The author.

THE DEVELOPMENT
OF
INSURANCE MATHEMATICS.

TWO LECTURES

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BY

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FIRST LECTURE.

Preliminary Development, Decimal System, Series, Logarithms, Permutations and Combinations, Probabilities.

SECOND LECTURE.

Evolution of Actuarial Science, Annuities Certain, Mortality Tables, Actuarial Science Proper.

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THE DEVELOPMENT OF INSURANCE MATHEMATICS.

FIRST LECTURE—PRELIMINARY DEVELOPMENT.

It is most interesting to study the growth and evolution of any branch of science and to observe how it sprang from some other branch of science, and what things had to be known as a preparation for the dawning of the new learning. It is a far cry, perhaps, from actuarial science to the decimal system of notation, and yet, had it not been for the invention of that system it may well be doubted whether arithmetic would ever have reached the stage which renders mathematical problems that would otherwise be difficult of operation, simple and easy. Addition, which is now so simple, albeit so laborious a matter, was exceedingly difficult under the old systems of notation. A student may readily form some conception of the difficulties if he will range a number of Roman numerals side by side, or one under the other—one arrangement is about as convenient as the other—and proceed to add. Multiplication, which is continued addition, was even harder. It follows that operations which must be performed upon a very large scale, in order that actuaries may do their tasks, were then almost impossible upon anything but a limited scale.

Almost equally absurd it may at first seem to declare that the new science could not have come into existence until after algebra was discovered; but it will be seen to be perfectly reasonable to say this when it is taken into account that actuarial science is merely a department of algebra, and that all its operations are algebraic. The form which all its formulas take is that of equations, and they are evolved from other equations by purely algebraic processes.

Algebra and decimal notation were both introduced into Europe by the Arabs through the Moorish schools of Spain and Africa, in the thirteenth century. The first mathematical books to be studied were those of the Arabian authors and Arabian translations of

Euclid, Archimedes, Appolonius and Ptolemy. The study was for a long time condemned by the church authorities, although one of the popes was a mathematician. It was connected in the minds of the people with astrology and alchemy and, indeed, most of the early mathematicians were astrologers also.

The following things needed to develop in mathematics before actuarial science could begin its evolution: Decimal notation, decimal fractions—desirable, not necessary—series and summation, logarithms, permutations and combinations, probabilities.

Decimal fractions were brought in by Pitiscus in 1617. Their use, however, did not become common until after 1700 and, indeed, so little was known for a long time of the discovery of Pitiscus that the honor of the invention was claimed by mathematicians of a later date.

Series were treated by the ancient mathematicians, but in a very fragmentary and imperfect manner, algebra being unknown. Euclid and Appolonius, among others, discussed the subject somewhat, but it was never given the attention that was necessary in order to make the development of annuities possible until in the seventeenth and eighteenth centuries, when Leibnitz, Montmort, James (or Jacob) and John Bernoulli, La Place, Gauss and Cauchy threshed out the matter thoroughly.

Even with a perfected decimal system, both of integral and fractional notation, and in spite of the perfection of algebra as a mathematical art, it would have been most difficult to perform the operations required in compound discount computations, as in annuities certain, without the aid of logarithms, which make it almost as easy to find the present value of a sum due in 1,000 years as if the term were a single year. In no branch of mathematical science, perhaps, has the invention of logarithms proven a greater blessing. Their invention, as is well known, is due to Napier, and dates from about 1610. They were soon afterward improved by Briggs, who also suggested the decimal base, which made logarithms much easier to employ and far more useful.

Annuities certain grew so inevitably out of the formulas for the summation of geometric series that it is impossible, perhaps, to mark out closely the period when the first mention of the subject is found. Indeed, although the topic has a deal of interest of itself, and is generally considered abstruse and difficult by students, problems in annuities certain, like all compound-interest problems, were originally merely set as exercises in series, and no thought was given that such might be the foundation for a new branch of the science. As a separate branch of mathematical research, we do not find that annuities certain appear before the beginning of the eighteenth century.

As series and, perhaps, logarithms, were necessary first steps to annuities certain, which is one of the three legs upon which actuarial science will be found to stand, so in like manner permutations and combinations had to precede probabilities, which is the second of the three supports of actuarial science.

Perhaps, as the subject first appeared in mathematics, nothing was ever more unpromising so far as practical results were concerned than permutations and combinations. That it could advantage anybody to know how many different arrangements could be made of n things taken k at a time, seemed too ridiculous to talk about. Often has it proved, however, that the topic which was considered mere mathematical sport has been found most useful in some field which was awaiting that very discovery to unlock new treasures of practically valuable learning. So it was in the case of the apparently idle speculations as to the number of arrangements of different things which one could make; what seemed to the schoolmen only elegant and frivolous speculation, highly amusing because of the frequently startling results, was the missing word to unseal the secrets of probabilities.

The subject was not unknown to the ancients, either of Greece or of India. Aristotle, whose death took place 322 B. C., suggested the solutions of such problems, and, before his time, Arjabhatta, the Indian savant, treated certain phases. The modern development began in the later years of the sixteenth century, in the researches of Cardan, which, however, were not printed until 1663. He was followed soon by Pascal, Leibnitz and James (or Jacob) Bernoulli.

Concurrently with the development of the theories of permutations and combinations came their application in the science of probabilities. For Cardan, who died in 1576, wrote the first known essay on the subject of probabilities, though the same was not published until 1663. It was known by the title "De ludo aleae"—"Concerning the game of dice."

In ancient times they evidently had the conception that the value of a chance or probability could be valued; for, as we shall see, the courts occasionally set values upon life annuities. Moreover, loans were made by venturesome capitalists upon ships and cargoes, repayable at more than the usual interest, but forgiven entirely if the property were destroyed. The similarity to insurance is clear, and it is also evident that there was some rough fashion by which what was believed to be the value of the hazard was ascertained. But if, as frequently in our days, there was empirical guessing at the value of hazards, there was no attempt made to calculate these values, so far as is now known.

Galileo, at some time during his life, which closed in 1642, also wrote briefly upon the same subject as Cardan.

But the first work of importance consists of a series of letters by Pascal in reply to certain inquiries of the Chevalier de Mere—by some thought to have been mythical—concerning the chances of play. The first of these letters was written in 1654, and they were published in 1679. In them some of the most intricate problems of probabilities have been worked out. The philosopher was not satisfied to deal merely with the simple questions that lay upon the surface, but boldly assailed the most complicated. In the course of his investigations he discovered the "arithmetical triangle" and "figurative numbers," which have puzzled the drones among college students from that day to this, but have served many good purposes in mathematics, and stood ready to welcome the binomial theorem at a later date. Merely to mention this one discovery, set forth in his letters, may serve to indicate that he did not indulge in mere child's play in his primer of probabilities. His contemporary, Fermat, also took part in the discussion.

The next work upon the subject was by Christian Huygens. Like the first writing dealing with probabilities, it was an attempt to compute the chances with dice. Its title was "De Ratiociniis in Ludo Aleae." The pamphlet appeared in 1657.

After this there was a long pause. The situation in mathematics was at the time about as follows: After the subject had been reintroduced it was for a long time strenuously opposed by church influences. Moreover, it was by the schools despised as forming no part of a liberal education. The universities for a long time knew it not. The connection of mathematics with astrology—all who studied the stars were esteemed astrologists in those days—with alchemists—all students of nature were considered alchemists—did not add to its repute. The church was suspicious of all learning which did not emanate from itself. But at about this time, as a result of the serious attention to mathematics, which some of the greatest minds of the age had paid, and also, perhaps, because the control of education was passing into secular hands, mathematics was beginning to make its way into the universities. But this was only in the more abstract forms. Most of its applications to chemistry, astronomy and other sciences were yet to be made. The least promising of these applications seemed to be to probabilities; for not only was it denied that there was or could be such a science, but the use of it, unless to encourage gaming, nobody could see. Therefore the first attempts to solve problems of chance were ridiculed by most scholars, who denied both that it could be done and that it would be of any use if it were done. Doubtless, therefore, the science would have been a long while getting a start, had not a recluse like Pascal, fond of all speculations that tested his skill in analysis and synthesis, and altogether careless whether his speculations were useful or not, been tempted to essay the task of estimating chances.

Notwithstanding this auspicious beginning, however, there was nothing further upon the subject for nearly forty years, and nothing of much consequence for more than fifty years. In 1693 the first study of the subject appeared in English; its title is "An Arithmetical Paradox Concerning the Chances of Lotteries." Thus but little over two hundred years ago actuarial science, which was first brought to its full development in Great Britain, found lodgment there.

This first English work was of minor importance. It merely treated some practical phases of the subject, and was by no means thorough. But there had been for many years preparing a really great work upon the subject, the first of all great works. "Ars Conjectandi," by Jacob (often called James) Bernoulli, was, of course, completed before the author's death in 1705, but it was not brought out until 1713. It was edited and published by the nephew of the author, Nicholas, son of John Bernoulli; and from this date the subject of probabilities afforded this family opportunities to distinguish themselves even unto the third generation.

That "Ars Conjectandi" marks the close of an old era and the beginning of a new is sufficiently attested by the following: It was written in Latin, a language until then practically always employed as a vehicle for learning, and it dealt with the newest application of what was itself a new science, algebra. It was at the turning point. Latin was going out; science was coming in. The book was not a mere sketch or study. Jacob Bernoulli had toiled during the last years of his life most indefatigably and without paying the least attention to the circumstance that the whole speculation was by most learned men deemed vain and unprofitable. When he laid down his pen, disappointed even in his hope that his manuscript should be published, he had developed the scraps of speculation into a demonstrated science. And although even at a later day persons could be found who were so dull as not to apprehend his truths, and so intensely prejudiced as to decide against him on a priori and most fanciful grounds, he had fixed the fundamental lines for the investigation of probabilities where they have ever since been found. His work dealt with all the simpler problems and with many of the compound problems, including some of the most intricate. It was long the storehouse from which later mathematicians drew their material, and what is owing to Bernoulli it would now be very difficult to overestimate. He found probabilities a subject for occasional and wholly tentative reasoning; he left it a science.

After the death of Jacob Bernoulli, but before the publication of his book, there appeared in France, in 1708, a pamphlet entitled "An Essay on the Analysis of the Game of Hazard." The author was the great Montmort. The work has value, but need only be here mentioned to show that the speculations of Pascal had not been fruitless in his own country. There for some time the topic was, however, following Pascal's lead, treated only as to the probabilities in various games of chance.

The next great mathematician to take up the matter was also a Frenchman, De Moivre; but he was an exile in London, and his books were written in English. His fame was also achieved in England, and he is known as a great English mathematician. I have said that his books were written in English, which also was the case as to all, save a pamphlet entitled "De Mensura Sortis," which was published in 1711. This was followed by his great work, "Doctrine of Chances," which was published in 1718, and was the second book to treat the entire subject of probabilities, and the first to appear in a modern language. This book remained the standard text-book upon the subject for a long time, and during the author's lifetime two revised editions, the last much enlarged, also appeared.

Meanwhile the Bernoullis were keeping interest in the subject alive in other countries. Nicholas Bernoulli even preceded the publication of his uncle's "Ars Conjectandi" by his own pamphlet, "Specimens of the Art of Conjecture, Applied to Questions of Law," which was printed in 1709. Daniel Bernoulli, his brother, followed this in 1738 with his "Specimens of a New Theory Concerning the Measurement of Chance."

In Great Britain, De Moivre reigned supreme in this branch of applied mathematics for many years. He enjoyed great repute and, as we shall see when we discuss actuarial science itself, he was in great demand to put into practice his own theories, in the form of computations of probabilities of various sorts. But in 1740 a rival arose, one Thomas Simpson, a self-taught mathematician, who had also been a necromancer—a trade by this time out of fashion in connection with mathematics—and who at first enjoyed no standing whatever, although before his life closed his accomplishments were such that the Encyclopedia Britannica gives him credit as the greatest non-academic mathematician that Great Britain has produced. Simpson's first book, published in 1740, was entitled "The Nature and the Laws of Chance." It was written in language as simple as the author could employ, because it was his purpose to render the subject popular and easy of comprehension by the ordinary untrained mind. This of itself was a twofold offense against De Moivre, because it was, in the view of the distinguished mathematician, both an attempt to dispute his unrivaled position and also an attempt to cheapen and lower his science. Consequently the younger man, who, as events have proved, deserves more credit for the growth and extension and perfection of actuarial science and practice than even De Moivre, was by no means welcomed by his great contemporary.

In France the discussion of probabilities yet turned upon particular hazards in games of chance. Thus, so late as 1751, we find the great Euler, one of the most wonderful mathematicians of all time, sending forth a pamphlet entitled "Calculations of the Probabilities of the Game of Rencontre." The time when France would again make a great contribution to the science of probabilities was thus yet a while deferred.

In 1773 La Grange began a work on "An Article on Probability," but it was many years before the subject was treated exhaustively by a Frenchman, which seems all the more remarkable since France had so many able mathematicians, and also since the first work of importance upon the subject, Pascal's letters, appeared in that tongue. But the slowness with which the science of probabilities in general developed there may, as we have seen, be ascribed with much plausibility to the circumstance that Pascal's letters were concerning certain peculiar gambling hazards. This view is borne out by the fact that, while discussions of probabilities in general were missing, discussions upon particular problems of gambling were frequent.

Between 1750 and 1760 three Englishmen contributed short but valuable works upon the subject, Hoyle in 1754, James Dodson in 1755, and Samuel Clark in 1758. In 1769 John Bernoulli, grandson of John Bernoulli, the elder, and grand nephew of Jacob Bernoulli, published a pamphlet, the last to bear the distinguished name which was attached to the first comprehensive work upon the subject.

France, which had so long dragged behind Great Britain in developing this science, began to bestir herself in the last years of the eighteenth century when, by reason of the impulse which the revolution gave all men, sciences and arts began to develop as never before. The first fruit of this travail was a pamphlet by Condorcet in 1785. But it reached its full development when, in 1812, La Place's "Analytical Theory of Probabilities" appeared. This book remained the classic text-book upon the subject for many years, and really exhausted all that was known concerning it, besides containing much that was original and of great value. It remains to this day one of the three or four really valuable books upon the science.

Well might it be such, also, for not only was La Placé one of the greatest mathematicians of his own or of any time, but he labored under an encouragement which had never before been given any student of the subject. Pascal's letters brought forth ridicule and abuse, almost without a dissenting voice, excepting from advanced mathematicians who were as much out of favor as himself. Much the bold speculator cared about that, however; but his successors did care. For, be it known, the heterodoxy of to-day becomes the orthodoxy of to-morrow as the world advances. When Pascal's work took its place in the accepted views of men of learning the orthodox set their line just beyond what Pascal had demonstrated. All beyond that was heterodox, and therefore, as ever, the ban was put upon the innovator. In France, too, there was rather more of this than elsewhere, though Jacob Bernoulli's great work could not see the light until its author was dead. In England alone, thanks to the mathematical turn which Newton's great achievements gave academical education, the reception of the new learning was warmer, but even there the one-time innovator, De Moivre, threw all sorts of obstacles, not sparing ridicule, in the way of Simpson.

With La Place it was different. Not only had the revolution broken up the theological and scholastic prejudices in France and installed instead a passion for innovation, but La Place received the direct support of the emperor, was rewarded directly for his services, and saw his book published with honors and at the nation's expense. In turn he dedicated it to the emperor who thus befriended him and encouraged science. Besides, La Place also had access, of course, to all that had already been written upon the subject, which had now been under the consideration of some of the brightest minds for more than a century and a half, and had been an established and demonstrated science for almost exactly a century; for Bernoulli's "Ars Conjectandi" appeared in 1713, and La Place's "Analytical Theory of Probabilities" in 1812.

Although after the publication of La Place's monumental work the evolution of the science of probabilities may be considered to have been completed in all essential particulars, there remained much to do in the matter of rendering the same lucid and readily intelligible, and also of adapting it to the practical purposes for which it was intended. This was the task of the nineteenth century especially, as will more fully appear when we take up actuarial science as a separate study, although this development also began early in the eighteenth century. Most of these productions have taken the form of works upon special applications of the science of probabilities, but there have been at least three books upon probabilities which deserve particular mention.

In 1838 Professor Augustus De Morgan published his "Essay on Probabilities," which, notwithstanding the modesty of the title, was a very thorough and complete work upon the subject, making a thick volume. It became the standard text-book upon probabilities in the English language, and held that place until this branch of mathematics began to be taught in the universities as one of the higher branches, when a work that is more technical, less diffuse, with more formulas and less of explanation and reasoning, and, above all, with many examples for solution, was demanded, and was supplied in Whitworth's "Choice and Chance," published within the present generation, and now the leading English text-book upon the subject. In America, although in our country the principles of the science of probabilities have been more widely applied in practical affairs than in any other country, nothing has been done toward making text-books upon the subject, and very little instruction upon it has been given in schools or colleges.

A work of the greatest importance upon the principles of the science of probabilities, and upon their application to many things, was published in French in 1845. It was the work of a Belgian, M. Quetelet, and was entitled "Letters on the Theory of Probabilities as Applied to Moral and Political Sciences." The writing of these letters began in 1837, without any thought of publishing the same. Their value and significance may be estimated, however, by the fact that almost immediately upon their appearance they were translated into English and other languages, and became the most valuable text-book upon the subject that its students could turn to. This was very largely because of the fullness and clearness with which the application of the law of probabilities to point the lessons of statistics was set forth, though the same lucidity marked the whole of the work.

The history of the development of the science has thus far been traced from the earliest times to the present. It is, as we have seen, essentially a purely modern science. This branch of mathematics could hardly develop in advance of algebra, decimal notation, decimal fractions, series, annuities certain and the like, for much of it, in theory or application, depends upon one or all of these. Singular is it, also, that the first great work upon the subject and the last, in the sense of an original and suggestive work, the contributions of Pascal and Quetelet, have both been in the form of letters, written with no intention originally of publishing the same. We shall see, when we come to consider the origin of mortality tables and the evolution of actuarial science, that the growth of the body of knowledge concerning probabilities has been accompanied by the application of the principles of the science to the practice of insurance, and especially of life insurance.

It remains, then, only to discuss what this science of probabilities is which we have been tracing through its different stages of historic development. The primary discovery was that when a thing had happened to a certain number out of a group, to each of which it was a priori equally likely to happen, the probability that it had happened to a particular one in the group, when one did not know which, could be expressed by a fraction, of which the number to whom it had happened was the numerator, and the number in the whole group, that is, the number to whom it had happened and to whom it failed to happen, combined, was the denominator. This was the fundamental discovery.

Following this at no great distance was the discovery that the chance that the thing had failed to happen to a particular one in the group could be measured by a fraction, with the number to whom it had failed to happen as a numerator, and with the same denominator as before. A little consideration sufficed to show that these were complementary values, the sum of which was unity, because adding them gave the number to whom it happened plus the number to whom it did not happen, both for numerator and for denominator. Thus it was determined that the sum of the chances that a thing happens and that it did not happen is always unity;

but, since such a summation also brings certainty, for as to a given individual it may with confidence be affirmed that the thing either has happened or has not happened, it has been customary to represent the mathematical value of certainty by unity. A very ingenious proof that this is the correct thing, and indeed the only thing to do, is given by Whitworth in "Choice and Chance," already referred to; it will be seen that actual proof offers difficulties, though it is not hard to see that the thing is true. The following is the passage:

"But it may be asked, is certainty a degree of probability at all, or can smaller degrees of probability be said to have any ratio of certainty? Yes. For if we refer to the instance already cited of the six passengers in the ship (from which one person had been lost) we observe that the chance of the lost man being a passenger is six times as great as the chance of his being our friend (one of the six). This is the case, however great our ignorance of the circumstances of the event; and it will evidently remain true until we attain some knowledge which affects our friend differently from his fellow passengers. But the news that the lost man was a passenger does not affect one passenger more than another. Therefore, after receiving this news, it will still hold good that the chance of the lost man being a passenger is six times as great as the chance of its being our friend. But it is now certain that the lost man was a passenger; therefore the probability that it was our friend is one-sixth of certainty."

Since by the fundamental rule of probabilities we will also find that it is one-sixth of unity, it follows that certainty is correctly represented by unity.

From these simple laws the science of probability extended to compound probabilities. One of the most remarkable of the early discoveries was that, when you know the chance that one thing will happen, and also the chance that another will happen, the chance that both will happen, unless one is prohibitive of the other, is not the sum of the chances, but their product. And from this the investigations proceeded to deal with all imaginable compound and complex probabilities, many of the most difficult being treated in advance of the development of principles for determining the simpler and easier values.

It was a long step, but one soon taken, from the probability that a thing, known to have happened to one of a group, has happened to a particular one, to the probability that events will occur in future, a probability based upon averages, depending upon their reliability, their extent, the correctness of classification and their applicability. These problems have been so difficult in operation that to this day an enormous amount of the theory of probabilities remains unappropriated by applied sciences because of the great practical difficulties in the way of so applying them. Many very interesting subjects of speculation in the laws of probabilities are therefore neglected by the actuary in his practice, and, indeed, only the simpler elements of the science are at the present time employed by him, though he would be a rash man indeed who would undertake to say what will be the limit of their practical application next year.

Though coming so late in the family of mathematical sciences, probabilities already rejoice in a biographer of no mean reputation. Todhunter, the great English mathematician, has written and published an extensive work, now unhappily out of print, upon the history of probabilities, in which he traces the growth of the true theories, the correction of errors and the development of new principles with great fullness and precision. His account begins with the earliest writers upon the subject and closes with La Place. From it much of the material for this lecture has been obtained, and particularly so concerning some of the smaller and less important pamphlets which are not to be found in the usual library of the actuary. As to the more important, including even Bernoulli's "Ars Conjectandi" and De Moivre's "Doctrine of Chances," as well as La Place's great work, references to such have been made directly.

SECOND LECTURE—MORTALITY TABLES, ANNUITIES CERTAIN AND ACTUARIAL SCIENCE PROPER.

We have seen how the fundamental mathematics, requisite before actuarial science could develop, evolved; we have now to consider the genesis and evolution of the application of the laws of probability to insurance, and especially to life insurance. And we shall find that, first of all, there needed to be something for the law of probability to act upon, viz., a mortality table, and also a handmaiden, in the form of a developed law of compound interest and discount and of annuities certain.

First, then, as to mortality tables. We find the first of them in times just back of mediæval, and just this side of ancient. The pretorian prefect, Ulpian, in 364 A. D., put forth a table of life expectancies or probable lives, invented for the purpose of estimating the value or annuities for probate purposes. The Roman law at that time provided that not more than a certain portion of the value of an estate could be willed away from the heirs. But testators left annuities, as charges upon their estates, the same being payable to beneficiaries for life; and so it became necessary to estimate the value of these annuities. The Ulpian method was simple. It assumed that a term of an annuity certain could be employed to express the value of a life annuity, and that this term was that of the probable life of the annuitant, or that term of years at the end of which the chances would be equal that he was living or dead.

Actuarial science could have no real genesis until this fallacy was dead and the truth had appeared. Ulpian's experiment was forgotten, and his theory was wholly lost when we next encounter something like a mortality table, and this time with true principles applied to ascertain the value of an annuity. The author was no less a person that John de Witt, Grand Pensionary of Holland and West Friesland, who turned aside from the onerous duties of state to consider the subject of life pensions or annuities. Such were already granted by states which were hard pressed for money and which could not negotiate loans in the usual way. De Witt formed the idea that the prices at which such were sold were utterly inadequate, and he proceeded to investigate the subject. His work was done before 1671, when the results of his labors were presented in a state paper. This was not many years after the appearance of the first pamphlets about probabilities, as, for instance, Pascal's letters. It was more than forty years before the publication, in 1712, of Jacob Bernoulli's "Ars Conjectandi," the first comprehensive work on probabilities. Therefore, it may be truly said, not merely that actuarial science began with de Witt, but also that he built much of the substructure for it; since in his work he developed newly, if not first of all, the fundamental principles of probabilities.

Most important of all else, though, in his work, was the discovery of the true theory of the operation necessary to compute the values of annuities; and this was: Take a large number of annuitants at a given age, the mortality for each age being known, and follow them down through life as they drop out and disappear, computing the amount which would need to be paid each year; reduce these sums to their present value; add them all together, and you have the aggregate present value of all sums payable on account of these annuities. The equal share of each annuitant in this aggregate is manifestly the value of his annuity.

The value thus found is not the same that would be found by treating the annuity as equivalent to an annuity certain for the term of the expectancy, or of probable life, according to the same table of mortality. But this value will prove by actually running out, on the basis of the interest and mortality assumptions, while the value from the annuity certain for a term equivalent to the expectation or the probable life will not. Moreover, when we come to examine into the matter, we cannot fail to be impressed with the circumstance that the expectation could only furnish an equivalent when there is no interest counted upon, for it is the average of years of life; while in effect, the value of the annuity is the average of the same years of life, discounted. But the discount is greater for payments which are longer deferred; from which fact it appears that the equivalence no longer holds, when discount is introduced.

The Ulpian theory died hard. It reappeared repeatedly after actuaries had learned to deal with the subject according to science and reason. Thus in 1738, after the true principles were clearly enunciated and had been adopted by mathematicians generally, a learned London barrister, Weyman Lee, published an ingenious book in which he undertook to prove the equivalence. The name of his book was "An Essay to Ascertain the Value of Leases and Annuities for Years and Lives." The following passage is illustrative of his reasoning:

"I suppose no one will controvert these points: That he who has an annuity for the life of a person, has an annuity for such a term of years as such person in fact shall live; and, when he buys it, the term of years to which any person's life shall be prolonged being uncertain, that he buys it for such a time as there is a chance or reasonable probability that the person may live whose life is nominated."

Lee also objected that the other theory made the value of an annuity by the same mortality table equivalent to the values for different terms of years, according as the interest assumed was one rate or another. We have seen that this is necessarily so, and that only when no interest is assumed is the value equal that of an annuity certain for the expectancy.

The false theory, dating back to Ulpian, and so ably championed by Lee, is a very natural generalization when one makes a hasty conclusion, and it is certainly extremely plausible. As if to emphasize that the human mind in its evolution ever travels over the same track, we find this idea reappearing every now and then by a strange sort of atavism, as it were, and in the most unexpected quarters. Thus within the year a leading financial paper of New York gravely announced editorially:

"Life insurance companies employ two factors in fixing premiums on policies of insurance. They are, first, the average expectation of life of the insured person, according to carefully prepared mortality tables; second, the rate of interest at which money can be invested. The company takes the case of a man at, say, 21 years of age, and finds from the mortality tables that his average expectation of life is, say, 30 years. By simple arithmetical calculations it computes the amount of money that such a man should pay, annually, semi-annually, or quarterly, as the case may be, in order that his money at the assumed rate of interest, say 3½ per cent, shall at least equal the amount of his policy at the end of thirty years. The amount, thus computed, with an allowance for expenses and profit, is his premium."

This apparent digression is really no digression at all. We are to follow the growth of actuarial science, and it is worth while, therefore, to know what it is, and particularly what it is not, and why, therefore, John de Witt must be considered its founder, and not Ulpian.

de Witt, in addition to discovering the fundamentally true theory of annuities for life, and applying the principles of probabilities and compound discount to the ascertainment of their values, also invented a mortality table. This he constructed on a plan of equal decrements, limited to from ages three to fifty, however, followed by other periods in which the law of equal decrement was also supposed to work. We shall encounter this supposed law, and modifications of it, again and again. It reappeared in the work of the earliest English actuary who, so far as we know, was not acquainted with de Witt's work at all. De Witt also foresaw that there would be a selection of the best lives for annuitants, and that a mortality much lower than in the general population was to be expected. Thus he foresaw a truth which it took the British government many years, involving great loss, to discover.

The next discovery was by the British philosopher, Halley, who, in 1693, published in the "Transactions" of the Royal Society the result of his investigations into the mortality of the city of Breslau, Germany, which had given out a complete statement of deaths, by ages, for a number of years. Assuming a stationary population, Halley proceeded to construct a mortality table from the records of deaths only. Halley thus produced a complete mortality table showing the diminution of the number living each year out of an original group. By means of this table he also proceeded to give the now common and always simple formulas for computing the values of annuities. The table which he made is the first that was made from population statistics. The development of actuarial science was in a large degree due to him. de Witt's work lay buried in the archives of Holland and West Friesland, and was only uncovered within the last fifty years.

I have said that Halley's Breslau table was the earliest that was drawn from actual experience, and this is true so far as a table showing the mortality at each year of age is concerned. But, as early as 1662, John Graunt had published an analysis of the mortuary returns of London, and had formulated a mortality table in a rough form, but not giving the rate of mortality at each year of age. His work, however, had attracted much attention, and passed through several editions.

The next great advance was when De Moivre published in 1725 his great work on life annuities. A French exile, living in London, he had already distinguished himself in mathematics, and especially in his works on probabilities. Now he proceeded to apply his learning to the practical purpose of determining the values of leases for life and other life annuities. Already the summation of series, when following some mathematical law, had been developed to a point which, he plainly saw, rendered it comparatively simple and easy to compute annuity values, whether for one or for more lives, if only a mortality table could be brought to follow a mathematical law. He investigated the Breslau table put forth by Halley, and discovered that within reasonable limits, for a long period of life at least, and possibly for the whole, the number surviving to each year of age out of a given number setting out from the earliest age of the table, was a term in a decreasing arithmetical series. That is, he found that if the number surviving be treated as a term of such a series, it would diminish very nearly as in the original table. This means that the number of deaths could be assumed to be the same for each year of age out of the number originally setting out. Thus, if the deaths, for instance, were 1,000 per annum out of an original 100,000, setting out at age 20, this would mean 1,000 out of 100,000 at age 20, 1,000 out of 99,000 at age 21, etc. So the probabilities of death would each year increase, not because of any change in the numerator of the fraction, but solely because of the steady and regular diminution of the denominator. Without going into details at this time, it must be readily seen that for the purpose of determining any value or function whatever, the arrangement of a mortality table as a mere regularly decreasing arithmetical series offered great advantages.

De Moivre also discovered, somewhat later in his career, that complete tables of annuities could be made by a continuous process, starting from the extreme limit of life, without computing all the values separately which go to make up the value of each annuity. This discovery applied, first, to annuities certain; but its application to life annuities was soon discerned as well. A word of explanation may be needed to make the matter clear to you, and I linger to give it because this is the first great labor-saving invention in the science, which now embraces so many.

Suppose, then, that you know the value of one, due at the end of one year, and desire to discover the value of an annuity of one for two years. This is plainly worth at the end of one year, and before the first annuity payment is made, one plus the value of one, due at the end of one year. Therefore, at the beginning of the first year it will be worth one plus the present value of one, due at the end of one year, the whole multiplied by the present value of one, due at the end of one year. And so, by a continuation of this process, a complete table of the values of annuities for each of long periods may be successively computed. The same principle, the discounts merely being made also to include the probability as well as the interest factor, applies to life annuities; and this was De Moivre's discovery. It was also independently seen by both the great mathematician Euler and also by Thomas Simpson, De Moivre's younger rival in England.

It was in 1742 that the first work of Thomas Simpson upon the subject, his "Doctrine of Annuities and Reversions," made its appearance. Simpson, as was stated in the previous lecture, was a self-taught, non-academic mathematician, believed to be the greatest of such that England has produced. He had but emerged from obscurity and ill fame, owing to conjuring and necromancy. That he should come forth as a rival of De Moivre was almost insupportable by the latter, who was now in the flower of his age and at the zenith of a well-earned reputation. Simpson's book, too, was an attempt to popularize the science, never a popular movement among those who hope to profit by keeping it exclusive. And, worst of all, he had little respect for the doctrine of equal decrements, and, indeed, did not spare it.

Simpson urged a return to Halley's method of taking the facts as one found them, instead of seeking to compel the mortality table to follow a rule to which it was indifferently suited. At the same time, on the basis of De Moivre's principle, he discovered an equal age formula for joint life computations, which greatly simplified the operations, and which bears his name to this day. But perhaps his greatest distinction is this, that he, first of all actuaries, fully realized the importance of the life insurance phase. Up to his day life insurance was known in but two forms, one being policies for short terms, without privilege of renewal, and the other, insurance for the whole of life for indeterminate amounts, to be determined by the proceeds of assessments. Simpson introduced the idea of whole life insurance for level annual premiums and, as we shall see, he also afterward introduced the practice.

At about the same time Des Parcieux, the elder, produced in France his work, "Probabilities of Human Life," published in 1746. This was practically the beginning of annuities in that country. It was followed in 1749 by Buffon's "Tables of Probabilities of Human Life," which was mainly a discussion of tables already published, but included also some new tables. Des Parcieux, in his work, gave tables which he had constructed from the experience of two tontine funds and from the mortality in fourteen monasteries. The last became the standard table for insurance in France and remained standard until within the last generation.

In 1753 the scientist Kersseboom of The Hague brought out a volume discussing the tables of Halley and Buffon, and developing some of the principles of the science.

James Dodson, an English university man of high standing, began in 1748 the preparation of a volume entitled "Analytical Solution of Problems Relating to Annuities." Mr. Dodson began to take an interest in the subject comparatively late in his busy life, and his interest was rather sharpened than otherwise by the discovery that, while he might procure life insurance at very high rates in a stock company, and for a very short period, he could get no whole life insurance at all, even in the Amicable on the assessment plan, because he was more than 45 years of age. He became one of the earliest and most active and influential of

Simpson's converts to the view that life insurance could be furnished on a whole life plan with equal annual premiums, and that safe premiums for the same could be computed. Together these men labored for several years to bring enough people together to organize a company on the mutual plan—they had no hope of interesting stockholders in so long and uncertain a venture, even in those speculative days. The result was the "Society for the Equitable Assurance of Lives and Survivorships," which, now generally known as "The Old Equitable," survives to this day. The organization was not without difficulties, and had to be effected as an unincorporated voluntary association, practically a copartnership, because the Parliament refused a charter on the ground that the undertaking was very risky.

The tables which were employed by the Equitable were based upon a mortality table constructed by Dodson from the mortuary statistics of London. Fortunately the experience proved an exaggeration of the deaths which the company experienced, and as it did much more life insurance than annuities, the result was favorable. Of course it would have been disastrous if the contrary had proven the case, or if the annuity business had exceeded the life insurances, for the same mortality table was employed for both.

Neither Thomas Simpson nor James Dodson, though they contributed so much to the foundation of the Equitable, and indeed were and are recognized to have been its founders, having stirred up interest in the matter by their publications and by lectures upon the subject, in London and elsewhere, was included in the management of the society; but it set forth upon its career on plans devised and furnished by them, and its success is theirs.

In the same year that the Equitable began operations there appeared a first edition of a most remarkable contribution to actuarial science, by Dr. Richard Price, a Unitarian preacher, who was destined to have much to do, both indirectly and directly, with the development of life insurance, and of the Equitable in particular. The work was entitled "Observations on Reversionary Payments." It passed into four editions, each much enlarged, during the succeeding twenty-one years, and was the authoritative work upon the subject for a very long time, a revised edition being published so late as 1806.

Dr. Price warmly supported Thomas Simpson's view that it was folly to construct mortality tables by purely mathematical principles, and urged that investigations of the actual death rates be made as the only means of making a reliable table of mortality. He also encouraged with enthusiasm the work of the Equitable, and in his later editions contrasted sharply its scientific and safe system, assuring solvency, with the unsafe plans and insolvent condition of all the assessment annuity concerns with which the country was then pestered. In addition to his demonstrations of these things he himself analyzed several bodies of population statistics, such as those of Sweden, which had recently been published, constructing mortality tables from them; and, which proved most important of all, he collected the data concerning deaths in several parishes of Great Britain, notably Northampton, and made mortality tables from these. The Northampton table enjoyed great repute, and was the standard for both life insurance and annuity computations for many years. It was adopted by the Equitable, and Dr. Price was invited to prepare monetary tables on that basis for that company. Later, his nephew, William Morgan, was made actuary of the company, a post which he occupied throughout his long life. The theories of De Moivre were, for the time, completely lost sight of, and it was conceded everywhere that the only correct system of developing a mortality table was to follow the facts of statistics closely, not expecting the same to express a mathematical law. William Morgan also distinguished himself by the publication of many books and pamphlets, including contributions to the Royal Society, of which he was an honored member. His most important work, perhaps, was entitled "Doctrine of Annuities and Assurances on Lives and Survivorships," published in 1779. To survivorships he devoted much study. He also edited and revised the edition of Richard Price's "Observations on Reversionary Payments," which appeared in 1806.

In France progress was also being made, although there annuities proved more popular by far than life assurances, which is true even to this day. In 1767 the great mathematician, Euler, published a work upon "General Researches into the Mortality and Increase of the Human Race," and in 1781 DesParcieux, the younger, contributed a "Treatise on Annuities." In Great Britain, Maseres, with a foreign-sounding name, published a large book, of no very great value, entitled "Principles of the Doctrine of Annuities," in 1783; in it he endeavored to make the whole thing a matter of arithmetic.

There was great progress in life insurance during the last part of the eighteenth century, but little was done in the direction of increasing either the material upon which actuarial science works, or the methods of doing the work. The Northampton table was now fully accepted in Great Britain as the standard, and the government even sold annuities based upon its accuracy.

In 1810 the monumental work of Francis Bailly upon "Life Annuities and Assurances" appeared. It was the first comprehensive and thorough treatise that had ever been published upon the subject. What was known had by this time clarified in men's minds, and it was now possible to treat the topic thoroughly and well. Bailly's equipment for this task was nearly as good as that of La Place for dealing with probabilities; and it is noteworthy that the two epoch-making works made their appearance at about the same time, La Place's "Probabilities" appearing in 1812. Bailly's book was so good and so comprehensive that as late as 1864, when by reason of the invention of commutation columns as a convenience for calculations and of a change in the symbols, the book had become antiquated, its worth caused it to be rewritten, edited and produced with modern symbols, so as to continue to be useful to the student. Yet, notwithstanding his abilities and accomplishments, some of which, of course, embraced the correction of others' errors, or perhaps, more accurately stated, because of these things, Bailly was anything but welcomed by Morgan, now aging, and now also the absolute master of things actuarial. In consequence a feud, now disguised, now breaking forth, raged between the men during their joint career.

In 1815 appeared Joshua Milne's "Valuation of Annuities and Assurances," also an epoch-making work; for, in addition to much that was new and clearer in the text, the author included a new mortality table, drawn from the experience of the parish of Carlisle. This table was the first to be graduated so that it seemed to express a regular force of mortality, although for the same no mathematical expression was known or sought. The influence of Richard Price was yet too powerful to permit turning one's attention in that direction anew. The graduation was by what is now known as the graphic method, and it was necessary, particularly in this case, because the number of lives dealt with was very small. This table showed a much lower mortality rate than the Northampton table and came far nearer, also, to expressing what the companies were finding to be their own experience. This was even confirmed by the experience of the Equitable itself. In consequence most of the younger companies promptly adopted the new table, especially as the Northampton was not merely not fitted for annuities but was positively dangerous. But the Northampton held its own with the Equitable and some of the older companies, and there was a bitter fight about the matter, the echoes of which have, even now, scarcely ceased to rumble.

In 1820 appeared Hendry's work on "Life Annuities," and in 1825 Griffith Davies' remarkable book, entitled "Tables of Life Contingencies." This work introduced into general use the second great labor-saving device of actuarial science, the commutation column. A suggestion of something of this sort had been made by William Dale as early as 1772 in his "Introduction to the Study of Annuities," and it had also been suggested by William Morgan in the introduction to his principal work, published in 1779. A much more completely developed system had also been devised by the German scientist, Tetens, and published in 1785; but his system was not known to actuaries until much later, having been buried in a work upon another subject. In 1812 Francis Bailly presented to the Royal Society a paper by George Barrett, a self-taught English mathematician, in which a system was outlined. The paper was read, but not published, by the Royal Society, an omission which does it little credit, which was attributed to William Morgan, who was upon the committee, and is supposed to have been actuated by envy of Bailly, and in any event did not see the importance of what he had himself suggested, and which called forth the bitterest criticism from Bailly, who included the paper in the second and subsequent editions of his own book. Unfortunately, as a result of the indifference of the Royal Society, Barrett was unable to print the tables which he had prepared, and his invention for the time was of no practical use.

Griffith Davies, himself a self-taught mathematician, who had risen by hard work and his talents to a responsible post as actuary of a leading London company, developed the idea along somewhat different lines from those employed by any of his predecessors, and by adding actual commutation tables, according to the Carlisle table and various rates of interest, he also made his ideas of practical use forthwith.

In the same year Benjamin Gompertz, a great mathematician, who otherwise gave little attention to actuarial problems, wrote a letter to Francis Bailly concerning a mathematical force of mortality which, according to his view, could be given expression in terms of the infinitesimal calculus and algebra, and which he believed to be at the foundation of all mortality tables. In his letter he explained that the natural law, which he considered to be expressing itself in the mathematical law, was that as men grow older the vitality wanes, and that the measure of this change caused the change in the death rate and could be evolved out by differential methods. Thus within the first twenty-five years of the new century the supremacy of the Northampton table, which at the beginning of that century seemed so securely fixed in that position, was threatened by two foes, one the Carlisle table, which disputed its data, and one Gompertz's discovery, which renewed the interest in De Moivre's original theory of a mathematical law for mortality, which theory had been regarded as hopelessly dead. But while the Carlisle table was making its way rapidly, the theory of Gompertz, being found to square with the facts of none of the tables, was for a time thrust aside.

In 1826 the eminent British mathematician, Charles Babbage, published a work entitled "A Comparative View of Various Life Offices," which showed a marked bias for the Equitable. The book is mainly non-mathematical, however.

In 1832 T. R. Edmonds published a book in which he further discussed the possibility of employing a mortality table in which, while it accurately represented experience, the graduation was according to a mathematical formula. His book was entitled "Life Tables, Founded upon the Discovery of a Numerical Law Regulating the Existence of Every Human Being, Illustrated by a New Theory of the Causes Producing Health and Longevity." The system proposed by Mr. Edmonds bears a striking resemblance to that suggested by the originator of actuarial science, John de Witt, whose essay upon the subject was at that time sleeping unknown to all actuaries in the archives of Holland and West Friesland. Edmonds' system was not adopted, but the ideas which are set forth in the book, and also are plainly indicated by the singular title, have survived, and, indeed, found lodgment in many minds. As Gompertz had thought of the thing, "force of mortality," Edmonds thought of the name, and while Edmonds' formulas are forgotten, his name for the tendency which he did not accurately measure has become a permanent addition to life insurance terminology. Already, before Edmonds, Thomas Young espoused the cause of the mathematical formula in a letter which, much to William Morgan's disgust, was accepted and printed by the Royal Society in 1826. It was entitled "A Formula for Expressing the Decrement of Life," and it led to a controversy with the nephew of the great Richard Price, who considered that the reputation of himself and his uncle depended upon refuting all arguments in favor of a mechanical system.

The Northampton table was giving way, however, before an enemy more formidable than the mathematical formula; that enemy was experience. For the British government found its experience with annuitants disastrous, having sold annuities at prices fixed by this table; and life insurance companies in general found the table to absurdly exaggerate the losses which they might expect. Consequently the Carlisle table steadily made its way into favor. The publication of the government annuity experience, tabulated by John Finlaison in 1823, and adopted by the government in 1829, completely displaced the Northampton table as to its use for annuities, and, indeed, it was thereafter retained by no company of importance excepting the Equitable, which stubbornly retains it, even to this day, or at least was doing so at a recent date.

Another consideration which occurred to de Witt at the very beginning forced itself upon men's notice in connection with annuitants, and that was that there is a very great measure of self-selection among annuitants, assuring that the average of longevity will be much higher than among the population, and also than among insured lives, in spite of the careful medical selection which companies had learned to make. This made all tables that are suitable for insured lives unsuitable, for that very reason, to measure the value of annuities. The fact was recognized in France before it was in Great Britain, for in France the Duvillard table, named after its author, was introduced so early as 1806, and being better suited to the purpose than DesParcieux's table, was adopted for annuity purposes. In Great Britain the government experience table of 1829 was employed until 1860, when a second government table appeared, and this in succession until 1883, when a third government table appeared, all the production of the Finlaisons, father and son. At this time an unadjusted table has just appeared, drawn from the joint experience with annuitants of the chief British companies; and while it confirms in a remarkable manner the lessons of the government tables, it shows even a greater longevity and will doubtless be mainly employed hereafter.

In Edmonds' work, in addition to discussing the force of mortality, he considered the force of sickness, i. e., the liability to disablement, which he deemed to be subject to the same law, being only another expression of it. There appeared in 1835 a work entitled "A Treatise on Friendly Societies," by Charles Ansell, which contained the experience of the Highland Society in relation to sickness, and also the combined experience of a number of Scottish friendly societies. Ansell supplied monetary tables for the same. These were adopted as standard by the friendly societies generally, and as they proved to be too low, much embarrassment was caused thereby. No trace of evidence that Edmonds' theory was correct was discernible, however, in this; nor, indeed, could its correctness be tested before there were several experiences at hand to compare. The publication of Henry Ratcliff's report upon the experience of the Manchester Unity of Odd Fellows for 1846-8, and of the report upon the experience of the Foresters, by Frederick G. P. Neison, at about the same time, opened the way for a thorough development of the subject. Two later reports of the Manchester Unity appeared during Mr. Ratcliff's lifetime, viz., one covering 1856-60, and the other 1866-70; and, as a result of careful comparison, it was developed that, so far as could be determined a priori, the liability to illness could be represented as a function of the force of mortality. The very recent investigation of the entire aggregate experience of the British friendly societies, during the present year, confirms also the theory that from a mortality table the liability to sickness may be ascertained with much accuracy. It is still, however, a matter of ratio, and the algebraic expression for a force of sickness has not as yet been developed. Yet another report upon later experience of the Manchester Unity is now in preparation.

With the advent of the commutation method of making computations there was naturally a great change in symbols, and a new notation began to come in. In consequence of this, and of the new formulas as well, that were found necessary, the old text-books were rapidly becoming antiquated. Naturally, therefore, Prof. de Morgan's "An Essay on Probabilities and on Their Application to Life Contingencies and Insurance Offices," published in 1838, and especially "A Treatise on the Value of Annuities and Reversionary Payments," by David Jones, published in 1840, were generally welcomed. The former remained for a long time the best English text-book on probabilities, though not so highly esteemed as to its actuarial content, while the latter was for more than a quarter century the book of books for the actuarial student, and is valued unto this day.

The years from 1840 to 1860 were fruitful in the elucidation of many questions which were very difficult to handle before the commutation system was introduced. They also saw the beginning of the great work of England's registrar-general, Dr. William Farr, author of English Life Tables Nos. 1, 2 and 3, the first two of which appeared during this period. Toward the close of the period the Institute of Actuaries was organized in London, and although this body was frowned upon by many of the actuaries of great reputation in that day, who refused to join, and organized the Actuaries' Club as a more exclusive and aristocratic society, the Institute, by setting its standard higher and higher, and also by excluding all considerations as to candidates except those of fitness, won a pre-eminent place for itself and also great respect for the entire profession. It also founded a journal in which from that time forward the most valuable discoveries and inventions have usually been first given to the world. The history of the progress of actuarial science may be read since its foundation in the volumes of this journal.

To this "Journal of the Institute of Actuaries" William Makeham, in 1860, contributed what has proved to be a solution of the disagreement between the school that followed De Moivre and the school that followed Dr. Price. In other words, he discovered and gave to the profession the extension of the principle laid down by Gompertz, which was needed to reconcile his theory as to the force of mortality with the facts of experience. Gompertz had guessed that increasing mortality was the consequence of waning vitality, and could be measured by a steadily increasing function. Makeham divined that deaths might be divided into two classes, viz., from causes that apply with almost equal force at all ages, such as accidents, and from causes which flow from diminished vitality. A modification of the formula of Gompertz to agree with this idea gives an expression for the force of mortality which has proved most useful, and also to measure with great accuracy the facts of experience.

Notwithstanding this discovery, so strong was the force of tradition and the influence of the school of Dr. Price that when a table of mortality was constructed from the experience of 20 British companies, under the auspices of the Institute, the work being completed and published in 1869, the same was graduated by a formula which was really merely a very smooth mathematical expression for the graphic method; and it was not before 1887 when the second volume of the "Institute of Actuaries' Text-book" was published, that tables were published in Great Britain which were graduated by Makeham's formula. It was found that such graduation gave results very close to the original data, and the system is now practically universally accepted. It presents all the advantages in the matter of facility of computing joint life functions that were originally foreseen by Simpson and others.

The 20 British Offices table was not the first combined experience table to be constructed in Great Britain. In 1843 a table, known in Great Britain as the 17 Offices, and in America as the Actuaries', or Combined Experience Table, was published, having been under construction for a number of years. It was hotly attacked in England by both the friends of the Carlisle table and the lingering champions of the Northampton table, and it never became standard there; though tables of monetary values were published by Jenkin Jones.

In the United States, on the contrary, this table, chancing to have come out at about the right moment, and appealing to Elizur Wright as a fair test of solvency, was introduced by him in 1859 as a standard for valuing the policies of companies doing business in Massachusetts, of which State he was commissioner. In consequence this table was pretty widely adopted by American companies and departments, and is yet in very common use.

Early in the sixties there was published a table drawn from the experience of the Mutual Life Insurance Company. This table was graduated by Sheppard Homans, actuary of that company. A few years later Mr. Homans published another table embracing a modification of the Actuaries' table, by the experience of the Mutual Life, and with some other changes. This table was given the name American Experience Table, and was adopted as the standard in New York and in other States, as well as by the Mutual Life and many companies. It is now also in common use.

As in Great Britain, so also in the United States, the first table giving the combined experience of the home companies has been discredited and little used. In 1870 a table covering the experience of 36 American offices, which had been prepared under the direction of the Chamber of Life Assurance and a committee of prominent actuaries, made its appearance. Although it was graduated according to Makeham's formula, and thus afforded many advantages in actuarial operations, and although it was accompanied also by a most elaborate set of monetary tables, it was adopted by no company of consequence, and has never been standard anywhere.

The principal contributions of America to actuarial mathematics have been three formulas for the computation of individual reserves by what is known as the retrospective method, which are known by the names of their authors, as follows: Wright's, Fackler's and McClintock's formulas. The first two are more commonly employed to compute values of policies involving one life only, and the last to compute the values of joint life policies, for which purpose it is singularly useful. Another important discovery was the contribution plan of dividing surplus, the honor of which is due to Sheppard Homans and David Parks Fackler, jointly.

The most recent developments in the mathematics of insurance are due to one man, who has by competent authority been pronounced the greatest living actuary, Dr. Thomas Bond Sprague. They are the discovery that a select mortality table, instead of the mortality tables commonly in use, represents most closely the facts of experience in life insurance companies. Thus the usual form of mortality table exhibits the average death rate at a certain age, for instance, without regard to whether the lives are freshly selected by medical examination or not. It has long been understood that the mortality during early years of insurance is comparatively light, and the theory of adverse selection by discontinuances had been exploited long before Dr. Sprague's time. Indeed, in the analysis which actuaries made of the mortuary experience of the Equitable Society when first published, they clearly distinguished the effects of adverse selection for twenty years after entry. But Dr. Sprague discovered that, in the later experience of companies, embodied in the data for the 20 Offices' table the effects of favorable selection appeared to wear off in about five years. So he recommended the use of select tables, composed of the actual experience during the first five years of insurance, combined with the general experience upon lives, admitted more than five years. He prepared and published the monetary tables necessary to give his views effect.

This was followed some years later by another discovery and suggestion of even greater practical importance, viz., that the net reserve system, as usually employed, was inaccurate, and should be modified. Thus he discovered that, owing to the circumstance that expenses are much higher the first year of insurance than later, the reserve which the usual methods brought out was neither just according to the fundamental meaning of the prospective nor of the retrospective formulas, because it would not be needed to make good the deficiencies of future premiums, the original premiums being adequate for all purposes; nor could it have been accumulated from past premiums, the conditions being as per assumption, and expenses being incident, as they actually are. He presented his views before British actuaries at various times, and at length before the first Congress of Actuaries in 1895. The issues which he raised are now being fought out by American companies and departments, mainly in the courts, in disputes as to construction of contracts, but also between opposing schools of actuaries. In France and some other countries the principles laid down by Dr. Sprague have already been adopted.

In the foregoing, upon looking it over, I find that I have forgotten to mention the invention of a system of conversion of annuities to insurance, and vice versa, and the production of a most remarkable series of conversion tables for annuities and life insurances, which was printed in 1856. It was the work of William Orchard, one of the most brilliant young mathematicians of Great Britain, whose early death was justly deplored. I have also forgotten to mention the work of David Chisholm, whose series of tables have been most valuable aids to actuaries.

There has been no lack of text-books in the last half of the nineteenth century. The Institute of Actuaries followed up its great work of collecting the combined mortality experience of twenty companies into a mortality table by putting forth a text-book which is now everywhere the standard authority for higher education upon the subject. It was published in two volumes, the first dealing with annuities certain and the second dealing with life contingencies. A second edition of the first volume is now about to appear, and a second edition of the second volume is in preparation. In America there have also been published text-books, though none which even pretends to the thoroughness of the Institute of Actuaries' publications.