, 5 of the 2d, 9 of the 3d, in one number. 8 units of the 1st, none of the 2d, 7 of the 3d. Three hundred and forty-five. Five hundred and five. Eight hundred and fifty. Nine hundred and ninety-nine; write the number formed by adding one unit to nine hundred and ninety-nine. Write, &c. &c. &c.

LESSON V.

1. The decimal system for numbers, not exceeding *hundreds*, being well understood, its extension to higher numbers presents no very great difficulty; since we can continue to set units ten times as great on the left of preceding ones, and thus reach to numbers of any magnitude.

2. Yet, if a new name was given to every new order of units, the relative value of so many different orders of units would not be readily understood.

On this account, it being easy to read, at once, three figures, and to form a ready idea of their relative value, the nomenclature has been still farther simplified by *dividing numbers into sections of three figures, called* PERIODS, as in the following example:

Trillions.	*Billions.*	*Millions.*	*Thousands.*	*Units.*
304,	120,	571,	054,	647
hundreds tens units	hundreds tens units	hundreds tens units	hundreds tens units	hundreds tens units

The first period being that of		Units.
The second,	"	Thousands.
The third,	"	Millions.
The fourth,	"	Billions.
The fifth,	"	Trillions.

Then *Quadrillions*, *Quintillions*, *Sextillions*, *Septillions*, *Octillions*, *Nonallions*, *Decallions*, &c. Beyond billions, however, numbers are too large for any practical use, and are seldom met with. We count commonly then by millions.

3. Each period, it will be seen, has its *units*, *tens*, and *hundreds*, and thus the manner of expressing numbers is rendered very simple:

After having separated the Periods by Commas, each Period, beginning at the left, is read off successively as a single number of three figures, to which the name of the period is then given, and the next one read in its turn to the end.

Thus, the number above would read,

Three hundred and four Trillions, *One hundred and twenty* Billions, *Five hundred and seventy-one* Millions, *Fifty-four* Thousands, *Six hundred and forty-seven* (Units). The word *units* is most generally omitted, it being understood. When some sections contain only zeros, their name is omitted, and the significant figures only enunciated.

Thus, 4,000,005,

reads 4 *millions* and 5 *units*, omitting the word *thousands*, because here that period has no significant figure.

Likewise, 4,005,000,

reads 4 *millions* and 5 *thousands*, omitting the period of *units*, which contains no significant figure.

4. In order that the mechanism of numbers may be thoroughly understood, the pupil should be exercised on several examples; and, after having divided, *by commas*, the proposed number into periods, he should be made to give to each figure its *proper name* and that of its *relative value*, as follows; using the same example as above.

1st Period, { 7—units, 4—tens, 6—hundreds.

2d Period, { 4—units of thousands, 5—tens of thousands, 0—hundreds of thousands.

3d Period, { 1—unit of millions, 7—tens of millions, 5—hundreds of millions,

and so on to the highest period. This is called *numerating.*

5. He should also be made to designate, at once, any of the units of the number, without going through the whole reading, and to repeat in order the numeration scale, as follows:

Units, Tens, Hundreds, Thousands (or units of thousands), *Tens of Thousands, Hundreds of Thousands, Millions* (or units of millions), *Tens of Millions,* &c.

Questions.—Why is a number divided into periods? How many figures in each period? What are their relative names? Name the successive periods. How is a number read? Also, when one or more periods have no significant figure? Is the word Units indispensable after reading the last period? Read 5,604,325. Read 900,807,006; 85,000,000,065; 900,000,000,000. Write them in words. Analyze them. Repeat the numeration scale. What is the name of the 5th order, of the 7th, of the 10th, of the 13th, &c.? In the first number, what is the relative value of 6? of 2? of 4? of 3? By what should periods be divided?

LESSON VI.

1. In order to write numbers: *Set down successively, as they are enunciated, the three figures which compose each period;* taking care to supply the place of absent units by 0, and not to omit any period; the place of those which are not enunciated being occupied by three Zeros.*

Exercises.—Write Five thousand; Fifty thousand; Five hundred thousand; Five millions.

One hundred and six thousand and nine.

Three hundred and four millions, fifty-four thousand, eight hundred and seven.

Six millions, one hundred.

Nine billions, one hundred millions and forty.

Five trillions, two hundred and four thousand and five.

Sixty-five quadrillions and seventy millions.

Twenty billions, six thousand and fifteen.

N. B.—It is a good habit, which I recommend, never to omit the commas which divide periods; it prevents mistakes, and greatly facilitates the reading and arrangement of numbers.

2. It may be noticed how rapidly the value of the successive orders of units increases. Between the simple unit *One* and *One ten*, there is only nine units.

Between the unit *ten*, of the second order, and the unit *one hundred* of the third order, there is ninety.

Between *one hundred* and *one thousand*, nine hundred.

Between *one thousand*, the unit of the fourth order, and *ten thousand*, that of the fifth order, there is nine thousand.

---

* The beginner will do well to prepare the number he has to write as follows. Let it be, for instance,

Forty billions, twenty-five thousand and ten,

write successively in a line the names of the periods thus:

Billions, *Millions,* *Thousands,* *Units,*

. . . , . . . , . . . , . . .

with three dots under each, separated by commas; then put the significant figures in their place, thus:

. 4 . , . . . , . 2 5 , . 1 .

and then complete with ciphers for the remaining dots,

4 0 , 0 0 0 , 0 2 5 , 0 1 0

Ciphers at the beginning are omitted as useless, since they do not fix the relative place of any figure.

Then ninety thousand between the two next; nine hundred thousand between the two following, &c.

So that very few figures suffice to write very large numbers.

3. It may also be remarked, *that the figure at the head of a number has a greater value than the part of the number which follows it*, since it is composed of superior units, each of which is greater than the largest number that can be formed with inferior units, which we know to be composed of 9s (IV., 4). Thus, in 19; 199; 1999, &c., 1 ten is greater than nine units; 1 hundred greater than 99; 1 thousand greater than 999, &c. Consequently, it is still greater than numbers composed of figures of inferior value. Hence, in 3,975, 2,986,725, 3 has a greater value than 975, and 2 than 986,725, from the place they respectively occupy.

Therefore, *of two numbers, that is the greater which has most figures, or whose first figure is the largest.*

4. *Not only single units, but any figure is worth ten times more for each place it is removed to the left;* and consequently, *ten times less for each place it is removed to the right;* for, since the individual units of which the figure is composed become ten times greater for each removal to the left (III., 4), their sum must likewise become ten times greater.

If, for instance, one large cake weighs as much as ten small ones, 4 of the large cakes will weigh ten times as much as four small ones.

5. From this remark we conclude that, *to increase* (multiply) *a figure ten, one hundred, one thousand times, &c., it suffices to add respectively, one, two, three zeros to it, and, in general, as many zeros as the number of times ten it is to be increased.*

Thus, 4 made ten times greater, is - 40.
one hundred times, 400.
one thousand times, 4000, &c.

6. Not only single figures, but also whole numbers,

will be thus increased by the addition of zeros on their right.

Thus, 57,630 is ten times larger than 5,763;
576,300, one hundred times;
5,763,000, one thousand times, &c.;
since each figure, displaced one rank to the left, is made ten times larger (4), and consequently also the whole number.

7. We will close the subject for the present with an important remark, viz:

Any part of a number taken by itself forms a number of units of the order of those of its last figure. For example, in the number 6,708,924, if we consider the relative value of the figures 8, 9, and 2, we see that 2 is a number of *units of the second order*, or *tens*.

9 is composed of units ten times as large, and 8 of units one hundred times as large.

Consequently, they stand in the regular decimal relation in regard to each other, and form a number of units of the second order, which may be read 892 *tens*.

Thus, again, in the above number, we have,

67 *hundred thousands.*
670 *ten thousands.*
6708 *thousands.*
67089 *hundreds*, &c.;

the units of the last figure always giving their *name* to the number.

I have dwelt at some length on Numeration, because experience has taught me the importance of teaching it thoroughly. The student cannot devote too much time to this subject; for it is only when he understands perfectly the principles and mechanism of numbers, that he can be expected to make rapid and permanent progress.

Questions.—How are large numbers written? How is the place of absent periods supplied? How many simple units are there between a unit of the 2d and of the 3d order? of the 3d and 4th? of the 4th and 5th? &c. Why is the first figure of a number worth more than the rest of the number? How much does a figure increase or diminish by change of place in a number? How is a number increased ten, one hundred, one thousand, ten thousand times? How may any section of a number be read by itself? Why? What name does it take?

LESSON VII.

ROMAN NOTATION.

1. Roman characters being frequently used for numbers, the student should be made acquainted with them.

The Romans expressed numbers by only seven capital letters of the alphabet, viz:

I for -	1,	C for -	100,
V " -	5,	D " -	500,
X " -	10,	M " -	1000.
L " -	50,		

By combinations of these letters, they formed all their numbers after the following manner:

I - for	1	XI - -	11	CI - -	101	MI - -	1001
II - - -	2	XII - -	12	CII - -	102	MII - -	1002
III - -	3	XIII -	13	&c., to		&c., to	
IIII or IV	4	XIV - -	14	CC - -	200	MM - -	2000
V - - -	5	&c., to		CCC -	300	MMM -	3000
VI - - -	6	XX - -	20	CD - -	400	$M\overline{V}$ or $\overline{IV}$	4000
VII - -	7	XXI - -	21	D - -	500	$\overline{V}$ - - -	5000
VIII - -	8	&c., to		DC - -	600	$\overline{VI}$ - - -	6000
IX - - -	9	XXX -	30	DCC -	700	$\overline{X}$ - -	10,000
X - - -	10	XL - -	40	DCCC -	800	$\overline{L}$ - -	50,000
		L - - -	50	CM - -	900	&c., to	
		LX - -	60	M - -	1000	$\overline{M}$ -	1,000,000
		LXX -	70			one million.	
		LXXX -	80				
		XC - -	90				
		C - - -	100				

In this system:

As often as a character is repeated, so many times is its value repeated.

A less character before a greater is taken from it.

A less character after a greater is added to it.

A bar over any figure increases it one thousand fold.

This is evidently a very imperfect system of numeration.

Questions.—How many characters were used by the Romans? What were they? How are numbers expressed? Write numbers between ten and twenty; between thirty and forty. Write 345; 456; 764, &c. What is the value of a character repeated? What is the effect of a less character before a greater? After the same? What is a bar over any character? Write Three millions, Five thousand, five hundred and fifty-five.

CHAPTER II.

CONTAINING THE FOUR FUNDAMENTAL RULES.

LESSON VIII.

1. In every question dependent on arithmetic, a certain number, called *the result*, is to be found.

2. The *result* is generally obtained by combining together certain given numbers, or *data*, by means of operations which are all reducible to four elementary ones, namely, *Addition*, *Subtraction*, *Multiplication*, and *Division.*

3. In these operations, several signs are used as abbreviations, which will be introduced and explained as they become necessary.

The sign of equality is =, and is read, *is equal to.*

Thus, one dollar = 100 cents, means and reads, *one dollar is equal to one hundred cents.*

SIMPLE ADDITION.

The sign is +, called *plus.*

Examples: 2 apples + 3 apples = 5 apples.
$2+3=5$, read, 2 plus 3 is equal to 5.

Symbols: $2a+3a=5a$. $a+b=c$.

Question.—John has two apples, and receives 3. How many has he? *Ans.* $2+3=5$.

4. Addition *is an operation by which several numbers are united into one, called their* Sum, *or total amount.*

There is no difficulty in adding together single figures; children at an early day learn to perform such additions on their fingers, and to give promptly their sum. Presuming that this readiness has been acquired, and that the student will be acquainted with practical addition, before a book on arithmetic is put into his hands, I omit the addition table.

5. Since a number is a collection of units of the same kind, it is clear that several numbers can be added together so as to form but one number, only when their units are of the same nature; as when 2 *horses* and 3 *horses* are added to make 5 *horses;* but 2 *horses* and 3 *oxen* could not be united into one number of either horses or oxen.

Sometimes, however, numbers may be presented with different names, and yet may be added together if they can be brought under *a common denomination.*

Thus, 4 *horses*, 5 *cows*, 6 *oxen*, may be joined under the same denomination, as 15 *heads of live stock;* 2 lions, 3 tigers, and 5 wolves, as 10 *wild animals*, &c.

6. *Different units*, therefore, *can be added only after they have been reduced to a common denomination.*

It is, therefore, always understood, in addition, that the numbers are formed of units of the same kind.

7. In order to add together several simple numbers,

Write the numbers to be added under each other, so that the units of the same order may be exactly under each other in the same column. That is, units under units, tens under tens, hundreds under hundreds, &c., and draw a line beneath the last.

Add successively, either upwards or downwards, the figures of each column, beginning with that of units, and set down, under the column so added, the figure which expresses the units of its order.

If the sum of the column consists of more than one figure, and therefore contains units of the next superior order, carry these to the next column, and add up this new column in the same way.

Under the last column to the left, write its entire sum.

8. Let us take, for example, the numbers: Be careful, if you would avoid mistakes, to place the figures exactly under each other.

You begin on the right, in order that the excess of each column may be at once added to the next.

3 12
5,947
859
3,407
846
11,059

Then you say 7 and 9 are 16, and 7 are 23, and 6 are 29; and the sum of the first column being 29 *units*, that is, 2 *tens* and 9 *units*, you set down the 9 under the column of units, and *carry* the 2 tens to the next column to which they belong.

The *carrying to the next column* is frequently done mentally; but it is a good practice for beginners, and in long additions, to set the number carried at the top of the next column, as shown in the example.

Passing on to the second column, you say, 2 *carried* and 4 are 6, and 5 are 11, and 0 are 11, and 4 are 15; that is, 15 *units of the 2d order, or tens*, equal to 1 *hundred* and 5 *tens:* set down the 5 under the column of tens, and carry 1 to that of hundreds.

Then you add the third column, 1 *carried* $+9+8+4+8=30$ *units of the* 3*d order, or hundreds*, which is 3 *units of the* 4*th order*, and *none of the* 3*d;* therefore, you set down 0 under the column of hundreds, and carry 3 to that of thousands.

The addition of the column of thousands gives now 11: we set down 1 and *advance* 1 to the column of ten thousands; or, in other words, write at once 11, as stated in the rule.

PROOF OF ADDITION.

9. Every method of proving addition is, after all, but a repetition of the same operation. The simplest and most common way to make this indispensable verification, and avoid the same errors, is to reverse the order in which you have summed up; if it was, for example, downwards at first, let it now be upwards.

10. Sometimes, also, one of the numbers is cut off and the others added; after which the single number is also added; but this method offers no advantage.

When the addition is very long, the column had better be divided into several smaller ones.

Questions.—What is the object of every operation in arithmetic? What are the four fundamental rules? What is to be found? What is given? What is the sign of equality? Give an example. What is the sign of addition? What is addition? What is its result called? What is the first condition, in order that several numbers may be added? How can units of different natures be added? Repeat the rule for addition. Why do you begin on the right? If the sum of a column consists of one figure, what is done with it? When it consists of more, what is done? What do you carry? Why? Give examples. How is addition proved?

EXAMPLES.

Examples for practice are so easily set, and the objection to giving the results so obvious, that they would be of but little use. They should be given as the following:

1st. $235+697+508+1016+674+590=3,720$.

2d. $5,709+23+79,807+606+5+709+678,753+201+7,900+49,722+46=$;

the student being left to arrange them in regular columns himself, and setting down the result after the sign = in the statement, which he should copy and show, together with the details of the operation.

LESSON IX.

SIMPLE SUBTRACTION.

1. Sign −, called *minus*.

Examples: 5 apples − 2 apples = 3 apples.

$5-2=3$ read, 5 *minus* 2 is equal to 3.

Symbols: $5a-2a=3a$. $a-b=c$.

Question.—John has 5 apples; he gives away 2. How many has he left? *Ans.* $5-2=3$.

2. Subtraction *is an operation by which the difference of two numbers is found.*

3. The result of subtraction is called *difference*, *remainder*, and sometimes *excess*, according to the manner of viewing the relation of the two numbers.

The smaller number is called the *subtrahend;* the larger, the *minuend*. These terms are but little used.

4. Subtraction is the reverse of addition; since, if two numbers be added together, one of them taken away from the sum, must leave the other. For example: John has 2 apples, and I give him 3; he has then 5. If, now, I take away from 5 the 3 last given, there remains evidently the 2 he had at first.

5. In subtraction, as in addition, the numbers must be composed of units of the same kind, or reduced to the same denomination, that the operation may be possible (VIII., 6). No subtraction could be made, for example, between 3 *horses* and 2 *oxen*, except under the common name of *live stock*. *Ten cents* could not be subtracted from *one dollar*, except by considering the latter as *one hundred cents*, &c.

6. There is no difficulty in subtraction, when all the figures of the smaller number are less than the corresponding ones of the larger.

Let it be proposed, for example, to subtract from if the *units*, the *tens*, the *hundreds*, &c., of the second number be respectively taken from the *units*, *tens*, *hundreds*, &c., of the first, each part will have thus been subtracted; and, consequently, the whole of the smaller number will have been taken from the larger. Commencing, then, at the right hand, we say successively, 5 from 7 leaves 2, 4 from 8 leaves 4, 3 from 5 leaves 2, and 0 from 9 leaves 9; and, setting down each separate remainder in its proper place, we find that 9,242 is the whole remainder; so that, with the conventional signs, 9,587−345=9,242.

$$\begin{array}{r} 9{,}587 \\ 345 \\ \hline 9{,}242 \end{array}$$

7. But, in most instances, it will happen that some of the figures of the number to be subtracted are greater than the corresponding ones of the other. This gives rise to the only difficulty in subtraction.

Let it be proposed, for example, to find the difference between the numbers - - - 3,756

and - - - - - - - - 1,894

After having set down the less number under the greater, as in addition, so that the units of the same order may be under each other, beginning at the right hand, we say: 1,862

4 *from* 6 *leaves* 2, and set down 2 under the units.

Then, observing that 9 is larger than 5, add mentally 10 to 5, and say:

9 *from* 15 *leaves* 6, *and* 1 *to carry;* set down 6.

Then 1 *carried and* 8 *are* 9; 9 *from* 17 *leaves* 8; set down 8 and carry 1.

Finally, 1 *carried and* 1 *are* 2; 2 *from* 3 *leaves* 1: set down 1, and the remainder is 1,862.

That is, 3,756−1,894=1,862.

8. The reason for *carrying* will be readily understood by considering that, since 9 cannot be subtracted from 5, we render the subtraction possible by adding to the upper figure 5 the convenient number 10, which increases it to 15; from which we subtract 9.

But these *ten* units, which we have added for the convenience of the operation, must now be deducted in some way. For this purpose, we observe that they are equal to *one* unit of the next higher order; and then we have to subtract, not only the figure 8 of the next order, but likewise this additional *one;* that is, 9.

The same process is applied to the new figure 9, which is larger than 7; we subtract from 17, and then deduct the 10 that we have mentally added, by carrying them as 1 unit of the higher order to the figure of this order in the lower number.

Subtraction, by carrying in this way, is less liable to errors than the old method, formerly taught, of *borrowing;* that is, diminishing the upper number by one superior unit, instead of increasing the lower one as much, which amounts to the same thing.

Questions.—What is subtraction? What is its sign? Give an example. What is its result called? What is the subtrahend? What is the minuend? What is the relation of subtraction to addition? What should be the nature of the units of both numbers? How are the numbers set down? Where do you begin the operation? How do you subtract? What is done when the lower figure is larger than the upper? Give a demonstration.

LESSON X.

1. From the explanations given in the preceding lesson, we deduce the following rule:

I. *Set down the less number under the greater,* so that the units of the same order in each shall be exactly under each other, and draw a line beneath them.*

II. *Then, beginning at the right hand, subtract successively each figure of the lower number from that immediately over it, and set down the remainder.*

III. *When the upper figure is the smaller, add in your mind* 10 *to it, and then subtract the lower one; set down the remainder, and carry* 1 *to the next figure of the lower number.*

IV. *Continue the operation in this way, until the whole of the less number has been subtracted; if then there are more figures in the larger number, bring them down to the left of the remainder.*

2. The last part of the rule is exemplified in the annexed subtraction, where, after 45 has been subtracted from 519, the remaining part 24, of the large number, since there is nothing to be taken from it, is brought down by the side of the remainder 474, already found.

```
24,519
    45
------
24,474
```

3. The case, when the upper number contains ciphers, must be noticed. Take this example: - - - - - -

```
200,010,025
     30,008
-----------
199,980,017
```

Here we say, 8 from 15 leaves 7; 1 carried from 2 leaves 1; then 0 from 0, twice, leaves 0; 3 from 11 leaves 8; then, carrying

* This is customary; but it will be well to exercise the pupil to subtract in any relative position of the numbers.

1, though there is no remaining figure in the lower number, we continue to say, 1 from 10, three times, leaving as many 9s as there are zeros above; and, finally, 1 from 2 leaves 1. Hence,

When there are zeros above and below (the absence of figures in the less number being equivalent to zeros), *if you do not carry* 1, *set down at once in the result as many* 0s*; and, if you carry, as many 9s as there are* 0s *in the upper number; taking care, in this case, to deduct* 1 *from the figure which precedes the* 0s.

4. It frequently happens that there is no remainder under the left-hand figures of the two numbers: in that case, do not set down ciphers in the remainder.

For example: - - - 24,506
24,497
. . . . 9

Simple dots, to show that no column has been omitted, are sufficient. Zeros at the beginning of a number are useless; the practice is awkward, and may, in some instances, cause mistakes to be made.

NOTE.—There is a manner of subtracting which beginners cannot well understand, but for which I cannot find a more suitable place. It is by means of the *Arithmetical complement.*

The Arithmetical complement of a number is its difference from 1, *with as many zeros as the number itself has figures.*

Thus the arithmetical complement of - - -	67,814
is - . - - - - - - - - - -	32,186
their sum being - - - - - - - -	100,000

The Arithmetical complement is obtained by subtracting the figure of units from 10, *and all the others from* 9.

Suppose, now, that you had to subtract from -	98,653
	67,814
	30,839

Add to the upper number successively each figure of the arithmetical complement, carrying as usual, and at the end strike out the superior unit carried.

Thus we would say,

instead of 4 from 13, - - - - - -	$6 + 3 = 9$
instead of 2 from 5, - - - - - -	$8 + 5 = 13$
instead of 8 from 16, - - -	1 carried, $+ 1 + 6 = 8$
Then, - - - - - - - - -	$2 + 8 = 10$
and - - - - - - -	1 carried, $+ 3 + 9 = 13$

The last 1 is struck out, and the answer is as above, 30,839.

The reason of this is that,

$$98{,}653 - 67{,}814 = 98{,}653 + 32{,}186 - 100{,}000;$$

which, for those who do not understand algebra, may be explained by the following illustration:—A man owes me 36 dollars, and I owe him 97; difference, 61.

I give him his 97 dollars, but he cannot pay back my 36 dollars otherwise than by giving me a 100 dollar note; for which I make the change 64, which is the arithmetical complement of 36, and then the account stands thus as received by him:

$$97 + 64 - 100 = 161 - 100 = 61.$$

This method is expeditious when several numbers are added, from which one is to be subtracted. Thus, - - 1,217,245
+ 612,940
+ 726,783
—1,675,321

=~~1~~0 881,647

It may be also used advantageously in subtracting several numbers at once from the sum of several others; only, in this case, strike out the last superior unit carried, for each number.

Example: - - -	768,403		
	+ 297,546		
	+ 829,427 :	that is, you add,	
	— 562,973	- - -	137,027
	— 428,465	- - -	571,535
	— 356,893	- - -	643,107
	— 263,563	- - -	736,437
			and cut off 4.
	=~~4~~ 283,482,		

PROOF OF SUBTRACTION.

Add the remainder to the less number; the sum must be equal to the greater number, if the work is right (IX., 4).

EXAMPLE.

From - -	5,386,427
take - -	4,258,792
Remainder,	1,127,635
Proof, -	5,386,427

Questions.—Repeat the rule of subtraction. What is done if there are more figures in the greater number than in the lower? When both contain ciphers? When the figures on the left leave no remainder? How do you prove subtraction? Why?

EXAMPLES FOR PRACTICE.

1. 24,607,801 — 15,608,819 = 8,998,982.
2. 1,234,567 — 702,973 =
3. 7,020,914 — 2,766,809 =
4. 95,000 — 1,006 =
5. 4,073,050,062 — 2,803,767,-086 =
6. 20,004,001,003 — 8,405,-128,605 =
7. 1,000 — 7 =
8. 4,000 — 1 =
9. 30,000 — 15 =
10. 10,000,000 — 10 =
11. 45,001 — 11 =

12. From fifty thousand and sixty-three take one thousand and sixty-five.

13. From two hundred millions and three thousand subtract one million and nine.

14. The flood happened about the year of the world 1656, and the birth of Christ 4004. How long was it before Christ?
How long before the present year?

15. Gunpowder was invented in 1330; printing in 1441. How long did the first precede the second?

16. America was discovered in 1492. How long ago is it?

17. The mariner's compass was invented in 1302. How long did it precede the discovery of America?

18. General Washington died in 1799. How long has he been dead?

19. The first settlement in the United States was made by the English, at Jamestown, Virginia, in 1607. How long after the discovery of America?
How long before the Declaration of Independence, in 1776?

20. I have lent a man 2,000 dollars; he has paid back at several times 250, 314, 457, and 510 dollars. How much does he owe yet?

21. A merchant buys 600 barrels of flour, for 3,600 dollars, and sells 450 barrels for 3,150, how many barrels are left? And how much more must he receive to get back his money?

22. A merchant buys 20,750 yards from one person, 16,184 from another, 3,066 from a third; he sells successively 1,200

yards, 7,965 yards, 10,704, and 671. How many yards has he left?

23. I have a revenue from rents of 1,570 dollars; from stock, 1,600. I spend for house-rent, 450; for fuel, 360; for provisions, 875; for the doctor, 75; servants' hire, 320; taxes, 87; insurance, 107; clothing, 550. How much have I left at the end of the year?

24. A man pays 350 dollars for 100 sheep, 60 for a pair of oxen, 200 for two horses, 450 for a carriage. He gives in return 200 bushels of wheat, for 185 dollars; two cows, worth 45; 50 barrels of corn, worth 275. How much has he to pay, besides?

25. A merchant finds himself, in his account

With	A,	in debt,	- 1,609,	and creditor,	-	346
"	B,	"	- 6,567,	"	-	12,612
"	C,	"	- 12,676,	"	-	9,709
"	D,	"	- 983,	"	-	28
"	E,	"	- 13,910,	"	-	5,600
"	F,	"	56,	"	-	4,315

Does he owe more than is due to him, and how much?

The last question is one of those which afford means of verification by varying the operation. For, it is clear that we may by subtraction find the balances (differences) in each individual account, add them on their respective sides, and then subtract.

Or else add the debts and credits, and subtract; both of which methods must produce the same answer. For, it is the same thing whether he pay all, and receive from all together, or settle with each separately.

In all the above questions, after having ascertained what the nature of the units of the answer will be, the numbers, in the operation, are considered as abstract numbers.

LESSON XI.

SIMPLE MULTIPLICATION.

1. *Sign* $\times$; read, *into* or *multiplied by;* sometimes, simply *by.*

Examples: $2 \times 4 = 8$, read, 2 *multiplied by* 4, *are equal to* 8.

Symbols: $a \times b = c$.

2. *Question.*—John gives 6 apples to each one of his 4 brothers. How many apples has he given in all? *Answer.*—24 *apples.*

Or, 1 yard of cloth costs 6 dollars. How much will 4 yards cost? *Answer.*—24 *dollars.*

Such questions as these give rise to multiplication. In both cases, 6 is to be repeated 4 times; which might be done by adding $6+6+6+6=24$. But it is more simple to say, 4 *times* 6 *are* 24. This is MULTIPLICATION.

3. As regards the manner of performing the operation, we have this

FIRST DEFINITION.

MULTIPLICATION *is a short method of Addition, by which a number is repeated as many times as there are units in another.*

In the above case, for instance, 4 is repeated 6 times, and we have, $4\times 6 = 24$.

4. But when, from the nature of a question, we wish to ascertain the character of the operation, we may use this

SECOND DEFINITION.

MULTIPLICATION *is an operation by which the amount for One unit being given, that for a certain number of the same units is obtained.*

In the above 1st Question, for example, we see that *One* brother has 6 apples; the 4 brothers, consequently, must have 4 times 6 apples.

In the 2d, *One* yard is worth 6 dollars; 4 yards must be worth 4×6.

5. The number to be repeated is called the MULTIPLICAND. It is the amount for *One* unit of the other.

The number by which it is multiplied, is the MULTIPLIER. One of the units of the multiplier is equivalent to the whole multiplicand.

The result of the operation is called the PRODUCT.

Both the Multiplicand and the Multiplier are also called FACTORS OF THE PRODUCT; because it is formed by multiplying them together.

In the above example, 6 is the Multiplicand;
4 " the Multiplier;
24 " the Product;
6 and 4 are the two Factors of the Product.

6. Sometimes, also, the product is called a *Multiple* of one of its factors, to express that it contains it an exact number of times, and can therefore be obtained from it by multiplication.

Thus, 24 is a multiple of 6, because 4 times 6 make 24; but 27 is not.

7. Since it is the multiplicand which forms the product, by being repeated a number of times, it follows *that the product is of the same nature as the multiplicand;* and, since the number of times it is repeated is marked by the units of the multiplier:

1. *That when the multiplier is* one, *the product is equal to the multiplicand.*

2. *That when the multiplier is more than* unity, *the product is larger than the multiplicand.*

3. *And by analogy, that when the multiplier is less than* unity, *the product is smaller than the multiplicand.* This occurs in fractions, as will be seen hereafter.

8. We will consider three cases in multiplication:

The first, when neither of the two factors exceeds 12.

The second, when one of them has several figures, and the other does not exceed 12.

The third, when both factors are composed of several figures.

CASE I.

9. This case requires only a ready knowledge of the following table of multiplication, which must be committed to memory:

1 time	0	is	0	Twice	0	are	0	3 times	0	are	0
1 time	1	is	1	Twice	1	are	2	3 times	1	are	3
1 time	2	is	2	Twice	2	are	4	3 times	2	are	6
1 time	3	is	3	Twice	3	are	6	3 times	3	are	9
1 time	4	is	4	Twice	4	are	8	3 times	4	are	12
1 time	5	is	5	Twice	5	are	10	3 times	5	are	15
1 time	6	is	6	Twice	6	are	12	3 times	6	are	18
1 time	7	is	7	Twice	7	are	14	3 times	7	are	21
1 time	8	is	8	Twice	8	are	16	3 times	8	are	24
1 time	9	is	9	Twice	9	are	18	3 times	9	are	27
1 time	10	is	10	Twice	10	are	20	3 times	10	are	30
1 time	11	is	11	Twice	11	are	22	3 times	11	are	33
1 time	12	is	12	Twice	12	are	24	3 times	12	are	36

4 times 0 are 0	7 times 0 are 0	10 times 0 are 0
4 times 1 are 4	7 times 1 are 7	10 times 1 are 10
4 times 2 are 8	7 times 2 are 14	10 times 2 are 20
4 times 3 are 12	7 times 3 are 21	10 times 3 are 30
4 times 4 are 16	7 times 4 are 28	10 times 4 are 40
4 times 5 are 20	7 times 5 are 35	10 times 5 are 50
4 times 6 are 24	7 times 6 are 42	10 times 6 are 60
4 times 7 are 28	7 times 7 are 49	10 times 7 are 70
4 times 8 are 32	7 times 8 are 56	10 times 8 are 80
4 times 9 are 36	7 times 9 are 63	10 times 9 are 90
4 times 10 are 40	7 times 10 are 70	10 times 10 are 100
4 times 11 are 44	7 times 11 are 77	10 times 11 are 110
4 times 12 are 48	7 times 12 are 84	10 times 12 are 120

5 times 0 are 0	8 times 0 are 0	11 times 0 are 0
5 times 1 are 5	8 times 1 are 8	11 times 1 are 11
5 times 2 are 10	8 times 2 are 16	11 times 2 are 22
5 times 3 are 15	8 times 3 are 24	11 times 3 are 33
5 times 4 are 20	8 times 4 are 32	11 times 4 are 44
5 times 5 are 25	8 times 5 are 40	11 times 5 are 55
5 times 6 are 30	8 times 6 are 48	11 times 6 are 66
5 times 7 are 35	8 times 7 are 56	11 times 7 are 77
5 times 8 are 40	8 times 8 are 64	11 times 8 are 88
5 times 9 are 45	8 times 9 are 72	11 times 9 are 99
5 times 10 are 50	8 times 10 are 80	11 times 10 are 110
5 times 11 are 55	8 times 11 are 88	11 times 11 are 121
5 times 12 are 60	8 times 12 are 96	11 times 12 are 132

6 times 0 are 0	9 times 0 are 0	12 times 0 are 0
6 times 1 are 6	9 times 1 are 9	12 times 1 are 12
6 times 2 are 12	9 times 2 are 18	12 times 2 are 24
6 times 3 are 18	9 times 3 are 27	12 times 3 are 36
6 times 4 are 24	9 times 4 are 36	12 times 4 are 48
6 times 5 are 30	9 times 5 are 45	12 times 5 are 60
6 times 6 are 36	9 times 6 are 54	12 times 6 are 72
6 times 7 are 42	9 times 7 are 63	12 times 7 are 84
6 times 8 are 48	9 times 8 are 72	12 times 8 are 96
6 times 9 are 54	9 times 9 are 81	12 times 9 are 108
6 times 10 are 60	9 times 10 are 90	12 times 10 are 120
6 times 11 are 66	9 times 11 are 99	12 times 11 are 132
6 times 12 are 72	9 times 12 are 108	12 times 12 are 144

Questions.—What is the sign of multiplication? Give an example. What questions give rise to multiplication? What is multiplication? What is the multiplicand? The multiplier? What is the result called? What are factors? To which does the given unit belong? What is it equal to? Of what nature is the product? When is it equal to the multiplicand? When larger? When smaller? How much is 7×8; 6×7; 9×5; 12×6; $2\times6\times7$; $3\times2\times11$, &c.?

EXERCISES.

1. How much are 12 pounds of sugar, at 6 cents a pound?

2. How much are 8 pounds of coffee, at 12 cents a pound?

3. How much are 7 yards of cloth, at 5 dollars a yard?

4. How much are 9 pencils, at 2 cents a pencil?

5. How much are 11 pairs of shoes, at 3 dollars a pair?

6. How much are one dozen pairs of stockings, at 4 shillings a pair?

7. How much are 10 chickens, at 9 cents apiece?

8. How much are 9 bushels of wheat, at 6 shillings a bushel?

9. There are 5 bushels of wheat in a barrel. How many in 9 barrels?

10. At 9 dollars a month, what will be my wages in a year?

11. 9 rows of trees; 9 trees in a row. How many trees in all?

12. A man hires a cart, horse, and driver; the cart and horse at 12 shillings, the man at 10 shillings a day. What has he to pay for 11 days?

13. Bought 11 yards, at 9 cents; 6 yards, at 8 cents; 7 yards, at 10 cents. What is to pay?

14. Bought 9 pieces of cloth, containing 11 yards in a piece; 6 pieces, containing 12 yards; 7 pieces, containing 8 yards. How many yards in all?

15. A man buys 11 sheep, at 3 dollars, and sells them at 4. What does he gain? What would he lose, if he sold them for 25 dollars?

In all these questions, the pupil will easily discover that a certain number is given as equivalent to the unit of another number; and that, after having determined from the data the character of the product, the multiplication proceeds as with abstract numbers.

Let the pupils answer, in each case, these

Questions.—What is the unit of comparison? What is the nature of the multiplicand? The multiplier? What is the nature of the product?

Such examples may be multiplied with beginners, and omitted, as well as this and other simple lessons, with more advanced pupils.

LESSON XII.

CASE II.

1. When one of the factors does not exceed 12:

I. *Set down the small number as multiplier under the large number, and draw a line beneath.*

II. *Multiply each figure of the multiplicand by the multiplier, setting down and carrying as in addition.*

FIRST EXAMPLE.

7 *houses were sold at* 5,869 *dollars each. What is the whole cost?*

Here, 5,869 is the multiplicand; and the product must be in dollars (XI., 7). Having ascertained this, we proceed to repeat 5,869, as an abstract number, 7 times.

2. The result might be obtained by the successive addition of each figure of the above number 7 times, with proper carrying. But we shorten the operation by multiplication, and say:

OPERATION.

```
 5,869
     7
------
41,083
```

7 times 9 are 63; set down 3 and carry 6.

7 times 6 are 42, and 6 carried, are 48; set down 8 and carry 4.

7 times 8 are 56, and 4 carried, are 60; set down 0 and carry 6.

Finally, 7 times 5 are 35, and 6 carried, are 41; which set down.

3. The arrangement of the successive units set down in the product, is the consequence of the principle (VIII., 5) *that the repetition of units of any kind or order, must give units of the same kind or order.*

4. After having ascertained, from the nature of the question, what the nature of the product will be, it has been said that the multiplication is made as if with abstract numbers.

5. Furthermore, to simplify the operation, we have recommended to consider the smaller number as the mul-

tiplier. This we are justified in doing, from the principle that, *in multiplying two abstract numbers together, the product is the same, whether the first be multiplied by the second, or the second by the first;* or, in other words, *it is indifferent which of the factors is made the multiplier.*

This fact may have been remarked in the Multiplication Table; and may be further understood, in general, as follows:

Suppose that we place 6 units in a horizontal row, and form four such rows; we shall have four times 6 units.

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

But the simple inspection of the diagram will show that the same might have been done by forming 6 vertical rows, of 4 units each, which is 6 times 4 units; equal, therefore, to 4 times 6.

Hence, $4 \times 6 = 6 \times 4$.

It is clear that this illustration would apply to any two numbers, and, consequently, that the principle is general.

6. The same may also be explained as follows:

I give 4 apples to each of 6 boys; it is in all 6 times 4 apples.

But, instead of this, I might give each boy 1 apple first, making in all 6 apples, and repeat it four times, which would be four times 6 apples; and, since each boy will then have 4 apples, as before; it is evident that 6 times 4 = 4 times 6.

SECOND EXAMPLE.

What is the cost of 7,006 *yards of cloth, at* 9 *dollars per yard?*

Here it is 9 dollars which is to repeated 7,006 times, and which is the multiplicand. But, availing ourselves of the preceding remarks, and considering the two factors as abstract numbers, we make 9 the multiplier, and say:

9 times 6 are 54; set down 4 and carry 5.

9 times 0 are 0, and 5 are 5.

9 times 0 are 0.

9 times 7 are 63.

$$\begin{array}{r} 7{,}006 \\ 9 \\ \hline 63{,}054 \end{array}$$ dollars.

Questions.—How do you multiply by one figure? What do units of any order, multiplied by simple units, give? Of two numbers, which is the multiplicand? Will it alter the result to take it for the multiplier? Prove this.

EXAMPLES FOR PRACTICE.

6,007×5=	123,456,789×2=	123,456,789× 8=
7,974×8=	123,456,789×3=	123,456,789× 9=
30,457×9=	123,456,789×4=	123,456,789×10=
60,812×6=	123,456,789×5=	123,456,789×11=
908,004×7=	123,456,789×6=	123,456,789×12=
287,815×9=	123,456,789×7=	67,400,000× 7=

LESSON XIII.

CASE III.

1. To multiply by a number composed of several figures.

I. *Choose of the two factors that which appears most convenient as multiplier* (XII., 5); *set it down under the other, considered as multiplicand, and draw a line beneath.*

II. *Form successively the product of the multiplicand by each of the significant figures of the multiplier, and write the individual products under each other, beneath the line, observing to place the first right-hand figure of each product in the column directly under its multiplier, and the other figures on the left, under preceding numbers, so as to form regular columns, as in addition.*

III. *Add up the partial products, as they are arranged; their sum will be the total product required.*

2. Let it be proposed, for example, to multiply

	657
by	849
product by 9,	5,913
" by 40,	26,280
" by 800,	525,600
total product,	557,793

In order to understand the second part of the rule, we must consider that we have to repeat the multiplicand, 657, as many times as there are units in the multiplier, 849; that is, 800 times +40 times+9 times.

In the first place, we know

already how to multiply by 9 simple units, and obtain the product, 5,913.

3. But how is the multiplication by 40 to be effected? Evidently by repeating the multiplicand 40 *times;* that is, *ten times* 4.

Hence, after having multiplied by 4; that is, repeated the multiplicand 4 times, and found 2,628, we must take this product 10 times. The result will be 40 times the multiplicand.

But we have seen, in Numeration (VI., 4, 5, 6), that to increase a number 10 times, we have only to add one 0 to it, by which the *units* become *tens.*

Hence, we multiply now the product 2,628 by ten, by adding a 0, which makes it 26,280, and advances the figure 8 of simple *units* to the column of *tens*, and thus places it under its multiplier 4, to the order of which it is raised by the zero.

In like manner, the multiplication by 800 is made by adding two 0s to the product of the multiplicand by 8; since this product must be one hundred times that by 8, and consequently of the order of hundreds.

4. From this explanation we conclude that: *The multiplication of a number of simple units by units of any order, gives in the product units of the same order:* that is, the multiplication of units by *tens*, gives *tens;* by *hundreds*, gives *hundreds*, &c.

The same may also be understood by remarking that, to multiply a number of units by hundreds, is the same thing as multiplying the hundreds by that number (XII., 5), and, consequently, must produce hundreds.

In practice, the 0s, which properly belong to each partial product, are omitted, as unnecessary, since the order of the first figure is sufficiently indicated by its place in the column of its order. The operation is commonly set down thus:

```
   657
   849
  ----
  5913
 2628
5256
------
557793
```

But beginners will do well to retain these 0s until they can multiply correctly without them.

5. It is customary to multiply in regular order, beginning with the units. This is more natural and convenient, but not indispensable; and any arrangement might be used, provided the figures of the partial products be placed in their proper column, we might, for example, multiply - - - 657
by 849

Thus, product by 4 tens, -	26280
" by 8 hundreds,	525600
" by 9 units, -	5913
Total product, -	557,793

6. The above rule applies to all cases of multiplication. It may not be amiss, however, to consider particularly the case, *when there are ciphers at the right hand of one or both of the factors.*

The rule applies as readily to this case; for then, also, the place of the units here occupied by a 0, must be advanced by the proper number of 0s, to the order of the first significant figure you multiply by. Let it be proposed, for instance, to multiply

47000
by 1900

42300000
47000

The result is 89,300,000.

That is, the product 423000 of the multiplicand by 9 units of the 3d order is advanced to the place of hundreds by two additional zeros; and that by 1 follows.

It will be readily observed that to the ciphers of the multiplicand there is thus added in the product those of the multiplier; so that there is in the product a number of ciphers equal to those of both the multiplicand and multiplier. Hence,

When there are ciphers at the end of one or both of the factors, multiply as if they were not there, and then add to this product as many ciphers as there are in both factors.

7. *Remark.*—When, as in the last example, one of the figures you have to multiply by is 1, do not go through the process of multiplication, but set down at once the whole multiplicand, since any number multiplied by 1, is evidently that number itself.

This remark is inserted here, because I have very frequently seen, even advanced pupils, go through an awkward regular multiplication by 1!

PROOF OF MULTIPLICATION.

8. The only proof of multiplication which can be understood at present, besides going over the same work, is to make *the multiplicand the multiplier; and if the products are alike, the operation may be presumed to be correct.*

Questions.—Give the general rule of multiplication. Explain why the right-hand figure of each partial product is to be set down under its multiplier. When you multiply simple units by a figure of any order, what is the order of the product? Why are the additional 0s of partial products omitted in practice? How do you multiply when there are ciphers at the end of one or both of the factors? Why? How do you multiply by 1? By 10? By 100? By 1000? What is the product of units by tens? By hundreds? Of tens by tens? By thousands? Of hundreds by hundreds? Of thousands by thousands? Of hundreds by tens of thousands? Of hundreds by tens? Of thousands by hundreds of thousands? Of thousands by millions? &c. How do you prove multiplication?

EXERCISES.

1. $36\times25=900$.
2. $9{,}664\times16=154{,}624$.
3. $365\times38=13{,}870$.
4. $41{,}364\times35=1{,}447{,}740$.
5. $421\times48=20{,}208$.
6. $34{,}293\times74=2{,}537{,}682$.
7. $895\times23=20{,}585$.
8. $91{,}738\times81=7{,}430{,}778$.
9. $5{,}296\times29=153{,}584$.
10. $576{,}781\times64=36{,}914{,}176$.
11. $2{,}564\times47=120{,}508$.
12. $718{,}328\times96=68{,}959{,}488$.
13. $540{,}042\times23=12{,}420{,}966$.
14. $9{,}378{,}964\times42=393{,}916{,}488$.
15. $1{,}345{,}894\times49=65{,}948{,}806$.
16. $12{,}345{,}679\times27=333{,}333{,}333$.
17. $46{,}123{,}101\times72=3{,}320{,}863{,}272$.
18. $14{,}565{,}869\times51=742{,}859{,}319$.

19. 5,004,000 × 60 =
20. 5,851 × 657 = 3,844,107.
21. 2,843,200 × 80 =
22. 37,864 × 209 = 7,913,576.
23. 814 × 951 = 774,114.
24. 40,058 × 342 = 13,699,836.
25. 594 × 437 = 259,578.
26. 527,527 × 285 = 150,345,195.
27. 1,345 × 108 = 145,260.
28. 749,643 × 695 = 521,001,885.
29. 1,055,054 × 570 = 601,380,780.
30. 40,200,050 × 320 = 1,286,401,600.
31. 87,468 × 5,847 = 511,425,396.
32. 5,401 × 3,004 = 16,224,604.
33. 34,960,078 × 8,405 = 293,839,455,590.
34. 529,600 × 2,900 =
35. 12,321 × 12,321 = 151,807,041.
36. 523,972 × 15,276 = 8,004,196,272.
37. 1,055,070 × 31,456 = 33,188,281,920.
38. 500,407 × 870,497 = 435,602,792,279.
39. 6,795,634 × 918,546 = 6,242,102,428,164.
40. 40,070,065 × 38,015,732 = 15,232,906,283,422,580.
41. 2,708,630,425 × 206,008,604 = 558,001,172,606,176,700.

These examples may also be used for Division, by giving to the pupil the product and one factor.

They may also serve for exercises with ciphers, by adding some at the end of the factors.

For additional questions, see Division and Appendix.

1. There are 360 degrees round the earth; each degree measures 121,519 yards. What is the circumference of the earth, in yards? *Ans.* 43,746,840 *yards.*

2. A man bought a farm of 235 acres, at 56 dollars an acre; a second, of 320, at 60 dollars; a third, of 628, at 27 dollars; and a fourth, of 497, at 39. How much did he pay for the whole? *Ans.* 68,699 *dollars.*

3. A locomotive runs, every day, 16 miles an hour, for twelve hours. How many miles has it travelled in three years? *Ans.* 210,240 *miles.*

4. 238 chests of tea, of 45 pounds each, at 2 dollars per pound.
98 bags of coffee, at 23 dollars per bag.
56 firkins of butter, at 19 dollars per firkin.
129 hogsheads of molasses, at 27 dollars.
What is the whole amount? *Ans.* 28,221 *dollars.*

5. 21 pieces of cloth, of 36 yards each, at 7 dollars a yard.

49	"	"	67	"	9	"
37	"	"	29	"	5	"
58	"	"	73	"	13	"
69	"	"	59	"	6	"

How many pieces, how many yards, and what is the whole cost?
Ans. 234 *pieces,* 13,417 *yards,* 119,672 *dollars.*

In the first question, the amount of yards in *one degree* is given, and that for 360 to be found; in the second, the cost of *one acre,* and that of many required; in the third, the run in *one hour* is given, and that in the number of hours contained in three years, required.

In one word, each question gives the amount for *one,* and requires it for *many;* showing the operations to be multiplications.

LESSON XIV.

SIMPLE DIVISION.

1. *Signs:* $\frac{24}{4}$, or 24 : 4 = 6; read, 24 *divided by* 4, *is* equal to 6.

Symbols: $\frac{a}{b}=c$. $a : b = c$; *a divided by b, is* equal to *c*.

Questions.—John has 24 apples, which he divides equally among his 4 brothers. How many does he give to each?
Answer. 6 *apples.*

Or, 4 yards of cloth have cost 24 dollars. How much is it a yard? *Answer.* 6 *dollars.*

2. Questions like these give rise to Division, which, it will be perceived, is the inverse rule of *Multiplication.* In that operation, two numbers are given, and the aggregate of their product required; in this, the aggregate is known as well as one of the numbers, and the other is to be found. In other words, we see that

FIRST DEFINITION.

3. Division *is an operation by which the product and one of its factors being known, the other factor is to be found.*

In questions like the above, for example, we must ascertain how many times the small number 4 is contained in the larger, 24.

This might be done by successive subtractions of 4, by which it would be found that 4 goes 6 times in 24; and that, consequently, 6 is the answer.

But it is shorter to say at once, *In 24, how many times 4?* 6 *times*, is the ready answer. This method changes subtraction into *Division*.

As regards, therefore, the mere mechanism of the operation, our first impression of its character is that

SECOND DEFINITION.

4. Division *may be considered as a short method of Subtraction, by which it is found how many times a number is contained in another.*

In the above case, 4 is contained 6 times in 24. Hence, we have $\frac{24}{4} = 6$, for the answer.

But when, from the nature of the question, we wish to ascertain what operation is to be performed, it is more readily understood by reference to the First Definition.

For, whenever an aggregate amount is given, as well as one of the numbers which have produced it, we know, at once, that it is by division that the other is to be found.

We know, for example, that 4 brothers have 24 apples among them, and we wish to find the share of each.

Or, we know that 24 dollars is the total cost of 4 yards, and wish to find the cost of one; the answer in both cases, repeated 4 times, must produce 24.

In these examples, 4 and 6 are the factors of 24.

5. The number to be divided is the Dividend. It corresponds to the Product of multiplication.

The number we divide by, the Divisor.

The result is called, the Quotient. Both are the factors of the Dividend.

6. Consequently, *the dividend contains the divisor as many times as there are units in the quotient.*

24 contains 4 as many times as there are units in 6.

Hence, 1st. *When the divisor is* one, *the quotient is a number equal to the dividend.*

2d. *When the divisor is greater than* unity, *the quotient is a number smaller than the dividend.*

3d. *When the divisor is smaller than* unity, *the quotient is a number larger than the dividend.*

7. Sometimes the divisor does not go an exact number of times in the dividend, and there is left a number smaller than the divisor. This is called the *Remainder.*

If, for example, we divide 27 by 4, we get 6 to 24, and 3 over. 27 is the *dividend,* 4 the *divisor,* 3 the *remainder.*

8. We will consider two cases in division: the first, *Short Division,* when the divisor does not exceed 12.

The second, *Long Division,* when the divisor exceeds 12.

They both require a ready knowledge of the following

DIVISION TABLE.

2	in	2	once	3	in	3	once	4	in	4	once
2	in	4	twice	3	in	6	twice	4	in	8	twice
2	in	6	3 times	3	in	9	3 times	4	in	12	3 times
2	in	8	4 times	3	in	12	4 times	4	in	16	4 times
2	in	10	5 times	3	in	15	5 times	4	in	20	5 times
2	in	12	6 times	3	in	18	6 times	4	in	24	6 times
2	in	14	7 times	3	in	21	7 times	4	in	28	7 times
2	in	16	8 times	3	in	24	8 times	4	in	32	8 times
2	in	18	9 times	3	in	27	9 times	4	in	36	9 times
5	in	5	once	6	in	6	once	7	in	7	once
5	in	10	twice	6	in	12	twice	7	in	14	twice
5	in	15	3 times	6	in	18	3 times	7	in	21	3 times
5	in	20	4 times	6	in	24	4 times	7	in	28	4 times
5	in	25	5 times	6	in	30	5 times	7	in	35	5 times
5	in	30	6 times	6	in	36	6 times	7	in	42	6 times
5	in	35	7 times	6	in	42	7 times	7	in	49	7 times
5	in	40	8 times	6	in	48	8 times	7	in	56	8 times
5	in	45	9 times	6	in	54	9 times	7	in	63	9 times

8	in	8	once	9	in	9	once	10	in	10	once
8	in	16	twice	9	in	18	twice	10	in	20	twice
8	in	24	3 times	9	in	27	3 times	10	in	30	3 times
8	in	32	4 times	9	in	36	4 times	10	in	40	4 times
8	in	40	5 times	9	in	45	5 times	10	in	50	5 times
8	in	48	6 times	9	in	54	6 times	10	in	60	6 times
8	in	56	7 times	9	in	63	7 times	10	in	70	7 times
8	in	64	8 times	9	in	72	8 times	10	in	80	8 times
8	in	72	9 times	9	in	81	9 times	10	in	90	9 times

11	in	11	once	12	in	12	once
11	in	22	twice	12	in	24	twice
11	in	33	3 times	12	in	36	3 times
11	in	44	4 times	12	in	48	4 times
11	in	55	5 times	12	in	60	5 times
11	in	66	6 times	12	in	72	6 times
11	in	77	7 times	12	in	84	7 times
11	in	88	8 times	12	in	96	8 times
11	in	99	9 times	12	in	108	9 times

Questions.

$\frac{30}{5} =$, $\frac{44}{11} =$

$\frac{72}{9} =$, $\frac{64}{8} =$

$\frac{56}{7} =$, $\frac{55}{11} =$

$\frac{63}{7} =$, $\frac{60}{12} =$

$\frac{96}{12} =$, $\frac{84}{12} =$

$\frac{35}{7} =$, $\frac{21}{3} =$

$\frac{25}{5} =$, $\frac{28}{4} =$

$\frac{49}{7} =$, $\frac{42}{6} =$

$\frac{48}{6} =$, $\frac{32}{4} =$

Questions.—What is the sign of division? Give an example. What questions lead to division? What is division? First definition? Second definition? What is the dividend? The divisor? The quotient? The remainder? Give examples. How many times does the dividend contain the divisor? What is the quotient when the divisor is 1? More than 1? Less than 1? How many cases in division? In . . . . how many times . . . .? What is . . . . divided by . . . .?

EXERCISES.

1. A man travels 12 miles a day; how long will it take him to travel 144 miles? The aggregate amount is given, and one of its factors; the other to be found. Hence the operation is *Division.*

2. A man gets 56 dollars for 8 sheep. How much a sheep? The amount for several is given, that for 1 to be found: *Division.*

3. How many oranges can be bought for 84 cents, at 6 cents apiece? The product and one factor known; the other to be found: *Division.*

4. A hatter has to pack 72 hats, and can put 9 in a box. How many boxes will he want? The whole amount, 72, is known; 9

must be one of its factors; the other factor is the answer: *Division.*

5. Four persons make up 132 years with their joint ages. What is the average of their ages?

6. 7 men bought 25 horses for 2000 dollars, and sold them for 1923. How much did each lose?

7. John has 47 apples, and gives 8 to each of 5 children. How many has he left?

8. Such questions as these are to be asked:
In 84, how many times 5? and how much over?

9. How much is 87 divided by 9? and how much over?
Varying the numbers in both.

10. A man paid 48 dollars for 8 sheep, and 99 for 9 cows. How much more did he pay for each cow than for one sheep?

LESSON XV.

SHORT DIVISION.

1. QUESTION.—*To divide* 4326 *dollars equally between* 7 *persons.*

In order to obtain a correct idea of the operation, we must consider this number as composed of the following parcels:

4 of one thousand dollars each.
+ 3 of one hundred.
+ 2 of ten.
+ 6 of one.

And now we have to ascertain how many of each parcel or order the share of each person will contain.

Beginning with the thousands, as there are but four of them for 7 persons, they cannot have any entire thousand in their individual shares.

We must, therefore, pass on to the next inferior order, and change the higher units into smaller ones by considering that 4 thousands = 40 hundreds. These we join to the 3 in the given number, which makes in all 43 hundreds. These divided by 7, give 6 hundreds for each share, and 1 hundred over.

This hundred also must be divided: for this purpose, it is changed into 10 tens, which, joined to the two tens in the given number, make 12 tens; and these divided by 7, give to each 1 ten, and there remain 5 tens over.

The remainder, 5 tens, joined to the 6 units of the number, makes 56; and 56 divided by 7, gives exactly 8 simple units to each. So that the share of each person is composed of 6 hundreds,
1 ten,
and 8 units.
That is, collectively, 618 dollars.

2. The operation is performed in the following manner:

Place the divisor *on the right** of the dividend. Separate them by a vertical line, and draw a horizontal line under the dividend.

4326	7
618	

And, commencing on the left, say: 7 *in* 4 *is nought.* Then take together the first two figures as one number, and say: 7 *in* 43, 6 *times, and* 1 *over* (set down the 6 under the hundreds, of which order it is);

7 *in* 12, 1 *time, and* 5 *over* (set down 1 under the tens).

7 *in* 56, 8 *times* (set down 8 under the units).

The answer is as above, 618.

REMARKS.

3. We see in this operation that we get out successively the units of the different orders, beginning with the largest, and that the division of *thousands, hundreds, tens*, and *simple units* by *units*, gives respectively in the quotient, *thousands, hundreds, tens*, and *units*, or, in general, that: *The division of a number of units of any order by any number of simple units, gives units of the same order in the quotient.*

* The divisor is more generally placed on the left, and the quotient on the right of the dividend. But the position of the divisor on the right and the quotient under it, is preferable, and I recommend it; because each partial product is more conveniently made by multiplying the divisor by a figure below it, than across the dividend. It has to be so arranged for the proof of division; and besides, this arrangement is used in algebra by all modern mathematicians.

4. In the preceding operations, we began at the right, because we might expect to have a large amount to carry to the next superior order. In *division*, on the contrary, we begin at the left because we may expect a remainder, which must be joined to the inferior order of units next to the right.

5. There is a case which must be particularly noticed. It is when some orders of units do not exist in the quotient. In that case, the place of the absent orders is occupied by zeros.

OPERATION.

3256048	by 8
407006	

Take, for example, the division of Since the first figure 3 of the order of millions is smaller than the divisor 8, there can be no millions in the quotient. Passing on then to the order of hundred thousands, we take the first two figures together, and say: *In* 32, *how many times* 8? 4 *times*, and we set down the 4 under the hundred thousands.

Then we say: 8 *in* 5 *is* 0, *and* 5 *over*. So that there is no ten thousands in the quotient; we set 0 in their place, and say:

8 *in* 56, 7 *times*. We set down 7 under the thousands.

Then, 8 *in* 0, 0; which we set down in the place of hundreds, to show that there is no figure of this order.

Again: 4 tens cannot be divided as such by 8. We set down 0 under the tens, and divide 48 units by 8, which gives 6 for the last figure.

So that the whole answer is 407,006, in which the figures 4 and 7 get their proper places and value by means of the 0.

6. Sometimes the division cannot be exactly made. Let us suppose that 35,681 dollars are to be divided between 6 persons.

OPERATION.

35681	6
quotient, $5946+\frac{5}{6}$	

In effecting the division, we find a quotient, 5946, and a remainder, 5. Now, it is clear that this remainder, 5 dollars, is also to be divided among the 6 persons. This could be done by converting it into smaller units, like cents, for instance. But, when this is not required, *the remainder is written after the quotient, and the divisor placed under it, with the sign of division between them, to show that the remainder also is to be divided.*

Questions.—What is short division? How should the number be decomposed? How are the numbers arranged in short division? Where do you commence to divide? Why? Explain the process. What is the order of the quotient, when you divide units of any order by simple units? When you divide millions? ten thousands? thousands? hundreds? &c., by units? When the units of any order do not contain the divisor, what is done with the quotient? Why? What is done with the remainder? Why?

EXERCISES.

1. Divide 16,992,360 by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
2. Also, 132,889,680 by the same.
3. " 36,005,472 by 2, by 3, by 4, by 8, by 9.
4. " 111,111,111 by 3, by 9.
5. " 1,430,400 by 7, = - - + Remainder 6.
6. " 6,730,214 by 10, = - - + " 4.
7. " 4,560,389 by 9, = - - + " 8.
8. " 6,080,519 by 12, =
9. " 66,297,439 by 11, =

LESSON XVI.

LONG DIVISION.

1. Long Division is performed as follows:

I. *Set down the divisor on the right* of the dividend. Separate them by a vertical line, and draw a horizontal line under the divisor.*

II. *Take at the left of the dividend just as many figures as will contain the divisor.*

* See preceding note. The operation thus arranged is more compact and convenient.

III. *Considering this number as a partial dividend, find how often it contains the divisor, and set the first figure of the quotient under the divisor.*

IV. *Multiply the whole divisor by this figure, and subtract the product from the first partial dividend.*

V. *Next to the remainder, bring down the following figure of the dividend; then find how often the divisor is contained in this second partial dividend, and write the new quotient on the right of the preceding one.*

VI. *Multiply the whole divisor by the new quotient, subtract from the second partial dividend, and bring down the following figure of the dividend; you have then a third partial dividend.*

VII. *Continue the operation in the same way until all the figures of the dividend have been brought down.*

VIII. *If there is a remainder, write it after the quotient, and the divisor under it* (XV., 6).*

2. To explain the above rule, let it be proposed:

To divide equally 557,793 *dollars between* 849 *persons.*

This amount may be considered, as was done in short division, to be composed of as many parcels as there are orders of units in it; and then we have to divide each parcel successively.

The first inquiry is, what will be the largest order of units or parcel in the quotient, and how many figures it will contain? It is evident that the quotient can contain *no hundred thousands,* since there are but 5 of them for 849; *no ten thousands,* because there are only 55 of them; and *no thousands,* since their number, 557, though composed of three figures, does not contain 849.

OPERATION.

```
557,793 | 849
5094    |----
-----   | 657
 4839
 4245
 -----
  5943
  5943
  ----
  0000
```

* The division of the remainder by the divisor thus indicated, is generally written in smaller figures than the quotient.

Hence, the first part of the number, which contains the divisor, must be composed of four figures, *one more* than the divisor. *This first partial dividend*, 5577, being of the order of *hundreds* (VI., 7), it follows, that the first figure in the quotient must also be of the order of *hundreds* (XV., 1 and 3), and that the quotient will contain three figures or orders of units, *hundreds*, *tens*, and *units*.

This first investigation is very important, and should always precede the operation, to guard against mistakes.

Now, dividing the first partial dividend, 5577 *hundreds*, by 849, we find for the first share 6 *hundreds*, which we set in the quotient.

Then the multiplication of 6 *hundreds* by the divisor, 849, gives a product of 5094 *hundreds*, for the first total share or parcel to be taken from 5577.

The subtraction leaves a remainder, 483 *hundreds*.

3. Having now disposed of the *hundreds*, let us proceed to ascertain how many *tens* the quotient will contain. For this purpose, we must convert the 483 *hundreds* into *tens*. We know that they are equal to 4830 *tens*, to which the 9 of the general dividend must be added; making, in all, 4839 *tens*.

This is the reason why 9 is brought down next to 483. Its annexation, according to the principles of numeration, makes at once a number, 4839, of tens (VI., 7).

The second partial dividend, 4839 *tens*, divided by the units of the divisor, will give us 5 *tens* in the quotient; which, multiplied by the divisor, make 4245 *tens*, the second share or parcel to be subtracted from the dividend; leaving a remainder, 594 *tens*.

By the side of this, now, we bring down the 3 units of the general dividend, and we form the number of 5943 *units*.

This last partial dividend divided by 849, gives exactly 7 *units* in the quotient; the 7 *units*, repeated 849 times and subtracted, leaving no remainder.

So that the whole quotient is 657.

Questions.—What is long division? Repeat the rules. How are the numbers set down? How many figures do you take for the first partial dividend? Of what order will the first figure of the quotient be? How many figures will it contain? Why is the next figure of the dividend brought down?

EXERCISES.—Use those given in Multiplication.

LESSON XVII.

1. THIS Lesson contains some practical methods to facilitate the operation of Division.

In the first place, we have implicitly admitted that the quotient of each partial dividend could be readily ascertained. But this is not always the case when the divisor is large; then,

The figure in the quotient for each partial dividend is found by trials as follows:

If the second figure of the divisor does not exceed 5, *see how often its first, or first two figures are contained in the first part of the dividend; the result will probably be the required figure of the quotient.*

If the second figure of the divisor is over 5, *add one unit to its first figure, and try it with this addition.*

2. The reason of this is, that, as it would seldom be possible to tell at once how often the whole divisor will go in a dividend, an approximation is obtained by taking one or two figures at the beginning of each for the whole number, since the size of a number depends chiefly on its first figure. (VI., 3)

In the annexed example we try 8 in 55; but it must be recollected that it is in reality 800 in 5500; and we try 8, because 849 is nearer to 800 than to 900.

But if the divisor were over 850, for example 879, its value being then nearer to 900 than to 800, the approximation would be greater with 9 than with 8.

OPERATION.

557793	849
5094	657
4839	
4245	
5943	
5943	
0000	

In many cases, like the present, there can be no doubt as to any figure in the quotient. It is evident that 8 will only go 6 times; for, 5 multiplied by 8 would leave a remainder out of 55 nearly double 8; and 7 would give the product 56.

In the second partial dividend, 4839, though 48 is 6 times 8, 6 is too large, because the product of 6 by 4 would give something to be carried in addition to 48.

In the third partial dividend, 5943, the trial with 8 in 59 gives evidently 7, since the product of 6 by 8 taken from 59 would leave a remainder much larger than 8, while 8 would give 64.

3. But the conclusions are not always so evident; and it frequently happens that the first trial is not successful. This is generally discovered after the divisor has been multiplied; and it is clear that:

I. *When the product of the divisor by the quotient is larger than the partial dividend, the quotient is too large.*

II. *When this product, subtracted from the partial dividend, does not leave a remainder smaller than the divisor, the quotient is too small.*

If, we had, for example, to divide 2046 by 256, we might be tempted to try 9, and then 8; both of which would be found to give products larger than the divisor.

If, applying the 2d rule, we divide 20 by 3, instead of 2, then the quotient 6 would give the product 1536 and remainder 510, larger than the divisor, showing that 6 is too small.

It is true, that, since the larger the parts of the dividend and divisor used in the trial, the more certain the quotient will be, we might have tried 25, which in 206 goes 8 times, and thus escaped the trial of 9; but, even then, the apparent quotient 8 would have failed.

4. In order to avoid these useless mutiplications and subtractions, it is always best, before multiplying, to verify, *by short division*, the quotient figure, as follows:

Divide the dividend by the largest probable quotient figure, comparing successively each figure of the result with the corresponding one of the divisor, until you get a figure either smaller or larger than in the divisor. In the first case, the quotient is too large; in the other it is right.

This method has over the common method the advantage that it is quicker, more certain, and does not require to set down any number, since every figure you get is the same as in the divisor, until you stop.

For instance, in the division of 1269231 by 158654, we try 9 first, and say: 9 in 12, 1 (as the divisor).

9 in 36; 4 (smaller than the second figure in the divisor). We are at once certain that 9 is too large, since it is contained in the dividend a number of times less than the divisor.

Trying, now, 8, the division gives us successively the following figures of the divisor: 1, 5, 8, 6, 5; but, for the last, we get 3 instead of 4. Then, again, the number which should multiply 8 to make the dividend, is smaller than the divisor; consequently 8 is also too large.

Passing on to 7, and dividing, we get 1, as in the divisor for the first figure; but, for the second, we get 8, which is larger than the second figure, 5, of the divisor, and shows at once that 7 is right; since a greater number than the divisor being contained 7 times, the divisor itself must be contained.

5. *Secondly.* A case frequently occurs in division, which, though comprised in the general rule, should be noticed separately. It is when some orders of units are absent in the quotient, as in this example:

261089975	43486
260916	$6004\frac{31}{43486}$.
... 173975	
173944	
.... 31	

In the first place, it requires six figures of the dividend to contain the divisor, the last of them being of the order of *thousands;* it follows that the first figure of the quotient will be of the order of *thousands,* and the quotient must have four figures.

Now in the first partial dividend, 261089, because 26 contains 4 at the most 6 times, and must contain it more than

5 times, since the product 20, of 5 by 4, would leave a remainder much larger than the divisor, it follows, without trial, that 6 is the amount of thousands in the quotient.

Multiplying, then, the divisor by 6, we have to subtract the first share, 260916, and find a remainder of 173 *thousands* still to be divided.

Now we have to find the hundreds of the quotient. For this purpose, we change the 173 *thousands* of the dividend into *hundreds*, by annexing the figure 9 of the hundreds (XVI., 3).

The second partial dividend, 1739 *hundreds*, does not contain the divisor, 43486; consequently we have no hundreds in the quotient, and show it by writing a 0 in the place of *hundreds*.

In order, now, to get the *tens*, a third partial dividend, 17397 *tens*, is formed by bringing down 7 tens next to the last number, and we see at once that this number is likewise smaller than the divisor. Hence, also, there are no *tens* in the quotient, which we show by another 0 in their place.

Finally, the last partial dividend of units is formed by bringing down the 5 units: 4 units are obtained in the quotient, and a remainder, 31, is left, whose division by the divisor is indicated, as appears in the result.*

FIRST REMARK.

6. *When the divisor is 1, followed by ciphers, as 10, 100, 1000, &c., cut off from the right-hand of the dividend as many figures as there are ciphers in the divisor. The figures on the left will be the quotient; those on the right, the remainder.*

Example.—Divide 9865 by 100. *Ans.* $98\frac{65}{100}$.

The reason of this is evident; for each figure is thus removed two places to the right, and becomes *one hun-*

* That 4 is the right quotient, is also obvious at the mere inspection of the number, since 5 times 4 would be larger than 17, and 3 times 4 would leave a remainder, 5, larger than the divisor.

dred times smaller; that is, the 8 *hundreds* are now 8 *units*, and the 9 *thousands* are 9 *tens*.

Our notation expresses that the remainder, 65, also is to be divided by 100.

Nothing is more calculated to produce an impression that a person is ignorant of the very elements of arithmetic, than to set a regular operation in this case.

Division by 1, is evidently the number itself.

SECOND REMARK.

7. *When the divisor is any other number followed by ciphers*, a similar rule is given in all arithmetics; but I have seen so many errors committed in applying it, that it had better be deferred until we come to Decimal Fractions (XXVII).

Questions.—How do you find the figure in the quotient? What do you do when the second figure of the divisor is 5 or more? Why? When the product of the divisor by the figure in the quotient exceeds the dividend? And when the remainder is equal to or larger than the divisor, what do you conclude? What is the best mode of trial in doubtful cases? What do you conclude in trials, when you come to a figure smaller than that in the divisor? When to one larger? Why? When a partial dividend is smaller than the divisor, what do you do? What is done with the remainder? When the divisor is 1, followed by ciphers, how do you divide? Why?

EXERCISES.

1. 13312 eggs are to be packed in 52 kegs. How many eggs in each keg? *Ans.* 256 *eggs.*

2. A person buys land for 17575 dollars, at 37 dollars an acre. How many acres does he buy? *Ans.* 475 *acres.*

3. What number, multiplied by 422, will produce 253622? *Ans.* 601.

4. A railroad receives 666490 dollars per year. How much is it a day? *Ans.* 1826 *dollars.*

5. The light of the sun passes from the sun to the earth, a distance of about 95,000,160 miles, in 8 minutes. How much is that per second? *Ans.* 197917 *miles.*

6. The whole pay of a number of soldiers is 238080 dollars; each soldier gets 48 dollars. How many soldiers are there? *Ans.* 4960 *soldiers.*

LESSON XVIII.

NATURE OF THE QUOTIENT.

1. This is the place to introduce a remark of some importance, in regard to the *nature of the quotient.*

It has been said in multiplication that the product is always of the same nature as the multiplicand (XI., 7).

2. In division, the *dividend* is *a product*, of which the *divisor* and *quotient* are the *factors.* According to the nature of the question, either of them may be the multiplicand, and therefore of the same nature with the dividend. It follows that,

I. *The quotient is of the nature of the dividend when the divisor is not.*

II. *But when the dividend and divisor are of the same nature, the units of the quotient may be anything else.*

EXAMPLE OF THE FIRST CASE.

For 24 *dollars* 4 *yards* were bought; how much is it a yard? *Ans.* 6 *dollars.*

Here the dividend and divisor are different, and the quotient necessarily of the nature of the dividend.

EXAMPLE OF THE SECOND CASE.

At 4 *dollars* a *yard*, how many yards can be bought for 24 *dollars?* *Ans.* 6 *yards.*

Here the dividend and divisor are both *dollars;* the quotient is different from both.

3. It is an error to imagine that in the first example we divide *dollars* by *yards*, and in the second *dollars* by *dollars.*

In both cases, after having ascertained, from the question, of what nature the units of the quotient are to be, we find their number by the division of *abstract numbers,* without further reference to their nature.

It is true that, in the first case, it would not be precisely an error to say that we divide 24 *dollars* by 4, and get 6 *dollars;* but it is by 4, and not 4 *yards*, we divide. *The divisor is invariably an abstract number.*

In the second case, however, it would be a real absurdity to say that we divide *dollars* by *dollars*, and get *yards*. We want *abstractly* to find a number of units, designated by the question, equal to the number of times that 4 is contained in 24, without reference to their nature.

Singular as it may appear, the question of dividing a number of units by units of the same kind, such as dollars by dollars, is by almost everybody answered by a number of units of the same nature; when, in fact, it is rarely so.

4. In the few applications which follow the 17th lesson, this principle is further elucidated. In the second and sixth, the dividend and divisor are of the same nature; the quotient differs from both.

In the others, the dividend and divisor being of different kinds, the quotient is of the nature of the dividend.

The third question relates only to abstract numbers.

It may be further remarked that, in practice, no question in multiplication or division is complete unless three quantities are given, one of which is a single unit, whose nature serves to determine that of the answer. In each of the preceding applications, this unit will readily be discovered.*

PROOF OF DIVISION.

5. *Multiply the quotient by the divisor, and to this product add the remainder, if there be any. The sum should be equal to the dividend.*

---

* Multiplication and division are in fact *rules of three*, in which one of the terms of the proportion is a unit of the nature of one of the numbers.

EXAMPLE.			PROOF.	
Dividend,	5,778	25 divisor	231	quotient.
	50	$231\frac{3}{25}$ quotient.	25	divisor.
	.77		1155	
	75		462	
	.28		+ 3	remainder.
	55		5,778 =	dividend.
Remainder	.3			

It is evident that, since the divisor is taken from the dividend a number of times equal to the quotient 231, and an excess 3 is left, the repetition of the divisor 231 times, adding in the remainder, must reproduce the dividend.

And, because it makes no difference which of two factors is taken for the multiplier, we prefer the divisor, in order that, in making new products, we may more certainly detect errors, which going over the same multiplication would not be as likely to do.

6. It must be observed, that $\frac{3}{25}$ is intrinsically a part of the quotient, and that we virtually multiply by that also when we add 3; for 3 divided by 25, is the same as the 25th part of 3, and a 25th part repeated 25 times, must make the whole; that is, 3.

PROOF OF MULTIPLICATION.

7. Division is the most certain proof of multiplication. *Divide the product by either the multiplicand or multiplier; the other must be the quotient.* For, either factor expresses how often the other is contained in the product.

EXAMPLE.

Multiplicand,	2564	
Multiplier,	47	
	17948	
	10256	
Product,	120508	47 divisor = multiplier.
	94	2564 quotient = multiplicand.
	265	
	235	
	300	
	282	
	188	
	188	
	000	

Questions.—Of what nature is the quotient, when the dividend and divisor are of different natures? When they are alike? Give examples. How, in the operation, are those numbers, and especially the divisor, considered? How many quantities must be known in multiplication and division, to determine the nature of the answer? How do you prove division? Why? How do you prove multiplication by division? Why? Is there any analogy between these operations and the rule of three? (See note.)

LESSON XIX.

QUICK DIVISION.

1. After the student has acquired sufficient facility in division, he may greatly abbreviate it, both as regards time and space, by subtracting at the same time that he multiplies.

RULE.

Multiply successively, by the figure in the quotient, each figure of the divisor, and subtract the product from the corresponding figure in the dividend.

If the product is larger than the figure in the dividend, subtract this product from the next number above it, ending with the figure in the dividend; and then carry the tens thus borrowed to the product of the quotient by the next figure of the divisor, to be subtracted together with this product, and go on with the operation in the same way to the end.

EXAMPLE.

557793	849
4839	657
5943	
....	

Say, in 55 how many times 8; 6 times, and set 6 in the quotient.

6 times 9 are 54; *from* 57 (which is the next number above 54, *ending with the figure* 7) *leaves* 3, *and* 5 *to carry.* Set down 3.

6 *times* 4 *are* 24, *and* 5 carried, *are* 29; 29 *from* 37, *leave* 8, *and* 3 *to carry.* Set down 8.

6 *times* 8 *are* 48, *and* 3 carried, 51; *from* 55, *leave* 4. Set down 4.

Now bring down 9, and say: 8 *in* 48; it goes only 5 times. Set down 5 in the quotient, and proceed as before.

5 *times* 9 *are* 45; *from* 49 *leave* 4, *and* 4 *to carry*, 5 *times* 4 *are* 20, *and* 4 carried, 24; 24 *from* 33, *leave* 9, *and* 3 *to carry*, &c., to the end. The final quotient is 657.

2. This method of subtracting and carrying, is only an extension of that used in subtraction. We have to subtract the first product, 54, from 7; and, in order to render the subtraction possible, we take the number ending with 7, next above 54, which is 57, and subtract.

But, in doing so, we have virtually added 50 to the figure 7 of the partial dividend; and, in order not to alter the remainder, we must also add the same amount to the number we subtract, which we do, as 5 units of the next higher order joined to the product 24.

Thus we get 29; and, as this cannot be subtracted from 3, we take the number ending in 3 next above it, which is 33, and subtract, &c.

A little practice will make this method familiar, and it will prove much more convenient to a correct calculator.

It will be used exclusively in subsequent examples.

Questions.—In what does quick division differ from common division? How do you subtract? What number do you subtract from? What do you carry? Give the demonstration. Repeat the rule. What is the advantage of this method?

EXAMPLES FOR PRACTICE IN DIVISION.

1. 8,760 : 24 = 365.
2. 7,629 : 32 = ... $+\frac{13}{32}$.
3. 40,200 : 75 = 536.
4. 274,036 : 48 = ... + 4 Rem.
5. 257,355 : 35 = 7,353.
6. 599,359 : 78 = ... + 7 Rem.
7. 4,280,822 : 91 = 47,042.
8. 750,969 : 75 = ... + 69 Rem.
9. 5,616,072 : 84 = 66,858.
10. 1,195,030 : 99 = ... + 1 Rem.

11. 15,076,944 : 72 = 209,402.
12. 12,864,175 : 32 = ... + 15 Rem.
13. 46,508,928 : 96 = 484,468.
14. 999,999,999 : 81 = . . .
15. 252,144 : 412 = 612.
16. 136,714 : 534 = ... + 10 Rem.
17. 2,777,848 : 487 = 5,704.
18. 807,000 : 394 = ... + 88 Rem.
19. 511,425,396 : 5,847 = 87,468.
20. 405,768,567 : 50,406 = . . . + 267 Rem.
21. 436,940,074 : 64,237 = 6,802.
22. 9,638,789 : 3,456 = . . . + 5 Rem.
23. 270,974,414,340 : 276,390 = 980,406.
24. 33,188,281,929 : 31,456 = . . + 9 Rem.
25. 103,805,467 : 38,446 = . . . + 1267 Rem.
26. 57,824 : 10 =
27. 694,359 : 100 =
28. 89,045,000 : 1000 =
29. 6,472,001 : 1000 =
30. 573,070,698 : 6,587 = . . . + 1698 Rem.
31. 921,253,442,978,025 : 918,273,645 =
32. 435,603,662,775 : 870,496 = . . + 500,407 Rem.
33. 352,107,028,000 : 1,672,940 = 210,472 $+ \frac{320}{1672940}$.

The examples with answers, may also serve for multiplication.

Those without answers are intended for exercise on some particular difficulty, which would be removed to the disadvantage of the student, if the quotients were given.

The above examples may be varied by changing remainders In the beginning, those given in multiplication may be used.

* * * *

PROOF OF QUICK DIVISION.

In proving *quick division,* you may multiply the divisor by the quotient; as, in this case, there is not the objection noticed in Lesson XVIII., 5; since here the partial products are not written down.

* * * *

CHAPTER III.

CONTAINING DECIMAL FRACTIONS.

LESSON XX.

DECIMAL FRACTIONS.

SUPPLEMENT TO NUMERATION.

1. In the preceding lessons we have considered only *whole numbers* or *integers;* that is, numbers composed of whole units.

This lesson will introduce the subject of fractions.

2. The word FRACTION implies, *a part of any thing.* The meaning is only *relative* to that *thing;* for, a fraction of a thing may be a whole number, in regard to another. For example:

A *ten cent* piece is a *fraction* of a *dollar;* it is a whole number, when compared to *one cent.*

A *company* of men is a whole number of *men;* it is a *fraction* of a *battalion.*

The *battalion* itself is a fraction of the *regiment,* and that only a fraction of the *army.*

An inch, a foot, a yard, are all *units;* yet the inch is a *fraction* of the *foot,* the *foot* of a *yard,* and the *yard* itself of a *mile.* They are all relative units.

3. *Decimal fractions* (or simply *decimals*) are those which express a division of any unit into ten *equal* parts.

4. They are formed by an extension of the decimal system of numeration and notation to a descending scale below simple units.

We have seen, in numeration, that units increase in value ten times for every place they are removed to the left.

Evidently, therefore, every order of units is ten times smaller for every place that it stands removed from the one on its left.

Thus, while one thousand is ten times larger than one hundred, this is ten times smaller than one thousand.

Again, one thousand is equal to one hundred tens; hence, one ten, the tenth part of a hundred, is the hundredth part of a thousand, &c.

The unit is the tenth part of one ten, the hundredth part of one hundred, the thousandth part of one thousand.

Viewed, therefore, in reference to superior units, inferior ones are truly *decimal parts.*

But there is nothing to prevent us from carrying the descending scale *by tenths*, below the *simple unit of comparison*, and, upon the same principle, to consider

The 1st order to the right of simple units as composed	-	of *tenths* of that unit.
The 2d,	- -	of *hundredths.*
The 3d,	- -	of *thousandths.*
The 4th,	- -	of *tenths* of thousandths.*
The 5th,	- -	of *hundredths of thousandths,*
&c.		&c.

5. The corresponding notation exhibited in the following table, is as simple as the principle itself:

Ten millions, &c.	20000000.	
Millions	2000000.	
Hundred thousands	200000.	
Ten thousands	20000.	
Thousands	2000.	
Hundreds	200.	
Tens	20.	
Units	**2.**	
	.2	tenths.
	.02	hundredths.
	.002	thousandths.
	.0002	tenths of thousandths.
	.00002	hundredths of thousandths.
	.000002	millionths.
	.0000002	tenths of millionths, &c.

* There exists frequently some ambiguity as regards the reading of decimal fractions. In some approved Arithmetics, we find *ten thousandths;* in others, *tens of thousandths.* Both of which I think incorrect; for the unit is certainly *one-tenth of one-thousandth,* but

6. The ascending and descending scales are separated by a dot, usually called the *decimal point*, but which I believe it would be better to call the *units' point.* I shall frequently use this expression.* *It is placed to the right of the units and before the tenths.*

7. *The value of decimal units, like that of whole numbers, depends on the place they occupy; that is, on their distance from the simple units.*

In order to fix the place of decimal figures, 0s are made to occupy the place of absent orders.

The successively decreasing decimals in the diagram explain this extension of the system of notation.

We can now read with ease any simple decimal fraction: 0.4 or .4 is 4 tenths.
0.06 or .06 is 6 hundredths.
0.009 or .009 is 9 thousandths.
0.0005 or .0005 is 5 tenths of thousandths, &c.

The 0 in the place of units is at pleasure used or omitted, since the decimal or units' point, made very distinct, is sufficient to fix the place of each order of figures.

Questions.—What is a fraction? Is a fraction absolute or relative? Give examples. What are decimals? How do the figures of decimals compare with each other? May a whole number be considered as a fraction of another? Give examples. What is the descending decimal scale? The 1st, 2d, 3d, &c., order? How are whole numbers separated from decimals? How is any order indicated when there are absent units? Read, 0.05; 0.00002; 0.0000003, &c. Is 0 indispensable, when simple units are absent? Why?

LESSON XXI.

1. Let us now read a decimal number composed of several figures; for example,

---

not ten times one thousandth. Besides, how could 400 *thousandths* in this way be distinguished from 4 hundred thousandths? The reading, 4 *hundredths of thousandths*, is not only correct, but removes all ambiguity.

* On the continent of Europe, the comma is used to separate decimals, and a dot to divide periods. In the United States, the decimal point is generally adopted, and the comma is employed to divide the periods of numbers. I think this notation preferable to the European.

0.0084593

This number contains:
- 0 tenths,
- 0 hundredths,
- 8 thousandths,
- 4 tenths of thousandths,
- 5 hundredths of thousandths,
- 9 millionths,
- 3 tenths of millionths.

But the remark of (VI., 7) enables us to read it off as a single number, by giving it the denomination of the last figure, without naming the others; that is, eighty-four thousand five hundred and ninety-three *tenths of millionths* (84,593 tenths of millionths).

For, the successive figures stand in regard to the respective value of their units, according to the decimal scale, and therefore may be considered as one number.

2. Sometimes a whole number is accompanied by decimals. It is then called a *mixed or fractional number;* as, for example, 43.29.

Which may be read, either 43 units and 29 hundredths, or 4,329 *hundredths;* for, it is evident that the decimal relation exists between all these figures,

9 being a number of units of the order of hundredths.
2 is equal to 2 tens of them,
3 " to 300 of them,
and 4 " to 4000 of the same.

N. B.—When an abstract decimal number is read into two parts, the word *unit* should never be omitted after the whole number; otherwise there would sometimes be ambiguity.

For example, how could 200.004 be distinguished from - 0.204?
both reading *two hundred and four thousandths;* unless, after 200, the word *units* be introduced, and we read 200 *units* and 4 *thousandths,* which removes all doubts.

3. In the case of denominate numbers, the denomination being always given after the units, there can never be any uncertainty. If they were yards, for instance, we would read

200 *yards* and 4 thousandths (of a yard).

4. The writing of decimal fractions does not present any greater difficulty than the reading:

I. *Write the decimal number as it is spoken, just as you would write a whole number; and cut off by the units' or decimal point as many figures as the denomination of the last figure requires.*

II. *If there be not as many figures, complete their number by ciphers between the first significant figure and the decimal point.*

EXAMPLES.

1st. *Write two hundred and thirty-four hundredths of thousandths.*

The denomination being of the 5th order of decimals, we must have five decimal figures.

We write, first, 234, and, to move 4 to the 5th place, we prefix two 0s, and then the decimal point, which gives - 0.00234, or .00234.

2d. *Write* 9 *thousand and* 35 *hundredths.*

Set down, first, 9035, and, because the last denomination is *hundredths*, cut off two figures, showing that the number is 90.35; equivalent to 90 whole units and 35 hundredths.

5. The first example shows that the introduction of each 0 between the decimal point and the significant part of a fraction, reduces it ten times; for, it removes all its figures one place to the right.

6. In general, in numbers attended with decimals, the position of the units' point determines the place of the entire units; and hence the place and value of every significant figure. Consequently,

I. *A number is increased ten times for each place the units' point is removed to the right.*

II. *A number is decreased ten times for each place the units' point is removed to the left.*

Since, in the first case, each figure ascends as much, and, in the second, descends as many places.

7. *Zeros*, in both cases, are added, when the number of figures is not sufficient.

EXAMPLE.

		23.56
increased	10 times, =	235.6
	100 times, =	2356.
	1000 times, =	23560. with the addition of 0.
	10,000 times, =	235600. with two 0s, &c.

23.56 decreased	10 times, =	2.356
	100 times, =	.2356
	1000 times, =	.02356, with one 0 prefixed.
	10,000 times, =	.002356, with two 0s, &c.

These examples show the great importance of putting the decimal point in its right place. It is, indeed, the only difficulty in operations with decimals.

8. Another very important principle is the following:

The addition of any number of 0s after a decimal fraction does not alter its value.

For, *it is clear that, so long as the position of the units' point is not changed, the value of all the figures also remains unchanged.*

	units		tenths	hundredths				
For example, to	9	.	7	5				
	units		**tenths**	**hundredths**				
Let us add several 0s; thus,	9	.	7	5	0	0	0	0.

The 9 which precedes the units' point, remaining *units*, the 7 remains *tenths*, and the 5 *hundredths*, as before.

The reading only of the number is changed from 9.75 hundredths to 9,750,000 millionths, which is the same thing, since 9 units = 9,000,000 millionths.

And 75 hundredths = 750 thousandths = 7500 tenths of thousandths = 75,000 hundredths of thousandths = 750,000 millionths.

In other words, the addition of four 0s makes the units of the fraction 10,000 times smaller, but compensates it by increasing their number 10,000 times.

9. *Conversely, zeros at the right hand of a decimal fraction may be struck out without altering its value.*

Questions.—How do you read decimal fractions? Why? What two ways are there to read a number accompanied by decimals? Why? What should not be omitted when read in two parts? Why? How do you write decimals? What do you do when the figures of the decimal number are not as many as the places of decimals? What is the effect of each 0 placed after the units' point, before the significant part of decimals? What is the effect of moving the decimal point to the left? To the right? Why? How do you increase a number 10, 100, 1000 times, &c.? How do you decrease it 10, 100, 1000, &c., times? What is done in either case, when the number of its figures is not equal to the requisite places? What effect have zeros added to the right of a decimal fraction on its value? On its reading? Why? What effect will the striking out of zeros, at the end of a decimal fraction, have on its value?

EXERCISES.

Read and write in words:

.6	.9661	.40789	.910457	.2100007
.29	.0753	.10101	.061042	.0102003
.04	.0107	.60606	.002004	.00001072
.106	.0025	.02149	.030303	.0000000209
.251	.0016	.00301	.000561	.0000600007
.019	.0009	.00036	.000022	.0000000008
.002	.3472	.00007	.000005	.00000000017
.090	.0106	.01009	.030034	.00700009005

Read both ways, viz., as whole and as mixed numbers:

45.05	3000.0009	400.036
70.007	.3009	.436
200.006	24.0008	6007.001001001
0.206	3650.009008	805.0009009

7

Write

1. Six tenths.

2. Four tenths and six hundredths.

3. Nine hundred and nine thousandths.

4. One thousand, five hundred and fifty-seven tenths of thousandths.

5. Nine hundred and eleven hundredths of thousandths.

6. Fifty-eight tenths of thousandths.

7. Four hundred and four hundredths of thousandths.

8. Six thousand and seven millionths.

9. Eight hundred thousand two hundred and seven millionths.

10. Six thousand two hundred and five tenths of millionths.

11. Four hundred and three millionths.

12. Six hundred and six thousand and six millionths.

13. Seven million two hundred and fifteen thousand and eight tenths of millionths.

14. Ten units and two hundredths.

15. Two hundred and one units and twenty-five thousandths.

16. Six hundred and forty-four tenths.

17. Nine thousand and ninety-nine thousandths.

18. Seven hundred thousand units and seven hundredths of thousandths.

19. Five million six hundred and sixty-seven thousand and eighty-seven tenths of thousandths.

20. Two million three hundred units and fifty-four tenths of millionths.

21. Sixty millions and five millionths.

22. Twelve billions of millionths.

23. Sixty-four units and seven hundredths of thousandths.

24. Six thousand units and two hundred and three tenths of thousandths.

25. Six thousand two hundred and three tenths of thousandths.

1. Increase 63.756 ten times, one hundred times, one thousand times.

2. Increase .6897, 10; 100; 1000; 10,000; 100,000 times.

3. Multiply 4.652 by 10; 100; 1000; 10,000; 100,000.

4. Increase .05 ten times, .06 hundred times, .07 one thousand times.

5. Multiply .006 by 10; .007 by 100; 0.005 by 1000; .003 by 10,000; 0.004 by one million.

6. Increase .000009, one hundred thousand times.

" .00005, one million of times.

7. Decrease 25.72, 10; 100; 1000; 10,000 times.

8. " .69, 10; 100; 1000 times.

9. Divide 1.05 by 100; 2.62 by one thousand.

10. " 0.007 by ten; 0.0008 by ten thousand.

LESSON XXII.

ADDITION AND SUBTRACTION OF DECIMALS.

1. The principles that only units of the same kind can be added or subtracted, applies evidently to *descending* as well as to *ascending* orders. Therefore,

In order to add or subtract numbers with decimals, set down the numbers so that the units' points may all be in the same column, which will place units of the same order under each other.

Then add or subtract, as with whole numbers; be careful to place the decimal point, in the result, between the units and tenths.

N. B.—In both operations, it will be convenient, though not indispensable, to *equalize the number of decimals;* that is, by the addition of zeros to make the number of decimal figures of all the numbers equal to the greatest. This we are authorized to do by the remark of Lesson XXI., 8.

EXAMPLE IN ADDITION.

To add 29.0146 + 3,146.5 + 2,109 + .62417 + 14.16:

Without equalizing decimals.	The decimals equalized.
29.0146	29.01460
3146.5	3146.50000
2109.	2109.00000
.62417	.62417
14.16	14.16000
5299.29877	5299.29877

In long additions, the second arrangement is clearly less liable to mistakes.

EXAMPLES IN SUBTRACTION.

(1.)

Not equalized.	Equalized.
1.9185	1.9185
$-$ .625	$-$ 0.6250
= 1.2935	= 1.2935

(2.)

214.81	214.81000
$-$ 4.90142	$-$ 4.90142
= 209.90858	= 209.90858

(3.)

2,714.	2,714.00000
$-$ .00916	$-$ 0.00916
= 2,713.99084	= 2,713.99084

The second arrangement is particularly recommended to beginners.

Questions.—How do you add numbers with decimal fractions? How do you subtract? How do you arrange the numbers? Why? Where do you place the decimal point in the result? What is meant by equalizing the decimals? What is the advantage of it?

EXERCISES.

1. 376.25 + 7.125 + 9762.0047 + .62 + 77.0005 + 41. = 10,264.0002.

2. 3.5 + 47.25 + 2.0073 + 927.01 + 1.5 + .7327 = 982.

3. 276 + 54.321 + 112 + 0.65 + 12.5 + .0463 = 455.5173.

4. 235 + .00092 + 2.0415 + .6 + .07 + .00018 = 237.7126.

5. Add five units and nine tenths; sixty-nine hundredths; two units and eleven thousandths; twelve units and six hundredths; seventy-five thousand and four tenths. *Ans.* 7521.061.

6. Add two hundred and two units and twelve millionths; one hundred thousand and three tenths of thousandths; forty-five units and ninety-nine thousandths; two hundred and sixty-five units and six hundredths of thousandths; ten million six hundred and twenty-five tenths of thousandths. *Ans.* 1,522.161872.

7. 62.09 — 23.0784 = 8. 10.25 — 2.4512 =
9. 8 — .006714 = 10. 2 — .000001 =
11. 25 — 0.0091 = 12. 15.1 — 14.99999 =
13. From 1.05 take 19 tenths of thousandths.
14. From 6.0001, take 23 hundredths of thousandths.
15. From 2 thousandths, take 2 millionths.
16. From one unit, take one hundred and eleven tenths of thousandths.
17. From one thousand, take one thousandth.
18. From one hundred thousand units, take one hundredth of thousandths.
19. From two hundred units and five thousandths, take two hundred and five thousandths.
20. From two hundred tenths, take one hundred and ninety-nine thousandths.

LESSON XXIII.

SUPPLEMENT TO MULTIPLICATION.

1. It is necessary to introduce here some additional considerations in regard to multiplication and division, to facilitate the understanding of these rules, as applied to decimal fractions.

PROPOSITION I.

I. *If the multiplicand or multiplier be made a certain number of times larger or smaller, the product will be larger or smaller the same number of times.*

In the first place, as regards the multiplicand, we know that the product is equal to as many multiplicands as there are units in the multiplier; and, of course, if there are several such multiplicands to be repeated, the repetition of each giving the original product, the aggregate will be equal to as many products as there are multiplicands.

When I multiply 4 by 5, it is the addition of 4 five times, and the product is 20. But, now, if I made the multiplicand 4 3 times larger, it would be

$$4 + 4 + 4,$$

which, repeated 5 times, would be

$$\begin{array}{c} 4 + 4 + 4 \\ 4 + 4 + 4 \\ 4 + 4 + 4 \\ 4 + 4 + 4 \\ 4 + 4 + 4 \\ \hline = 20 + 20 + 20 \end{array}$$

the whole amount being equal to 3 times the original product; that is, as many times as the number of multiplicands.

If there are fewer multiplicands, there is less to repeat, and the product must be smaller in proportion.

For example,—I give 4 apples to each of 3 boys; it is in all the multiplicand 4 repeated 3 times, = 12 apples.

If, instead of this, I were to give but 2 to each, the new product would be the same as if I took away one-half of the share of each; which would reduce the whole amount to one-half of 12, or to 6 apples.

2. *Secondly*, as regards the multiplier, the principle may be understood to be the same from the remark (XII., 5) that if it be taken for the multiplicand, the product is the same.

Or, by considering that the product, being equal to as many multiplicands as there are units in the multiplier, the *increase* or *reduction* of this number of units, produces a corresponding *increase* or *reduction* in the product.

If, for instance, I have to give 4 apples to each boy that presents himself, I would give 12 apples to 3 boys; but, if 5 times this number of boys were to come, I should have to do the same thing 5 times; that is, to give $5 \times 12 = 60$ apples.

3. Hence, *when more convenient, we may multiply or divide the product instead of its factors.*

EXAMPLES.

I. I buy 125 yards of cloth, at 4 dollars = 500 dollars. Now, I want 7 times that quantity; instead of multiplying 7 by 125, and then by 4, I observe that 4 times 125 is 500, and I multiply at once $500 \times 7 = 3{,}500$ dollars.

II. Or else, I wish to change it for an equal quantity, 7 times dearer. Here, again, instead of multiplying 4 by 7, and then by 125, I say at once $7 \times 500 = 3{,}500$ dollars.

PROPOSITION II.

4. *If one of the factors be multiplied by a certain number, and the other be divided by the same, the product will remain unchanged.*

This practically useful principle is almost evident from what precedes, since one operation destroys the other.

Instead of buying 6 yards, at 4 dollars, you prefer to buy only 2 yards, at 12 dollars. In both cases, the cost is 24 dollars; for, in the second, you divide the quantity by 3, but increase the price 3 times.

5. This principle is useful in simplifying multiplication, in some cases, as will be seen hereafter (XLV.), and may also be employed to prove multiplication. Thus: I find $594 \times 437 = 259,578$; and remarking, that 594 can be divided by 3, I divide it, and multiply the other factor by 3, which gives me $198 \times 1311 = 259,578$, as before.

Questions.—If the multiplicand is made a number of times larger, how will the product be? Why? Give an example. Same questions for a smaller multiplicand; also for the multiplier. Is the result the same if you multiply the product instead of the multiplicand? Ditto, instead of the multiplier? When should either be preferred? How is the product when one of the factors is multiplied, and the other divided, by the same number?

LESSON XXIV.

SUPPLEMENT TO DIVISION.

1. Since, in division,

the dividend,	correspond,	*the product,*
divisor,	respectively,	*multiplicand,*
and *quotient*	to	and *multiplier*

of multiplication, it may be anticipated that the principles enunciated above in *multiplication*, have corresponding ones in *division*.

PROPOSITION I.

The divisor remaining the same, if the dividend be made a certain number of times larger or smaller (that is, if it be multiplied or divided by a certain number) *the quotient will be made the same number of times larger or smaller.*

For, a larger dividend must contain the same divisor a greater number of times, and a smaller dividend contain it a less number.

Also, the dividend being the product of the divisor by the quotient, must contain the divisor as many times as there are units in the quotient (XI., 3); consequently, the number of units of the quotient must be increased or decreased as many times as the dividend, in order that the new dividend may still be produced by multiplying the same divisor.

Example.—A basket containing 12 apples, divided among 3 boys, gives each 4 apples as a quotient. 5 such baskets would be 5 dividends; the division of which would give 5 times as many apples to each boy—that is, 5 times the first quotient 4, or 20 apples.

Again, one-half the number of apples gives each only one-half his former share; that is, 2 instead of 4 apples.

2. Hence, *we may multiply or divide the quotient, instead of the dividend, when it is more convenient.*

PROPOSITION II.

3. *The dividend remaining the same, if the divisor be made a certain number of times larger, the quotient will be made as many times smaller; and if it be made smaller, the quotient will be made as many times larger.*

For a larger divisor must be multiplied by a smaller quotient, and a smaller divisor by a larger quotient, to produce the same dividend.

In other words, the same dividend contains a larger divisor a less number of times, and a smaller one a greater number of times.

Example.—24 dollars will buy 6 yards, at 4 dollars a yard; while they will buy only 3, at 8 dollars; but will buy 12, at 2 dollars.

Hence, *instead of multiplying or dividing the quotient, we may, when more convenient, divide or multiply the divisor, and vice versa.*

Example.—4 pieces of cloth, containing each 25 yards, cost 4,200 dollars. How much is it a yard? We might divide by 4, first, to find the cost of a piece, and then by 25, for that of a yard; but it is simpler to multiply 4 by 25 first, and then divide 4200 by the product 100.

PROPOSITION III.

4. *If the dividend and the divisor be each multiplied or divided by the same number, the quotient will not be changed.*

For, a number of dividends will contain the same number of divisors as many times as a single dividend contains a single divisor.

Example.—There is a bag of silver, containing 200 dollars for each company of 100 men. The division will give each soldier 2 dollars. Now, it matters not whether the number of companies be increased or diminished, provided the number of bags be increased or diminished in the same proportion, and there continues to be one bag to each company; the division of the whole sum of money among all the soldiers must give to each the same amount, 2 dollars, as the division of a single bag by a single company.

Otherwise, the dividend is the product of the divisor by the quotient; and, consequently, the multiplication of the same quotient by a divisor a number of times larger or smaller, must produce a dividend as many times larger or smaller.

Questions.—How is the quotient, when the dividend is increased a number of times? How, when it is decreased? Why? To what do the dividend, divisor, and quotient correspond in multiplication? May the quotient be multiplied or divided, instead of the dividend? Why? When is it preferable? If the divisor be increased, how is the quotient? Why? If it be decreased? Why? Give examples. May the divisor be multiplied, instead of dividing the dividend? May it be divided instead of multiplying the dividend? Why? If the dividend and divisor be either divided or multiplied by the same number, how is the quotient? Why? Give examples.

LESSON XXV.

MULTIPLICATION OF DECIMALS.

1. Let it be proposed, for example, to multiply - - - - - - 5.407 by - 2.54

OPERATION.

```
   5.407
    2.54
 -------
   21628
  27035
 10814
 -------
13.73378
```

In the first place, let us consider, for a moment, the multiplier as the whole number, 254; it is clear that the multiplicand, 5.407, being (XXI., 1 and 2) the same thing as 5407 *thousandths*, the repetition of this number 254 times, would give a product of the same nature (XI., 7); that is, 1373378 thousandths, which, expressed decimally, is

1373.378.

2. So that, *in the multiplication of a decimal number by a whole number, the product contains as many decimal places as the multiplicand.*

3. But, in considering the multiplier 2.54 as a whole number, we have increased it one hundred times; or, in other words, ten times for each decimal place it contains (XXI., 6). The product

1,373.378

is, therefore, one hundred times too large; and it must now be divided by 100, to reduce it to its true value,

13.73378.

This is done by removing the units' point (XXI., 6) two additional places to the left, which makes in all as many places as there were decimals in both the multiplicand and multiplier. Hence:

I. *Multiply decimal numbers as if there was no decimal point; and, in the product, point off as many figures as there are decimals in both the multiplicand and the multiplier.*

II. *If there be not so many figures in the product, the deficiency is supplied by zeros, between the product and decimal point.*

As in the following example:

	OPERATION.
Let it be proposed to multiply -	0.03054
by -	0.023
	9162
	6108
	0.00070242

If we multiply without regard to the units' point, we get the product,

70242,

which is much too large; for, it should, in the first place, contain five decimals, if the multiplier were the whole number, 23: and, in the next, since the multiplier is one thousand times smaller, we must cut off three additional decimals in the product; in all, *eight decimal figures;* namely, *five* on account of the multiplicand, and *three* on account of the multiplier.

The last figure must, therefore, be of the *eighth order* of decimals. We move it to that place by prefixing the requisite number of zeros between the product and decimal point, as shown above.

REMARK.

4. The multiplication of decimal fractions, unconnected with whole numbers, presents an apparent contradiction, which is calculated to embarrass beginners; namely, that the product of decimals is smaller than either factor.

Thus, for instance, when we multiply - -	0.3
by -	0.02
we get for the product - - - -	0.006

which is of the third descending order, and consequently smaller than the multiplicand, which is of the first, and the multiplier, which is of the second. Hence, the pupil is at a loss to understand whether the operation is a multiplication or a division.

The fact is, that there is both a division and multiplication in it. In order to understand this, it must be recollected that *the object of multiplication is, knowing the value corresponding to one unit, to find the value corresponding to several;* as in this example :

One yard of cloth costs 6 dollars. What will 4 yards cost? The answer is evidently obtained *by repeating the multiplicand,* 6 dollars, *as many times as there are units in the multiplier,* 4; because the multiplicand is equal in value to each one of these units.

But if the question were:

One (yard) *costs* 6 (dollars), *what will* 4 *hundredths* (of a yard) *cost?*

The case would be different; for, then, the multiplicand, 6 dollars, is no longer given as the value of one of the units in which the multiplier is expressed, which are *hundredths*, but it is the value of a whole unit (a yard) one hundred times larger. Consequently, were we to repeat *this multiplicand,* 6 *dollars,* as many times as the number, 4, of units in the multiplier, we would get a product one hundred times too large.

A preparatory operation must, therefore, be introduced before the multiplication can be correctly performed.

This operation is a division. It consists in reducing the multiplicand, so that its value may correspond to the smaller fractional unit, *one hundredth,* in the multiplier; that is, *since one whole unit* (yard) *is equivalent to the given multiplicand* (6 dollars), *the one hundredth part of the whole unit* must be equivalent to *the one hundredth part of the given multiplicand,* 6.

Which, by the principle of decimal numeration, is 0.06 (XXI., 6); and, this being the true amount equivalent to the unit, *one hundredth* of the multiplier may now, with propriety, be considered as the true multiplicand; and, as such, be repeated as many times as there are units in the multiplier (in this case, four times. The operation is thus reduced to a genuine

multiplication of - - - - -		0.06
	by -	4
and the true product is - - - -		0.24

which is now, as usual, larger than its true multiplicand.

This product, 24 hundredths of a dollar, is the same that would have been obtained by the application of the above rule of decimal multiplication.

It will be seen, therefore, that the whole difficulty arises from the fact, which is not generally noticed, that this is a compound operation; the first part of which is in fact a division, to reduce down the given multiplicand from its value, corresponding to a large unit, to the standard of the smaller units of which the multiplier is composed.

The second, which is the true multiplication, to repeat as usual this altered multiplicand as many times as there are units in its multiplier.

In practice, the multiplication and the preparatory division are blended in one operation; and the proper reduction, instead of being previously made in the multiplicand, is made in the product, because it is simpler and amounts to the same thing.

5. The above explanation shows how we must understand the principle that *the multiplication, by descending units* 10, 100, 1000, *&c., times smaller, gives products decreasing also* 10, 100, 1000, *&c., times, like the orders multiplied by.*

It is an analogy, in regard to the decreasing of products by descending units, corresponding to the increase by ascending orders; and which implies that the given multiplicand, in regard to whole units, is first reduced down to correspond to *one* of the fractional units.

N. B.—These remarks, applied to our first example,	35.407
multiplied by . . .	12.54
	= 444.00378

may also serve to explain, in another way, the principle for cutting off decimals. For, the multiplicand, 35.407, is here given as equivalent to *one whole unit* of the integral part, 12, of the multiplier, which is a mixed number.

Consequently, in order to multiply by 1254 as a single number, we must determine first that reduced multiplicand which corresponds to one of its last units (viz: *hundredths*).

This is done by cutting off as many additional decimals in the multiplicand as there are in the multiplier, which here makes it .35407. By this preparatory division, the multiplier becomes a single abstract number; and the new multiplicand containing as many decimals as there were in both multiplicand and multiplier, we must cut off the same number in the product, since it is of the same nature as the multiplicand.

Questions.—What is the rule for the multiplication of decimals? How many decimals do you cut off? Why? What is done when the number of figures in the product is not sufficient for decimals? Give examples. When is the product of two decimal numbers smaller than either factor? Is the operation in that case truly a multiplication? Is it a single operation? How can the statement be rectified? How is the product of a number by a multiplier composed of descending units? Explain the cutting off of decimals by the previous transformation of the multiplicand.

EXERCISES.

In the following exercises, the decimal point is purposely omitted; the fixing of its place being the only difference between common and decimal multiplication.

1. $12.34 \times 5.6 = 69104.$
2. $0.123 \times 0.45 = 5535.$
3. $0.321096 \times .2465 = 7150164.$
4. $79.347 \times 23.15 = 183688305.$
5. $0.0253 \times 345 = 87285.$
6. $0.0010001 \times .001 = 10001.$
7. $724.623 \times 5.56 = 402890388.$
8. $11785 \times .027 = 318195.$

It is unnecessary to multiply examples here. The teacher may use any of the exercises previously given in multiplication and division, by introducing decimal points in any place he may choose.

LESSON XXVI.

DIVISION OF DECIMALS.

CASE I.

1. *The divisor being a whole number.*

If we had to divide a decimal by a whole number, as

80.64 by 9,

the quotient would be 8.96; for, it is clear that the units of the quotient being always, in such a case (XV., 3), of the same nature as those of the dividend, which here is 8064 *hundredths*, the quotient must be a number of *hundredths:* that is, 896 *hundredths;* or, decimally, 8.96.

Hence, *the quotient contains as many decimal places as the dividend, when the divisor is a whole number.*

CASE II.

2. *When the divisor contains decimals.*

In this case, all we have to do is to make the divisor a whole number, and to divide, as in the preceding case.

Let us take the annexed

FIRST EXAMPLE.

OPERATION.

36/8.64	4/8
3 2 6	7.68
384	
. . .	

We make, at once, the divisor a whole number by striking off its decimal point.

But, in doing this, we increase the divisor ten times for each decimal place it contains (XXI., 6); and consequently the quotient would be made the same number of times smaller, if the dividend remained unchanged (XXIV., 3).

In order, therefore, that the quotient may not be altered, we must increase the dividend as many times as the divisor.

This we do by removing its units' point as many places to the right as there were places in the divisor.

Then the operation is performed as usual; and the units' point is placed in the quotient when the operation reaches it in the new dividend; or else, we may, after having found the whole quotient,

768

fix its character by cutting off the same number of decimal places as were left in the dividend, (XV., 3).

GENERAL RULE.

3\. I. *Make the divisor a whole number by striking off its units' point, and remove the units' point of the dividend as many places to the right as there were decimal figures in the divisor, annexing ciphers, if necessary.*

II. *Divide, then, as in whole numbers, and place the units' point after the quotient-figure of units.*

III. *If the first partial dividend is decimal, set down a zero for units and one for each vacant order after it.*

N. B.—This rule embraces all the cases usually separated in Arithmetics, and is more easily understood and applied by beginners.

No confusion can result from the two decimal points in the dividend, because it is always that on the right which determines the order of the quotient.

SECOND EXAMPLE.

4\. *When the number of decimals is the same on both sides,* though the rule would apply, it is evident that no preparation is necessary; and we may divide at once, without regarding the decimal point; for, this amounts to increasing equally the dividend and divisor. Then the quotient, to the end of the dividend, is a whole number.

OPERATION.

368.64	1.92
1766	192
384	
...	

THIRD EXAMPLE.

5\. *When the number of decimals is less in the divi-*

dend, the removal of the units' point requires the addition of some 0s; it is the case mentioned in the rule.

Let it be proposed to divide 3686.4 by 1.152

OPERATION.

```
3686/400 | 1/152
. . 230 4 |------
 . . .    | 3200
```

FOURTH EXAMPLE.

.025 by 0.5

```
/0.25 | 0/5
      |-----
   .. | 0.05
```

FIFTH EXAMPLE.

0.25 by 0.00005

```
Which is changed into 0/25000 by | 0/00005
                                 |---------
                                 |  5000.
```

6. In these two examples, and especially in the last, we see that the quotient is larger than the dividend; the last being 5,000 whole *units*, while the dividend was only the small fraction, 25 *hundredths*.

This result is more readily understood than the decrease of the product in the multiplication of decimal fractions. For, however small a quantity may be, there may always be conceived a much smaller one, which will be contained in it a great number of times.

Questions.—Give the rule for dividing decimal fractions, when the divisor is a whole number; when it contains decimals. What do you make the divisor, if you do away with its units' point? What must be done, after this, with the dividend? Why? If the number of decimal places in the dividend is smaller, what is done? How many decimals must there be in the quotient? When the number of decimal places is the same in both dividend and divisor, how do you divide? Is the quotient ever larger than the dividend? Explain it.

EXERCISES.

The units' points and annexed ciphers, which constitute the principal difficulty of the following exercises, are purposely omitted in the answers.

1. $482.2635 \div 32.1509 = 15.$
2. $559.650 \div 2.5 = 22386.$
3. $234.70525 \div 3.653 = 6425.$
4. $0.7854 \div 1.4 = 561.$
5. $0.87275718 \div 0.162 = 538739.$
6. $0.00070242 \div 0.023 = 3054.$
7. $116.0435 \div 0.29 = 40015.$
8. $7254.06 \div 0.0758 = 957.$
9. $118.5 \div 0.125 = 948.$
10. $0.5822 \div 142 = 41.$
11. $0.00102048 \div 31.89 = 32.$
12. $0.99 \div 0.00225 = 44.$

What is one tenth divided by one tenth? One tenth by one hundredth? One hundredth by one tenth? By one thousandth? By one hundredth? Ten units by one hundredth? One hundredth by ten units? One thousandth by one thousandth? One tenth by thousand units? &c. How often are ten units contained in one tenth? One tenth in ten units? One hundredth in one unit? Fifteen units in fifteen hundredths? Two hundred units in two hundredths? Sixty-five tenths in sixty-five units? &c. &c.

LESSON XXVII.

OF DECIMAL REMAINDERS.

1. ***When a division of decimals by a whole number leaves a remainder, it is a number of units of the same order as the last units of the dividend.***

Set it over the divisor, and join it to the quotient, as in simple division.

EXAMPLE.

Let it be proposed to divide 3.723944 by 0.145.

After having prepared the operation, as usual (XXVI., 3), we see that the altered dividend is of the order of *thousandths*. The remainder is consequently 54 *thousandths*, which are yet to be divided by 145. Therefore, the expression, $\frac{54}{145}$ of this division, annexed to the quotient, completes it.

OPERATION.

```
3/723.944 | /145
 ·823     |------------
   989    | 25.682 54/145
  1194    |
   ·344   |
     54   |
```

EXTENSION OF THE QUOTIENT.

2. Had we stopped at the whole number, 25, for the quotient, the remainder would have been 98 whole units, and the quotient itself would not have been so exact as that we have obtained by changing the remainder into tenths, then hundredths, &c., and getting corresponding figures in the quotient.

3. We may, in the same way, get a still greater approximation by changing the final remainder into smaller orders, and dividing these also by the divisor. Thus the quotient may be extended to orders as small as may be desired.

For this purpose, one cipher is added to the remainder, 54 *thousandths*, which makes it 540 *tenths of thousandths;* and thus 3 *tenths of thousandths* are obtained in the quotient.

In the same way, to the new remainder, 105, add a new cipher, and you get 7 *hundredths of thousandths* in the quotient, with a remainder, 35. The addition of a new zero to this, makes it 350 *millionths*, which, divided by 145, gives 2 *millionths* in the quotient. The next figure is 4 *tenths of millionths.*

OPERATION.

3/723.944	/145
823	25.6823724$\frac{20}{145}$
989	or
1194	25.6823724 +
344	
540	
1050	
350	
600	
20	

If, from the nature of the operation, the approximation is deemed sufficient, we may stop here.

4. N. B.—To give the complete quotient, we have annexed to it the indicated division, $\frac{20}{145}$, of the remainder; but, in practice, the desired approximation being obtained by the diminutive size of the last units, the remainder is generally neglected, and the quotient given without this small addition. Sometimes, in that case, the sign + is added, as shown above, to indicate that the quotient is not exact, and might be extended.

5. *It is better, when the requisite degree of approxi-*

mation is fixed beforehand, to add, at once, to the dividend as many zeros as are necessary to make the last decimal place of the dividend of the desired order.

Wishing, in the present instance, to extend the quotient to tenths of millionths, seven decimal places are requisite; and we may add at once four zeros to complete the number of decimals and prepare the operation thus:

$$3723.9440000 \,|\, \underline{145}$$

This arrangement secures room for the operation; a thing which should always be attended to for the sake of neatness and accuracy.

6. The dividend being now a number of *tenths of millionths*, the quotient is, of course, of the same order; and the result is said to be *obtained true*, or *carried to, within one tenth of millionth*, or *to the seventh decimal place;* because, what might follow, if the division were extended, is less than one tenth of millionth (VI., 3).*

7. N. B.—Divisions carried out in that way will very frequently never end, and give rise to *infinite decimals.*

REMARK I.

8. *When a number, accompanied by decimals, is to be divided by* 1, *followed by zeros*, such as 10, 100, 1000, &c., *remove the units' point as many places to the left as the divisor contains zeros; the result is the quotient.*

EXAMPLES.

25.37 divided	by	10,	is	-	2.537
"	"	100,	"	-	0.2537
"	"	1000,	"	-	0.02537
"	"	10,000,	"	-	0.002537

This results from Lesson XXI., 6.

* When the figure after the last decimal retained, is either 5 or over, it is customary to add one unit to the last decimal, because the rejected figure is nearer to *ten* than to *zero*. For instance, in the above division, had we stopped at the *tenths of thousandths*, we should have taken for the quotient 25.6824 instead of 25.6823.

If the dividend should end with 0s, strike them out when they pass to the right of the decimal point, since they are of no use there (XXI., 8 and 9). 2500, for example, divided by 1000 is 2.5.

REMARK II.

9. *If the divisor, being an integer, ends with zeros, they may be struck off, and the decimal point removed as many figures to the left in the dividend, when the division may proceed as usual.*

For, in doing this, you decrease both equally, which does not alter the quotient (XXIV., 4).

EXAMPLE.

Let it be proposed to divide 996.03 by 2700.

If the dividend were an integer, the case would be the same; some of its last figures then would be made decimals.

OPERATION.

```
9.96/03 | 27ØØ
1 86    |-------
  240   | 0.3689
   243  |
    ..  |
```

For example: 684 to be divided by 20, would be changed into - - - - 68.4 to be divided by 2.

This is the remark adverted to in Lesson XVII., 7; which I have seen strangely and yet frequently misapplied, because it is usually given for whole numbers, and omitted in connexion with decimal fractions. I have seen boys, in examples like this,

15.·96738 to be divided by 20,

apply the usual rule, and prepare the operation by cutting off in the divisor the 0, and in the dividend the figure 8; as here exhibited:

```
15.9673)8 | 2Ø,
          |-----
```

and then divide by 2, without reflecting that their object should be to decrease the dividend 10 times, as well as the divisor, and that this can be done only by moving the units' point; thus:

1.5/96738 | 2$\not{0}$.

10. I have seen also this case. Having to divide 15.96738 by .200, the operation was prepared thus:

15.967)38 | .2$\not{0}\not{0}$,

by striking off the two ciphers, and cutting off two figures in the dividend, forgetting that the two 0s struck out do not alter the divisor, nor does the separation of the decimal figures in the dividend change its value. Here there would be in fact no error in the result, but a mere awkward evidence of a misapprehension of principles.

REMARK III.

11. When the dividend and divisor, being both integers, end with 0s, an equal number may be struck out of each, and then the operation will be performed as usual.

EXAMPLE.

	612400	divided	by	12000.
is the same as	6124	"	by	120.
and as	612.4	"	by	12.

Since both numbers are decreased equally in each case.

Such a transformation is not indispensable, but merely preferable.

N. B.—Here, again, recollect that the zeros struck off, if they come after a decimal fraction, do not require a corresponding operation in the other number.

For example:

If 612400 were to be divided by 1.2000,

the removal of the three 0s of the divisor would not affect its value; and, therefore, the dividend must not be touched, though we divide by 1.2 instead of 1.2000.

Questions.—Of what order is the remainder of a decimal division? What is done with it? How may the quotient be extended? How do you show that the division is not ended? If you add a 0 to the decimal remainder, what do you make it? What are infinite decimals? How would you prepare the dividend to obtain a quotient to within one hundredth? One thousandth? One tenth of thousandth? &c. What is meant by this expression, to within one hundredth? &c. How do you divide decimal numbers by 10, 100, 1000, &c.? If the dividend ends with zeros, what is done when the divisor is 1, followed by zeros? If the divisor, being an integer, ends with zeros, how do you proceed? If the decimals of the divisor end with zeros, what is done? If the dividend and divisor, being whole numbers, end with ciphers, what may be done?

EXERCISES.

The decimal point and additional ciphers are purposely omitted in the answers.

1. $6345.923 \div 54.23 = 117018 +$
2. $27845.96 \div 9.8732 = 28203581 +$
3. $10 \div 563 = 177 +$
4. $7 \div 365 = 19178 +$
5. $0.00026253 \div 0.175 = 150 +$
6. $256.6 \div 0.057 = 450175 +$
7. $1 \div 0.159 = 6289308 +$
8. $0.2 \div 23.2 = 862068 +$
9. Divide .00636056 by .86 to within one millionth.
10. Divide 61 by 0.825 to the 6th place of decimals.
11. Get the quotient $\frac{8792}{937.6567}$ true to the 5th place.
12. Divide 37.96416 by 0.156 to one hundredths.
13. Divide 0.59 by 79800 to the 12th place of decimals.
14. Divide 23 by 0.000579 to within one hundredth of millionth.

N. B.—It being desirable that the learner should perform readily divisions, without any assistance from answers, the teacher may use, also, some of the preceding examples in multiplication and division, with proper and to him easy modifications, such as the introduction of decimal points, ciphers, and remainders, which will still leave a key to the teacher, unknown to the pupil.

CHAPTER IV.

CONTAINING VULGAR FRACTIONS.

LESSON XXVIII.

NATURE OF VULGAR FRACTIONS.

1. Decimal fractions have already explained what fractions are; but units smaller than the *unit of comparison* are not always formed by successive subdivisions by 10.

In a great many cases, the relation between the principal and the subordinate units is expressed by other numbers than 10 or its multiples 100, 1000, &c.

For example: I want to measure the length of a board. *The unit of comparison*, for this purpose, is *the foot*. But I find 12 feet and something over, less than a foot, which I must necessarily measure by some smaller unit, connected with the foot by some known relation; as, for example, *the inch*, 12 of which are equal to one foot. The inch is therefore, *a fraction* of a foot, equal to one twelfth part of it. It is a *relative unit*.

Again: I want for a coat 2 yards and a piece less than a yard; that is, a fraction of a yard, the measure of which must be expressed in smaller units, bearing a known relation to the whole yard. If four such pieces, for instance, are equal to one yard, then I would ask the merchant for 2 *yards* and a *quarter*, or *fourth*, which would indicate the relation of the small additional piece to the standard of measure; *the quarter* is *a relative unit.*

We see, in these cases, that the number mentioned is expressed, as was done in decimals, by two kinds of units; the one, the *unit of comparison;* the other, a smaller *relative unit.*

It will be readily understood, that the relation of small to large units may be expressed in a variety of ways, as infinite as numbers themselves; for, a single unit may be conceived to be composed of any number of small parts.

2. This variety of subdivisions of a unit, requires a separate nomenclature and notation.

When it takes 2	small units to make up a large one, *the small relative unit* or *fraction* is called	*one half*, written - - $\frac{1}{2}$
" " 3		*one third*, " - - $\frac{1}{3}$
" " 4		*one fourth* or one quarter, - - $\frac{1}{4}$
" " 5		one fifth, " - - $\frac{1}{5}$
" " 6		one sixth, " - - $\frac{1}{6}$
&c.		&c. &c.

These expressions convey to the mind, in an abstract manner, the relation between the small units and the whole one.

NOTATION.

3. Since the value or magnitude of each small unit is fixed by the number of parts into which the principal unit has been divided, it follows that the notation used for division is correctly applied here; for,

The fraction	$\frac{1}{2}$	is *one*, divided by -	2
"	$\frac{1}{3}$	" " -	3
"	$\frac{1}{4}$	" " -	4
	&c.		&c.

The fractional part of a thing may contain more than one of the inferior units into which the principal unit has been divided.

Instead of *one* third, we may have *two* thirds.

" of *one* fourth,	"	*two* or *three* fourths or quarters.
" of *one* fifth,	"	2, 3, or 4 fifths.
" of *one* sixth,	"	2, 3, 4, or 5 sixths,
&c.		&c.

The notation adopted in this case is like the preceding. The number of parts is written above, and the relation of value of one of them to the *principal unit* is written below, the line of division.

Thus: $\frac{2}{3}$ for two thirds.
$\frac{3}{4}$ for three fourths or quarters.
$\frac{4}{5}$ for four fifths.
$\frac{6}{7}$ for six sevenths.
$\frac{9}{12}$ for nine twelfths, &c.

These are termed Vulgar Fractions, to distinguish them from decimal fractions.

DEFINITIONS.

4. The two numbers used to write a fraction, are its *two terms.*

The upper number is the *numerator*, so called because it expresses the number of parts.

The lower number is the *denominator*, because it expresses the relative denomination of the inferior unit, with reference to the principal one.

READING OF VULGAR FRACTIONS.

5. *Read the numerator as a simple number, and then the number in the denominator, with the addition of the proper termination.*

This termination beyond *thirds* is generally *th*, except with numbers over twenty ending with *one*, which is then called *first*, as $\frac{1}{21}$ (*one twenty-first.*)

With those ending with 2, which is then called *second*, as $\frac{9}{22}$ (*nine twenty-seconds*).

And with those ending with 3, which is then called *third*, as $\frac{7}{43}$ (*seven forty-thirds*).

But $\frac{16}{29}$ would read *sixteen twenty-ninths*; $\frac{11}{47}$ *eleven forty-sevenths*, &c.

IDENTITY OF FRACTIONS AND THE QUOTIENTS OF DIVISION.

6. The notation adopted for fractions is the same as that for division; because, in fact, *a fraction is nothing but the expression of the division of its numerator by its denominator.*

Let us take, for example, the fraction $\frac{3}{8}$; which means 3 units, each of which is the *one eighth part* of the principal unit; that is, 3 times $\frac{1}{8}$.

It will be easily understood that 3 times $\frac{1}{8}$ is the same thing as 3 divided by 8: for, let us suppose that we have 3 things, say 3 apples, and wish to divide them equally among 8 boys. Having divided each into 8 parts, it will make no difference whether we give to a boy $\frac{1}{8}$ of each apple, making in all, for his share, 3 times $\frac{1}{8}$; or else 3 pieces of the same apple, which are $\frac{3}{8}$ of it.

7. From this it also appears that 3 is equal to 8 times $\frac{3}{8}$. For, 8 eighths of anything, is the thing itself.

8. Hence we see here only a change of names, and that a fraction is nothing but the expression of a division, in which

The Dividend	is called	*the Numerator;*
the Divisor	"	*the Denominator;*
and *the Quotient*	"	*the Fraction* itself.

So that $\frac{3}{4}$	is the quotient	of 3	divided	by 4.
$\frac{3}{8}$	"	of 3	"	by 8.
$\frac{4}{5}$	"	of 4	"	by 5.
$\frac{5}{12}$	"	of 5	"	by 12.
$\frac{16}{99}$	"	of 16	"	by 99.
&c			&c.	

This is the true character of fractions; and much of the obscurity which learners find in this subject, arises from this difference of names to express the same thing. This is not the last example of the imperfection of arithmetic in this respect.

Hence, when we placed the remainder of division over the divisor, we used a fraction in reality to indicate a small addition to the quotient necessary to complete its exact value.

If we divide, for example, 19 by 4, the exact quotient is $4\frac{3}{4}$; that is, 4 whole units and 3 smaller relative units, equal each to only one fourth part of the principal one.

9. We may notice here that decimal fractions may be put under the form of common fractions, and that

0.1	is the same thing	as	$\frac{1}{10}$.
0.02	"	as	$\frac{2}{100}$.
0.125	"	as	$\frac{125}{1000}$.
0.0003	"	as	$\frac{3}{10,000}$.
&c.			&c.

DEFINITIONS.

10. The word FRACTION, in its most extended sense, is applied to all divisions, which, when merely indicated, are called *fractional expressions* or *fractional numbers.*

11. The different kinds of fractional expressions are distinguished sometimes by the following names, which are not, however, of much importance. A *proper fraction* is that whose numerator is smaller than its denominator; as, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{9}{11}$, &c.

An *improper fraction* is that whose numerator is equal to, or larger than, its denominator; such as, $\frac{5}{4}$, $\frac{6}{6}$, $\frac{125}{16}$, &c. It is another name for a common division, merely indicated by the conventional sign of the operation.

A *mixed number* is composed of a whole number and a fraction joined together; such as, $2\frac{1}{3}$, $4\frac{1}{4}$, $7\frac{3}{8}$, &c.: also, 4.07, 6.25, &c. It is nothing but the usual form of the complete quotient of a division; as, for example, $\frac{7}{3} = 2\frac{1}{3}$.

A *simple fraction* is a single fraction, like $\frac{1}{4}$, $\frac{2}{5}$, $\frac{7}{6}$, &c.

A *compound fraction* is a *fraction of fractions;* or, in other words, several fractions connected, with the word *of* between them; such as, $\frac{1}{2}$ of $\frac{1}{4}$; $\frac{2}{3}$ of $\frac{2}{5}$ of $\frac{5}{6}$.

A *complex fraction* is that of which one or both terms are mixed numbers; as,

$$\frac{2\frac{1}{4}}{3}; \quad \frac{3\frac{2}{3}}{6\frac{1}{4}}; \quad \frac{2}{4\frac{1}{5}}, \text{ \&c.}$$

It will be seen that a *proper fraction* is less than 1, and that an *improper fraction* is equal to 1, when its two terms are the same, and more than 1, when the numerator is the larger.

Questions.—What is the origin of fractions? What is the difference between the unit of a fraction and the unit of comparison? What is a relative unit? How many kinds of vulgar fractions are there? How do you write a fraction? What is the numerator? The denominator? Both together? What are vulgar fractions? How do you read a fraction? What do you call one part of a thing divided into 2? 3? 4? 5? 6? &c., parts? An apple is cut into six parts: how do you call 2; 3; 4; 5, of them? How many fifths in 2? in 3? in 4? in 5 apples? How many thirds in two apples and one third; how many fourths in three and three quarters? &c.

How do fractions compare with division? Show that a fraction is the quotient of its numerator by its denominator. Give examples. What, in a fraction, corresponds with the dividend? Divisor? Quotient? Can decimal fractions be written as vulgar fractions? What is the quotient of 6 by 7? of 11 by 12? of 19 by 20? Write 0.06; 0.175; 0.0061, &c., as vulgar fractions.

What are fractional numbers? fractional expressions? proper, improper, simple, compound, complex fractions? What is a mixed number? Give examples. What are fractions of fractions? What fractions are smaller than 1? equal to 1? larger than 1? How many times is $\frac{1}{8}$ smaller than 1? $\frac{3}{8}$ than 3? $\frac{5}{18}$ than 5? &c.

LESSON XXIX.

FUNDAMENTAL PROPOSITIONS.

1. It being well understood that a fraction is the expression of a division, and its value that of the quotient, we may repeat and apply to fractions the propositions relative to division, given in Lesson XXIV., by merely changing the words:

Dividend	into	Numerator;
Divisor	"	Denominator;
Quotient	"	Fraction.

PROPOSITION I.

2. *The denominator remaining the same, if the numerator of a fraction be multiplied*, or divided, *by a certain*

number, the fraction itself is made as many times larger, or smaller, *as there are units in the number we multiply*, or divide, *by*.

Because the *numerator* is another name for *dividend;* and we know that, in multiplying the dividend, we increase the quotient, and, in dividing, decrease it (XXIV., 1).

EXAMPLES.

If we multiply by 2, the numerator of the fraction $\frac{3}{4}$, we get $\frac{6}{4}$, which is double $\frac{3}{4}$.

Again: twice 3 *quarters of a yard* is 6 *quarters*.

If, in the fraction $\frac{9}{10}$, we divide 9 by 3, we get $\frac{3}{10}$, which are 3 small units, instead of 9.

In decimals, $\frac{9}{10}$ would be 0.9; which, divided by 3, is $0.3 = \frac{3}{10}$; a result identical with the preceding.

PROPOSITION II.

3. *The numerator remaining the same, if the denominator be multiplied*, or divided, *by a certain number, the fraction itself is made as many times smaller*, or larger, *as there are units in the number we multiply*, or divide, *by*.

For the *denominator* is another name for the *divisor;* and we know (XXIV., 3) that, in multiplying the divisor, we decrease the quotient; and, in dividing, increase it.

EXAMPLES.

If, in the fraction $\frac{3}{4}$, we multiply the denominator by 2, we get $\frac{3}{8}$, which is twice as small; for, each *eighth* is one half of one *quarter*, and *three* of them one half of *three quarters*.

Again: a quarter of an apple cut in two, makes *two eighths;* and if I give a boy *three of these pieces*, that is, 3 *eighths*, I give him half as much as if he had 3 *quarters*.

If, in $\frac{9}{10}$, we divide 10 by 5, we get $\frac{9}{2}$, which is 5 times as large; since each half is worth $\frac{5}{10}$, and $\frac{9}{2}$ is consequently $\frac{45}{10}$. Decimally, 4.5 is 5 times 0.9.

One half of an apple, cut into 5 pieces, would make 5 tenths. $\frac{9}{100}$, by the decimal notation, would be 0.09; and, if we divide the denominator, 100, by 10, we get $\frac{9}{10}$, or, decimally, 0.9, which is ten times as much.

4. From the two foregoing propositions, we conclude that:

I. *A fraction may be multiplied by a given number in two different ways; either by multiplying its numerator or by dividing its denominator.*

II. *A fraction may be divided by a given number, also, in two different ways; either by dividing its numerator or by multiplying its denominator.*

The nature of the question will determine, in each case, which of the two methods is to be chosen. In general, however, the method by division, when it is practicable, is preferable, because it reduces the size of the numbers.

EXAMPLES.

Let it be proposed to multiply the fraction $\frac{5}{16}$ by 3. The only way here would be to multiply its numerator; and the result would be $\frac{15}{16}$.

But, if it were to be multiplied by 4, since the denominator can be divided by 4, division should be preferred, and the result would be $\frac{5}{4}$; which is simpler than $\frac{20}{16}$.

Let it now be asked to divide $\frac{6}{7}$ by 5: there would be no other way than to multiply the denominator, 7, by 5, since the numerator cannot be divided. Thus, the result would be $\frac{6}{35}$.

But, if the fraction were to be divided by 3, then the operation by dividing the numerator, would be better; since it gives $\frac{2}{7}$, a simpler result than $\frac{6}{21}$, which the other method would produce.

Sometimes both methods are combined; of which we shall find numerous examples in the sequel.

5. *As a consequence of the above rule, we change a fraction into a whole number, equal to its numerator, by merely striking out its denominator.*

For example, in $\frac{3}{8}$, if we divide the denominator by 8, we get $\frac{3}{1}$, or 3 units, since a number divided by unity is equal to itself; and, in fact, we know that 3 is equal to 8 times $\frac{3}{8}$ (XXVIII., 7); for, 3 being the dividend, 8 its divisor, and $\frac{3}{8}$ the quotient, the product of 8 by $\frac{3}{8}$ must give the dividend again.

Questions.—What effect is produced on the value of a fraction by multiplying its numerator? By dividing the same? By multiplying its denominator? By dividing the same? Why? Give examples. Repeat each proposition. How many times is $\frac{6}{7}$ larger than $\frac{2}{7}$? $\frac{14}{15}$ than $\frac{2}{15}$? $\frac{4}{5}$ than $\frac{4}{15}$? $\frac{6}{7}$ than $\frac{6}{35}$? How many times is $\frac{2}{5}$ smaller than $\frac{4}{5}$? $\frac{9}{25}$ than $\frac{18}{25}$? $\frac{7}{16}$ than $\frac{7}{8}$? $\frac{3}{64}$ than $\frac{3}{4}$? &c. In how many ways can a fraction be multiplied by a given number? In how many ways divided? Which of the methods is generally preferred, when practicable? Why? Give examples. What is the effect of striking out the denominator of a fraction?

EXERCISES.

Multiply, in the simplest manner,

By 5, - $\frac{2}{7}$; $\frac{3}{4}$; $\frac{3}{5}$; $\frac{4}{9}$; $\frac{11}{25}$; $\frac{12}{35}$; $\frac{2}{11}$; $\frac{7}{35}$; $\frac{9}{45}$; $\frac{1}{10}$, &c.

By 7, $\frac{4}{5}$; $\frac{6}{7}$; $\frac{3}{28}$; $\frac{9}{11}$; $\frac{1}{14}$; $\frac{6}{17}$; $\frac{3}{49}$; $\frac{4}{65}$; $\frac{2}{29}$; $\frac{6}{42}$; $\frac{12}{84}$, &c.

Divide, in the simplest way,

By 6, - $\frac{1}{12}$; $\frac{12}{17}$; $\frac{4}{5}$; $\frac{18}{19}$; $\frac{13}{7}$; $\frac{24}{4}$; $\frac{1}{11}$; $\frac{30}{31}$; $\frac{5}{6}$; $\frac{6}{13}$, &c.

By 9, - $\frac{2}{3}$; $\frac{9}{11}$; $\frac{81}{40}$; $\frac{63}{7}$; $\frac{2}{5}$; $\frac{4}{9}$; $\frac{18}{19}$; $\frac{27}{28}$; $\frac{85}{7}$; $\frac{72}{8}$, &c.

LESSON XXX.

TRANSFORMATION OF FRACTIONS.

1. I have reserved, for a separate lesson, the third proposition, corresponding to that in division, on account of its great importance and frequent application in arithmetical operations. It forms the basis of several systems of calculations, especially *proportions* and *cancelling.*

PROPOSITION III.

If the numerator and denominator of a fraction be multiplied, or divided, *by the same number, the value of the fraction itself will remain the same.* Its form alone will be changed.

This is the same proposition as that in Lesson XXIV., 4, by changing the words *dividend*, *divisor*, and *quotient*, into *numerator*, *denominator*, and *fraction*.

EXAMPLES.

If both terms of the fraction $\frac{2}{4}$ be multiplied by 2, it becomes $\frac{4}{8}$.
If they be divided by 2, it becomes $\frac{1}{2}$.
And it is evident that $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.
You get one half of an apple, whether as *one half*, *two quarters*, or *four eighths*.

2. In order, however, to explain this important principle fully, independently of other propositions, let us take, as an example, the fraction

$$\frac{3}{4},$$

and multiply its two terms by 3; we will change it thus into

$$\frac{9}{12}.$$

The first *form* of the fraction expresses that the principal unit, an apple, for instance, has been divided into 4 small units or parts, and that three of them are the share taken.

But, in the second *form*, each fourth itself is now subdivided into 3 parts, so that the whole makes 12 parts; out of which the three pieces, constituting the particular share taken, would form 9 parts, or $\frac{9}{12}$ of the whole, which is, therefore, the same thing as $\frac{3}{4}$ expressed in units made 3 times smaller, but whose reduced size is compensated by their number; there being 3 of the smaller units, for each one of the larger.

3. This proposition shows that *the value of a fraction does not depend at all upon the actual magnitude of its terms, but only upon their numeral relation.*

4. This relation, which is *the quotient of the second by the first,* is usually called RATIO.

Thus the fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{6}{12}$, $\frac{5}{10}$, ... $\frac{5,000}{10,000}$, are all equivalent, because the denominator of each is equal to twice its numerator, though the magnitudes of the corresponding terms are so different. This *ratio* is 2 for every one of them.

Likewise, $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, $\frac{15}{20}$, &c., are all equivalent to $\frac{3}{4}$, because they result from the multiplication of its two terms respectively by 2, 3, 4, 5, &c. The *ratio* here is $\frac{4}{3}$.*

5. This shows also that *the number of forms under which a fraction may be expressed, without altering its value, is as infinite as the numbers by which both of its terms may be multiplied.*

A change of form is frequently necessary to facilitate arithmetical operations upon fractions.

This change or transformation may be made in two ways, either by *augmentation* of the terms or by *reduction.*

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* Beginners are apt to think that *adding* the same number to, or *subtracting* it from, the two terms of a fraction, does not change its value. This is a mistake.

Adding the same number to both terms, *increases the value of the fraction.*

Subtracting lessens it.

Take, for example, $\frac{7}{12}$; it is $\frac{5}{12}$ less than unity, which is $\frac{12}{12}$. Now add any number, 6, to both terms of the given fraction; you get $\frac{13}{18}$, which is larger than $\frac{7}{12}$, since it differs only $\frac{5}{18}$ from unity, which is $\frac{18}{18}$.

On the contrary, deduct 6 from both terms, and you get $\frac{1}{6}$, which is smaller than $\frac{7}{12}$.

It will be readily perceived, in general, that the difference between any fraction and unity, is always a supplemental fraction, *having the same denominator*, and whose *numerator is the difference between the denominator and the numerator of the original fraction.* But now this difference is not changed by an equal addition to both terms; so that the numerator of the supplemental fraction is invariable, while its denominator is increased. It follows that the supplemental fraction is diminished; and, consequently, the given fraction, by addition to both its terms, approaches nearer to *unity*, and thereby is increased.

A similar mode of reasoning would show that equal subtraction from both terms diminishes a fraction. In the case of an improper fraction, the effect is reversed: addition decreases, and subtraction increases it; and it would be demonstrated in the same way, by comparison to unity.

Both cases may be comprised in one, by saying that *equal addition to both terms of any fraction makes it approach nearer* to unity; *and equal subtraction makes it recede further from it.*

FIRST TRANSFORMATION—BY AUGMENTATION.

6. The object in enlarging the terms of a fraction, without altering its value, is generally to express it with a larger denominator, or, in other words, in smaller units.

As when $\frac{1}{2}$ is called $\frac{4}{8}$; the eighth part of a thing being only one fourth part of its half.

Or, also, when $\frac{1}{3}$ of a *foot* is called $\frac{4}{12}$; that is, 4 *inches*, which are smaller units than the foot.

The enlarged form of a fraction is obtained by multiplying both terms by such a number as will change the first denominator into the second.

The number we must multiply by, is evidently the quotient of the second denominator by the first.

If, for instance, we wish to express the fraction $\frac{2}{3}$, with the denominator, 36, we multiply both terms by 12, which is the quotient of 36 by 3.

The transformed fraction is then $\frac{24}{36} = \frac{2}{3}$.

7. It follows, also, from this, that *the corresponding terms of two or more equal fractions may be added together, viz., numerators to numerators, and denominators to denominators, and the result will still be the same fraction merely transformed.*

For example, take $\frac{1}{2}$ and $\frac{1}{2}$; it is clear that $\frac{1+1}{2+2} = \frac{1}{2}$ still.

" $\frac{2}{3}$ and $\frac{4}{6}$, by such an addition, will make $\frac{2+4}{3+6} = \frac{6}{9}$; which is another form for $\frac{2}{3}$.

This evidently amounts to multiplying both terms by the same number; in the first instance by 2, in the second by 3; or, in other words, dividing the sum of several equal dividends by the same number of divisors, which must give the same quotient as one dividend by one divisor.

8. Sometimes it is useful *to put a whole number under fractional form. This is done by multiplying it by the intended denominator, over which the product is placed as a numerator.*

5 may thus be made $\frac{15}{3}$; since 15 divided by 3, is 5. This amounts to changing the size of the units; 5 *yards*, for example, into 15 *feet*.

SECOND TRANSFORMATION—BY REDUCTION.

9. The object of reducing the form of a fraction is generally to simplify it. *The reduction of a fraction to simpler terms, is made by dividing both terms by some common factor.*

This amounts to expressing the value of the fraction in larger units.

As when $\frac{2}{4}$ is called $\frac{1}{2}$; the half of a thing being double one quarter.

Or, also, when 6 inches, otherwise $\frac{6}{12}$ of a foot, are called $\frac{1}{2}$ a foot.

In the fraction $\frac{60}{72}$, for example, it is easily discovered that 2 is a common factor. We may, therefore, divide both terms by it, and thus reduce the fraction to

$$\frac{30}{36},$$

a simpler form than the preceding. This also, by a second division of both terms by 2, may be reduced to

$$\frac{15}{18};$$

and, finally, by striking out the common factor, 3, from both numerator and denominator, we simplify still further to

$$\frac{5}{6},$$

which is not susceptible of reduction, since there is no common factor between 5 and 6. This is consequently the simplest form of the fraction.

10. *The change of form of a fraction is properly a change of the relative units in which it is expressed, in reference to a principal unit of comparison.*

When we transformed $\frac{2}{3}$ into $\frac{24}{36}$, we expressed, in *thirty-sixth* parts of the principal unit, the value of the fraction, which was at first given in *thirds* of the same.

In the example of reduction, the units were, at first, each one *seventy-second* part of unity, and the same value was at last expressed in *sixths;* $\frac{5}{6}$ being equal to $\frac{60}{72}$.

11. Sometimes it may be desired to transform a fraction, in reference to a given numerator. In that case, the rule is similar to that in reference to the denominator.

Multiply, or divide, ***both terms by the number which would change the first numerator into the second*** (that is, ***by their quotient***).

For example, to change $\frac{24}{36}$ into a fraction having the numerator 72, multiply both terms by 3: the transformed fraction will be

$$\frac{72}{108}.$$

To reduce it to one having the numerator 4, divide both terms by 6, because $4 \times 6 = 24$. The result will be

$$\frac{4}{6};$$

and it is evident that $\frac{4}{6}$; $\frac{24}{36}$; $\frac{72}{108}$, are all equivalent.

Questions.—If the two terms of a fraction are multiplied by the same number, what effect has it on its value? What, if divided by the same number? Why? Give examples. Repeat the proposition. Does the value of a fraction depend on the magnitude of its terms? What does it depend on? What is meant by ratio? Can a fraction be expressed in different ways? How many? In how many ways may a fraction be transformed? What are they? What does transformation by augmentation amount to? How do you give to a fraction a larger denominator without changing its value? Give examples. How do you transform a fraction by reduction? Give examples. When is a fraction reduced to its lowest terms? To its simplest expression? How would you increase or diminish the numerator, without altering the value of the fraction? What is the ratio of 9 to 18? *Ans.* 2. Of 16 to 48? Of 46 to 16? Of 11 to 15? Of 15 to 11? Of 25 to 5? Of 5 to 25? &c.

EXERCISES.

1. Transform the fraction $\frac{2}{5}$ into fractions having for denominators 15; 20; 35; 55; 60; 75; 85; 105; 250; 155; 2550, &c.

2. Transform the fraction $\frac{4}{7}$ so that its denominator may be successively 4, 5, 9, 15, 20, 24 times larger.

3. Change 7 into a fractional expression, with the denominator 4, 7, 11, 25.

4. Express 8 in units, 5, 6, 8, 10, 12 times smaller.

5. Express the fraction $\frac{5}{8}$ in units, 2; 3; 5; 6; 10; 11; 12 times smaller.

6. Reduce the fractions $\frac{4}{6}$; $\frac{10}{15}$; $\frac{9}{12}$; $\frac{13}{13}$; $\frac{20}{25}$; $\frac{6}{24}$; $\frac{16}{48}$; $\frac{49}{84}$; $\frac{9}{27}$; $\frac{96}{16}$; $\frac{12}{132}$; $\frac{11}{121}$, to their simplest expressions.

7. Express the fractions $\frac{6}{12}$; $\frac{10}{30}$; $\frac{21}{24}$; $\frac{18}{39}$, in units 3 times larger.

8. Express the fractions $\frac{5}{15}$; $\frac{15}{40}$; $\frac{25}{50}$; $\frac{35}{55}$; $\frac{15}{75}$, in units 5 times larger.

9. Express the fraction $\frac{3}{11}$ with the numerators 12, 21, 39, 54.

10. Express the fraction $\frac{30}{60}$ with the numerators 15, 10, 5, 3, 1.

CHAPTER V.

CONTAINING RULES RELATIVE TO THE FACTORS AND DIVISORS OF NUMBERS, AND CANCELLING.

N. B.—The whole of this chapter may be omitted with beginners.

LESSON XXXI.

PERMUTATION OF FACTORS AND DIVISORS.

THE answer to many questions in arithmetic, requires several simultaneous multiplications and divisions. It is, therefore, important to establish clearly the following propositions, in regard to the order in which they may be performed, before we introduce combined operations.

PROPOSITION I.

1. *When several numbers are to be multiplied together, the product will be the same, in whatever order the multiplications are made.*

This principle has already been demonstrated for two factors; it may be illustrated for three, by the following example.

Question.—There are 5 rows, of 4 baskets each, and in each basket 6 peaches: how many peaches are there in all?

We may arrive at the result in several ways.

1st. We may get at the number of baskets by multiplying the 4 baskets in one row by the number, 5, of rows,

which is . . .	4×5	and then multiply the product by the 6 peaches in each basket.	$4 \times 5 \times 6$ peaches.
or the number of rows, 5, by the 4 baskets in each	5×4		$5 \times 4 \times 6$

2d. We may count the peaches in a row of baskets,

equal to	4×6	peaches, and multiply this number by the number of rows.	$4 \times 6 \times 5$
or	6×4		$6 \times 4 \times 5$

3d. We may count the peaches in the baskets, at the head of each of the 5 rows,

equal to	5×6	and then repeat it by the number, 4, of such rows.	$5 \times 6 \times 4$
or	6×5		$6 \times 5 \times 4$

All of which must, of course, give the same number of peaches, 120.

2. A popular illustration like this, however, is not always practicable; and it is proper to demonstrate every principle by abstract and more mathematical methods. Here we may consider that, according to what has been previously demonstrated (XII., 5), the same result is finally obtained in a multiplication, whether we multiply the multiplicand, the multiplier, or the product, by a given number.

Therefore, having to multiply the product 4×5 by 6, we may make, first,

the product	4×5	and then multiply it	by 6, to get	$4 \times 5 \times 6$
or its equal,	5×4	"	" or	$5 \times 4 \times 6$

Otherwise, we may first

multiply the multiplicand .	4 by 6	and then multiply	by 5, to get	$4 \times 6 \times 5$
which is the same as	6 by 4	"	" or	$6 \times 4 \times 5$

Or, finally, we may begin with multiplying

the multiplier 5 by 6	and then multiply	by 4, to get	$5 \times 6 \times 4$
which is the same as . . . . 6 by 5	"	" or	$6 \times 5 \times 4$

Showing every possible combination of three factors for the same product.

3. The principle being now established for three, may be easily extended to any other number of factors.

Let us take, as an example, the product of

$$2 \times 4 \times 5 \times 7 \times 8 \times \mathbf{6};$$

in which **6** multiplies the product of all the preceding factors.

If, now, we transfer **6** to any other place; the third, for example, we shall have

$$2 \times 4 \times \mathbf{6} \times 5 \times 7 \times 8.$$

Let us make the products which precede and follow **6**. We shall have, then, three numbers; and, according to what has just been explained,

$$8 \times \mathbf{6} \times 280 = 8 \times 280 \times \mathbf{6};$$

or, by restoring the factors,

$$2 \times 4 \times \mathbf{6} \times 5 \times 7 \times 8 = 2 \times 4 \times 5 \times 7 \times 8 \times \mathbf{6};$$

and, since the same mode of reasoning would apply to any other place chosen for **6**, and also to any other factor, we might form, in that way, all the possible changes the number of factors admits of, without altering the product. Therefore,

The permutation of the factors does not alter the product.

And we may, in practice, adopt the order which will best facilitate the operations.

For instance, if we had to multiply

$$3 \times 125 \times 5 \times 7 \times 8 \times 2,$$

the order in which the numbers are presented would be unfavorable; but the operation might be greatly simplified by a change of this order; for, it would be readily perceived that $2 \times 5 = 10$; $8 \times 125 = 1000$; $3 \times 7 = 21$, and that the entire product is 210,000. In this way, the operation could be easily performed mentally.

CONTRACTION IN MULTIPLICATION.

4. It follows from this proposition, that, *instead of multiplying by the whole multiplier, we may multiply successively by its component factors; since the multiplicand and all the factors of the multiplier must form the product in whatever order we may choose to take them.*

For example, if we had to multiply 365 by 96,

we might multiply, first, - - 365

by - - 8

= 2920

and then this product by - - 12

= 35,040

This is called *contraction in multiplication.* It may facilitate mental operations.

Questions.—What is the effect of permutations of the factors upon the product? Demonstrate it for three factors. For any number. Repeat the first proposition. What is contraction in multiplication? Give an example.

EXERCISES ON MULTIPLICATION BY CONTRACTION.

1. $43,102 \times 66 = 2,844,732$.
2. $871,075 \times 42 = 36,585,150$.
3. $526,473 \times 144 = 75,812,112$.
4. $12,071 \times 99 = 1,195,029$.
5. $47,701 \times 242 = 11,543,642$.
6. $24,683 \times 375 = 9,256,125$.
7. $2,947 \times 336 = 141,456$.
8. $3,726 \times 132 = 491,832$.
9. $3,748 \times 147 = 550,956$.
10. $2,164 \times 153 = 331,092$.

LESSON XXXII.

PROPOSITION II.

1. *When the divisor is composed of several factors,*

the quotient will be the same, whether we divide by the whole divisor at once, or successively by its component factors, in any order whatever.

For, the quotient is another factor which, multiplied by those of the divisor, must produce the dividend. If, then, we divide by any one of the factors, the result contains yet the quotient and the other factors; and a succession of divisions, by the various factors, will reduce the result at last to the quotient alone.

Let it be proposed, for instance, to divide 168 by 24. The quotient would be 7, and 24 being 6×4, we have

$$168 = 6 \times 4 \times 7.$$

Consequently, if we divide 168 first by 4, this factor is removed from the dividend; then 6 and 7 remain, and the result of the first division is

$$42 = 6 \times 7.$$

Another division, by 6, will now reduce the first result to the required quotient, 7.

This, in fact, amounts to dividing the dividend and divisor by the same number, which we know does not alter the quotient (XXIV., 4).

2. Divisions are sometimes performed in this way, by the successive component factors of the divisor.

For instance, 1728 might be divided by 72, by means of two divisions; the first by 8, the second by 9.

OPERATION.

Dividend .	1728	8	1st divisor.
1st quotient .	216	9	2d divisor.
last quotient .	24		

This method is called *Contraction in Division.*

Contraction is more useful in division than in multiplication, because it substitutes short for long division.

3. It is evident that the order of the divisions does not

affect the quotient; since permutation of its factors will not change the amount of the divisor.

4. From what precedes, we may also conclude that, *whatever may be the place of a factor, in a product to be formed, we divide the product by this factor by striking it out.*

Thus, if we want to divide by **6**, the indicated product,

$$4 \times \mathbf{6} \times 5 \times 7,$$

the quotient will be $4 \times 5 \times 7$; since this multiplied by **6**, would reproduce the same amount.

5. Observe that, in striking out a factor, you divide by it; and that, since a number divided by itself gives 1, it is understood that 1 remains in lieu of the factor struck out. But this figure, as a factor, is generally omitted, because it has no effect on the product, and

$4 \times 5 \times 7$ is the
same as . . . $4 \times 5 \times 7 \times 1$.

PROPOSITION III.

6. *When, in an operation, a number is to be multiplied and also divided, it is immaterial whether you multiply or divide first.*

For, the numbers to be multiplied form a product; and it amounts to the same thing, whether you divide either of the factors or the product (XXIII., 3).

EXAMPLE.

Suppose that we have to multiply 144 by 17, and divide by 24. The operation will be indicated thus:

$$\frac{144 \times 17}{24} = 102.$$

We might, by multiplication, get first 2,448, and then divide by 24; but, we may also divide by 24, and then multiply the quotient 6 by 17.

Generally, to begin by division is preferable, because it simplifies the work; and it is absurd to enlarge a number first, to reduce it afterwards.

PROPOSITION IV.

7. *When the product of several factors is to be divided by a number composed also of factors, instead of dividing the product itself, we may divide successively any one of its factors by any one of those of the divisor, until the division is made as far as practicable.*

Because we divide a product by one of its factors, when we strike out this factor (XXXII., 1 and 4).

GENERAL PROPOSITION.

8. The four preceding propositions may be condensed into the following single one:

When several multiplications and divisions are combined in the same operation, it is immaterial in what order you multiply and divide.

OF CANCELLING.

9. The subject of this lesson is very important. It is one of the principles on which are based the simplifications called *cancellings;* which consist in striking out from a *dividend and divisor*, or, in other words, *numerator and denominator*, all common factors.

For cancelling, we must be able to discover the simpler factors of some of the numbers. There are, for this purpose, several rules on the divisibility of numbers, which it is important to be thoroughly conversant with, and which form the subject of the thirty-fourth lesson.

In cancelling, if you strike a factor out of a number, cross the number, and write above or below it the quotient which remains in lieu of the number.

EXAMPLE.

$\frac{72 \times 45 \times 16}{20 \times 12 \times 24}$. Cancelling 4 out of 20 and 72; 5 out of 20 and 45; and 8 out of 16 and 24: the arrangement will be

$$\begin{array}{c} 18 \quad\; 9 \quad\; 2 \\ \cancel{72} \times \cancel{45} \times \cancel{16} \\ \hline \cancel{20} \times 12 \times \cancel{24} \\ \qquad\qquad\quad 3 \end{array}$$

Now, we may still cancel 6 out of 18 and 12; then the remaining 2 of 12, with 2 of the numerator, and also 3 with 9.

This second change will give

$$\begin{array}{c} 3 \quad\; 3 \qquad\; \\ \cancel{18} \quad \cancel{9} \quad\; \cancel{2} \\ \cancel{72} \times \cancel{45} \times \cancel{16} \\ \hline \cancel{20} \times \cancel{12} \times \cancel{24} \\ \qquad \cancel{2} \quad\; \cancel{3} \end{array} = 9.$$

With a little practice, the whole cancelling might have been done, and 9 obtained at once.

Questions.—Repeat the second proposition; the third; the fourth. Will the quotient be the same, if you divide successively by the component factors of the divisor? Why? What is contraction in division? Give an example. Will the order of the factors, in contracted division, affect the quotient? Why? What effect is produced on the indicated product of several factors, if one is struck out? What is left in its place? Is it necessary to write down the quotient 1 as a factor? If, in the same question, several multiplications and divisions are to be made, what will be the effect of changing the order of these operations? May we divide any factor of the dividend by any factor of the divisor? Why? Repeat the general proposition for all the preceding. What is cancelling? Where do you set down the quotient of a number, out of which one or more factors have been cancelled? If a whole number is cancelled, what do you set down in its place? What is understood to occupy its place?

EXERCISES.

1. Prove the multiplications of the preceding lesson by contraction in division.

Divide by contraction—

2. 1,141,365,312 by 36 = 31,704,592.
3. 2,142,273,888 by 72 = 29,753,804.
4. 684,323,328 by 96 = 7,128,368.
5. 17,326,526,400 by 108 = 160,430,800.
6. 81,623,150,400 by 1320 = 61,835,720.
7. 999,999,999 by 81 = 12,345,679.

Perform the following multiplications and divisions by cancelling:

1. $\frac{2 \times 3}{2 \times 3 \times 4} = \frac{1}{4}$.

2. $\frac{2 \times 3 \times 4}{2 \times 3 \times 4 \times 5} = \frac{1}{5}$.

3. $\frac{2 \times 3 \times 10}{3 \times 5 \times 11} = \frac{4}{11}$.

4. $\frac{20 \times 15 \times 8}{30 \times 10} = 8$.

5. $\frac{2 \times 3 \times 5 \times 7 \times 8}{3 \times 4 \times 7 \times 9 \times 11} = \frac{20}{99}$.

6. $\frac{7 \times 11 \times 27 \times 5}{18 \times 10 \times 14 \times 2} = \frac{33}{16}$.

7. $\frac{2 \times 4 \times 7 \times 3}{2 \times 3 \times 5 \times 8 \times 4 \times 7} = \frac{1}{40}$.

8. $\frac{6 \times 1 \times 4 \times 2 \times 7 \times 18}{7 \times 2 \times 3 \times 11 \times 9 \times 25} = \frac{16}{275}$.

9. $\frac{12 \times 11 \times 10 \times 9 \times 8}{2 \times 3 \times 4 \times 5} = 792$.

10. $\frac{3150 \times 15 \times 10}{45 \times 7} = 1500$.

11. $\frac{45 \times 25 \times 40 \times 10 \times 7 \times 9}{20 \times 15 \times 8 \times 6 \times 11} = 178\frac{43}{44}$.

12. $\frac{500 \times 12 \times 2900 \times 12 \times 5 \times 3 \times 57}{860 \times 10 \times 1800 \times 7 \times 3} = 549\frac{51}{301}$.

13. $\frac{216 \times 192 \times 11 \times 7 \times 975 \times 50 \times 10 \times 20}{140 \times 9 \times 11 \times 546 \times 45 \times 15 \times 25} = 243.81$.

LESSON XXXIII.

Before presenting the rules on the divisibility of numbers, it is necessary to introduce some additional definitions and remarks.

ADDITIONAL DEFINITIONS.

1. *An even number* is one that can be divided exactly by 2.

All numbers ending with 0, 2, 4, 6, 8, are even numbers.

2. *An odd number* is one that cannot be divided by 2.

All numbers ending with 1, 3, 5, 7, 9, are odd numbers.

3. *A prime number* is one which cannot be divided

without a remainder, by any other number than itself or unity.

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, &c., are prime numbers.

About one fourth of the odd numbers are prime. They are all one unit above or one unit below a multiple of 6.

$6 \times 12 = 72$ one less is 71, one more 73 } both prime numbers.

$6 \times 11 = 66$ $66 + 1 = 67$, a prime number. $66 - 1 = 65$ is not one.

There are but few products of 6, which are not within one unit of at least one prime number.*

$6 \times 20 = 120$, is one of the exceptions. $120 + 1 = 121$, which is 11×11. $120 - 1 = 119$, " 7×17.

4. Two numbers are said to be *prime to each other*, when no common factor will divide both.

25 and 49 are prime to each other.
25 and 45 are not, because 5 divides both.
Nor are 21 and 49, which have the common factor 7.

5. A factor common to two numbers, is also called their *common divisor*.

4 is a common divisor of 48 and 72.

The greatest number which will divide two or more numbers, is their *greatest common divisor*.

In the preceding example, 4 is a common divisor; but not the greatest between 48 and 72, which is 24. The quotients being respectively 2 and 3, which have no factors, 24 is evidently the greatest between 48 and 72.

In the preceding lesson, instead of reducing $\frac{60}{72}$ to its lowest terms, $\frac{5}{6}$, by successive operations, it might have been done quicker by dividing both terms by their *greatest common divisor*, 12.

It is generally easy to reduce a fraction by successive divisions; but sometimes it requires the finding of the

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* Algebraically expressed, $6n \pm 1$, it contains all the prime numbers.

greatest common divisor, for which a method will be given hereafter (Lesson XXXVIII.).

6. We know that *a fraction is reduced to its simplest expression, or lowest terms, when its two terms are prime to each other.*

7. A fraction thus reduced, and which cannot be simplified, is called an *irreducible fraction.*

8. *A multiple*, it has been already said, is a number which contains another as a factor.

The factor is sometimes also called an *aliquot part* of its multiple.

12 is a *multiple* of 2, of 3, of 4, and also of 6, because each of them will divide it exactly.

2, 3, 4, 6, are *aliquot parts ;* that is, factors of 12.

It will be remarked that the same number may have an infinite number of multiples: 6, 24, 90, 600, 6,666, &c., are all multiples of 6.

But it has a limited number of aliquot parts: 12, for instance, has only the aliquot parts 2, 3, 4, 6.

9. The same number may be the multiple of many others: 12 has already served as an example.

A number which is the multiple of several numbers is called their *common multiple.*

Thus, 12 is the common multiple of 2, 3, 4, and 6.

60, 120, 240, 360, are all common multiples of 2, 3, 4, 6, 12, 15, &c.

Among the common multiples of several numbers, there is evidently one smaller than all the others, and which is just large enough to contain them. This is called the *least common multiple.*

In the last example, 60 is the smallest common multiple; for no smaller number can be found to contain 4 and 15; and, containing these, it of course contains their factors, 2 and 3.

10. The number which expresses the *relation of a multiple to its factor, is also called their* RATIO (XXX., 4).

11. The term *power* is frequently used to express the

number of times that the same factor appears in a product. Thus:

2×2 is the second *power* of 2.	It is generally written 2^2.
3×3 is the second *power* of 3.	" " 3^2.
$3\times 3\times 3\times 3$ is the fourth *power* of 3.	" " 3^4.
$7\times 7\times 7\times 7\times 7$ is the fifth *power* of 7.	" " 7^5.

Questions.—What is an even number? An odd number? A prime number? Are there many prime numbers? If you multiply a number by 6, and add or subtract 1, will it often make a prime number? When are numbers prime to each other? Which are the prime numbers in 43, 45, 51, 59, 61, 63, 67, 73, 77, 81, 89, 91, 93, 99? Are 5 and 7 prime to each other? Are 6 and 9? 9 and 11? 12 and 15? 6 and 25? 23 and 32? 45 and 49? What is a common divisor? The greatest common divisor? When is a fraction reduced to its simplest expression? What is it then called? What is a multiple? Of what numbers are 24; 36; 63; 96, &c., multiples? What is the least common multiple of 2, 3, 4? Of 2, 5, 6? Of 4, 8, 6, 3? Of 5, 3, 6, 2? &c.? What is an aliquot part? What are the aliquot parts of 12, 36, 48? What term may express the relation between multiple and factor? What number expresses the relation between 10 and 2? 12 and 3? 18 and 9? 28 and 7? 66 and 11? 90 and 10? &c. What is the ratio of 4 and 8; 5 and 20; 6 and 42? &c. What is a power?

LESSON XXXIV.

DIVISIBILITY OF NUMBERS.

To simplify and facilitate, in many cases, multiplication, division, the reduction and combination of fractions; to detect errors in those operations; and, in general, to acquire readiness and dexterity in numbers, it is necessary that the arithmetician should be able to discover with ease the simple factors of numbers. I will present here, therefore, some of the most useful rules for this purpose.

1. 2 *divides every even number.*

2. 3 *divides exactly any number when the sum of its figures is divisible by* 3.

Examples.—42; 72; 252 are divisible by 3, because $4+2=6$; $7+2=9$; $2+5+2=9$; and 6 and 9 are multiples of 3.

3. 4 *divides any number the last two figures of which make a multiple of* 4.

624; 476; 1548, are all divisible by 4; because 24; 76; 48, are thus divisible: 146 is not, because 46 is not a multiple of 4.

4. 5 *divides any number which ends with* 0 *or* 5.

20; 35; 60; 175; &c., are all divisible by 5.

5. 6 *divides any even number divisible by* 3 (2).

36; 42; 96; 12,456, &c., are divisible by 2 and 3; and, consequently, by 6.

6. 7 *divides any number when twice the figure of units, subtracted from the other part of the number, leaves for a remainder* 0, *or a multiple of* 7.

Thus:	84 is divisible by	7, because $2 \times 4 = 8$.
	91	"	7, because twice 1 subtracted from 9, leaves 7.
	343	"	7, because $2 \times 3 = 6$, and $34 - 6 = 28$, a multiple of 7.

In 2,401, we might subtract twice 1 from 240, and then try also if the remainder, 238, is divisible by 7, by saying $2 \times 8 = 16$, and $23 - 16 = 7$; which would prove, at last, the divisibility of 2,401.

But this tedious process is not necessary with large numbers; and we may bring down the test to three figures, by using the following rule:

When the number exceeds three figures, divide it into periods of three figures, and subtract the sums of the alternate periods from each other; then, if the remainder is 0, *or a multiple of* 7, *the number is divisible by* 7.

If, after the test has been reduced to three figures, the divisibility is not apparent, use the first rule.

EXAMPLES.

The number 801,801 is divisible by 7, because the two periods are equal.

60,315,325

also is divisible, because $60 + 325 - 315 = 70$; which is an evident multiple of 7.

Again, try . . 39,655
by the second rule, $655 - 39 = 616$, whose divisibility is not evident.

Then by the first rule, $2 \times 6 = 12$, and $61 - 12 = 49$, a multiple of 7. Hence, 39,655 is divisible by 7.

7. 8 *divides any number the three last figures of which form a multiple of* 8.

1,568 is divisible by 8, because 568 is evidently divisible, since its parts 560 and 8 are separately so.

8. 9 *divides any number the sum of whose figures is divisible by* 9.

81; 243; 747; 28,548, are all divisible by 9, because the sums of their figures are respectively 9; 9; 18; 27: which are multiples of 9.

The reason of this rule is so simple that it may be explained here. It is well known that the remainder of the division of 10; 100; 1000, &c., by 9, is 1: consequently, the remainder

Of 20; 200; 2000, &c.,	. .	is 2.
Of 30; 300; 3000, &c.,	. .	" 3,
&c.		&c.

Or, in other words, *the remainder of the division of a number of any order by* 9, *is its significant figure.*

Thus, in 28,548, which is $20{,}000 + 8000 + 500 + 40 + 8$,
the remainders of the division of each order by 9 are separately $\Big\}$ 2 , 8 , 5 , 4 , 8,
consequently the *aggregate remainder* is their sum 27, which being divisible by 9, it follows that the given number itself is so likewise.

When the aggregate remainder is not divisible by 9, its own remainder is also that of the given number. *This second remainder is generally the sum of the figures of the aggregate remainder.*

Thus, in 87,865, the aggregate remainder is $8 + 7 + 8 + 6 + 5 = 34$; which, divided by 9, gives a final remainder, $3 + 4 = 7$, which is also that of the division of 87,865 by 9.

All these remarks apply also to 3. We will make use of this property of 9 and 3, in the Appendix.

9. 10 *divides any number which ends with* 0.
100 " " " *with two* 0*s*.
1000 " " " *with three*, &c.

10. 25 *and* 50 *divide any number, the last two figures of which form a number divisible by them.*

275; 34,275; 9,350, are all divisible by 25.
250; 550; 34,750; 9,350, are divisible by 50.

11. 11 *divides a number when the difference between the sums of its alternate figures is either* 0 *or a multiple of* 11.

405,812 and 290,718,076 are both divisible by 11; because, in the first, $4+5+1=8+2$; and, in the second, $2+0+1+0+6=9$; $9+7+8+7=31$; and $31-9=22$; which is divisible by 11.

12. 12 *divides any number divisible by both* 3 *and* 4.

288 is divisible by 12; because the sum of its figures is 18, divisible by 3 (2), and the two last make a number divisible by 4 (3).

Questions.—How do you know when a number is divisible by 2? 3? 4? 5? 6? 7? 8? 9? 10? 100? 1000? &c. 25? 50? 11? 12? Give a demonstration for 9 and 3.

EXERCISES.

1. Which of the numbers 352; 456; 315; 1,247; 2,677; 5,544; 7,921,851; 4,817,628, are divisible by 3 and by 9?

2. Which of 622; 728; 6,982; 4,176; 918,578; 782,496; 34,926, are divisible by 4? and by 8?

3. Which of 245; 373; 504; 924; 2,897; 9,254,154; 3,457,923; 834,461, are divisible by 7?

4. Which of 465; 270; 975; 3,275; 4,050; 3,000; 925; 7,600, are divisible by 5? by 10? by 25? by 50? by 100? by 1000?

5. Which of 649; 7,285; 315,788; 3,259; 7,821,472; 61,809,132; 9,145,389, are divisible by 11?

6. Which of 168; 296; 8,034; 1,236; 2,358, are divisible by 12? by 24?

LESSON XXXV.

1. 13 *divides a number when the sums of its alternate periods of three figures, subtracted from each other, leave for a remainder* 0 *or a multiple of* 13.

1,001 is divisible by 13, because
1 (of the period of thousands) — 1 (of the period of units) = 0.
236,210 " " because 236 — 210 = 26, a multiple of 13.
657,177,209 " " because 657 + 209 — 177 = 689, which is divisible by 13.

In a case like the last, when the subtraction of the alternate periods has reduced the test to a number of three figures only, it may not always be readily perceived whether the remainder is divisible by 13. We must then do as was done for 7; use a separate rule for three figures, as follows:

Multiply the figure of units by 9, *or, when you can, take one third of it, and subtract between the result and the other part of the number; the remainder must either be* 0 *or a multiple of* 13.

In the remainder, 689, of the preceding example, $9 \times 9 = 81$, and $81 - 68 = 13$; or else, $\frac{9}{3} = 3$, and $68 - 3 = 65 = 5 \times 13$.
Again: 104 is divisible by 13, because $4 \times 9 - 10 = 26$.

Again: *when the last two figures form a number divisible by* 4, *it is sometimes shorter to divide this part by* 4, *and subtract between the quotient and the hundreds.*

In 104, divide 4 by 4, and subtract; you get $1 - 1 = 0$.
In 936, divide 36 by 4, and you get . . . $9 - 9 = 0$.
In 884, divide 84 by 4, which gives 21, and $21 - 8 = 13$.
Showing that all these numbers are divisible by 13.

2. *As regards* 17, *cut off two or three figures to the right of the number, and then divide the part thus cut off by the number of its figures; and, if the subtraction between the quotient and the other part of the given number leaves* 0, *or a multiple of* 17, *the given number is divisible by* 17.

In 297,636, for example, cut off 636, one third of which is 212; then subtract $297 - 212 = 85$, and you see that the whole number is divisible by 17.

N. B.—If the three last figures were not divisible by 3, you could always make them so, by adding or subtracting 17; since the remainder of a division by 3, is always 2 or 1.

In 1,666, cut off 66; of which, take the half, 33; then $33 - 16 = 17$; and the number is divisible. If the given number were not even, make it so, by the addition or subtraction of 17.

In 1,581, for example, add 17 to the last two figures; you get 98, one half of which is 49; and, $49 - 15$ being 34, the number is divisible by 17.

N. B.—In some cases it will be more simple to double the first part of the number. For example, in 1,581, you may double 15, and subtract 30 from 81, which is $51 = 3 \times 17$.

3. 19 *divides a number when twice its last figure, added to the other part, is a multiple of* 19.

114 is divisible by 19, because $2 \times 4 + 11 = 19$.
798 " " because $2 \times 8 + 79 = 95 = 5 \times 19$.
95 itself, because $2 \times 5 + 9 = 19$.

I do not give demonstrations of all these properties of numbers; they would appear too abstruse at present, and in fact belong rather to algebra.

I have added to the rules frequently met with in arithmetics, those for 7, 13, 17, and 19, which I deem sufficiently simple to be performed mentally. Beyond 19, the rules become too complicated for practical purposes.

Questions.—What are the rules of divisibility by 11? 12? 13? 17? 19?

EXERCISES.

1. Which of the numbers 104; 695; 9,099; 615,602; 304,672; 92,416,493; 21,859,838; 2,649,257, are divisible by 13?

2. Which of 189; 266; 2,394; 4,368; 7,728, are divisible by 63; and which by 14? (XXXII., 1.)

3. Which of 225; 540; 1,296; 1,485; 396, are divisible by 15? 36? 44? 99? 55? 495? (XXXVI., 1.)

4. Which of 255; 363; 4,284; 6,001; 724; 816, are divisible by 17?

11 *

5. Which of 171; 215; 361; 589; 617; 931; 1,026, are divisible by 19?

6. How many times is 2 a factor in 8; in 46; in 64?
" 3 " 27; in 99; in 783?
" 7 " 35; in 49; in 84; in 343?

7. What are the simple factors of 324; 640; 826; 375; 2,625; 588; 207,025; 20,025; 8,119; 32,476; 357,236; 224,939?

LESSON XXXVI.

This lesson will contain some general remarks concerning the divisibility of numbers, which will facilitate the application of the foregoing rules.

1. *A number is exactly divisible by a multiple when it contains all its component factors.*

Thus, 12 divides 192, because it is divisible by the component factors, 3 and 4, of 12; and we have seen (XXXII., 1) that we may divide by the factors instead of the whole divisor.

2. *Every number which divides another, divides any one of its multiples.*

In other words, *a number which divides one factor, divides the product.*

Ex.—Since 4 divides 12, it divides also 192.

This follows, also, from Lessons XXXII. and XXXIII. We may get 4 out of each 12 by three subtractions, and thus take out successively the sixteen 12s contained in 192.

3. *A number, prime to others, cannot divide their product.*

60 is equal to 6 × 10, and 5 divides it, because it divides 10; but it does not divide 63, because it divides neither 7 nor 9.

For, by reference to the table of Pythagoras, it will appear evident that the repetition of 7 can make the same number as the repetition of 5, only where the horizontal column of the one meets the vertical column of the other; whereas, the multiple 63 is at the meeting of the column 7 and 9.

4. *A number which divides exactly each part, divides also their sum.*

36 is equal to 12 + 24; both of which are divisible by 4 as well as 36.

For, we may take out the first part by successive subtractions; then the second part will remain, and may be divided exactly.

5. *A number which divides another and one of its parts, divides also the other part.*

Or, in other words, *a number which divides two others, divides their difference.*

4 divides 36 and 12; consequently, also, 36 — 12 = 24.

For, we can take out the first given part by successive subtractions of the divisor; and if, after continuing the subtractions, the last part left a remainder, it would evidently follow that the whole was not divisible, as admitted in the premises.

6. From the two preceding propositions, we conclude that:

When you wish to test the divisibility of a number by a factor, you need not consider any figure or part of the number which is evidently divisible by the given factor.

If it were required to ascertain whether 714,273 is divisible by 7, we might omit the two 7s and 14; so that the trial would depend only on 203, and because 2 × 3 = 6 and 20 — 6 = 14, the divisibility would be proved (XXXIV., 6).

Again: take 264,654, to be divided by 13: we may take out 26 and 65, which are multiples of 13, and try only 4,004; which is divisible by 13, because the two periods are equal.

Here, the parts being separately divisible, the whole is so likewise (4).

7. By the application of the same principle, *we may also frequently ascertain whether a number is divisible or not by another* (even a large one), *by adding or subtracting, at any place of it, such a convenient auxiliary number as will make it divisible by the given factor.*

Then, *if this auxiliary number is not divisible, the* *whole number itself is not so; if it is, the whole number is divisible.*

343 is divisible by 7, because the addition of 7 makes it 350, which is 7 times 50. Hence, $343 = 7 \times 49$.

241 is not divisible by 7, because you must take from it the indivisible number, 31, to make it the multiple, 210, of 7.

13 does not divide 255, because it takes only 5 more units to make it the multiple $260 = 13 \times 20$; so that $255 = 13 \times 19 + 8$.

But 13 divides 234, because the addition of the multiple, 26, makes it 260. Hence, $234 = 13 \times 18$.

Even the large prime number, 367, may be readily discovered to be a factor of 3,303, and also of 4,037; because the addition of 367 to the first, makes it $3,670 = 367 \times 10$: so that $3,303 = 9 \times 367$.

As regards the second number, very little practice in numbers will enable any one to see at a glance that, if 367 be subtracted from the first part, 403, the remainder would be 36 tens, which, joined to the last figure, makes it 367. Hence, $4,037 = 11 \times 367$, being the sum of the two multiples 3670 and 367.

The number 30,927 will be discovered to be divisible by 13, more promptly, by taking away 91 from the middle part, 92; when, in the remainder, 30,017, it will be at once discovered that the difference of the periods is 13 (XXXV., 1).

I merely point out this manner of discovering the factors of numbers, which, with some practice, is applicable in a great many cases; but no precise rule can be given. The successful application of the method depends altogether on the degree of readiness and dexterity of each individual. The young arithmetician should exercise himself in applying it, and it will not be long before he has acquired much expertness in it. This will greatly improve his facility in calculations.*

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* The trials to discover the prime factors of a number, need not be carried beyond that number (square root) which, multiplied by itself, would give the nearest product of equal factors below the given number.

For example: we wish to find whether 1,373 has any factors; we try successively every prime number until we come to 37; and observing that $37 \times 37 = 1,369$, and that the next prime number, 41, multiplied by itself, will exceed 1,600, and, consequently, our given number, we stop the trial at 37; because, evidently, any number above it would require to be multiplied by a factor smaller than 37, and therefore already tried. Hence, we conclude that 1,373 is a prime number.

Questions.—If the component factors divide a number, will the multiple divide it also? If a number divides one factor, does it divide the product? Why? When may a number divide the product of two others? When will it not? If two numbers have a common factor, will it divide their sum and difference? In testing the divisibility of a number, what parts of it may be neglected? How may an auxiliary number be used to simplify the test of divisibility?

END OF PART I.

PART II.

The First Part contains all the principles and rules necessary to understand and execute with facility every elementary operation in arithmetic. Their applications to practical questions, which require generally a combination of several of the rules, will form the subject of the second part.

When a question is proposed, the first step, which is commonly the most embarrassing part of the process of solution, consists in reducing, by a course of reasoning and analysis, the conditions of the question to a *numerical statement*, indicating, by means of the usual signs, the various elementary operations to be performed.

The next step is more mechanical, and consists merely in working out the result from the numerical statement, by means of the elementary rules laid down in the first part.

The analysis of practical questions, as well as the arrangement and simplifications of numerical statements, constitute the chief object of the Second Part; in which no new operation will be found, but only applications of those already known.

For the final execution of the operations, reference will be made to the first part.

I would advise the student never to omit, and the teacher always to require, the setting down at the head of the operations the numerical statement of each question, in the manner that will be shown in the various examples I shall have occasion to explain. There are great advantages in this method; it requires and shows a correct investigation of the data; it enables the operator to discover cancellings and other simplifications, which the combination of operations frequently allows of; and, finally, in case of some error having been committed, it facilitates its detection and the revision of the whole process.

By being careful in the beginning to arrange his work systematically and neatly, the student will soon acquire the faculty to embrace in his mind the numerical statements and the simplifications they lead to; and thereby he will obtain great mental readiness in calculations.

CHAPTER VI.

CONTAINING OPERATIONS IN VULGAR FRACTIONS AND PROPORTIONS.

LESSON XXXVII.

TRANSFORMATION OF FRACTIONS AND NUMBERS, COMMONLY CALLED REDUCTION.

I HAVE placed the complement of the doctrine of fractions in the Second Part, because what follows involves no new principle; and each question, when properly investigated, is a mere repetition or application of what precedes. The operations on fractions belong, therefore, to the practical part of the work; and, like the rest, should not be entered upon until the theoretical principles and elementary rules have been completely mastered.

CASE I.

1. To transform (reduce) an improper fraction into its equivalent whole or mixed number.

In conformity to a general practice, I here make use of the word improper fraction, and introduce the present rule, though the case is nothing more than common division.

Let us take, for example, as an improper fraction, $\frac{12}{4}$. What is it, if not 12 divided by $4 = 3$?

Again: why should $\frac{751}{17}$ be called an improper fraction; and what need is there of a new rule to find out that

751 divided by 17, is equal to $44\frac{3}{17}$?

The rule for division has taught how to obtain the integral part, 44, of the quotient, as well as its fractional part, $\frac{3}{17}$, by setting the remainder over the divisor.

2. I consider it, therefore, almost useless to state that, *to transform an improper fraction into its equivalent whole or mixed number, you divide the numerator by the denominator, as in common division.*

CASE II.

3. To transform a whole or mixed number into its equivalent improper fraction.

We have seen (XXX., 8) how a whole number may be put under the form of a fraction, with a given denominator.

In the case of mixed numbers, therefore, all we have to do is to put the whole number under a fractional form, having the same denominator as the fraction, and then to add the numerators. Let us take, for example,

$$44\tfrac{3}{17}.$$

We know that $44 = \frac{44 \times 17}{17} = \frac{748}{17}$, and this added to $\frac{3}{17}$, is $\frac{751}{17}$.

So that this is not a new operation.

4. *Multiply the whole number by the denominator of the fraction; to the product add the numerator of the fraction, and place the result over the said denominator.*

This is evidently the reverse of the preceding case, and requires exactly the same operations as the proof of division (XVIII., 5). For, since 751 divided by 17, is $44\frac{3}{17}$, if we wished to go back to the *dividend*, we would multiply the *divisor*, 17, by the *quotient*, 44, and add the *remainder*, 3.

N. B.—This rule may also be understood by considering that 1 unit is 17 *seventeenths*, and, consequently, 44 is equal to 44 times this; that is, $44 \times 17 = 748$ seventeenths, and that 3 seventeenths more make 751 seventeenths or $\frac{751}{17}$. But the above mode of reasoning, in this case, is more systematical, inasmuch as it retains the same mathematical uniformity of investigation.

CASE III.

5. *To reduce a fraction to its simplest terms, apply the rules for the divisibility of numbers, and divide both terms by all the common factors you may discover* (XXX., 9).

Or, if the numbers are large, *find their greatest common divisor, and divide both terms by it.*

The first method, with a competent knowledge of the divisibility of numbers, will generally suffice, and prove the shortest.

6. Sometimes, however, a large divisor, common to both, may not be apparent. In that case, resort must be had to the method of the *greatest common divisor*, which will be the subject of the next lesson.

Questions.—How is an improper fraction transformed into its equivalent whole or mixed number? How is a whole or mixed number put under a fractional form? What simple rules does each operation correspond to? How is a fraction reduced to its lowest terms?

EXERCISES.

1. Transform $\frac{56}{7}$; $\frac{192}{16}$; $\frac{1362}{25}$; $\frac{2918}{16}$; $\frac{1325}{101}$; $\frac{5871}{130}$, to equivalent whole or mixed numbers.

2. Change into fractions, $12\frac{1}{2}$; $12\frac{7}{9}$; $14\frac{7}{10}$; $151\frac{2}{7}$; $183\frac{5}{21} = \frac{3848}{21}$; $1782\frac{25}{27} = \frac{48139}{27}$; $6579\frac{291}{1000}$.

3. Reduce to their simplest terms, by cancelling factors,

$\frac{48}{56}$; $\frac{60}{125}$; $\frac{84}{170}$; $\frac{238}{252} = \frac{17}{18}$; $\frac{1344}{1536} = \frac{7}{8}$; $\frac{240}{288} = \frac{5}{6}$; $\frac{192}{240} = \frac{4}{5}$; $\frac{1750}{3150} = \frac{5}{9}$; $\frac{38,220}{176,400} = \frac{13}{60}$.

4. Change 15 into a fractional expression whose denominator shall be . . 7

23	"	"	10
45	"	"	25
51	"	"	500
65	"	"	272

LESSON XXXVIII.

THE GREATEST COMMON DIVISOR.

1. The operation for finding the greatest common divisor, is a series of divisions. It depends on the principle explained in Lesson XXXVI., 5, that,

A number which divides another and one of its parts, divides also the other part.

2. Let it be supposed, for instance, that, in order to reduce the fraction

$$\frac{276}{360}$$

to its lowest terms, we wish to find the greatest common divisor between its numerator and denominator: we remark, first, that this number cannot be greater than 276; and, therefore, it is proper to try whether 276 itself might not be the greatest common divisor. With this view, we divide 360 by 276, and get a quotient, 1, which we set above the divisor, 276.

OPERATION.

	1	3	3	2
360	276	84	24	12 = *Ans.*
84	24	12	0	

Because there is a remainder, 84, we conclude that 276 is not the greatest common divisor.

But, whatever this greatest common divisor may be, since it must divide 360 and the part, 276, now taken out of it, it must also divide the other part, 84 (XXXVI., 5).

The question is, therefore, reduced to finding the greatest common divisor between 276 and 84.

For this purpose, upon the same principle as before, we divide 276 by 84, and get a quotient, 3, which we set likewise above the new divisor.

There being still a remainder, 24, we conclude that 84 is not yet the greatest common divisor sought.

But, whatever it may be, we know that it divides 84 and 276, and therefore, also, the new remainder; and the question is further reduced to finding the greatest common divisor between 84 and 24.

By a new division, we have now a quotient, 3, and another remainder, 12; which shows that 24 itself is not the number sought, and that we must look for it in the last remainder, 12.

Accordingly, we divide 24 by 12; and, getting an exact quotient, 2, we conclude that 12 *is the greatest common divisor.*

3. For, it divides 24 and all the preceding remainders, which are composed of multiples of 24 and 12, as can easily be verified.

$$24 = 12 \times 2.$$
$$84 = 24 \times 3 + 12.$$
$$276 = 84 \times 3 + 24 = 23 \times 12.$$
$$360 = 276 + 84 = 30 \times 12.$$

4. Furthermore, it is evidently the greatest, because 12 is one of the remainders, which must contain it.

We conclude, therefore, that the simplest form of the fraction $\frac{276}{360}$, is $\frac{23}{30}$.

The example shows the best manner of arranging the operation.

5. Hence, *to find the greatest common divisor between two numbers:*

I. *Divide the greater number by the less; this by the remainder; then, the first remainder by the second; the second by the third, and so on, until one of them divides exactly the preceding. This last remainder will be the greatest common divisor.*

II. *If the last remainder be* 1, *we conclude that the numbers have no other common measure than unity; or, in other words, that they are prime to each other, and the fraction irreducible.*

As regards the last case, let us take the fraction $\frac{317}{873}$.

OPERATION.

	2	1	3	15	1	1	2
873	317	239	78	5	3	2	1
239	78	5	3	2	1	0	

The operation shows that no other number than unity divides both terms; consequently, the fraction is in its simplest form, and there is properly no common divisor.

6. It furnishes also another useful remark. *When, in the course of the operation, one of the remainders is a* prime number, *which does not divide the preceding remainder, it is certain that no common divisor exists, and it is unnecessary to continue the operation.*

In the last example, 5 does not divide 78; and, consequently, since it is a prime number, no divisor can be found for it, but unity, and it would be wrong to carry the operation further.

7. *To find the greatest common divisor between several numbers, find it between the two smallest of them; then, between this and the third number in size; between the second divisor, thus obtained, and a fourth number; and so on, to the last.*

It is evident that the greatest common divisor between several numbers, cannot exceed the common divisor between any two of them.

Therefore, if we knew them all for pairs, there would only remain to find the number common to them all.

For this reason, to proceed systematically, we begin with the two smallest numbers, and continue as above stated.

Let us take, as an example, 504, 756, 1260, and 2,058.

In the first place, we perform the operation between 504 and 756, and find that their greatest common divisor is 252. We know that this particular greatest divisor must contain the general one.

FIRST OPERATION.

	1	2
756	504	252
252	..	

Therefore, we now operate upon it and 1260, and find that 252 answers also for 1260.

SECOND OPERATION.

	5
1260	252
·00	

Lastly, we operate with 2058, and find 42 for the final result.

THIRD OPERATION.

	8	6
2058	252	42
42	..	

REMARKS.

The operation of the greatest common divisor should always be simplified, when practicable, as follows:

8. REMARK I.—*Strike out any prime factor contained in one of the numbers, and not in the other.* For, it is evident that it can make no part of a common divisor.

For example, if we take 249 and 377:

We observe that 249 is divisible by 3; while 377 is not. Consequently, 3 can be no part of the greatest divisor. I, therefore, suppress it out of 249, which leaves 83; which, being a prime number, leads me at once to the conclusion that 83 must be the greatest common divisor, if any.

But, 377 is not divisible by 83; therefore, without further operation, it is made apparent that 249 and 377 are prime to each other, and $\frac{249}{377}$ irreducible.

9. REMARK II.—*Take out any factor common to both, and set it aside as part of the greatest common divisor, by which to multiply the final result of the operation.*

For example, in 2,150 and 3,612, I see at once that 25 is a factor of the first number, but not of the second; and, according to the first remark, I strike it out; by which the trial is now to be made on the smaller number, 86, with 3,612.

Now, I see that 2 divides both; hence, 2 is part of the greatest common divisor. Dividing 86 by it, the quotient is 43; and, because this is a prime number, it must be the reduced greatest common divisor, if any. A trial proves it to be so; consequently, the whole divisor is 2×43, and the first number, 2,150, is equal to 86×25; the second, to 86×42.

Questions.—What is the greatest common divisor? Give the rule for finding it. What principle does the rule depend on? How is the operation arranged? Where do you put the quotient? How do you know that there is no common divisor? What is the character of the fraction in that case? If you come to a prime number for a remainder, and it does not divide the preceding one, what do you conclude? How is the greatest common divisor between several numbers found? Why? What may be done, if one number contains a factor not in the other? If it contains one common to both?

EXERCISES.

Find the greatest common divisor of

1. 246 and 372. . . . . . *Ans.* 6.
2. 9,024 and 3,760. . . . . *Ans.* 752.
3. 637 and 143. . . . . . *Ans.* 13.
4. 999 and 592. . . . . . *Ans.* 37.
5. 3,072 and 912. . . . . . *Ans.* 48.
6. 138,174 and 49,365. . . . . *Ans.* 3.
7. 132,568 and 920,712. . . . . *Ans.* 1,816.
8. 16,784 and 3,144. . . . . *Ans.* 8.
9. 101,549 and 40,103. . . . . *Ans.* 7.

Reduce to their lowest terms, by the greatest common divisor,

1. $\frac{192}{576}$. *Ans.* $\frac{1}{3}$.
2. $\frac{252}{364}$. *Ans.* $\frac{9}{13}$.
3. $\frac{1344}{1536}$. *Ans.* $\frac{7}{8}$.
4. $\frac{512}{4096}$. *Ans.* $\frac{1}{8}$.
5. $\frac{5665}{5720}$. *Ans.* $\frac{103}{104}$.
6. $\frac{7944}{8916}$. *Ans.* $\frac{662}{743}$.
7. $\frac{8904}{4494}$. *Ans.* $\frac{212}{107}$.
8. $\frac{2433}{13787}$. *Ans.* $\frac{3}{17}$.
9. $\frac{314175}{100005}$. *Ans.* $\frac{355}{113}$.
10. $\frac{7992}{11544}$. *Ans.* $\frac{9}{13}$.
11. $\frac{156933}{19557}$. *Ans.* $\frac{329}{41}$.
12. $\frac{100110}{31866}$. *Ans.* $\frac{355}{113}$.

13. $\frac{54369}{73355}$. *Ans.* $\frac{63}{85}$.

14. $\frac{42,614,574,994,432}{149,720,237,927,424} = \frac{16,807}{59,049}$.

15. $\frac{7,241,379,310,344,827,586,206,896,551}{9,999,999,999,999,999,999,999,999,999} = \frac{21}{29}$.

1. What is the greatest common divisor between 336; 720; 1736? *Ans.* 8.

2. " " 2961; 799; 564; 94? *Ans.* 47.

3. " " 210; 462; 2157; 189? *Ans.* 21.

LESSON XXXIX.

TRANSFORMATION OF VULGAR INTO DECIMAL FRACTIONS.

1. A great many operations are much simplified by this transformation. A common fraction, it has been said, is nothing but the expression of the quotient of the numerator by the denominator.

2. Hence, to change a vulgar into a decimal fraction: *Divide the numerator by the denominator, annexing ciphers to the numerator as far as necessary, to give the requisite number of decimals.*—For example,

3. Let it be proposed to transform $\frac{7}{8}$ into a decimal fraction. Since 7 *units* do not contain 8, there are no units in the quotient, and we change the 7 into 70 *tenths*, by adding a 0 to it, and continue the division by adding 0s to the remainders, as in the case of the extension of the quotient, (XXVII., 3.)

OPERATION.

```
70  |8
 60 |-----
  40|0.875
```

4. Two distinct cases will occur in this transformation.

I. The division may terminate, and an exact quotient be obtained.

II. The division may never end: then the quotient will be *an infinite decimal.*

CASE I.

5. *The division will always end when the denominator does not contain any other factor than 2 and 5.*

For, it is readily seen that either 2 or 5 will divide exactly any number as many times as there are 0s at the end of it. Thus:

70	will be divided	once	by either	2 or 5.
700	"	twice	"	"
7,000	"	three times	"	"

The last for example, divided three times by 2, might be written

$$\frac{7 \times 10 \times 10 \times 10}{2 \times 2 \times 2};$$

where, evidently, each 2 may be cancelled out of the corresponding 10. And the same thing may be said of 5, in

$$\frac{7 \times 10 \times 10 \times 10}{5 \times 5 \times 5}.$$

Therefore, $\frac{7}{8}$, because 2 is three times factor in 8, will require the addition of three 0s to the numerator, changing the fraction into $\frac{7000}{8}$, the quotient of which will end at the third decimal place; and we get $\frac{7}{8} = 0.875$.

$\frac{13}{25}$, in which 5 is twice factor, will require the addition of only two zeros, and the exact quotient will have but two decimals, and we get $\frac{13}{25} = 0.52$.

In the fraction $\frac{317}{1250}$, which is equal to $\frac{31}{2 \times 5 \times 5 \times 5 \times 5}$, the factor, 5, being found four times in the denominator, the complete quotient will contain 4 decimals, and we shall find $\frac{317}{1250} = 0.2536$.

6. In general, *the quotient will contain a number of figures equal to the greatest number of times that either 2 or 5 is factor in the divisor* (that is, to the greatest power of either; XXXIII., 11).

For, it is only the factor which is found the greatest number of times in the denominator, that determines the number of decimal places; so that, in the last example, we have four, and not five, of them. The reason of this is, that each 10 is formed of 5 and 2 as joint factors; therefore, the exact division by 5 implies that by 2.

The above remark will enable the operator to determine how far the division is to be carried, if he wishes to get a complete quotient: it will also be a check against mistakes.

CASE II.

7. *The division cannot end when any other prime factor than* 2 *or* 5 *exists in the denominator.*

For, no other prime factor divides exactly 10; and, consequently, none can divide its decimal multiples or powers 100, 1000, &c. (XXXVI., 3.)

In this case, therefore, the decimal fraction will be infinite; and, furthermore, if carried far enough, the same figures will be repeated periodically. For, since the quotient cannot end, we must at last exhaust all the forms of the remainder, and come back to one of those that have preceded, when the same rotation of quotients will return. Thus we find:

$\frac{1}{3} = 0.3333$, &c. $= 0.\overline{3}$.

$\frac{3}{11} = 0.272727$, &c. $= 0.\overline{27}$.

$\frac{12}{11} = 1.090909$, &c. $= 1.\overline{09}$.

$\frac{13}{37} = 0.351351$, &c. $= 0.\overline{351}$.

$\frac{6}{7} = 0.857142857142$, &c. $= 0.\overline{857142}$.

$\frac{11}{12} = 0.91666$, &c. $= 0.91\overline{6}$.

$\frac{7}{24} = 0.291666$, &c. $= 0.291\overline{6}$.

These infinite decimal fractions are called *repeating* or *periodical* fractions.

N. B.—To show that the figures must be conceived to be indefinitely repeated, a conventional sign is made to separate the period. Some use a dot over each figure of the period; but it is not as distinct as the angular line (┌—) here exhibited.

8. If the denominator of the fractions produces a long period, and it is not important to carry the operation so far, the sign + may be placed after the last decimal figure (XXVII., 4.)

Thus, we may write $\frac{6}{7} = 0.857 +$, if we do not wish to obtain a nearer approximation than one thousandth.

9. This sign is omitted when the decimal part is judged to be sufficiently accurate for the purpose intended; then it is customary to add 1 to the last figure, if the following is 5 or more (XXVII., note.) Thus, we would write:

$\frac{2}{3} = 0.67$; $\frac{3}{11} = 0.273$; $\frac{6}{7} = 0.86$;

$\frac{11}{12} = 0.917$, or $= 0.9167$, } according to the degree of
$\frac{7}{24} = .2917$, or 0.29167, } approximation wanted.

It may be observed that sometimes, as in the case of $\frac{11}{12}$, $\frac{7}{24}$, the period does not begin at once.

10. *The period generally begins after a number of decimals equal to the greatest number of times that either* 2 *or* 5 (not both) *is factor in the denominator.*

Thus, $\frac{4}{75} = \frac{4}{3 \times 5 \times 5} = 0{\cdot}0\overline{53}$; the period begins after two figures.

$\frac{7}{24} = \frac{7}{3 \times 2 \times 2 \times 2} = 0.291\overline{6}$; the period begins after three figures.

11. This can be readily demonstrated by multiplying both terms of the fraction by as many equal factors as will change each complement factor of its denominator into 10.

Thus, if we had to transform $\frac{2}{225}$, which is equal to $\frac{2}{9 \times 5 \times 5}$, the multiplication of both terms by twice 2, would change it into $\frac{2 \times 2 \times 2}{9 \times 10 \times 10} = \frac{8}{9 \times 100}$. Now, $\frac{8}{9} = 0.\overline{8}8$, and evidently the division by 100 would push the period two places to the right, and make it $0.00\overline{8}8$, which begins at the third figure.

TRANSFORMATION OF DECIMALS INTO VULGAR FRACTIONS.

12. *Place the decimal fraction, as a numerator, over the number which gives its name to its last units;* or, in other words, *over* 1, *with as many* 0s *as there are decimal figures.*

Thus, 0.2; 0.25; 0.009; 4.19; 5.0007, will be, respectively, $\frac{2}{10}$; $\frac{25}{100}$; $\frac{9}{1000}$; $4\frac{19}{100}$ or $\frac{419}{100}$; $5\frac{7}{10,000}$ or $\frac{50,007}{10,000}$.

Then, if practicable, you may reduce the fraction to its lowest terms, as in the first two examples, which become $\frac{1}{5}$ and $\frac{1}{4}$.

13. *In the case of a repeating fraction, we may return to the original fraction by placing the period over as many 9s as there are decimals, and then reducing the fraction down to its lowest terms* (XXXVII., 5.)

Thus:

$0.\overline{3} = \frac{3}{9} = \frac{1}{3}$; $0.\overline{27} = \frac{27}{99} = \frac{3}{11}$; $0.\overline{9} = \frac{9}{9} = 1$;
$0.\overline{351} = \frac{351}{999} = \frac{13}{37}$; $0.\overline{857142} = \frac{857142}{999999} = \frac{6}{7}$.

This is easily demonstrated by considering that:

$\frac{1}{9} = 0.111$, &c.; $\frac{1}{99} = 0.0101$, &c.; $\frac{1}{999} = 0.001001$, &c.;
or, in general, that 1 divided by a number composed of 9s, forms a period of as many figures as there are 9s; this period consisting of 1 preceded by 0s. Now, any other period may be considered as the product of the number which forms it, by one of the above. For example:

$$0.351351, \&c. = 351 \times 0.001001, \&c. = \frac{351}{999} = \frac{13}{37}.$$

If the period does not begin next to the units' point, transform it first, reduce and annex the fraction to the decimal part which precedes; then change the mixed number, thus formed, into a vulgar fraction, by dividing it as above (12), *by the decimal denominator of its order, and reducing.*

Examples: $0.91\overline{6} = 0.91\frac{6}{9} = \frac{91\frac{2}{3}}{100} = \frac{275}{300} = \frac{11}{12}$.

$$0.07\overline{954} = 0.79\frac{54}{99} = \frac{79\frac{6}{11}}{1000} = \frac{875}{11000} = \frac{7}{88}.$$

Addition, subtraction, multiplication, and division of these fractions, when perfect accuracy is desired, are best performed by changing them into vulgar fractions.

Questions.—How do you transform a vulgar fraction to a decimal one? When will you get a limited number of decimals? How many decimals will you have in that case? What is an infinite fraction? What is a repeating one? Will an infinite division always produce a repeating fraction? How do you indicate the period? If you do not reach the end of the period, how do you show that the decimals are not complete? Does the period always begin after the decimal point? When does it not? How far will it begin from it? How do you change a decimal fraction into a vulgar one? How, a simple repeating one? How, a repeating fraction which does not begin at the decimal point? How many decimals will $\frac{3}{32}$; $\frac{1}{512}$; $\frac{9}{80}$; $\frac{11}{625}$, &c. give? When will the period begin in the decimals of $\frac{11}{28}$; $\frac{2}{29}$; $\frac{1}{340}$; $\frac{1012}{582}$; $\frac{355}{113}$? &c.

EXERCISES.

Transform into decimal fractions:

1. $\frac{1}{16} = .0625.$ 2. $\frac{16}{289} = .0553+$
3. $\frac{700}{793} = .8827+$ 4. $\frac{653}{633} = 1.03159557+$
5. $\frac{447}{257} = 1.73930$, within one hundredth of thousandth.
6. $\frac{1}{361}$ . . to the tenth place of decimals.
7. $\frac{1}{913}$ . . to within one millionth.
8. $\frac{63}{64}$ . . until the division ends.
9. $\frac{77}{148}$ . . until the period is formed.
10. $\frac{7}{143}$ . . " "
11. $\frac{1}{81} = 0.\overline{012345679}.$
12. Change 0.625; 0.550; 0.7250; 0.21, into the simplest equivalent vulgar fractions.
13. What fractions are equal to the periods $0.\dot{1}62$; $0.\dot{1}8$; 0.72; $0.\overline{83}$; $\overline{.027}$; $.\overline{02439}$; $0.0\overline{307692}$; $.03\overline{205128}$?

LESSON XL.

REDUCTION OF FRACTIONS TO THE SAME DENOMINATOR.

1. This transformation has for its object to express several fractions, having different denominators, in units of the same kind. It is a necessary operation, preparatory to the addition and subtraction of fractions.

For instance, we can add $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$, because they are 1, 2, and 3 small relative units of the same kind, namely, 1 *seventh*, 2 *sevenths*, 3 *sevenths*, whose relation to a principal unit is expressed by the denominator, 7; and we get $\frac{1}{7}+\frac{2}{7}+\frac{3}{7}=\frac{6}{7}$, by merely adding the numerators; $\frac{2}{7}$ more would be $\frac{8}{7}$, and so on.

2. But, if we had to add $\frac{1}{2}$ and $\frac{1}{3}$, the case would be different.

The units being dissimilar, no single amount could be formed of them (it is neither 2 halves nor 2 thirds); and some transformation is necessary to reduce them to expressions of similar units.

If, for example, they were parts of feet:

$\frac{1}{2}$ might be called . . . . 6 inches.
$\frac{1}{3}$ " " . . . . 4 "

And the sum of the $\frac{1}{2}$ and the $\frac{1}{3}$ of a foot might be expressed under the name of inches, as . . . . . } 10 inches.

3. With abstract numbers, however, the relation of units being altogether numerical, the fractions must be transformed into expressions numerically equal.

This is readily accomplished by choosing, for a common denominator, any one of the common multiples of the given denominators; such, as in this case, as 6, 12, 18, 24, &c., either of which may be assumed for the common denominator. The transformation of the fractions will then be made (as in Lesson XXX., 6) *by multiplying both terms of each by the factor which will change its own denominator into the common one.*

This factor is evidently the quotient of the new by the old denominator.

Thus, according to the object in view,

$\frac{1}{2}$ and $\frac{1}{3}$ may be transformed respectively into

$\frac{3}{6}$ and $\frac{2}{6}$	and then added thus	$\frac{5}{6}$	which are all equivalent.
or $\frac{6}{12}$ and $\frac{4}{12}$		$\frac{10}{12}$	
or $\frac{9}{18}$ and $\frac{6}{18}$		$\frac{15}{18}$	
&c.	&c.		&c.

LEAST COMMON MULTIPLE.

4. Generally, however, the simplest form is preferred, and *the least multiple* chosen for the new denominator.

In the preceding example, it would be 6.

But the least common multiple of several numbers is not always so evident, and a systematic method must be used to find it.

I recommend the following, which depends upon this principle, that a number is exactly divisible by another when it contains all its factors (XXXVI., 1). Therefore,

5. *Decompose the largest number into its prime fac-*

tors; then examine another number; and, if it contain any factors beside those of the first number, annex them to it with the sign of multiplication; do the same thing, successively, with all the numbers.

The product of all the factors, thus connected, will be the least common multiple of the given numbers.

For, it is evident that it must contain every number; since a number which has among its factors those of another, contains it: 216 contains 24, because it has 3 and 8 among its factors.

It is the least common multiple, because it is composed of only such factors as are indispensable to make it contain every individual number.

6. The same factor is frequently found several times in the same number, and should consequently appear, in the least common multiple, as many times (and no more) as it is found in the number which contains it the greatest number of times *as a factor.*

In other words, the highest power (XXXIII., 11) of any one factor must always appear in the least common multiple.

FIRST EXAMPLE.

Let it be proposed to find the least common multiple between

2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 48.

Take 48 as the largest, and decompose it into its simple factors, $3\times2\times2\times2\times2$; or, for those who understand algebra, ... 3×2^4.

Now, consider the second number,

$36=3\times3\times2\times2$, or . . . . $3^2\times2^2$.

Since it contains the factor, 3, once more than 48, we must annex it to the first series of factors, which increases it to

$3\times3\times2\times2\times2\times2$, or . . . $3^2\times2^4$,

which now contains both 48 and 36.

A short examination shows that all the other numbers are contained in the last compound multiple, which we conclude to be the smallest.

The multiplications being effected, this number is only 144.

That it is the least common multiple sought is evident, since it contains the factors of all the numbers, and each factor only as many times as it is contained in that number which contains it most (that is, raised to its highest power).

SECOND EXAMPLE.

Let us now take the numbers

49; 21; 16; 75; 120; 840; 108; 112.

The largest, decomposed into factors, is
$7\times3\times2\times2\times2\times5$, or . . $7\times3\times2^3\times5$.

Now, 120 is evidently part of it already.

112 being $7\times2\times2\times2\times2$, or 7×2^4, contains 2 once more as a factor. Therefore, the first number must be multiplied by it, and we get
$7\times3\times2\times2\times2\times2\times5$, or . $7\times3\times2^4\times5$.

Considering, now, 108, which is $3\times3\times3\times4$, or $3^3\times4$, we have to annex to the common multiple, the factor 3, twice, by which it is increased to
$7\times3\times3\times3\times2\times2\times2\times2\times5$, or $7\times3^3\times2^4\times5$.

75 being $5\times5\times3$, we have for it an additional factor, 5, which raises the common multiple to
$7\times3\times3\times3\times2\times2\times2\times2\times5\times5$, or $7\times3^3\times2^4\times5^2$.

To contain 49, it requires an additional factor, 7, and becomes
$7\times7\times3\times3\times3\times2\times2\times2\times2\times5\times5$,
or . . . . $7^2\times3^3\times2^4\times5^2$,*
which completes the least common multiple, since 16 and 21 are evidently already in it. The product being made, gives for it . . . . . . 75,600.

7. In practice, so minute a decomposition is not necessary. The young arithmetician will soon learn to make

* The simplicity of this notation recommends it to those who understand it.

most of the analysis mentally, and discover readily, not only the simple prime factors, but also the number of times they are to appear in the least common multiple (in other words, the power to which each is to be raised). This method will be found more expeditious and improving than that usually adopted.

Both methods require a ready perception of the divisibility of numbers.

Questions.—What is the object of the reduction of fractions to the same denominator? Can fractions, having the same denominator, be added? Why? Can fractions, with different denominators, be added? What must be done first? What numbers will answer for common denominators? Which of them is preferable? How is the least common multiple found? What is necessary in order that a number, composed of factors, may contain another?

EXERCISES.

Find the least common multiple,

1. Of 3; 6; 21.
2. " 3; 5; 21; 35.
3. " 8; 63; 560. . . . *Ans.* 5,040.
4. " 8; 11; 13; 25; 43. . . *Ans.* 1,229,800.
5. " 15; 11; 20; 24; 33; 55; 120. *Ans.* 1,320.
6. " 4; 8; 12; 18; 24; 36. . *Ans.* 72.
7. " 9; 12; 18; 21; 25; 45; 54; 105; 225. . . . . . *Ans.* 6,300.
8. " 60; 28; 240; 225; 490; 720. *Ans.* 176,400.
9. " 20; 48; 280; 960; 1,800; 5,040; 6,860. . . . . . *Ans.* 4,939,200.
10. " 11; 17; 19; 21; 7; 51; 187; 133; 399. . . . . . *Ans.* 74,613.

LESSON XLI.

REDUCTION OF FRACTIONS TO THE LEAST COMMON DENOMINATOR.

1. After having found the least common multiple of the denominators of several fractions, it is very easy to

transform them, so that all may have the least common multiple for their denominator.

This is a transformation by augmentation, which it has been taught, in Lesson XXX., is made by multiplying both terms of the fraction by the number which will raise its denominator to the size of the common denominator; *this number is the quotient of the least common multiple by the denominator of the fraction* (XXX., 6).

Let us take, for example, $\frac{2}{3}$; $\frac{3}{4}$; $\frac{7}{8}$; $\frac{5}{24}$; $\frac{11}{48}$; $\frac{5}{32}$.

The least common multiple is readily discovered to be 96; and then the terms of the various fractions must be multiplied each as follows:

$$\frac{2 \times 32}{3 \times 32};\ \frac{3 \times 24}{4 \times 24};\ \frac{7 \times 12}{8 \times 12};\ \frac{5 \times 4}{24 \times 4};\ \frac{11 \times 2}{48 \times 2};\ \frac{5 \times 3}{32 \times 3}.$$

The operation thus indicated, gives at last, by effecting the multiplications,

$$\frac{64}{96};\ \frac{72}{96};\ \frac{84}{96};\ \frac{20}{96};\ \frac{22}{96};\ \frac{15}{96}.$$

for the fractions reduced to their least common denominator.

2. The decomposition into factors of the least common multiple, facilitates the transformation of the fractions, when the numbers are too large to see, at once, as in the preceding example, by what number each denominator is to be multiplied. For, then the factors of each denominator being known, it will be easily discovered by what factors it must be multiplied, to be raised up to the common multiple, and it will be sufficient to

3. ***Multiply both terms of each fraction by all the factors of the common denominator, which are not in its own.***

Let us take, for example, the fractions

$$\frac{17}{28};\ \frac{11}{24};\ \frac{173}{225};\ \frac{319}{490}.$$

The least common multiple will be found to be

$7^2 \times 3^2 \times 5^2 \times 2^3$;

or $7 \times 7 \times 3 \times 3 \times 5 \times 5 \times 2 \times 2 \times 2 = 88{,}200$;

in which 3, 5, and 7 are twice factors, and 2 three times.

Now, if we compare with it each denominator, we find that

$$\left.\begin{array}{r} 28 = 2\times 2\times 7 \\ 24 = 2\times 2\times 2\times 3 \\ 225 = 5\times 5\times 3\times 3 \\ 490 = 7\times 7\times 5\times 2 \end{array}\right\} \text{or} \left\{\begin{array}{l} 2^2\times 7 \\ 2^3\times 3 \\ 5^2\times 3^2 \\ 7^2\times 5\times 2 \end{array}\right.$$

$$\text{And that } \left.\begin{array}{r} 28 \\ 24 \\ 225 \\ 490 \end{array}\right\} \text{wants the factors} \left\{\begin{array}{l} 2\times 5\times 5\times 3\times 3\times 7 = 3{,}150. \\ 7\times 7\times 3\times 1\times 5\times 5 = 3{,}675. \\ 7\times 7\times 2\times 2\times 2 = 392. \\ 3\times 3\times 5\times 2\times 2 = 180. \end{array}\right.$$

by which the respective fractions must be multiplied, and which changes them into

$$\frac{53{,}550}{88{,}200};\quad \frac{40{,}435}{88{,}200};\quad \frac{67{,}816}{88{,}200};\quad \frac{33{,}420}{88{,}200}.$$

Though these results would be obtained by a skilful calculator without so much detail, I would even recommend that the student be made at first to show the full arrangement of the operations; placing the least common divisor, indicated by its factors as above, at the head of his work, and then

$$\frac{17\times 2\times 5\times 5\times 3\times 3\times 7}{28\times 2\times 5\times 5\times 3\times 3\times 7};\quad \frac{11\times 7\times 7\times 3\times 5\times 5}{24\times 7\times 7\times 3\times 5\times 5};$$

$$\frac{173\times 7\times 7\times 2\times 2\times 2}{225\times 7\times 7\times 2\times 2\times 2};\quad \frac{319\times 3\times 3\times 5\times 2\times 2}{490\times 3\times 3\times 5\times 2\times 2};$$

He will thus understand the operation better, and be made sooner sufficiently familiar with it, to dispense with the dissection of the denominators.

Questions.—How do you reduce fractions to the least common denominator? Repeat the rule for large numbers. By what do you multiply each denominator? Is it always necessary to decompose the fractions into factors?

EXERCISES.

Reduce to the least common denominator:

1. $\frac{3}{4}$; $\frac{5}{6}$; $\frac{7}{8}$; $\frac{6}{7}$.
2. $\frac{2}{11}$; $\frac{7}{9}$; $\frac{3}{5}$; $\frac{2}{3}$; $\frac{4}{33}$.

3. $\frac{2}{5}$; $\frac{4}{7}$; $\frac{3}{10}$; $\frac{6}{35}$; $\frac{7}{24}$; $\frac{9}{40}$.

4. $\frac{3}{13}$; $\frac{4}{39}$; $\frac{2}{15}$; $\frac{6}{65}$; $\frac{7}{26}$; $\frac{11}{30}$.

5. $\frac{7}{15}$; $\frac{3}{11}$; $\frac{1}{20}$; $\frac{13}{24}$; $\frac{2}{33}$; $\frac{17}{55}$; $\frac{43}{120}$.

6. $\frac{2}{9}$; $\frac{5}{18}$; $\frac{11}{21}$; $\frac{24}{25}$; $\frac{11}{45}$; $\frac{6}{105}$; $\frac{53}{54}$; $\frac{103}{225}$.

7. $\frac{11}{20}$; $\frac{4}{13}$; $\frac{5}{26}$; $\frac{61}{65}$; $\frac{49}{52}$; $\frac{3}{4}$; $\frac{4}{5}$; $\frac{7}{10}$.

8. $\frac{3}{22}$; $\frac{6}{121}$; $\frac{4}{33}$; $\frac{2}{99}$; $\frac{12}{11}$; $\frac{7}{18}$; $\frac{23}{198}$; $\frac{1}{363}$.

LESSON XLII.

ADDITION OF FRACTIONS.

1. *The object of the addition of fractions is to find a single amount equal in value to the sum of several fractions.*

Question.—$\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$ *of a yard have been successively cut from a piece of cloth : how many yards and parts of a yard have been taken out of the piece?*

RULE.

2. I. *If the fractions have the same denominator, add the numerators together, and place their sum over the common denominator.*

II. *If they have not the same denominator, begin with reducing them to the least common denominator.*

3. It has been said, in Lesson XL., that we may add together

$$\tfrac{1}{7} + \tfrac{2}{7} + \tfrac{3}{7},$$

because they are units of the same magnitude; and, consequently, we add

1 *seventh,*	as we would add	1 *apple.*
2 *sevenths,*	"	2 *apples.*
3 *sevenths,*	"	3 *apples.*

The result being 6 *sevenths* in one case, and 6 *apples* in the other.

The only difference being that, in the first case, the name of the units has reference only to their size, as parts of the same thing, or unit of comparison, and is expressed by their numerical relation to that thing.

While, in the second case, the name of the units expresses merely their nature, without reference to their size.

4. But if, instead of units of equal magnitude, it were proposed to add

$$\tfrac{2}{3} + \tfrac{3}{4} + \tfrac{5}{6} + \tfrac{7}{8},$$

the numbers to be added being composed of *thirds, fourths, sixths, eighths*—all units of different sizes—could not be added into one single number. Therefore, a transformation must precede the addition, in order to give to the units of each fraction the same *numerical denomination.*

5. The least common denominator being . . . .

Arrange the operation as follows, and then form each numerator according to Lesson XXX., by multiplying the old one by the number which would make its denominator 24.

OPERATION.

$$3 \times 2 \times 4 = 24$$

$$\begin{array}{l|l} \tfrac{2}{3} \;.\; 2 \times 2 \times 4 = 16 & \\ \tfrac{3}{4} \;.\; 3 \times 3 \times 2 = 18 & \\ \tfrac{5}{6} \;.\; 5 \times 4 = 20 & \\ \tfrac{7}{8} \;.\; 7 \times 3 = 21 & \\ \qquad\qquad\qquad 75 & 24 \\ & 3\tfrac{3}{24} = 3\tfrac{1}{8} \end{array}$$

The new numerators being written under each other, may now be added. Their sum, in the example, is 75; and the result might be written $\frac{75}{24}$.

But, generally, it is changed by division into a mixed number; and the fractional part reduced, when practicable, to its simplest expression, as we have reduced here $\frac{75}{24}$ to $3\frac{1}{8}$.

After the pupil has gained some experience, he may omit the intermediate formation by factors of the new numerators; but, in the beginning, the complete arrangement should be insisted on.

REMARK.

6. In reducing the sum of the fractions, the student must observe that he is to try, in the numerator, such factors only as he knows, by its formation, to be contained in the denominator.

Thus, in the preceding example, if he wished to reduce $\frac{75}{24}$, he would know that 3 is the only factor to be tried. This is important in large numbers.

7. Sometimes the fractions are not alone, and form part of mixed numbers, to be added. For example:

QUESTION.

A man has sold the following number of yards of cloth: $3\frac{1}{4}$; $\frac{2}{3}$; $5\frac{5}{6}$; $9\frac{3}{5}$; $6\frac{1}{2}$. How much has been taken out of the piece?

OPERATION.

Least common denominator $2 \times 2 \times 3 \times 5 = 60$

Arrange the numbers in this way . . . . . . Add the fractions as directed above; their sum is here $\frac{171}{60}$; take out the integral part, 2, and carry it, to be added with the other whole numbers, and set down the sum, 25, to which join the fractional part, $\frac{17}{20}$. Thus, the number of yards sold is ascertained to be $25\frac{17}{20}$.

$3\frac{1}{4}$	$1 \times 15 = 15$	
$\frac{2}{3}$	$2 \times 20 = 40$	
$5\frac{5}{6}$	$5 \times 10 = 50$	
$9\frac{3}{5}$	$3 \times 12 = 36$	
$6\frac{1}{2}$	$1 \times 30 = 30$	
$25\frac{17}{20}$	171	60
		$2\frac{51}{60} = 2\frac{17}{20}$

To add mixed numbers, add the fractions first; reduce the sum; set down the fractional part, and carry the whole number that may result from their addition to the column of whole numbers.

Some direct to reduce the mixed numbers to improper fractions; but this complicates the operation, and it is always absurd to form large numbers, to reduce them afterwards.

SUBTRACTION OF FRACTIONS.

8. *The object of the subtraction of fractions is to find a single expression for the difference of two fractions.*

What has been said, in regard to the addition of fractions, applies to subtraction, and the rule is similar.

I. *If the fractions have the same denominator, subtract the less from the greater numerator, and set the difference over the common denominator.*

II. *If they have not the same denominator, begin with reducing them to the least common denominator.*

Thus, $\frac{6}{7} - \frac{2}{7} = \frac{6-2}{7} = \frac{4}{7}$; 2 *sevenths* taken from 6 *sevenths*, leaves 4 *sevenths*; and $\frac{7}{8} - \frac{2}{3} = \frac{21}{24} - \frac{16}{24} = \frac{5}{24}$, by reducing the two given fractions to the same denominator, 24, and then subtracting the numerator, 16, from 21.

9. *When the fractions are joined to whole numbers, subtract the fraction first and then the whole number.*

EXAMPLE.

From	$13\frac{3}{4}$	Fraction transformed	$\frac{6}{8}$
take	$3\frac{5}{8}$	"	$\frac{5}{8}$
The result is . . .	$10\frac{1}{8}$	Difference	$\frac{1}{8}$

In the subtraction of mixed numbers, if the fractional part of the number to be subtracted is larger than the other, the operation must be performed by means of an auxiliary unit, in a manner similar to what was taught in simple subtraction.

OPERATION.

Let us suppose that, from	$5\frac{2}{3}$	$\frac{8}{12}$	$1+\frac{8}{12}=\frac{12}{12}+\frac{8}{12}=\frac{20}{12}$
1 carried, we have to subtract	$3\frac{3}{4}$	$\frac{9}{12}$	. . . . $\frac{9}{12}$
	$1\frac{11}{12}$		$\frac{11}{12}$

We reduce the fractions to a common denominator; and, because the lower one, $\frac{9}{12}$, is larger than the upper one, $\frac{8}{12}$, we add, mentally, to the fraction of the upper number, a whole unit, which, converted into a fraction, is $\frac{12}{12}$. This, with $\frac{8}{12}$, makes $\frac{20}{12}$; from which $\frac{9}{12}$ may now be subtracted, leaving $\frac{11}{12}$.

But, because we have thus increased the upper number by 1, it is necessary to increase also the lower number by the same amount, in order that the difference may not be altered. This is done by carrying *one* to the 3 units; which makes 4 to be subtracted from 5, and the final remainder is $1\frac{11}{12}$.

It is not necessary, however, to add $\frac{12}{12}$ to $\frac{8}{12}$. In practice, especially when the fractions are large, it will be found more simple to subtract the lower fraction, $\frac{9}{12}$, from the transformed unit, $\frac{12}{12}$, and then add the remainder, $\frac{3}{12}$, to the upper fraction, $\frac{8}{12}$.

10. Here, as in addition, it is wrong to reduce to improper fractions and complicate numbers, which are to be simplified afterwards.

When several fractions have to be subtracted, add them together first, and subtract their sum.

EXAMPLE.

$$\frac{5}{6}+\frac{3}{4}-\frac{1}{8}-\frac{2}{3}. = \frac{20}{24}+\frac{18}{24}-\frac{3}{24}-\frac{16}{24}=\frac{38-19}{24}=\frac{19}{24}.$$

Questions.—What is the object of addition of fractions? Of subtraction? Repeat the rule of addition of fractions. Why do you add the numerators? Why not the denominators? If the fractions have the same denominator, what is the nature of their units? How can you add $\frac{1}{3}$ and $\frac{1}{5}$? How should mixed numbers be added? Give the rule for subtraction. How do you subtract fractions having the same denominator? Having different denominators? How do you subtract mixed numbers? How, when the fraction of the number to be subtracted is the larger? Is it proper to put mixed numbers under fractional form, either for addition or subtraction? How do you subtract several fractions from others?

EXERCISES.

1. $\frac{2}{3}+\frac{3}{4}+\frac{5}{6}+\frac{7}{12}+\frac{23}{36}=3\frac{17}{36}$.
2. $\frac{1}{8}+\frac{1}{63}+\frac{1}{560}+\frac{1}{5040}=\frac{1}{7}$.
3. $\frac{6}{7}+\frac{3}{4}+\frac{5}{7}+\frac{1}{3}+\frac{1}{4}=2\frac{19}{21}$.
4. $\frac{3}{4}+\frac{7}{8}+5\frac{1}{2}+7\frac{4}{5}+4=18\frac{37}{40}$.

5. $9\frac{3}{4} + 7\frac{2}{3} + 5\frac{7}{8} + 11\frac{5}{6} = 35\frac{1}{8}$.

6. $7\frac{1}{2} + 11\frac{3}{4} + 25\frac{7}{9} + 9\frac{3}{5} = 54\frac{113}{180}$.

7. $\frac{19}{20} - \frac{13}{17} = \frac{63}{340}$.

8. $\frac{16}{21} - \frac{8}{15} = \frac{8}{35}$.

9. $14\frac{2}{3} - \frac{153}{247} =$

10. $8\frac{2}{3} - 4\frac{1}{5} = 4\frac{7}{15}$.

11. $5\frac{3}{8} - 2\frac{1}{2} = 2\frac{7}{8}$.

12. $3\frac{7}{9} - 2\frac{10}{11} = \frac{86}{99}$.

13. $15 - \frac{3}{7} =$

14. $\frac{1}{3} + \frac{1}{5} + \frac{6}{7} - \frac{4}{15} = 1\frac{13}{105}$.

15. $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{23}{60}$.

16. $\frac{5}{42} + \frac{3}{154} - \frac{2}{231} = \frac{10}{77}$

17. $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} = \frac{1}{32}$.

18. $14\frac{7}{18} - \frac{3}{4} - \frac{5}{9} - \frac{3}{5} - \frac{1}{3} = 12\frac{3}{20}$.

19. $27\frac{1}{2} - 3\frac{1}{2} - 4\frac{3}{4} - 7\frac{4}{5} - \frac{7}{8} = 10\frac{23}{40}$.

20. $234\frac{7}{8} - 5\frac{3}{4} - 3\frac{1}{2} - 4\frac{3}{5} - 117\frac{5}{6} - 10\frac{11}{12} = 92\frac{11}{40}$.

21. Which of the fractions $\frac{8}{35}$ and $\frac{3}{7}$, is the largest; and what is their difference? *Ans.* $\frac{3}{7} - \frac{8}{35} = \frac{1}{5}$.

22. A merchant sells, out of a piece of cloth measuring $30\frac{7}{8}$ yards, at different times, $7\frac{3}{4}$; $9\frac{2}{3}$, and $11\frac{5}{12}$: how much must remain? *Ans.* $2\frac{1}{24}$ *yards.*

23. There is a pole, $\frac{3}{8}$ of it in the mud, $\frac{2}{5}$ in the water: how much of it is above the water?

24. Out of a firkin of butter, weighing $86\frac{1}{4}$ pounds, there was sold $2\frac{1}{2}$; $4\frac{3}{4}$; $5\frac{3}{8}$; $9\frac{2}{3}$, and 16 pounds. The balance is weighed, and found to be only $37\frac{1}{3}$ pounds: how many have been stolen? *Ans.* $10\frac{5}{8}$.

LESSON XLIII.

MULTIPLICATION OF FRACTIONS.

1. This operation is an immediate consequence of the Twenty-ninth Lesson.

CASE I.

To multiply a fraction by a whole number.

RULE.

Multiply the numerator or divide the denominator.

Question.—One yard costs $\frac{7}{8}$ of a dollar: how much will 5 yards cost?

They will cost $\frac{7}{8} \times 5 = \frac{7 \times 5}{8} = \frac{35}{8} = 4\frac{3}{8}$.

Second example: $\frac{11}{18} \times 9 = \frac{99}{18}$; and, by reduction, would be $\frac{11}{2} = 5\frac{1}{2}$.

Which may be obtained at once by cancelling 9 with 18; or, in other words, applying the second part of the rule.

This has been explained in Lesson XXX., 9.

CASE II.

2. To multiply a whole number by a fraction, the rule is the same as the preceding; which may be understood by considering that $\frac{7}{8} \times 5$, is the same thing as $5 \times \frac{7}{8}$.

3. However, as the principle that we may transpose the multiplier, has been proved only for whole numbers, it might not be admitted for fractions, without some demonstration.

Let us multiply, first, 5 by 7 whole units; the product is either . . 5×7 or 7×5.

But this product is too large, since we have made use of a multiplier whose units should be divided by 8.

Now, we know that the result is the same, whether we divide the multiplier or the product (XXIII., 3). We do the latter; and, dividing by 8, get either

$$\frac{5 \times 7}{8} \text{ or } \frac{7 \times 5}{8} \text{ for the correct result.}$$

CASE III.

4. To multiply two fractions together.

Transform mixed numbers, if there be any, into equivalent fractions; then multiply the numerators together and the denominators together.

(Algebraically, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$).

DEMONSTRATION.

Let it be proposed to multiply $\frac{3}{4}$ by $\frac{5}{8}$.

If we had to multiply $\frac{3}{4}$ by the whole number, 5, we would multiply the numerator, and the product would be (1) $\frac{3\times 5}{4}$.

But, in multiplying by 5 whole units, instead of 5 *eighths*, we make the multiplier 8 times too large.

The product, $\frac{3\times 5}{4}$, is, consequently, also 8 times too large, and must now be divided by 8; which is done by multiplying its denominator, 4, by 8: and thus, the final result is,

$$\frac{\text{The product of the numerators} \quad 3\times 5}{\text{by the product of the denominators} \quad 4\times 8} = \frac{15}{32}.$$

Which proves the rule.

This is the general case. The cases of multiplication of a fraction and a whole number might be given as consequences; since the product by the whole number would require no rectification.

5. Though made part of the rule, the transformation of mixed numbers into fractions is not indispensable. It is frequently preferable to *multiply by parts.*

For example, to multiply	$16\frac{2}{9}$
by . . .	$18\frac{3}{8}$
Multiply, first, the two whole numbers, 18 by 16, and you get	288
Secondly, $\frac{2}{9}$ by 18, by cancelling 9 out of 18,	4
Thirdly, 16 by $\frac{3}{8}$, by cancelling 8 out of 16,	6
And, finally, $\frac{3}{8}$ by $\frac{2}{9}$, which, by cancelling 2 and 3, is	$\frac{1}{12}$
Answer .	$298\frac{1}{12}$

In cases like this, when the denominators divide the whole numbers, and also when the numbers are large, this mode of *multiplying by parts* is preferable. The choice between the two methods must be regulated by the judgment of the operator.

6. In the multiplication of vulgar fractions, we ob-

serve, as in decimals, that the product is smaller than the factors. We can explain this more easily still, and show that the rule is a double operation, by the analysis of one of the questions which lead to multiplication of fractions.

For example:

One yard of stuff costs $\frac{3}{4}$ of a dollar: what will $\frac{2}{3}$ of a yard cost? If *two-thirds of a yard* were called *two feet*, there is no one who would think of multiplying $\frac{3}{4}$ by 2; since $\frac{3}{4}$ of a dollar is not the value of *one foot*, but of *one yard;* and, consequently, not the *true multiplicand* of 2 feet.

But the value of one foot would be found first, by taking one third of that of a yard; and then, the multiplication being properly stated, could be regularly performed, and the product would be found, as usual, larger than the true multiplicand.

$\frac{1}{3}$ of $\frac{3}{4}$ of a dollar is $\frac{1}{4}$: *this is the value of $\frac{1}{3}$ of a yard;* and *twice* that is $\frac{2}{4}$ or $\frac{1}{2}$ of a dollar, which is the answer.

Evidently, whether the piece of stuff be called $\frac{2}{3}$ of a yard, or two feet, the principle is the same; the value of $\frac{1}{3}$ must be obtained first.

7. Therefore, the operation is compound; there is in it both a *division* of the multiplicand, to make it the value of one unit of the multiplier, and an ordinary *multiplication*, which then brings a result larger than the true multiplicand.

In the present example, the product, $\frac{1}{2}$ *of a dollar*, is larger than the altered multiplicand, $\frac{1}{4}$ of a dollar.

In practice, both operations are blended together, because, dividing the product gives the same result as dividing the multiplicand; but the leading operation is a multiplication.

The reading of the numerical statement of the successive operations to be performed shows all this plainly. It is

$$\frac{3\times 2}{4\times 3}=\frac{1}{2}, \text{ (by cancelling 3, and then 2 out of 4).}$$

Which reads: $\frac{3}{4}$ *multiplied* by 2, and *divided* by 3, is equal to $\frac{1}{2}$.

FRACTIONS OF FRACTIONS.

In the last question, we have taken $\frac{1}{3}$ of $\frac{3}{4}$; which is $\frac{3}{4}$ divided by 3; equal to $\frac{3}{4\times 3}$, and then twice that, or $\frac{3}{4}\times\frac{2}{3}$, is evidently taking $\frac{2}{3}$ of $\frac{3}{4}$; and, when so expressed, it is termed a *fraction of fraction;* and, since the same operations are performed as in the multiplication of fractions, it follows that $\frac{2}{3}$ of $\frac{3}{4}$ is the same thing as $\frac{3}{4}\times\frac{2}{3}$.

Hence, *fractions of fractions*, or *multiplication of fractions*, are identical operations. They are also called *compound fractions;* three names for the same thing.

In the same way that we take $\frac{2}{3}$ of $\frac{3}{4}$, we may take $\frac{5}{8}$ of the result; that is,

$$\frac{5}{8} \text{ of } \frac{2}{3} \text{ of } \frac{3}{4} = \frac{5\times 2\times 3}{8\times 3\times 4},$$

and then another part, $\frac{4}{5}$, of the result, making it $\frac{4}{5}$ of $\frac{5}{8}$ of $\frac{2}{3}$ of $\frac{3}{4}$, &c. All of which will be another mode of expression for multiplication; that is,

$$\frac{4}{5} \text{ of } \frac{5}{8} \text{ of } \frac{2}{3} \text{ of } \frac{3}{4} = \frac{4\times 2\times 5\times 3}{5\times 3\times 8\times 4} = \frac{1}{4}.$$

8. Therefore: *A compound fraction is reduced to a simple one by multiplying the numerators together and the denominators together.*

9. In practice, indicate all the products and divisions first, as in the above example, and then cancel all common factors between the compound numerator and the denominator, before executing the multiplications.

As a general rule, always simplify data in preference to results; it will shorten the operations.

Here you see that

4×2; 5 and 3 of the numerator cancel respectively

8, 5 and 3 of the denominator, and thus the operation is found at once to be $\frac{1}{4}$, without going through any multiplications.

N. B.—Observe that cancelling a factor means dividing by it, leaving in its place the quotient 1; which is omitted as a factor, because it does not alter the product of the others: but it must, of course, be set down as a numerator, if all the factors happen to be cancelled, as in $\frac{4\times2\times5\times3}{5\times3\times8\times4}=\frac{1}{4}$. But, in the denominator, it would be useless, since division by 1 gives a quotient equal to the dividend. Thus, $\frac{2\times3\times4}{2\times3}$ is reduced by cancelling 2 and 3, to 4, and not $\frac{4}{1}$.

It is not uncommon for pupils, when cancelling a factor, to say that *nothing* remains. This is erroneous; they should say that *one* is left, though not set down.

Questions.—How do you multiply a whole number and a fraction together? Why? Show that the result is the same, whether you multiply a whole number by a fraction or the fraction by the whole number. How do you multiply two fractions together? Demonstrate the rule. How do you multiply mixed numbers? Is multiplication of fractions a simple or compound rule? Explain it. What are fractions of fractions? How are they reduced to a single one? What should be done before performing the multiplications? If you strike off a factor, what remains in its place? Why is it not set down? What is $\frac{2}{3}$ of $\frac{3}{4}$; $\frac{4}{5}$ of 4; $\frac{3}{5}$ of 5; $\frac{11}{12}$ of $\frac{12}{11}$; $\frac{6}{5}$ of $\frac{10}{12}$? &c.

EXERCISES.

1. Multiply 9 by $\frac{3}{4}$.
2. Multiply $\frac{2}{3}$ by 6.
3. Multiply $\frac{3}{8}$ by $\frac{4}{5}$.
4. Multiply $4\frac{1}{2}$ by 6.
5. Multiply 7 by $5\frac{2}{5}$.
6. $6\frac{3}{4}\times7\frac{4}{5}=52\frac{13}{20}$.
7. $\frac{5}{6}\times\frac{2}{3}\times\frac{6}{7}=\frac{10}{21}$.
8. $\frac{2}{3}\times\frac{13}{4}\times5\times\frac{3}{4}\times\frac{3}{5}=4\frac{7}{8}$.
9. $\frac{2}{9}$ of $\frac{3}{5}$ multiplied by $\frac{5}{8}$ of $3\frac{2}{7}=\frac{23}{84}$.
10. $5\times\frac{2}{3}\times\frac{2}{7}$ of $\frac{3}{5}$ multiplied by $4\frac{1}{6}=2\frac{8}{21}$.
11. $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{8}$ of $\frac{6}{7}$ of $12=3\frac{3}{14}$.
12. $48\frac{3}{5}\times13\frac{5}{6}=672\frac{3}{10}$.
13. What will $13\frac{1}{2}$ gallons of wine cost, at $4\frac{3}{4}$ dollars per gallon? *Ans.* $64\frac{1}{8}$ *dollars.*
14. A man owns $\frac{4}{15}$ of a cargo; $\frac{3}{8}$ of which are lost: what is his share of the loss? *Ans.* $\frac{1}{10}$.

15. A man owns $\frac{3}{25}$ of a capital, and sells $\frac{5}{12}$ of this part: what part of the capital does he sell? *Ans.* $\frac{1}{20}$.

16. One pound of sugar costs $\frac{11}{12}$ of a dollar: what will $17\frac{3}{5}$ pounds cost? *Ans.* $16\frac{2}{15}$ *of a dollar.*

17. One yard costs $6\frac{9}{10}$ of a dollar: what will $\frac{2}{3}$ cost? *Ans.* $4\frac{3}{5}$ *dollars.*

18. A man failing in trade, can pay only $\frac{2}{5}$ of a dollar on each dollar: how much will he pay on $12\frac{1}{2}$ dollars? *Ans.* 5 *dollars.*

LESSON XLIV.

DIVISION OF FRACTIONS.

CASE I.

1. *To divide a fraction by a whole number, divide the numerator, or multiply the denominator* (with algebraical symbols $\frac{a}{b} : c = \frac{a}{bc}$).

This is a repetition of what was said in the First Part, Lesson XXIX.

1*st Question.*—3 yards have cost $\frac{3}{4}$ of a dollar, how much is it a yard? Here you divide the numerator. *Ans.* $\frac{1}{4}$ *of a dollar,*

2*d Question.*—5 yards have cost $\frac{3}{4}$ of a dollar, what is it a yard? Here you cannot divide the numerator, and must multiply the denominator. *Ans.* $\frac{3}{20}$ *of a dollar.*

CASE II.

2. *To divide a whole number by a fraction, multiply the number by the denominator and divide by the numerator* (with algebraical symbols $a : \frac{b}{c} = \frac{ac}{b}$).

DEMONSTRATION.

Let it be proposed to divide 12 by $\frac{7}{9}$; that is, to divide 12 by 7 units of a size referred by the denominator 9 to another known unit.

If we had only to divide 12 by 7 the result would be

$\frac{12}{7}$;

but the quotient is not 7 whole units, but a number of units 9 times smaller. The divisor we have used is consequently 9 times too large; and, consequently, (XXIV., 3) the quotient would be 9 times too small without a correction: in order to obtain the right result, therefore, we must now make this quotient 9 times larger by multiplying the numerator, which gives for the final result

$$\frac{12 \times 9}{7} = \frac{108}{7} = 15\tfrac{3}{7}.$$

If the question had been to divide 12 by $\frac{4}{5}$, the numerical statement, either written or mentally arranged, would be $\frac{12 \times 5}{4}$. In this case 4 can be cancelled, and the result would be $3 \times 5 = 15$.

3. The operation evidently amounts to *multiplying the whole number by the fraction inverted* (algebraically $a : \frac{b}{c} = a \times \frac{c}{b}$).

4. In both cases, the quotient is larger than the dividend, because a thing smaller than *one* must be contained in any number oftener than one itself.

5. To divide one fraction by another, *multiply the dividend fraction by the divisor inverted* (algebraically $\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$).

DEMONSTRATION.

Let it be proposed to divide, for example, $\frac{3}{5}$ by $\frac{8}{11}$.

We might reason as in the preceding case, but we will vary the demonstration, and show another way to arrive at the result.

Make the divisor a whole number by striking off its denominator 11. This amounts to multiplying the divisor by 11 (XXIX., 5).

Having thus multiplied the divisor, we must also multiply the dividend by 11, in order to get the same quotient (XXIV., 4), which gives us the new dividend

$$\frac{3 \times 11}{5}$$

to be divided by 8; and the final division is effected as above by multiplying the denominator by 8, so that the result is

$$\frac{3 \times 11}{5 \times 8} \text{ or } \frac{3}{5} \times \frac{11}{8};$$

the latter being the divisor inverted.

6. Before effecting the operations, be careful to cancel the common factors:

If we had $\frac{16}{25} : \frac{8}{15}$; or, with the other sign of division, $\frac{\frac{16}{25}}{\frac{8}{15}}$, the indicated operations would be

$$\frac{16 \times 15}{25 \times 8} = \text{(by cancelling) } \tfrac{6}{5}.$$

Observe that *cancelling, in the division of two fractions reduced to their simplest terms, takes place between numerator and numerator, and between denominator and denominator.*

In the preceding example, 8 is cancelled out of 16, and 5 out of both 15 and 25.

7. Had the divisor been $\frac{4}{5}$ instead of $\frac{8}{15}$, the result would have been

$$\tfrac{16}{25} \times \tfrac{5}{4} = \tfrac{4}{5}.$$

Which shows that *the division of fractions may be made by dividing numerator by numerator, and also denominator by denominator, when they will divide.*

Hence, if the denominators are the same, we have only to divide the numerators; since equal numbers cancel.

8. *If either the dividend or the divisor, or both, are mixed numbers, transform them first into fractional expressions, and then divide as above.*

If we had to divide $12\frac{3}{4}$ by $6\frac{2}{3}$, the successive operation would be

$$12\tfrac{3}{4} : 6\tfrac{2}{3} = \tfrac{51}{4} : \tfrac{20}{3}; \text{ or, } \frac{\frac{51}{4}}{\frac{20}{3}} = \frac{51 \times 3}{4 \times 20} = \tfrac{153}{80} = 1\tfrac{73}{80}.$$

9. It will be observed that the four fundamental rules in fractions require no new principles, and are mere

combinations of the simple rules of the First Part. They are, in fact, *compound operations.*

In division, as well as in multiplication by a fraction, we remark that the operation requires both a multiplication and a division. For, the division of 12 by $\frac{7}{9}$, for example, is indicated thus: $\frac{12 \times 9}{7}$; which reads, 12 *multiplied* by 9, and *divided* by 7.

10. Here we observe a similarity of features, which makes it sometimes difficult for learners to distinguish, in combinations of fractions, whether they have to multiply or to divide. They will remove all doubt by reading the question with whole numbers. The operation in fractions will be analogous to that with integers; which will be recognised by considering that:

1st. If *one* being given, *many* are to be found, it is *multiplication.*

2d. If a *whole* being given, a *part* is either given or required; in other words, if a product and one factor being given, the other is required, it is *division.*

For instance:

1st Question.—One yard costs $\frac{3}{4}$ of a dollar; how much will $\frac{2}{3}$ of a yard cost? If we read without the denominators, as follows: *One yard costs* 3 *dollars; how much will* 2 *yards cost?* we see at once that it is a multiplication.

2d Question.—How much stuff can be bought for $\frac{3}{4}$ of a dollar, at $\frac{2}{3}$ of a dollar a yard? Read without the denominators, and you easily perceive that the amount to be expended is a dividend, and the cost per yard its divisor. Consequently, the analogy shows that the operation is a division.

3d Question.—$\frac{4}{5}$ of the income of a man amounts to 400 dollars: what is his income? Read, without the denominator, 4 *times the income of a man is* 400 *dollars: what is the income?* Evidently, you must divide 400 by 4; consequently, you must also divide 400 dollars by $\frac{4}{5}$.

Questions.—How is a fraction divided by a whole number? Why? A whole number by a fraction? Why? A fraction by a fraction? Why? What must be done before effecting the operations? When the numerator of the dividend is a multiple of that of the divisor, what is done? When the denominator of the dividend is a multiple of that of the divisor, what is done? How do you divide mixed numbers? Is division of fractions a simple or compound rule? How will you ascertain whether the question is a multiplication or a division? Give examples. Is $\frac{1}{4}$ of 36 a multiplication or division?

EXERCISES.

All the results must be reduced to their simplest expression.

1. Divide $\frac{3}{4}$ by $\frac{2}{5}$.
2. $\frac{12}{49}$ by $\frac{4}{7}$.
3. $\frac{5}{28}$ by $\frac{15}{14}$.
4. $\frac{25}{9}$ by $\frac{5}{3}$.
5. $\frac{7}{16}$ by $\frac{3}{4}$.
6. $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{2}{9}$.
7. $7\frac{1}{3}$ by $9\frac{5}{9} = \frac{33}{43}$.
8. $\frac{2}{3}$ of $\frac{3}{5} \div \frac{5}{7}$ of $7\frac{3}{5} = \frac{7}{171}$.
9. $\dfrac{2\frac{2}{3}}{3\frac{1}{4}} = \frac{32}{39}$.
10. $\dfrac{3\frac{1}{4}}{4\frac{1}{4}} =$
11. $\dfrac{2\frac{1}{2}}{4\frac{1}{3}} =$
12. $\dfrac{\frac{10}{11}}{\frac{4}{30}} =$
13. $\dfrac{100}{4\frac{7}{8}} = 20\frac{20}{39}$.
14. $6\frac{1}{8} : 2\frac{1}{3} =$
15. $5{,}205\frac{1}{5} : \frac{4}{5}$ of $91 = 71\frac{1}{2}$.

16. At $\frac{3}{4}$ dollar a bushel, how many bushels can be had for 5 dollars? *Ans.* $6\frac{2}{3}$ *bushels.*

17. By how much must $3\frac{1}{4}$ be multiplied, that the product may be $29\frac{1}{4}$? *Ans.* 9.

18. What part of $\frac{3}{4}$ is $\frac{3}{5}$?

19. What part of $7\frac{2}{9}$ is $2\frac{4}{5}$? *Ans.* $\frac{126}{325}$.

20. At $\frac{3}{4}$ of a dollar per bushel, how much can you buy for $\frac{2}{5}$ of a dollar?

21. If a man can build $5\frac{3}{4}$ yards in a day, in how many days will he build $18\frac{2}{5}$? *Ans.* $3\frac{1}{5}$ *days.*

22. If it take $\frac{2}{3}$ of a bushel to sow an acre, how many acres will 16 bushels sow?

23. If $\frac{3}{4}$ of a gallon cost $3\frac{3}{5}$ dollars, how much is it a gallon?

MISCELLANEOUS QUESTIONS IN FRACTIONS AND CANCELLING.

1. A yard costs $47\frac{2}{5}$: what is the value of $12\frac{7}{8}$ yards? *Ans.* $610\frac{11}{40}$ *dollars.*

2. A person has bought $23\frac{5}{12}$ yards, for $745\frac{13}{20}$ dollars: how much is it for a yard? *Ans.* \$31.86+

In the following exercises, recollect that cancelling may take place between numerators and between denominators of fractions, which are to be divided by each other; and, therefore, cancel where you can before combining the numbers. Example:

$$\frac{\frac{\cancel{8}}{9}\times\frac{3}{\cancel{4}}\times\frac{\cancel{15}^{3}}{49}}{\frac{\cancel{5}}{7}\times\frac{1}{\cancel{4}}\times\frac{\cancel{8}}{21}} = \frac{\frac{\cancel{3}^{1}}{\cancel{9}}\times\frac{\cancel{3}}{\cancel{49}}}{\frac{1}{\cancel{7}\times\cancel{21}_{3}}} = \frac{1}{\frac{1}{3}} = 3.$$

3. $\dfrac{\frac{7}{6}\times\frac{1}{2}\times\frac{3}{10}}{\frac{6}{11}\times\frac{7}{8}\times\frac{9}{2}} = \frac{11}{135}.$

4. $\dfrac{\frac{1}{7}\times\frac{4}{9}\times 3}{\frac{6}{11}\times\frac{8}{9}\times 4} = \frac{11}{112}.$

5. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}} = \frac{10}{7}.$

6. $\dfrac{2\frac{1}{2}+\frac{1}{6}}{3\frac{1}{2}-\frac{1}{8}} = \frac{64}{81}.$

7. $\dfrac{4\frac{1}{7}-2\frac{1}{4}}{6\frac{1}{2}-2\frac{1}{7}} = \frac{53}{122}.$

8. $\dfrac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}{1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}} = \frac{7}{11}.$

9. $\dfrac{\frac{7}{8}+0.453}{0.46} = 2.8869+$

10. $\dfrac{5\frac{4}{5}+7.35}{4.375-\frac{1}{2}} = 3.39354+$

11. $\dfrac{4.37-1.75}{4\frac{3}{4}-2\frac{1}{2}} = 1.1\overline{64}.$

12. $\dfrac{\frac{0.12}{2.75}+1.314}{2} = 0.67881+$

13. In a book is found $\frac{4}{5}+ \quad\quad + \frac{6}{7}+\frac{3}{8}+\frac{8}{15} = 3\frac{67}{168}$. The second number being defaced, what should it be? *Ans.* $\frac{5}{6}$.

14. There is also found $\frac{9}{10}$ of $\frac{2}{\ }$ of $\frac{3}{4}$ of $\frac{5}{6} = \frac{3}{8}$: what is the missing denominator?

15. Likewise, $9\frac{1}{6}$ divided by $\frac{1}{2}$ of . . . . $= 2\frac{13}{21}$: what number is omitted? *Ans.* 7.

16. Of what number is 176 the $\frac{11}{23}$ part?

17. *A* owns $\frac{2}{3}$ of $\frac{3}{4}$ of a ship, and *B* $\frac{3}{8}$ of $\frac{4}{5}$: which is the greater share, and by how much? *Ans. A has* $\frac{1}{5}$ *more.*

18. A farmer had his sheep in five fields; in the first, he had $\frac{1}{4}$ of them; in the second, $\frac{1}{6}$; in the third, $\frac{1}{8}$; in the fourth, $\frac{1}{12}$; and in the fifth, 450: how many had he in all? *Ans.* 1200 *sheep.*

19. $\frac{7}{8}$ of the income of a gentleman are \$4,593; what is the whole income? *Ans.*

20. A merchant loses $\frac{2}{7}$ of his private income, which amounts to £283$\frac{1}{2}$, but he receives from his business £568$\frac{1}{4}$, which is a profit of $\frac{2}{5}$ on his capital: how much does he get in all; and what is his capital in trade? *1st Ans.* £770$\frac{3}{4}$. *2d Ans.* £1420$\frac{5}{8}$.

21. A grocer sold $\frac{1}{8}$ of a gallon of wine for $\frac{3}{10}$ of a dollar: what was it a gallon? Is this multiplication or division?

Ans. 2$\frac{2}{5}$ *dollars.*

22. A man travels 4 miles in $\frac{5}{6}$ of an hour: how much does he travel an hour? *Ans.* 4$\frac{4}{5}$ *miles.*

23. $\frac{3}{4}$ of a pole is 6$\frac{2}{3}$ ft. long: how long is the pole? *Ans.*

24. A pole is $\frac{2}{5}$ in the mud, $\frac{2}{7}$ in the water, and 4$\frac{7}{12}$ ft. above the water: how long is the pole? *Ans.* 12$\frac{7}{24}$ *ft.*

25. A person having spent $\frac{1}{2}$ of his money at one time, and $\frac{1}{3}$ at another, had 26$\frac{2}{3}$ dollars left: how much had he?

Ans. 160 *dollars.*

26. 135$\frac{11}{12}$ is $\frac{9}{16}$ of what number? *Ans.* 241$\frac{17}{27}$.

27. A man owns $\frac{4}{5}$ of a ship, and sells $\frac{1}{3}$ of his share for 4000 dollars: what is the value of the ship? *Ans.* 15,000 *dollars.*

LESSON XLV.

(Which may be omitted with beginners.)

SHORT METHODS IN MULTIPLICATION.

The following transformations will greatly simplify both written and mental operations.

1. *To multiply by* 5, *multiply by* 10 *and divide by* 2. Because $5 = \frac{10}{2}$.

Examples: $673 \times 5 = \frac{6730}{2} = 336.5$; $97.57 \times 5 = \frac{975.7}{2} = 487.85$; $4.17 \times 0.05 = \frac{4.17 \times 0.10}{2} = 0.2085$.

In this and following examples, it will be remarked that the multiplication by 10; 100; 1000, &c., requires merely the displacing of the units' point, either before or after multiplying. In this consists the advantage of these methods.

2. *To multiply by* 25, *multiply by* 100 *and divide by* 4. Because $25 = \frac{100}{4}$.

Examples: $347 \times 25 = \frac{34700}{4} = 8675$; $67.4 \times 25 = \frac{6740}{4} = 1685$; $16.77 \times 0.025 = \frac{16.77 \times 0.100}{4} = .41925$.

3. *To multiply by* 50, *multiply by* 100 *and divide by* 2. This is nearly the same thing as the first case.

4. *To multiply by* 75, *multiply by* 100 *and deduct one-fourth of the product.* Because $75 = 100 - \frac{100}{4}$.

Examples: $35 \times 75 = 3500 - \frac{3500}{4} = 2{,}625$; $0.29 \times 0.75 = .29 - 0.0725 = .2175$.

5. *To multiply by* 125, *multiply by* 1000 *and divide by* 8. Because $125 = \frac{1000}{8}$.

Examples: $38 \times 125 = \frac{38000}{8} = 4{,}750$; $4.9 \times 1.25 = \frac{49}{8} = 6.125$.

6. *To multiply by* $12\frac{1}{2}$, *multiply by* 100 *and divide by* 8. This is about the same rule as the preceding, by writing $12\frac{1}{2}$ decimally, 12.5.

7. *To multiply by* $37\frac{1}{2}$, *multiply by* 100 *and take* $\frac{3}{8}$ *of the product.* Because $37\frac{1}{2} = \frac{3}{8}$ of 100.

Example: $457 \times 37\frac{1}{2} = \frac{137100}{8} = 17{,}137.5$.

8. *To multiply by* $62\frac{1}{2}$, *multiply by* 1000 *and divide by* 16. Because $62\frac{1}{2} = \frac{1000}{16}$.

Examples: $129 \times 62\frac{1}{2} = \frac{129000}{16} = 8{,}062.5$; $27.35 \times 6.25 = \frac{27.35 \times 100}{16} = 170.9375$.

9. *To multiply by* $87\frac{1}{2}$, *multiply by* 100 *and take* $\frac{7}{8}$ *of the product ;* or, *deduct* $\frac{1}{8}$ *of the same.*

Example: $139 \times 87\frac{1}{2} = \frac{13900 \times 7}{8} = \frac{97300}{8} = 12{,}162.5$; or, $13900 - \frac{13900}{8} = 12{,}162.5$.

10. *To multiply by* $16\frac{2}{3}$, *multiply by* 100 *and divide by* 6. Because $16\frac{2}{3} = \frac{100}{6}$.

Example: $36 \times 16\frac{2}{3} = \frac{3600}{6} = 600$.

11. *To multiply by* $33\frac{1}{3}$, *multiply by* 100 *and divide by* 3. Because $33\frac{1}{3} = \frac{100}{3}$.

Examples: $48 \times 33\frac{1}{3} = 1{,}600$; $7.67 \times 0.33\frac{1}{3} = 2.556$.

12. *To multiply by* $66\frac{2}{3}$, *multiply by* 100 *and take* $\frac{2}{3}$ *of the product.*

Example. $78 \times 66\frac{2}{3} = \frac{15600}{3} = 5{,}200$; or, $= 2{,}600 \times 2 = 5{,}200$.

13. *To multiply by* 15, 35, 45, or 55, *you may multiply respectively by* 30, 70, 90, 110, *and take one half of the product.*

Or else, you may take one-half of the multiplicand before multiplying.

Examples: $38 \times 15 = 19 \times 30 = 570$; $475 \times 35 = \frac{475 \times 70}{2} = \frac{33{,}250}{2} = 16{,}625$.

$76 \times 45 = 38 \times 90 = 3{,}420$; $57 \times 55 = \frac{57 \times 110}{2} = \frac{6270}{2} = 3{,}135$.

You may also multiply by 15, by adding one-half of the number to it and multiplying by 10 (or inversely).

Thus: $86 \times 15 = (86 + 43) \times 10 = 1{,}290$; $79 \times 15 = 790 + 395 = 1{,}185$.

14. As regards multiplications by such numbers as 18,

24, 36, 48, 63, &c., which can be decomposed into factors, I refer to *contraction in multiplication* (XXXI., 4).

15. *To multiply by* 11, *you may add each figure to that on its right.*

To avoid mistakes in applying this rule, beginners will do well to conceive a zero to be placed at each extremity of the number, and use them as figures of the number.

Example: 789×11. Conceive the number to be written 07890, and then say . . . . . . $9 + 0 = 9$.
$8 + 9 = 17$; carry 1 and set down . . . . 7.
1 carried $+ 7 + 8 = 16$; carry 1 and set down . 6.
1 carried $+ 0 + 7$ . . . . . . $= 8$.
So that the whole product is 8,679.

The reason of this operation will be easily discovered by performing it in the regular way. A direct multiplication by 11 will, however, be nearly as expeditious.

16. *To multiply by any number between* 12 *and* 20, *you may multiply successively each figure of the multiplicand by the last of the multiplier, and add to each product the figure of the multiplicand on the right of that you multiply.*

Example: 423×17.
$7 \times 3 = 21$, carry 2 and set down . . . . . . 1.
$7 \times 2 + 2$ carried $+ 3 = 19$; carry 1 and set down . . 9.
$7 \times 4 + 1$ carried $+ 2 = 31$; carry 3 and set down . 1.
3 carried $+ 4$, the extreme left-hand figure . . . 7.
So that the product is 7,191.

In the case of the multiplier, 19, it might be preferred to multiply by 20, and deduct the given multiplicand.

Thus: $365 \times 19 = 365 \times 20 - 365$.

17. This remark applies also to 29, 39, 49, &c. When you have such multipliers, you may multiply by 30, 40, 50, &c., and subtract one multiplicand.

MULTIPLICATION BY MULTIPLES.

18. Suppose we had to multiply 914,785 by 246,144. It is indifferent by which figure we multiply first (XIII., 5). Therefore, remarking that 24 is a multiple of 6, and 144 a multiple of 24, we multiply first by 6, observing to set the product in its right place; then we multiply this first product by 4, which gives that by 24; which, finally, we multiply by 6, to get that by 144.

OPERATION.

```
    914785
    246144
  ------------
   5488710 . . .   product by 6.
  21954840 . . . . this by . . 4.
   131729040       and then by 6.
  ------------
  225168839040
```

If the last part had been 145, instead of 144, we might still have proceeded in the same way, and added one multiplicand; if 150, the product by 6 should have been added.

This example will suffice. The skilful application of the method will depend on the readiness of the operator.

SHORT METHODS IN DIVISION.

Shorter methods are less numerous, and also less useful, in division than in multiplication.

19. *To divide by* 5, *multiply by* 2 *and divide by* 10.

Examples: $\frac{176}{5} = \frac{176 \times 2}{10} = 35.2$; $\frac{379.7}{0.05} = \frac{379.7 \times 2}{0.10} =$ 7,594.

20. *To divide by* 25, *multiply by* 4 *and divide by* 100.

Examples: $\frac{288}{25} = \frac{288 \times 4}{100} = 11.52$; $\frac{6.36}{0.025} = \frac{6.36 \times 4}{0.100} =$ 254.4.

21. *To divide by* 125, *multiply by* 8 *and divide by* 1000.

22. " *by* $12\frac{1}{2}$, *multiply by* 8 *and divide by* 100.

23. *To divide by* $62\frac{1}{2}$, *multiply by* 16 *and divide by* 1000.

24. " *by* $16\frac{2}{3}$, *multiply by* 6 *and divide by* 100.

25. " *by* $33\frac{1}{3}$, *multiply by* 3 *and divide by* 100.

26. " *by* $37\frac{1}{2}$, *multiply by* 8 *and divide by* 3 *and* 100.

27. " *by* $87\frac{1}{2}$, *add one-seventh and divide by* 100. Because $87\frac{1}{2} + \frac{87\frac{1}{2}}{7} = 100$.

Example: $\frac{686}{87\frac{1}{2}} = \frac{686+98}{100} = 7.84$.

28. *To divide by* $66\frac{2}{3}$, *add one-half and divide by* 100. Because $66\frac{2}{3} + \frac{66\frac{2}{3}}{2} = 100$.

Example: $\frac{68}{66\frac{2}{3}} = \frac{68+34}{100} = 1.02$.

29. *To divide by* 15, 35, 45 *or* 55, *double the dividend and divide respectively by* 30, 70, 90, 110.

30. *To divide by* 75, *add one-third and divide by* 100. Because $75 + \frac{75}{3} = 100$.

Example: $\frac{237}{75} = \frac{237+79}{100} = 3.16$.

I have introduced these rules here, in order that they may be used in subsequent operations. Algebra will furnish other ready ways of simplifying operations.

Questions.—What simple way is there to multiply by 5; 25; 50, &c. . . . ? How do you multiply when multiples are discovered in the multiplier? What short method is there to divide by 5; 25, &c. . . . ?

EXERCISES.

Exercises on the preceding methods can be performed with advantage only under the eye of the teacher, who can readily supply them. The pupil may, however, perform the following by himself:

Multiply, by making use of such multiples as may exist in the multiplier,

1. 6,927 by 567 =
2. 79,088 by 18,729 =
3. 687,846 by 168,287 =
4. 963,792 by 1,296,133 =
5. 942,566 by 64,816 =
6. 6,720,701 by 3,361,216 =
7. 265,603 by 525,625 =
8. 345,784 by 6,752,459 =

LESSON XLVI.

PROPORTIONS.*

3 : 6 : : 18 : 36 *read* 3 *is to* 6 *as* 18 *is to* 36.

Symbols.—*a* : *b* : : *c* : *d* " *a* " *b* " *c* " *d*.

Question.—If 3 yards cost 6 dollars, how much will 18 yards cost? *Ans.* 36 *dollars*.

1. This subject is introduced here, because proportions are nothing but fractions in disguise; and another confusion of terms for things already known, serves to make a separate theory, which does not add an iota to the knowledge of arithmetic.

The doctrine of proportions might be altogether dispensed with, even in geometry, where they are most useful.

2. It has been said (XXX., 5) that the same fraction might be transformed in an infinite number of ways; such as

$$\tfrac{1}{2};\ \tfrac{2}{4};\ \tfrac{3}{6};\ \tfrac{18}{36};\ \tfrac{56}{112},\ \&c.;$$

which are all equal in value; that is, the numerator and denominator of every one have the same *ratio* to each other. So that, for example,

* Formerly two sorts of proportions were used; *arithmetical* and *geometrical* proportions. The first has been pretty nearly abandoned. Here we consider only geometrical proportions, which will probably in time be restored to the doctrine of fractions, to which they belong.

$$\tfrac{3}{6} = \tfrac{18}{36};$$

or, the ratio of 3 to 6 is the same as that of 18 to 36.

3. Let it be remembered that the *ratio* of two quantities is the quotient of the second by the first (XXX., 4); in the examples just preceding, the common ratio is 2.

4. Two equal ratios, or fractions, written with the other sign of division; thus,

3 : 6 = 18 : 36, or 3 : 6 : : 18 : 36,

are called a *proportion;* which is read as above, 3 *is to* 6 *as* 18 *is to* 36; or, *as* 3 *is to* 6, *so is* 18 *to* 36. Hence,

5. A PROPORTION *is the expression of the equality of two ratios.*

Would not $\frac{3}{6} = \frac{18}{36}$, be just as plain and more in accordance with preceding rules; and could it not be read exactly in the same way? I recommend, indeed, to pupils to accustom themselves to write proportions so.

6. This is not all; new names are also given to the numbers which compose the proportion.

The 1st number is called .	1st *antecedent.*
the 2d " " .	its *consequent.*
the 3d " " .	the 2d *antecedent.*
and the 4th " " .	the 2d *consequent.*

It is therefore said that the 1st antecedent is to its consequent as the 2d antecedent is to the 2d consequent.

The four numbers are the *terms* of the proportion; the first and last are the *extremes;* and the two middle ones the *means.*

So that,

the *dividend,*	which become in fractions respectively	the *numerator,*	are now in proportions	the *antecedent,*
the *divisor,*		the *denominator,*		the *consequent,*
and the *quotient,*		and the *fraction,*		and the *ratio.*

7. Besides their fundamental property, that the two ratios are equal, proportions have another equally essential property, viz:

The product of the extremes is equal to the product of the means.

Thus,

in the proportion, $3 : 6 :: 18 : 36$, we have $3 \times 36 = 6 \times 18$.
and with symbols, in $a : b :: c : d$, " $a \times d = b \times c$.

DEMONSTRATION.

This property can be easily demonstrated by restoring the fractional form,

$$\frac{3}{6} = \frac{18}{36}.$$

It is evident that, since the fractions are equal, if we reduce them to the same denominator, the numerators must be equal. The reduction to the same denominator gives

$$\frac{3 \times 36}{6 \times 36} = \frac{18 \times 6}{36 \times 6};$$

and because the numerators must be equal,

$$3 \times 36 = 18 \times 6.$$

The first product is that of the *extremes;* the second, that of the *means.*

8. The same may also be understood by considering that, since the fractions are equal, their quotient is *one;*

that is, $\frac{\frac{3}{6}}{\frac{18}{36}} = 1$; or, by inverting the second fraction (XLIV., 5), $\frac{3}{6} \times \frac{36}{18} = 1$. The inversion brings the two extremes together, and also the two means together, and proves the equality of their products.

9. Proportions have other properties, which may be of some use in algebra and geometry, but of little importance in practical arithmetic.

10. If any one term of a proportion is unknown, it can readily be found by means of the three others; for, this term must be absent from one of the products, and it will suffice to *divide the known product by the remaining factor.*

If, for example, we knew the three terms, 3, 36, and 6, we could find 18 by dividing the product 3×36 by 6; that is, $\frac{3 \times 36}{6} = 18$.

If it was 6 that was unknown, the product, 3×36, divided by 18, would give 6. The same thing, of course, would obtain in regard to the other terms, 3 and 36.

11. This operation also has a distinct name; it is

THE RULE OF THREE.

The object of which, it will appear, is, *having given three terms of a proportion, to find the fourth term.*

12. According to the operation just explained,

If the unknown term is an extreme, *divide the product of the* means *by the other* extreme.

If it is a mean, *divide the product of the* extremes *by the other* mean.

13. It is clear that there was no need of new expressions and signs for this. For, having given as a condition the equality of two fractions,

$$\tfrac{3}{6} = \tfrac{18}{36}, \text{ for example:}$$

if either the numerator or denominator of one of them was unknown, it might easily be found by means of the other fraction.

Let us suppose, for instance, that the numerator 18 is not known. It is clear, that since the second fraction results from the first by multiplying both its terms by the same number, all we have to do is to find what that number is. This is done by dividing 36 by 6, and then multiplying 3 by it. Which gives $\frac{3 \times 36}{6}$, as above.

In like manner, if the denominator 6 was unknown, it would be found by dividing the known denominator, 36, by the same number which should divide 18 to change it into 3; that is, by their quotient; and the operation would be

$$36 \div \tfrac{18}{3} = \frac{3 \times 36}{18} = 6, \text{ as above.}$$

So that what we knew before was sufficient, without a new theory and new terms.

14. Since a ratio cannot be established between things which are not of the same nature, it follows that, in prac-

tical questions, the first two terms of a proportion are of the same nature, *dollars*, *yards*, *pounds*, or anything else.

The last two must likewise be of the same nature:

As, when we say, 2 *yards* cost 4 *dollars;* consequently, 6 *yards* must cost 12 *dollars:* from which we form the proportion 2 yards : 6 yards : : 4 dollars : 12 dollars.

15. Consequently, in the *rule of three*, two of the three numbers given will necessarily be of the same nature; and the third, of the nature of the number sought.

It is natural, therefore, to form the first ratio with the two similar numbers, and the second with the two others.

16. The last being unknown, it is customary to represent it by one of the last letters of the alphabet, x, y, or z. So that, if the question were,

2 *yards cost* 4 *dollars: what will* 6 *yards cost?*

The statement, by proportion, would be $2 : 6 :: 4 : x$, and the result $x = \frac{6 \times 4}{2} = 12$.

17. In this case, we find that a greater number of yards requires a greater number of dollars. The rule of three is then said to be *direct*.

But if the question were,

A piece of work was done in 8 *days, by* 6 *men: in how many days would* 12 *men do the same?*

It is evident that more men would require less days, in the ratio of 12 to 6. The statement by proportion would be

$$12 : 6 :: 8 : x, \text{ and the result } x = \frac{6 \times 8}{12} = 4.$$

In this case, the rule of three is *inverse*. Therefore,

If *more* requires *more*, or *less* requires *less*, the proportion is *direct*.

If *more* requires *less*, or *less* requires *more*, it is *inverse*.

18. According to this distinction, in order to state the question numerically by proportions,

I. *Place in the third term the number which is of the same nature with the answer.*

II. *Consider, from the nature of the question, whether the answer must be greater or less than the third term, and then arrange the first two numbers in the same order of magnitudes.*

The proportion being thus formed, the unknown term will be found as directed above. After stating the operations to be made as above,

$$x = \frac{6 \times 8}{12}, \text{ for example,}$$

multiply, divide, or cancel, as is found most convenient.

The distinction between *direct* and *inverse* rules, is of some use when proportions are resorted to, but unnecessary when the question is solved by simple fractions, which some call *by analysis.* We will compare these two methods in various practical questions. For the present we will give a few simple exercises, to familiarize the pupil with proportions.

Questions.—Are proportions indispensable in arithmetic? What is a ratio? What is a proportion? How could it be otherwise expressed? Give examples. By what single name are the four numbers of a proportion called? What is the first? The second? The third? The fourth? What do they correspond to in fractions? What are the means? The extremes? What relation exists between them? Prove it. If one term of a proportion is wanting, how can it be found? What is the rule of three? Direct? Inverse? Give examples. How do you state a question by proportions? How could any term be found by means of the fractional form? What is the ratio of 1 to 4? Of 4 to 1? Of 3 to 12? Of 6 to 42? Of 10 to 100? Of 10 to 1? Of 15 to 5? Of 6 to 3? Of 12 to 48? Of 48 to 12? Of 7 to 9? Of 9 to 7? In a proportion, what is the ratio of the last two terms equal to? Does the ratio of two numbers refer to their absolute magnitude? What is the third term in a rule of three? How is the fourth represented?

EXERCISES.

1. 25 pounds have cost 650 dollars: what will 384 pounds cost? *Ans.* 9,984 *dollars.*

2. It took 20 days for 135 men to do a piece of work: how long will it take 300 men? *Ans.* 9 *days.*

3. 45 men have built 280 yards of masonry: how much would 76 men have built in the same time? Find the result in decimals to within one hundredth. *Ans.* 472.89 *yards.*

4. If 100 men can finish a work in 12 days, how many will do it in 3 days? *Ans.* 400.

5. If 3 paces of a person be equal to 2 yards, how many yards will 160 paces be equal to? *Ans.* $106\frac{2}{3}$.

6. A garrison, consisting of 10,000 men, has provisions at the rate of 39 ounces a day, for each man, for 50 days; 3000 men are added: what should now be the ration of each to last the same time? *Ans.* 30 *ounces.*

7. $\frac{3}{8}$ of a yard of cloth cost $2\frac{1}{2}$ dollars; what will $7\frac{1}{2}$ yards cost? *Ans.* 50 *dollars.*

8. If 7 pounds cost $1\frac{1}{2}$ dollars, what will 3 pounds cost? *Ans.* $\frac{9}{14}$ *of a dollar.*

9. If $72\frac{1}{2}$ yards cost $38\frac{2}{3}$ dollars, what will $99\frac{3}{8}$ cost? *Ans.* 53 *dollars.*

10. If 0.3 of a house cost 100.75 dollars, what will .95 cost? *Ans.* 319.04+*dollars.*

The 9th question is a remarkable example of cancelling.

CHAPTER VII.

CONTAINING DENOMINATE NUMBERS.

LESSON XLVII.

DENOMINATE NUMBERS.

1. All the preceding principles and rules have had reference almost exclusively to *abstract numbers*, and are applicable to all kinds of units. If they have been well understood, their application to *denominate numbers* will require but little explanation.

2. It has been said, at the beginning of this work, that a denominate number is one whose unit of comparison is designated by a particular name.

But, *arithmetic considers only denominations which express relative numerical values;* such as *one foot* relatively to *one yard; one ounce* in regard to *one pound*, &c.

3. So that smaller units are fractions of the larger ones, but the numerical relation between them is expressed by a name instead of a number. One foot is *one third* of a yard; one ounce, $\frac{1}{16}$ of a pound; and, if the relative value of each inferior unit was expressed by a fraction of the larger unit, instead of a name; or, if the higher units were given as a whole number of smaller ones, the preceding operations could be applied at once, without any further explanation.

But, as they are of very frequent use, and their relations are generally given in convenient numbers, practice has introduced some simple rules, in combining them, which it will be advantageous to examine.

The first thing to be done is to commit carefully to memory the different measures and their subdivisions in use in the United States. For this purpose, the following tables are introduced here.

UNITED STATES CURRENCY, OR FEDERAL MONEY.

TABLE.

				E.	$	d.	cts.	mills.
Mills, ...		marked	*m.*					
10 mills make	1 cent,	"	*ct.*	1 =	10 =	100 =	1000 =	10,000.
10 cents "	1 dime,	"	*d.*		1 =	10 =	100 =	1000.
10 dimes "	1 dollar,	"	$.			1 =	10 =	100.
10 dollars "	1 eagle,	"	*E.*				1 =	10.

This currency is very convenient; it will be recognised as belonging to the *decimal system.* Hence, to make any calculation in dollars, it is only necessary to recollect that

Eagles	.	correspond to	.	*tens.*
Dollars	.	"	.	*units.*
Dimes	.	"	.	*tenths.*
Cents	.	"	.	*hundredths.*
Mills	.	"	.	*thousandths.*

and consequently perform every operation in Federal money as in decimal numbers and fractions.

The coins of the United States are of gold, silver and copper, as follow:

1st. The gold coins:

The Double Eagle,	worth	Twenty Dollars.
The Eagle,	"	Ten "
Half Eagle,	"	Five "
Quarter Eagle,	"	Two and a half Dollars.
Three Dollar piece,	"	Three "
Gold Dollar,	"	One Dollar.

2d. The silver coins—The Dollar = 100 cents.
Half Dollar = 50 cents.
Quarter of a Dollar = 25 cents (frequently called a Quarter).
Dime = 10 cents (called also Ten-Cent Piece).
Half Dime = 5 cents (called Five-Cent Piece; and at the South, Picayune).
The Trime, or Three-Cent Piece.

3d. The copper coins—The Cent; the Half Cent (now very rare.)
The Mill is an imaginary coin, used merely in calculations for greater accuracy.

Additional details relative to coins and currencies will be found in the Appendix.

NUMERATION AND NOTATION OF FEDERAL MONEY.

1. From the decimal relation of the successive denominations in Federal money, it will appear evident, that any amount of that money might be written by placing the different denominations, side by side, in their order, as in common decimal numbers; using 0*s* in the place of absent denominations.

Thus 18 eagles, 5 dollars, 6 dimes, 7 cents, and 2 mills might be written,

E.	*$.*	*d.*	*ct.*	*m.*
18	5	6	7	2

and agreeably to the remark of Lesson VI., 7, read in various ways; as for example:
185,672 *mills;* 18,567 *cents* and 2 *mills;* 1,856 *dimes* and 72 *mills;* or again, 185 *dollars* 67 *cents* and 2 *mills*, &c., according as the unit of comparison adopted might be the *mill, cent, dime* or *dollar.*

Likewise, 21 *eagles* 6 *cents* and no mills might be written either 21,006 *cents,* if we wanted the amount in *cents;* or 210,060 *mills,* if the unit of comparison was the *mill.*

the vacant places of units of *dollars, cents* and *mills* being occupied by 0*s* as usual.

2. But this is not the customary way of writing and reading Federal money. The *dollar* being its *unit of comparison,* the other denominations are referred to it; and accounts kept in *dollars, cents* and *mills,* or *fractions of cents* instead of mills.

Eagles, though real coins, do not appear in reading amounts of money, otherwise than as *tens of dollars;* and likewise *dimes* are introduced only as *tens of cents.*

Accordingly, the first of the above amounts would be
written $185.672 or 185.672 dollars,
and read 185 *dollars* 67 *cents* and 2 *mills,*
or 185 *dollars* 672 *mills.*

The second would be written
$210.06 that is, 210 dollars and 6 cents.

The sign $ preceding the whole, or the word *dollars* following it, and the units' point being placed between the dollars and dimes, to fix the place of the UNIT of comparison, THE DOLLAR. Hence,

RULE.

To write Federal money, write down the amount of dollars; place the units' (decimal) *point after it; and next to this write the cents and mills* (or fractions of cents instead of mills); *Fill the places of absent denominations by zeros; and put the sign $ before the number, or write the word* dollars *after it.*

Thus 2 dollars and 5 cents would be written		$2.05	
3 " and 7 mills	"	$3.007	
6¼ cents referred to the dollar		$0.06¼	or $ .06¼
5 cents and 2 mills	do.	$0.05	or $ .052
9 mills	do.	$0.009	or $ .009

RULE.

3. *To read Federal money, read the number to the left of the units' point as* dollars; *the next two figures as* cents, *and the third as* mills

Dimes are not named in the reading of Federal money, and it is customary to change them into cents by the addition of a 0; thus:

2 dollars and 4 dimes—that is, decimally,	\$2.4
is written in preference	\$2.40

which is the same thing; and reads 2 *dollars* and 40 *cents*, or also 240 *cents*.

4. Since the figures, which compose an amount of Federal money, are decimally connected, it follows (VI., 7,) that it may be read, not only in the usual way as *dollars*, *cents* and *mills*, but also as *cents* and *mills*, or as *all mills*; thus:

\$2.654 may be called, 2 *dollars*, 65 *cents* and 4 *mills*,
265 *cents* and 4 *mills*,
or even 2,654 *mills*.

For the same reason	2 *dollars* or \$2.00
may be called	200 *cents*
and also	2000 *mills*

using in place of the vacant denominations additional zeros, which we know do not alter the value of numbers after decimal fractions, (XXI., 8); for they do not change the relative position of the denominations.

5. Very frequently, especially in mercantile business, fractions of cents are used instead of mills, as for instance,

\$10.12½	which reads	ten *dollars* 12½ *cents*,
0.18¾	"	18¾ *cents*,
25.33⅓	"	twenty-five *dollars* 33⅓ *cents*.

Sometimes the *cents* are written *as vulgar fractions of dollars* in smaller figures, as $\$10\frac{12\frac{1}{2}}{100}$; $\$25\frac{33\frac{1}{3}}{100}$; $\$265\frac{75}{100}$; and often also in the bills of merchants simply

$\$10^{\underline{12\frac{1}{2}}}$; $\$25^{\underline{33\frac{1}{3}}}$; $\$265^{\underline{75}}$.

The last mode, however, is liable to cause errors.

In deeds, drafts and important documents in general, amounts of money are written at full length in words: In that case, it is a good practice to introduce also the numbers in figures between brackets, as an additional security against errors.

In accounts and bills, the decimal point is not used: The dollars and cents are separated in two different columns by a vertical line, as in the following example.

Richmond, August 25th, 1856.
Wm. Smith, to Ths. Jones, Dr.

		$	cts.
Jan'y 6th	To 25 pounds of Sugar at $12\frac{1}{2}$ c	3	$12\frac{1}{2}$
March 7	" 2 " Tea " 1.15	2	30
"	" 2 " Coffee " $.18\frac{3}{4}$		$37\frac{1}{2}$
April 20	" 20 " Bacon " $16\frac{2}{3}$	3	$33\frac{1}{3}$
July 9	" $1\frac{1}{4}$ gallon of Wine " $\$3^{25}$	4	$6\frac{1}{4}$
	Rec'd. Paym't. Sept. 1st 1856. Ths. Jones by A. Grant, cl'k.	$ 13	$19\frac{7}{12}$

(Cheque to pay for the above.)

Richmond, Sept. 1st, 1856.

Cashier of the Bank of Virginia pay to Ths. Jones or bearer Thirteen $\frac{20}{100}$ dollars.

$\$13.\frac{20}{100}$* Wm. Smith.

Questions.—What are denominate numbers? What kind of denomination does arithmetic consider? Give examples. Could these names be expressed by numbers? By what kind of numbers would you express inferior units? By what kind of numbers would higher units be expressed in regard to lower ones? What is Federal money? Repeat the table. What are the gold coins? The silver coins? The copper coins of the United States? How are amounts of Federal money written? How read? How are cents written in fractional form?

EXERCISES.

1. Read, as dollars, cents, and mills, $18.30; $55.367; $5.4.
2. Read, as cents, $16.51; $3.40; .07.
3. Write fourteen dollars sixteen cents and three mills.
4. " seven dollars seven cents and seven mills.
5. " six dollars and six mills.
6. " twenty dollars and four cents.
7. " six cents and a quarter.
8. " twelve and one half cents.
9. " eight and two thirds cents.
10. " thirty-one and one fourth cents.
11. " ten dollars twenty-five cents.
12. " one hundred and six cents.
13. " two hundred and thirty-one cents and a quarter.

* It is customary to omit fractions smaller than $\frac{1}{2}$; but to add *one* where they exceed it; as to $\frac{1}{2}$, it is alternately omitted and made 1 in accounts.

14. In 35 dollars and 9 mills, how many mills?

15. In 6 dollars 25 cents and 1 mill, how many mills?

16. In $150, how many cents? How many mills? How many dimes?

17. How many mills in $9.67; in $6.5; in $5; in 75 cents; in 9 cents?

18. How many cents in $500.29? In 60.2?

II. ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF FEDERAL MONEY.

6. The *dollar* being the *unit of comparison* of Federal money, *dimes* are *tenths* of that unit,
cents " *hundredths*,
mills " *thousandths*,
and consequently, ADDITION, SUBTRACTION, MULTIPLICATION and DIVISION in *Federal money are mere operations in decimal fractions*, which were thoroughly explained in Chapter III., and therefore a few examples will suffice in this place.

EXAMPLES IN ADDITION.

7. I. If it was proposed to add $13 and 25 cents; $105 and 6 cents; $78 and 15 mills; 33 cents; the operation would be arranged and performed as an ordinary addition, as may be seen in the annexed operation.

OPERATION.

$13.25
105.06
78.015
.33
$196.655

II. The bill in No. 5 is an example of addition with *fractions instead of mills:* The fractions are added first, and give $\frac{7}{12}$ and 1 to be carried. The rest of the operation as usual.

EXAMPLES IN SUBTRACTION.

8. I. Let it be proposed, for instance, to subtract 56 dollars 25 cents and 9 mills from 105 dollars 19 cents.

The operation would be as shown here.

OPERATION.

105.19 0
56.25 9
$48.93 1

II. The annexed operation is an example with fractions of cents.

OPERATION.

From	$\$2,014.12\frac{1}{2}$	$=\frac{3}{6}$
To subtract	$927.66\frac{2}{3}$	$=\frac{4}{6}$
	$\$1,086.45\frac{5}{6}$	

Since $\frac{2}{3}$ or $\frac{4}{6}$ is greater than $\frac{1}{2}$ or $\frac{3}{6}$, we borrow $1=\frac{6}{6}$, from which $\frac{4}{6}$ is subtracted, leaving $\frac{2}{6}$, which added to the upper $\frac{3}{6}$ (Lesson XLII., 9), makes $\frac{5}{6}$; this we set down and carry 1 to the lower figure 6; 1 and 6 are 7, and 7 from 12 leaves 5 and 1 to carry, &c. The rest of the operation as usual.

III. From $110 to subtract $6\frac{1}{4}$ cents.

In such a case it is better to equalize the denominations as shown here, by the addition of 0*s*, though it is not indispensable, (XXII.)

OPERATION.

$$
\begin{array}{r}
\$110.00 \\
0.06\frac{1}{4} \\
\hline
\$109.93\frac{3}{4}
\end{array}
$$

EXAMPLES IN MULTIPLICATION.

9. I. *The multiplier a whole number.* Let it be proposed to multiply 25 dollars 56 cents and 8 mills by 12.

It is clear that the multiplication of *mills* and *cents* will give respectively *mills* and *cents*, just as the repetition of *thousandths* and *hundredths* gives *thousandths* and *hundredths* in the product; therefore we multiply as in decimal fractions, and cut off by the units' point as many denominations or decimals as are contained in the multiplicand.

OPERATION.

$$
\begin{array}{r}
\$25.568 \\
12 \\
\hline
\$306.816
\end{array}
$$

II. *The multiplier containing decimal and other fractions.* For instance:

A bookseller orders books for an individual, and to cover all expenses is to get $\$1.37\frac{1}{2}$ to the dollar on the invoice, which amounts to $\$216.62\frac{1}{2}$—how much is he to receive?

This multiplication may be performed in two different ways—either by retaining the fractions of cents, or more simply, *by changing them to decimals, that is, in this case, mills.*

OPERATION.

(1)

	$216.62½
	1.37½
product of ½ by 137½	68¾
product by ½	10831
" by 7	151634
" by 3	64986
" by 1	21662
	$297.8593¾

(2) Changed to decimals.

$216.625
1.375
1083125
1516375
649875
216625
$297.859375

In either case, cut off, according to the rules in decimal fractions, (Lesson XXV., 3), as many decimal figures as there are in both the multiplicand and multiplier. The result is 297 *dollars* 85 *cents* 9 *mills* and $\frac{375}{1000}=\frac{3}{8}$ of a mill.

Recollect that the multiplier 1.37½ is considered as an abstract number.

EXAMPLES IN DIVISION.

10. Since, in our mode of proceeding in the division of decimals, we make the divisor a whole number by striking off the units' point (XXVI., 2), we will at once consider the case of division by decimal fractions.

Question. 140 dollars 90 cents 6¼ mills have been paid for 41¾ yards of cloth; how much is it a yard?

The fractions being written decimally, the operation will be as follows by quick division: we strike off the decimal point in the divisor to make it a whole number, and remove that of the dividend the same number of decimals to the right; and thus see at once that the result must be a number of thousandths or mills; so that the result is 3 dollars 37 cents 5 mills.

OPERATION.

$140/90.625	41/75.
15 656	3.375
3 1312	=$3.37½
20875	
....	

11. If the fractions were not conveniently transformed into decimals, and perfect accuracy was required, the division should be made as in mixed numbers (XLIV., 8). But this is rarely the case, and the change of small fractions to decimals furnishes generally a sufficient approximation.

EXERCISES IN THE FOUR RULES.

1. Multiply 375 dollars and 6 cents by 62. *Ans.* \$23,253.72.

2. From 296 dollars 4 cents 2 mills take 117 dollars 59 cents 6 mills. *Ans.* \$

3. From ten dollars and 6 mills take 9 mills. *Ans.* \$

4. Bought 338 sheep at 2 dollars 69 cents. *Ans.* \$909.22.

5. What will $89\frac{1}{4}$ yards cost at 34 cents 6 mills per yard? *Ans.* \$30.88$\frac{1}{20}$.

6. Bought 96 yards of cloth at 5 dollars 67 cents per yard. *Ans.* \$544.32.

7. From one hundred dollars take 65 cents. *Ans.* \$

8. Bought 484 lbs. of tallow at 7 cents 3 mills. *Ans.* \$35.33.2.

9. From one thousand dollars take sixteen cents. *Ans.* \$

10. How much are 315.25 cubic yards of building stones at \$8.62$\frac{1}{2}$ a yard? *Ans.* \$2719.031$\frac{1}{8}$.

11. Divide 276 dollars 75 cents by 12. *Ans.* \$23.062$\frac{1}{2}$.

12. Divide 2754 dollars by 216. *Ans.* \$12.75.

13. Sold 48 bales of cotton, each containing 397 pounds, at 13 cents 7 mills per pound. *Ans.* \$2,610.67.2.

14. A man received 51 dollars and 52 cents for 16 days. How much did he earn a day? *Ans.* \$3.22.

15. Bought $7\frac{3}{4}$ bales of cotton, each 327 pounds, at \$9.37$\frac{1}{2}$ cts. per hundred pounds; what is the whole cost? \$244.851$\frac{9}{16}$.

N. B. In such questions observe that the division by 100 pounds gives decimals in the multiplier.

16. A railroad has cost \$159,799.89 for $12\frac{3}{4}$ miles; how much is it per mile? *Ans.* \$12,533.32 $4\frac{12}{17}$.

17. What will be the cost of carrying 459.75 cubic yards of earth 1569 feet at 9 mills per yard for 100 feet? *Ans.* \$64,92.1+

18. There was paid \$1,433.16 to 27 men, how much was it to each man? *Ans.* \$53.08.

19. What is the cost of 832 bushels of wheat at 1 dollar $18\frac{3}{4}$ cents per bushel? *Ans.* \$988.

20. How much are 357 pounds of coffee at $13\frac{3}{4}$ cents? *Ans.* \$

21. How much are $18\frac{3}{4}$ yds. of calico at 25 cts.? *Ans.* \$4.68$\frac{3}{4}$.

22. How much are $125\frac{3}{4}$ acres of land at \$18.75? *Ans.* \$2,357.81$\frac{1}{4}$.

23. Bought a farm of $266\frac{1}{3}$ acres for \$8,115.65, how much is it an acre? *Ans* \$30.47.2 (nearly).

24. What will 96.675 feet of board cost at \$11.65 a thousand? *Ans.* \$

25. What will $376\frac{756}{1000}$ M. of bricks come to at \$6.87$\frac{1}{2}$ per M.? (M. means 1000 bricks.) *Ans.*

26. What will 17,625 ft. of plank cost at \$12.75 per thousand? *Ans.* \$224.718$\frac{3}{4}$.

27. Bought 42 barrels of apples, each containing 3 bushels, for \$39.37$\frac{1}{2}$; how much is it a bushel? *Ans.* \$

28. Bought	12 lbs.	coffee	at	$16\frac{2}{3}$ cts. a pound	$
	4 "	tea	"	$\$1.31\frac{1}{4}$ "	
	60 "	sugar	"	$12\frac{1}{2}$ "	
	15 "	cheese	"	11 "	
	$105\frac{1}{2}$ "	butter	"	17 "	
	4 gallons	wine	"	$4.25 a gallon	
					$51.335.

29. Sold	28 yds.	linen	at	$62\frac{1}{2}$ cts.	$
	8 "	calico	"	$18\frac{3}{4}$ "	
	24 "	gingham	"	$37\frac{1}{2}$ "	
	11 "	silk	"	$\$1.18\frac{3}{4}$ "	
	$\frac{3}{4}$ "	vesting	"	$3.00 "	
					$\$43.31\frac{1}{4}$.

LESSON XLVIII.

ENGLISH CURRENCY, OR STERLING MONEY.

TABLE.

	£.	sh.	d.	far.
2 farthings make 1 halfpenny, marked $\frac{1}{2}$.	1 =	20 =	240 =	960.
4 farthings " 1 penny, " *d.*		1 =	12 =	48.
12 pence " 1 shilling, " *sh.*			1 =	4.
20 shillings " 1 pound sterling, " £.				
21 shillings " 1 guinea.				

Farthings are generally expressed by fractions; $\frac{1}{4}$d. for 1; $\frac{1}{2}$ for 2; $\frac{3}{4}$ for 3.

Note.—This table is important, on account of the great intercourse with England, and because it was used in this country before the revolution. It is still used by some people, but the value of the pound varies in different states.

	Where one shilling is	And one pound
Thus, *one dollar* is 5 *shillings* in Canada and Nova Scotia,	20 cents	$4.00
" 8*sh.* in New York and Ohio,	$2\frac{1}{2}$ cents	$2.50
" 10*sh.* in North Carolina,	10 cents	$2.00
" 6*sh.* in New England States, Virginia, Kentucky, Tennessee,	$16\frac{2}{3}$ cents	$\$3.33\frac{1}{3}$
" 7*sh.* 6*d.* in New Jersey, Pennsylvania, Delaware, Maryland,	$13\frac{1}{3}$ cents	$2.66\frac{2}{3}$
" 4*sh.* 8*d.* in South Carolina and Georgia.	$21\frac{3}{7}$ cents	$\$4\frac{2}{7}$

Federal money is rapidly driving away this troublesome variety of currencies.

TROY WEIGHT.

Grains, . . .	marked . .	*gr.*	*lb.*	*oz.*	*dwt.*	*gr.*
24 grains make	1 pennyweight,	*dwt.*	1 =	12 =	240 =	5,760.
20 pennyweights	1 ounce,	*oz.*		1 =	20 =	480.
12 ounces	1 pound,	*lb.*			1 =	24.

By this weight precious metals and jewels are weighed. Troy weight is also frequently used by chemists.

AVOIRDUPOIS WEIGHT.

Drams,	. . . . .	. . . . . .	marked	*dr.*
16 drams	make	1 ounce,	"	*oz.*
16 ounces	"	1 pound,	"	*lb.*
28 pounds	"	1 quarter,	"	*qr.*
4 quarters	"	1 hundred weight,	"	*cwt.*
20 hundred weight	"	1 ton,	"	*ton* or *T.*

T.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
1 =	20 =	80 =	2,240 =	35,840 =	573,440.
	1 =	4 =	112 =	1,792 =	28,672.
		1 =	28 =	448 =	7,168.
			1 =	16 =	256.
				1 =	16.

By this weight are weighed all coarse articles; such as hay, grain, groceries, and all baser metals.

NOTE.—1 *lb.* avoirdupois	=	14 *oz.*	11 *dwt.*	$15\frac{1}{2}$ *gr. Troy.*
1 *oz.* "	=	0	18	$5\frac{1}{2}$.
1 *dr.* "	=	0	1	$3\frac{1}{2}$.

The pound avoirdupois contains $6,999\frac{1}{2}$ grains.

The pound Troy, *U. S. Standard*, 5,760 "

175 Troy pounds are equal to 144 avoirdupois.

175 Troy ounces = 192 avoirdupois.

In some states, the hundred weight is reckoned only 100 lbs., and the ton 2,000 lbs. This is certainly preferable.

In England, the standard of weight is obtained from a cubic inch of water which weighs 252.458 grains; of such grains 5,760 make a pound Troy, and $6,999\frac{1}{2}$ a pound avoirdupois.

APOTHECARIES' WEIGHT.

				℔	℥	ʒ	℈	gr.
Grains,	.	.	marked *gr.*					
20 grains	make	1 scruple,	" *sc.* or ℈.	1 =	12 =	96 =	288 =	5,760.
3 scruples	"	1 dram,	" *dr.* or ʒ.		1 =	8 =	24 =	480.
8 drams	"	1 ounce,	" *oz.* or ℥.			1 =	3 =	60.
12 ounces	"	1 pound,	" *lb.* or ℔.				1 =	20.

Apothecaries and physicians make use of this weight in compounding medicines; but they buy and sell their drugs by avoirdupois weight.

The pound, ounce, and grain are the same as in Troy weight.

LONG MEASURE.

3 barley corns	make	1 inch, . .	marked	*in.*
12 inches . .	"	1 foot, . .	"	*ft.*
3 feet . .	"	1 yard, . .	"	*yd.*
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ feet	"	1 rod, perch, or pole,	"	*rd.*
40 rods . .	"	1 furlong, . .	"	*fur.*
8 furlongs or 320 rods	"	1 mile, . .	"	*mile, ml.*
3 miles . .	"	1 league, . .	"	*lea.* or *L.*
60 geographical, or $69\frac{1}{15}$ statute miles	"	1 degree, . .	"	*deg.* or °.

360 degrees the circumference of the earth.

mile.	*fur.*	*rd.*	*yds.*	*ft.*	*in.*
1 =	8 =	320 =	1,760 =	5,280 =	63,360.
	1 =	40 =	220 =	660 =	7,920.
		1 =	$5\frac{1}{2}$ =	$16\frac{1}{2}$ =	198.
			1 =	3 =	36.
				1 =	12.

A fathom is six feet, and is used to measure the depth of water.

A hand is four inches, and is used to measure the height of horses.

One span is nine inches.

Six points are one line.

Twelve lines, one inch: sometimes the inch is subdivided into ten, sometimes into eight lines.

The *standard yard* has been fixed in England by the length of the pendulum vibrating seconds; in London, at a temperature of 62° Fahrenheit, the yard is $\frac{360000}{391393}$ of the pendulum, which is 39.1393 inches in length.

CLOTH MEASURE.

$2\frac{1}{4}$ inches . .	make	1 nail, . .	marked	*nl.*
4 nails . .	"	1 quarter of a yard,	"	*qr.*
4 quarters .	"	1 yard, . .	"	*yd.*
3 quarters .	"	1 ell Flemish, .	"	*E. Fl.*
5 quarters .	"	1 ell English, .	"	*E. E.*
4 quarters $1\frac{1}{5}$ inches	"	1 ell Scotch, .	"	*E. S.*
6 quarters . .	"	1 ell French, .	"	*E. Fr.*

yd.	*qr.*	*nl.*	*in.*
1 =	4 =	16 =	36.
	1 =	4 =	9.
		1 =	$2\frac{1}{4}$.

This measure is used for all kinds of stuffs; the English, Flemish, &c., are used for goods imported from the respective countries from which they take their names.

LAND OR SQUARE MEASURE.

144 inches .	make	1 square foot,	marked .	*sq. ft.*
9 square feet	"	1 square yard,	" .	*sq. yd.*
$30\frac{1}{4}$ square yards	"	1 square pole,	" .	*P.*
40 square poles	"	1 rood, .	" .	*R.*
4 roods .	"	1 acre, .	" .	*A.*
640 acres .	"	1 mile, .	" .	*M.*

A.	*R.*	*P.*	*sq. yd.*	*sq. ft.*	*sq. in.*
1 =	4 =	160 =	4,840 =	43,560 =	6,272,640.
	1 =	40 =	1,210 =	10,890 =	1,568,160.
		1 =	$30\frac{1}{4}$ =	$272\frac{1}{4}$ =	39,204.
			1 =	9 =	1,296.
				1 =	144.

By this measure land and all surfaces are measured; such as boards, glass, pavements, plastering, &c.

In measuring land, a 4-pole chain, or 66 feet in length, is used. It is divided into 100 links, each of which is 7.92 inches in length.

10 square chains make 1 acre.

There are 640 acres in a square mile.

A square inch has its side 1 inch long.

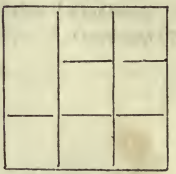
A square foot has its side 1 foot long.

A square yard has its side 1 yard long.

The difference between 3 *square feet* and 3 *feet square*, is explained by the annexed diagram.

3 square feet.

3 feet square.



3 *square feet* means 3 squares, each of which has its four sides one foot long.

3 *feet square* means a square whose sides are 3 feet long. It contains, as the figure shows, 9 *square feet*; and, in general, a square contains a number of square feet equal to the product of its side by itself.

So that 12 feet square = 144 square feet; and the number of small squares contained in a square figure, is equal to the product of its two sides. Thus, a figure 4 feet in length and 3 feet in breadth, would contain 12 square feet.

SOLID OR CUBIC MEASURE.

1,728 solid inches .	make	1 cubic foot,	marked	*C. ft.*
27 solid feet .	"	1 cubic yard,	"	*C. yd.*
40 C. feet of round, or 50 C. feet of hewn timber	"	1 ton, . .	"	*ton.*
128 feet, solid feet; that is, a pile 8 feet long, 4 feet wide, and 4 feet high	"	1 cord of wood,	"	*C.*

4 feet wide, 4 feet high, and 1 foot in length, make 1 cord foot of wood.

By this table the solid contents of bodies are determined; such as stone, timber, earth, boxes of goods, &c.

A cube is a solid, having all its sides equal and its faces squares.

1 cubic inch has sides 1 inch long.
1 cubic foot " 1 foot long = 1,728 inches.
1 cubic yard " 1 yard long = 27 cubic feet.

In general, solid contents are the number of cubic units formed by the product of the height by the breadth and thickness. Thus it is that a cord is $8 \times 4 \times 4 = 128$ cubic feet.

Masonry is also measured by the perch, which is 1 perch in length, $1\frac{1}{2}$ feet in breadth, and 1 foot high; that is, $16\frac{1}{2} \times 1\frac{1}{2} \times 1 = 24\frac{3}{4}$ cubic feet. The cubic yard should be, and is very frequently, preferred.

WINE MEASURE.

This is used for all liquors, except beer, ale and milk. The U. S. Standard gallon contains 231 cubic inches.

Gills,			marked	*gi.*
4 gills	make	1 pint,	"	*pt.*
2 pints	"	1 quart,	"	*qt.*
4 quarts	"	1 gallon,	"	*gal.*
$31\frac{1}{2}$ gallons	"	1 barrel,	"	*bl.*
63 gallons	"	1 hogshead,	"	*hhd.*
2 hogsheads	"	1 pipe,	"	*pi.*
2 pipes	"	1 tun,	"	*tun.*

tun.	*pi.*	*hhd.*	*bl.*	*gal.*	*qt.*	*pt.*	*gi.*
1 =	2 =	4 =	8 =	252 =	1,008 =	2,016 =	8,064.
	1 =	2 =	4 =	126 =	504 =	1,008 =	4,032.
		1 =	2 =	63 =	252 =	504 =	2,016.
			1 =	$31\frac{1}{2}$ =	126 =	252 =	1,008.
				1 =	4 =	8 =	32.
					1 =	2 =	8.
						1 =	4.

This gallon weighs 8.3389 pounds. There are 7.48 in a cubic foot.

Besides these,

42 gallons make	make	1 tierce,	marked	*ticr.*
84 gallons "	"	1 puncheon,	"	*pun.*

ALE AND BEER MEASURE.

This is used in measuring ale, beer, and milk.

Pints,			marked	*pt.*
2 pints	make	1 quart,	"	*qt.*
4 quarts	"	1 gallon,	"	*gal.*
36 gallons	"	1 barrel,	"	*bar.*
$1\frac{1}{2}$ barrels = 54 gallons	"	1 hogshead,	"	*hhd.*
2 barrels	"	1 puncheon,	"	*pun.*
2 hogsheads	"	1 butt,	"	*butt.*
2 butts	"	1 tun,	"	*tun.*

butt.	*pun.*	*hhd.*	*bar.*	*gal.*	*qt.*	*pt.*
1 =	$1\frac{1}{2}$ =	2 =	3 =	108 =	432 =	864.
	1 =	$1\frac{1}{2}$ =	2 =	72 =	288 =	576.
		1 =	$1\frac{1}{2}$ =	54 =	216 =	432.
			1 =	36 =	144 =	288.
				1 =	4 =	8.
					1 =	2.

A gallon, beer measure, contains 282 cubic inches; and it is remarkable that the wine and ale gallons have the same proportion to each other as the troy and avoirdupois pound.

Besides the above, there is the kilderkin, which is $\frac{1}{2}$ barrel, and the firkin, which is $\frac{1}{4}$ barrel.

DRY OR CORN MEASURE.

This is used in measuring all dry articles.

Pints,	. . . . .		marked . .	*pt.*
2 pints	.	make	1 quart, " . .	*qt.*
4 quarts	.	"	1 gallon, " . .	*gal.*
2 gallons or 8 quarts		"	1 peck, " . .	*pk.*
4 pecks	.	"	1 bushel, " . .	*bu.*
36 bushels		"	1 chaldron, " . .	*ch.*

ch.	*bu.*	*pk.*	*gal.*	*qt.*	*pt.*
1 =	36 =	144 =	288 =	1,152 =	2,304.
	1 =	4 =	8 =	32 =	64.
		1 =	2 =	8 =	16.
			1 =	4 =	8.
				1 =	2.

The gallon, dry measure, contains $268\frac{4}{5}$ cubic inches. A Winchester bushel is $18\frac{1}{2}$ inches in diameter, and 8 inches deep; it contains $2,150\frac{2}{5}$ cubic inches. It is *the United States Standard.*

But the coal bushel must be $19\frac{1}{2}$ inches in diameter: 36 of these bushels make a London chaldron of coal, the weight of which is 3,156 pounds, or nearly 1 ton 8 cwt.; and a bushel, 88 pounds. The barrel, for measuring unshelled Indian corn, contains 5 bushels.

Besides the above,

2 quarts	make . .	1 pottle, .	marked	*pot.*
8 bushels	" . .	1 quarter,	"	*qr.*
5 quarters	" . .	1 wey or load,	"	*wey.*
2 weys	" . .	1 last, .	"	*last.*

By an act of parliament, passed in 1824, and carried into execution 1st January, 1826, a greater uniformity has been established; and now only one gallon measure, whether for grains or liquors, is allowed in the United Kingdom. This *standard imperial gallon* contains 277.274 cubic inches; its weight, 10 pounds of water, at 62°.

8 such gallons make the bushel, which is 2218.192 cubic inches.

The United States gallon contains 9.7034 pounds of water, avoirdupois: a cubic yard contains 21.69 bushels.

TIME.

Seconds,	. . . .	marked .	*sec.* or ″.
60 seconds make .	1 minute,	" .	*m.* or ′.
60 minutes " .	1 hour,	" .	*hr.*
24 hours " .	1 day,	" .	*day.*
7 days " .	1 week,	" .	*wk.*
30 days " .	1 month,	" .	*mo.*
52 weeks 1 day 6 hours, or $365\frac{1}{4}$ days, make	1 common or Julian year,	" .	*yr.*

yr.	*mo.*	*wk.*	*day.*	*hr.*	*m.*	*sec.*
1 =	12 =	52 =	$365\frac{1}{4}$ =	8,766 =	525,960 =	31,557,600.
		1 =	7 =	168 =	10,080 =	604,800.
			1 =	24 =	1,440 =	86,400.
				1 =	60 =	3,600.
					1 =	60.

The exact solar year is 365 days 5 hrs. 48′ 48″. In reckoning time, only the 365 days are counted as one year: the odd six hours, by accumulating for 4 years, make one day, which is then added, and gives 366 days to every fourth year, called *bissextile* or *leap year.*

This additional day is given to the month of February, and makes it 29 days long, in every year which is a multiple of 4.

CIRCULAR MEASURE.

This is used in geometry, astronomy, and geography.

Seconds,	. .	. . . . .	marked	″.
60 seconds	. make	1 minute, . . .	"	′.
60 minutes	. "	1 degree, . . .	"	°.
30 degrees	. "	1 sign, . . .	"	*S.*
12 signs or 360°	"	1 circumference of a circle,	"	*Cir.*

Cir. S.	°	′	″
1 = 12 =	360 =	21,600 =	1,296,000.
1 =	30 =	1,800 =	108,000.
	1 =	60 =	3,600.
		1 =	60.

MISCELLANEOUS TABLE.

12 things .	make	1 dozen.
12 dozen .	"	1 gross.
12 gross or 144 dozen	"	1 great gross.
20 things .	"	1 score.
112 pounds .	"	1 quintal (of fish).
24 sheets of paper	"	1 quire.
20 quires .	"	1 ream.
56 pounds of flour	"	1 bushel.
196 pounds of flour	"	1 barrel.
200 pounds of salt meat	"	1 barrel.

A sheet folded in

2 leaves, which make	4 pages,	a folio.
4 " "	" 8 "	a quarto or 4to.
8 " "	" 16 "	an octavo or 8vo.
12 " "	" 24 "	a duodecimo or 12mo.
18 " "	" 36 "	an octodecimo or 18mo.

REMARKS ON THE PRECEDING TABLES.

I have not divided these tables into lessons, because they are to be used chiefly for reference; nor do I think it necessary to set down questions. The tables have to be committed to memory, and the questions will suggest themselves readily to the teacher.

It will be observed that the scale between the several denominations is very irregular, though the ratios are generally simple numbers. It would be a great deal better if the decimal scale of division could have been adopted for all.

There is, however, an advantage in having distinct names for the successive fractional subdivisions of the same unit.

If we were to use only the smaller unit, we would frequently have occasion to employ very large numbers.

If, on the contrary, we were to refer numerically the subdivisions to the largest unit, calculations and the operations of commercial business would be encumbered with fractions.

Even in Federal Money, where the regularity of the decimal system is introduced, each subdivision of the dollar, for this reason, has a special name different from the abstract scale of numeration.

Questions.—Is the scale between the different subdivisions of the same unit regular? Would the decimal scale have any advantage? Is there any advantage in giving different names to the successive subdivisions of the same unit? And, if so, what are they? In the decimal system of federal money, is the change of name an advantage?

LESSON XLIX.

TRANSFORMATION (REDUCTION) OF DENOMINATE NUMBERS.

1. In operations with denominate numbers, it is frequently advantageous to express a number of units of a certain denomination either *in a greater number of relative units of a lower denomination*, or *in a smaller number of relative units of a higher denomination.*

In the first case, fractions are got rid of.

In the second, numbers of an inconvenient size are avoided.

The first is *transformation descending;* the second, *transformation ascending.*

2. In order to perform readily such transformations of denominate numbers into expressions of the same value, the pupil, instead of learning by rote and practising mechanically certain rules applicable to the case, will do better to reason the nature of the question, and consider,

I. That there is, between the relative units of the same kind of measure, a dependence, which is expressed by particular names: such as pounds, shillings, pence; or, yards, feet, inches, &c.

II. That these names express a *numerical relation* or *ratio*, between the lower and the higher unit; that is, *how many units of the lower denomination it takes to make one of the higher.*

The number expressing the ratio of one shilling to one pound, is 20; of one penny to one shilling, 12; to one pound, 240; of one foot to one yard, 3; of one inch to one foot, 12, &c.

III. That an inferior unit is nothing but a fraction of the superior one; and might be expressed, instead of a name, by a fraction having for its denominator the number of small units it takes to make a large one.

Thus, 1 shilling is $\frac{1}{20}$ £; 1 penny is $\frac{1}{12}$ *sh.*, or $\frac{1}{240}$ £.
1 *ft.* is $\frac{1}{3}$ *yd.*; 1 inch is $\frac{1}{12}$ *ft.*, or $\frac{1}{36}$ *yd.*
1 *qr.* is $\frac{1}{4}$ *cwt.*; 1 pound is $\frac{1}{28}$ *qr.*, or $\frac{1}{112}$ *cwt.*
&c. &c. &c.

3. He will, therefore, discover the identity between the change of unit to be made in this case, and the transformations of fractions.

For example, let us suppose that we want to change £5 into shillings: since 1 shilling is $\frac{1}{20}$ of a pound, it is transforming the whole number, 5, into *twentieths*, which gives the fractional number, $\frac{100}{20}$ (XXX., 8), or, by using the word *shilling*, instead of *twentieth*, into 100 shillings. In both cases, 5 is multiplied by the ratio 20 of the shilling or fraction to the pound sterling.

If we wished to express a number of shillings; 6, for example, by the denomination of a pound, we would consider that 1 *sh.* = £$\frac{1}{20}$. Consequently, 6 *sh.* = £$\frac{6}{20}$. Here we would divide by the number expressing the ratio between the two.

Likewise, 120 *sh.* would be £$\frac{120}{20}$ = £6.

4. The identity of the two transformations, in fractions and denominate numbers, being well understood, if the pupil recollects or revises what he has learned in fractions, he can have no difficulty in performing the operations, either in the case of *augmentation* or *reduction*, of the number of units for the same value; which are commonly called *descending* and *ascending reduction.*

5. To transform an amount of units from one denomination into another.

Multiply the given amount by the value of one of its units, in terms of those of the new denomination.

This rule applies to *ascending* and *descending* cases, and to fractional expressions as well as to whole numbers. For, similar parts bear to each other the same ratio as the wholes.

Thus, for example, $\frac{1}{3}$ of a pound is 20 times larger than $\frac{1}{3}$ of a shilling: consequently,

£$\frac{1}{3}$ is $20 \times \frac{1}{3}$ of a shilling, or $\frac{20}{3} = 6\frac{2}{3}$ shillings.
Or, in decimals, . . . $6.6\overline{6}$

For the same reason, $\frac{1}{3}sh. = £\frac{1}{3 \times 20} = £\frac{1}{60}$; or, in decimals, $0.01\overline{66}$.

In like manner, $\frac{1}{10}$ of a yard is 3 times larger than $\frac{1}{10}$ of a foot; and also $\frac{7}{10}$ of a yard is 3 times larger than $\frac{7}{10}$ of a foot; since each individual unit is 3 times larger in the first than in the second expression.

Hence,
$\frac{7}{10}$ *yd.* is $3 \times \frac{7}{10}$ of a foot; or, $\frac{21}{10} = 2\frac{1}{10}$*ft.*; or, again, 2.1*ft.*
And,

$\frac{7}{10}$*ft.* is $\frac{7}{10 \times 3} = \frac{7}{30}$ *yd.*; or, in decimals, $0.02\overline{3}$.

By these considerations, we have but one rule for all cases.

Questions.—What is the object of changing the denomination of a number from high to low? from low to high units? How is the dependence of denominate units expressed? What do the names indicate? How could lower units be expressed numerically, in regard to high ones? Give examples. Is there any difference between the change of denomination and transformation of fractions? Give examples. What two cases have you to consider? How do you proceed, in the first case? How, in the second? Does the same rule apply also to fractions? Show it by examples.

EXERCISES IN FEDERAL MONEY.

Since the scale of subdivision is the decimal scale, the removing of the decimal point to the right, as was done in decimals, will multiply by 10, 100, or 1000, and thus lower the denomination. On the contrary, the removing of the decimal point to the left, will raise the denomination.

Thus, $15.235, which is 15 *dollars* 235 *mills*, is changed into 1523.5 cents—fifteen hundred and 23 cents and 5 mills, by removal of the decimal point two places to the right.

546 cents, by placing a decimal point two places to the left, will be changed into its equivalent, $5.46—5 dollars and 46 cents.

1. How many mills in 3 cents? *Ans.* 30 *mills.*
2. " " 10 cents? *Ans.* 100 "
3. " " in $65? *Ans.* 65,000 "
4. In 2 cents and 3 mills, how many mills? *Ans.* 23 "
5. How many cents in $2 and 15 cents?
6. How many in $3 and 50 cents?
7. In $500, how many dimes? cents? mills?
8. How many mills in 200 dollars 37 cents and 9 mills?
9. How many dollars, cents, and mills, in 178,854 mills? *Ans.* 178.85 4.
10. How many " in 1,906,783 mills? *Ans.* $
11. Change 19 cents; 28 cents; 45 cents, and 9 mills, into mills.
12. " 28 mills to cents. *Ans.* 2 *cts.* 8 *mills.*
13. " 209 mills to cents.
14. " 657 cents to dollars. *Ans.* $6.57.
15. " $50 and 9 cents to cents. *Ans.* 5,009 *cents.*
16. " $75 and 21 cents to cents.
17. " 10,104 cents to dollars. *Ans.* $101.04.
18. Express 16 cents relatively to dollars. *Ans.* $0.16.
19. " 65 mills " " *Ans.* $0.065.
20. " 5 cents " " *Ans.*

EXERCISES IN ENGLISH CURRENCY.

1. Change £17 into shillings. *Ans.* $17 \times 20 = 340$ *sh.*
   " " into pence. *Ans.* 4,080 *d.*
   " " into farthings. *Ans.* 16,320 *far.*
2. " £256 to shillings.
3. " 60 shillings into pence.
4. " 350 pence to farthings.

5. Express 3 shillings in reference to a pound. *Ans.* £$\frac{3}{20}$.
6. " 6 pence in the denominations of shillings.
Ans. $\frac{1}{2}$*sh.*
7. " 15 shillings in £. *Ans.* £
8. " 7 pence in *sh.* *Ans.* *sh.*
9. Change £$\frac{6}{7}$ into shillings. *Ans.* $\frac{6}{7} \times 20 = 17\frac{1}{7}$*sh.*
10. " £$\frac{4}{9}$ into shillings. *Ans.*
11. " $\frac{2}{3}$ shillings into pence. *Ans.* 8*d.*
12. " $\frac{3}{5}$*sh.* $\frac{4}{7}$*sh.* $\frac{1}{9}$*sh.* into pence. *Ans.*
13. " $\frac{3}{5}$*sh.* to the denomination of a pound.
Ans. $\frac{3}{5} : 20 =$ £$\frac{3}{100}$.
14. " $\frac{2}{7}$ penny to the denomination of a shilling. *Ans.*
15. " $\frac{1}{5}$ penny to the denomination of a pound.
Ans. $\frac{1}{5 \times 12 \times 20} =$ £$\frac{1}{1200}$.

This is a case of fractions of fractions, $\frac{1}{20}$ of $\frac{1}{12}$ of $\frac{1}{5}$.

16. Change £$\frac{4}{9}$ into shillings, with decimals.
Ans. $\frac{4 \times 20}{9} = 8.8\overline{8}$.
17. " $\frac{5}{7}$ shilling to pence, with decimals. *Ans.*
18. " 13 shillings to the decimal of £1.
Ans. $\frac{13}{20} =$ £0.65.
19. " 5 pence to the decimal of 1 shilling. *Ans.*
20. " £0.16 to the decimal of shillings.
Ans. $0.16 \times 20 = 3.2$*sh.*
21. " 0.19 shilling to the decimal of a penny. *Ans.*
22. " 0.25 shilling to the decimal of a pound.
Ans. $\frac{0.25}{20} =$ £0.0125.
23. " 0.17 of a penny to the decimal of a shilling. *Ans.*
of a pound. *Ans.* $\frac{0.17}{20 \times 12} =$ £$0.000708\overline{3}$.
24. " $\frac{2}{3}$ of a shilling to the decimal of a pound.
Ans. $\frac{2}{3 \times 20} = 0.03\overline{3}$.
25. " $\frac{3}{5}$ of a penny to the decimal of a shilling. *Ans.*

These examples embrace all the cases relative to English currency. Similar examples might be given in other measures; but these are sufficient to show the mode of operation. The exercises in compound numbers will complete the subject.

LESSON L.

COMPOUND NUMBERS.

Properly speaking, all numbers are compound, since they are composed of different orders of units, connected together by a known numerical relation.

Thus, 3,756, for example, is composed of *thousands*, *hundreds*, *tens*, and *units*.

But the name of *compound numbers* is more particularly reserved for *numbers composed of different denominate units, whose relative values are expressed by arbitrary names;* such as *pounds*, *shillings*, and *pence*, not connected by the decimal scale.

Compound numbers are also susceptible of transformations, or changes of the units of comparison, which, without altering their value, render them more convenient for calculations and commerce.

For instance, it will be simpler to introduce into accounts,

£21 6*sh.*, . . than its equivalent, . 5,112 *pence.*

In other cases, for the purpose of dividing, for instance, it may be more convenient to use 5,112 pence, than its equivalent in pounds and shillings.

A moment's consideration will show that these transformations are the same as those previously explained in mixed numbers. For, we may put a compound number under the form of a mixed one, since the smaller units are but fractions of the larger.

Thus, £5 7*sh.* may be written £$5\frac{7}{20}$.

In that form, being a mixed number, the rules of Lesson XXXVII., 3, may be applied to it, and

$$£5\tfrac{7}{20}$$

will be transformed into $£\frac{5\times20+7}{20} = \frac{107}{20}$ of a pound;

which, by restoring the name *shilling*, instead of the *denominator*, 20, is

107 shillings.

If the number were composed of several descending units; as, for example, in

£5 7*sh.* 4*d.* 3*far.*

The same operation might be continued by considering that, after having changed the pounds into shillings, the new integer, 107*sh.* might now, together with the next denomination, be taken as a second mixed number, $107\frac{4}{12}$ shillings, and transformed, in its turn, into *twelfths* or *pence.*

OPERATION.

£5 7*sh.* 4*d.* 3*far.*
20
107*sh.*
12
1288*d.*
4
5155*far.*

Thus, $107\frac{4}{12} = \frac{12 \times 107 + 4}{12} = \frac{1288}{12}$ shillings.

or, with the denomination *pence* instead of *twelfths*, 1,288 pence.

Finally, this, with the 3 farthings, would constitute a last mixed number,

$1,288\frac{3}{4}$ pence;

whose transformation into farthings would complete the operation, and give

5,155 farthings = £5 7*sh.* 4*d.* 3*far.*

The manner of arranging the operation is shown above. It must have been noticed that it is only a succession of transformations into the consecutive orders forming the scale, and which may be stated for practice as follows:

To transform a compound number into one of its lower denomination.

Beginning with the higher denomination, transform successively each number into one of the next descending denomination (XLIX., 5), *adding all along to the transformed number the integers of the same denomination, until you reach the denomination required.*

The reverse of this operation, or *transformation ascending*, is nothing more than what was taught in the preceding lesson. It is the same as the ordinary division by which we change an improper fraction into a mixed number.

Let it be proposed, for instance, to change

1264 grains troy weight,

into higher denominations: it may be considered as the improper fraction,

$$\tfrac{1264}{24} \text{ of a } dwt.,$$

OPERATION.

1264	24	
.64	52	20
16*gr.*	12*dwt.*	2*oz.*

which becomes, by effecting the division, the mixed number

$$52\tfrac{16}{24} dwt.;$$

or, by restoring the name *grain*, instead of the *denominator*, 24, the compound number,

$$52 dwt.\ 16 gr.$$

But, 52*dwt.* being more than one ounce, we may wish to obtain out of it the units of this denomination. Considering that

$$52 dwt. = \tfrac{52}{20} \text{ of an ounce},$$

and that the improper fraction,

$$\tfrac{52}{20} oz. = 2\tfrac{12}{20} oz.;$$

or, as a compound number,

$$2 oz.\ 12 dwt.,$$

we get, finally, $1{,}264 gr. = 2 oz.\ 12 dwt.\ 16 gr.$

The manner of arranging the operation is shown above, and it appears that the last quotient and the successive remainders compose the result. Here, again, we see only a repetition of known operations; and we have hardly occasion for the following rule:

To transform a large number of a lower denomination into a compound number of its higher denominations.

1. *Divide the number by the ratio of its units to those of the next higher denomination.*

2. *Then divide the quotient in the same way, and each successive quotient until you reach the higher denomination required.*

3. *The answer will be the last quotient, with all the remainders in their order.*

Questions.—What is a compound number? May a compound number be expressed as a mixed number? How? Give examples. In what does the change of units, in compound and in mixed numbers, differ? Give the rule. What name is given to the transformation from a higher to a lower unit? From a lower to a higher? What simple operation is the last? Give an example. What is each quotient; what, each remainder? Is the result a compound or mixed number? Repeat the rule.

EXERCISES.

1. In £91 11*sh.* $3\frac{1}{2}$*d.*, how many farthings? *Ans.* 87,902*far.*
2. In 19*sh.* $8\frac{3}{4}$*d.*, " " *Ans.* 947*far.*
3. Reduce 38*sh.* $4\frac{1}{2}$*d.* to halfpence. *Ans.* 921 *halfpence.*
4. How many pounds, &c., in 3,175*far.*? *Ans.* £3 6*sh.* $1\frac{3}{4}$*d.*
5. " " in 1,206*d.*? *Ans.* £5 0*sh.* 6*d.*
6. How many shillings in 9,752*d.*? *Ans.* 812*sh.* 8*d.*
7. How many pounds in 143*d.*? *Ans.* £0 11*sh.* 11*d.*
8. In £121 0*sh.* $9\frac{1}{2}$*d.*, how many halfpence? *Ans.* 58,099.

TROY WEIGHT.

1. In 15*lb.*, how many grains? *Ans.* 86,400*gr.*
2. How many ounces in 5,749*dwt.*? *Ans.* 287*oz.* 9*dwt.*
3. Reduce 11*oz.* 13*dwt.* 13*gr.* to grains. *Ans.* 5,605*gr.*
4. How many grains in 15 spoons, each weighing 6*dwt.* 15*gr.*? *Ans.* 2,385*gr.*
5. Reduce 47*lb.* 10*oz.* 15*dwt.* to *dwt.* *Ans.* 11,495*dwt.*
6. In 76,773*gr.*, how many pounds? *Ans.* 13*lb.* 3*oz.* 18*dwt.* 21*gr.*

AVOIRDUPOIS.

1. In 19*T.* 14*cwt.* 2*qr.* 19*lb.* 11*oz.* 13*dr.*, how many drams? *Ans.* 11,316,157*dr.*
2. How many cwt. in 9,563*lb.*? *Ans.* 85*cwt.* 1*qr.* 15*lb.*

3. In 13*cwt.* 21*lb.*, how many pounds? *Ans.* 1,477*lb.*

4. In 3*T.* 25*lb.*, " " *Ans.* 6,745*lb.*

5. Reduce 7*cwt.* 3*qr.* 11*lb.* to ounces. *Ans.* 14,064*lb.*

6. Change 8,994,384 ounces to tons, &c.
Ans. 250*T.* 19*cwt.* 0*qr.* 21*lb.*

APOTHECARIES' WEIGHT.

1. Reduce 9℔ 2℥ 3ʒ 1℈ 7*gr.* to grains. *Ans.* 53,007*gr.*

2. " 16℔ 6ʒ to ounces. *Ans.* 192℥ 6ʒ.

3. " 15℔ to grains. *Ans.* 86,400*gr.*

4. " 2,925℈ to pounds, &c. *Ans.* 10℔ 1℥ 7ʒ.

5. " 2,107*gr.* to ounces, &c. *Ans.* 4℥ 3ʒ 0℈ 7*gr.*

6. " 58,478*gr.* to pounds. *Ans.* 10℔ 1℥ 6ʒ 1℈ 18*gr.*

WINE MEASURE.

1. In 25 tuns, how many pints? *Ans.* 50,400.

2. How many hogsheads in 4,935 quarts?
Ans. 19*hhd.* 36*gal.* 3*qt.*

3. In 3*hhd.* 13*gal.* 2*qt.*, how many half pints? *Ans.* 3,240.

4. In 52 gallons, how many gills? *Ans.* 1,664.

5. Reduce 2 hogsheads 50 gallons to pints. *Ans.* 1,408.

6. Reduce 5 hogsheads to pints. *Ans.* 2,520.

ALE OR BEER MEASURE.

1. Change 5 hogsheads to pints. *Ans.* 2,160.

2. In 6 hogsheads 2 quarts, how many pints? *Ans.* 2,596.

3. In 665 pints, how many hogsheads?
Ans. 1*hhd.* 29*gal.* 0*qt.* 1*pt.*

4. Change 37*bar.* 6*gal.* 3*qt.* to pints. *Ans.* 10,710.

DRY MEASURE.

1. Change 72 bushels into pints. *Ans.* 4,608.

2. " 15 bushels 3 pecks 2 quarts into pints. *Ans.* 1,012.

3. In 16 chaldrons 15 bushels, how many pecks? *Ans.* 2,364.

4. Change 1,024 pints into bushels. *Ans.* 16.

5. " 5,328 pints into chaldrons. *Ans.* 2*ch.* 11*bu.* 1*pk.*

6. In 6,225 bushels, how many barrels of corn? *Ans.*

LONG MEASURE.

1. In 57 miles 2 furlongs, how many poles? *Ans.* 18,320.

2. How many furlongs in 19,753 yards? *Ans.* 89*fur.* 173*yd.*

3. In 590,057 inches, how many leagues?
Ans. 3*lea.* 2*fur.* 110*yd.* 1*ft.* 5*in.*

4. In 38,396 rods, how many miles? *Ans.* 119*ml.* 7*fur.* 36*rd.*

5. In 75 degrees, how many miles? *Ans.* 5,180.

6. In 2,085 miles, how many degrees? *Ans.*

7. Reduce 4 yards 5 feet 11 inches to inches. *Ans.* 215*in.*

8. In 4,652 yards, how many miles? *Ans.* 2*ml.* 1,132*yd.*

CLOTH MEASURE.

1. In 158 yards, how many nails? *Ans.* 2,528.

2. How many ells English in 5,932 nails? *Ans.* 296*E.* 3*qr.*

3. In 29 pieces of holland, each of 36 ells Flemish, how many yards? *Ans.* 783.

4. Change 42 English ells 3 quarters to quarters. *Ans.* 213.

5. In 17 yards 2 quarters 2 nails, how many nails? *Ans.* 282.

6. In 65 ells French, how many yards? *Ans.* 97*yd.* 2*qr.*

LAND OR SQUARE MEASURE.

1. In 41 acres 2 roods 14 perches, how many rods?
Ans. 6,654*P.*

2. How many square rods in 7,752 square feet?
Ans. 28*P.* 129*Sq. ft.*

3. In 5,972 perches, how many acres? *Ans.* 37*A.* 1*R.* 12*P.*

4. Change 12 square yards to square inches. *Ans.* 15,552.

5. Change 6,480 square inches into square yards. *Ans.* 5.

6. In 25*A.* 3*R.* 12*P.*, how many square poles? *Ans.* 4,132.

SOLID OR CUBIC MEASURE.

1. In a pile of wood 96*ft.* long, 5*ft.* high, 4*ft.* wide, how many cords? *Ans.* 15.

2. In 82 tons of round timber, how many inches?
Ans. 5,667,840.

3. How much wood in a load 6*ft.* long, 4*ft.* high, $2\frac{1}{2}$*ft.* wide?
Ans. $\frac{15}{32}$ *cord*, or $3\frac{3}{4}$ *cord feet.*

4. In 6 cubic yards, how many inches? *Ans.* 279,936.

5. In 64,000 cubic inches, how many cubic yards? *Ans.* 1*yd.* 10*ft.* 64*in.*

6. How many cubic inches in a cord of wood? *Ans.* 221,184.

TIME.

1. How many hours in 57 years? *Ans.* 499,662.

2. In 57,953 hours, how many weeks? *Ans.* 344*wk.* 6*da.* 17*hr.*

3. How many days from 19th March to 23d September? *Ans.*

4. How many days from 19th November, '43, to 14th May, '44? *Ans.*

5. How many seconds in a year? *Ans.* 31,557,600.

6. How many in a day? *Ans.* 86,400.

7. If a man can count 100 per minute, how long will it take him to count 1,000,000, at the rate of 10 hours per day? *Ans.*

CIRCULAR MEASURE.

1. Reduce 6° 9′ to minutes. *Ans.* 369′.

2. " 9° 40′ 44″ to seconds. *Ans.* 34,844″.

3. How many minutes in the circle? *Ans.*

4. How many seconds? *Ans.*

5. How many degrees in 3,315″? *Ans.* 0° 55′ 15″.

6. How many signs in 2,021′? *Ans.* 1*S.* 3° 41′.

LESSON LI.

1. Compound numbers are sometimes subjected to a kind of transformation, which it may be proper to notice; though, if properly investigated, it also falls under some of the rules previously explained; and a little consideration would enable the pupil to perform the operations by means of the principles he is already acquainted with.

These transformations consist:

1st. In expressing a denominate fraction of a higher unit in integers of its lower denominations.

2d. In the inverse operation, which consists in expressing a compound number as a denominate fraction of a higher denomination.

CASE I.

2. Let us take, as an example of the first case, the denominate fraction,

$$£\tfrac{5}{6},$$

to be changed into integers of a lower denomination: by turning to Lesson XLIX., the pupil will recognise here a case of *descending transformation* of denominate fractions; and, by a known rule, he will get

$$£\tfrac{5}{6} = \tfrac{100}{6}sh. = 16\tfrac{4}{6}sh.$$

But the fraction, $\frac{4}{6}$ or $\frac{2}{3}$ *sh.*, in its turn, may be changed into pence,

$$\tfrac{2}{3}sh. = \frac{2\times 12}{3}\,sh. = 8d.$$

Hence, the complete transformation is

$$£\tfrac{5}{6} = 16sh.\ 8d.$$

The operation is evidently the same as would take place in the extension of a division to lower units. If, for instance, the remainder of a division by 6 had been £5, and we wished to get the part $\frac{5}{6}$ of the quotient in smaller units, the operation would be as follows:

1st. Multiply £5 by 20, to change it into shillings.

2d. Divide the product, 100, by 6, and the quotient is 16*sh.*, with a remainder, 4.

3d. Multiply this remainder by 12, to lower to pence.

4th. Divide the product, 48, by 6, and the quotient, 8*d.*, completes the answer, which is composed of the successive quotients.

OPERATION.

```
  5     | 6
 20     |-----------
-----   | 16sh. 8d.
100sh.
  4
 12
-----
 48d.
```

If there was, again, a remainder, you might proceed in the same way, to farthings. This is analogous to what was done in decimals; and it is hardly necessary to state that, in order to change a denominate fraction to integers of a lower denomination, *Transform the numerator of the fraction into lower units and divide. If there is a fractional part in the quotient, transform it in the same manner until you reach the lower denomination required.*

The operation would be the same, though rather easier, with *decimal fractions.*

Let us take, as an example,

0.7 of a bushel.

By the principles of Lesson XLIX., you change this into pecks by multiplying by 4, and thus get

2.8 *pk.*, or 2 *pks.* and 0.8 of a *pk.*

Now, in its turn, the fractional part,

0.8 multiplied by 2, is 1.6 *gal.*

OPERATION.

$$\begin{array}{l} 0.7 \text{ bushels.} \\ \underline{\ \ 4\ \ } \\ 2.8 \text{ pecks.} \\ \underline{\ \ 2\ \ } \\ 1.6 \text{ gallons.} \\ \underline{\ \ 4\ \ } \\ 2.4 \text{ quarts.} \\ \underline{\ \ 2\ \ } \end{array}$$

Then the fractional part, .6, multiplied by 4, gives

2.4 quarts;

the fractional part of which contains no integer of pints. Therefore, the answer is, by collecting all the integers of the successive products,

$$0.7 bush. = 2 pks.\ 1 gal.\ 2.4 qt.$$

What renders here the operation easier is, that the use of the decimal point dispenses with divisions.

N. B.—Be careful to multiply only the decimal part of each product, which is the only one to be transformed.

CASE II.

3. In the transformation of a compound number into a fraction of a higher denomination, the pupil will readily recognise an addition of fractions.

Let it be proposed, for example, to convert

9*oz.* 6*dwt.* 16*gr.*

into the fraction of a pound troy. It is clear that, if instead of employing names, we express the successive numbers by their fractional values, relative to the pound, the compound number will become

$$\left(\tfrac{9}{12} + \tfrac{6}{20} \text{ of } \tfrac{1}{12} + \tfrac{16}{24} \text{ of } \tfrac{1}{20} \text{ of } \tfrac{1}{12}\right) \text{ of a pound.}$$

$$\text{or,} \quad \tfrac{9}{12} + \frac{6}{20\times12} + \frac{16}{24\times20\times12} = \tfrac{7}{9}lb.$$

So that the operation is nothing else than an addition of fractions, and may easily be performed as such.

The usual method, however, is as follows:

Reduce the compound number to its lowest denomination, and set the result, as a numerator, over the number which expresses the ratio of the last units to the higher denomination required.

Reduce the fraction to its lowest terms.

OPERATION.

$$\begin{array}{l} 9oz.\ 6dwt.\ 16gr. \\ \underline{20} \\ 186dwt. \\ \underline{24} \\ 4480gr. = \tfrac{4480}{5760} = \tfrac{7}{9}lb. \end{array}$$

In the example we have chosen, the compound number is changed to 4,480*gr.*; and 1*gr.* being

$$\frac{1}{24\times20\times12} \text{ of a } lb.\text{; or, } \tfrac{1}{5760}lb.,$$

the answer is $\frac{4480}{5760}lb.$, which can be transformed into $\frac{7}{9}lb.$

4. The transformation of a compound number into a *decimal fraction* of a higher unit, might be made in two different ways:

I. Either *find the vulgar fraction equivalent to the compound number, and change it into a decimal fraction;*

II. Or, *change successively each part of the compound number into a decimal fraction of the denomination just above it, until you reach the upper one.*

This second method will generally be found the shortest. Let us take the same example as above, and consider each denominate part successively.

The third, . $16gr. = \frac{16}{24}dwt. = 0.\dot{6}dwt.$

This, together with $6dwt.$, is $6.'6dwt. = \frac{6.'6}{20}oz. = 0.3'\overline{3}oz.$; which, added to $9oz.$, gives $\frac{9.3'\overline{3}}{12}lb. = 0.7778'\overline{6}.$

The operation is given here in all its details, but may be performed almost at a glance, with but little practice.

Questions.—How do you change a denominate fraction into a compound number of lower units? What simple operation is it? How is the same done in decimals? Give examples. How do you transform compound numbers into a denominate fraction of a higher denomination? What simple operation is it? What is the usual way? How is the same transformation made in decimals?

EXERCISES IN TRANSFORMATION OF DENOMINATE FRACTIONS, BOTH VULGAR AND DECIMAL.

N. B.—Look to the rules and examples in Sterling Money.

1. What part of a *dwt.* is $\frac{1}{300}$ of a *lb.* troy? *Ans.* $\frac{4}{5}$.
2. Change $\frac{1}{196}$*cwt.* to the fraction of a *lb.* *Ans.* $\frac{4}{7}$.
3. Change $\frac{2}{3}$ of a £ into smaller units.
   Ans. $\frac{2}{3} \times 20sh. = 13\frac{1}{3}sh.$
   And because . . $\frac{1}{3}sh. = 4d.$ . . $= 13sh.\ 4d.$
4. Change $\frac{2}{5}$ of a shilling to smaller denominations.
   Ans. $\frac{2}{5} \times 12d. = 4\frac{4}{5}d.$
   And because . $\frac{4}{5}d. = \frac{4 \times 4}{5}far.$. $= 4d.\ 3\frac{1}{5}far.$
5. Express $\frac{3}{5}$ *lb.* troy in smaller integers. *Ans.* 7*oz.* 4*dwt.*
6. Express $\frac{4}{5}$ mile in smaller denominations. *Ans.* 6*fur.* 16*P.*
7. Reduce 2*qr.* $3\frac{1}{3}$*na.* to the fraction of an ell English. *Ans.* $\frac{5}{9}$*E. E.*
8. Reduce 4*sh.* $6\frac{1}{2}$*d.* to the fraction of a £. *Ans.* £$\frac{109}{480}$.
9. Express 14*sh.* $5\frac{1}{2}$*d.* in decimals of a £. *Ans.* £$0.72291'\overline{6}$
10. " 15*sh.* in decimals of a £. *Ans.* £0.75.
11. " 3*qr.* 18*lb.* in decimals of a *cwt.* *Ans.* 0.910714 +
12. " 2*qr.* 2*na.* in decimals of a *yd.* *Ans.* 0.625.
13. " 14*gal.* 3*qt.* of wine in decimals of a hogshead. *Ans.* 0.2341 +
14. " 13*sh.* $9\frac{3}{4}$*d.* in decimals of a £. *Ans.* 0.691.
15. What is the value of 0.9 shilling? *Ans.* $10\frac{4}{5}$*d.*

16. Change 0.592 of a *cwt.* into a compound number.
Ans. 2*qr.* 10*lb.* 4*oz.* 13*dr.*+

17. Express 0.258 tun of wine in lower integers.
Ans. 1*hhd.* 2*gal.*+

18. " 0.12785 of a year. *Ans.* 46*da.* 15*hr.* 57*m.* 57″+

19. " 2*ft.* 3*in.* in the fraction of a yard. *Ans.* $\frac{3}{4}$*yd.*

20. What is $\frac{5}{6}$ of a hogshead? *Ans.* 52*gal.* 2*qt.*

21. What is $\frac{4}{5}$ of a guinea? *Ans.* 16*sh.* 9$\frac{3}{5}$*d.*

22. What part of a mile is 9*ft.* 9*in.*? *Ans.* $\frac{13}{7,040}$*ml.*

23. What part of a day is 4*hr.* 3*m.*? *Ans.* $\frac{27}{160}$*da.*

24. Change 1*lb.* 4*oz.* 12*dwt.* 16*gr.* to the decimal of a pound.
Ans. 1.3861*lb.*

25. Express 6,600*ft.* in decimals of a mile. *Ans.* 1.25*ml.*

26. What is in integers 0.625 of a gallon? *Ans.* 2*qt.* 1*pt.*

27. What is .025 of an acre? *Ans.* 4*P.*

28. What is .125 of a year, leaving out the 6 hours?
Ans. 45*da.* 15*hr.*

29. What is 2.16 miles? *Ans.* 2*ml.* 51*rd.* 1*yd.* 0*ft.* 3$\frac{3}{5}$*in.*

30. What is 0.1125 of a ton? *Ans.* 2*cwt.* 1*qr.*

31. What decimal fractions of 1*cwt.* is 1*lb.*? *Ans.* .00892857.

32. What decimal fraction of a pound is 1 farthing?
Ans. $.00104\overline{16}$.

LESSON LII.

COMPOUND ADDITION.

RULE.

1. I. *Set down the units of the same denomination under each other.*

II. *Take the sum of the column of the lowest denomination; find by division how many units of the next higher order there may be in this sum; set down the remainder under the column just added up, and carry the units of the quotient to be added with the next column.*

III. *Proceed in the same way with all the columns.*

FIRST EXAMPLE.

2. For a first illustration of this rule, let us take the annexed addition in avoirdupois weight.

OPERATION.		
cwt.	*qr.*	*lb.*
76	3	14
37	2	15
14	1	11
128	3	12

The addition of the pounds gives 40, that is 1 quarter and 12 pounds; I set down 12 and carry 1 quarter.

Now 1 quarter added to the others makes 7 quarters, which are 1 cwt. and 3 quarters; I set down 3 and carry 1. Finally, adding 1 together with the cwt. gives 128 cwt.

SECOND EXAMPLE.

Let it be proposed to add the following numbers:

	£	*sh.*	*d.*
		33	
	765	19	7
	1,279	17	6
	915	13	11
	2,594	19	8
	589	8	6
Sum total,	£6,145	19*sh.*	2*d.*

Adding, first, the column of pence, I get 38 pence, which contain 3 shillings and 2 pence.

I set down 2 under the column of pence, and carry the 3 shillings to the column of units of shillings.

Adding up the first column in shillings, I find 39; I set down 9 and carry 3 tens, to be added with the second column of shillings.

I now add the tens of shillings and get 7; and, because it takes 2 tens of shillings to make one pound, I take one half of 7, which is 3, with a remainder, 1. I set down this remainder and carry the £3 to the column of pounds.

Then I complete the addition, and the result is as above.

These two examples will suffice to explain the rule for the addition of compound numbers.

3. This addition is proved like simple addition, by reversing the operation.

Questions.—What is compound addition? How do you set down the numbers? Where do you begin to add? What do you do with each successive sum? What do you set down? What do you carry? How do you prove addition?

EXERCISES.—FEDERAL MONEY.

(1.) \$75 3*c.* 6*m.* + \$25 7*m.* + \$256 15*c.* + \$618*m.* + 278*m.* + 96*c.*

(2.) \$209 5*m.* + 65*m.* + 273 *cts.* + \$6 3*c.* + \$44 + \$9,729 68 *c.*

(3.) \$719 2 *c.* + 6,921*m.* + 3,456*c.* + \$0.15 + \$900 + 67 *cts.* + \$2,966.

ENGLISH CURRENCY.

£	*sh.*	*d.*
1	3	5
1	0	9
0	7	$11\frac{3}{4}$
20	13	9
0	1	$3\frac{3}{4}$
23	7	$2\frac{1}{2}$

£	*sh.*	*d.*
149	14	$6\frac{3}{4}$
387	19	$8\frac{1}{2}$
456	15	$7\frac{1}{4}$
622	0	8
734	9	$9\frac{1}{2}$

£	*sh.*	*d.*
28	9	$10\frac{1}{4}$
605	3	$7\frac{3}{4}$
100	4	0
77	2	$8\frac{1}{2}$
0	13	$11\frac{3}{4}$

TROY WEIGHT.

lb.	*oz.*	*dwt.*	*gr.*
6	2	11	19
64	9	17	22
0	6	0	17
100	7	19	23
172	2	10	9

oz.	*dwt.*	*gr.*
37	9	3
9	5	3
3	16	21
17	7	8
5	9	0
3	0	19

lb.	*oz.*	*dwt.*	*gr.*
83	11	15	22
16	7	19	20
10	0	0	9
32	1	8	10
44	2	17	14
18	1	16	7

AVOIRDUPOIS WEIGHT.

tons.	*cwt.*	*qr.*	*lb.*	*oz.*
2	2	1	17	10
12	10	0	2	2
0	2	1	18	3
0	0	0	9	11
14	14	3	19	10

cwt.	*qr.*	*lb.*
624	1	19
207	0	25
159	3	27
211	0	0
0	7	8

cwt.	*qr.*	*lb.*	*oz.*	*dr.*
91	2	15	14	1
25	3	21	0	9
19	0	6	2	15
33	1	16	15	2
10	3	4	10	11

APOTHECARIES' WEIGHT.

℔	℥	ʒ	℈	*gr.*
11	6	2	1	15
25	11	7	2	19
44	0	5	0	11
36	9	2	2	18
3	0	6	0	17
121	5	1	0	0

℔	℥	ʒ	℈	*gr.*
25	7	7	2	0
9	10	0	1	19
26	2	1	2	6
34	11	6	0	12
16	7	5	1	17

℔	℥	ʒ	℈	*gr.*
34	0	2	1	6
15	7	7	2	9
165	8	3	1	16
22	6	6	0	0
49	5	5	2	18

CLOTH MEASURE.

(1.)

yd.	qr.	na.
150	3	2
219	1	0
501	2	3
61	3	2
0	1	1
934	0	0

(2.)

yd.	qr.	na.
29	2	1
36	1	3
456	0	2
35	1	3
9	0	0

(3.)

yd.	qr.	na.
210	3	3
25	1	1
97	2	3
0	1	2
37	2	3

LONG MEASURE.

lea.	ml.	fur.	P.	yd.	ft.
16	2	7	25	3	1
25	1	6	17	2	2
10	0	1	0	1	0
44	2	5	39	5	1
0	1	4	31	4	2
98	0	1	35	0	0½

ml.	fur.	P.
29	3	14
19	6	29
5	4	20
9	1	37
7	0	3
4	5	9

yd.	ft.	in.
127	2	5
12	2	9
0	2	0
3	1	11
213	2	10
25	0	0

SQUARE MEASURE.

M.	A.	R.	P.
6	500	2	36
19	21	3	15
20	375	2	17
11	6	0	19
0	615	1	39
58	239	3	6

M.	A.	R.	P.
7	320	1	13
10	10	0	10
75	415	2	31
6	3	0	15
19	18	3	36
15	575	1	17

Sq. yd.	Sq. ft.	Sq. in.
19	5	141
25	7	126
9	8	117
21	4	19
16	3	100
7	0	109

CUBIC MEASURE.

C. yd.	C. ft.	C. in.
51	15	1678
76	5	1000
33	20	567
1	19	1062
3	0	9
15	26	1708
182	7	840

Cords.	C. ft.	C. in.
15	127	1600
2	61	1521
4	72	304
21	78	1411
44	106	319
16	125	0

C. yd.	C. ft.	C. in.
15	6	1500
21	21	0
697	14	212
4322	19	699
96	0	247
149	26	1461

WINE MEASURE.

T.	pi.	hhd.	bar.	gal.	qt.
21	1	1	1	30	3
72	0	1	0	25	2
21	1	1	0	31	1
4	1	0	1	7	2
3	1	0	0	16	3
124	0	1	1	17	1

T.	pi.	hhd.	bar.	gal.	qt.
6	1	1	1	31	2
15	0	1	0	26	3
26	0	1	1	15	1
31	0	1	1	3	0
71	1	1	0	18	3
27	1	1	0	9	3

hhd.	gal.	qt.	pt.
17	60	3	1
2	51	2	0
5	25	1	1
9	17	0	1
7	44	2	1
16	19	3	1

DRY MEASURE.

(1.)

ch.	bu.	pk.	qt.	pt.
16	26	3	3	1
10	35	2	7	1
0	16	1	6	0
2	0	1	2	1
4	19	0	4	0
34	26	1	7	1

(2.)

ch.	bu.	pk.	qt.	pt.
19	30	2	6	1
21	18	3	1	0
36	17	0	7	1
0	14	1	0	1
7	32	3	5	0
9	0	3	4	0

(3.

bu.	pk.	gal.	qt.	pt.
6	2	1	2	1
21	3	1	3	0
17	0	0	0	0
18	1	1	2	1
26	3	0	3	1
0	3	1	3	1

TIME.

yr.	mo.	da.	hr.	′	″
4	10	15	7	9	17
7	9	29	16	51	49
10	6	13	9	0	0
15	7	22	14	56	40
6	0	1	2	24	59
44	10	22	2	22	45

yr.	mo.	da.	hr.	′	″
115	0	0	0	0	0
17	7	17	17	41	19
91	5	6	1	0	48
29	11	28	22	52	24
10	3	6	17	25	26
0	0	1	4	1	7

mo.	da.	hr.	′	″
1	4	6	2	17
26	7	3	40	19
6	19	17	14	45
2	2	2	1	0
41	17	3	18	53
21	21	21	21	21

CIRCULAR MEASURE.

°	′	″
6	15	59
9	14	41
10	32	26
14	41	32
2	0	1
7	42	11
50	26	50

°	′	″
21	3	8
1	16	19
31	3	2
6	8	59
7	11	30
9	39	17

°	′	″
17	32	57
6	17	8
3	9	1
4	50	50
7	9	10
8	19	34

LESSON LIII.

COMPOUND SUBTRACTION.

RULE.

1. I. *Place the less number below the greater, so that like units may be exactly under each other.*

II. *Begin with the lowest order, and subtract each figure from the one above it, if possible.*

III. *But, if a number of any order is larger than that above it, make the subtraction possible by adding to the* *upper number as many units as are equal to one of the next higher order; subtract and carry* 1 *to the lower number of the next higher order; go on, in the same way, to the end.*

2. It will be observed that this is analogous to the subtraction of simple numbers, and exactly the same as the rule for subtracting mixed numbers, as it should be; since a compound number is in fact a mixed number, and all the operations with it should be performed in the same way.

To illustrate this rule by an example, let us suppose that we have,

	£	*sh.*	*d.*	*far.*
From	25	0	5	2
to subtract	17	11	10	1
	£7	8*sh.*	7*d.*	1*far.*

In this example, we subtract 1 farthing from 2, and set down the remainder, 1.

Then, as we cannot take 10*d.* from 5, we render the subtraction possible by adding to 5 the value of 1 shilling; that is, 12*d.*, and say, 10 from 17, leaves 7.

Now, having added the value of 1 shilling to the upper number, we must add the same to the lower one, in order that the difference may not be changed. We therefore say, 1 shilling carried and 11 are 12; 12 from 0 we cannot. Here, again, to render the subtraction possible, we add to the shillings the value of 1 pound; that is, 20 shillings, and say, 12 from 20 leaves 8, which we set down.

Finally, the pound, added as 20 shillings to the upper number, must also be added to the lower one; and we conclude by saying, 1 carried and £17 are 18, which, taken from 25, leaves £7.

PROOF.

3. Compound subtraction is proved as in simple numbers, by adding the smaller number and remainder.

EXAMPLE.	OPERATION.				
	mo.	*day*	*h.*	*m.*	*sec.*
Let it be proposed from	6	16	13	27	19
to subtract	1	22	16	41	37
42 sec. and 37 are one min. and 19 sec.					
46 min. and 41 are one hour and 27 min.	4	23	20	45	42
21 h. and 16 are one day and 13 h. Proof	6	16	13	27	19
24 days and 22 are one mo. and 16 days.					

Finally, 5 mo. and 1 are 6 mo. which proves the operation.

Questions.—How are numbers set down in subtraction? Where do you begin? If the lower number of any denomination is larger than the upper one, how do you subtract? Why? Give an example. What do you add? What do you carry? Repeat the rule. How do you prove subtraction?

EXERCISES, WITH PROOFS.—FEDERAL MONEY.

1. From 215 dollars 6 cents, subtract 119 dollars 55 mills.
2. From 16 dollars and 20 mills, subtract 7 dollars 25 cents.
3. From 6,728 cents, subtract 50,739 mills.

ENGLISH CURRENCY.

1. From £1 2*sh.* 1*d.*, take 3*sh.* 4*d.* *Ans.* 18*sh.* 9*d.*
2. From £5 2*sh.* 1*d.*, take 9*sh.* 6*d.* *Ans.* £4 12*sh.* 7*d.*
3. From £8, take 20*sh.* and 3*far.* *Ans.* £6 19*sh.* 11¼*d.*

TROY WEIGHT.

1. From 7*lb.* 3*oz.* 11*gr.*, take 3*lb.* 7*oz.* 5*dwt.* 19*gr.*
2. From 4*lb.* 9*oz.* 1*dwt.* 13*dr.*, take 3*lb.* 8*oz.* 16*dwt.* 12*gr.*
3. From 73*lb.*, take 26*lb.* 7*oz.* 2*dwt.* 16*gr.*

AVOIRDUPOIS WEIGHT.

1. From 32*T.* 10*cwt.* 1*qr.* 27*lb.*, take 14*T.* 19*cwt.* 3*qr.* 16*lb.*
2. From 3*cwt.* 2*qr.* 1*lb.* 6*oz.* 15*dr.*, take 2*cwt.* 3*qr.* 25*lb.* 7*oz.* 14*dr.*
3. From 16*T.* 16*lb.*, take 5*cwt.* 15*oz.*

N. B.—Let these also be done when the hundred weight is only 100 pounds.

APOTHECARIES' WEIGHT.

1. From 10℔ 12*gr.*, take 9℔ 6℥ 1℈ 13*gr.*
2. From 112℔ 7ʒ 2℈ 19*gr.*, take 97℔ 11℥ 6ʒ 2℈ 18*gr.*
3. From 16℔, take 9℔ 10℥ 7ʒ 1℈ 15*gr.*

CLOTH MEASURE.

1. From 934*yd.*, take 150*yd.* 3*qr.* 2*na.*
2. From 69*E.Fl.* 2*qr.* 1*na.*, take 50*E.Fl.* 3*na.*
3. From 51*E.Fr.* 5*qr.* 2*na.*, take 51*yd.* 3*qr.* 3*na.*

LONG MEASURE.

1. From 16*ml.* 5*fur.* 31*rd.* 4*yd.*, take 5*ml.* 5*yd.*
2. From 35*rd.* 2*yd.* 2*ft.* 2*in.*, take 16*rd.* 3*yd.* 2*ft.* 10*in.*
3. From 17*ml.*, take 39*rd.* 4*yd.* 1*ft.* 11*in.*

SQUARE MEASURE.

1. From 19*M.* 10*A.* 3*R.* 36*P.*, take 7*M.* 325*A.* 13*P.*
2. From 6*A.*, take 2*R.* 16*P.*
3. From 19*sq.yd.* 5*sq.in.*, take 18*sq.yd.* 143*sq.in.*

CUBIC MEASURE.

1. From 320*C.yd.* 20*C.ft.*, take 309*C.yd.* 21*C.ft.* 1,400*C.in.*
2. From 15*cords* 100*C.ft.*, take 10*cords* 116*C.ft.*
3. From 21*C.yd.*, take 65*C.ft.* 895*C.in.*

WINE MEASURE.

1. From 6*T.* 1*hhd.* 39*gal.* 2*qt.*, take 2*T.* 45*gal.* 1*qt.*
2. From 16*gal.* 1*qt.* 1*pt.* 2*gi.*, take 15*gal.* 3*gi.*
3. From 45*gal.*, take 15*qt.* 2*pt.* 1*gi.*

DRY MEASURE.

1. From 34*ch.* 26*bu.* 7*qt.*, take 16*ch.* 26*bu.* 3*pk.* 3*qt.* 1*pt.*
2. From 38*bu.* 7*gal.* 3*qt.* 1*pt.*, take 31*bu.* 1*pt.*
3. From 16*bu.*, take 2*qt.* 1*pt.*

TIME.

1. From 30*yr.* 29*da.* 40″, take 28*yr.* 6*mo.* 8*hr.* 51′ 45″.
2. From 10*yr.* 6*mo.*, take 15*da.* 6*m.*
3. From 1*yr.*, take 29*hr.* 17′ 35″.

CIRCULAR MEASURE.

1. From 50° 26′ 50″, take 49° 31′ 55″.

2. From 9*S*. 25°, take 51° 51″.

3. From 315°, take 59′ 35″.

N. B.—In the sequel, examples illustrative of each lesson will be given at the end of it. But questions relative to it, or combining it with others, will be found in the Appendix, for more advanced students, revising arithmetic. Those included in the lessons, are for beginners.

LESSON LIV.

ADDITION AND SUBTRACTION OF DENOMINATE FRACTIONS.

1. Fractions of different denominations of the same measure have frequently to be added or subtracted; as, for example,

$\frac{1}{3}$ of a pound and $\frac{1}{4}$ of a shilling.

All that is necessary is to reduce them to the same denomination; which may be done in several ways, either

2. *By transforming the fractions into integers of smaller denominations.*

By the rule of Lesson L., we may, for example,
change . . . . . £$\frac{1}{3}$ into 6*sh*. 8*d*.
and . . . . . $\frac{1}{4}$*sh*. into 0*sh*. 3*d*.

Their sum, under this form, would be . 6*sh*. 11*d*.
and their difference . . . . 6*sh*. 5*d*.

3. Or else, *by transforming the fractions into others having all the same denomination.*

By the rule of Lesson XLIX., we may change £$\frac{1}{3}$
into . . . . $\frac{20}{3}$*sh*. $= \frac{80}{12}$.
Which may be added, as usual, to $\frac{1}{4}$ *sh*. $= \frac{3}{12}$.

The sum being . . . . . $\frac{83}{12}$*sh*.

Or, we may change the second fraction into the fraction of a pound, by making it . . . £$\frac{1}{80} = \frac{3}{240}$.
Which, added to . . . . £ $\frac{1}{3} = \frac{80}{240}$.

would give . . . . . £$\frac{83}{240}$.

These three sums, $6sh.\ 11d.$; $\frac{83}{12}sh.$; $£\frac{83}{240}$, are equivalent.

The form to be preferred depends on the nature of the question proposed. The corresponding differences in fractional form would be . . . . $\frac{77}{12}sh.$

Or, ... $£\frac{77}{240}$.

Both equivalent to $6sh.\ 5d.$

4. If the fractions were decimal, instead of vulgar, the operation would be similar. Thus,

$$0.65lb. \text{ and } .75dwt.$$

might be added as follows:

OPERATION.

$$0.65 \times 20 = 13dwt.$$
$$\underline{0.75}$$
$$13.75dwt.$$

Or, because $.75dwt. = 18gr.$, it would also be $13dwt. + 18gr.$

The subtraction between the same would give

$$12.25dwt.$$

Or, . . $12dwt.\ 6gr.$

These are mere applications of what has been taught in Lessons XLIX. and L.

Questions.—How do you add fractions of the same measure, but of different denominations? How do you subtract them? What determines the form to be chosen? How do you change fractions into integers of lower denominations? How do you transform a fraction into another of a higher denomination? Of a lower denomination?

EXERCISES IN THE THREE FORMS.

1. $£\frac{7}{9} + \frac{3}{10}sh. = 15sh.\ 10\frac{4}{15}d. = \frac{1427}{90}sh.$, or $£\frac{1427}{1800}$.
2. $\frac{2}{3}yd. + \frac{3}{4}ft. + \frac{7}{8}ml. = 1{,}540yd.\ 2ft.\ 9in.$, or, fractionally, . or decimally,
3. $\frac{1}{3}wk. + \frac{1}{4}da. + \frac{1}{2}hr. = 2da.\ 14\frac{1}{2}hr.$ or, . or "
4. $£\frac{7}{9} - \frac{3}{10}sh. = 15sh.\ 3\frac{1}{15}d.$ or, . or "
5. $£\frac{3}{4} - \frac{3}{4}sh. = 14sh.\ 3d.$ or, . or "

6. $\frac{1}{6}yd. - \frac{2}{3}in. = 5\frac{1}{3}in.$

7. $£\frac{2}{3} + \frac{5}{9}sh. = 13sh.\ 10d.\ 2\frac{2}{3}far.$, or $\frac{125}{9}sh.$, or decimally ? . .

8. $7wk. - 9\frac{7}{10}da. = 5wk.\ 4da.\ 7hr.\ 12m.$

9. $£\frac{1}{7} + \frac{2}{9}sh. + \frac{5}{12}d. = 3sh.\ 1d.\ 1\frac{20}{21}far.$, or $\frac{3139}{1008}sh.$, or decimally ? . . . .

10. $£\frac{5}{9} - \frac{2}{3}$ of $\frac{3}{4}sh. = 10sh.\ 7d.\ 1\frac{1}{3}far.$, or $\frac{191}{18}sh.$, or . . . . ?

11. $\frac{2}{7}$ of $£5\frac{1}{6} - \frac{3}{5}sh. = £1\ 8sh.\ 11d.\ 0\frac{12}{35}far.$, or $£\frac{3037}{2100}$, or . . . . ?

12. What will remain, if $13\frac{5}{6}sh.$ and $7\frac{5}{8}d.$ be taken from £1 ?
Ans. $5sh.\ 6\frac{3}{8}d.$

13. Which is greater, $\frac{1}{5}$ of $6sh.$ or $1sh.\ 2\frac{1}{2}d.$?
Ans. $1sh.\ 2\frac{1}{2}d.$ by $\frac{1}{10}d.$

14. $\frac{1}{2}cwt. + 15lb. + \frac{1}{3}cwt. + 9oz. + \frac{3}{4}lb. = 3qr.\ 25lb.\ 10\frac{1}{8}oz.$

15. $\frac{2}{5}oz. - \frac{5}{8}dwt. = 7dwt.\ 9gr.$, or (fraction) or . . (decimal).

16. $£0.5 + 1.25sh. + 0.12d. = 11sh.\ 3.12d.$, or $135.12d.$

17. $0.16cwt. + 2qr. + .75lb. = 2qr.\ 18.67lb.$

18. $0.75ml. + 0.9P. + 3.25yd. = 241P.\ 2yd.\ 2.1ft.$

19. $£0.75 - .90sh. = 14sh.\ 1d.\ 0.8far.$

20. $0.35lb. - .85oz.$ (troy) $= 3oz.\ 7dwt.$

21. $0.65A. - 21.75P. = 82.25P.$

LESSON LV.

MULTIPLICATION OF COMPOUND NUMBERS.

1. Compound numbers being, in reality, only mixed numbers, we know already a method by which they might be multiplied. For this purpose, it would suffice to put the lower denominations under the form of fractions; and, indeed, in many cases, this might be the best way.

2. If we had, for instance, to multiply

£7 15*sh.* by 1*ft.* 3*in.*,

we might change them into $£7\frac{3}{4}$ and $1\frac{1}{4}ft.$

Or, decimally, £7.75 and 1.25*ft.*,

and then multiply.

But, in general, this is not the most expeditious method.

3. The operation, as usually performed, is much more

difficult, though shorter. It requires much attention, and can be explained clearly only by examples. But, before giving these, it is important to present some remarks on the nature of multiplication.

4. MULTIPLICATION is the addition of a quantity to itself a certain number of times; so that, whatever the multiplicand may be, *the multiplier is always made an abstract number, and the product is invariably of the same nature as the multiplicand* (XI., 7).

5. It is therefore a great absurdity to speak of multiplying together denominate quantities; as *pounds* or *dollars* and *yards*. When, for instance, the cost of so many yards of cloth, at so much a yard, is required, we do not multiply money by yards; but we take the value of a yard as many times as there are units in the number of yards, and the amount is a sum of money.

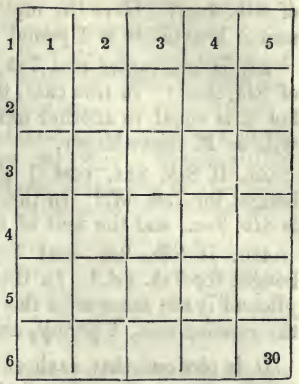
Thus, 4 yards of cloth, at 3 dollars a yard, are 3 dollars taken 4 times; that is, 12 dollars.

Again: if *one dollar* will buy a certain number of *yards*, and we wish to know how many yards we can buy with a certain sum of dollars, we repeat the number of yards as often as there are dollars in this sum; and the amount is a number of yards, with which the name of the unit we multiply by has nothing to do.

Let it be supposed, for instance, that 4 yards cost 1 dollar; if it be asked how many yards can be had for 3 dollars, we merely repeat 4 yards 3 times; the denomination *dollar* having nothing to do with the result.

6. There is one seeming exception to this, namely: when we multiply *yards* by *yards*, *feet* by *feet*, or any other measure of length by itself, and get *square yards*, *square feet*, or some other square measure. But this, in reality, is a mere short mode of expression, which should not be understood literally.

Suppose, for instance, that we have a square board, 5 inches in length by 6 in breadth, and we wish to ascertain how many square inches its surface contains; we divide its sides into inches, and, by drawing lines across,



form squares, each one square inch; and then, instead of counting each square, we count the inches on both edges, and, multiplying them together, call their product 30 *square inches.*

This, however, if properly investigated, is not multiplying linear inches by linear inches. But we observe that, along the upper row, there are as many square inches as there are divisions in it; and, also, that there are as many such rows as there are inches along the other side. Consequently, we repeat the number in one row, as many times as there are rows, which is here 5 multiplied by 6. *This evidently is not multiplying inches by inches, but repeating a number of small squares, equal to that of the inches along one line, by a number of units equal to the number of inches on the other line.*

7. Nor is it correct to consider the multiplication of feet by feet, as producing always square feet; it may sometimes give linear feet, as some subsequent examples will show (LVIII.). In this, as in other cases, a precise statement alone can determine the nature of the product.

8. We cannot properly, therefore, multiply one compound number by another. If we were asked, for example, to multiply 8*lb.* 9*oz.*, troy, by 7*sh.* 6*d.*, the question would be undetermined; for nothing tells us which is the multiplicand or the multiplier. And even if either number was designated as the multiplicand, we do not know to what unit of the multiplier it is equal. To illustrate this fully, we will frame four different questions upon the same example:

1st. If one pound cost 7*sh.* 6*d.*, what will be the cost of *8lb. 9oz.*? Here the multiplicand is *7sh.* 6*d.*, and the unit it is equal to is 1 pound weight.

2d. If one ounce cost *7sh.* 6*d.*, what will be the cost of *8lb. 9oz.*? In this case, the multiplicand is the same, but it is equal to another unit, 1 ounce, and the product will be 12 times larger.

3d. If *8lb. 9oz.* cost 1 shilling, how much can be bought for *7sh.* 6*d.*? In this third case, the multiplicand is *8lb. 9oz.*, and the unit of the multiplier is 1 shilling.

4th. If *8lb. 9oz.* cost 1 penny, how much can be bought for *7sh.* 6*d.*? In this fourth question, the multiplicand is the same as in the preceding, but it is equal to the smaller unit, 1 penny, and the product is larger.

It is obvious that each of these questions will furnish a different answer. The first two will give pounds, shillings, and pence, in the product; the last two, pounds and ounces.

9. It must be clear, therefore, that we cannot proceed, in any case, to multiply denominate numbers, unless it be distinctly understood which unit of the multiplier the multiplicand is equal to. Then, the number of those units, considered as an abstract number, constitutes the multiplier.

This remark does not apply to pounds, shillings, and pence, alone; it is equally applicable to dollars and simpler numbers. If it were proposed, for instance, to multiply $4\frac{1}{4}$ yards by $1.25, the question would be fully as incomplete; for, one *dollar*, one *dime*, or one *cent*, may be the value of the yard; or it may be either the yard or the quarter which is worth $1.25. Therefore, to any such question, the pupil must answer by asking for a clear statement, by which he can ascertain his data.

10. I have enlarged upon this point, because important mistakes are frequently made in compound multiplications, for want of due consideration, as regards the unit of comparison; and I have often heard such a question as this propounded: ***What is the product of* £2**

19sh. 6d. by £2 *19sh. 6d.*? and seen wonder expressed that the answers given by different persons did not agree.

Persons conversant with the correct principles of arithmetic, will not discuss seriously such a question, which is indefinite, even with dollars.

Let it be proposed, for instance, to multiply $1.11 by $1.11: the answer may be $123.21; $12.321; or $1.2321, or even other numbers, according to the question, which might be,

1st. A man has adventured $1.11, and made $1.11 for each cent: how much has he made in all? *Ans.* $123.21.

2d. His gain has been $1.11 for each dime. *Ans.* $12.321.

3d. His gain has been $1.11 for each dollar. *Ans.* $1.2321.

This is sufficient to show what a great variety of answers may be obtained by changing the unit of comparison. The intelligent pupil, therefore, should not accept as a question a multiplication so incompletely stated.

11. In fact, *multiplication* is always the consequence of a species of *rule of three;* in which one of the terms is *one unit*, of the nature of the multiplier; so that, in general, the statement would be,

1 : the multiplier : : the multiplicand : the product.

For example:

One yard has cost 4 *dollars:* how much will 3 *yards* cost? would give the proportion

$$1yd. : 3yd. :: \$4 : \text{the answer} = \$4 \times \frac{3yd.}{1yd.}$$

That is, the answer would be $4, multiplied by the ratio of 3 yards to 1 yard; which is nothing but the abstract number 3, the nature or denomination of the numbers having nothing to do with their ratio. This shows why the multiplier is always an abstract number.

If, in all cases, this is kept in view, multiplications indefinitely stated as the above, will no longer be regarded as arithmetical puzzles.

The question is incomplete until the unit to which the multiplicand is equivalent is given.

12. *The general principle of compound multiplication is to multiply successively each part of the multiplicand by each part of the multiplier;* having regard, of course, to the dependence of the various orders of units.

We will, however, consider three cases.

CASE I.

13. When the multiplier is a small simple number:

Let it be proposed, for example, to

multiply	£247 17*sh.* 11*d.*
by	9
The product will be .	£2,231 1*sh.* 3*d.*

In this case, *you multiply severally the parts of the multiplicand, beginning with the lowest denomination, and carry successively, to their proper column, the integers of higher denominations, furnished by the inferior products.*

Thus, the product of 11*d.* by 9, is 99*d.*, equal to 8*sh.* 3*d.*; you set down 3*d.*, and carry 8*sh.* to their proper column.

Then, 9 times 17*sh.* are 153*sh.*, and 8 carried, 161*sh.* . . . . . . = £8 1*sh.*; you set down 1*sh.* and reserve £8 to be carried.

Finally, 9 times £7, and £8 carried, are £71, &c.

Questions.—How is the multiplier considered in all multiplications? Can you multiply dollars by yards; yards by dollars; or any denomination by another? What is meant when feet, multiplied by feet, are said to make square feet? Is the question definite, when it is asked to multiply a compound number by another? What is required to make it definite? How do you multiply a compound number by a simple small number? Is there any analogy between multiplication and the rule of three?

EXERCISES.

1. How much are 8*lb.* of tea, at 5*sh.* 8½*d.*? *Ans.* £2 5*sh.* 8*d.*

2. " 9*cwt.* of cheese, at £1 11*sh.* 5*d.* per *cwt.*? *Ans.* £14 2*sh.* 9*d.*

3. How much are 12*cwt.* of sugar, at £3 7*sh.* 4*d.* per *cwt.*?
Ans. £40 8*sh.*

4. Multiply 14*lb.* 9*oz.* 14*dwt.* 17*gr.* by 5.
Ans. 74*lb.* 0*oz.* 13*dwt.* 13*gr.*

5. " 2*da.* 6*hr.* 30*m.* by 19. *Ans.* 43*da.* 3*hr.* 30*m.*

RICHMOND, 19th May, 1846.

Thomas Leslie

Bought of *John Richards*,

$1\frac{3}{4}$ pounds of tea,	at	4*sh.* 6*d.*	£0 7*sh.* $10\frac{1}{2}$*d.*
4 bushels of corn,	at	5*sh.* 4*d.*	
5 quarts of brandy,	at	8*sh.* 4*d.* per gallon.	
6 do. rum,	at	7*sh.* 6*d.* per gallon.	
$7\frac{1}{4}$ yards of chintz,	at	5*sh.* 5*d.*	
			£6 13*sh.* $4\frac{3}{4}$*d.*

CHARLESTON, 2d January, 1847.

William Hunt

Bought of *James Nichols*,

2 pieces muslin,	at	30*sh.*	£3 0*sh.* 0*d.*
25 yards linen,	at	2*sh.*	
$28\frac{1}{2}$ yards calico,	at	2*sh.* 6*d.*	
$28\frac{1}{2}$ yards flannel,	at	2*sh.* 2*d.*	
1 piece bombazette,	at	56*sh.*	
2 pieces blue calico,	at	57*sh.* 6*d.*	
$50\frac{1}{2}$ yards dimity,	at	2*sh.* 6*d.*	
3 pieces drilling,	at	84*sh.*	
			£39 11*sh.* 2*d.*

LESSON LVI.

CASE II.

1. When one of the factors, though simple, is a large number, the reduction of the inferior products cannot be effected mentally. In that case, the method of the first case is not the most expeditious. The operation will be more readily performed by means of *aliquot parts*, commonly called PRACTICE.

MULTIPLICATION BY ALIQUOT PARTS, OR PRACTICE.

	OPERATION.		
2. Let it be proposed to multiply	£784	15*sh.*	9*d.*
by	857		
	£5488		
	3920		
	6272		
for 10*sh.*	428	10*sh.*	
for 5*sh.*	214	5*sh.*	
for 6*d.*	21	8*sh.*	6*d.*
for 3*d.*	10	14*sh.*	3*d.*
	£672,562	17*sh.*	9*d.*

Here, after having multiplied the whole numbers together, as usual, we pass on to the multiplication of the fractional part, 15 shillings by 857.

In order to this, we separate the number of shillings into the most convenient *aliquot parts* of a pound, which here are 10 and 5, and multiply separately 10 shillings and 5 shillings; the sum of the two products being evidently the same as that of 15 shillings.

Now, the product of 1 pound by the multiplier being £857, it is evident that the product of *half a pound* must be *one half of* £857. So that we have,

For 10*sh.* . . £428 10*sh.*

And because 5 shillings are one half of 10 shillings, the product of 5 shillings by 857, must be one half of that of 10 shillings. Hence we get at once,

For 5*sh.* . . £214 5*sh.*

Having now completed the multiplication of the shillings, we pass on to that of the 9 pence. These we also decompose into the most convenient aliquot parts, namely, 6 and 3.

6 pence is one half of a shilling; consequently, $\frac{1}{10}$ of 5 shillings. Hence, the product of 6 pence by 857, must be $\frac{1}{10}$ part of the product of 5 shillings. Thus we get

For 6*d*. . . £21 8*sh*. 6*d*.

Finally, the product of 3 pence is one half of the product of 6 pence by 857; which is,

For 3*d*. . . £10 14*sh*. 9*d*.

This method is expeditious and simple. Each individual product is made to depend, by a simple relation of aliquot numbers, upon some preceding product. As regards the choice of the most convenient aliquot parts, and the progress of dependence of the successive products, no positive rule can be given; practice alone will make it familiar.

3. As a second example, let us propose this

QUESTION.—*At* 65*lb*. 14*oz*. 13*dr*. *for* £1, *how much can be had for* £59?

OPERATION.

	65*lb*.	14*oz*.	13*dr*.	
	59			
	585*lb*.			
	325			
½ of 59 . .	29	8*oz*.	. .	for 8*oz*.
½ of last . .	14	12	. .	for 4.
½ . . .	7	6	. .	for 2.
¼ of last . .	1	13	8*dr*. . .	for 8*dr*.
½ . . .	0	14	12 . .	for 4.
¼ . . .	0	3	11 . .	for 1.
	3,889*lb*.	9*oz*.	15*dr*.	

After having multiplied the two integers together, we decompose 14*oz*. into 8, 4, and 2, and multiply them severally by the multiplier, as shown in the operation.

Then we pass on to the 13*dr*., which we decompose into 8, 4, and 1. And then, because 8*dr*. is ½*oz*., it is ¼ of 2*oz*.: we therefore take ¼ of the product of 2*oz*., and proceed with the other aliquot parts according to their relative values.

4. If the question had been:

The cost of 1*lb. being* £59, *what will be that of* 65*lb.* 14*oz.* 13*dr.?*

The multiplicand would no longer be the compound number; it would be £59, and the product must be in £ *sh.* and *d.*, but the operation would be nearly the same.

After having multiplied the two integers together as usual, we decompose the fractional parts in the same way as before, and get for 8*oz.* one half of £59, that is £29 10*sh.* Then, the other subdivisions depending on this, according to the same scale as in the preceding question, give the results set down in the operation.

OPERATION.

	£59				
	65*lb.* 14*oz.* 13*dr.*				
	£585				
	325				
for 8*oz.*	29	10*sh.*			
4	14	15			
2	7	7	6*d.*		16
8*dr.*	1	16	$10\frac{1}{2}$	.	8
4	0	18	$5\frac{1}{4}$	.	4
1	0	4	$7\frac{5}{16}$	.	5
	£3,889	12*sh.*	$5\frac{1}{16}$*d.*		$\frac{17}{16} = 1\frac{1}{16}$

5. In these operations it will be observed that the nature of the multiplier is not considered, but merely the abstract numerical relation of its parts.

6. Sometimes it may even be preferred to multiply in this way by a fraction.

OPERATION.

Let it be proposed to multiply . .	\$59
by . . .	$\frac{13}{16}$
We have for $\frac{8}{16}$	29.50
for 4	14.75
for 1	$3.56\frac{1}{4}$
	\$$47.81\frac{1}{4}$

Questions.—When one of the factors is a large simple number, how do you multiply? What are aliquot parts? How are the integers multiplied? How the smaller parts? What aliquot parts should be preferred? How do you form individual products from others? If the single number is the multiplicand, does the operation differ materially? Are the denominations considered in the multiplier? How are they considered? Can you multiply with fractions by aliquot parts?

EXERCISES.

1. How much are 53 loads of hay, at £3 15*sh.* 2*d.* per load? *Ans.* £199 3*sh.* 10*d.*

2. " 79 bushels of wheat, at 11*sh.* $5\frac{3}{4}$*d.* per bushel? *Ans.* £45 6*sh.* $10\frac{1}{4}$*d.*

3. " 94 casks of beer, at 12*sh.* 2*d.* per cask? *Ans.* £57 3*sh.* 8*d.*

4. " 109 barrels of fish, at 14*sh.* 6*d.* per barrel? *Ans.* £79 0*sh.* 6*d.*

5. " 345 barrels of pork, at £1 3*sh.* 9*d.*? *Ans.* £409 13*sh.* 9*d.*

6. " 47 casks of rice, each weighing 2*cwt.* 1*qr.* 23*lb.*? *Ans.* 115*cwt.* 1*qr.* 17*lb.*

7. In the lunar cycle of 19 years, of 365*da.* 5*hr.* 48′ 49″, how many days and parts of a day? *Ans.* 6,939*da.* 14*hr.* 27′ 31″.

8. How much water will be contained in 57 cisterns, each containing 455*gal.* 3*qt.* 1*pt.*? *Ans.* 25,984*gal.* 3*qt.* 1*pt.*

9. A father gives each of his 7 sons 150*A.* 3*R.* 12*P.*: how many acres in all? *Ans.* 1,055*A.* 3*R.* 4*P.*

10. What is the whole weight of 11 ingots of silver, each weighing 4*lb.* 1*oz.* 15*dwt.* 22*gr.*? *Ans.* 45*lb.* 7*oz.* 15*dwt.* 2*gr.*

11. At 1.22\frac{1}{2}$ an ounce, what will be the cost of a vase weighing 3*lb.* 5*oz.* 17*dwt.* 19*gr.*? *Ans.* 51.314\frac{71}{96}$.

12. At the rate of 12*dwt.* 7*gr.* for $1, of what weight could a vase be made for $148? *Ans.* 7*lb.* 6*oz.* 19*dwt.* 4*gr.*

LESSON LVII.

CASE III.

1. The last case is when both the multiplicand and multiplier are compound numbers.

Let us begin with the simple example of a fractional multiplier.

1st Question.—*A yard of a certain work has cost* £65 17*sh.* 11*d.*: *how much will* $39\frac{7}{8}$ *yards cost?*

OPERATION.

	£65	17*sh.*	11*d.*		
	$39\frac{7}{8}$				
	£585				
	195				
For 10*sh.* . .	19	10*sh.*			
5 . .	9	15			
2 . .	3	18			
For 6*d.* . .	0	19	6*d.*		
3 . .	0	9	9		
2 . .	0	6	6		8
For $\frac{4}{8}$*yd.* . .	32	18	$11\frac{1}{2}$	.	4
$\frac{2}{8}$ . .	16	9	$5\frac{3}{4}$	.	6
$\frac{1}{8}$ . .	8	4	$8\frac{7}{8}$	.	7
	£2,627	11*sh.*	$11\frac{1}{8}$*d.*		$\frac{17}{8} = 2\frac{1}{8}$

We multiply, first, by the integer, 39, as in the preceding case, which gives the first eight partial products.

There remains, then, to multiply by $\frac{7}{8}$: we decompose $\frac{7}{8}$ into $\frac{4}{8}$, $\frac{2}{8}$, and $\frac{1}{8}$, and observe that, since the product by 1, would be £65 17*sh.* 11*d.*, the product by $\frac{4}{8}$ must be one half of it; so that we get,

For . $\frac{4}{8}$ . . . £32 18*sh.* $11\frac{1}{2}$*d.*
Then, for $\frac{2}{8}$ one half of this, or £16 9*sh.* $5\frac{3}{4}$*d.*
And, finally,
For . $\frac{1}{8}$ one half again, or £8 4*sh.* $8\frac{7}{8}$*d.*

which is the last partial product. The sum of all the partial products is the answer.

In this example, we might have multiplied by 40, and subtracted the product by $\frac{1}{8}$.

2. 2d Question.—*The yard of a certain piece of work has cost £25 19sh. 5d.: what will 69yd. 2ft. 11in. cost?*

OPERATION.

	£25	19*sh.*	5*d.*	
	69*yd.*	2*ft.*	11*in.*	
	£225			
	150			
For 10*sh.* . .	34	10*sh.*		
5 . . .	17	5		
4 ($\frac{1}{5}$ of 69) .	13	16		
For 5*d.* ($\frac{1}{12}$ of product by 5*sh.*)	1	8	9*d.*	36
For 1*ft.* . . .	8	13	$1\frac{2}{3}$	. 24
1	8	13	$1\frac{2}{3}$	. 24
For 6*in.* . . .	4	6	$6\frac{5}{6}$	. 30
3	2	3	$3\frac{5}{12}$	. 15
2	1	8	$10\frac{5}{18}$	. 10
	£1,817	4*sh.*	$8\frac{31}{36}$	$\frac{103}{36} = 2\frac{31}{36}$

In this example, the yard is the unit, equal to the multiplicand.

We multiply, first, the cost of 1 yard by 69, as in Case II. We will only remark that, 5 pence being the twelfth part of 5 shillings, we take $\frac{1}{12}$ of this product.

Then we come to the fractional parts of a yard; and, since 1 foot is $\frac{1}{3}$ of a yard, we take $\frac{1}{3}$ of the value, £25 19*sh.* 5*d.*, of a yard, and get,

For 1*ft.* . . £8 13*sh.* $1\frac{2}{3}$*d.*,

which we repeat a second time.

Now, for 6*in.* we take $\frac{1}{2}$ of this, or £4 6*sh.* $6\frac{5}{6}$*d.*
Then, for 3, . $\frac{1}{2}$ again, or 2 3 $3\frac{5}{12}$.
And, finally, for 2, $\frac{1}{3}$ of the product by 6, or 1 8 $10\frac{5}{18}$.
And then add up the partial products.

3. If the multiplicand was the value of 1 foot instead of a yard, the product would be three times as much. The 69 yards must be first changed into feet.

Ans. £5,451 14*sh.* $2\frac{7}{12}$*d.*

4. If it was the value of 1 inch, both yards and feet should first be converted into inches; and the product would be 36 times larger. *Ans.* £65,420 10*sh.* 7*d.*

5. It might also be the cost of a rod. In this case, the product would be still different; the number of rods in 69 yards must be extracted, and then the product would be found, in a similar way, to be £330 8*sh.* $1\frac{11}{18}$.

6. These different suppositions, which I propose as exercises, show that, in compound numbers, we cannot propose simply to multiply two compound numbers, but must designate the unit of value.

7. As a last example, let us take a question, in which the unit of value is larger than the whole multiplier.

3d QUESTION.—*3cwt. 3qr. 27lb. cost* £1: *how much can be had for* 11*sh.* 7*d.*?

OPERATION.

For £1 . . .	*3cwt.*	*3qr.*	*27lb.*		
	£0	11*sh.*	7*d.*		
For 10*sh.* . .	1*cwt.*	3*qr.*	27*lb.*	8*oz.*	
1 . .	0	0	22	5	$9\frac{3}{5}$*dr.*
For 6*d.* . .	0	0	11	2	$12\frac{4}{5}$
1 . .	0	0	1	13	$12\frac{4}{5}$
	2	1	6	14	$3\frac{1}{5}$

Here, the value of £1 being the multiplicand, the value of 10 shillings is one half of it; and, from this first product, we obtain all the others, as the operation shows.

8. Operations so complicated are seldom met with; but they are good exercises, depending almost exclusively on the judgment of the operator, and in which he cannot proceed by blind rote. As such, they are well calculated to improve the readiness of the student in figures, and his quickness in the combination of different units.

PROOF.

9. The simplest way to prove compound multiplication is to double either multiplicand or multiplier, and proceed thus with the operation, which must give a double product.

The proof by division would generally be too complicated.

Questions.—How do you multiply two compound numbers together? What parts do you multiply first? Is it sufficient to give the multiplicand and multiplier? What else must be known? Give examples. How do you proceed, when the unit of value is an inferior order of the multiplier? When it is superior to the orders in the multiplier? Give examples. What is the simplest mode of proving compound multiplication? Could it be proved by division?

EXERCISES.

1. If one rod of a certain work costs 25£ 19*sh.* 5*d.*, how much will 69*yd.* 2*ft.* 11*in.* cost? *Ans.* £330 8*sh.* $1\frac{1}{18}$*d.*

2. If one foot costs £25 19*sh.* 5*d.*, how much will 69*yd.* 2*ft.* 11*in.* cost? *Ans.* £5,451 14*sh.* $2\frac{7}{12}$*d.*

3. If one inch costs £25 19*sh.* 5*d.*, how much will 69*yd.* 2*ft.* 11*in.* cost? *Ans.* £65,420 10*sh.* 7*d.*

4. If one yard costs £3 2*sh.* 6*d.*, what will 5*yd.* 2*ft.* 3*in.* cost? *Ans.* £17 19*sh.* $4\frac{1}{2}$*d.*

5. A star describes an arc of 2° 35′ in a year: what arc will it describe in 27*yr.* 9*mo.*? *Ans.* 71° 41′ 15″.

6. In a certain business, the capital has produced £15 11*sh.* 5*d.* for each pound invested: how much must a partner receive whose share is £31 17*sh.* 9*d.*? *Ans.* £496 10*sh.* $3\frac{47}{80}$*d.*

7. At 6*sh.* 8*d.* an ounce, what will a vase, weighing 2*lb.* 10*oz.* 14*dwt.* cost? *Ans.* £11 11*sh.* 4*d.*

8. What is the tax on £745 14*sh.* 8*d.*, at 3*sh.* 6*d.* in the pound? *Ans.* £130 10*sh.* 0*d.* $3\frac{1}{5}$ *far.*

9. How many hundred weight of raisins in $7\frac{3}{5}$ casks, each containing 2*cwt.* 0*qr.* 25*lb.*? *Ans.* 16*cwt.* 3*qr.* 16*lb.* $6\frac{2}{5}$*oz.*

10. How many pounds of coffee in $13\frac{3}{4}$ bags, each containing 1*cwt.* 3*qr.* 15*lb.*? *Ans.* 25*cwt.* 3*qr.* 17*lb.* 4*oz.*

LESSON LVIII.

MULTIPLICATION OF MEASURES OF LENGTH, AND DUODECIMALS.

1. For some practical purposes, as, for instance, to measure work, the foot is divided into 12 *inches* or *primes*, marked ′; the inch or prime is subdivided into 12 *seconds*, marked ″; the seconds into 12 *thirds*, marked ‴, and so on. These are called *duodecimals*, because it is a regular scale, in which the ratio between the successive orders of units is 12 instead of 10.

2. The addition and subtraction of duodecimals are like those of other compound numbers. See LI. and LII.

3. As regards multiplication, let it be proposed to multiply . . .

The operation, performed by the method of aliquot parts, would be as shown here.

The answer being .

OPERATION.

	24*ft.*	7′	6″	
by	2*ft.*	5′	4″	
	48			
For 6*in.* .	1			
For 1 .	0	2*in.*		
For 6*sec.* .	0	1		
For 4*in.*	8	2	6*sec.*	
For 1 .	2	0	7	6‴
For 4″ .	0	8	2	6
	60*ft.*	2′	4″	0‴

4. But of what nature is it? Are the 60 feet linear, square, or cubic? It is what nothing in the statement enables us to determine.

The question might be, for instance,

1st. The trimming of a dress is 24*ft.* 7′ 6″ in length, and each foot requires 2*ft.* 5′ 4″ of ribbon: how much ribbon is wanted for the whole?

Ans. 60*ft.* 2′ 4″, *linear measure.*

2d. A foot-bridge is 24*ft.* 7′ 6″ long, and 2*ft.* 5′ 4″ wide: how many square feet of plank are required for its floor? *Ans.* 60*ft.* 2′ 4″, *square measure.*

3d. A surface of ground, measuring 24*ft.* 7′ 6″, square measure, is to be excavated 2*ft.* 5′ 4″ deep: how many cubic feet of earth will be dug out?

Ans. 60*ft.* 2′ 4″, *cubic measure.*

5. These examples show that it is not correct to affirm that feet multiplied by feet, or inches by inches, give always square feet or square inches in the product. And again, I repeat, let the intelligent pupil to whom such

a multiplication is proposed, ask for a more precise statement.

Whatever may be the nature of the answer, the operation may be performed readily by aliquot parts, as above; and there does not appear to be any necessity for any additional rule.

Arithmeticians, however, have thought proper to introduce another method, which is sometimes used by mechanics to measure the square or solid feet of their work. This is the

MULTIPLICATION OF DUODECIMALS.

6. Taking the same example as above, the operation is performed as follows:

OPERATION.

			24*ft.*	7′	6″
			2*ft.*	5′	4″
Product by 4″ . .		8′	2″	6‴	0⁗
" by 5′ .	10*ft.*	3	1	6	
" by 2*ft.* .	49	3	0		
	60*ft.*	2′	4″	0‴	0⁗

If we consider that

$1' = \frac{1}{12}$ of a foot.

$1'' = \frac{1}{12}' = \frac{1}{12} \times \frac{1}{12}$ of a foot.

$1''' = \frac{1}{12}'' = \frac{1}{12}$ of $\frac{1}{12}' = \frac{1}{12} \times \frac{1}{12} \times \frac{1}{12}$ of a foot.

$1'''' = \frac{1}{12}''' = \frac{1}{12}$ of $\frac{1}{12}'' = \frac{1}{12}$ of $\frac{1}{12}$ of $\frac{1}{12}' = \frac{1}{12} \times \frac{1}{12} \times \frac{1}{12} \times \frac{1}{12}$ of a foot.

It will be seen that each additional index (′) signifies a division by 12, and that the product, $1' \times 1'$ will be

$$\frac{1}{12} \times \frac{1}{12} = 1''.$$

$$1' \times 1'' = \frac{1}{12} \times \frac{1}{12 \times 12} = 1'''.$$

$$1'' \times 1'' = \frac{1}{12 \times 12} \times \frac{1}{12 \times 12} = 1''''.$$

&c. &c.

7. That is, *the product of two units will give a unit of an order designated by the sum of the indices of the two factors.*

Consequently, *there will be in the product as many duodecimals as there are in both factors together.*

This is a rule similar to that observed in decimal fractions (XXV., 3), as might have been anticipated.

Therefore,

I. In the multiplication of duodecimals, *multiply successively all the orders of the multiplicand by the orders of the multiplier, beginning with the lowest.*

II. *The index of each product will be the sum of the indices of the factors.*

III. *Set each product in the column of its order; and, if it exceed* 12, *carry* 1 *to the next superior order for every* 12 *it contains.*

IV. *Continue to reduce and carry until you reach the highest denomination.*

V. *The sum of the several amounts will be the answer.*

This method is certainly a good exercise; but, if the two are well understood, that by aliquot parts will be found fully as easy in practice and less liable to error.

8. If there was another factor, 5*ft.* 11′, for example, the duodecimal multiplication would be

60*ft.*	2′	4″	
5*ft.*	11′		
55*ft.*	2′	1″	8‴
300	11	8	
356*ft.*	1′	9″	8‴

But it does not follow that the product would be solid measure; it might be *linear, square,* or *cubic* measure, according to the character of the question, of which I need not give any example, after the preceding explanations.

9. In the measurement of work, it is usual to limit the subdivisions to inches. In that case, I would advise the use of fractions, and more especially of decimal fractions, into which any number of inches can readily be transformed, near enough for practical purposes.

Let it be proposed, for instance, to measure a floor whose sides are 18*ft.* 6*in.* and 14*ft.* 3′.

	OPERATION.
You can change these readily into	18.50
and	14.25
	9250
	3700
	7400
	1850
The product of which is .	263.6250*sq. ft.*

The advantage of this transformation will be still greater, if the product has to be multiplied by a third factor; and when you have, besides, to multiply the measurement by the price. If the price were $4\frac{1}{2}$ cents a square foot, you would find it easier to multiply 263.625 by 0.045, than 263*ft.* 7′ 6″ by $4\frac{1}{2}$ cents.

Questions.—In measuring work, how is the foot divided? How many seconds in a prime; thirds in a second? &c. What are duodecimals? How would you add and subtract them? Do feet by feet always give square feet? When do they not? What may they give? Give examples. Would the method by aliquot parts answer to multiply duodecimals? What is the method used generally? Which do you prefer? What is the rule of indices in the multiplication of duodecimals? How many duodecimals will there be in the product? Give the rule of multiplication of duodecimals. Could decimals be used in the calculations of measurements of work? How? Would it be as convenient? As liable to error as duodecimals?

EXERCISES.

1. Multiply 9*ft.* 6′ by 4*ft.* 9′. *Ans.* 45*ft.* 1′ 6″.

2. What is the price of a marble slab, 5*ft.* 7′ by 1*ft.* 10′, at one dollar per square foot? *Ans.* $10.22.

3. There is a house with three tiers of windows, three in a tier: the height of the first tier is 7*ft.* 10′; of the second, 6*ft.* 8′;

of the third, *5ft.* 4′; and the breadth of each *3ft.* 11′: what will the glazing come to, at 14 pence per foot?

Ans. £13 11*sh.* 10½*d.*

4. A bale measures *7ft.* 6′ by *3ft.* 3′, and *1ft.* 10′: what are its solid contents? *Ans.* 44*ft.* 8′ 3″.

5. A merchant imports 6 bales, of the following dimensions:

	Length.		Height.		Depth.	
No. 1.	*2ft.*	10′	*2ft.*	4′	*1ft.*	9′
2.	2	10	2	6	1	3
3.	3	6	2	2	1	8
4.	2	10	2	8	1	9
5.	2	10	2	6	1	9
6.	2	11	2	8	1	8

What is the whole solid content and the freight, at 20 cents per foot? *Ans. Content*, 71.64*ft.*
Freight, $14.328.

6. How many solid feet of timber in a piece of timber 24*ft.* 6′ long, 1*ft.* 2′ 6″ wide, and 10′ 10″ thick?

Ans. 26*C. ft.* 8′ 8″ 6‴ 6⁗.

Decimally, $24.5 \times 1.20\dot{8}\dot{1} \times .903 = 26.72 +$

7. 2*ft.* 4*in.* of one kind of timber are exchanged for each running foot of another kind: how many running feet must be given for 48*ft.* 9*in.*? *Ans.* 113*ft.* 9′ *linear feet;*
or, 113.75.

8. On each square foot 3*ft.* 5*in.* of strips of wood are used for ornament: how many feet of strips will be wanted for 105*sq. ft.* 6′?

Ans. 360*ft.* 5′ 6″ *linear feet;*
or, 360.46, or $360\frac{11}{24}$

N. B.—Here is an example of square feet by linear feet, which give linear feet in the product.

9. What will be the plastering, at 20 cents a yard, of a ceiling 7*yd.* 8*in.* by 4*yd.* 2*ft.* 10*in.* *Ans.* $7.14.

10. What will be the cost of paving a surface 19*yd.* 1*ft.* 6*in.* by 18*yd.* 9*in.*, at 4¾ pence per yard? *Ans.* £7 0*sh.* 10*d.*

11. How much wood in a load 7*ft.* 6′ long, 3*ft.* 3′ wide, 1*ft.* 10 high? *Ans.* 44*ft.* 8′ 3″.

12. For one solid foot of timber 3*sq. ft.* 26*sq. in.* of plank are given: how many running feet of plank, one foot wide, must be given for 275 solid feet 1,448 cubic inches?

Here is a multiplication of solid by square feet, giving running feet in the answer. What answer would be given if, without further explanation, solid by square feet were to be multiplied? Here the answer may be obtained by decimal fractions, by vulgar fractions, or by duodecimals.

LESSON LIX.

COMPOUND DIVISION.

1. Let us first recollect that the *dividend* being the product of the *divisor* and *quotient*, one of these must be of the nature of the dividend, and that the other may be anything else (XVIII., 2).

2. In fact, division may also, like multiplication, be considered as a species of rule of three, in which one of the terms is *one unit* of a known nature; so that we have either

The divisor : the dividend : : 1 : the quotient.
Or, The divisor : 1 : : the dividend : the quotient.

In the first case, the divisor and dividend being of the same nature, the quotient is of the nature of the known unit. For example:

Cloth has been bought for 60 dollars, at 12 dollars a yard: how many yards have been bought? Here we have \$12 : \$60 : : $1yd.$: the answer $= 1yd. \times \frac{\$60}{\$12} = 5yd.$

The answer being one yard, taken as many times as the ratio of \$60 to \$12; that is, repeated as many times as the abstract quotient, $\frac{60}{12}$; it follows that the quotient assumes the character of the given unit, which is one *yard*, though the numbers divided are *dollars*.

3. In the second case, the divisor and known unit are of the same kind; and, consequently, the quotient compares with the dividend, and is of the same nature. Example:

12 yards of cloth have been bought for 60 dollars: how much is it a yard?

$$12yd. : 1yd. : : \$60 : \$60 \times \frac{1yd.}{12yd.}$$

Which shows that the answer is \$60, multiplied by the *abstract quotient of* 1 *to* 12; or, in other words, \$60 divided by the abstract number, 12; and the quotient is of the nature of the dividend.

4. It will appear, from what has just been explained, that before proceeding to divide, the nature of the quotient must be inquired into from the data of the question. We will, therefore, distinguish two cases.

CASE I.

5. When the dividend and divisor are of the same nature. Then the quotient is of the nature of the single unit, referred to in the question as equivalent to the divisor.

In this case,

I. *Change both numbers into units of the lowest denomination they contain.*

II. *Divide the two results, taking care to convert successively the remainders into smaller denominations of the kind indicated by the nature of the question.*

FIRST EXAMPLE.

One yard of a certain work is worth £23 19*sh.* 8½*d.*: how much can be executed for £2,728 17*sh.* 10*d.*?

It is clear that the required number of yards, multiplied by £23 19*sh.* 8½*d.*, must produce £2,728 17*sh.* 10*d.* Therefore, the aggregate must be divided by the part, and the operation is a DIVISION.

OPERATION.

£2728 17*sh.* 10*d.*	£23 19*sh.* 8½*d.*
20	20
54577	479
12	12
654934	5756
2	2
1309868	11513

1309868 | 11513
15856
43438 | 113*yd.* 2*ft.* 3*in.* 9′ $\frac{10539}{11513}$.
8899
3

26697 feet.
3671
12

44052 inches.
9513
12

114156 lines or seconds.
10539

The two numbers, being reduced to their lowest denomination, become the fractions

$\frac{1309868}{480}$ of a pound, and $\frac{11513}{480}$ also of a pound;

and we shall have as many yards for the answer as the former fraction contains the latter; the original denomination of the numbers having nothing to do with the character of the quotient, which must be in yards, though pounds are apparently divided.

The operation is consequently reduced to a simple division of abstract fractions; and, because they have the same denominator, we have only to divide their numerators (XLIV., 7), since it amounts to increasing both the dividend and divisor 480 times, which does not change the quotient. Therefore, the result is,

$$1yd. \times \frac{1309868}{11513}; \text{ or, } \frac{1309868}{11513}yd.:$$

which we tranform into integers, according to the method of Lesson LI.

6. It is clear that, in this division, *the nature of both the dividend and divisor* disappears, and only *their abstract numerical ratio* remains, by which the *given unit* being multiplied, *a new denomination is substituted.*

I introduce these remarks here because I have seen pupils frequently express astonishment that a remainder of dollars or pounds (as they thought) should be converted into feet, ounces, or other denominations than money.

SECOND EXAMPLE.

One pound sterling has been given for 15*lb.* 4*oz.*: how much will 329*lb.* 5*oz.* 14*dr.* cost?

OPERATION.

lb.	*oz.*	*dr.*	*lb.*	*oz.*		
329	5	14	15	4	84318	3904
16			16		·6238	
1979			244		2334	£21 11*sh.* 11*d.* $\frac{59}{122}$.
329			16		20	
5269			3904		46680	shillings.
16					·7640	
31628					3736	
5269					12	
84318					44832	pence.
					5792	
					1888	

By changing here both numbers into drams, we reduce the operation to the division of

$\frac{84318}{256}$ of a pound, by $\frac{3904}{256}$ of a pound;

or, as in the preceding case, to the abstract ratio,

$\frac{84318}{3904}$ multiplied by £1, which is at last £$\frac{84318}{3904}$.

7. Evidently, in this case, if one of the numbers was not a compound number, it must nevertheless be reduced to the same lowest denomination as the other, since the ratio of the dividend and divisor must be between units of the same numerical dimension, in order that both may be made whole numbers, by striking out their equal denominators.

For example:

If one hundred weight of sugar costs £12, how much should be given for £55 5*sh.*?

The dividend and divisor are both changed into shillings, and then the result is in hundred weight, quarters, &c.; the value of the fraction, $\frac{1105}{240} \times 1cwt. =$ 4*cwt.* 2*qr.* 11$\frac{2}{3}$*lb.*

Questions.—In division, of what nature may the quotient be? When is it of the nature of the dividend? What else may it be? When? Might a division be stated as a proportion? How? How will a proportion show the nature of the quotient? What inquiry should precede the operation of division? How many cases are there in compound division? What is the first case? What is the nature of the quotient in this case? What operation should precede the division? Give the rule for the operation. How do you get smaller integers in the quotient? Explain how the denomination is changed from one measure to another. If one of the numbers was not compound, should it also be reduced? Why?

EXERCISES.

1. At £3 15*sh.* 6*d.* per hundred weight, how much sugar can be bought for £654 9*sh.* 9$\frac{3}{4}$*d.*? *Ans.* 173*cwt.* 1*qr.* 14*lb.*

2. At $14.40 per hundred weight, how much can be bought for $544.50? *Ans.* 37*cwt.* 3*qr.* 7*lb.*

3. At $18.50 per acre, how much land can be purchased for $4,252.455? *Ans.* 229*A.* 3*R.* 18*P.*

4. How many feet of plank, at 38*sh.* 6*d.*, per thousand, can be bought for £16 17*sh.* 6*d.*? *Ans.* 8,767*ft.*

5. At \$2.25 per rod, what length of road can be made for \$9,676? *Ans.* 13*ml.* 140*R.* 2*yd.* $1\frac{1}{2}$*ft.*

LESSON LX.

CASE II.

1. When the dividend and divisor are of different natures, the quotient is of the nature of the dividend.

In questions like the following,

35 yards of stuff have cost £57 3*sh.* 7*d.*: what is the price of one yard?

2. *The divisor being a whole number* of the denomination of the single unit included in the question, which in this instance is the yard, it must be considered as an abstract number (LIX., 3), since the individual price of *one yard* must be $\frac{1}{35}$ of the cost of 35 *yards*, a ratio with which the precise nature of the articles compared has nothing to do.

The division is here performed in a manner similar to simple division.

You divide, first, the units of the highest order; the quotient is £1, with a remainder, £22.

You change this remainder into units of the next lower denomination, taking care to add the units of this new denomination, contained in the original dividend; and thus get 443 shillings.

OPERATION.

```
£57 3sh. 4d. | 35
 22          | £1 12sh. 8d.
 20
 ---
440 shillings.
  3
 ---
443
 93
 23
 12
 ---
276 pence.
  4
 ---
280
```

This new dividend of inferior units gives you, in the quotient, the smaller integers, 12 shillings, for the second part of the quotient, with a remainder, 23 shillings.

In the same way, the second remainder is changed into the next lower order of units, to which is added the number of similar units of the original dividend; making, here, 280 pence, which, divided by 35, give, for the last integers of the quotient, 8 pence.

3. *When the divisor also is a compound number*, the operation must be brought to the preceding case by a transformation of the divisor, as follows:

I. *Reduce the divisor to its lowest denomination.*

II. *Then multiply each term of the dividend by the ratio of the lowest denomination of the divisor, to its unit of comparison.*

III. *Divide this altered dividend by the new divisor, considered as an abstract number.*

EXAMPLE.

9*cwt.* 1*qr.* 5*lb.* have cost £3,257 9*sh.* 3*d.*: how much is it per hundred weight? The statement of the operation would be

$$\pounds 3{,}257\ 9sh.\ 3d. \times \frac{1cwt.}{9cwt.\ 1qr.\ 5lb.}\ \text{(LIX., 3)}.$$

In order to establish the ratio by which we multiply, we change both its terms into pounds; that is, one hundred weight is 112 pounds, and the divisor 1,041 pounds; so that the statement is changed into

$$\pounds 3{,}257\ 9sh.\ 3d. \times \tfrac{112}{1041}.$$

And all we have to do is to multiply each part of the dividend by 112, and then divide by 1,041. Thus:

OPERATION.

```
 £3,257     9sh.    3d.
    112     112     112
 ---------  -----   ----
£364784  1008sh.  336d. | 1041
   5248                 |-------------
   •434                 | £350 9sh. 4d.
     20
 --------
   9688 shillings.
    319
     12
 --------
   4164 pence.
   ....
```

4. This manner of proceeding, it will be perceived, amounts to reducing the divisor to its lower denomination, 1041*lb.* = $\frac{1041}{112}$ of one hundred weight.

And then making it a whole number, as usual in division by fractions, by striking out its denominator; which operation rendering it 112 times larger, we must also multiply the dividend by 112, that the quotient may not be altered.

5. It will be remarked that, in multiplying by 112, I do not reduce the lower products and carry to the higher denominations. This, though generally done, is lost labor; since the division of the successive products will generally give small quotients, as in this example.

But, even if, in some rare case, the number of units of some of the lower denominations should exceed the value of an integer of the next higher order, the reduction would always be readily made afterwards.

If, for instance, the dividend were £146 16*sh.* 3*d.*, and the divisor 1*cwt.* 1*qr.* 5*lb.*, the quotient, by our method, would be

£112 27*sh.* 12*d.*;

which would, without difficulty or hesitation, be changed at once into

£113 8*sh.*

6. N. B.—In such questions, be careful to observe which of the units of the divisor is the unit of comparison, whose value is required; because it is in regard to this unit that the divisor is to be transformed.

For instance, if, in the first question, it was a foot of stuff, the value of which were required, the 35 yards must be changed into 105 feet.

If, in the second, it was the ton whose value were required, the denominator of the transformed divisor should be 2,240, instead of 112.

If it were the *quarter*, then the denominator should be 28.

If the *pound*, the reduction to the lowest denomination would be the only preparatory operation necessary.

Questions.—What is the second case in compound division? How do you divide, if the divisor is a whole number? How, when it is a compound nnmber? By what number should the terms of the dividend be multiplied? Why? Explain it by the arithmetical statement. Explain it by making the divisor a whole number. Is it necessary or proper to reduce the individual products of the dividend? What attention should be had in regard to the unit of comparison? Give examples.

EXERCISES.

1. Divide £227 10*sh.* 4*d.* by 11. *Ans.* £20 13*sh.* 8*d.*

2. " £31 2*sh.* 10½*d.* by 99. *Ans.* £ *sh.* *d.*

3. " £315 3*sh.* 10¼*d.* by 135. *Ans.* £ *sh.* *d.*

4. " 23*lb.* 7*oz.* 6*dwt.* 12*gr.* by 7. *Ans.* *lb.* *oz.* *dwt.* *gr.*

5. " 1,061*cwt.* 2*qr.* by 28. *Ans.* *cwt.* *qr.* *lb.*

6. " 375*ml.* 2*fur.* 7*P.* 2*yd.* 1*ft.* 2*in.* by 39. *Ans.* 9*ml.* 4*fur.* 39*P.* 2*ft.* 8⅔*in.*

7. If 72 bushels cost £20 9*sh.* 6*d.*, what is it per bushel? *Ans.* *sh.* *d.*

8. If 1 hundred weight cost 23*sh.* 4*d.*, what is it per pound? *Ans.* 2½*d.*

9. When 2 hundred weight of sugar cost £8 17*sh.* 4*d.*, what is it per pound? *Ans.* 9½*d.*

10. If an ingot of gold, weighing 9*lb.* 9*oz.* 12*dwt.*, be worth £411 12*sh.*, what is that per grain? *Ans.* 1¾*d.*

11. A person breaking, owes £1,490 5*sh.* 10*d.*, and has £784 17*sh.* 4*d.*: what will each creditor get on the pound? *Ans.* 10*sh.* 6¼*d.* $+\frac{20993}{35767}$.

12. If 42*cwt.* 1*qr.* 14*lb.* cost $63.56¼, what is it per hundred weight? *Ans.* $1.50.

13. If 16*cwt.* 2*qr.* 17*lb.* of sugar were sold for £46 11*sh.* 1*d.*, how much is that per hundred weight? *Ans.* £2 15*sh.* 2*d.*

14. If $17,113.637 have been paid for 578*A.* 3*R.*, how much is it per acre? *Ans.* $29.57.

15. If $69\frac{7}{12}$ yards have cost £2,728 17*sh.* 9*d.*, how much is it a yard? *Ans.* £39 4*sh.* $4\frac{176}{835}$*d.*

16. If $\frac{5}{7}$ of an ounce cost £$\frac{11}{12}$, what will 1 ounce cost? *Ans.* £1 5*sh.* 8*d.*

17. A printer uses 1 sheet of paper for every 16 pages of an 8vo. book: how much paper will be necessary to print 500 copies of a book containing 336 pages, allowing 2 quires of waste paper in each ream? RIGHT ANSWER. 24 *reams* 6 *quires* 2⅔ *sheets.*

CHAPTER VIII.

CONTAINING PRACTICAL QUESTIONS WHICH DEPEND ON PROPORTIONS.

LESSON LXI.

1. We have explained, in Lesson XLVI., how questions depending on simple proportion should be stated, and how the *rule of three* is reduced to plain multiplication and division. It may have been remarked that the answer is always of the denomination of one of the given numbers, and is obtained by changing this number, according to the ratio of two other numbers.

2. It frequently happens, however, that practical questions are more complicated, and that the answer depends on the known number of the same denomination, by more than one condition, and that the combination of several ratios is necessary. Such questions we will now proceed to examine. They may all be classed under the head of

COMPOUND PROPORTION, OR DOUBLE RULE OF THREE.

3. The following is a simple example of questions of this sort:

Example 1st.—20 men have built 50 feet of wall in 18 days: how many men will build 120 feet of the same kind of work in 12 days?

All such questions may be performed in two different ways; either by the method of *ratios*, or by the method of *units*.

METHOD OF RATIOS.

I. ***Write down the number of the same denomination as the answer.***

II. ***Multiply it successively by the direct or inverse*** *ratio, as the case may be, of each pair of given numbers of the same kind; the final product will be the answer.*

Observe that both terms of a ratio, though of the same kind, may not be of the same denomination. When this is the case, they should be reduced to one and the same denomination.

In establishing a ratio, consider whether the answer should be increased or diminished by it. In the first case, place the larger number in the numerator; in the second, place it in the denominator.

In the question before us, for instance; since a number of men is required, we have to determine how the given number of men, 20, is to be modified by the conditions of the question, so as to be changed into the answer.

Beginning with the modification produced by the number of days, we remark that it would take a greater number of men to execute the *same work* in a less number of days. Hence, *the ratio is inverse*, and we multiply 20 by $\frac{18}{12}$.

But now, since *the work is different* and greater in quantity, the number of men must be increased in the *direct ratio* of the quantity of work; that is, in the ratio of $\frac{120}{50}$. Hence, the final answer is $20 \times \frac{18}{12} \times \frac{120}{50} =$ 72 men.

METHOD OF UNITS.

4. Find the answer for *one unit of each denomination, and then that for a number of each kind will easily be obtained, either by multiplication or division, as the nature of the question may require.*

In the above example, we must first find how many men it would take to build *one foot* of wall in *one day.*

Evidently, if it take 20 men to build a wall in 18 days, it will take 18 times as many to build the same in one day; that is,

$$20 \times 18.$$

This being the number necessary to build 50 feet in one day, it will require only the fiftieth part of it to build one foot; that is,

$$\frac{20 \times 18}{50},$$

which is the number of men requisite to build *one foot* in *one day.*

And now, passing on to the second series of numbers, it is clear that the number of men necessary to build one foot of wall in 12 days, instead of one day, will be $\frac{1}{12}$ of this; that is,

$$\frac{20 \times 18}{50 \times 12},$$

and that it will take 120 times this new number to build 120 feet, instead of one foot; or, finally,

$$\frac{20 \times 18 \times 120}{50 \times 12} = 72 \text{ men, as above.}$$

5. The same might be obtained by a series of proportions multiplied into each other; but the preceding methods are preferable, and proportions are no longer used in such questions.

The following more complicated example would be as readily calculated:

Example 2d.—If 9 laborers, working 8 hours a day, have spent 24 days in digging a ditch 54 yards long, 14 wide, and 5 feet deep, how many days will it take 72 laborers, working 11 hours a day, to dig a ditch 275 yards long, 18 broad, and 7 feet deep?

FIRST METHOD.

Here, a number of days being required, we multiply the known number of days, 24, as follows:

	THE RATIOS of laborers.	of hours.	of lengths.	of breadths.	of depths.	
24 days	by $\frac{9}{72}$	by $\frac{8}{11}$	by $\frac{275}{54}$	by $\frac{18}{14}$	by $\frac{7}{5}$	$= 20$ days.

The answer is readily obtained by cancelling 9 and 8, with 72; 11 and 5 out of 275; 18 out 54; 7 out of 14; and, finally, 3 and 2 out of 24.

SECOND METHOD.

One laborer, working *one hour* a day, would dig a ditch *one yard* long, *one yard* wide, and *one foot* deep, in

$$\frac{24 \times 9 \times 8}{54 \times 14 \times 5} \text{ days;}$$

and, consequently, 72 laborers, working 11 hours a day, would dig a ditch 275 yards long, 18 broad, and 7 feet deep, in

$$\frac{24 \times 9 \times 8}{54 \times 14 \times 5} \times \frac{275 \times 18 \times 7}{72 \times 11} = 20 \text{ days, as above.}$$

Remark that, when you have obtained the amount for all the units of the data, the corresponding numbers of each kind are evidently on the opposite side of the line of division, and may be written at once. Thus, 9 laborers of the first series being in the numerator, the corresponding number, 72 laborers, is in the denominator of the second series: 8 hours of the first is in the numerator; and consequently 11 of the second in the denominator. 54 yards of the first is in the denominator; and, therefore, 275 yards of the second must be in the numerator, &c.

Questions.—What is compound proportion? By how many methods can questions in compound proportion be solved? Explain the method by ratios. What is done when two numbers of the same nature are not of the same denomination? How do you determine which of the numbers of a ratio is to multiply, and which to divide? Explain the methods by units. In this method, after having found the answer for *one* of each denomination, how should each one of the corresponding numbers of the second series be placed?

EXERCISES.

1. 15 men, working 10 hours a day, have taken 18 days to construct 450 yards of a certain work: how many men, working 12 hours a day, would make 480 yards in 8 days? *Ans.* 30 *men.*

2. 1,200 yards of cloth, $\frac{5}{4}$ of a yard wide, will clothe 500 men: how many yards, $\frac{7}{8}$ wide, are necessary for 960 men?
Ans. 3,291$\frac{3}{7}$ *yards.*

Here multiply 1,200 by the ratio of the widths and by that of the men.

3. A man, walking 15 hours a day, has travelled 375 miles in 20 days: how many hours a day must he travel to go 400 miles in 18 days? *Ans.* 17$\frac{7}{9}$ *hours.*

4. If a family of 9 persons spend 450 dollars in 5 months, how much would be sufficient to maintain them 8 months, if 5 more were added to the family? *Ans.* \$1,120.

Either multiply \$450 by the ratios, or find first what will support one person one month, and, from that, what will support 14 persons 8 months.

5. If 100 dollars gain 5 dollars in 12 months, how much will 750 dollars gain in 7 months? *Ans.* \$21.875.

6. If 120 bushels of corn can serve 14 horses 56 days, how many days will 94 bushels serve 6 horses? *Ans.* $102\frac{16}{45}$ *days.*

7. If 7*oz.* 5*dr.* of bread can be bought at $4\frac{3}{4}$ pence, when wheat is at 4*sh.* 2*d.* per bushel, what weight may be bought for 1*sh.* 2*d.*, when the price of wheat is 5*sh.* 6*d.* per bushel?
Ans. 1*lb.* 5*dr.* $\frac{51}{209}$*dwt.*

8. If the transportation of 13*cwt.* 1*qr.*, a distance of 72 miles, cost £2 10*sh.* 6*d.*, what will be the transportation of 7*cwt.* 3*qr.* on 112 miles? *Ans.* £2 5*sh.* 11*d.* $1\frac{77}{159}$ *far.*

LESSON LXII.

N. B.—Those who intend learning Algebra, had better begin it now.

FELLOWSHIP OR PARTNERSHIP.

1. The preceding lesson completes all that is strictly necessary to solve, with proper care and reflection, all questions depending on proportion.

The following rules, however, being of great practical utility, will be noticed separately, and at some length, more on account of their frequent applications in business transactions than of any real difficulty they present.

2. Fellowship is a rule by which a given amount is divided into a number of parts, in a certain proportion.

By this rule, which is an application of compound proportion, are adjusted the gains, losses, and charges of partners in company; the effects of bankrupts; the shares of prizes, &c.

3. Fellowship may be single or double. It is *single* when the *shares* or *dividends* depend only on one condition, and are consequently proportioned to one number only.

It is *double* when each share depends on several conditions; it is then a case of compound proportion.

SINGLE FELLOWSHIP.

4. *Example* 1*st*.—Three merchants have formed a copartnership: *A* put in \$15,000; *B*, \$22,540; *C*, \$25,600. At the end of a year their joint profit amounts to \$12,000: what is the share of each?

It is evidently conformable to justice to divide the proceeds in proportion to the sum adventured by each. Upon this principle, the process of distribution is very simple.

Find first what is the profit made on each dollar, and multiply it by the stock in trade of each person.

Now, it is evident that the profit on one dollar is equal to the whole profit divided by the aggregate capital; that is, to

$$\$\tfrac{12000}{63140}.$$

Consequently, the statement of the successive shares will be,

1st share, . $\frac{12000 \times 15000}{63140} = \$2,850.81.$

2d share, . $\frac{12000 \times 22540}{63140} = \$4,283.81.$

3d share, . $\frac{12000 \times 25600}{63140} = \$4,865.38.$

Being in all, *to prove the work*, . \$12,000.00.

In business, the money advanced is called the *capital*, or *stock*, and the sum to be distributed among the partners, the *dividend*.

Example 2*d*.—A ship, worth \$4,800, is lost; of which $\frac{1}{8}$ belongs to *A*, $\frac{1}{3}$ to *B*, and the balance to *C*: what is the loss of each?

We must first find the proportional share of *C*, which is $\frac{13}{24}$.

Then we have to divide the amount in the proportion of the three shares, $\frac{3}{24}$, $\frac{8}{24}$, and $\frac{13}{24}$; that is, of the numbers . . 3, 8, 13.

Now, it is evident that each unit loses a part of the whole amount, equal to \$4,800, divided by the sum, 24, of the three numbers; that is, to . . . \$200;
and, consequently, the loss for 3 units is . \$600
for 8 " . \$1,600
for 13 " . \$2,600

Verification, ... \$4,800

DOUBLE FELLOWSHIP.

5. *Example* 1*st.*—*A* and *B* trading together, *A* has in the business \$500 for 4 months, and *B* \$600 for 5 months; the profits are \$240: how are they to be divided?

It is clear that each dollar has produced a certain profit in each month, which, multiplied by the number of months, will give its whole profit while in the business.

Hence, it is just that the division should be in proportion, not only of each capital, but likewise of the time; that is, of the compound ratio of

$$500 \times 4 \text{ to } 600 \times 5, \text{ or of } 2{,}000 \text{ to } 3{,}000.$$

6. In fact, it is easy to understand that \$2,000 in one month will yield as much profit as 500 in 4 months, and 3,000 as much as 600 in 5 months.

By thus *multiplying the various proportional conditions, the compound ratio is reduced to a simple one,* and the rest of the *operation is as in single fellowship.*

In this case, the ratio being as 2 to 3, the result is,

For the first, . . \$96.
For the second, . . \$144.

Example 2*d.*—Three persons hold a pasture in common, for which they are to pay £30 per annum. *A* puts in 7 oxen for 3 months; *B*, 9 oxen for 5 months; *C*, 4 oxen for 12 months: what part of the rent must each person pay?

Each ox, of course, is supposed to consume an equal quantity per month; and it is evident that

7 oxen consume as much in 3 months as 7×3 in one.
9 " " as much in 5 months as 9×5 in one.
and 4 " " as much in 12 months as 12×4 in one.

The question is thus reduced to single fellowship; and the division made in the proportion of the numbers 21, 45, 48, gives,

.	for *A*, .	£5	10*sh.*	$6\frac{6}{19}d.$
	for *B*, .	11	6	$10\frac{2}{19}$
	for *C*, .	12	12	$7\frac{11}{19}$

7. Questions like the following, though differing in some particular, may be ranked with fellowship:

A man, with the force he employs, may complete a certain work in $2\frac{1}{2}$ months; another could do it in $3\frac{3}{4}$ months: if they form a partnership, when can they finish the work?

To solve this question, let us find what part of the work each will accomplish in one month.

The first will make, in one month, $\frac{1}{2\frac{1}{2}} = \frac{2}{5}$ of the work.

The second, " " $\frac{1}{3\frac{3}{4}} = \frac{4}{15}$.

Hence, both together will make $\frac{2}{5} + \frac{4}{15} = \frac{10}{15}$ of the same in one month; that is, $\frac{1}{15}$ in $\frac{1}{10}$ part of a month; and, consequently, the whole in $\frac{15}{10}$ of a month, or $1\frac{1}{2}$ months.

8. The same may also be understood in this way: Multiply the two times together; it is evident that, in the time $2\frac{1}{2} \times 3\frac{3}{4}$, the first would do the work $3\frac{3}{4}$ times, and the second $2\frac{1}{2}$ times.

Hence together, in this multiple time, they would do it $2\frac{1}{2} + 3\frac{3}{4}$ times; therefore, they will do it once in

$$\frac{2\frac{1}{2} \times 3\frac{3}{4}}{2\frac{1}{2} + 3\frac{3}{4}} = 1\frac{1}{2} \text{ months.}$$

Questions.—What is fellowship? Single? Double? Dividend? How are the shares found in single fellowship? How, in case of fractional proportions? How do you proceed in double fellowship? What other questions may be classed with fellowship?

EXERCISES.

1. A ship's company take a prize of £1000, which is to be divided among them, according to their pay and the time they

have been on board. The officers and midshipmen have been on board 6 months, the sailors 3 months. The officers receive 40 shillings a month, midshipmen 30 shillings, sailors 22 shillings. There are 4 officers, 12 midshipmen, 110 sailors: what is the share of each? *Ans. Each officer*, . . £23 2*sh.* $5\frac{23}{173}$*d.*

" *midshipman*, 17 6 $9\frac{147}{173}$.

" *sailor*, . . 6 7 $2\frac{2}{173}$.

2. 1,200 horses are to be distributed to three regiments, in proportion to their numerical strength. The number of men in the first is to that in the second as 11 to 8, and that of the first to the third as 9 to 7: how many horses must each regiment receive? *Ans.* 1*st*, . . $479\frac{1}{31}$.

2*d*, . . $348\frac{12}{31}$.

3*d*, . . $372\frac{18}{31}$.

1,200 0.

In a case like this, which occurs frequently, when the distribution does not allow of fractions, the unit is allowed to the largest fraction. The surplus horse would therefore go to the third regiment.

3. A bankrupt is indebted to *A*, \$277.33; to *B*, 305.17; to *C*, \$152; and to *D*, \$105: his estate being worth \$677.50, how must it be divided? *Ans.* (*Omitting fractions*) *A*, \$223.81.

B, \$246.28.

C, \$122.67.

D, \$84.74.

4. An individual engages in a certain enterprise, with \$25,000; five months afterwards he takes in a partner, with \$40,000; then, after six more months, he takes a second partner, with \$60,000. After two years, the profits amount to \$80,000. It has moreover been stipulated, that the first individual is to be allowed 5 per cent. on the profits, for his management of the concern: what is the share of each? *Ans.* 1*st*, . \$25,308.41.

2*d*, . \$26,990.65.

3*d*, . . \$27,700.94.

5. *A* can mow 5 acres in 3 days; *B* can mow 7 acres in 4 days; and *C*, 9 acres in 5 days: how long will it require them, working together, to mow 15 acres? *Ans.* 2.875.

Find what each can do in one day, and solve by No. 7.
Or, find in what time each will mow 15 acres, and solve by No. 8.

6. A mound of earth, containing 60 cubic yards, can be removed with a cart in 3 days, of 10 hours work; with a two-horse

wagon, in two days; and, with a four-horse wagon, in one day: in what time will the three together do it? *Ans.* $5hr.\ 5\frac{5}{11}m.$

7. A reservoir has two cocks to supply it; by the first, it may be filled in 40 minutes; by the second, in 50. It has also a discharging cock, by which it may be emptied in 25 minutes. These three cocks being open together, in what time would the cistern be filled? *Ans.* 3*hr.* 20*m.*

LESSON LXIII.

ALLIGATION.

1. This rule, which is closely allied to fellowship, serves to find the mean value or average of several things.

Example 1*st.*—A wine merchant mixes together 25 bottles of wine, at 12 cents; 18, at 15 cents; and 20, at 16 cents: what is the value of one bottle of the mixture?

It will readily occur, that the	25 bottles cost	$3.00.
"	18 "	2.70.
"	20 "	3.20.
Which gives, for the aggregate value of	63 bottles, .	$8.90.

And, consequently, for one bottle, $\$\frac{8.90}{63} = 14cts.\ 1m.$

Example 2*d.*—500 men are employed, of whom 160 are paid $2 a day; 200, $1.75; and 140, $1.50: what is the average price of this labor? *Ans.* $1.76.

2. This operation is called *alligation medial.*

The average is obtained by multiplying the individual values by the respective quantities, and dividing the sum of the products by the aggregate quantity.

3. The reverse, by which the mean value or average being given, the proportion of the component parts is obtained, is distinguished by the name of *alligation alternate.*

Example 1*st.*—How much wine, at $2 per gallon, should be mixed with wine at $3, so as to make a mixture worth $2.75 per gallon?

It is clear, that each gallon of the first wine will gain

75 cents in the mixture, and each gallon of the second will lose 25 cents; and also, that the quantities mixed must be such that the gain by the one may be equal to the loss by the other; a result which will be accomplished by taking quantities in reverse proportion of the loss and gain; that is, in the proportion of 25 to 75; since the products will be equal; that is, in this case, the gain 75 cents $\times$ 25 $=$ 25 cents $\times$ 75 the loss.

So that 0.25 or $\frac{1}{4}$ of a gallon of the dearer wine, and 0.75 or $\frac{3}{4}$ of the cheaper will make one gallon of the mixture.

4. This process of reasoning proves that the proportion of the component parts is found by taking *the difference between the price of each and that of the compound: the quantities are in an inverse ratio of these differences.*

5. *When the quantity of the compound, besides, is fixed, each component will be obtained by multiplying the quantity of the compound by the proportional quantity of each component part.*

Thus, if 60 gallons of wine were to be made by the above conditions, . $\frac{1}{4}$ of 60 $=$ 15,
and . . . $\frac{3}{4}$ of 60 $=$ 45,
would be the respective quantities.

6. *When the quantity of one of the parts is given together with all the prices:*

I. *After having found the proportion of the components, get the second part by multiplying the quantity of the first by the ratio between the two.*

II. *Their sum will be the quantity of the compound.*

Example.—How many gallons of water must be mixed with 30 gallons of wine, at \$6 per gallon, to make a mixture worth \$5 per gallon?

In the first place, we find the ratio between the quantities to be as 5 : 1.

Therefore, we must add $\frac{1}{5}$ of water, or 6 gallons.

7. Alligation alternate may sometimes be proposed between a greater number of ingredients; as, for example,

between tea, at \$1, at \$1.50, and at 75 cents, to be worth \$1.25.

It is very clear, that the mixture may be made in a great variety of ways; since, if we combine two of them in any way (provided only that the general mixture may still remain between the partial mixture and the remaining ingredient), these two may be mixed by the regular rule.

The operation would be more indefinite and arbitrary still, if the number of ingredients were increased; in fact, such a problem is of no importance in practice.

Questions.—What is alligation? Medial? Alternate? How is the average found when the quantities and values are given? How, the proportion of components, when both prices and their average are given? How, the two quantities, when, besides prices, the quantity of the compound is given? How, the second part and the aggregate, when prices and the quantity of one of the components are known? How is alligation alternate, in case of several compounds?

EXERCISES.

1. A goldsmith mixes 8*lb.* $5\frac{1}{2}$*oz.* of gold, of 14 carats fine, with 12*lb.* $8\frac{1}{2}$*oz.*, of 18 carats: what is the fineness of the mixture? *Ans.* $16\frac{51}{127}$ *carats.*

2. If I mix 27 bushels of wheat, at 5*sh.* 6*d.* the bushel, with the same quantity of rye, at 4*sh.* per bushel, and 14 bushels of barley, at 2*sh.* 8*d.* per bushel, what is the worth of a bushel of the mixture? *Ans.* 4*sh.* $3\frac{3}{4}+\frac{23}{68}$.

3. A contractor wishes to bid for 20 miles of road; he estimates 3 miles at \$875 per mile; 5, at \$1,500; 6, at \$650; 4, at \$1,100; and 2, at \$2,000: what must be his average bid per mile? *Ans.* \$1,121.25.

4. In grading a road up a mountain, by an experimental survey, it is found that the rise of 45 rods is at 10°; 22 rods, at $9\frac{1}{2}$°; 38, at 9°; 60, at 5°; 110, at 4°; and 55, at 3°: what should be the uniform grade of the road, allowing a level platform of 2 rods in every 60 rods? *Ans.* 5° $57\frac{3}{8}$′.

5. A survey is run through woods; the first course is 120 poles long; the second, 85, and deviates 3° to the right of the first; the third, 95 poles, and deflects from the first 5°, also to the right; the fourth, 125 poles, and deflects 6° to the left of the first; the fifth, 75 poles, deflecting 1° to the right; finally, the sixth, 50

poles, deflecting 2° to the left: what is the straight direction between both ends, in regard to the first course?

Ans. 0° $4\frac{10}{11}'$ *to the left of first course.*

Multiply distances by grades, and subtract between the products of the left and right; then divide by the whole distance: the result will show the deflection on the side of the larger product.

N. B.—The two preceding questions are convenient and correct in practice, for angles not exceeding 12° or 15°.

6. On a railroad there were carried 15,750 tons 173 miles; 6,820 tons, 82 miles; 9,719 tons, 32 miles. The cost of repairs of cars has been $17,188.60: what is it per ton per mile?

Ans. 0.478 *ct.*

7. A manufacturer wants to make, with a mixture of cotton, linen that will sell at $1 per yard, while pure linen would sell at $1.50, and cotton stuff at 25 cents, all of the same fineness: what proportion of cotton and flax must be used?

Ans. 2 *of cotton to* 3 *of flax.*

8. A man can finish a piece of work in 30 days, with 240 men, working 10 hours a day. He engages 100 men, who agree to work 12 hours; but all those who are yet for hire, consent to work only 9 hours: how many of these must be engaged to finish in time? *Ans.* 133 *men* (*omitting the fraction* $\frac{1}{3}$).

9. Make 120 gallons worth 6*sh.* per gallon, with wines worth 5*sh.* and 8*sh.* *Ans.* 80 *at* 5*sh.*
40 *at* 8*sh.*

10. A grocer would mix teas, worth 8 and 13 shillings, so as to sell the compound at 11 shillings per pound: in what proportions should they be mixed? *Ans.* 2 *at* 8*sh.*
to 3 *at* 13*sh.*

11. A trader bought one hogshead of rum, of 115 gallons, at $1.10 per gallon: how many gallons of water must be put into it to make $5, by selling it at $1?

Ans. $16\frac{1}{2}$ *gallons of water.*

LESSON LXIV.

PERCENTAGE.

1. Under this head are included interest, discount, commission and brokerage, insurance, loss and gain, tare and tret, exchange, equations of payments, and in one word, all questions in which a certain proportional allowance is to be calculated.

2. The *rate* of this allowance is generally referred to *one hundred*, in order to avoid small fractions. Thus, for certain services or advantages, an allowance of $5 in every $100 is made.

The *rate*, in this case, is said to be 5 *per cent.* It is sometimes referred to *one*, and then is written $\frac{5}{100}$, or 0.05. The first mode of expressing it, however, is more general, though not preferable. We will use both ways, for practice.

SIMPLE INTEREST.

3. i being the interest,
r the rate for 1,
t the time,
p the principal,
s the sum or total amount.

The algebraical formulæ for interest are, $i = prt$, and $s = p(1 + rt)$.

4. *Interest* is an allowance made for the use of money borrowed, in proportion to the time and at a certain *rate* agreed upon between the parties, or fixed by law.

The *principal* or *capital* is the sum lent, and on which the interest is paid.

The word *amount* is particularly used to designate the principal and interest added together.

The interest of 100 for one year determines the rate of interest; it is *the rate per cent. per annum.*

When the rate is established by law, it is styled the *legal rate.* This, in most of the United States, is 6 per cent.; and, therefore, in such states, when no rate is mentioned, 6 per cent. is understood.

In New York, however, it is 7 per cent., and 8 in Louisiana.

Interest exacted at a rate exceeding legal interest, is considered *usury.*

Interest may be either *simple* or *compound.* We consider only simple interest in this lesson.

5. All questions in interest are like the following:

What is the interest on $4,500, for 3 years and 6 months, at 7 per cent. per annum?

This is evidently a question in compound proportion, since the *interest* depends on two ratios: 1st. That of the *principal* to 100 dollars; 2d. That of the *time* to the unit of time, *one year*.

The statement is made by considering, in the first place, that since \$7 is given for the use of 100 dollars one year, as many times \$7 must be paid for the use of \$4,500 one year as there are 100 dollars in this capital; that is,

$$\$7 \times \tfrac{4500}{100}; \quad \text{or,} \quad \$0.07 \times 4500.$$

And, in the next, that this interest for one year must be multiplied by the time during which the money was borrowed; making the answer

$$7 \times \tfrac{4500}{100} \times (3yr.\ 6mo.) = 7 \times \tfrac{4500}{100} \times 3.5 = \$1,102.50;$$

which shows that *interest is calculated*, in general, *by multiplying the capital by the rate of interest and by the time, and dividing by* 100.

It must be observed that the number 3.5, which stands for the time, 3*yr.* 6*mo.*, or $3\frac{1}{2}$ years, decimally expressed, is, in fact, the abstract number resulting from the ratio of the time, 3*yr.* 6*mo.*, to its unit of comparison, one year. So that the above statement should in reality be

$$\$7 \times \tfrac{4500}{100} \times \frac{3yr.\ 6mo.}{1yr.}$$

The unit of comparison is generally omitted in the denominator, when its nature is well understood; because it does not affect the numerical result, and merely changes the denominate quantity, 3.5 *years*, into the abstract number, 3.5.

6. Sometimes the rate of interest is not reckoned by the year, but by some other unit of time; *the month*, for example.

The difference in the *unit of time*, however, produces none in the application of the rule. Thus, in the following question:

What is the interest on \$6,000 for 1 year 1 month and 15 days, at the rate of $\frac{3}{4}$ per cent. per month?

The statement would be $\$\frac{3}{4} \times \frac{6000}{100} \times \frac{(1yr.\ 1mo.\ 15da.)}{1mo.}$

In which the compound time may be reduced to the standard of comparison, the month, in order to make the ratio a simple abstract number; making thus the statement

$$\$\tfrac{3}{4} \times \tfrac{6000}{100} \times 13.50 = \$607.50;$$

in which the rate, principal, and time, reduced to its standard, are multiplied together, and divided by 100, according to the rule.

Questions.—What is percentage? Interest? Rate of interest? Principal? Amount? Legal rate, in general? What is it in New York? In Louisiana? In the other states? What is called usury? What rule of arithmetic do questions in interest belong to? What is the rule for calculating interest? What should be done with compound time? How is this factor considered? Explain it by ratios.

EXERCISES.

In the following questions, it must be recollected that beginners should always indicate, in a complete statement, all the operations to be performed, whereby they will very frequently simplify considerably their execution.

In most cases, it will be best to reduce compound time to decimals.

1. How much is the interest of $187.25, for 16 months, at 6 per cent. per annum?

Statement: $\frac{\$6 \times 187.25}{100} \times \frac{16}{12} = 8 \times 1.8725 = \$14.98.$

Here, you cut off two figures of the capital, and cancel 12, with 6, and 2, out of 16.

2. What is the interest of $694.84, for 9 months, at 10 per cent. per annum? *Ans.* $52.11 3*m.* *or,* $52.113.

3. How much is the *amount* (capital and interest) of $985, for 5 years and 8 months, at .06 per annum? Calculate the interest for $5\frac{2}{3}$ years, or reduce all to months. *Ans.* $1,319.90.

4. What is the interest of $126.46, for 9 months, at 6 per cent.?

5. Find the interest of 547.80, for 3 years, at 0.05 per annum. *Ans.* $

6. Find the interest of $223.20, for 9 years and 3 months, at 7 per cent. *Ans.* $144.522.

7. What is the interest on \$498.50, for 6 years, at $4\frac{1}{2}$ per cent. per annum? *Ans.* \$134.595.

8. What is the interest on \$4,988.75, for 8 years, at 0.08 per annum? *Ans.* \$3,192.80.

9. What is the interest of \$912, for *3yr. 5mo.* 18*da.*, at 5 per cent.? *Ans.* \$158.08.

10. What is the interest on \$648, for 1*yr.* 10*mo.* 19*da.*, at 0.10? *Ans.* \$122.22.

11. A merchant has purchased cloth for \$3,859.25; for which he offers a note, payable in 18 months, including interest at $\frac{3}{4}$ per cent. per month: what should be the amount on the face of the note? *Ans.* \$4,380.25.

LESSON LXV.

1. Although, with a little practice, simplifications applicable to particular cases will be readily discovered, there are some methods, which recur so frequently in practice, that they may with propriety be introduced here.

Simple interest, for a number of months, is readily calculated, when the rate per cent. is an aliquot part or factor of 12; that is, when it is 6, 4, or 3 per cent.

INTEREST AT SIX PER CENT. PER ANNUM.

2. Let it be proposed to find the interest of \$375, for 2 months, at 6 per cent. The statement would be

$$\frac{\$375 \times 0.06 \times 2}{12}, \text{ or } \frac{\$375 \times 6 \times 2}{100 \times 12} = \$3.75.$$

In which 6×2 cancels 12; and the *interest for two months is thus obtained by simply cutting off two figures of the capital;* and, from this, *the interest for any number of months may be calculated, by multiplying that for two months by half the number of months.*

Thus would the interest of the above sum, \$375, for 8 months, be . . $\$3.75 \times 4 = \$15.00$.
For 9 months, . $\$3.75 \times 4\frac{1}{2} = \$16.875$.

INTEREST AT FOUR PER CENT. PER ANNUM.

3. If it were asked to get the interest of $448.50, for 3 months, the statement would be

$$\frac{\$448.50 \times 4 \times 3}{100 \times 12} = \$4.485.$$

Which shows that, at 4 *per cent. per annum, the interest for three months is obtained by merely removing the unit's point two places to the left.*

For any other *time, the interest would be found by multiplying by one third of the number of months, and displacing the decimal point two places to the left.*

INTEREST AT THREE PER CENT. PER ANNUM.

4. *The interest for four months, at* 3 *per cent., would be found by displacing the decimal point two places to the left.*

For any number of months, the interest would be obtained, after having displaced the units' point, by multiplying by one fourth the number of months.

Example: What is the interest on $4,768, for 16 months, *at* 3 *per cent.?* *Ans.* $\$47.68 \times 4 = \$190.72.$

What is it for 9 months?

$$Ans.\ \frac{\$47.68 \times 9}{4} = \$107.28.$$

INTEREST FOR A NUMBER OF DAYS.

5. In the computation of interest in Federal money, it is customary to reckon the year at 12 months, and each month at 30 days; considering thus the year to contain only 360 days instead of 365 days. This method makes but an inconsiderable difference in the amount of interest, and is very convenient in practice, especially at rates like 6, 4, 3, 9, which are factors of 360. Thus, for an amount of $696, at 6 per cent. per annum, for 37 days, we would get

$$\frac{\$696 \times 37 \times 6}{360 \times 100} = \frac{0.696 \times 37}{6} = \$4.292.$$

Which shows that, *to get the interest for a number of days, at* 6 *per cent. per annum, you must cut off three figures, multiply by the number of days, and divide by* 6.

At 4 *per cent., you would have to divide by* 9.

At 3, *by* 12.

At 9, *by* 4.

And at 12, *by* 3.

6. The most usual rate of interest being 6 per cent., other rates are frequently calculated from it, on account of the facility with which it is computed.

5 per cent., by subtracting $\frac{1}{6}$ from it; 7, 8, and 9, by adding respectively $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, to the interest at 6 per cent.

Example: \$156, at 7 per cent., for 4 months, $= 3.12 + \frac{3.12}{6} = 3.64$.

7. Simplifications of this kind, with a little practice, will be readily discovered in all cases, without reference to stated rules.

8. In general, questions which involve years, months, and days, are performed as in compound numbers, which the preceding remarks serve to simplify, in the cases referred to.

EXAMPLES.

I. What is the interest of \$418, for 1 year 7 months and 17 days, at 6 per cent.?

	\$418
	1*yr.* 7*mo.* 17*da.*
For 1 year, . .	25.08
For 6 months, . .	12.54
For 1 month, . .	2.09
For 15 days, $\frac{1}{2}$ of 1 month,	1.045
For 2 days, $\frac{1}{15}$ of 1 month,	0.139 +
	\$41.894 +

II. How much is the interest of \$268.44, for 3 years 5 months 26 days, at 6 per cent. per annum?

For 3 years, 18 times 2.6844, . .	\$48.3192
For 5 months, $\frac{5}{2}$ or $\frac{10}{4}$ of do. .	6.7110
For 15 days, $\frac{1}{10}$ of 5 months, .	.6711
For 10 days, $\frac{1}{15}$ of 5 months, .	.4474
For 1 day, $\frac{1}{10}$ of preceding, . .	.04474
	\$56.19344

III. What is the interest of $910.50, for 3 years 9 months 26 days, at 7 per cent.?

For 3 years, . .	$21 \times 9.1050 =$	$191.2050
For 9 months, $\frac{1}{4}$ of 3 years,	. .	47.80125
For 18 days, $\frac{1}{15}$ of 9 months,	. .	3.18675
For 6, $\frac{1}{3}$,	. . . . . .	1.06225
For 2, $\frac{1}{3}$,	. . . . . .	.35408$\frac{1}{3}$
		243.60933\frac{1}{3}$

Or else,

For 1 year, .	$\$9.1050 \times 7 = 63.7350.$	
For 3 years,	. . . .	$191.2050
For 6 months, .	$\frac{1}{2}$ of 1 year,	31.8675
For 3 " .	$\frac{1}{2}$ . .	15.93375
For 15 days, . .	$\frac{1}{6}$ . .	2.655625
For 10 days, . .	$\frac{1}{9}$ of 3 months,	$1.770416\overline{6}$
For 1 day, . .	$\frac{1}{10}$ . .	$.1770416\overline{6}$
		$\$243.60933\overline{3}$

Questions.—How is the interest on Federal money, at 6 per cent., calculated for 2 months? For any number of months? At 4 per cent. for 3 months? For any number? At 3 per cent. for 4 months? For any number of months? For a number of days, how is it customary to reckon the year? How is the interest for a number of days calculated, at 3? 4? 6? 9? 12 per cent.? How may interest be calculated for any rate, from that at 6 per cent.? How, when the time is a compound number?

EXERCISES.

1. What is the interest of $230.50, for 9 months, at 0.04 per annum? *Ans.*

2. On $95, for 2 months, at 6 per cent.? *Ans.*

3. On $194, for 4 months and 12 days, at 0.06? *Ans.* $4.268.

4. On $265.48, for 2 months and 21 days, at 6 per cent.? *Ans.* $3.584.

5. On $318, for 10 months and 16 days, at 0.06 per cent.? *Ans.* $16.748.

6. On $1, for 18 days, at 6 per cent.?

7. On $47,984, for 15 months, at 3, 4, 6 per cent. per annum?

8. On $618.96, for 41 days, at 3, 4, 6, 9, 12 per cent. per annum?

9. On $240, for 2 years 6 months and 13 days, at 0.06? *Ans.* $36.52.

10. On $615.75, for 2 years 7 months and 16 days, at 4; 5; 6; 7; 8; 9; 10 per cent. per annum?

11. On $241.60, for 3 years 4 months and 15 days, at 6 per cent.? *Ans.* $48.924.

12. On 1,440, for 5 years 11 months and 28 days, at 0.08? *Ans.* $690.56.

LESSON LXVI.

1. In the preceding lesson, we have considered the year, according to a general practice, to consist of 360 days. Some persons, however, calculate by days at the more correct rate of 365 in a year.

Example.—What is the interest on $7,086, at 6 per cent., for 39 days?

Statement: . $\$7{,}086 \times \frac{6}{100} \times \frac{39}{365} = \$45.43.$

It will be readily perceived that this mode of computing, which is less favorable to the lender, is not so simple in practice, and admits only of occasional cancellings.

The interest, in this case, will be 63 cents less than when the year is considered as only 360 days.

2. When the sum on which the interest is to be calculated, is in pounds, shillings, and pence, the operation is not so simple as in Federal money.

3. When the fractions are convenient, you may,

I. *Reduce the shillings and pence to decimals of a pound.*

II. *Then find the interest, as above.*

III. *And, finally, reduce the decimal part of the answers to shillings and pence.*

Example 1st.—What is the interest on £150 15*sh.* 6*d.*, at 6 per cent., for 2 years and 6 months?

The principal, in decimals, is £150.775, and the interest on it £1.50775 × 15 = £22.61625 = £22 12*sh.* 3$\frac{3}{4}$*d.*

4. In other cases, when both the time and principal

being compound numbers, the fractions are not readily changed into decimals, it will be found shorter to perform the operation by aliquot parts. The first step of the operation will, in general, be to multiply the rate per cent. by either the capital or time, as may appear more simple.

Example 2d.—What is the interest on £70 19*sh.* 4*d.*, at 6 per cent., for 2 years 8 months and 16 days?

Statement: $\dfrac{(£70\ 19sh.\ 4d.) \times 6 \times (2yr.\ 8mo.\ 16da.)}{100}$

In this case, it will be best to multiply the capital by 6, and then the operation is reduced to multiplying

	£425	16*sh.*	
by $\frac{1}{100}$ of	2*yr.*	8*mo.*	16*da.*
For 2 years, take $\frac{1}{50}$ . .	£8	10*sh.*	3.84*d.*
8 months, $\frac{1}{3}$ . .	2	16	9.28
16 days, $\frac{1}{15}$ . .	0	3	9.418$\frac{2}{3}$
	£11	10*sh.*	10.538$\frac{2}{3}$

5. If the year was reckoned at 365 days, the interest for the 16 days should be taken separately, as follows:

$$\frac{(£425\ 16sh.) \times 16}{100 \times 365} = \frac{£425.80 \times 16}{100 \times 365} = \frac{68.1280}{365} = 3sh.\ 8.796\tfrac{1}{2}d.$$

Showing a difference of only 0.622 of a penny between the two modes of reckoning interest for 16 days; a difference too inconsiderable to be noticed.

Example 3d.—To find the interest, at 7$\frac{1}{2}$ per cent. per annum, of £119 7*sh.* 7*d.*, for 4 years 7 months and 23 days.

Here, instead of multiplying either the time or capital by the rate, it will be found easier to calculate first the interest for one year.

And then to get from it, by aliquot parts, that for the given time.

OPERATION.

			£119 7*sh.* 7*d.*
			4*yr.* 7*mo.* 23*da.*
For 5 per cent., take	$\frac{1}{20}$.	£5 19*sh.* 4.55*d.*	
For $2\frac{1}{2}$, "	$\frac{1}{2}$.	2 19 8.275	
Interest for 1 year,	.	£8 19*sh.* 0.825*d.*	
" for 4 years,	. . . .		£35 16*sh.* 3.3*d.*
" for 6 months,	$\frac{1}{8}$, . . .		4 9 6.4
" for 1 month,	$\frac{1}{6}$, . . .		0 14 11.1
" for 15 days,	$\frac{1}{2}$, . . .		0 7 5.5
" for 5 "	$\frac{1}{3}$, . . .		0 2 5.8
" for 3 "	$\frac{1}{10}$ of 1 month, .		0 1 5.9
			£41 12*sh.* 2.0*d.*

smaller fractions being neglected.

Questions.—How should interest be calculated, if the year be reckoned at 365 days? To whom is this mode of computing favorable? How is interest on sterling money calculated, when the fractions are simple? In other cases?

EXERCISES.

The year being considered 365 days:

1. What is the interest on \$87.56, for 72 days, at 6 per cent. for a year? *Ans.* \$1.036.

2. On \$2,962.19, for 254 days, at 6 per cent. *Ans.* \$123.678.

3. On \$1,733.97, for 102 days, at 8 per cent.? *Ans.*

4. On £355 15*sh.*, for 4 years, at 4 per cent.? *Ans.* £56 18*sh.* $4\frac{3}{4}$*d.*

5. On £32 5*sh.* 8*d.*, for 7 years, at $4\frac{1}{2}$ per cent.? *Ans.* £10 3*sh.* 4*d.*

6. On £319 6*d.*, for $5\frac{3}{4}$ years, at $3\frac{3}{4}$ per cent.? *Ans.* £68 15*sh.* $9\frac{1}{2}$*d.*

7. The year being 365 days, what is the interest on £107, for 117 days, at $4\frac{3}{4}$ per cent. per annum? *Ans.* £1 12*sh.* 7*d.*

8. On £17 5*sh.*, for 117 days, at $4\frac{3}{4}$ per cent.? *Ans.* 5*sh.* 3*d.*

9. On £712 6*sh.*, for 8 months, at $7\frac{1}{2}$ per cent.? *Ans.* £35 12*sh.* $3\frac{1}{2}$*d.*

10. The year being 360 days, on £107 16*sh.* 10*d.*, at 5 per cent., for 7 years and 12 days? *Ans.* £37 18*sh.* 5*d.*

11. How much is the amount of \$298.59, from the 19th May, 1847, to the 11th August, 1848, at $5\frac{3}{4}$ per cent.? *Ans.* \$319.67.

12. *Richmond*, June 14th, 1846.

For value received, I promise to pay, on the 29th day of April, 1847, to Henry Lee or order, the sum of one hundred and ninety-six dollars, bearing interest from date, at $5\frac{3}{4}$ per cent.

$196. JAMES OGDEN.

What is the amount on day of payment? *Ans.* $205.86.

13. *Richmond*, January 9th, 1845.

$658.

Nine months after date, I promise to pay to Thomas Watkins or order, with interest from date, at 6 per cent. per annum, six hundred and fifty-eight dollars, for value received.

JEREMIAH NICHOLS.

Amount, ... $

LESSON LXVII.

1. Having now explained different methods of computing interest, it may not be amiss to complete the subject with several questions which simple interest may give rise to.

2. It will be readily perceived that *the principal*, *the rate*, *the interest*, *the time*, and *the amount*, are so connected that any one of them may be obtained by means of the others.

CASES I. AND II.

3. The principal, rate, and time being known, to find the interest and the amount.

These cases form the subject of the preceding lessons.

CASE III.

4. The principal, rate, and interest being given, to find the time. Algebraical formula (see LXIV.): $t = \frac{i}{pr}$.

Divide the given interest by the interest for one year, since the interest is the product of the interest for one year, by the time.

Example.—How long must the interest, at 8 per cent., on \$560 be collected by a certain individual, in order that he may be paid a debt of \$106.40?

$$\text{Statement: } \frac{\$106.40}{560 \times 0.08} = 2yr.\ 4mo.\ 15da.$$

The numerator is the aggregate interest to be collected; the denominator, the interest of \$560 for one year, at the rate 0.08, or 8 per cent.

CASE IV.

5. The principal, interest, and time being given, to find the rate. Formula: $r = \frac{i}{pt}$.

Divide the interest by the time and by the principal; since the aggregate interest is the product of the principal and time by the rate.

Example.—A person has borrowed \$3,750, for 2 years and 6 months, and paid for the use of it \$719.25: at what rate did he pay interest?

$$\text{Statement: } \frac{\$719.25}{3,750 \times 2\frac{1}{2}yr.} = 0.0767, \text{ or } 7.67 \text{ per cent.};$$

an illegal interest in many states.

CASE V.

6. The *time, rate,* and *interest* being given, to find the *principal.* Formula: $p = \frac{i}{rt}$.

Divide the interest by the time and the rate; since the interest would be produced by multiplying together the capital, rate, and time.

Example.—On what principal should the interest be given up for 16 months, so as to pay up a debt of \$124, the rate of interest being 0.06?

$$\text{Statement: } \frac{\$124}{0.06 \times \frac{16}{12}} = \frac{12400}{8} = \$1,550.$$

N. B.—If, in this case, the amount instead of the principal were required, find the principal first and add it to the interest.

CASE VI.

7. In some questions, the amount may be given: if, then, either the principal or the interest is also known, the other is readily obtained by subtracting the known one from the amount; and thus the question will fall under one of the preceding cases: as, for example, in this

Question.—A man, for the loan of $200, requires a note of $500, payable in 10 years: what is the rate of interest thus virtually exacted?

The difference, $300, is the interest; and, by Case IV.,

$$\frac{300}{10 \times 200} = 0.15,$$

or 15 per cent. the rate required.

8. The only case which requires a separate investigation, is that when

The *time*, *rate*, and *amount* being given, it is required to find the *interest* and the *principal*.

This case is properly the rule of *discount*, which, on account of its importance, we will consider in a separate lesson.

Questions.—The principal, rate, and time being given, how is the interest found? How, the amount? The principal, rate, and interest being given, how is the time found? The principal, interest, and time being given, how is the rate found? Having given the time, rate, and interest, how is the principal found? How, the amount? If the interest and capital are known, how is the amount found? How would you proceed, in questions in which the amount and either the principal or the interest are given together, to find the other? What rule do questions belong to, in which the time, rate, and amount are given, and either the interest or principal required?

EXERCISES.

1. A person paid $94.902 on $981.75, at 6 per cent. per annum: what was the time? *Ans.* 1*yr.* 7*mo.* 10*da.*

2. In what time will $1,800 amount to $1,853.70, at 6 per cent.? *Ans.* 5*mo.* 29*da.*

3. \$423.20 have been paid for the loan of \$920, for 8 years what is the rate? *Ans.* 0.0575, *or* $5\frac{3}{4}$ *per cent.*

4. For a sum, \$3,650, a man exacts a note of \$4,790.62$\frac{1}{2}$, payable in $2\frac{1}{2}$ years; what is the rate of his usury? *Ans.* $12\frac{1}{2}$ *per cent.*

5. £9 12*sh.* 1*d.* have been paid for the use of £32 5*sh.* 8*d.*, for 7 years: what is the rate of interest?

6. For the loan of £873 15*sh.*, for $2\frac{1}{2}$ years, a note of £977 10*sh.* $1\frac{3}{4}$*d.* is taken: is the rate usurious? *Ans.* $4\frac{3}{4}$ *per cent.*

7. \$922.14 having been paid, at an interest of 8 per cent., for 3 years and 11 months, what was the capital? *Ans.* \$2,943.

8. The interest of a note bearing interest at 5 per cent., amounts, in 3 years, to £82 3*sh.* 3*d.*: what was the sum on the face of the note? *Ans.* £547 15*sh.*

LESSON LXVIII.

SIMPLE DISCOUNT.

Algebraical formulæ of discount: $p = \frac{s}{1+rt}$, $d = s - p$, or $d = \frac{srt}{1+rt}$. (See LXIV., 3)

1. *Discount* is a deduction made from a debt when it is paid before it becomes due.

Discount may be like interest, *simple* or *compound*. In this lesson, we consider only Simple Discount. Compound Discount will be the subject of a subsequent lesson.

If I owe, for instance, a note of \$106, payable at the end of the year, and I am asked to discharge this debt now, it cannot be expected that I would agree to it without receiving some advantage for this anticipation of payment; and this advantage should be exactly equal to the regular interest I would receive by retaining the use of the money I pay out. It is in fact the same thing as if I, in my turn, were to lend money to my lender to cancel his claim of \$106. In this case, the principal I give him must be such that, together with the interest for the time the note has to run, the aggregate amount may be precisely equal to the sum due at the expiration of the time.

If the interest is at 6 per cent., the note being for \$106, the principal would evidently be 100.

2. Hence, *the rule of discount consists in finding what principal would produce the amount due in a given time.* This is *the present value* of the note.

The discount is the difference between the face of the note and its present value.

This rule, therefore, as was said before, is the same as Case VI., of the preceding lesson; where,

The *amount*, *rate*, and *time* being given, it is proposed to find *the principal.*

Divide the given amount by the amount of one dollar, at the given rate, and for the given time.

Example.—What is the present worth and discount of a note of $450, due 6 months hence, at 6 per cent.?

In the first place, it is evident that $1 would become $1.03 in 6 months, and consequently as many dollars must be paid now as the number of times the amount, $450, contains 1.03; that is,

$\frac{450}{1.03} = \$436.893$, and, for the discount, $450 - 436.89 = \$13.107$.

3. The same may be obtained by proportions, as follows:

Since $100, with the addition of interest, amounts to $103, in 6 months, we have the proportion

$$103 : 100 :: 450 : \frac{450 \times 103}{100}.$$

The correctness of the operation may now be verified by calculating the interest on $436.893 and the amount, which would be found to be $450.

Questions.—What is discount? What case in questions on interest does it belong to? How is the capital found? How, the discount? By division? By proportions?

EXERCISES.

1. What is the discount of £308 15*sh.*, due in 18 months, at 8 per cent. per annum? *Ans.* £33 1*sh.* 7$\frac{2}{7}$*d.*

2. What is the present worth of \$5,150, due in $4\frac{1}{2}$ months, at 8 per cent. per annum, and allowing 1 per cent., besides, for prompt payment? *Ans.* \$4,950.

3. *A* is to pay \$5,927 on the 19th of April, and \$5,989 on the 19th of July following. He prefers to pay all on the 19th of January: how much will discharge both sums, at 8 per cent.?
Ans. Reckoning by months, . \$11,569.438.
By days, and 365 *in the year,* \$11,572.820.

4. What is the discount on \$4,590.39, at 6 per cent., for 47 days of a year of 365 days? *Ans.* \$35.19.

5. What is the principal of the amount \$2,202.34, for 125 days, the year of 365 days, at 6 per cent.? *Ans.* \$2,158.

6. What is the present value of a note for \$4,850, payable in $13\frac{1}{2}$ months, at $\frac{3}{4}$ per cent. per month? *Ans.* \$4,404.09.

7. What is the discount on a note for \$2,850.45, payable in 2 years and 8 months, at $8\frac{3}{4}$ per cent. per annum? *Ans.* \$539.27.

8. Bought goods to the amount of \$950, at 90 days: what ready money will discharge it, the interest being at 6 per cent.?
Ans. \$935.75, *for* 3 *months.*
\$935.95, *for* 90 *days.*

9. What is the present value of three notes, payable at one, two, and three years, interest at 5 per cent., each note being for \$478.17? *Ans.* \$1,305.90.

10. What amount of ready cash will discharge a debt of \$1,950, of which \$190 are payable in 6 months; \$270 in one year; \$490 in 18 months, and \$1000 in 2 years, interest at 6 per cent.?
Ans. \$1,781.581.

LESSON LXIX.

BANK DISCOUNT.

Formulæ: $p = s(1 - rt)$ . . $d = srt$.

1. The above method of calculating discount is the only one by which neither party suffers a loss. It is, however, not always the discount allowed in ordinary transactions, and especially by bankers.

2. When money is obtained from a bank, it is customary to deduct and retain as discount the interest on the amount of the note for the time it has to run, and also for *three days of grace*, which are usually allowed.

Thus, if a note for $100 be discounted at a bank, for 30 days, the interest for 33 days, which, at 6 per cent., is 55 cents, is deducted for discount; and the holder of the note receives $99.45 for the $100 which appears on the face of the note.

3. This manner of discounting is evidently to the advantage of the banker; since he exacts interest for more than the sum he advances; that is, not only for the money you receive, but likewise on the interest he withholds. So that *you give up the interest of the interest.*

To make this perfectly clear by a simple case, let us suppose that you hold a note for $106, payable in one year: it is plain that it is equivalent to $100 now; yet the banker deducts the interest not only on the $100, but also on the $6, which are the interest to accrue during the year.

This shows that the *discount*, and not the *interest*, is the just deduction for anticipation of payment. Yet the last method is in common use, especially for small accounts, not only because it is more expeditious, but probably also because the lender has the power to make his own terms.

This mode of discounting being only, after all, a mere calculation of simple interest, it is unnecessary to give any example or exercise on it.

4. As to discount allowed for prompt payment, or other considerations which have no reference to time, the following simple examples will suffice:

1. How much is the discount of $853, at 2 per cent.? *Ans.* $17.06.

2. An agent sells property for $985.75, and is to receive 4 per cent. for his trouble: how much is he to pay? *Ans.* $946.32.

COMMISSION OR BROKERAGE, AND INSURANCE.

5. *Commission or Brokerage* is an allowance made to an agent for buying or selling. The allowance is generally a certain percentage or rate per hundred, on the amount paid or received.

6. *Insurance* is an allowance of a certain rate per

cent., made to an individual or company, who, in consideration thereof, will make good the loss sustained by fire, navigation, storms, &c., up to the amount or *risk* insured for.

The written agreement is called the *policy.*

The sum paid for insurance is the *premium.*

The allowance made, in all these cases, is readily calculated by the preceding rules of interest.

Questions.—What is the usual rule of discount of bankers? In what does it differ from true discount? What is commission? Brokerage? Insurance? Policy? Premium?

EXERCISES.

1. What is the commission on \$2,716.50, at $2\frac{1}{2}$ per cent.?
Ans. \$54.412.

2. The sale of certain goods amounts to \$1,873.40: what is to be received for them, allowing $2\frac{1}{2}$ per cent. for commission, and $\frac{1}{4}$ per cent. for prompt payment of the net proceeds?
Ans. \$1,821.999.

3. What is the present worth of \$9,150, due in $7\frac{1}{2}$ months, discounting at the rate of 5 per cent. per annum, and allowing $1\frac{1}{2}$ per cent. for prompt payment? *Ans.*

4. What is the commission on £1,371 9*sh.* 3*d.*, at 5 per cent.?
Ans. £68 11*sh.* $5\frac{1}{2}$*d.*

5. What is the brokerage on \$1,853, at $\frac{3}{4}$ per cent.?
Ans. \$13.897.

6. " " on £874 15*sh.* 3*d.*, at $\frac{1}{4}$ per cent.?
Ans. £2 3*sh.* $8\frac{3}{4}$*d.*

7. " " on £1,321 11*sh.* 4*d.*, at $1\frac{1}{8}$ per cent.?
Ans. £14 17*sh.* 4*d.*

8. A factor receives \$988, to lay out in goods, allowing his commission of 4 per cent. on the purchase: how much does he lay out? *Ans.* \$950.

9. A factor has in his hands \$3,960, which he is desired to lay out in iron, reserving $2\frac{1}{8}$ per cent. on the purchase; the iron being at \$95 per ton, how much can he purchase?
Ans. 37*T.* 17*cwt.* 3*qr.* $16\frac{4}{19}$*lb.*

10. What is the premium on \$1,873, at $\frac{1}{8}$ per cent.?

11. " " on £924, at 7 per cent.?
Ans. £64 13*sh.* 7*d.*

12. What must be paid for insurance on a house valued at

$7,000, the cost of the policy being $1.50, and the rate of insuring ½ per cent.; and the insurance being effected on only ¾ of the valuation? *Ans.*

LESSON LXX.

TARE AND TRET.

1. These are allowances made in selling goods by weight.

Tret is an allowance to the buyer for waste, dust, &c., generally at the rate of 4 pounds per 104 pounds.

2. *Tare* is a deduction for the weight of the case or envelope containing the commodity.

3. *Draft* is an allowance on the gross weight; it is always deducted before the tare.

4. *Gross weight* is the whole weight of the commodity, together with the hogshead, barrel, bag, or box, which contains it.

5. *Net weight* is what remains after the allowances have been deducted, and which is paid for.

EXAMPLES.

1. At $1.25 per pound, what will 3 chests of hyson tea come to, weighing 96*lb.*, 97*lb.*, and 101*lb.*; the tare being 20*lb.* per chest?

$96 + 97 + 101 - 3 \times 20 = 234lb.$, and $\$1.25 \times 234 = \$292.50$.

2. What is the cost of a firkin of butter weighing gross 132*lb.*, at 19 cents; the tare being 15*lb.* for the first 50*lb.*, and 1*lb.* for every ten pounds over fifty? *Ans.* $20.71.

N. B.—Fractions less than 5 pounds are not considered; but, from 5 up to 10, 1 is added to the tare: so that, if the firkin were 129 instead of 132 pounds, the tare would be the same.

PROFIT AND LOSS.

6. This is an application of the rule for interest, by which the amount gained or lost in a mercantile transaction is ascertained, and which also serves to determine

at what price goods should be sold to realize a certain profit or sustain a certain loss.

These operations require no new rule, and a few examples will suffice.

Questions.—What is tare; tret; draft; gross weight; net weight? What is profit and loss? How is it calculated? How do you find the rate per cent. of either profit or loss? How would you find at what price an article must be sold, to lose or gain a certain percentage on the first cost? The price an article sold for, and the percentage realized or lost being known, how would you find the first cost?

EXERCISES.

1. I purchased 80 yards of cloth, at \$3.75 per yard, and sold at \$4.50: how much did I gain, and what is it per cent. on the purchase?

2. A trader bought one hogshead of strong rum, containing 115 gallons, at \$1.10 per gallon: how many gallons of water must be put in it to gain \$5, by selling it at \$1 per gallon?

3. A merchant bought 2,750 bushels of wheat, for \$3,300; but, finding it damaged, he agreed to sell it at a discount of 10 per cent.: what will it be per bushel? *Ans.* \$1.08.

4. A merchant sells his goods at 2½ pence profit on the shilling: how much is it per cent.?

5. Bought a piece of baize, of 42 yards. for £4 14*sh.* 6*d.*, and sold it at 2*sh.* 6*d.* per yard: what is the profit or loss?
Ans. 10*sh.* 6*d.* *profit.*

6. Which is the better bargain, in purchasing fish at 17 shillings per quintal, at 4 months credit, or 16*sh.* 8*d.*, cash; interest at 6 per cent. per annum?

7. Bought 4 hogsheads of rum, containing 450 gallons, at \$1 per gallon, and sold it at \$1.20 per gallon, on a credit of 3 months. While in my possession, the rum lost 10 gallons by leakage: what did I gain or lose, interest being at 6 per cent.?
Ans. \$70.19 *gain.*

8. A man buys 596 gallons of wine, at 6*sh.* 3*d.* per gallon, cash, and sells it immediately at 6*sh.* 9*d.* per gallon, payable in 3 months, interest at 6 per cent. per annum: what does he gain?
Ans. £11 17*sh.* 8*d.*

9. A distiller is offered 1000 gallons of molasses, at 48 cents, cash, per gallon, or 50 cents, with 2 months credit, the interest being at 8 per cent.: which is the most advantageous bargain, and by how much? *Ans. He will gain* \$13.6 *by cash payment.*

10. If 14 pieces of stuff be bought at \$9.60, and 5 of them

sold at $14.40, and 4 at $12: at what price must the rest be disposed of to gain 20 per cent. on the whole?

11. Sold 10 casks of alum, weighing gross 33*cwt.* 2*qr.* 15*lb.*, tare 15*lb.* per cask: what is the whole amount of sale, at 23*sh.* 4*d.* per hundred weight? *Ans.* £37 13*sh.* 6½*d.*

12. What is the cost of 32 boxes of soap, weighing 31,550 pounds, at 8 cents per pound, allowing 4 pounds per box for draft, and 12 per cent. for tare? *Ans.* $301.211.

13. A man buys 4 hogsheads of tobacco, weighing 38*cwt.* 2*qr.* 8*lb.* gross, tare 94 pounds per hogshead, at $9 per hundred weight, ready money, and sells them at 11¼ pence per pound, allowing tare at 14 pounds per hundred weight. He is to receive two-thirds in cash; and, for the remainder, a note at 90 days. His gain or loss is required, supposing he discounted the note at 60 days, at 6 per cent. per annum. *Ans.* $283.43 *profit.*

LESSON LXXI.

EXCHANGE AND THE REDUCTION OF CURRENCIES.

1. *Exchange* is that mercantile operation by which remittances of money are made from one place or country to another.

It is called exchange because, when the trade is well regulated, the settlement of accounts is made by means of *drafts*, without any actual transmission of money.

Thus: *A*, of Richmond, sells tobacco to *B*, of Liverpool. Now, if *B* were to send, in return, goods for a like amount, it is very clear that their account would be readily balanced on their books, without transmission of money.

But the intricacies of commerce do not allow of so easy a settlement. Goods are imported by different merchants: *C*, of Richmond, for example, may receive some from another person, *D*, of Liverpool. Then he applies to *A* for a draft on *B*, of Liverpool, and sends it to *D*, who collects the amount from his neighbor, *B*. So much of the account between the two places is settled by this draft; and the whole may be so settled unless the balance on either side is considerable; in which case, coin has to be remitted.

Sometimes the operation is more complicated, and drafts have to pass through several hands: *C*, for instance, may find no other channel of remittance than through New York and Paris; that is, he may be compelled to procure, from a merchant in New York, a draft upon a merchant of Paris, which he remits to *D*. The draft then finds its way from Liverpool to Paris, in some mercantile operation between the two places.

2. If an exact balance of accounts between several countries existed at all times, settlements by drafts would always suffice, and nothing more would be requisite than to reduce, in the accounts, the currency of one country to the standard of the other, in order to establish the equality of amounts in the books.

3. But exchange is affected by the increase or diminution of the bills drawn in one country on the other, which, like other commodities, rise or fall according to the demand; that is, when, at any particular time, funds on a particular place are in great demand, there is an advance on their intrinsic value, and the holder of a draft is allowed a *premium*, the limit of which is evidently the cost of conveying the specie from the debtor to the creditor. When, on the contrary, the indebtedness is on the other side, drafts are readily obtained; they are then at a *discount*.

4. The *premium* or *discount* for 100, is the *rate of exchange*.

When there is neither premium nor discount, but an exact intrinsic equivalency exists between the currencies of two countries, exchange is said to be at *par*.

5. The money or currency of a country has, therefore, two distinct values: its *intrinsic value* and its *commercial value*.

The *intrinsic value* is that which a coin has, when compared to a certain standard, according to the quantity of each metal it contains.

Tables of the relative values of coins are prepared in every country, for the use of commercial men, generally in reference to the particular standard of their own country. I will only give, at the end, a table of the coins, the value of which has been fixed by Congress.

The *commercial value* varies with the fluctuations of trade, and is equal to the intrinsic value, with the addition or subtraction of the exchange, according to the current rate at the time.

6. In order, therefore, to transfer an amount of money from one country to another, *the relative intrinsic value* in both and the *rate of exchange* must be combined in a compound proportion. The operation is one of great simplicity.

I. *You change the sum of money into another currency by multiplying it into the intrinsic ratio of the two currencies.*

II. Then, *you add or subtract the amount of exchange, according to the current rate.*

7. The first part of the operation is known as the

REDUCTION OF CURRENCIES.

Example 1st.—Let it be proposed to estimate a sum of £165 9*sh.* 6*d.*, Georgia currency, in Virginia currency.

By reference to the note in Lesson XLVIII., we find that $1 is equal to 6*sh.* Virginia, and 4*sh.* 8*d.* Georgia currency. Hence we have

$$4sh.\ 8d. : £165\ 9sh.\ 6d. :: 6sh. : (£165\ 9sh.\ 6d.) \times \frac{6sh.}{4sh.\ 8d.}$$

That is, we must multiply the given sum by the ratio, $\frac{6sh.}{4sh.\ 8d.}$ or $\frac{9}{7}$, and the answer is, £212 15*sh.* $0\frac{6}{7}d.$

The same might also be obtained in multiplying by the ratio of the respective currencies, expressed in terms of the pounds, reduced to Federal money; which is,

$$\frac{4\frac{2}{7}}{3\frac{1}{3}} = \frac{9}{7}, \text{ as above (see same note).}$$

Example 2d.—If the same amount was to be reduced to Federal money, a simple multiplication by the value $\$4\frac{2}{7}$ of the Georgia pound would suffice:

Observing that $4\frac{2}{7} = \frac{30}{7}$, and reducing the inferior denominations to decimal fractions of the pound (LI., 4), we would get

$$\$\tfrac{30}{7} \times 165.475 = \$709.178.$$

These operations are too simple to need further examples.

EXCHANGE.

8. Exchange may take place between two countries which use the same currency; as, for example, between New York and Richmond.

If, for instance, exchange were 10 per cent. in favor of New York; that is, if \$1 in New York were worth \$1.10 in Richmond, and I wished to place \$1000 in the former place, I must pay \$1,100 in Richmond for a draft.

If, on the contrary, exchange was 10 per cent. against New York; that is, if \$1 in Richmond were worth \$1.10 in New York, New York funds would be at a discount of 10 per cent.; and the draft on New York, for \$1000, would cost only \$909.09, as results from the proportion $1.10 : 1 :: 1000 : \frac{1000}{1.10} = 909.09$.

This is the correct mode of computation; but, generally, the premium is simply deducted in a manner similar to bank discount.

Exchange, when the currencies are different, combines the two preceding operations.

Example 1st.—Exchange on London being at $7\frac{1}{2}$ per cent. premium, I wish to remit £1,120: how much is to be paid here for a draft of this amount?

Ans. $\$4.44 \times 1120 \times 1.07\frac{1}{2} = \$5,345.76$.

N. B.—By the laws of Congress, the pound sterling was reckoned at \$4.44; but, more recently, Congress having lowered the standard of gold coin in the United States, the par or intrinsic value of the pound sterling has been increased. It is now about \$4.85; yet, in computing exchange, \$4.44 is retained for the value of the pound sterling, but a certain percentage is added to equalize the exchange: so that an addition of $9\frac{1}{4}$ per cent., which raises $4.44 to $4.85, establishes par; and exchange at $9\frac{1}{4}$ is therefore considered equal to par.

Frequently the rate of exchange is given by quoting the value of the unit of currency of the foreign country, instead of a percentage. This is the case in regard to exchange upon France, as in the following question:

Example 2d.—The exchange on France being quoted at *francs* 5.35, what should be paid for a draft of 6,519 francs?

This quotation means that, according to the rate of exchange, the dollar is worth 5.35 francs. We therefore divide the amount of francs, 6,519, by 5.35.

Ans. $1,218.50.

Example 3d.—A purchase has been made in Russia, to an amount of 2,480 rubles, by a merchant of Richmond, who has no direct means of making payment. He is obliged to purchase in New York a draft on London, which he remits to St. Petersburg. The exchange on New York is at a premium of $\frac{3}{4}$ per cent.; the exchange on London is at $9\frac{3}{4}$ premium in New York; and, finally, the rate of exchange between St. Petersburg and London is, ruble $2sh.\ 9\frac{1}{2}d.$: what should be the amount of the draft in sterling, and how much should be paid in Richmond?

1*st. Ans.* $2480 \times (2sh.\ 9\frac{1}{2}d.) = £346\ \ 3sh.\ 4d.$
(face of the note.)

2*d. Ans.* $\$4.44 \times 1.0075 \times 1.0975 \times (£346\ \ 3sh.\ 4d.)$ $= \$1{,}699.487$, *to be paid in Richmond.*

Questions.—What is exchange? How is it made? What are drafts? What is rate of exchange? When is it at a premium? At a discount? How many kinds of values has any currency? What is its intrinsic value? Its commercial value? How is currency reduced? How, exchange calculated?

EXERCISES.

1. Change $237.50 to English; New York, New England, and Virginia; Pennsylvania, Georgia currency.

2. Change 6,245 thalers of Prussia to Federal money (see at the end.)

3. Change 4,243 thalers to rubles of Russia.

4. Change \$647.50 to florins of Austria.

5. When exchange on London is at $8\frac{1}{4}$, what must be paid for a draft of £4,670 10*sh.*?

6. Exchange being at francs 5.44, what is to be paid for 10,250.75 francs?

7. New York funds being at a premium of $\frac{3}{4}$ in Richmond, and at a discount of $1\frac{3}{4}$ in New Orleans, what should be given in Richmond to remit \$7,455.75 to New Orleans, in New York funds?

8. Cincinnati funds being at a discount of $1\frac{1}{4}$ per cent. in New York, and New York funds at a premium of $\frac{1}{8}$ in New Orleans, a merchant of Cincinnati directs his correspondent in New York to pay \$4,560 for him in New Orleans: what must he pay in Cincinnati for it?

LESSON LXXII.

COMPOUND INTEREST.

Algebraical formulæ: $s = p\,(1 + r)^t$, and $i = s - p$.

1. Compound interest is that which arises from adding unpaid interest to the principal, and taking interest on the aggregate amount.

The same operation may be repeated succcessively at each period when interest becomes due.

It will be perceived, therefore, that compound interest is calculated by the ordinary rules. The following is an example of the process:

What are the compound interest and the amount of \$500, for 6 years, at 8 per cent. per annum?

OPERATION.

	Principal, 1st year, . .	\$500.
Amount of the	1st year, $500 \times 1.08 = 540$.	principal, 2d year.
"	2d year, $540 \times 1.08 = 583.20$	do. 3d year.
"	3d year, $583.20 \times 1.08 = 629.856$	do. 4th year.
"	4th year, $629.856 \times 1.08 = 680.244+$	do. 5th year.
"	5th year, $680.244 \times 1.08 = 734.664+$	do. 6th year.
"	6th year, $734.664 \times 1.08 = 793.437$,	am't required.

2. *The compound interest is the difference between the*

last amount and the original principal; in this example, $293.437.

The simple interest for the same period would be only $240. The increase, by compounding interest, is only $53.437; but, though compound interest increases slowly, at first, it swells the capital rapidly when the computation is carried to a considerable period.

3. Owing to this rapid accumulation of principal, the exacting of compound interest is forbidden by law.

4. The computation of compound interest is very laborious by plain arithmetical operations, but presents otherwise no great difficulty. Algebra furnishes simpler methods to calculate it. For the convenience, however, of those who are not acquainted with algebra, and the use of logarithms, the following table is introduced here:

TABLE,

Showing the amount of ONE, *for any number of years up to* 30, *at the rates of* 5 *and* 6 *per cent. per annum.*

Years.	5 per cent.	6 per cent.	Years.	5 per cent.	6 per cent.
1	1.050000	1.060000	16	2.182874	2.540351
2	1.102500	1.123600	17	2.292018	2.692772
3	1.157625	1.191016	18	2.406619	2.854339
4	1.215506	1.262476	19	2.526950	3.025599
5	1.276281	1.338225	20	2.653297	3.207135
6	1.340095	1.418519	21	2.785962	3.399563
7	1.407100	1.503630	22	2.925260	3.603537
8	1.477455	1.593848	23	3.071523	3.819749
9	1.551328	1.689478	24	3.225099	4.048934
10	1.628894	1.790847	25	3.386354	4.291870
11	1.710339	1.898298	26	3.555672	4.549382
12	1.795856	2.012196	27	3.733456	4.822345
13	1.885649	2.132928	28	3.920129	5.111686
14	1.979931	2.260903	29	4.116135	5.418387
15	2.078928	2.396558	30	4.321942	5.743491

The following are added here, merely to show the rapid increase of compound interest, for long periods and higher rates.

50	11.467392	18.420147	200	17,292.51	115,125.84
100	131.501	339.302	300	2,273,982.75	39,062,430.21

This table can easily be extended: *To find the amount for any number of years greater than the number in the table, multiply together the amounts for two numbers of years, whose sum is equal to the period for which the amount is required.*

Thus, 50 may be obtained by multiplying the amount for 25 years by itself, or that for 24 and 26 together, or any other two amounts whose periods together make 50.

5. The use of this table is very simple: *In order to find the amount for any sum of money for a number of years, multiply* the amount standing *against the period by the given principal;* for it is evident, from the manner of obtaining the amount, that it is proportional to the principal.

6. Questions like those given in simple interest, may also be proposed in compound interest; such as finding the *rate, time,* &c. But they require a knowledge of algebra and logarithms, and are too complicated for arithmetic.

7. Though compound interest is not allowed by law, it is frequently important to calculate it, as a basis for speculations, contracts, bargains, notes, investments, the value of stocks, the cost and character of constructions, especially public works, in reference to their duration, &c.

8. Although I must refer to algebra, and recommend its study for a complete knowledge of this subject, I will, in the following lessons, give a few examples of its applications to the investigation of certain transactions.

It is frequently desirable to know when a capital will double itself, at a certain compound interest. The following method is simple and sufficiently correct for practical purposes, at the usual rates of interest:

The number of years in which a capital is doubled, multiplied by the rate of interest, makes nearly 72. Thus, at 5 per cent., it will double in about 14 years; at 6, in about 12, &c.

Questions.—What is compound interest? How is it computed? Does it raise the amount rapidly? How is the table used? Is it allowed by law? What questions may be proposed in compound interest? Can they all be solved by arithmetic? Though compound interest cannot be exacted, is it useful to consider it?

In what cases? How will you ascertain when the capital will be doubled, at compound interest?

EXERCISES.

1. What is the amount and compound interest of \$629, for 7 years, at 6 per cent.? *Ans. Amount*, \$945.78. *Interest*, \$316.78.

2. What is the amount and compound interest of \$1,256, for 15 years, at 0.06 per annum? *Ans. Amount*, \$1,754.066. *Interest*, \$498.066.

3. What is the amount of \$12,500, for 6 years, at 5 per cent.? *Ans.* \$16,751.20.

4. What is the compound interest of £246 14*sh.* 6*d.*, for 3 years, at 6 per cent.? *Ans.* £47 2*sh.* 6*d.*

5. What is the compound interest of £760 10*sh.*, for 4 years, at 0.06 per annum? *Ans.* £199 12*sh.* 2*d.*

6. What is the compound interest of £370, for 6 years, at 4 per cent.? *Ans.* £98 3*sh.* 4¼*d.*

7. What is the amount of \$76.75, for 2¾ years, at 0.03 per annum? *Ans.* \$83.257.

8. What is the amount of \$217, for 2¼ years, at 5 per cent. per annum, the interest payable quarterly? *Ans.* \$242.669.

9. The population of a country is 10,000, and increases at the rate of 0.01 per year: what will it be in 10 years? *Ans.* 11,046.

LESSON LXXIII.

COMPOUND DISCOUNT.

Algebraical formulæ: $p = \frac{s}{(1+r)^t}$. $d = s\,(1 - \frac{1}{(1+r)^t})$

1. It has already been said that, though the law does not allow of compound interest, it is, nevertheless, frequently necessary to compute it, with a view to ascertain the propriety or relative advantages of various transactions.

The fundamental principle of all computations where compound interest is introduced is, that the holder of the money has the power, by retaining it, to invest and re invest successively its proceeds, and thus to compound interest; and that *he would not consult his own advantage by entering into any agreement which would not place him, at the end of a fixed time, in the same situation as if he had retained the use of his funds.*

Let us suppose, for instance, that some property has been sold for \$18,000, payable in 10 years, and that the seller should wish to have it discounted, the rate of interest being 8 per cent.

The purchaser would naturally consider that, by the rule of simple discount, he would have to pay \$10,000 cash; whereas the 10,000 dollars, if retained in his hands, would, by the accumulation of semi-annual interest for 10 years, reach to an amount of . \$21,911.22.

He must, therefore, decline an arrangement at simple interest, by which he would be a loser, in the end, to the amount of . . . . \$3,911.22.

By the computation of *compound discount*, however; that is, *by finding the principal which, at semi-annual compound interest (the usual period for dividends) would produce the amount to be paid*, the purchaser would ascertain that the sum he may pay, without either loss or profit, for \$18,000, due in 10 years, at a rate of 8 per cent., is only \$8,214.96

That is, as much as . . . \$1,785.04 less than the principal resulting from the computation by simple discount. A transaction upon such a basis is the only one which would do justice to both parties.

2. Hence, *compound discount is the only correct mode of calculation*, especially for long periods. As regards short periods and small sums, the difference is generally too inconsiderable to be noticed; it would only embarrass ordinary transactions.

3. Compound discount is calculated in a manner similar to simple discount.

I. *To find the present value, or principal, divide the whole amount by the compound amount of* ONE *for the given time.*

II. *Subtract the result from the given amount; the difference will be the compound discount.*

4. Questions like the following belong to compound discount, and show some of its applications:

1. What is the present value of $30,000, payable in 7 years, at 6 per cent. interest? *Ans.* $19,951.713.

2. A man proposes to buy land for $4,587, payable in 7 years: to what cash price is it equivalent, interest being at 5 per cent.? *Ans.* $3,259.897.

3. How much should be invested at present, at 4 per cent. per annum, to make up £569 6*sh.* 8*d.*, to be paid in 9 years? *Ans.* £400.

LESSON LXXIV.

ANNUITIES.

1. An annuity is a sum of money payable at regular periods, either for a limited time or for ever.

2. *The amount of an annuity*, forborne for some time, is the sum of all the payments due, with the addition of the interest on each.

3. The *present value* of an annuity is that sum which, being properly invested, will be exactly sufficient to pay the annuity.

4. *Interest on annuities* may be calculated *at simple* or *at compound rates.*

ANNUITIES AT SIMPLE INTEREST.

Algebraical formula: $s = a\left(t + \frac{rt(t-1)}{2}\right)$ { s being the final amount; a the annuity; t the time; r the rate.

5. *Example.*—What is the amount of an annuity of $500, unpaid for 5 years, at 5 per cent. per annum?

OPERATION.

1st. Amount of $500, for 4 years, . . .	$600
2d. " " for 3 years, . . .	575
3d. " " for 2 years, . . .	550
4th. " " for 1 year, . . .	525
5th. " " just due, ...	500
Total amount, ...	$2,750

This operation is too simple to require further explanation.

6. As regards the present value of an annuity, when only simple interest is allowed, most writers on the subject have defined it to be: A sum such that, if put out at interest, at the rate allowed, its amount, at the end of the time of its duration, will be the same with the amount of the annuity.

This view of the question, though supported by even able mathematicians, is undoubtedly erroneous: for, each payment of the annuity does not differ in any way from the payment of a note, subscribed beforehand, for the same amount; and certainly no one would think of estimating the present value of several notes, by computing their amount to a period beyond their maturity, and then discounting this amount instead of the notes themselves: for, such a calculation would evidently introduce the discount of the interest added, and consequently some compound interest.

Take this simple example: Two notes, of $100 each, are due in one and two years: what is their present value? It is clear that, by the above method, the first note, at the end of the second year, would amount, at 6 per cent. interest, to $106, and that, in discounting this amount, we would include also the discount of the interest, 6, which is added to the note.

7. The correct way is evidently to discount each payment separately, as would be done for notes: for, it is obvious that the aggregate present value of all the payments is the sum of the present value of each.

It is but rarely, however, that such an estimate can be useful, it being erroneous to value annuities otherwise than with compound interest.

ANNUITIES AT COMPOUND INTEREST.

Algebraical formulæ: $s = a \cdot \frac{(1+r)^t - 1}{r}$ $p = \frac{a}{r} \cdot \frac{(1+r)^t - 1}{(1+r)^t}$

8. Computations relative to annuities at compound interest, are very laborious, by the common rules of arith-

metic. It is then necessary to calculate the compound interest for each year, separately, as in the following operation, which is applied to the preceding example:

1st.	Amount of	$500,	at 5 per	cent.,	for 4 years,	.	$607.75
2d.	"	"	"	"	for 3 years,	.	578.81
3d.	"	"	"	"	for 2 years,	.	551.25
4th.	"	"	"	"	for 1 year,	.	525.00
5th.	"	"	"	"	just due,	.	500.00
	Total amount,					.	$2,762.81

Algebra furnishes means to get as readily the amount and present value for long as well as for short periods.

9. But, with the table of Lesson LXXII., these calculations may be made with tolerable facility as follows: I. *To get the total amount of an annuity, find, in the table, the compound amount of* one *for the given time; deduct a unit from it; divide by the rate, and multiply by the annuity.* II. *To get the present worth, find the amount of the annuity, and divide it by the compound amount of* one *for the time.*

Thus, in the above example, the compound amount of ONE for 5 years, less 1, being 0.276281; this, divided by the rate 0.05, and multiplied by 500, gives 2,762.81 for the total amount of the annuity, as we had found it.

For the present worth, divide this total by the compound amount 1.276281, and you get 2,164.81.

10. The present value of an annuity which is to continue for ever, is evidently the principal whose annual interest is equal to the annuity. (Lesson LXVII., 6.)

Tables of the amounts of an annuity of ONE, and also of its present value, are met with in several arithmetics; but for the purposes of this work, the table of the compound amount of ONE will be sufficient, and may be used as just explained; the tables of annuities being very rarely wanted in practical transactions.

EXERCISES.

1. What will an annuity of $800 amount to, at 5 per cent. interest, in 25 years? *Ans.* $38,181.679.

2. A person wishes to purchase an annuity which shall give him, at 6 per cent., an income of $500, for 10 years: how much must he pay for it? *Ans.* $3,680.04.

The following questions belong also to annuities and compound discount, and are very common applications of them. The general principle which must guide in solving them, is to compare the amounts produced at the end of the stipulated times:

3. What is the intrinsic value of government stock, paying semi-annually 5 per cent. interest, and redeemable in 20 years, when common interest is at 6 per cent.?

The intrinsic value must produce, at the compound interest of 6 per cent., an amount equal to the capital invested, together with the accumulation of interest at 6 per cent. on the 5 per cent. dividends.

4. What is the comparative value of stock bearing 4 per cent. interest, redeemable in 15 years, and stock bearing 6 per cent., redeemable in 25 years, when common interest is at 5 per cent., both dividends paid quarterly?

5. What annuity should be laid aside, to rebuild a structure which has cost $25,000, and is estimated to last 15 years, interest at 6 per cent.?

6. What annuity should a man pay, who insures his life for $40,000, interest being at 5 per cent., and his probability of life 15 years; so that the company may clear 2 per cent. on the amount they will have then to pay?

7. What annuity should be paid to a man for an amount of $35,000, given up by him, his probability of life being 12 years, and interest at 6 per cent.?

8. How much should a man pay at present, so as to receive $1,500 at the end of every year, for 12 years, the interest being reckoned at $7\frac{1}{2}$ per cent.? *Ans.* $11,602.91.

9. What should a man buy the reversion of a lease of 20 years, to pay him $100 a year, so that he may make 8 per cent. per annum on his money? *Ans.* $981.815.

10. What is the comparative advantage, as regards ultimate expense, of a wooden and of a brick house, the wooden house costing $2,000; the brick house $2,500; and there being in the frame house $120 of exterior framing, to be renewed every 20 years; $150 of weather-boarding, to renew every 8 years; $60 of additional painting, to renew every 5 years; and, finally, the insurance of the wooden house being $1\frac{1}{4}$ per cent.; that of the brick house $\frac{3}{4}$ per cent.?

LESSON LXXV.

EQUATION OF PAYMENTS.

1. The object of this rule is to find the *mean* or *equated time* of payment of several sums, due at different times; so that, by paying the aggregate amount of all the notes at that time, there may be no loss of interest to either party.

RULE.

I. *Find, by simple discount, the present worth of each note.*

II. *Then find in what time the aggregate principal thus calculated will, by the addition of interest, become equal to the sum total of all the notes.*

This will be the *true equated time* for the payment of the whole.

To illustrate this rule by a simple example, let us suppose that, the rate of interest being 6 per cent.,

$106	are due	in one year;
112	"	in two years;
118	"	in three years;
124	"	in four years;
130	"	in five years;

and that it is proposed to reduce them all to one note of the aggregate amount, $590, payable at a time such that neither party shall lose interest:

It is clear that the present value of each note, in this simple case, is $100; and that, consequently, $500 ready money should cancel all obligations.

Now, the question is, when will $500 become $590, by the addition of interest? The interest being $90, all we have to do is to find the time, by Case III., of Lesson LXVII.; the answer will be

$$\frac{90 \times 100}{6 \times 500} = 3 \text{ years.}$$

When simple interest alone is admitted, this is the correct and equitable method to calculate the equated payment, though it is not the rule usually adopted among merchants, which is as follows:

Multiply the amount of each note by the time it has to run, and divide the aggregate of the products by the whole amount due; the quotient is taken for the mean time.

That this rule is incorrect, will readily appear by reference to the preceding example: for, by multiplying each sum by its time, we get

$$\begin{aligned} 106 \times 1 &= 106 \\ 112 \times 2 &= 224 \\ 118 \times 3 &= 354 \\ 124 \times 4 &= 496 \\ 130 \times 5 &= 650 \\ \hline & 1830 \end{aligned}$$

And 1830, the sum of all the products, divided by 590, gives $3\frac{6}{59}$ years for the equated time; by which the lender loses, on the present value, \$2.57; since the whole amount, \$590, discounted for $3\frac{6}{59}$ years, gives only \$497.43.

Again, by this difference, he would be in possession, at the end of the second time, of \$3.60 less than by the other mode of computation; since \$3.60 is the interest on \$590 for $\frac{6}{59}$ of a year.

The only recommendation in favor of the customary rule, seems to be its greater simplicity; but this consideration is hardly sufficient to induce the lender to submit to a loss which may be considerable for large amounts and distant payments.

2. Indeed, neither of these rules does justice to both borrower and lender. The consideration of compound interest is the only mode by which this end could be readily attained: for, it is evident that the lender, having the right to insist upon payment at the time each note falls due, and having, from that time, the management of his money, may, by proper investments, obtain the inte-

rest of every sum which comes into his hands, and that nothing can prevent his thus compounding interest.

3. Let us now apply these considerations to the same example.

According to the common rule for the equation of payments, the lender receiving his money in $3\frac{6}{59}$ years, would, by the accumulation of interest, in $1\frac{53}{59}$ years, be in possession, at the expiration of 5 years, of $658.708.

Whereas, if he had received his money for each note, when due, the following result would have been produced, at the expiration of 5 years, by the accumulation of interest:

By the	1st note,	of $106,	the amount	$133.823
"	2d "	of 112,	"	133.394
"	3d "	of 118,	"	132.585
"	4th "	of 124,	"	131.440
"	5th "	of 130,	"	130.000
	Total amount,			$661.242

Being a difference of $2.534 for the small amount of $500.

4. These calculations show that, when compound interest is not allowed, a person who consents to equating payments by the common rule, may subject himself to a considerable loss, on large sums.

The method by discount, which has been given at the beginning of this lesson, is more nearly correct.

Questions.—What is equation of payments? What is the correct method, at simple interest? What method is generally in use? Which of the parties does the latter favor? Is either of these methods mathematically correct? What is the true and equitable method? Explain it.

EXERCISES.

1. *A* owes *B* $2000, whereof $400 are to be paid in 3 months; $600 in 5 months, and the remainder in 10 months: at what time may the whole be paid, without injustice to either?

Ans. By the usual rule, 7*mo.* 3*da.*

2. £1,200 are due, $\frac{1}{2}$ in 3 months; $\frac{1}{4}$ in 6 months, and $\frac{1}{4}$ in 9 months: what is the equated time of payment for the whole?

Ans. 5*mo.* $7\frac{1}{2}$ *da.*

3. *C* owes to *D* $1,400, to be paid in 3 months; but *D* being in want of money, *C* pays him, at the expiration of 2 months, $1,000: how much longer than 3 months ought *C*, in equity, to defer the payment of the rest? *Ans.* $2\frac{1}{2}$*mo.*

In the preceding examples, the times are so short that the method by discount would make but little difference. Let the pupil verify it.

4. There fell due,	on January 13th, . . .	$700
" "	on February 15th, . .	1,300
" "	on March 6th, . . .	4,500
" "	on March 21st, . . .	2,200
" "	on May 10th, . . .	5,620
" "	on July 17th, . . .	2,410
		$16,730

From what equated time should interest, at 6 per cent., be charged on the whole amount, the settlement taking place on the 1st of September? *Ans. By the common rule,*
By discount,

5. An individual has bought an estate for $90,000, payable in 15 equal annual instalments, the first of which in 12 months; the parties agree to reduce the fifteen payments to one note: at what time should it be made payable—

Ans. 1*st, by the common rule,* . 8 *years;*
2*d, by simple discount,* . 7.219;
3*d, by compound interest,* . 7.46.

What is the loss or gain to the purchaser by each of these methods?

LESSON LXXVI.

PARTIAL PAYMENTS.

1. It frequently happens that partial payments are made at different times, on bonds and notes bearing interest. The question then occurs: How is a final settlement to be made?

The leading principle of such transactions is, I believe, in every state, that interest is not to be compounded; and, accordingly, usage and the decisions of the courts of law have established rules for the purpose, which vary in different places. Whatever these rules may be, they are but simple applications of what precedes.

2. The rule most commonly in use, and which has

especially been sanctioned by the courts of Massachusetts and Virginia, is as follows:

Deduct each successive payment (if it exceed the interest) *from the amount of both the principal and interest due at the date of such payment, and take the remainder for a new principal; upon which perform a similar operation at the next payment; and so on, until the time of final settlement.*

But, if the partial payment be less than the interest, do not deduct it, but merely reserve it, to be added to subsequent payments, until the aggregate sum paid in exceeds the interest up to the last payment, when the whole is deducted, as above.

Example.—A note, dated January 1st, 1830, promises to pay $10,000 in 6 months, with interest thereon, from date, at 6 per cent.

This note is not paid at maturity, but partial payments are made, which are endorsed on the note, as follow:

Received,	April 1st, 1830,	$240
"	August 1st, 1830,	40
"	December 1st, 1830,	60
"	February 1st, 1831,	600
"	July 1st, 1831,	400
"	June 1st, 1834,	3,000
"	September 1st, 1834,	120
"	January 1st, 1835,	150
"	October 1st, 1835,	500

Judgment is to be entered on the 1st of December, 1840: for what amount should it be rendered?

According to the preceding rule, the answer is $10,619.62. The operations are too simple to be exhibited here; they are left for an exercise.

3. *When several notes are due, the sums paid are applied to the payment of the first; and the others are considered in their order, after the principal and interest have been settled in full.*

4. The practice in Connecticut differs from the above, in the following particulars:

1st. All sums paid, whether large or small, are deducted from the amount of principal and interest.

2d. When the remainder exceeds the preceding principal, interest is computed only on this principal.

3d. No interest is reckoned for less than one year. If payment be made within a less period, interest is added to both this payment and the principal, to the end of the year, and the difference between them taken.

This rule avoids compound interest more completely than the first, which is not altogether free from it.

5. Among merchants, in open accounts, interest is frequently added, on both sides, to the sums paid and received up to the day of settlement.

6. But, after all, no rule which is not based upon compound interest, can be strictly correct or regular in its application. The above rule is an example of this; for, it will be readily perceived that the aggregate amount ultimately received will depend on the partial payments made; and, above all, on the periods at which they are made. This should not be the case; and, whatever be the amount equitably due, it should be received in full, and the same in every way the payments may be effected.

If compound interest were not rejected, this irregularity would not exist. The rejection of it is mathematically erroneous, and especially unfavorable to the lenders of money and holders of notes; for, it plainly amounts to preventing a man, who is kept out of his money, from obtaining the advantages which would have accrued to him, had he retained the management of it.

7. Hence, as a natural consequence, must result, on the part of the debtor, a disposition to delay payment, and, on the part of the creditor, less indulgence towards his deserving debtor: for, returning to the above example, it will be easily ascertained that the person who had a right to exact payment on the 1st of July, 1830, if he had received it then, might, by successive reinvestments of the proceeds, even only every six months, have in-

creased the amount, in nine years and eleven months, to $17,972.40

Whereas, by the common rule, he would be allowed, at the end of the time, if no payments had been made, only . . . $15,950.00

Surely, it will not appear just, that the debtor should be benefited to so great an amount, while the creditor is kept out of his funds; and such a result will prompt the latter to insist upon payment.

If it be considered, as the table in Lesson LXXII. shows, that compound interest does not accumulate rapidly at first, would it not be preferable to allow it for a limited period; ten years, for example? The policy of limitation might be applied, in this case, with as much propriety as in others.

Questions.—How are partial payments generally settled? Is this rule just and correct? Is it favorable to the creditor or the debtor? What is its natural tendency? What is the only way by which uniform results can be obtained?

EXERCISE.

Lynchburg, Va., June 20th, 1841.

For value received, I promise to pay Henry Johnson or order, on demand, three thousand eight hundred and forty-nine dollars and thirty-five cents, with interest at 6 per cent. per annum.

JOHN HAYS.

$\$3,849\frac{35}{100}$.

On this note are the following endorsements:

March 4th, 1842: Received three hundred and seventy-seven dollars and fifty cents.

June 14th, 1843: Received eight hundred and twenty-five dollars.

September 18th, 1843: Received eight hundred and five dollars.

June 24th, 1844: Received two hundred and thirty-six dollars and twenty-five cents.

March 8th, 1845: Received sixty dollars and eighty-five cents.

December 9th, 1845: Received four hundred and ninety dollars.

July 10th, 1846: Received eight hundred and forty-five dollars.

What remains due on September 28th, 1847? *Ans.* $1,131.485.

CONCLUSION.

I HAVE now completed what constitutes pure arithmetic. Indeed, some parts of what precedes, will be much better understood with the assistance of algebra.

I omit several subjects which do not appear to belong properly to arithmetic, because it is incompetent to teach them thoroughly.

In the first place, *the rules of position*, in arithmetic, are indefinite and complicated; whereas, they are of the simplest order of questions in algebra, by which they can be solved with great ease, after a few lessons in this science.

As regards *involution* and *evolution*, particularly the extraction of *square* and *cube roots*, and also *progressions*, they are applied chiefly to questions in geometry and algebra; and, therefore, belong properly to the latter science, an elementary course of which is the shortest way to learn them, and, indeed, to obtain a thorough knowledge of arithmetic. Such a course will ultimately prove a saving of time.

The same thing may be said of *mensuration*, which should be preceded by the study of geometry, and learned in special treatises. The introduction of this subject in arithmetics, will rather delay than accelerate the progress of the student.

For similar reasons, I omit *book-keeping* and all that relates to *mercantile forms*. There are excellent special and comprehensive works on these subjects, which, for study and reference, are far preferable to the necessarily limited articles to be found in arithmetics.

To crowd so much incomplete and imperfect knowledge in elementary works, appears to me calculated to do more harm than good, by deceiving the student into a belief that he has really learned that which he cannot well understand, and still less apply practically.

Should this small volume be favorably received, it is my intention to go on with a complete condensed course of mathematics, confining myself altogether to what is practically useful.

APPENDIX.

OF COINS.

The purity of gold is generally estimated, not by our weights, but by an Abyssinian weight, called *Carat*.

The carat is subdivided into 4 *grains*, and these into *quarters*.

A carat is equal to $2\frac{1}{2}$ pennyweights: pure gold is 24 carats fine.

English standard gold is 22 carats fine, and 2 carats alloy, which is copper.

In *the United States*, the standard of both gold and silver has been simplified by the law of the 18th of January, 1837, which makes it . . . . $\frac{9}{10}$ of pure metal,
and . . . . $\frac{1}{10}$ of alloy.

In gold coins, the alloy consists of pure silver and copper; the silver never to exceed one half of the alloy.

The weight of the eagle remains as before, 258 grains; of which 232.2 is pure gold.

That of the dollar, ... $412\frac{1}{2}$ grains; of which 371.25 is pure silver.

And that of the cent, ... 168 grains of copper.

This new standard is also that of *France*. So that the comparative value of the coins of the two countries is now in the exact proportion of their weights.

By a subsequent law of March 3d, 1853, in consequence of the great influx of gold from California and Australia, and the appreciation of silver, the weight of the fractions of the dollar were reduced as follows: the half dollar from $206\frac{1}{4}$ to 192 grains; the quarter, dime, half dime, and three cent piece in proportion. No silver dollar is now coined.

CURRENCY OF ENGLAND.

The mint or standard price of gold in England is, per pound, troy weight, ... £46 14*sh.* 6*d.*

A pound of standard silver contains 11*oz.* 2*dwt.* of silver,
and . . . . 0 18 of copper.

The proportional value of silver and gold is as $15\frac{1}{4}$ to 1.

The sovereign of England contains, of pure
gold, ... 113.001 grains;
Of standard gold, ... 123.274 "

So that, compared to the present standard of the United States, it is intrinsically worth . . . . . . $4.85.

Though, by the latest law of Congress on the subject (March 3d, 1843), it is made . . . . $4.859.

So that, to the former value of the pound sterling, $4.44, fixed by the law of Congress, of the 31st of July, 1789, and continued for many years, there must now be added 9¼ per cent., to give it its present value in United States gold coin.

Hence, exchange on England at 109¼ per cent., is now considered *par*, among mercantile men, who reckon still by the old value, $4.44, of the pound sterling.

The English shilling contains . 80.727 grains of silver, and . . . . 87.27 of standard silver.

Consequently, compared to the new standard in the United States, the value of the silver shilling is . . 21¾ cents.

And the dollar is worth, intrinsically, 4 silver shillings and 7⅕ pence, though it is generally reckoned at 4 shillings and 6 pence; while the shilling, as the twentieth part of a sovereign, is 24½ cents.

There is in all this a discrepancy, which affects exchange unfavorably to the United States; the standard of which is silver, while that of England is gold.

CURRENCY OF FRANCE.

The standard of French coin is, like that of the United States, 0.9 pure metal and 0.1 alloy.

155 gold pieces, of 20 francs, weigh 1 kilogramme, equal to . . . . 2.68027*lb*.

This makes the single piece of 20 francs, weighing 6.45161 grammes = 99.601 grains, worth . . $3.86.

One silver piece, of 2 francs, weighs 1 decagramme = 154.3835*gr*.
" of 5 francs " 2.5 " = 385.95888*gr*.

Hence, the intrinsic value of the 5 franc piece is . $0.93573, and of the United States dollar . . . 5.343 francs.

The law of March 3d, 1843, makes the 20-franc piece $3.855, and the 5-franc piece . . . . 0.93.

It will be remarked here, also, that 4 five-franc pieces are only $3.74, instead of the equivalent value, 3.86, of a gold piece of 20 francs; showing that in France, as well as in England, the proportional rates of gold and silver differ from that adopted in the United States.

VALUE OF FOREIGN COINS MADE RECEIVABLE BY CONGRESS.

The Law of March 3d, 1843,

Fixes the value of the gold coins—

Of Great Britain, of not less than	0.915½ in fineness, at	$0.946 per *dwt*.
Of France, of not less than	0.899 " at	0.929 "

This makes the Sovereign of England worth	$4.859
the Guinea,	5.102
and the Napoleon, or 20 franc piece,	3.855

It also fixes the value of the following silver coins:

The Spanish Pillar Dollar, The Dollar of Mexico, The Dollar of Peru, and The Dollar of Bolivia,	when not less than 0.897 in fineness, and 415 grains weight,	at 100 cents.

The Five-franc Piece, of not less than 0.9 in fineness, and 384 grains in weight, at	$0.93
Milreis of Portugal,	1.12
Rix Dollar of Bremen,	0.78¾
Thaler of Bremen, of 72 grotes,	0.71
Milreis of Madeira,	1.00
Milreis of Azores,	0.83⅓
Marcbanco of Hamburg,	0.35
Ruble of Russia,	0.75
Rupee of British India,	0.44½

By the law of May 22d, 1846, for computation at the custom house,

The dollar of Norway and Sweden,	$1.06
The dollar of Denmark,	1.05
Thaler of Prussia and northern States of Germany,	0.69
Florin of southern States of Germany,	0.40
Florin of Austria and of Augsburg,	0.48½
Lira of Lombardo-Venitian Kingdom,	0.16
Franc of France and Belgium; Lira of Sardinia,	0.186
Ducat of Naples,	0.80
Ounce of Sicily,	2.40
Pound of Nova Scotia, N. Brunswick, New Foundland and Canada,	4.00

These laws repeal every part of former laws, inconsistent with them; and, consequently, the law of July 27th, 1842, which had fixed the value of the pound sterling at $4.84.

It leaves in force, of the law of June 28th, 1834, the rate of

The gold of Portugal and Brazil, of 22 carats, at $0.948 per pennyweight.
" Spain, Mexico, and Colombia, of 20 carats 3 7-16 grains, at 0.899 "

Calculated by these rates, the following is the legal value of the coins

Of Portugal—	Dobraon, of 24,000 Rees,	$32.706
	Dobra, of 12,800 Rees,	17.301
	Johannes,	17.064
	Moidore (half in proportion),	6.557
	Milree,	0.78
	Piece of 16 testoons, or 1,600 rees,	2.121
	" of 12 testoons, or 1,200 rees,	1.574
	" of 8 testoons,	1.12
	Old Crusado, of 400 rees,	0.588
	New Crusado, of 480 rees,	0.637
	New Dobra,	16.253
	Johannes (double in proportion),	8.763
	" (half in proportion),	4.371

Of Brazil—Dobraon, Dobra, Johannes, Moidore, New Crusado, as in Portugal.

Of Spain—	Quadruple Pistole or Doubloon, of 1772; double and single, and also shares in proportion,	$16.038
	Doubloon, 1801,	15.535
	Pistole, 1801,	3.884
	Coronilla, Gold Dollar, or Vintern, 1801,	.983

Of Mexico and Colombia—Doubloons; shares in proportion, 15.535

No other coin is legal tender.

A great many laws have been passed by Congress, in regard to the currency, since 1789; but the above only are now in force.

APPLICATION OF THE RULES OF DIVISIBILITY TO PROVE MULTIPLICATION AND DIVISION.

1. In Lesson XXXVI., 2, it has been said that a number which divides one factor, divides also the product.

Hence, *if the multiplicand or multiplier contain a certain factor, verify if the product is divisible by the same.* If it is not, you have made some mistake in the operation. If it is divisible, there is much probability that the operation is correct, though it is not a positive proof.

2. In Division,

I. *If a factor exists in the dividend, and either in the quotient or in the divisor, it must be found in the remainder.*

II. *If there is a factor in the divisor or in the quotient not contained in the dividend, subtract the remainder from the dividend, and the difference must be divisible by this factor.*

These are mere tests, but not certain proofs.

3. The most usual application of the divisibility of numbers, however, is in the

PROOF OF MULTIPLICATION BY CASTING OUT NINES.

Let us suppose that we have multiplied 2,653,294 by 872, and found the product 2,313,672,368. In order to test the correctness of the operation:

Find, by casting out nines, the remainder of each factor divided by 9 (XXXIV., 8); *multiply the two remainders together; cast the nines out of this product; the remainder of this operation must be the same as that of the given product, after casting out the nines.*

Thus, in the above example, the remainder of the multiplicand, divided by 9, is . . .	4
of the multiplier	8
Their product is 32; the sum of whose figures is the remainder, which is the same as that of the given product.	5

Hence we conclude that the operation is very probably correct. But this is not a positive proof; for, after all, it merely establishes the equality of the remainders, which may exist between numbers entirely unconnected, and may occur in an incorrect multiplication, by errors compensating each other. This is, nevertheless, a convenient test, and which may be generally relied on, with a usually correct calculator.

Demonstration.—The reason of this rule is obvious; for, both the multiplicand and the multiplier are each composed of two parts; namely: a multiple of 9 and the remainder. Consequently, in multiplying them together, the product of the two remainders is the only part of the general product not divisible by 9. Therefore, when nines are cast out of it, there must be left the remainder of the whole product, divided by 9.

Remark that, in taking the sum of the figures of a number, with a view of casting out nines, you need not introduce any part evidently divisible by 9: (XXXVI., 6.) Thus, in the above multiplicand, we sum up only 2 + 2, because 6 + 3, 5 + 4, and 9, are each divisible by 9; in 872, we reject 72 for the same reason, and get at once the remainder, 8; finally, in the product, 2,313,672,368, we reject 36; 72; 36, and sum up only 2 + 3 + 1 + 8, or even 2 + 3, since 1 + 8 = 9.

PROOF OF DIVISION.

Division may also be tested by casting the nines out of the divisor and quotient, and of the dividend, which is their product.

If there is a remainder, what is left of it, after the nines have been cast out, must be added to the remainder of the product of the divisor and quotient, and the nines cast out of the sum, if necessary; then the final remainder must be equal to that of the dividend.

EXAMPLE.

14,716,925, divided by 5,375, gives a quotient 2,738, and a remainder 175.

Now, the product of the remainders of the divisor and quotient leaves	4
and the casting out of the nines out of the remainder leaves . . .	4
which, added to the above, gives	8

Which is also the remainder of the dividend, after casting out nines.

N. B.—This test fails altogether in an exact division, when the divisor is divisible by 9; since, in that case, any number in the quotient, multiplied by the divisor, would give a product divisiblo by 9.

ABBREVIATED METHOD OF APPROXIMATION IN THE MULTIPLICATION OF DECIMALS.

Decimals are most frequently used to make calculations on numbers that have been obtained by observations of some kind, such as measuring, weighing, &c.; and, since we seldom can depend on the accuracy of these observations, to within some small fraction, it is evidently needless to carry the calculation to a degree of accuracy greater than the observations or measurements themselves.

In operations with decimals, therefore, it is frequently unnecessary to carry the result beyond a limited number of decimal places. The operations of multiplying decimals may be much shortened in such cases.

For example, to multiply 2.753 by 2.313, carrying the product to only three places of decimals, we may proceed in three different ways.

OPERATIONS.

I.	II.		III.
2.753	2.753		2.753
2.313	2.313		2.313
8259	5506		5506
2753	825	9	826
8259	27	53	28
5506	8	259	8
6.367689	6.367	689	6.368

The first multiplication is in the usual way; each successive product is shifted one step to the left.

The second is in reversed order; each partial product is shifted one step to the right.

In the third, which is the shortened form, we proceed as in the second, only we never write any digit to the right of the column of the last required decimal place; we are only careful to add *one unit* to the extreme digit when the next appears to be 5 or upwards. This shortened process is particularly convenient when a great number of decimals is to be multiplied.

N. B.—Some arithmeticians invert the figures of the multiplier, but the method appears to be more liable to error.

Decimal division is susceptible of a similar abbreviation; but it requires a great deal more care to guard against mistakes. On this account, and because its applications are rare, it is unnecessary to notice it here.

PREPARATORY TABLES IN MULTIPLICATION AND DIVISION.

In both these operations, when the numbers are large, it is often convenient to make a special table of successive products, from 1 to 9. This is particularly advantageous in decimal divisions, carried to a great number of places.

Let us suppose, for instance, that we have to divide 453,994,781.2346 by 73,809, to six decimal places: it would then be convenient to prepare the following table of products:

By 1,	73809	
2,	147618	
3,	221427	
4,	295236	
5,	369045	
6,	442854	
7,	516663	
8,	590472	
9,	664281	
10,	738090	

Every one of which may be used as it is wanted, as exhibited in the annexed operation. This preparatory arrangement has the additional advantage, that it gives at once, without trials, the exact figure of the quotient for each partial dividend.

N. B.—In preparing such a table, it is well to extend it to 10, as a verification.

OPERATION.

```
453994781.2346 | 73809
442854         |-------------
-------        | 6150.940687+
 111407
  73809
 ------
 375988
 369045
 ------
   694312
   664281
   ------
    300313
    295236
    ------
     507746
     442854
     ------
      648920
      590472
      ------
       584480
       516663
       ------
        67817
```

THE END.

THE END.