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FACULTY WORKING PAPER NO. 1473

Capital Deepening and the Rate of Interest

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July 1988

Capital Deepening and the Rate of Interest

Hans Brems, Professor
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CAPITAL DEEPENING AND THE RATE OF INTEREST

By HANS BREMS

Abstract

Using the same method and notation, the paper simulates the Böhm-Bawerk model of circulating capital and the Åkerman -Wicksell model of fixed capital. They have three things in common. First, technology was assumed to be stationary. Second, a lower rate of interest would always lengthen the time span of capitalist production. Indeed, with constant-elasticity production functions the interest elasticities of both Böhm-Bawerk's optimal period of production and Åkerman-Wicksell's optimal useful life are shown to be <u>minus</u> one. Third, with its lengthening time span, a wealthier economy would enjoy a larger physical net national product.

CAPITAL DEEPENING AND THE RATE OF INTEREST

By HANS BREMS

I. INTRODUCTION

Interest is the price of time, and the Böhm-Bawerk (1889) centenary is an excellent occasion to ask to what it is that takes time. The answer differs between circulating and fixed capital.

Ricardo [1821 (1951, I: 31)] saw circulating capital as "capital ... employed in the payment of wages, which are expended on food and clothing." What takes time in the case of circulating capital is the maturing of output in slow organic growth such as in agriculture, cattle raising, forestry, and winery or in time-consuming construction jobs. Ricardo saw fixed capital as "buildings and machinery [which] are valuable and durable." What takes time in the case of fixed capital is the utilization of durable plant and equipment over their useful lives.

Böhm-Bawerk [1889 (1923: 339-357)] was aware of the existence of fixed capital but gave it no place at all in his grand equilibrium of interest and wages. For a third of a century serious theorizing failed to go beyond Böhm-Bawerk and come to grips with fixed capital. The first to do so were Åkerman (1923) and Wicksell [1923 (1934)].

The purpose of the present paper is to compare the Böhm-Bawerk and the Åkerman-Wicksell models. We shall set them out in the simplest modern forms which will rigorously deliver their respective conclusions. We shall use the following notation.

Variables

a_1 ≡ labor absorbed in constructing one physical unit of producers' goods

C ≡ aggregate physical consumption

I ≡ aggregate physical gross investment

J ≡ present net worth of an endless succession of investments

L ≡ labor employed

P ≡ price of consumers' goods

p ≡ price of producers' goods

S ≡ aggregate physical capital stock of producers' goods

u ≡ useful life of producers' goods

X ≡ physical output of consumers' goods

y ≡ period of production

Parameters

a_2 ≡ labor absorbed per annum in operating one physical unit of producers' goods

α ≡ elasticity of physical output per annum per man with respect to period of production

β ≡ elasticity of construction labor per producers' good with respect to useful life

F ≡ available labor force

g ≡ rate of inflation

m ≡ multiplicative factor in production function (1)

n ≡ multiplicative factor in production function (15)

r ≡ nominal rate of interest

ρ ≡ real rate of interest

w ≡ money wage rate

Symbols v, τ, and t are used for vintage, specific time, and general time, respectively.

II. CIRCULATING CAPITAL: BÖHM-BAWERK

1. The Book

Positive Theory was not an easy book to read. Its language was clear enough, and Smart's translation[1] conveys it well. But it was not well organized. One fourth of the first edition (books III and IV) and one third of the last edition (book III) were long interruptions on value and price. What mattered--indeed all that mattered--were the closing 44 percent of the first edition (books V through VII), condensed to 30 percent of the last edition (book IV). To Wicksell (1911: 42) even the closing one-eighth of the book could be read separately. Here the variables were quantitative: the period of production, the rate of interest, and the wage rate. Here they were determined in a grand equilibrium using nothing but arithmetic. It is a tribute to Böhm-Bawerk's logic that it survived Wicksell's [1893 (1954)] verbatim translation into mathematics, simple interest and all.

2. Our Own Simulation

Let us build the simplest modern model which will rigorously deliver Böhm-Bawerk's conclusions, i.e., first that the lower rate of interest of a wealthier economy would lengthen the optimal period of production of circulating capital, second, as a result, the wealthier economy would enjoy a larger net national product. Our model will need only a single good, i.e., a maturing consumers' good, and only one kind of labor. The economy will be stationary: its available labor force hence its physical output will be stationary.

3. Microeconomics: What to Maximize?

Always referring to a rate of "profit" but never to a rate of "interest," English classicists thought of entrepreneurs who were their own capialists. Böhm-Bawerk [1889 (1923: 329 and 381)] knew better and separated his capitalists from his entrepreneurs. As lenders and borrowers, respectively, capitalists and entrepreneurs would meet in a capital market. Facing the resulting market rate of interest, would not an individual entrepreneur consider it a constant, use it to capitalize his future opportunities, and try to maximize their present net worth?

Occasionally Böhm-Bawerk [1889 (1923: 342-343)] does mention such capitalization. But in his tables on pp. 388-397, entrepreneurs are maximizing their rate of profit in a single production run. A left-over from English classicists? And why only a single production run? Maximizing present net worth of an endless succession of production runs would be in better accordance with Böhm-Bawerk's inherent logic and would be the straightforward thing to do. So we shall do it.

Let an individual entrepreneur plan an endless succession of production runs: every yth year the physical output X of a production run matures, and a new production run is started.

Define present net worth of such an endless succession as present worth of all its future revenue <u>minus</u> present worth of all its future labor.

4. Present Worth of All Future Revenue

A production run employing L men and producing every yth year the physical output X is producing an average physical output $X/(Ly)$ per annum per man. Let $X/(Ly)$ be a rising function of the period of production y but rising in less than proportion, as Böhm-Bawerk suggested. Let α be the elasticity of $X/(Ly)$ with respect to y, and let us adopt, as Böhm-Bawerk never did, a constant-elasticity production function

$$X/(Ly) = my^{\alpha} \tag{1}$$

where $0 < \alpha < 1$.

Inflation was never mentioned by Böhm-Bawerk but we easily accommodate it to show that the interest rate that mattered was the real one. So let the price of output be inflating from present time v to a later time t at the rate g per annum:

$$P(t) = e^{g(t - v)}P(v) \tag{2}$$

Define the real rate of interest as the nominal one <u>minus</u> the rate of inflation:

$$\rho \equiv r - g \tag{3}$$

Occurring every yth year, revenue is $P(t)X$ whose present worth is $e^{-r(t - v)}P(t)X$ or, with (2) and (3) inserted:

$$e^{-\rho(t - v)}P(v)X \tag{4}$$

Write (3) successively for $t = v + y$, $v + 2y$, ... Summing over production runs, find the present worth of an endless succession of future revenue

$$(1 + e^{-\rho y} + e^{-2\rho y} + \ldots)e^{-\rho y}P(v)X$$

The parenthesis is an endless geometrical progression with first term 1, common ratio $e^{-\rho y}$, and sum $1/(1 - e^{-\rho y})$. As a result, using (1), the present worth of the endless succession of future revenue is

$$\frac{e^{-\rho y}}{1 - e^{-\rho y}} P(v)X = \frac{LmP(v)y^{\alpha + 1}}{e^{\rho y} - 1} \tag{5}$$

5. Present Worth of All Future Labor

For circulating capital we adopt Böhm-Bawerk's flow-input, point-output scheme and assume our entrepreneur to employ L men uniformly. Such uniformity will rule out reswitching [Samuelson (1966)]. Let the men be employed at a money wage rate inflating at the rate g per annum:

$$w(t) = e^{g(t - v)}w(v) \tag{6}$$

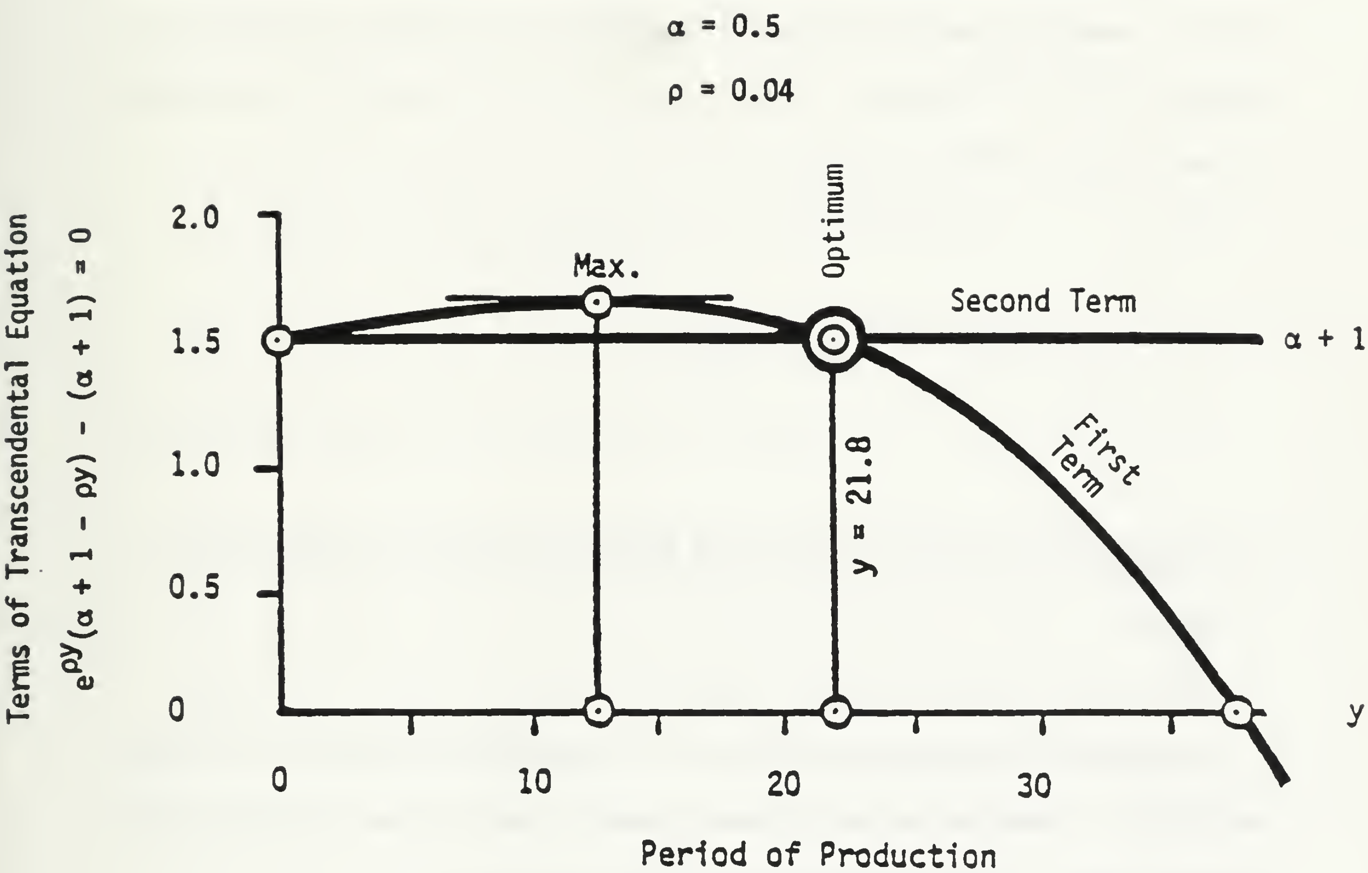

FIGURE 1

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Occurring continuously, then, labor cost during a small fraction dt of a year located at time t is Lw(t)dt. As seen from time v the present worth of that is $e^{-r(t - v)}Lw(t)dt$ or, with (3) and (6) inserted, $e^{-\rho(t - v)}Lw(v)dt$. The present worth of an endless succession of such future labor cost is

$$\int_v^\infty e^{-\rho(t - v)}Lw(v)dt = \frac{Lw(v)}{\rho} \qquad (7)$$

6. Maximizing Present Net Worth of Endless Succession of Production Runs

We defined present net worth of our endless succession as present worth (5) of all its future revenue minus present worth (7) of all its future labor. Call that present net worth J(v):

$$J(v) = [\frac{mP(v)y^{\alpha + 1}}{e^{\rho y} - 1} - \frac{w(v)}{\rho}]L \qquad (8)$$

Finally maximize present net worth $J(v)$ of the endless succession of production runs with respect to the length of the period of production y and write the first-order condition for such a maximum:

$$\frac{\partial J(v)}{\partial y} = \frac{e^{\rho y}(\alpha + 1 - \rho y) - (\alpha + 1)}{(e^{\rho y} - 1)^2} LmP(v)y^{\alpha} = 0 \qquad (9)$$

which will be zero if the numerator is zero:

$$e^{\rho y}(\alpha + 1 - \rho y) - (\alpha + 1) = 0 \qquad (10)$$

Let us solve our transcendental equation (10) graphically. For, say, $\alpha = 0.5$ and $\rho = 0.04$ figure 1 shows its terms. The first term does contain y and will be a curve starting at the value $\alpha + 1$ for $y = 0$, rising to a maximum for $y = \alpha/\rho$, declining to the value zero for $y = (\alpha + 1)/\rho$, and being negative thereafter. The second term does not contain y, hence is a horizontal line. Satisfying (10), curve and line intersect twice, at $y = 0$ and $y = 21.8$. Appendix I proves that the latter intersection satisfies the second-order condition and represents optimal period of production.

7. Sensitivity of Optimal Period of Production to Real Rate of Interest

How sensitive is optimal period of production to the real rate of interest ρ? Differentiate our first-order condition (10) implicitly with respect to ρ treating y as a function of ρ and find the simple elasticity

$$\frac{\rho}{y}\frac{dy}{d\rho} = -1 \tag{11}$$

or, in English, the elasticity of optimal period of production with respect to the real rate of interest equals minus one: a lower real rate of interest will lengthen the optimal period of production.

8. Demand for Capital in Our Single Vintage

To prepare for our macroeconomics let us conclude our microeconomics by asking how much capital has been borrowed, in our single vintage v, up to a certain time τ? The entrepreneur was employing L men uniformly at the money wage rate (6). During a small fraction dt of a year located at time t, then, he was borrowing the sum $Lw(t)dt$.

As seen from time τ, the present worth of it is $e^{r(\tau - t)}Lw(t)dt$ or, with (6) inserted, $e^{\rho(\tau - t)}Lw(\tau)dt$. The present worth at time τ of all such sums borrowed from time v to time τ is the integral:

$$Dv(\tau) \equiv \int_v^\tau e^{\rho(\tau - t)}Lw(\tau)dt = \frac{e^{\rho(\tau - v)} - 1}{\rho} Lw(\tau) \tag{12}$$

And now for Böhm-Bawerk's macroeconomics.

9. Macroeconomics: Aggregate Demand for Capital

In Böhm-Bawerk's [1889 (1923: 382)] macroeconomics all industries produced the same good in the same production function. For the economy as a whole we may assume that new vintages are started continuously. In Böhm-Bawerk's language [1889 (1923: 326-327)] production is "staggered": at time τ, y vintages v are in operation, $\tau - y \leq v \leq \tau$. To find the present worth at time τ of the sums borrowed in all vintages in operation at that time we must integrate (12) once more, this time with respect to vintage:

$$D(\tau) \equiv \int_{\tau - y}^{\tau} D_v(\tau)dv = \frac{e^{\rho y} - 1 - \rho y}{\rho} \frac{Lw(\tau)}{\rho} = \frac{e^{\rho y} - 1 - \rho y}{\rho y} \frac{Lw(\tau)}{\rho} y \quad (13)$$

Under our special production function (1) the elasticity (11) will hold and make the product ρy a constant. But then the entire first factor of the form (13) will be a constant, and at a given real rate of interest ρ the aggregate demand for capital (13) will be in direct proportion to the wage bill $Lw(\tau)$ per vintage and to the number y of vintages in operation.

10. Macroeconomics: Demand Meets Supply

Böhm-Bawerk [1889 (1923: 329 and 381)] separated capitalists and entrepreneurs. His capitalists were lenders described as "all owners of wealth who do not consume but 'save' it." Entrepreneurs were borrowers described as "every one who wishes to produce in capitalist methods," i.e., time-consuming methods. Capitalists and entrepreneurs would meet in an aggregate capital market described as "the market in which present goods are exchanged against future goods," i.e., lenders would supply present goods and demand future ones; borrowers would demand present goods and supply future ones. For three reasons the meeting of capitalists and entrepreneurs would establish a premium on present goods _vis a vis_ future goods. First, men typically expect to be better off in the future, so why save? Second, men typically do not feel future wants as intensely as present ones, so why save? Third, time-consuming methods are more productive. Böhm-Bawerk called his percentage premium on present goods an "agio" and eventually "the rate of interest."

11. Macroeconomics: The Grand Equilibrium

Everything now falls into our laps: in a grand equilibrium the rate of interest would induce a period of production y just long enough to absorb the entire available capital stock and employ the entire available labor force. The profitability of the last extension of the period of production thus permitted, would equal the rate of interest. A consideration of nonequilibria will illuminate the mechanism of Böhm-Bawerk's macroeconomic equilibrium.

A rate of interest higher than its equilibrium value would make a shorter period of production y optimal and for two reasons lower the aggregate demand for capital (13). First, a vintage may still employ L men, but with a shorter period of production y, fewer vintages y would be in operation, so entrepreneurs would be hiring less labor than actually available. Second, such excess supply in the labor market would lower the wage rate. At a lower wage bill per vintage and fewer vintages in operation, entrepreneurs in accordance with (13) would demand less capital than actually available. There would be excess supply in the capital market, and competition among lenders would lower the interest rate to its equilibrium level.

A rate of interest lower than its equilibrium value would make a longer period of production y optimal and for two reasons raise the aggregate demand for capital (13). First, a vintage may still employ L men, but with a longer period of production y, more vintages y would be in operation, so entrepreneurs would be trying to hire more labor than actually available. Second, such excess demand in the labor market would raise the wage rate. At a larger wage bill per vintage and more vintages in operation, entrepreneurs in accordance with (13) would demand more capital than actually available. There would be excess demand in the capital market, and competition among borrowers would raise the interest rate to its equilibrium level.

A modern reader would agree with Böhm-Bawerk's [1889 (1923: 401)] conclusion: "We have, then, over the sphere of our investigation so far, to record three elements or factors which act as decisive determinants of the rate of interest: the Amount of the national subsistence fund, the Number of workers provided for by it, and the Degree of productivity in extending production periods."

12. Macroeconomics: Net National Product

A Böhm-Bawerk economy was stationary and knew no fixed capital. Consequently it knew neither net nor gross investment and knew no capital consumption allowances: its net national product was simply its output of consumers' goods per annum. The Böhm-Bawerk economy was always fully employed: its labor employed Ly always equaled its available labor force.

But a wealthier Böhm-Bawerk economy had a larger available capital stock, hence a lower rate of interest, hence a longer period of production y. Consequently its labor was more productive: according to our (1) physical output $X/(Ly)$ per annum per man would be higher. The wealthier economy would then enjoy a larger physical net national product.

13. Synchronized Input and Output

Does the length of the period of production matter, then?

It does. Think of Böhm-Bawerk's "staggered" production. Here, as we saw, at time τ, y vintages v were in operation: $\tau - y \leq v \leq \tau$. True, in a starting vintage $v = \tau$ labor will remain invested for y years. True, in a maturing vintage $v = \tau - y$ labor will remain invested for no more years. True, in an average vintage labor will remain invested for an average period of production approximately equaling y/2 years. While this is all true, it is also true that at any moment the labor input of a starting vintage and the output of a maturing one are synchronized. Under such synchronization is the length y or its average y/2 of no consequence--as some early critics said? Of course not. As we have just shown, synchronized output is the <u>larger</u> the longer the period of production y. We agree with Schumpeter's (1954: 907) comments on such synchronization: "Böhm-Bawerk's period of production would, if his assumptions be accepted, express one of the most meaningful characteristics of an economic process, however 'cycleless' it may be."

III. FIXED CAPITAL: ÅKERMAN-WICKSELL

1. Why Did it Take so Long?

"Professor Åkerman," Schumpeter (1954: 908) says, "dealt in one of the most important works in this field, with the problems of fixed capital, which is so curiously absent from Böhm-Bawerk's schema."

Was it absent because it was unimportant? When in 1821 Ricardo added his new chapter "On Machinery," he did so because he considered it "a subject of great importance." The second industrial revolution made it even more important. Was fixed capital absent because it was more difficult? Åkerman's treatment certainly made it appear more difficult.

2. The Book and the Book Review

What takes time in the case of fixed capital is the utilization of a durable producers' good over its useful life. Åkerman's and Wicksell's contribution was to consider the length of that useful life an economic variable: better-constructed producers' goods would be longer-lasting, thus delivering their output over more years, but also cost more interest. Exactly how long should they last? Each in his own way, both were able to answer that question.

Åkerman's answer was precise but always in the form used by Böhm-Bawerk, i.e., arithmetical examples. On his way to it, Åkerman got lost. Böhm-Bawerk got lost in value and price. Åkerman got lost in amortization procedures: how much construction cost is "left" in a producers' good of a certain age? Which amortization procedure is the "correct" one?

Åkerman's (1923) book was a thesis, and Wicksell was his adviser. Wicksell's [1923 (1934)] review consisted of a verbal evaluation of Åkerman's work and a mathematical restatement of it.

3. Our Own Simulation

We can hardly improve upon Wicksell. What we can do is to build the simplest modern model which will rigorously deliver his conclusions, i.e., first, that the lower rate of interest of a wealthier economy would lengthen the optimal useful life of durable producers' goods. Second, as a result, the wealthier economy would enjoy a larger net national product. Our model will need only two goods, a consumers' good and a durable producers' good, and only two kinds of labor, construction labor and operating labor. The economy will be stationary: its available labor force, physical capital stock, and physical output will be stationary.

As before we simplify things by using Fisher's present-net-worth maximization. Then we need not worry, as Åkerman did, about how much construction labor is "left" in a producers' good of a certain age, indeed we need not even refer to amortization! As before, let an individual entrepreneur plan an endless succession of replacements of durable producers' goods: every uth year a retired producers' good is replaced by a new one physically identical to it.

Define present net worth of such an endless succession as present worth of all its future revenue minus present worth of all its future construction labor minus present worth of all its future operating labor.

4. Present Worth of All Future Revenue

Assume physical output X of consumers' goods per annum per producers' good to remain uniform throughout the useful lives of all replacements. Such uniformity, assumed by Wicksell [1923 (1934): 274)], will rule out reswitching [Samuelson (1966)].

Inflation was never mentioned by Wicksell [1923 (1934)] either. Again we easily accommodate it to show that the interest rate that mattered was the real one. So as in (2) above let the price of output be inflating at the rate g per annum.

Occurring continuously, then, revenue during a small fraction dt of a year located at time t is $P(t)Xdt$. As seen from time v the present worth of that is $e^{-r(t - v)}P(t)Xdt$ or, with (2) and (3) inserted, $e^{-\rho(t - v)}P(v)Xdt$. The present worth of all such future revenue is then the integral

$$\int_v^\infty e^{-\rho(t - v)} P(v)Xdt = \frac{P(v)X}{\rho} \qquad (14)$$

5. Present Worth of All Future Construction Labor

Let the scrap value as well as the length of the construction period of producers' goods be negligible. Borrowing a leaf from Ricardo, Wicksell [1923 (1934: 285-288)] revived the distinction between construction ("renewal") labor and operating ("co-operating") labor. Let the construction of durable producers' goods require labor--indeed let us be truly Ricardian and Wicksellian and assume it to require nothing else. Let a_1 be the labor absorbed in constructing one physical unit of producers' goods, and let a_1 be a rising function of useful life u but rising in less than proportion. Let β be the elasticity of a_1 with respect to u, and let us adopt, as Wicksell [1923 (1934: 276)] did, a constant-elasticity production function

$$a_1 = nu^\beta \qquad (15)$$

where $0 < \beta < 1$.

Let producers' goods be priced p and sold under pure competition and freedom of entry and exit. Then their price will equal their cost of production:

$$p(t) = a_1 w(t) \tag{16}$$

As before in (6), let the money wage rate be inflating at the rate g per annum.

Occurring every uth year, construction-labor cost is $a_1 w(t)$ whose present worth is $e^{-r(t - v)} a_1 w(t)$ or, with (3) and (6) inserted,

$$e^{-\rho(t - v)} a_1 w(v) \tag{17}$$

Write (17) successively for $t = v, v + u, v + 2u, \ldots$ Summing over replacements, find the present worth of an endless succession of future construction labor

$$(1 + e^{-\rho u} + e^{-2\rho u} + \ldots) a_1 w(v)$$

The parenthesis is an endless geometrical progression with first term 1, common ratio $e^{-\rho u}$, and sum $1/(1 - e^{-\rho u})$. Using (15) find the present worth of the endless succession of future construction labor

$$\frac{a_1 w(v)}{1 - e^{-\rho u}} = \frac{n u^{\beta} w(v)}{1 - e^{-\rho u}} \tag{18}$$

6. Present Worth of All Future Operating Labor

As for operating labor, Wicksell [1923 (1934: 286)] used a Cobb-Douglas function to introduce substitution between operating labor and the producers' goods operated. Instead, we simplify things by using a fixed coefficient between them: throughout its useful life, but regardless of its length, let a_2 be labor absorbed uniformly per annum in operating one physical unit of producers' goods. Such uniformity will rule out reswitching [Samuelson (1966)].

Occurring continuously, then, operating labor cost during a small fraction dt of a year located at time t is $a_2 w(t)dt$. As seen from time v the present worth of that is $e^{-r(t - v)} a_2 w(t)dt$ or, with (3) and (6) inserted, $e^{-\rho(t - v)} a_2 w(v)dt$, and the present worth of all such future operating labor is

$$\int_v^{\infty} e^{-\rho(t - v)} a_2 w(v)dt = \frac{a_2 w(v)}{\rho} \tag{19}$$

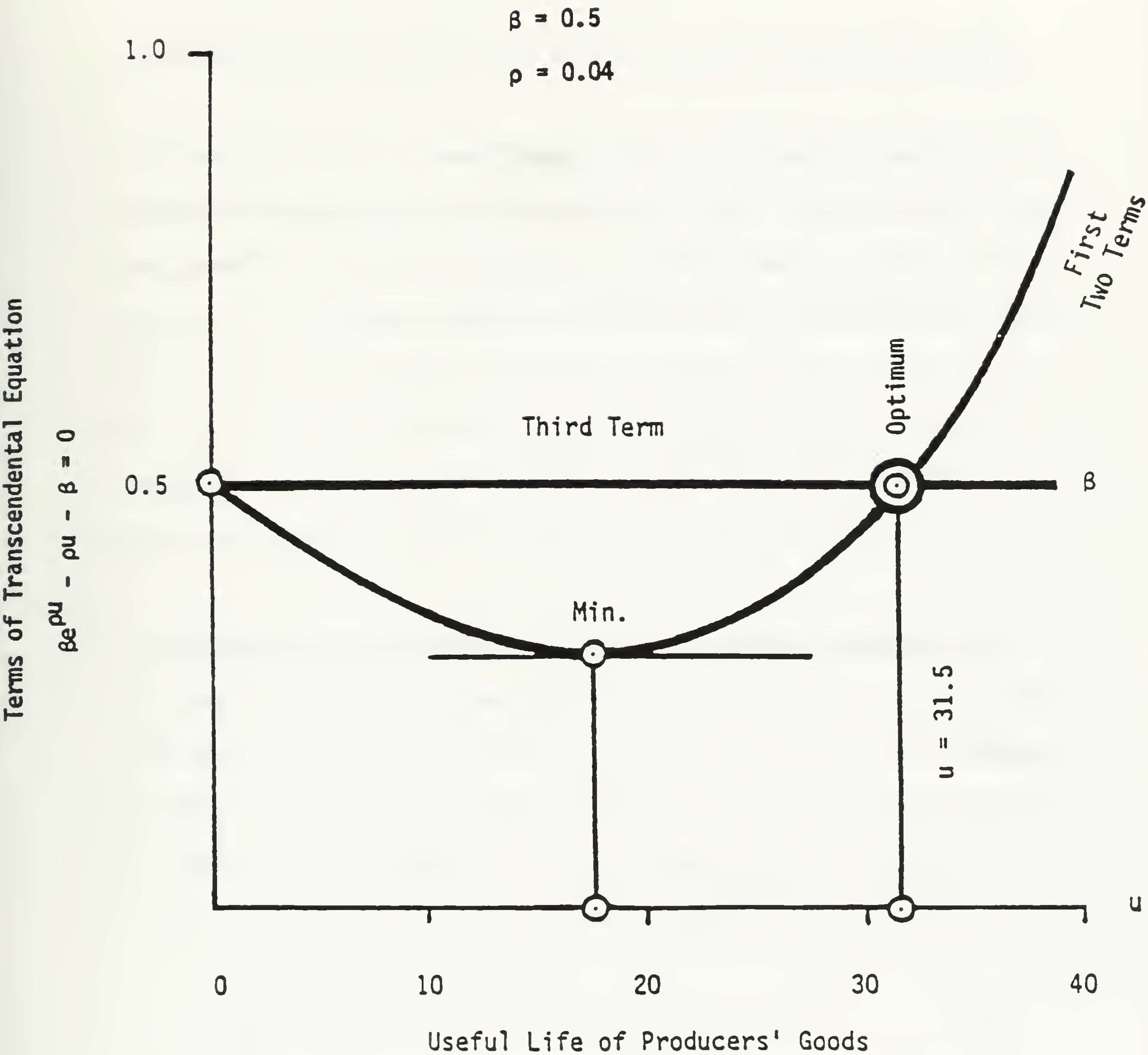

FIGURE 2

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7. Maximizing Present Net Worth of Endless Succession of Replacements

We defined present net worth of our endless succession of replacements as present worth (14) of its future revenue minus present worth (18) of its future construction labor minus present worth (19) of its future operating labor. Call that present net worth J(v):

$$J(v) = \frac{P(v)X}{\rho} - \frac{nu^{\beta}w(v)}{1 - e^{-\rho u}} - \frac{a_2 w(v)}{\rho} \qquad (20)$$

Finally maximize present net worth J(v) with respect to useful life u and write the first-order transcendental condition for such a maximum:

$$\frac{\partial J(v)}{\partial u} = \frac{(\beta + \rho u)e^{-\rho u} - \beta}{(1 - e^{-\rho u})^2} nu^{\beta - 1} w(v) = 0 \qquad (21)$$

If zero, the numerator would remain zero if multiplied by minus $e^{-\rho u}$. So such multiplication would permit us to write our first-order condition in its simplest possible transcendental form

$$\beta e^{\rho u} - \rho u - \beta = 0 \tag{22}$$

Let us solve our transcendental equation (22) graphically. For, say, $\beta = 0.5$ and $\rho = 0.04$ figure 2 shows its terms. The first two terms contain u and will be a curve starting at the value β for $u = 0$, declining to a minimum for $u = 17.5$ and rising forever after. The third term does not contain u, hence is a horizontal line. Satisfying (22), curve and line intersect twice, at $u = 0$ and $u = 31.5$. Appendix II proves that the latter intersection satisfies the second-order condition and represents optimal useful life of producers' goods.

8. Sensitivity of Optimal Useful Life to Real Rate of Interest

How sensitive is optimal useful life to the real rate of interest ρ? Differentiate our first-order condition (19) implicitly with respect to ρ treating u as a function of ρ and find the simple elasticity

$$\frac{\rho}{u}\frac{du}{d\rho} = -1 \tag{23}$$

or, in English, the elasticity of optimal useful life with respect to the real rate of interest equals minus one--as found by Wicksell [1923 (1934: 278)]: a lower real rate of interest will lengthen the optimal useful life of producers' goods.

9. Macroeconomic Full-Employment Equilibrium

Aggregate labor employed per annum in construction is

$$L_1 = a_1 I \tag{24}$$

Aggregate labor employed per annum in operation is

$$L_2 = a_2 S \tag{25}$$

Let physical gross investment I be stationary. Then physical aggregate capital stock S of producers' goods will consist of u vintages, each of size I:

$$S = Iu \tag{26}$$

Let there be full employment:

$$F = L_1 + L_2 \tag{27}$$

10. Macroeconomics: Capital Stock

The wealthier economy has the lower rate of interest, hence the longer useful life of its producers' goods. Insert (15), (24), (25), and (26) into (27) and find its physical capital stock to be

$$S = \frac{F}{nu^{\beta - 1} + a_2} \tag{28}$$

which is always up if u is up, because $\beta - 1 < 0$. So the wealthier economy with its lower rate of interest and longer useful life of its producers' goods has the larger physical capital stock. Physical capital stock is larger simply because it consists of more solidly built, hence longer-lasting producers' goods.

11. Macroeconomics: Net National Product

In money terms gross national product is CP + Ip. With our stationary physical capital stock net investment is zero, so physical net national product is simply C. What can we say about C? Each producers' good produced a physical output of consumers' goods X per annum. The aggregate physical capital stock S of such producers' goods will then produce the aggregate physical output of consumers' goods

$$C = SX \tag{29}$$

As we just saw, S was always up if u was up. We conclude, as Wicksell [1923 (1934: 298-299)] did, that with its larger and longer-lasting physical capital stock the wealthier economy will enjoy a larger physical net national product.

IV. CONCLUSION

Using the same method and notation, we have simulated the Böhm-Bawerk model of circulating capital and the Åkerman-Wicksell model of fixed capital. They had three things in common. First, technology was assumed to be stationary. Second, a lower rate of interest would always lengthen the time span of capitalist production. Third, with its lengthening time span, a wealthier economy would enjoy a larger physical net national product.

The similarity was well expressed by Wicksell [1919 (1934: 240)] himself even before he wrote his review of Åkerman: "A farmer has to choose between two ploughs, one of which lasts ten years, and the other, equally useful, lasting eleven. If he chooses the more durable (and dearer) plough, he has the benefit of an extra year's service, which, however, ... must ... replace the difference in price between the two ploughs accumulated by the total interest for the eleven years. Similarly, the price of old wine must exceed the price of newly-pressed wine by the interest for the years of storage." "From an economic point of view the difference is therefore unessential."

What came later?

Blitz (1958) and Kleiman-Ophir (1966) referred to and agreed with the Åkerman-Wicksell result that a lower rate of interest would always lengthen useful life. But between 1958 and 1966 reswitching attracted wide attention--although, as Velupillai (1975) has observed, Fisher noticed it already in 1907. Reswitching referred to the possibility within already known technologies that a time span might be optimal at a high rate of interest, inoptimal at a medium rate of interest, and optimal once again at a low rate of interest.

For circulating and fixed capital alike, Samuelson (1966: 571-574) found reswitching possible. "Possible" is one thing; empirically plausible is another. Ripening wine or growing timber may not employ labor uniformly throughout their period of production. But as observed by Barna (1961: 80) and Domar (1961: 98), durable producers' goods are typically designed to produce their physical output uniformly--as we assumed them to be.

Beyond that, we cannot pursue the matter here--but hope to do so elsewhere.

APPENDIX I. SECOND-ORDER CONDITION FOR THE BÖHM-BAWERK CASE

To prove that the second-order condition for a maximum (8) is satisfied, differentiate (9) once more with respect to y, and use (10):

$$\frac{\partial^2 J(v)}{\partial y^2} = \frac{(\alpha - \rho y)e^{\rho y}\rho}{(e^{\rho y} - 1)^2} LmP(v)y^{\alpha} \tag{31}$$

Write (10) as

$$\alpha = \frac{1 - e^{\rho y}(1 - \rho y)}{e^{\rho y} - 1} \tag{10}$$

Subtract ρy on both sides and find

$$\alpha - \rho y = \frac{1 + \rho y - e^{\rho y}}{e^{\rho y} - 1}$$

which is negative for all $\rho y > 0$. As a result the second derivative (31) is negative.

APPENDIX II. SECOND-ORDER CONDITION FOR THE WICKSELL CASE

To prove that the second-order condition for a maximum (20) is satisfied, differentiate (21) once more with respect to u, and use (22):

$$\frac{\partial^2 J(v)}{\partial u^2} = \frac{1 - (\beta + \rho u)}{(1 - e^{-\rho u})^2} e^{-\rho u} \rho n u^{\beta - 1} w(v) \tag{32}$$

Write (22) as

$$\beta = \frac{\rho u}{e^{\rho u} - 1} \tag{22}$$

Add ρu on both sides and find

$$\beta + \rho u = \frac{\rho u}{1 - e^{-\rho u}}$$

which for all $\rho u > 0$ will be greater than one, because $\rho u > 1 - e^{-\rho u}$. As a result the second derivative (32) is negative.

FOOTNOTES

[1]The first translation of Positive Theory followed the first edition and was done by William Smart, Lecturer on Political Economy in Queen Margaret College, Glasgow, who observed that "a translator who does his duty must pass the work he renders through his own mind." He did and did it well. The second translation came 68 years later, having the advantage of following the fourth (posthumous) edition, and was done by George D. Huncke, formerly Head of the Foreign Language Department of the Andrew Jackson High School of New York City with Hans F. Sennholz as a consulting economist. Newer is not always better. The Huncke translation isn't.

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