PRINCIPLES OF ELEMENTARY ALGEBRA

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THE PRINCIPLES

OF

ELEMENTARY ALGEBRA

BY

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PREFACE.

In the following pages I have endeavored to put into form what in my opinion should constitute an Intermediate Algebra, intermediate in the sense that it is not intended for absolute beginners, nor yet for the accomplished algebraist, but as a stepping-stone to assist the student in passing from the former stage to the latter.

The work covers pretty well the whole range of elementary algebraic subjects, and in the treatment of these subjects fundamental principles and clear ideas are considered as of more importance than mere mechanical processes. The treatment, especially in the higher parts, is not exhaustive; but it is hoped that the treatment is sufficiently full to enable the reader who has mastered the work as here presented, to take up with profit special treatises upon the various subjects.

Much prominence is given to the formal laws of Algebra and to the subject of factoring, and the theory of the solution of the quadratic and other equations is deduced from the principles of factorization.

The Sigma notation is introduced early in the course, as being easily understood, and of great value in writing and remembering important symmetrical algebraic forms.

Synthetic Division is commonly employed, and the principles of its operation are extended to the finding of the highest common factor.

Except in the case of surds, no special method is given for finding the square root of an expression which is a complete square, as the operation is only a case of factoring, and a simple case at that. For expressions which are not complete squares, the most rational method is by means of the binomial theorem or of undetermined coefficients, both of which are amply dealt with.

Probably the most distinctive feature of the work is the importance attached to the interpretation of algebraic expressions and results. Algebra is an unspoken language written in symbols, of which the manipulation is largely a matter of mechanical method and of the observance of certain rules of operation. The results arrived at have little interest and no special meaning until they are interpreted. This interpretation is either Arithmetical, that is, into ideas involving numbers and the operations performed upon numbers; or Geometrical, that is, into ideas concerning magnitudes and their relations. Both interpretations necessitate observation and the exercise of thought; but the geometrical offers the wider scope for ingenuity, and is the better test of mathematical ability. In several cases, as in that of the quadratic equation, the solution frequently gets its complete explanation only through its geometric interpretation. Hence geometrical problems are freely introduced, and the relations between the symbolism of Algebra and the fundamental ideas of Geometry are discussed at some length.

The Graph is freely employed both as a means of illustration and as a medium of independent research; and through these means an effort is made to connect Algebra with Arithmetic upon the one hand, and with Geometry upon the other.

The exercises are numerous and varied, and I trust that they will be found to be fairly free from errors.

N. F. D.

CONTENTS.

For a more detailed statement consult the Index at the end of the book.

CHAPTER I.

Symbols, Definitions, and Formal Laws.

1. Arithmetic is pure or concrete. Pure arithmetic deals with abstract number or numerical quantity. Concrete arithmetic has relation to numbers of concrete objects or things.

Thus 3 is an abstract number, but 3 days is concrete.

Algebra is primarily related to pure arithmetic, but its extension to concrete arithmetic is an easy matter.

The quantities which are the subject of arithmetic are of three kinds:

(1) Whole numbers or integers;

(2) Symbolized operations called fractions;

(3) Numerical quantities which cannot be exactly expressed as integers or fractions, but whose values may be expressed to any required degree of approximation. Such are the square roots of the non-square numbers, the cube roots of the non-cube numbers, etc. This third class goes under the general name of *incommensurables.*

The expression *numerical quantity,* and frequently the word *number,* will be taken to denote any of the three classes.

2. Numbers are fundamentally subject to two operations — increase and diminution; but convenience, drawn from experience, has led us to enumerate four elementary operations, viz.: Addition, Subtraction, Multiplication, and Division.

All higher operations on numbers are but combinations of the four elementary ones.

3. Algebra originated in arithmetic, and elementary algebra is arithmetic generalized, the generalization being effected by employing symbols, usually non-numerical, to stand for and represent not only numbers or numerical quantities, but also the operations usually performed upon numbers.

Thus algebra becomes a symbolic language in which numbers and the operations upon them are written.

The symbols of algebra are thus primarily of two kinds:

(1) *Quantitative* symbols, which represent numerical quantities, and

(2) *Operative* symbols, which indicate operations to be performed upon the quantity denoted by the quantitative symbol.

A third class, called *verbal* symbols, may be enumerated, in which the symbol is a convenient contraction for a word or phrase.

4. The quantitative symbols are usually letters. The operative symbols, especially in elementary algebra and in arithmetic, are mostly marks or signs which are not letters. Relative position is employed to denote some operations, and in higher algebra very complex operations are often denoted by letters.

The verbal symbols do not denote quantity, and they cannot be said, in general, to denote operations.

The principal verbal symbols are:

(1) $=$ and $\equiv$, either of which denotes that all that precedes the symbol, taken in its totality, *is equal to* or *is the same as* all that follows the symbol, taken also in its totality.

(2) $>$ and $<$. The first denotes that all that precedes the symbol, taken in its totality, is greater than all that follows the symbol, taken in its totality; and the second is like the first with *less* put for *greater*.

Other verbal symbols will be introduced as required.

5. From Art. 3 it is seen that operations in arithmetic must be special cases of more general operations in algebra. And hence it follows that arithmetic and algebra must proceed on similar principles, and must be subject to the same formal operative laws.

That the generalizing process of algebra should introduce new ideas into arithmetic is to be expected; and that this generalization should carry us beyond the necessarily limited field of arithmetic is also to be expected. Illustrations will occur hereafter.

6. The operative symbol $+$ (*plus*) denotes addition, and tells us that the quantity before which it stands, and to which it belongs, is to be added to whatever precedes.

Thus, $5+3$ tells us that 3 is to be added to 5, and $0+3$ is the same as the arithmetical number 3.

Similarly, $+a$ is the same as a, whatever a stands for; and for this reason the sign $+$ is seldom written whenever it can be dispensed with without producing ambiguity.

$a+b$ is the same as $+a+b$, and indicates that the number denoted by b is to be added to the number denoted by a.

7. Any interpretable combination of quantitative and operative symbols is an **algebraic expression.** We shall, in the meantime, confine ourselves to expressions written in a single line, as

$$3ab + 2c + d^2, \text{ etc.}$$

In arithmetic we know that $3 + 5$ is in its sum the same as $5 + 3$, and $3 + 5 + 8$ is the same as $3 + 8 + 5$, the same as $8 + 5 + 3$, etc. And as this must be a particular case of algebra (Art. 5), we must have $a + b = b + a$, $a + b + c = a + c + b = b + c + a =$ etc.

This is the *Commutative Law for Addition*, and is expressed by saying that *the order of adding quantities is arbitrary, or the sum is independent of the order of the addends.*

8. The symbol $-$ (*minus*) placed before a quantity indicates that the quantity is to be subtracted from whatever precedes the symbol.

Thus, $5 - 3$ tells us that 3 is to be subtracted from 5; and $a - b$ tells us that the quantity denoted by b is to be subtracted from that denoted by a. Now, a and b denoting any numerical quantities, as long as a is greater than b the subtraction is arithmetically possible, and the result is an arithmetical quantity. But if a is less than b, the operation symbolized is not arithmetically possible. The expression $a - b$ is then a symbolic representation of an operation that cannot be arithmetically performed, and the result of the operation, whatever it may be, is not arithmetical.

Thus, $0-3$, which is simply written -3, and which is called a **negative** number, does not belong to pure arithmetic, but is an idea introduced into algebraic arithmetic by generalizing the operation of subtraction.

And thus to every pure number, called now a positive number, corresponds an algebraical negative number, the relation between corresponding numbers being that their algebraic sum is zero or nothing.

Negative numbers are important in their relations to concrete arithmetic, and especially where geometric ideas are concerned. This matter will be dealt with in Chapter VIII.

9. It is said in Art. 2 that numbers are fundamentally capable of only increase or diminution. Hence $+$ and $-$ symbolize the two great operations in arithmetic and algebra. These are distinctively **the signs** of algebra.

By the *sign* of a quantity is meant that one of these two signs which precedes the quantity; and to *change signs* is to change $+$ to $-$ and $-$ to $+$ throughout.

Also, two quantities have like signs when both are preceded by $+$ or both by $-$; otherwise they have unlike signs.

When no sign is written, $+$ is understood.

10. That part of an expression included between two consecutive signs is called a **term**.

To indicate that any portion of an expression lying between two non-consecutive signs is to be taken in its totality as a single term, we enclose the portion within brackets.

Thus, in the expression $a+2bc-4(3c+2ab)$, a, $2bc$

are single or simple terms, and $3c + 2ab$ is to be considered as one complex term.

The sign $-$ and the number 4 preceding the brackets, apply to its contents in their totality.

Instead of brackets, we often employ a line called a *vinculum*, drawn above the portion indicated, as, $\overline{3c + 2ab}$; and sometimes, for special reasons, this line is placed beneath instead of above.

11. The symbol $\times$ or $\cdot$ (*into* or *by*) indicates that the quantity following the symbol is to act as a multiplier upon the quantity preceding the symbol.

With numerical symbols, as 4, 7, etc., it is evident that we cannot dispense with the symbol, as 34 is not the same as 3×4; but this difficulty does not exist with letters, and hence we usually write ab instead of $a \times b$ or $a \cdot b$.

In this case relative position or juxtaposition becomes a symbol of multiplication.

12. The parts which make up a term, or an expression, by multiplication only, are **factors** of the term or expression.

Thus, 3, a, b, and c are factors of $3abc$, 3 being a numerical factor, and a, b, c literal factors. So, also, 4, a, $b + c$, and $c + a$ are factors of the expression $4a(b + c)(c + a)$.

13. In arithmetic we know that 4×6 is the same in value as 6×4; $3 \times 2 \times 5$ is the same as $2 \times 5 \times 3$, etc.; and as this must be a particular case of algebra, we must agree that $ab = ba$, that $abc = bca =$ etc.

This is the *Commutative Law for Multiplication*, and is expressed by saying that *a product is independent of the order of its factors.*

Thus, ab means, indifferently, that b multiplies a or that a multiplies b.

14. Since multiplication by $+1$ effects no change in the quantity multiplied, we have

$$(+1)(+a)=+a;$$

Or, $+$ multiplied by $+$ gives $+$ in the product.

Again, $a-a=0=a+(-a)=a+(+1)(-a)$, as $-a$ may be taken in its totality by placing it within brackets.

Hence $(+1)(-a)$ must be $-a$;

Or, $+$ multiplied by $-$ gives $-$, and, by the commutative law for multiplication, $-$ multiplied by $+$ gives $-$ in the product.

Again, $$a-a=0,$$

and writing $-b$ for a, we have

$$-b-(-b)=0;$$

and hence $-(-b)$, or $(-1)(-b)$ must be the same as $+b$;

Or, $-$ multiplied by $-$ gives $+$ in the product.

Collecting results, we have as the *Law of Signs:* The multiplication of two like signs gives $+$, and the multiplication of two unlike signs gives $-$ in the product.

15. It is readily established in arithmetic that

$$3(4+2+5)\equiv 3\times 4+3\times 2+3\times 5.$$

And as this must be a particular case of algebra, we must assume that

$$a(b+c+d)\equiv ab+ac+ad.$$

We have here three terms, b, c, and d, which are placed in brackets and taken as a complex term, and we have a multiplier a which operates upon this complex term. And we see that we may distribute this operator so as to act separately upon each of the terms of which the complex term is composed.

This is the *Distributive Law* for multiplication.

Some other operations—like multiplication—are distributive, and some are not. The case for each operation must be worked out and learned by itself.

16. A term, such as $aaabbc$, containing repeated letters, is simplified in form by writing it a^3b^2c, in which the small numerals placed to the right and above a and b show how many times each of these letters enters, respectively, as a factor.

The symbol a^3 is read 'a cubed,' and b^2 is read 'b squared.'

The letters a and b are **subjects** or **roots**, and the 3 and 2 are **exponents** or **indices**.

Similarly, a^n, where n denotes any integer, is the nth power of a, read 'a nth-power,' or 'a-to-the-nth,' and denotes that a is to be taken n times as a factor.

Now if $n = p + q$, *i.e.* if n be separated into two integers denoted by p and q, we have

$$a^n \equiv a \cdot a \cdot a \cdots \text{ to } n \text{ factors.}$$

$$\text{Or } a^{p+q} \equiv a \cdot a \cdot a \cdots \text{ to } p \text{ factors} \times a \cdot a \cdot a \cdots \text{ to } q \text{ factors}$$

$$\equiv a^p \cdot a^q.$$

This expresses the *Index Law*, its statement being that *the product of any powers of the same subject is that power of the subject which is denoted by the sum of the exponents.*

Thus, $a^4 \cdot a^3 = a^7$; $a^2 \cdot a^5 \cdot a^8 = a^{15}$, etc.

Of course a is the same as a^1.

17. The Commutative laws for addition and multiplication, the Distributive law for multiplication, and the Index law are the great formal laws of elementary algebra, and its symbolism, its principles of operation, and its results are applicable to any subject which by any consistent process of interpretation can be shown to be governed by these laws.

A proper conception of this fact opens the way to important extensions in the applications of algebra.

The foregoing laws belong to elementary algebra *because* this subject is a generalization of arithmetic, in which these laws hold; but that they are not all essential to all kinds of algebra, we know, as we have a special algebra, Quaternions, in which the commutative law for multiplication does not in general hold true. But this latter algebra is *not* generalized arithmetic.

18. A single letter as a quantitative symbol is said to be of one **dimension**, and the number of dimensions of a term, which consists of multiplications only, is the number of letters, either expressed or implied, which the term contains.

Thus, abc, a^2b, a^3 are each of three dimensions; and $3a^2bcd$, $4a^2b^3$, a^2b^2c are each of five dimensions, a numerical factor, as 3 or 4, having no dimensions.

The number of dimensions of a term constitutes its *degree;* thus the first three of the preceding terms are of the third degree, and the last three of the fifth degree.

Usually a term is said to be of a certain degree in some particular letter or letters.

Thus, $3ab^2x^3$ is of the first degree in a, of the second in b, and of the third in x.

When the degree of each term of an expression has reference to the same particular letter, this letter is called a **variable**, and any other letters occurring in the expression are **constants**.

Thus, in the expression $ax^2 + bx + c$, ax^2 is of the second degree in x, bx is of the first, and c, not containing x, is the *absolute* or *independent* term. In this case x is the variable, and a, b, c are constants.

The term *dimension* is derived from geometry, and its significance will be more fully seen hereafter.

19. An expression of one dimension in each term is a **linear** expression, as $a + b + c$.

An expression which contains a variable in the first degree only is linear in that variable.

Thus, $3abx$ is linear in x, and $3ab$ is the *coefficient* of x.

Similarly, $x - a$ and $ax + bc$ are both linear in x, although the first is of one dimension, and the second of two, in regard to all the letters.

An expression which is of the same dimensions in every term is *homogeneous*. Such expressions are specially important.

Thus, $a + b + c$, $ab + bc + ca$, $a^3 + 3a^2b + 3ab^2 + b^3$ are each homogeneous with respect to all the letters.

20. A quantitative symbol stands for any numerical quantity whatever, and operations upon such general quantities can be only symbolically indicated. Thus a and b being any quantities, we denote their sum by $a + b$, and their product by ab.

Herein lies the advantage of algebraic symbolization, its great powers being due to two things:

(1) The universal significance of the quantitative symbol, and

(2) That the operations performed, unlike those in arithmetic, are not lost sight of, so that a chain of consecutive operations may be so reduced by transformations, as to depend upon the smallest possible cycle of such operations.

21. As algebra is generalized arithmetic, every algebraic relation, which is arithmetically interpretable, expresses some general relation amongst numbers.

Thus, $ab(a+b)$ gives a^2b+ab^2 by distributing ab, or $ab(a+b)\equiv a^2b+ab^2$.

This interpreted gives the arithmetical theorem:

The sum of two numbers multiplied by their product is equal to the sum of the numbers formed by multiplying each number by the square of the other.

It may be remarked that theorems like the foregoing cannot, in general, be proved by an arithmetical process. Repeated trials with different numbers would give a sort of moral proof, but not a mathematical one, since we could not possibly try all numbers. On the other hand, the quantitative symbol standing at once for any, and hence for every number, gives a proof which is both rigid and universal.

EXERCISE I. a.

1. The following are identities arising from distribution; interpret them as arithmetical theorems.

i. $a(a+b)\equiv a^2+ab$.   ii. $ab(a-b)\equiv a^2b-ab^2$.

iii. $(a + b)(a - b) \equiv a^2 - b^2$. v. $(a-b)(a-b) \equiv a^2 + b^2 - 2ab$.

iv. $(a+b)(a+b) \equiv a^2 + b^2 + 2ab$. vi. $(a + b) + (a - b) \equiv 2a$.

vii. $(a + b) - (a - b) \equiv 2b$.

viii. $(a + b)^2 + (a - b)^2 \equiv 2(a^2 + b^2)$.

ix. $(a + b)^2 - (a - b)^2 \equiv 4ab$.

x. $(a + b + c)^2 \equiv a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

2. Reduce to a single number —

i. $1 - \{-2(-1 + \overline{1 - 2})\}$.

ii. $3\{4 - 5(6 - 7[8 - 9])\}$.

iii. $\frac{1}{3}\{\frac{1}{4} - \frac{1}{5}(\frac{1}{6} - \frac{1}{7}[\frac{1}{8} - \frac{1}{9} \cdot \overline{\frac{1}{10} - \frac{1}{11}}])\}$.

3. Condense as much as possible —

i. $2a - \{3a - (a - \overline{b - a})\}$.

ii. $a - b\{1 - b(1 - a \cdot \overline{1 - b})\}$.

4. Distribute the following —

i. $\{(a - b) - 2(b - c)\} \cdot \{(a + b) + 2(b + c)\}$.

ii. $\{(m + 1)a + (n + 1)b\} \cdot \{(m - 1)a + (n - 1)b\}$
$+ \{(m + 1)a - (n + 1)b\} \cdot \{(m - 1)a - (n - 1)b\}$.

5. If n is an integer, $2n$ is an even integer, and $2n + 1$ is an odd integer.

6. The product of two odd integers is an odd integer.

7. The sum of two odd integers is an even integer.

8. The square of an odd integer is an odd integer.

9. The square of an even integer is divisible by 4.

10. What power of 2 is $2^n \times 2^2 \times 2^{1-n} \times 2$?

11. What power of a is $a^{m-n} \cdot a^p \cdot a^{n-p} \cdot a^{p-m}$?

12. According to the index law $a^4 \times a^{-1} \equiv a^{4-1} \equiv a^3$.
Hence interpret a^{-1}.

22. The symbols $\equiv$ and $=$. The symbol $\equiv$ placed between two expressions denotes that one of the expressions may be transformed into the other by the formal operations of algebra. The whole is then called an *Identical equation*, or an **Identity**, and the connected expressions are the *members* of the identity.

Thus, $ab(a-b) \equiv a^2b - ab^2$ is an identity, since the left-hand member is transformed into the right-hand one by distribution.

The symbol $=$ between two expressions tells us that the expressions are to be equal in their totalities, although, in general, no transformation can change one into the other.

That this condition of equality may exist, some relation must hold amongst the quantitative symbols, and, usually, one of these, called the variable, is to have its value so adjusted as to bring about an identity.

Thus, $4a + x - 3 = 6a - 1$ is an **equation**, or a *conditional equation*, which is true on the condition that x takes such a value as will make the whole an identity. We readily see that if x stands for, or is replaced by $2a + 2$, the condition is satisfied. We say then that $2a + 2$ is the value of x, and that x is the variable of the equation.

The variable is often called the *unknown*, and it is manifest that any letter occurring in the equation may be taken as the variable. If a be so taken, its value is found to be $\frac{1}{2}x - 1$.

23. Evidently an identity is not affected by performing the same operation upon each of its members; and, as a conditional equation is to be brought to an identity, the relation existing amongst its quantitative symbols is not changed by performing the same operations upon each member.

Thus in the equation $3x-2a+1=x+a-4$, we may subtract x from each side, add $2a$ to each side, and subtract 1 from each side. We then have

$$2x=3a-5.$$

Then dividing each side by 2, we get as the value of x,

$$x=\tfrac{3}{2}a-\tfrac{5}{2}.$$

We notice that we transfer any term from one member to the other by writing it with a *changed sign* in the other.

Thus, $\quad x+a=b$ gives $x=b-a$.

The determination of the value of the variable, in terms of the constants of an equation, is called the *solution* of the equation. And although the solution of equations does not constitute the whole of algebra, it undoubtedly forms a very important part of it.

The following examples are given by way of illustration:

Ex. 1. To find the value of x in the equation

$$3\{a-4(1-x)\}=2\{a+3(x-1)\}.$$

Performing all the distributions,

$$3a-12+12x=2a+6x-6.$$

Transferring $6x$ from right to left, and $3a-12$ from left to right, we have

$$6x=6-a.$$

Dividing by 6, $\quad x=1-\tfrac{1}{6}a.$

Ex. **2.** To find a number which exceeds the sum of its third and fourth parts by 10.

Let x denote the number; the statement of the problem is algebraically expressed as

$$\tfrac{1}{3}x + \tfrac{1}{4}x = x - 10.$$

Multiplying by 12, $4x + 3x = 12x - 120,$

whence $x = 24.$

Ex. **3.** A lends to B one third of a dollar more than $\frac{1}{3}$ of his money, and to C one half a dollar more than $\frac{1}{2}$ of what he has left. A has then \$6. How much had he at first, and how much did he lend to B, and to C?

Let x denote A's money at first.

He lends to B, $\frac{1}{3}x + \frac{1}{3}$ dollars.

He has left, $x - (\frac{1}{3}x + \frac{1}{3})$, or $\frac{2}{3}x - \frac{1}{3}$ dollars.

He lends to C, $\frac{1}{2}(\frac{2}{3}x - \frac{1}{3}) + \frac{1}{2}$, or $\frac{1}{3}x + \frac{1}{3}$ dollars.

He has left, $(\frac{2}{3}x - \frac{1}{3}) - (\frac{1}{3}x + \frac{1}{3})$, or $\frac{1}{3}x - \frac{2}{3}$ dollars;

and this is \$6.

Whence $x = 20$; and B's loan = C's loan = \$7.

EXERCISE I. b.

1. Prove the following identities —

i. $$(a + b)^2 \equiv a^2 + b^2 + 2ab.$$

ii. $$(a^2 + b^2)(c^2 + d^2) \equiv (ac + bd)^2 + (ad - bc)^2.$$

iii. $$(a^2 - b^2)(c^2 - d^2) \equiv (ac - bd)^2 - (ad - bc)^2$$
$$\equiv (ac + bd)^2 - (ad + bc)^2.$$

iv. $$(a + b + c)^2 + (a + b - c)^2 + (b + c - a)^2 + (c + a - b)^2$$
$$\equiv 4(a^2 + b^2 + c^2).$$

v. $$(a^2 + b^2)^2 \equiv (a^2 - b^2)^2 + (2ab)^2.$$

2. Interpret ii. and iii. of 1 as theorems in numbers.

3. By means of v. of 1 find two numbers such that the sum of their squares shall be a square. Make a table of such numbers.

4. Find the value of x in each of the following —

i. $x + \frac{1}{2}x + \frac{1}{3}x = 2x - 2.$

ii. $3(1 - 2 \cdot \overline{3 - x}) = 2\{1 + 2(x - 2)\}.$

5. Simplify $1 - \{1 - (1 - x)\} + 1 + \{1 - (1 + x)\} + x - \{x - (x - 1)\}.$

6. Find x in the equation $(x - 4)(x + 6) = (x + 8)(x + 2).$

7. Find a number whose half exceeds the sum of its fourth and fifth parts by 40.

8. Find a number such that if it be increased by a, and if it be diminished by b, one third of the first result is equal to one half the second.

9. The sum of the ages of A, B, and C is 108 years. A is twice as old as B, and twice C's age is equal to A's and B's together. Find their ages.

10. After paying 2 % taxes on my income I have $1078 left. What is my income ?

11. I pay $33\frac{1}{3}$ % duty on the cost price of a horse. I keep him 2 months at an expense of $16, and I then sell him for $200, making 20 % profit on the cost price. What did the horse cost ?

12. In a certain school $\frac{2}{5}$ of the pupils are in the first form, $\frac{1}{3}$ in the second, $\frac{3}{20}$ in the third, and 14 in the fourth. How many pupils are in the school ?

13. A market-woman sells to A half an egg more than half she has, to B half an egg more than half she has left, and 10 eggs to C, and she then has 6 eggs left. How many had she at first ?

CHAPTER II.

THE FOUR ELEMENTARY OPERATIONS.

ADDITION AND SUBTRACTION.

24. The addition of a and b is denoted by $a+b$, where a and b stand for any quantities whatever.

If, however, $a=5$ and $b=-3$, the expression becomes $5-3$, and we have a case of subtraction.

Thus, symbolically, addition and subtraction are one and the same; for in the expression $a+b$ we cannot know whether an addition or a subtraction is to be performed, until we know something about the quantities for which the letters stand.

Moreover, any subtraction may be put into the form of an addition, and *vice versa;* for $a-b$ is the same as $a+(-b)$, and $a+b$ is the same as $a-(-b)$.

Thus the subtraction of one expression from another may be expressed as an addition, by changing all the signs of the subtrahend. Hence the rule for Algebraic Subtraction:

Change the signs of the subtrahend, and then perform addition.

Ex. To subtract $3a-2b+3$ from $6a+3b-4$ is the same as to add $-3a+2b-3$ to $6a+3b-4$; and the result is $3a+5b-7$.

25. Since a series of terms connected by $+$ and $-$ signs may be written as one connected by $+$ signs only, such a series is called an **Algebraic Sum**. Thus $5-4+3-1$ has 3 as its algebraic sum, and may be written

$$5+(-4)+3+(-1).$$

We are not justified in speaking of this as an *Arithmetic Sum,* for -4 and -1 have no meaning in pure arithmetic, and if we put it into the form $8-(+5)$, it becomes an arithmetic difference.

26. Symmetry. When the interchanging of two letters of an expression leaves the expression unchanged, except as to the order of the letters in a term, or the order of the terms in the expression, the expression is symmetrical in the two letters.

Thus, by interchanging a and b in

$$ab-ac+ad-bc+bd-cd,$$

we get $$ba-bc+bd-ac+ad-cd,$$

an expression the same as the former, except as to the order of the terms and of the letters in some of the terms.

Hence the expression is symmetrical in a and b.

It is readily seen that the expression is not symmetrical in c and d.

An expression which is symmetrical in every pair of two letters is symmetrical in all the letters.

Thus $ab+bc+ca$ and $abc+abd+acd+bcd$ are each symmetrical in all the letters which they contain.

Some special kinds of symmetry will be considered in a more advanced stage of the work.

27. When we know the letters which enter into an expression symmetrical in them all, and we are given a **type-term**, we can write the full expression by building up the *form* of the type in every possible way from the given letters, and taking the algebraic sum of all the terms so produced.

Thus from the letters a, b, c:

i. with type ab^2 we have —

$$ab^2 + bc^2 + ca^2 + ba^2 + cb^2 + ac^2.$$

ii. with type $a(bc - a)$:

$$a(bc - a) + b(ca - b) + c(ab - c).$$

iii. with type abc^2:

$$abc^2 + bca^2 + cab^2.$$

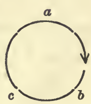
iv. with type $(b - c)(a^2 - bc)$:

$$(b - c)(a^2 - bc) + (c - a)(b^2 - ca) + (a - b)(c^2 - ab).$$

v. with a, b, c, and d, and type ab^2:

$$ab^2 + ba^2 + ac^2 + ca^2 + ad^2 + da^2 + bc^2 + cb^2 + bd^2 + db^2 + cd^2 + dc^2.$$

It will be noticed that in examples ii. and iii. each term is formed from the preceding one by changing a to b, b to c, and c to a. This is called a *cyclic* or circular substitution; for if we write the letters in a circle, as in the margin, we pass from one term to the next by commencing with a letter one step further around the circle until the whole is completed.



Many substitutions, where three letters are concerned, are of this character, the distinctive feature being that we do not interchange any two letters without, at the same time, interchanging every two in circular order.

A cyclic change with 4 letters and type ab gives $ab + bc + cd + da$. This lacks the terms ac and bd to make it completely symmetrical.

In examples i. and v. a circular change is not sufficient; for from the type ab^2 we must have a term ba^2, which is not given by a mere circular substitution. In other words, we must interchange two letters without affecting the third.

A little care and observation are all that are required in writing out such expressions from a given type.

28. The symbol Σ (sigma), amongst other uses, is conveniently employed to denote expressions consisting of algebraic sums, written from a type.

These are symmetrical in all the letters employed, and when written out are frequently of inconvenient length.

The notation Σa^2b, with three letters involved, stands for $a^2b + b^2a + b^2c + c^2b + c^2a + a^2c$.

With four letters involved it stands for v. of the preceding article.

$\Sigma(\overline{b - c} \cdot \overline{a^2 - bc})$ stands for iv. of the preceding article.

As employed hereafter, 3 letters will be understood unless a different number is indicated, or in cases where misunderstanding is not possible. Σ_4 will serve to indicate 4 letters, and generally Σ_n to indicate n letters.

This is known as the **Sigma Notation**.

EXERCISE II. a.

1. With 3 letters write in full—

i. $\Sigma(a^2 - b)$.

ii. $\Sigma(a + b - c)^2$.

iii. $\Sigma\frac{a^2b}{c}$.

iv. $(\Sigma a)^2$.

v. $\{\Sigma(a - b)\}^2$.

vi. $\Sigma a \times \Sigma ab$.

vii. $(\Sigma a)^2 + \Sigma(a + b - c)^2 - 4\Sigma a^2 + 2\Sigma ab$.

viii. $x^3 - x^2\Sigma a + x\Sigma ab - abc$.

2. With 4 letters write in full—

i. Σab.

ii. Σab^2c.

iii. $\Sigma a(b-c)$.

iv. $(\Sigma a)^2$.

v. $\Sigma(a-b)(c-d)$.

vi. $\Sigma(a^2-b)(c^2-d)$.

MULTIPLICATION.

29. The multiplication of one expression by another may be effected by a series of distributions, and when the expressions are not too complex this is usually the best method.

Thus $(a+b)(a-b+c) \equiv a(a-b+c)+b(a-b+c)$

$$\equiv a^2-ab+ac+ab-b^2+bc \equiv a^2-b^2+ac+bc.$$

By remembering and applying a few elementary product forms, the operation may often be much curtailed. As convenient fundamental forms we may take the following, although any simple form that can be remembered may be equally useful:

(1) $(a+b)^2 \equiv a^2+b^2+2ab$.

(2) $(a-b)^2 \equiv a^2+b^2-2ab$.

(3) $(a-b)(a+b) \equiv a^2-b^2$.

(4) $3(a+b)(b+c)(c+a) \equiv (a+b+c)^3-a^3-b^3-c^3$

$$\equiv (\Sigma a)^3 - \Sigma a^3.$$

(5) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$$\equiv a^3+b^3+c^3-3abc,$$

or $$\Sigma a(\Sigma a^2 - \Sigma ab) \equiv \Sigma a^3 - 3abc.$$

The mark $\therefore$ is a verbal symbol for 'therefore' or 'hence.'

Ex. 1. $(a+b-c+d)(a+b+c-d)$

$$\equiv(\overline{a+b}-\overline{c-d})(\overline{a+b}+\overline{c-d})$$
$$\equiv(a+b)^2-(c-d)^2$$
$$\equiv a^2+b^2-c^2-d^2+2ab+2cd.$$

Ex. 2. $(a+b+c+d)^2\equiv(a+b)^2+2(a+b)(c+d)+(c+d)^2$
$$\equiv a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd).$$

Ex. 3. To distribute

$s(s-a)(s-b)+s(s-b)(s-c)+s(s-c)(s-a)-(s-a)(s-b)(s-c)$, where $2s\equiv a+b+c$. s, being symmetrical in a, b, and c, is not altered by a circular substitution of these letters.

But this substitution brings $s(s-a)(s-b)$ to $s(s-b)(s-c)$, and $s(s-b)(s-c)$ to $s(s-c)(s-a)$.

Hence having the expansion of $s(s-a)(s-b)$, the expansions of the two following terms may be immediately written down by a cyclic interchange of letters.

Now $$s(s-a)(s-b)\equiv s^3-s^2(a+b)+s\cdot ab.$$
$$\therefore\ s(s-b)(s-c)\equiv s^3-s^2(b+c)+s\cdot bc,$$
and $$s(s-c)(s-a)\equiv s^3-s^2(c+a)+s\cdot ca.$$

The algebraic sum is
$$3s^3-2s^2(a+b+c)+s(ab+bc+ca),$$
and as $2s=a+b+c$, this becomes
$$-s^3+s(ab+bc+ca)\quad\ldots\ldots\quad A$$

Again the expansion of $(s-a)(s-b)(s-c)$ is
$$s^3-s^2(a+b+c)+s(ab+bc+ca)-abc,$$
or $$-s^3+s(ab+bc+ca)-abc.$$

And subtracting this from A gives abc as the final result.

This example furnishes a good illustration of the remark in Art. 20, for the long series of operations symbolized in the statement of the exercise is just equivalent in its totality to the two multiplications symbolized in the final result.

Ex. 4. To prove the identity

$$8(\Sigma a)^3 - \Sigma(a+b)^3 \equiv 3(2a+b+c)(2b+c+a)(2c+a+b).$$

Either of two methods may be adopted — (1) to transform one member to the other by the rules of operation; or (2) to transform each to the same third expression by distribution. We shall adopt the first way.

$$8(\Sigma a)^3 \equiv (\Sigma 2a)^3 \equiv (a+b+b+c+c+a)^3.$$

Now put $a+b=p$, $b+c=q$, $c+a=r$, and the identity reduces to

$$(p+q+r)^3 - p^3 - q^3 - r^3 \equiv 3(p+r)(r+q)(q+p),$$

which is true by Art. 29, (4).

30. Expansion of Symmetrical Homogeneous Expressions.

This form of expression is of frequent occurrence, and its properties of symmetricality and homogeneity enable us to expand it with some facility.

Ex. 1. To expand $(a+b+c)^3 - (a+b)^3 - (b+c)^3 - (c+a)^3$.

Being homogeneous and of 3 dimensions, the type terms in its expansion can only be a^3, a^2b, and abc. Taking the type a^3, we see that its coefficient is -1. Taking the type a^2b, its coefficient is readily found to be zero. And the coefficient of the type abc is 6.

$\therefore$ The expansion is $6abc - a^3 - b^3 - c^3$.

Ex. 2. To expand $(\Sigma a)^3 + (a+b-c)(b+c-a)(c+a-b)$.

The expansion being homogeneous of 3 dimensions and symmetrical, must be of the form

$$m\Sigma a^3 + n\Sigma a^2b + p\cdot abc.$$

The coefficient of a^3 is $1-1$ or 0; $\therefore m=0$.

The coefficient of a^2b is $3+1$ or 4; $\therefore n=4$.

The coefficient of abc is $6-2$ or 4; $\therefore p=4$,

and the expansion is $4\Sigma a^2b + 4abc$.

EXERCISE II. b.

1. Write out the type terms in the following symmetrical and homogeneous expressions —

i. $(a+b+c+d)^3$.

ii. $(a+b)^4$.

iii. $(a+b+c)^4$.

iv. $(a+b+c+d)^4$.

v. $(a+b+c+d)^5$.

2. Show that

$$(a+b-c)(b+c-a)(c+a-b)$$
$$\equiv ab(a+b)+bc(b+c)+ca(c+a)-a^3-b^3-c^3-2\,abc.$$

3. $\Sigma a \cdot \Sigma ab-(a+b)(b+c)(c+a)\equiv abc$.

4. Show that $\Sigma\{(a-b)(2\,b-c)\}\equiv(\Sigma a)^2-3\,\Sigma a^2$.

5. Show that

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$\equiv 2\,\Sigma a^2b^2-\Sigma a^4\equiv a^2(2\,b^2-a^2)+b^2(2\,c^2-b^2)+c^2(2\,a^2-c^2)$$
$$\equiv 4\,b^2c^2-(b^2+c^2-a^2)^2.$$

6. Expand —

i. $\Sigma(a+b)(a-b)$.

ii. $(\Sigma a)^4-\Sigma(a+b)^4+\Sigma a^4$.

iii. $\Sigma a^3(b-c)-\Sigma a\cdot\Sigma a^2(b-c)$.

iv. $a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$
$+2(s-a)(s-b)(s-c)$, where $2\,s=a+b+c$.

7. If $2\,s=a+b+c$, show that the three following expressions are identical in value —

$$s(s-a)(b+c)+a(s-b)(s-c)-2\,bcs,$$
$$s(s-b)(c+a)+b(s-c)(s-a)-2\,cas,$$
$$s(s-c)(a+b)+c(s-a)(s-b)-2\,abs.$$

8. Show that $(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)+(x-a)(x-b)(a-b)+(a-b)(b-c)(c-a)\equiv 0$.

9. Show that $(\Sigma a)^2 = \Sigma a^2 + 2\,\Sigma ab$, with any number of letters.

10. Multiply $\Sigma a^2 + 2\,\Sigma ab$ by Σa, 3 letters.

11. Multiply $\Sigma a^2 + \Sigma ab$ by Σab, 3 letters.

12. Multiply $\Sigma a^2 + 2\,\Sigma ab$ by Σa, 4 letters.

This gives $(a+b+c+d)^3$.

Prove the following theorems in numbers —

13. The difference between the square of the sum and the square of the difference of two numbers is the product of twice the numbers.

14. The sum of the squares of the sum and of the difference of two numbers is one-half the sum of the squares of twice the numbers.

15. If two numbers be each the sum of two squares, their product is the sum of two squares.

16. If two numbers be each the difference between two squares, their product is the difference between two squares.

17. If the sum of two numbers is 1, their product is equal to the difference between the sum of their squares and the sum of their cubes.

18. If the product of two numbers is 1, the square of their sum exceeds the sum of their squares by 2.

31. The distribution of $(x+a)(x+b)(x+c)\cdots$, the product of a number of binomial factors with one letter the same in each factor, is very important.

The dominant letter, x, is taken as the variable, and the expansion is arranged according to the powers of x.

With three factors we readily see that taking the x from every factor gives x^3; taking it from every factor but one, and taking the other letter from that one gives $x^2(a+b+c)$; taking x from one factor and the other letter from the other two gives $x(ab+bc+ca)$; and lastly, taking the second letters only gives abc.

Thus the expansion is

$$x^3 + x^2(a+b+c) + x(ab+bc+ca) + abc,$$

or $$x^3 + x^2\Sigma a + x\Sigma ab + abc.$$

Similarly, with 4 factors it is readily seen that the expansion is

$$x^4 + x^3\Sigma a + x^2\Sigma ab + x\Sigma abc + abcd.$$

In a similar manner it is shown that with n factors the expansion is

$$x^n + x^{n-1}\Sigma a + x^{n-2}\Sigma ab + x^{n-3}\Sigma abc + \cdots + abc\cdots.$$

It will be noticed in every case that the last term is the continued product of all the letters except the variable. This is important.

If the signs are negative in all the factors, as $(x-a)$, etc., then Σa, Σabc, etc., involving an odd number of letters in each term, will be negative, and Σab, etc., involving an even number, will be positive.

Thus, $(x-a)(x-b)(x-c)(x-d)\cdots$ to n factors

$$\equiv x^n - x^{n-1}\Sigma a + x^{n-2}\Sigma ab - + \cdots$$

Ex. 1. $(x+a)(x+b)(x+c)$

$$= x^3 + x^2(a+b+c) + x(ab+bc+ca) + abc;$$

and making $c = b = a$ gives

$$(x+a)^3 \equiv x^3 + 3x^2a + 3xa^2 + a^3.$$

Ex. 2. $(x+a)(x+b)(x+c)(x+d)$

$$= x^4 + x^3\Sigma a + x^2\Sigma ab + x\Sigma abc + abcd.$$

Now Σa contains 4 terms, Σab contains 6, Σabc contains 4, and $abcd$ is one term.

Therefore making $d = c = b = a$ gives

$$(x + a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4.$$

Ex. **3.** Similarly, we find

$$(x + a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5.$$

The coefficients of the several terms in these and higher powers are exhibited in the following table, which may be extended at pleasure —

The coefficients form the diagonals, up to the 8th power.

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	
1	3	6	10	15	21	28		
1	4	10	20	35	56			
1	5	15	35	70				
1	6	21	56					
1	7	28						
1	8							
1								

Ex. **4.** $(x + a - b)(x + b - c)(x + c - a) \equiv x^3 + x^2\Sigma(a - b)$
$$+ x\Sigma\{(a - b)(b - c)\} + (a - b)(b - c)(c - a).$$

Now $\quad \Sigma(a - b) = 0,\ \Sigma\{(a - b)(b - c)\} = \Sigma ab - \Sigma a^2,$

and $\quad (a - b)(b - c)(c - a) = \Sigma ab(b - a),$

and the expansion is

$$x^3 + x(\Sigma ab - \Sigma a^2) + \Sigma ab(b - a)$$

EXERCISE II. c.

1. Expand $(x - 1)(x - 2)(x - 3)(x - 4)$.

2. Expand $(x - 1)(x + 2)(x - 3)(x + 4)$.

3. Expand $(4x + 1)(3x + 2)(2x + 3)(x + 4)$.

4. Expand $(x+a+b-c)(x+b+c-a)(x+c+a-b)$.

5. Expand $(a+b+c)^5$.

Write it $\{a+(b+c)\}^5$ and pick out the coefficients of the type terms.

6. Expand $(a+b+c+d)^4$.

Write it $\{(a+b)+(c+d)\}^4$ and pick out the coefficients of the type terms.

7. Write in Σ notation the expansion of $(a+b+c+d)^5$.

32. Function. An expression such as a^2+ab changes value when a changes value or when b changes value. It is accordingly called a *function of a and b.*

When we wish to consider a alone as a variable, and regard b as being constant, we speak of the expression as a function of a, and we symbolize it as fa or $f(a)$, where f is a *functional* symbol.

This symbol merely denotes that a enters into an expression as a variable, without any regard to the other letters in the expression, and $f(x)$ stands for any expression in which x enters as a variable.

In general algebra the *form* of f is given, *i.e.* we are given an expression of the form required.

Thus, if $f(a)$ stands for $a^2+2ab+c$, then $f(x)$ stands for $x^2+2xb+c$, where x is substituted for a in the type-form. Similarly, $f(a-x)=(a-x)^2+2(a-x)b+c$, etc.

Ex. **1.** If $f(a)=a^2+2a+1$, $f(a-1)=(a-1)^2+2(a-1)+1\equiv a^2$.

Ex. **2.** If $f(x)=\dfrac{x}{1+x}$, $f\left(\dfrac{x}{1-x}\right)=\left(\dfrac{x}{1-x}\right)\Big/\left(1+\dfrac{x}{1-x}\right)\equiv x$.

EXERCISE II. d.

1. If $f(x)\equiv x^3+3x-10$, find $f(3)$, also $f(-3)$.

2. If $f(x)\equiv x^2-5x+6$ and $y=3-x$, find $f(y)$ in terms of x.

3. If $f(a) \equiv 1 - a$, show that $f\{f(a)\} \equiv a$.

4. If $f(x) \equiv a - x$, show that $f^3(x) = f(x)$.

f^3 stands for $f\{f(f)\}$.

5. If $f(x) \equiv x^2 + x + 1$ find $f(x-1)$.

6. If $f(x) \equiv 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} +$ etc., find the numerical value of $f(1)$ to 5 decimal places.

7. If $f(x) = x^4 - 3x^3 + 2x^2 + 3x - 3$, find $f(x+1)$.

33. An expression such as

$$x^4 + 2x^3 - 3x^2 - 2x + 1,$$

or

$$1 - 2x - 3x^2 + 2x^3 + x^4,$$

in which the exponents of the variable, x, are all positive integers, is called a *positive integral function of x*, or simply an **integral** function of x.

The first of these is written in descending powers of the variable, and the second in ascending powers.

The coefficients may be numerical or literal.

The function is *complete* when all the powers of the variable in consecutive order are represented; and any integral function may be made complete in form by writing zero coefficients to the missing terms. Thus,

$$x^5 + 0x^4 + 2x^3 + 0x^2 - 3x + 1$$

is complete in form.

34. In multiplying together two integral functions of the same variable, it is advantageous to operate upon the coefficients alone, as the proper powers of the variable are readily supplied to the result.

The functions, if not complete, should be made complete in form, and there is some advantage in writing them in ascending order of the variable.

Ex. 1. To multiply $a + bx + cx^2 + dx^3$ by $p + qx + rx^2$.

Coefficients $\begin{cases} a \quad +b \quad +c \quad +d \\ p \quad +q \quad +r \end{cases}$

Product with variable supplied $\left\{ \begin{array}{l|l|l|l|l} pa + pb & x + pc & x^2 + pd & x^3 + qd & x^4 + rdx^5 \\ \quad + qa & \quad + qb & \quad + qc & \quad + rc & \\ & \quad + ra & \quad + rb & & \end{array} \right.$

By observing in this typical case, how the coefficients in the product are made up, *i.e.* by a sort of cross-multiplication, as $pb + qa$, $pc + qb + ra$, etc., we can perform such multiplications with numerical coefficients with considerable facility.

Ex. 2. To multiply $1 - 2x + 3x^2 + x^3$ by $2 - x + 2x^2$.

$$\begin{array}{ll} 1 - 2 \;\; + \;\; 3 \quad + 1 \;\; . \; . \; . \; . \; . \; . \; . & \text{Multiplicand.} \\ 2 - 1 \;\; + \;\; 2 \qquad\quad . \; . \; . \; . \; . \; . \; . & \text{Multiplier.} \\ \hline 2 - 5x + 10x^2 - 5x^3 + 5x^4 + 2x^5 \;\; . & \text{Product.} \end{array}$$

Ex. 3. To multiply $x^2 + \frac{1}{2}x + \frac{1}{3}$ by $x^2 - \frac{1}{2}x - \frac{1}{3}$.

Arranging in ascending powers —

$$\begin{array}{l} \quad \frac{1}{3} + \frac{1}{2} \quad + 1 \\ - \frac{1}{3} - \frac{1}{2} \quad + 1 \\ \hline - \frac{1}{9} - \frac{1}{3}x - \frac{1}{4}x^2 + 0x^3 + x^4 \end{array}$$

Ex. 4. To expand $(1 + x - 2x^2 + 3x^3)^3$.

1st operation $\begin{cases} 1 + 1 - 2 + 3 \\ 1 + 1 - 2 + 3 \end{cases}$

2d operation $\begin{cases} 1 + 2 - 3 + 2 + 10 - 12 + 9 \quad \text{1st product.} \\ 1 + 1 - 2 + 3 \end{cases}$

$$1 + 3 - 3 - 2 + 24 - 15 - 17 + 63 - 54 + 27$$

Result, $1 + 3x - 3x^2 - 2x^3 + 24x^4 - 15x^5 - 17x^6 + 63x^7 - 54x^8 + 27x^9$.

Ex. 5. To multiply $x^5 - 2x^3 + x + 1$ by $x^2 + 1$.

Ordering in descending powers —

$$\begin{array}{l} 1+0+0-2+0+1+1 \\ 1+0+1 \\ \hline 1+0+1-2+0-1+1+1+1 \end{array}$$

Result, $x^8 + x^6 - 2x^5 - x^3 + x^2 + x + 1$.

35. A circulating decimal, or the arithmetical approximation to the value of any incommensurable, is an example of a series of arithmetical figures which is non-terminating. Similarly, in algebra we may have a series of terms, arranged in ascending powers of the variable, such that the series has no last term. Such a series is called an **infinite series**, and is indicated by writing a few terms at the beginning with three points, ⋯, with or without ad inf.; as

$$a + bx + cx^2 + dx^3 + \cdots \text{ ad inf.}$$

As we cannot write all the terms of an infinite series, we cannot, in general, write all the terms of any multiple of it. In some cases, however, certain multiples may become finite by the vanishing of all the terms after the first few.

We have the arithmetical analogue in a repeating or circulating decimal, such as $1 \cdot 2333 \cdots$, which gives a finite product, $3 \cdot 7$, when multiplied by 3, but another infinite series when multiplied by 4 or 5.

To multiply two infinite series together, we take the same number of terms in both multiplicand and multiplier, and retain that number of terms in the product.

Ex. 1. To multiply $1 + x + x^2 + x^3 + x^4 + \cdots$
by $1 - x + x^2 - x^3 + x^4 - \cdots$

$$\text{Operation}\begin{cases}1 + 1 + 1 + 1 + 1 \cdots \\ 1 - 1 + 1 - 1 + 1 \cdots\end{cases}$$
$$\overline{1 + 0 + 1 + 0 + 1 \cdots}$$

Product, $1 + x^2 + x^4 + \cdots$, an infinite series.

Ex. 2. To multiply $1 - x^2 + 2x^3 - 3x^4 + - \cdots$
by $1 + 2x + x^2$.

$$\text{Operation}\begin{cases}1 + 0 - 1 + 2 - 3 \cdots \\ 1 + 2 + 1\end{cases}$$
$$\overline{1 + 2 + 0 + 0 + 0 \cdots}$$

Product, $1 + 2x$, a finite result as far as the series extends.

36. To square the series $a + bx + cx^2 + dx^3 + ex^4 + \cdots$

$$\text{Operation}\begin{cases}a + b + c + d + e + \cdots \\ a + b + c + d + e + \cdots\end{cases}$$

$$\text{Coefficients}\begin{cases}a^2 + 2ab + 2ac & + 2ad & + 2ae & + \cdots \\ \qquad\quad + b^2 & + 2bc & + 2bd & \\ & & + c^2 & \end{cases}$$

The operation is simply multiplication, but owing to the identity of the multiplier and the multiplicand, the coefficients in the result consist of double products and squares; and we notice that a square appears as the first coefficient, and then in every alternate one.

By observing how the coefficients are made up, we may write the square of a series with great ease.

Ex. 1. To square $1 + 2x + 3x^2 + 4x^3 + \cdots$ to the term containing x^3.

$$\begin{array}{l|l|l} \multicolumn{3}{l}{1 + 2 + 3 \ + \ 4} \\ \hline 1 + 4 + 6 & + \ 8 & + \cdots \\ \quad\ \ + 2^2 & + 12 & \end{array}$$

$\therefore\ 1 + 4x + 10x^2 + 20x^3 + \cdots$ is the required square.

Ex. 2. To square $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-+\cdots$

$$\begin{array}{l} \underline{1+\frac{1}{2}-\frac{1}{8}\ \ +\frac{1}{16}} \\ 1+1-\frac{1}{4}\ \big|+\ \frac{1}{8}\ \big|+\cdots \\ \qquad\ \ +\frac{1}{4}\ \big|-\ \frac{1}{8}\ \big| \end{array}$$

$\therefore$ Square $=1+x$.

EXERCISE II. e.

1. Multiply $\quad 1+2x+3x^2+4x^3+5x^4$

by $\quad 1-2x+3x^2-4x^3+5x^4$.

2. Multiply $\quad 1-x+4x^2-7x^3+19x^4-40x^5+\cdots$

by $\quad 1+x-3x^2$,

to the term containing x^5.

3. Multiply $\quad -1-x+3x^2-2x^3-x^4+3x^5-2x^6-\cdots$

by $\quad 1+x+x^2$,

to the term containing x^6.

4. In the complex series

$$1+m(ax+bx^2+cx^3+\cdots)+n(ax+bx^2+cx^3+\cdots)^2$$
$$+p(ax+bx^2+cx^3+\cdots)^3+\cdots$$

find the coefficient of x^3.

5. Find the coefficient of x^5y^4 in the product of

$$(x+ax^3+bx^5+\cdots)\text{ by }\left(1-\frac{y^2}{2}-\frac{8a+1}{4}y^4\cdots\right).$$

6. Square the series, $\quad 1+\frac{x}{2}-\frac{1}{2}\left(\frac{x}{2}\right)^2+\frac{1}{2}\left(\frac{x}{2}\right)^3\cdots$

7. Multiply $\quad 1+x(1-2x)+x^2(1-2x)^2+\cdots$

by $\quad 1-x+2x^2$,

to the term containing x^3.

Let $y=x-2x^2$.

8. Find the square of $x - \frac{1}{2x} - \frac{1}{1 \cdot 2} \cdot \frac{1}{2^2 x^3} - \frac{1 \cdot 3}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2^3 x^5} - \cdots$

The law of formation of the terms is evident, and any required number of terms may be written down.

9. Show that

$$(1 - 2x + 3x^2 - 4x^3 + \cdots)(1 + 2x + 3x^2 + 4x^3 + \cdots)$$
$$\equiv (1 + x^2 + x^4 + \cdots)^2.$$

10. Find the coefficient of x^n in the product

$$(1 + c_1x + c_2x^2 + c_3x^3 + \cdots)(x^n + c_1x^{n-1} + c_2x^{n-2} + c_3x^{n-3} + \cdots).$$

11. Find the coefficient of x^{n-2} in 10.

12. Find the coefficient of linear x in

$$1 + mx + \frac{x(x-1)}{1 \cdot 2}m^2 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}m^3 + \cdots$$

13. If $y = ax + bx^2 + cx^3 + \cdots$ and $x = Ay + By^2 + Cy^3 + \cdots$ find A and B in terms of a and b, on the condition that the coefficient of each power of x, in the result of substituting for y, is zero.

14. If $f(x) \equiv 6\left\{\frac{1}{x} + \frac{1}{1} \cdot \frac{1}{3x^4} + \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{1}{5x^7} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{1}{7x^{10}} + \cdots\right\}$,

calculate $f(2)$ to four decimal places.

15. If $f(x) \equiv x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$, calculate, to four decimal places, the value of $8\{f(\frac{1}{5}) + f(\frac{1}{8})\} + 4f(\frac{1}{7})$.

In order to obtain 4 decimals exact, the calculation should be carried to at least 5 places.

16. In any multiplication, write the terms of the multiplier in an inverted order, and the partial products are not formed by a *cross-multiplication*. How are they formed?

DIVISION.

37. Division, in algebra as in arithmetic, is indicated in several ways. Thus, $a \div b$, $\frac{a}{b}$, and a/b all mean that a is to be divided by b.

In any case, a is the dividend and b the divisor, where a and b stand for any numerical quantities or algebraic expressions.

We define the operation indicated by $\frac{a}{b}$ as the inverse of multiplication, such that $\frac{a}{b} \times b = a$.

Denoting $\frac{a}{b}$ by q, we have $a = bq$, where q is the quotient. Hence the theorem:

The dividend is the product of the divisor and the quotient, and the quotient and the divisor are reciprocals in the sense that if either be made the divisor the other is the quotient.

Division thus consists in separating the dividend into two factors, one of which is the divisor; and any process which accomplishes this effects the division.

38. Index law in division. Assume $\frac{a^m}{a^n} = a^p$, and multiply both members by a^n.

Then $$a^m = a^n a^p = a^{n+p}, \text{ by the index law.}$$

Therefore $$m = n + p, \text{ or } p = m - n.$$

Whence $$\frac{a^m}{a^n} = a^{m-n};$$

and this must hold for all integral values of m and n.

Hence the quotient from dividing any integral power by another integral power of the same root is that power of the root whose index is found by subtracting the index of the divisor from that of the dividend.

Cor. 1. If $m = n$, $\quad \frac{a^n}{a^n} = 1 = a^{n-n} = a^0.$

Hence the zero power of any finite quantity is to be interpreted as meaning $+1$.

Cor. 2. Making $m = 0$, $\quad \frac{a^0}{a^n} = \frac{1}{a^n} = a^{0-n} = a^{-n}.$

Hence a negative exponent is to be interpreted as the reciprocal of the same root with the corresponding positive exponent.

Thus, $\frac{a}{b} \equiv ab^{-1}$; $1 + \frac{1}{x} + \frac{1}{x^2} \equiv 1 + x^{-1} + x^{-2}$; etc.

39. The most important cases of division, where any special process is required, are those involving a variable in an integral function.

Let $ax^2 + bx + c$ be a divisor, and $px^2 + qx + r$ be the quotient, and let $Ax^4 + Bx^3 + Cx^2 + Dx + E$ be the dividend.

By multiplying the divisor and quotient together, we obtain as coefficients in the dividend,

$$\left.\begin{array}{ccccc} A & B & C & D & E \\ ap & +bp & +cp & +cq & +cr \\ & +aq & +bq & +br & \\ & & +ar & & \end{array}\right\} \quad \ldots\ldots \quad X$$

And operating upon coefficients only, when we divide X by $a + b + c$ we should get $p + q + r$; or, in other words, we are given the coefficients in X, and also a, b, and c, and we are to obtain p, q, and r.

Let us see how it is to be done:

1. Dividing ap or A by a gives p, and p becomes known.

2. Multiply p by b, and subtract the product from B, leaving aq. Divide aq by a, and we have q.

3. Multiply q by b and p by c, and subtract the sum of these products from C, leaving ar. Divide ar by a, and r becomes known.

Thus p, q, and r are obtained.

In the foregoing, we notice — (1) that the only quantity by which we divide is a, so that if a be 1 there is no real division.

(2) If we change the signs of b and c, the partial product bp, bq, br, cp, cq, and cr all become additive, so that the only operations involved will be multiplication and addition.

(3) That the partial products, which form any coefficient in the dividend, as C, are made up by a cross-multiplication, as explained in Art. 34, Ex. 1.

This process is known as **synthetic division**, because we build up the terms of the dividend by getting the partial products which enter into their composition, and through this synthesis we obtain the terms of the quotient.

The following examples will illustrate:

Ex. 1. To divide $2x^4 + x^3 - 8x^2 + 17x - 12$

by $2x^2 - 3x + 4$.

Here $a=2$, $b=-3$, $c=4$; and we are to find p, q, and r such that—

$$2p=2, \quad -3p+2q=1, \quad 4p-3q+2r=-8,$$

whence, $\quad p=1, \qquad q=2, \qquad r=-3,$

and the quotient is x^2+2x-3.

This operation is systematically carried out as follows—

2	+ 3 − 4	. . Divisor, signs of b and c changed.
2 + 1 − 8	+ 17 − 12	. . Dividend.
3 6	− 9 + 12	. . Partial products.
− 4	− 8	
1 + 2 − 3	0 0	. . Quotient.

Here we change the sign of the 3 and 4 of the divisor, and thus have only additions. We then divide each sum by 2 as we proceed. Thus the multipliers in forming the partial products are 3 and − 4, and the divisor is 2.

Ex. **2.** To divide x^9-3x^5+6x-4 by x^2-2x+1.

The coefficient of the first term of the divisor being 1 need not be written. Making the functions complete in form, we have

	+ 2 − 1
1 + 0 + 0 + 0 − 3 + 0 + 0 + 0	+ 6 − 4
+ 2 + 4 + 6 + 8 + 4 + 0 − 4	− 8 + 4
− 1 − 2 − 3 − 4 − 2 + 0	+ 2
1 + 2 + 3 + 4 + 2 + 0 − 2 − 4	0 0

and the quotient is $x^7+2x^6+3x^5+4x^4+2x^3-2x-4$.

If desired, the functions may equally well be arranged in ascending order of the variable, as

$$
\begin{array}{r|r}
 & 2-1 \\ \hline
-4+6+0+0+0-3+0+0 & +0+1 \\
-8-4+0+4+8+6+4 & +2+0. \\
+4+2+0-2-4-3 & -2-1 \\ \hline
-4-2+0+2+4+3+2+1 & 0\quad 0
\end{array}
$$

or $$-4-2x+2x^3+4x^4+3x^5+2x^6+x^7.$$

40. The preceding are examples of exact division. In arithmetic, when the dividend is greater than the divisor, we can obtain an integral quotient and a remainder, where there is one; or, we may expand the remainder into a decimal series which, in general, is non-terminating.

Now, a higher degree in algebra corresponds to a greater quantity in arithmetic; so that, when the dividend is of a higher degree than the divisor, and the division is not exact, we may obtain a quotient and a remainder, or we may expand the remainder into an infinite series.

Ex. 1. To divide $x^7-x^5+5x^3+10x^2-5x+1$ by $x^4-2x^3+x^2-2$, obtaining the quotient and the remainder.

$$
\begin{array}{r|l l}
 & +2-\ 1+0+2 & \text{. . Divisor.} \\ \hline
1+0-1+0 & +5+10-5+1 & \text{. . Dividend.} \\
+2+4+4 & +4-\ 2+4+4 & \\
-1-2 & -2+\ 4 & \text{. . Partial products.} \\
 & +2 & \\ \hline
1+2+2+2 & +9+12-1+5 & \text{. . Result.}
\end{array}
$$

Quotient, $x^3+2x^2+2x+2.$

Remainder, $9x^3+12x^2-x+5.$

It will be noticed that a vertical line is drawn to the left of that part of the divisor which is used in forming the partial products.

In a case of even division, all the terms of the result to the right of this line are zeros, and when we wish to obtain the remainder we treat these terms as if they were zeros in forming the partial products.

If we employ the terms to the right of the vertical line in forming partial products, the quotient will extend into a series, and all the terms to the right of the line will contain negative powers of x, and the series will thus be arranged in descending powers of x.

If we wish the series to be in ascending powers, we must arrange our functions in that order before beginning the division.

Series so produced, like circulating decimals in arithmetic, have their coefficients connected by a fixed law of formation. Sometimes this law is obvious from simple inspection, and at all times it can be exactly determined.

This law is of great importance in investigations connected with Recurring Series.

Ex. 2. To divide $1 + x - x^2$ by $1 - 2x + x^2$ to a series in ascending powers of x.

$$
\begin{array}{r|rrrrrl}
 & +2 & -1 & & & & \\
\cline{2-3}
1 & +1 & -1 & & & & \\
 & +2 & +6 & +8 & +10 & +12 & +\cdots \\
 & & -1 & -3 & -4 & -5 & -\cdots \\
\hline
1 & +3 & +4 & +5 & +6 & +7 & +\cdots
\end{array}
$$

Here the law of the coefficients is obvious, and the series is

$$1 + 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5 + \cdots$$

If we arrange the dividend and the divisor in descending powers of x, the quotient coefficients are —

$$-1-1+0+1+2+3+4+\cdots$$

and the series is —

$$-1-\frac{1}{x}+\frac{1}{x^3}+\frac{2}{x^4}+\frac{3}{x^5}+\frac{4}{x^6}+\cdots$$

or $$-1-x^{-1}+x^{-3}+2x^{-4}+3x^{-5}+4x^{-6}+\cdots$$

Ex. **3**. To divide $1+x$ by $1-x+x^2$ to a series in ascending powers.

The series is $1+2x+x^2-x^3-2x^4-x^5+x^6+\cdots$

Here also the law of the coefficients is readily brought out, for the series may be written

$$(1+2x+x^2)(1-x^3+x^6-x^9+-\cdots).$$

Ex. **4**. Divide 1 by $1-2x+3x^2$.

	$2-3$
1	$2+4+2-8-22-20\cdots$
	$-3-6-3+12+33\cdots$
1	$+2+1-4-11-10+13\cdots$

The series is $1+2x+x^2-4x^3-11x^4-10x^5+13x^6\cdots$, and the law is not apparent from inspection.

41. The following results are frequently required, and should be committed to memory:

i. $$\frac{1}{1-x}\equiv 1+x+x^2+x^3+x^4+\cdots$$

ii. $$\frac{1}{1+x}\equiv 1-x+x^2-x^3+x^4-\cdots$$

iii. $$\frac{1}{(1-x)^2}\equiv 1+2x+3x^2+4x^3+5x^4+\cdots$$

iv. $$\frac{1}{(1+x)^2}\equiv 1-2x+3x^2-4x^3+5x^4-\cdots$$

Ex. 1. To expand $\frac{1+z}{(1-z)^2}$ to a series.

By iii. this is $(1+z)(1+2z+3z^2+4z^3+\cdots)$, which by distribution becomes

$$1+3z+5z^2+7z^3+\cdots$$

Ex. 2. To find from what division the series

$$1+x-2x^2-x^3-x^4+2x^5+x^6+x^7-2x^8\cdots$$

has been derived.

The series may be written

$$(1+x-2x^2)(1-x^3+x^6-x^9\cdots);$$

and, by ii., $1-x^3+x^6-x^9\cdots\equiv\frac{1}{1+x^3}.$

$$\therefore\ \frac{1+x-2x^2}{1+x^3} \text{ is the division.}$$

EXERCISE II. f.

1. Divide $x^5+x^3+2x^2-2x+8$ by x^2-2x+2.

2. Divide $6a^6+2a^5+9a^4-2a^3-a^2-3a+1$ by $2a^3+3a-1$.

3. Divide $1+x^2+2x^3-2x^4+8x^5$ by $1+2x+3x^2+4x^3$.

4. Divide $a-1$ by $a+1$ to 5 terms in descending powers of a.

5. Divide $x^5-x^3+x^2-2x-1$ by x^3+2x^2-3x+1, giving quotient and remainder.

6. Divide $a+2a^2-3a^3+4a^4$ by $a-a^2+a^3-a^4$ to a series, and obtain the law of the coefficients.

7. If $y=1-z+(1-z)^2+(1-z)^3+\cdots$, and $z=1+x+x^2+x^3+\cdots$, show that $y=-x$.

8. Divide $1+2x+3x^2+4x^3+\cdots$ by $1+3x+5x^2+7x^3+\cdots$ to 6 terms of a series.

9. Expand $x \div (x^2 - 2x + 1)$ into a series, first in ascending, and second in descending powers of x.

10. What must be added to $x^4 - x^3 + x^2 - x + 1$ to make it an even multiple of $x^2 - x + 1$?

11. Divide $1 + \frac{5}{2}x + \frac{13}{3}x^2 + \frac{77}{12}x^3 + \frac{7}{2}x^4 + \frac{25}{12}x^5 + x^6$

by $1 + 2x + 3x^2 + 4x^3$

12. Divide $x^3 + y^3 + 3xy - 1$ by $x + y - 1$.

Take x as variable and the functions are

$$x^3 + 0x^2 + 3yx + (y^3 - 1) \text{ and } x + (y - 1).$$

	$-(y-1)$
$1 + 0 \qquad + 3y$	$+(y^3 - 1)$
$\quad -(y-1) + (y-1)^2$	$-(y^3 - 1)$
$1 - (y-1) + (y^2 + y + 1)$	0

Quotient is $x^2 - x(y - 1) + (y^2 + y + 1)$.

13. Divide $a^3(b - c) + b^3(c - a) + c^3(a - b)$ by $a + b + c$.

Take a as variable and proceed as in Ex. 12.

14. Divide $x^5 + 2x^3y^2 - x^2y^3 + xy^4 - y^5$ by $x^3 + xy^2 - y^3$.

The functions being each homogeneous, put $y = zx$; this reduces the division to $x^2(1 + 2z^2 - z^3 + z^4 - z^5) \div (1 + z^2 - z^3)$.

15. Find the simplest division that will give the series

$$x + 3x^2 + 2x^3 - x^4 - 3x^5 - 2x^6 + x^7 + \cdots$$

16. What expression added to $(a + b + c)(ab + bc + ca)$ will make it exactly divisible by $a + b$?

17. Multiply $a + 3 + 3a^{-1} + a^{-2}$ by $1 - a^{-1}$.

18. Multiply $(x + x^{-1})^3 + (x + x^{-1})^2 + (x + x^{-1}) + 1$ by $x - x^{-1}$.

19. Divide $4x^2 - 7x + 3 - x^{-2} + x^{-3}$ by $x^{-1} - 2x^{-2} + x^{-3}$.

20. If $x - a$ be a factor of $x^2 + 2ax - 3b^2$, then $a = \pm b$.

21. If $1 \div (a^2 - ax + x^2)$ be expressed as $A + Bx + Cx^2 + Dx^3 + f(x)$, find A, B, C, D and the form of f.

22. If $x^3 - ax^2 + bx + c$ be divided by $x - z$, the remainder is $z^3 - az^2 + bz + c$.

23. If the terms of the divisor are written in an inverted order, by what arrangement of multiplication are the partial products formed?

CHAPTER III.

Factors and Factorization.

42. In a case of even division, we separate the dividend into two factors (Art. 37). One or both of these might be again separated into two factors, and so on, until the whole expression was separated into factors which should be linear in the variable or else numerical.

Thus $x^3 - bx^2 - a^2x + a^2b \equiv (x - a)(x + a)(x - b)$.

We thus see that Factorization, as an operation, is the inverse of Distribution.

In the factored expression written above the factors are each linear and binomial.

In $6a(a + b + c)(ab + bc)(x^2 + x + 1)$, 6 is a numerical factor, a is linear and monomial, $a + b + c$ is linear and trinomial, and $ab + bc$ and $x^2 + x + 1$ are both quadratic factors.

Theoretically, any integral function of a variable can be separated into factors linear in that variable; but the cases in which we can make the separation practically are limited to but a few classes, out of all the possible integral functions. Frequently, however, these cases are of most importance.

43. Factorization may sometimes be effected by making use of the standard forms of Art. 29.

Thus, because $a^2 - b^2 = (a + b)(a - b)$, we can always put into factors that which can be expressed as the difference between two squares.

Ex. 1. $(a^2+b^2)^2-(a^2-b^2)^2$

$$\equiv (a^2+b^2+a^2-b^2)(a^2+b^2-a^2+b^2)\equiv 4a^2b^2.$$

Ex. 2. $x^2+2a-a^2-1\equiv x^2-(a-1)^2\equiv (x-a+1)(x+a-1).$

44. An expression of the form x^2+ax+b can be factored at sight if we can discover two quantities such that their sum is a and their product is b whatever a and b may stand for. For if p and q be such quantities, the factors are $(x+p)(x+q)$.

Ex. 1. $x^2-2x-3\equiv(x+1)(x-3).$

Ex. 2. $a^2+2ab+b^2-a-b-6\equiv(a+b)^2-(a+b)-6$

$$\equiv(a+b-3)(a+b+2).$$

Ex. 3. $8x^2+8x-6\equiv 2(\overline{2x}^2+2\cdot\overline{2x}-3)$

$$\equiv 2(2x+3)(2x-1).$$

No particular rules can be laid down for this kind of factoring. Success is to be attained only by observation and practice.

EXERCISE III. a.

1. Put $(x^2+3x+2)(x^2-3x+2)$ into four linear factors.

2. Put $ab+2a^2-3b^2-4bc-ac-c^2$ into linear factors.

3. Express $x^4-(p+q)x^3+pqx^2+(p-q)x-1$ as the product of two quadratics in x.

4. Factor $(x+y)^2+(x+y)(a+b)+ab$.

5. Factor $(a+x)^2-3(a+x)+2$.

6. Factor $6(2x+3y)^2+5(6x^2+5xy-6y^2)-6(3x-2y)^2$.

7. Express $4a^2b^2-(a^2+b^2-c^2)^2$ in the form of four linear factors.

8. Factor $12x^2 - 7x + 1$, and $6x^2 - 21x + 18$.

9. Express $x^4 - px^3 + (q - 1)x^2 + px - q$ as two linears and a quadratic.

45. Let $f(x)$ be any integral function of x, and let p, q, r, s, etc., denote its factors, so that

$$f(x) \equiv p \cdot q \cdot r \cdot s \cdots$$

Now, assuming that all the factors are finite, if any factor becomes zero, the whole expression, $f(x)$, becomes zero. And, conversely, the expression cannot become zero unless one of its factors becomes zero.

If, then, we suspect that a certain expression is a factor of $f(x)$, we put that expression equal to 0; from this we find the value of x in terms of the other quantities concerned, and we substitute this value for x in the given function.

If the function becomes zero, or vanishes, the suspected expression is a factor; and if the function does not vanish, the suspected expression is not a factor.

Ex. **1**. Is $a + 1$ a factor of $6a^2b + 3a^2 + 12ab^2 + 12ab + 3a + 12b^2 + 6b$?

Put $a + 1 = 0$; this gives $a = -1$. Substitute -1 for a in the given function, and it vanishes.

Hence $a + 1$ is a factor.

Similarly, we find $a + 2b$ to be a factor; and by dividing by these we get $6b + 3$ as the third factor; and the expression becomes

$$(a + 1)(a + 2b)(6b + 3).$$

Ex. **2**. To factor $a(b + bc - c) + b(c + ca - a) + c(a + ab - b)$.

As the expression may have a monomial factor, try $a = 0$, *i.e.* put 0 for a.

The expression vanishes, and hence a is a factor.

But the expression is symmetrical in the three letters, and hence b and c must also be factors.

Therefore abc is a factor.

By the index law the dimensions of the expression must be the sum of the dimensions of its factors.

But abc is of three dimensions, and so also is the expression; hence there are no other literal factors.

There may be a numerical factor, since such a factor has no dimensions. The coefficient of the type abc from the expression is readily seen to be 3; therefore the whole expression factors to $3abc$.

Ex. **3**. To factor $ab(b^2 - a^2) + bc(c^2 - b^2) + ca(a^2 - c^2)$.

We readily discover that there are no monomial factors.

Since $b - a$ is a factor of one of the terms, let us try if it be a factor of the whole.

Put $b - a = 0$, or write b for a in the expression. It vanishes.

Therefore $b - a$, and from symmetry, $a - c$, and $c - b$, are all factors.

Therefore $(b - a)(a - c)(c - b)$ is a factor of 3 dimensions. But the expression is of 4 dimensions. Hence there is a fourth factor, symmetrical in a, b, and c, and linear. $a + b + c$ is the only such factor that can occur here, and

$$(b - a)(a - c)(c - b)(a + b + c)$$

is a factor, and comprises all the linear factors.

For the numerical factor take any type term that occurs in both the expression and the factored result, as a^3b. Its coefficient from the expression is $-$, and from the factored result it is $+$. Therefore -1 is the numerical factor, and the expression becomes

$$-(b - a)(a - c)(c - b)(a + b + c),$$

or

$$(a - b)(b - c)(c - a)(a + b + c).$$

Ex. **4**. To factor $ab(c - d) + bc(d - a) + cd(a - b) + da(b - c)$.

We find a to be a monomial factor, and by symmetry b, c, and d are factors.

Therefore $abcd$ is a factor of 4 dimensions.

But the expression is of only 3 dimensions, and should have only 3 literal factors, unless it can have any number of factors. The only expression that admits anything as a factor is 0. Hence the expression $\equiv 0$, identically.

46. If we have an integral function of x, and if its factored form be

$$(x-a)(x-b)(x-c)\cdots,$$

the independent term of the function, T say, is equal to $abc\cdots$ (Art. 31).

Hence, in trying to factorize the function, since a, b, c, etc., are all factors of T, the factors of T are the only quantities to be substituted for x in our trial.

By substituting the rational factors of T, any rational linear factors of the function will be discovered.

Of course it must be understood that factors are not necessarily discoverable in this way, since it is only in special cases that such functions have all or any of their linear factors rational.

Ex. 1. To factorize $x^4-3x^3-3x^2+7x+6$.

Here $T=6$, and its factors are ± 1, ± 2, ± 3, and ± 6.

Put 1 for x; the function does not vanish, and $x-1$ is not a factor.

Put -1 for x; the function vanishes and $x+1$ is a factor.

Similarly, try 2, -2, 3, -3, etc., successively until all the factors discoverable by this means are found.

Otherwise, having found $x+1$ to be a factor, divide the function by $x+1$. The quotient is

$$x^3-4x^2+x+6.$$

Of this new function $x+1$ is again a factor. Divide by $x+1$, and the quotient is

$$x^2-5x+6,$$

which factors into $(x-3)(x-2)$.

$$\therefore\ x^4-3x^3-3x^2+7x+6\equiv(x+1)^2(x-3)(x-2).$$

In employing this latter method, it will be more expeditious to try the higher factors of T first.

EXERCISE III. b.

1. Factorize the following —

i. $\Sigma a^2b + 2\,abc$.

ii. $\Sigma\,(ab^2 - a^2b)$.

iii. $\Sigma\,(a \cdot \overline{b^2 - c^2})$.

iv. $\Sigma\,\{a(b^3 - c^3)\}$.

v. $\Sigma\,\{b(c - d)(ab - cd)\}$.

vi. $\Sigma\,\{a(b^4 - c^4)\}$.

vii. $\Sigma\,(ab^2 - a^2b) + \Sigma ab - \Sigma a^2 + 1$.

viii. $(a + b - c)(b + c - a)(c + a - b) - \Sigma ab(a + b) + \Sigma a^3$.

ix. $\Sigma\,(ab \cdot \overline{c - d})$ four letters.

2. Put $\Sigma a^4bc - \Sigma a^3b^3$ into quadratic factors.

3. If the 6th power of a number be diminished by 1 and the 5th power of the same number be increased by 1, the difference of the results is divisible by the next greater number.

4. If any even power of a number be diminished by 1, and any odd power of the same number be increased by 1, the results have a common factor.

5. Factorize $x^4 + 8\,x^3 - 10\,x^2 - 104\,x + 105$.

6. Express $(x + 1)(x + 3)(x + 5)(x + 7) + 15$ as the product of two linears and a quadratic.

7. Factorize $x^4 - 5\,x^2 + 6$.

8. Factorize $(a+b+c)^3 - (b+c-a)^3 - (c+a-b)^3 - (a+b-c)^3$.

9. Factorize $\Sigma a\{bc + ac + c^2 - a^2\} - 5\,abc$.

10. Factorize

$$(ac + bd)^2 - abc(a - b + c) - bcd(b - c + d) - cda(c - d + a) - dab(d - a + b).$$

11. Factorize

$$(a + b - x)(b + c + x) + (b + c - x)(c + a + x) + (c + a - x)(a + b + x) - 3(ab + bc + ca) + abc - a^2 - b^2 - c^2 + 3\,x^2.$$

12. Factorize

$$(a-b)(b+c)+(b-c)(c+a)+(c-a)(a+b)+(a-c)^2.$$

13. Factorize $x^3+x(\Sigma ab-\Sigma a^2)-(a-b)(b-c)(c-a)$.

14. If the 4th power of a number be increased by 4 and diminished by 5 times the square of the number, the result multiplied by the number itself is the product of 5 consecutive numbers.

47. The symbol $\sqrt{\ }$ is defined by the relation $\sqrt{a}\times\sqrt{a}=a$, where a denotes any numerical quantity or algebraic expression.

Thus $\sqrt{(a^2+b^2+2ab)}\equiv a+b$, and $\sqrt{16}\equiv 4$.

The expression $\sqrt{a}$ is read the **square root** of a, or, more concisely, the *root* of a, and it denotes that a is to be separated into two identically equal factors, and that one of these is to be taken.

Hence when it is possible to separate a quantity or an expression into two such factors, it is possible to express *exactly* the square root of the quantity or expression.

Thus $\Sigma a^2+2\,\Sigma ab$ can be separated into the identically equal factors $\Sigma a\times\Sigma a$; and hence $\Sigma a\equiv\sqrt{(\Sigma a^2+2\Sigma ab)}$, for any number of letters.

48. The expression $\sqrt{a}$ requires careful consideration.

(1) If a is a positive square number it is the product of two identically equal factors, and $\sqrt{a}$ denotes one of these factors.

The factors may be both $+$ or both $-$, since in either case the product is $+$.

Therefore $\sqrt{a}$ has two signs and is often written $\pm\sqrt{a}$. Thus $\sqrt{x^2}$ is either $+x$, or $-x$; and $\sqrt{16}$ is $+4$, or -4.

The double sign, whether written or not, must always be mentally attached to a square root, being frequently of very great importance.

(2) If a is a positive non-square number, no two identically equal factors can be found for it.

The $\sqrt{a}$ then symbolizes one of that class of numerical quantities called *incommensurables*, or **irrational** quantities (Art. 1).

In this case $\sqrt{a}$ cannot have its value exactly expressed; but it may, under the form of a non-terminating decimal, be expressed to any degree of approximation we please, by the arithmetical process of 'extracting the square root.'

Thus	$(1.41)^2$ differs from 2 by 0.0118				
	$(1.414)^2$	"	" 2	"	0.000604
	$(1.4142)^2$	"	" 2	"	0.000039
	$(1.41421)^2$	"	" 2	"	0.0000002
	etc.				etc.

And the successive squares become closer and closer approximations to 2, the degree of approximation depending upon the extent of the decimal series.

This series, unlike circulants produced by division, has no arrangement of its digits which would indicate any law governing the order of their succession.

(3) Let a denote a negative number.

As like signs produce only + in multiplication, it is not possible to find, or to approximate to, or to conceive of a quantity, which multiplied once by itself will give the sign −.

The symbol $\sqrt{a}$ is then called an **imaginary** in contradistinction to the quantities of (1) and (2), which are *real.*

Thus $\sqrt{-3}$ is imaginary, while $\sqrt{3}$ is real.

49. As in arithmetic, so in algebra, an expression may be a complete square and be capable of having its square root exactly expressed, or it may be a non-square and admit only of having its root approximated to by an infinite series.

Thus

$$\sqrt{1+x} \equiv 1 + \frac{x}{2} - \frac{1}{1 \cdot 2} \cdot \frac{x^2}{2^2} + \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{x^3}{2^3} - \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{x^4}{2^4} + \cdots$$

General methods for this approximation will be considered hereafter.

It may be remarked that an algebraic expression cannot in itself be imaginary, as the character of real or imaginary is wholly due to the interpretation of the quantitative symbols.

Thus $\sqrt{a-b}$ is real, if a and b are both positive numbers, and b is less than a, but imaginary if b is greater than a. If a is positive and b negative, the expression is always real; and if a is negative and b positive it is always imaginary.

50. If a denotes a positive quantity, $\sqrt{a}$ can be expressed to any degree of approximation that we please, and hence $\sqrt{a}$ must, like other quantitative symbols, be subject to the commutative and distributive laws.

Hence

(1) $\sqrt{a}+\sqrt{b}=\sqrt{b}+\sqrt{a}$, and $\sqrt{a}\cdot\sqrt{b}=\sqrt{b}\cdot\sqrt{a}$.

(2) $\sqrt{a}(b+\sqrt{c})=b\sqrt{a}+\sqrt{a}\cdot\sqrt{c}$, etc.

51. The symbol $\sqrt{}$ is introduced here as a special operative symbol, having a relationship to the exponent, as will appear hereafter, and it is necessary that we should investigate the limits of its operation, and discover in how far it obeys the great formal laws of algebra.

(1) As $(\sqrt{a}\cdot\sqrt{b})(\sqrt{a}\cdot\sqrt{b})\equiv\sqrt{a}\cdot\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{b}\equiv ab$ by definition; and as $\sqrt{ab}\cdot\sqrt{ab}\equiv ab$ by definition, therefore $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$; and *the operative symbol* $\sqrt{\ }$ *is distributive over the factors of a product.*

$$\text{Thus}\quad \sqrt{2}\cdot\sqrt{a}\cdot\sqrt{2b}\equiv\sqrt{(4ab)}\equiv\sqrt{4}\cdot\sqrt{ab}\equiv 2\sqrt{ab}.$$

$$p\sqrt{q}\equiv\sqrt{p^2}\cdot\sqrt{q}\equiv\sqrt{p^2q}.$$

$$\sqrt{(45a^3bc^2)}\equiv\sqrt{(9a^2c^2\cdot 5ab)}\equiv 3ac\sqrt{5ab}.$$

etc. etc. etc.

(2) Since $\sqrt{(a+b)}\sqrt{(a+b)}\equiv a+b$ by definition, and $(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{b})\equiv a+b+2\sqrt{ab}$ by distribution, therefore $\sqrt{(a+b)}$ is not the same as $\sqrt{a}+\sqrt{b}$.

Or, *the symbol* $\sqrt{\ }$ *is not distributive over the terms of a sum.*

The statements of (1) and (2) form the working principles of this symbol, and should be carefully remembered.

52. Since $\sqrt{-a}\cdot\sqrt{-a}=-a$ by definition, whatever a may be, we assume that $\sqrt{a}$ is subject to the same general laws of operation whether a be positive or negative, *i.e.* whether the expression be real or imaginary.

$$\text{Hence}\quad \sqrt{-ab}=\sqrt{-a}\cdot\sqrt{b}=\sqrt{a}\cdot\sqrt{-b},$$

$$\text{and}\quad \sqrt{-a}=\sqrt{a(-1)}=\sqrt{a}\cdot\sqrt{-1},$$

where $\sqrt{a}$ is real.

Thus every imaginary number can be reduced to depend upon the symbol $\sqrt{-1}$, which is called the **imaginary unit**, and is usually symbolized by i.

If, then, x denotes any real number, ix denotes the corresponding imaginary; the relation between these being that the square of the first is $+x^2$, and of the other it is $-x^2$.

This generalization introduces us to a new set of numbers, the symbolic numbers or imaginaries.

All whole numbers, positive, negative, and imaginary, may be represented in the general scheme,

$$\cdots -5,\ -4,\ -3,\ -2,\ -1, 0, 1, 2,\ 3,\ 4,\ 5,\ \cdots$$
$$\cdots -5i, -4i, -3i, -2i, -i,\ 0,\ i, 2i, 3i, 4i, 5i, \cdots$$

53. The powers of the symbol i, which occur very frequently, are given in the scheme,

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1, \text{ etc.},$$

the powers repeating their values in cyclic order.

A quantity which is the algebraic sum of a real and an imaginary is called a *complex* quantity; but being arithmetically inexpressible on account of its imaginary part, it ranks with imaginaries.

Thus, $2+3i$ is a complex, and so also is $2-3i$. The square of $2+3i$ is $2^2+(3i)^2+12i$, or $12i-5$, another complex; but the product $(2+3i)(2-3i)$ is $2^2-(3i)^2$, or 13, a real.

$a+bi$ represents any number whatever; for if $b=0$, it is real; if $a=0$, it is imaginary; and if neither a nor b be zero, it is a complex.

54. The expression x^2+px+q can always be separated into two factors linear in x.

$$x^2+px+q = x^2+px+\frac{p^2}{4}-\left(\frac{p^2}{4}-q\right)$$

$$=\left(x+\frac{p}{2}\right)^2-\left(\frac{\sqrt{p^2-4q}}{2}\right)^2$$

$$=\left(x+\frac{p+\sqrt{p^2-4q}}{2}\right)\left(x+\frac{p-\sqrt{p^2-4q}}{2}\right).$$

As these factors both contain the same square root part, they will be both real or both complex, *i.e.* imaginary. But in any case they will, upon distribution, reproduce the original expression.

Thus the factors of x^2+2x+3 are

$$(x+1+\sqrt{-2})(x+1-\sqrt{-2}),$$

both being complex quantities.

55. The expression ax^2+bx+c can always be separated into a monomial factor, a, and two factors linear in x.

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right).$$

The part within brackets is the same expression as x^2+px+q, if we write $\frac{b}{a}$ for p and $\frac{c}{a}$ for q.

Making this substitution in the result of Art. 54, gives, after reducing,

$$a\left\{x+\frac{b+\sqrt{b^2-4ac}}{2a}\right\}\left\{x+\frac{b-\sqrt{b^2-4ac}}{2a}\right\}.$$

This factorization may also be done independently as follows:

$$ax^2+bx+c=\frac{1}{4a}\{4a^2x^2+4abx+b^2-(b^2-4ac)\}$$

$$=\frac{1}{4a}\{(2ax+b)^2-(\sqrt{b^2-4ac})^2\}$$

$$=\frac{1}{4a}\{2ax+b+\sqrt{b^2-4ac}\}\{2ax+b-\sqrt{b^2-4ac}\}$$

$$=a\left\{x+\frac{b+\sqrt{b^2-4ac}}{2a}\right\}\left\{x+\frac{b-\sqrt{b^2-4ac}}{2a}\right\}$$

Ex. **1.** The factors of $3x^2+5x-1$ are

$$3\left\{x+\frac{5+\sqrt{37}}{6}\right\}\left\{x+\frac{5-\sqrt{37}}{6}\right\}.$$

Ex. **2.** The factors of $2x^2-3x+2$ are

$$2\left\{x-\frac{3+\sqrt{-7}}{4}\right\}\left\{x-\frac{3-\sqrt{-7}}{4}\right\}.$$

The two factorizations, of Art. 54 and the present Article, are very important, and the forms of the factors should be carefully mastered.

Taking ax^2+bx+c as being the most general in form, the square root part of each factor is $\sqrt{b^2-4ac}$. The character of the factors will depend upon that of this part of them.

If a and c are $+$, and $4ac$ is greater than b^2, the factors will be complex quantities.

If $4ac$ be less than b^2, the factors will be real and unequal; and if $4ac=b^2$, the expression $\sqrt{b^2-4ac}$ becomes zero, and the factors are real and equal.

Cor. The expression ax^2+bx+c is a complete square when $b^2=4ac$.

The finding of the square root of an algebraic expression is equivalent to the separation of the expression into two identical factors, and hence requires no special process.

If the expression is a complete square, the factorization ranks with the simpler cases. But if it is not a complete square, the only practical method is by means of the binomial theorem or undetermined coefficients, to be given hereafter, and the root is expressed as an infinite series.

Thus, in $a^2 - 2ab + a + 4b^2 - 4b + 1$, we readily see that $\pm a$, $\pm 2b$, and ± 1 must be terms of the root, and the least observation shows that the root is $a - 2b + 1$.

Ex. **3.** The factors of $2x^2 - 3x - 4$ are

$$2\{x + \tfrac{1}{4}(-3 + \sqrt{41})\}\{x + \tfrac{1}{4}(-3 - \sqrt{41})\}.$$

Ex. **4.** The factors of $x^2 + 2x + 10$ are

$$(x + 1 + 3i)(x + 1 - 3i).$$

Ex. **5.** Determine the value of m so that $3x^2 + 4mx + 12$ may be a complete square. We must have

$$b^2 = 4ac, \text{ i.e. } (4m)^2 = 4 \cdot 3 \cdot 12.$$

Whence $m = +3$ or -3.

56. The expression $x^4 + bx^2 + c$ can be separated into four linear factors. Let $\alpha, \beta, \gamma, \delta$ denote these factors. Any two of these multiplied together gives a quadratic factor of the expression.

But these may be multiplied two together in three different ways, namely,

$$\alpha\beta, \gamma\delta;\ \alpha\gamma, \beta\delta;\ \text{and } \alpha\delta, \beta\gamma.$$

Hence $x^4 + bx^2 + c$ can be separated into a pair of quadratic factors in three different ways.

Ex. $$x^4 + 10x^2 + 9 = (x^2 + 9)(x^2 + 1)$$
$$= (x + 3i)(x - 3i)(x + i)(x - i).$$

Thence the pairs of quadratic factors are —

1. $(x^2+9)(x^2+1)$.
2. $(x^2+4ix-3)(x^2-4ix-3)$.
3. $(x^2+2ix+3)(x^2-2ix+3)$.

57. The factorization of x^3-1 is important.

$$x^3-1=(x-1)(x^2+x+1),$$

and by Art. 54,

$$x^2+x+1=\left(x+\frac{1-i\sqrt{3}}{2}\right)\left(x+\frac{1+i\sqrt{3}}{2}\right).$$

$$\therefore\ x^3-1=(x-1)\left(x+\frac{1-i\sqrt{3}}{2}\right)\left(x+\frac{1+i\sqrt{3}}{2}\right).$$

If any one of these factors becomes zero, the expression vanishes; *i.e.* $x^3-1=0$, and $x^3=1$.

Hence, when $x=1$, or $-\frac{1-i\sqrt{3}}{2}$, or $-\frac{1+i\sqrt{3}}{2}$, $x^3=1$.

And since the cube of each of these three values of x is 1, these are the *three cube roots of unity*, one of which is real, and the other two complex.

The complex roots are generally denoted by ω and ω^2, because the square of either of them is equal to the other.

Then $\omega^3=1$, $\omega^4=\omega^3\cdot\omega=\omega$, $\omega^5=\omega^2$, $\omega^6=1$, etc.

Since $\omega^3-1=0$, its equivalent $(\omega-1)(\omega^2+\omega+1)=0$; and as $\omega-1$ is not zero, we must have

$$\omega^2+\omega+1=0.$$

And this is the fundamental relation connecting the cube roots of 1; *i.e.* the sum of the three roots is zero.

Ex. Multiply together $x + \omega y + \omega^2 z$ and $x + \omega^2 y + \omega z$.

Distributing $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$ gives

$$x^2 + y^2\omega^3 + z^2\omega^3 + xy(\omega^2 + \omega) + yz(\omega^4 + \omega^2) + zx(\omega^2 + \omega).$$

But $\omega^3 = 1$, $\omega^2 + \omega = -1$, and $\omega^4 + \omega^2 = \omega + \omega^2 = -1$.

$\therefore\ (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) \equiv x^2 + y^2 + z^2 - xy - yz - zx.$

EXERCISE III. c.

1. Find $\sqrt{\{a^2 + b^2 + 2ab - 2a - 2b + 1\}}$.

2. Separate $x^2 + 4y^2 + 4xy - 4x - 8y + 4$ into two identically equal factors.

3. Express $\sqrt{ab^4} + \sqrt{ac^4} - \sqrt{4ab^2c^2}$ as a multiple of a single irrational factor.

4. Show that $(a + bi)(c + di)$ has the form $A + Bi$, and express A and B in terms of the small letters.

5. Distribute $(x - a + bi)(x - a - bi)$.

6. Show that $ix - \frac{1}{2}(ix)^2 + \frac{1}{3}(ix)^3 - \frac{1}{4}(ix)^4 + - \cdots$
$\equiv i(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - + \cdots) + \frac{1}{2}x^2(1 - \frac{1}{2}x^2 + \frac{1}{3}x^4 - + \cdots)$.

7. Factorize the following —

i. $x^2 - 3x - 3$.	v. $ax^2 + x + a$.
ii. $x^2 - 2x + 5$.	vi. $px^2 - (p + 1)x + 1$.
iii. $4x^2 - 4x - 2$.	vii. $(a^2 - b^2)x^2 + 2ax + 1$.
iv. $5x^2 + 3x - 6$.	viii. $2 + ax + \dfrac{1}{ax}$.

8. Resolve $x^4 - 11x^2 + 10$ into four linear factors.

9. Resolve $x^4 - 3x^2 + 1$ into linear factors.

10. Resolve $x^4 + x^2 - 2$ into its three pairs of quadratic factors.

11. Resolve $4x^2 - 4x - x^4 + 1$ into linear factors.

12. Resolve $x^3 - 2x^2 - x + 2$ into linear factors.

13. Distribute $(1 + x)(1 + \omega x)(1 + \omega^2 x)$.

14. Show that $(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) = \Sigma\alpha^2 - \Sigma\alpha\beta.$

15. Show that $(x + \omega y + \omega^2 z)^3 + (x + \omega^2 y + \omega z)^3$
$$\equiv 2\,\Sigma x^3 - 3\,\Sigma x^2 y + 12\,xyz.$$

16. Distribute $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)(x + y + z).$

17. Show that $(x + \omega y + \omega^2 z)^3 - (x + \omega^2 y + \omega z)^3$
$$\equiv -3\sqrt{-3}\,(x - y)(y - z)(z - x).$$

18. Find the relation between a and b when

$$(a + b)x^2 - (a^2 - b^2)x + a^2b + ab^2$$

is a complete square.

58. The integral function of x,

$$x^4 - 10\,x^3 + 35\,x^2 - 50\,x + 24,$$

factors into

$$(x - 1)(x - 2)(x - 3)(x - 4).$$

The numbers 1, 2, 3, 4 are the **roots** of the function, because from these, and the variable x, we may build up the function by multiplication, or, so to speak, cause it to grow up.

If any of these roots be put for x, the substitution will cause the function to vanish, since it makes one of the factors zero. And, conversely, the only single substitution that will make the function vanish must make one of the factors vanish, or is the substitution of one of the roots for x.

Hence the roots of an integral function of any variable are those quantities which, when put for the variable in the function, cause it to vanish. And reciprocally any quantity, which put for the variable will cause the function to vanish, is a root.

Thus the roots of $x^3 + x^2\Sigma a + x\Sigma ab + abc$ are $-a$, $-b$, and $-c$; for the function factors into

$$(x + a)(x + b)(x + c).$$

59. The expression $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ is a *conditional equation*, or simply an *equation* in which x is to have such a value as will make the expression an identity (Art. 22).

We have seen, in the preceding article, that this will be effected by making x equal to any one of the roots of the function, namely, 1, 2, 3, or 4.

It is readily seen that the same principle applies to functions of any degree.

Hence: (1) In the equation formed by putting an integral function of a variable equal to zero, we obtain the **roots** of the equation by separating the function into factors linear in the variable.

The determination of any one of these factors is *a solution* of the equation, and the determination of all these factors is the *complete solution*.

(2) The whole number of solutions, or the number of roots which the equation has, is the number of linear factors into which the function is theoretically separable, and this is the same as the degree of the function in the variable.

The solution of an equation is thus equivalent to the factorization of the function into factors linear in the variable.

60. When the roots of an integral function or of the corresponding equation are all real and all rational, they can generally be found.

Also, the methods of factoring now at our disposal are sufficient for the linear factorization of all integral functions of a single variable of not more than two dimensions; but these methods are not sufficient for the *general* factorization of functions of *more* than two dimensions. They suffice, however, for many special and particular cases.

Ex. 1. To find all the solutions of $x^4 - x^3 - 2x^2 - 2 + 4 = 0$.

By trial we readily find $x - 1$ and $x - 2$ to be factors.
Dividing by these gives $x^2 + 2x + 2 = 0$.
The factors of this are $(x + 1 + i)(x + 1 - i)$.
And the four roots are $1, 2, -1 - i$, and $-1 + i$.

Ex. 2. To solve the equation $x^3 - 5x + 2 = 0$.

Factorization gives $(x - 2)(x^2 + 2x - 1)$.
The factors of $x^2 + 2x - 1$ are $\{x + (1 + \sqrt{2})\}\{x + (1 - \sqrt{2})\}$.
And the roots are $2, -(1 + \sqrt{2}), -(1 - \sqrt{2})$.

A linear equation in any variable is simply a linear factor of unity. By proper transformations such an equation may always be brought to the form

$$Ax - B = 0,$$

or

$$x = \frac{B}{A},$$

which is the only solution, x having but a single value.

EXERCISE III. d.

1. Solve the following equations —

i. $x^3 + 4x^2 - x - 4 = 0$.

ii. $x^5 + x^3 - x^2 - 1 = 0$.

iii. $x^3 + 3x^2 + 4x + 2 = 0$.

2. Find values of x that will make $(b^2 - a^2)x + \frac{3ab - x}{2}$ equal to $(x - 1)(a - b)$.

CHAPTER IV.

Highest Common Factor. — Least Common Multiple.

61. The expressions $2a^2bc$ and $6ab^2$ have 2, a, and b as factors common to both, and the product of these, $2ab$, is the highest common factor of the expressions.

The name Highest Common Factor is contracted to *H. C. F.*, and sometimes *G. C. M.* (greatest common measure) is used in its stead.

The expressions $x^3 - 7x + 6$ and $x^4 + 2x^3 - 9x^2 + 8$ factor respectively into $(x-1)(x-2)(x+3)$ and $(x-1)(x+1)(x-2)(x+4)$. They accordingly have the binomial factors $(x-1)(x-2)$ or $x^2 - 3x + 2$ as their *H. C. F.*

Common monomial factors, where they exist, are readily detected by inspection. To detect binomial factors, we may factor the expressions and pick out the common factors, as in the preceding example, or we may proceed upon the principle now to be established.

62. Theorem. If two expressions have a common factor, the sum and the difference of any multiples of the expressions have the same common factor.

Let A and B denote the two expressions, and let f denote their common factor, so that $A = Pf$ and $B = Qf$, where P and Q denote all the factors remaining in A and B respectively after the removal of f.

Let a and b be any numerical multipliers.

Then, $aA \pm bB = aPf \pm bQf = (aP \pm bQ)f$;
and as this last expression contains the common factor f, the theorem is proved.

63. Now let A and B be two integral functions of x, and let them have the common factor f, which we will suppose to be quadratic, as their *H. C. F.*

By taking the sums or differences of proper multiples of A and B, we may reduce the dimensions of each by unity, and obtain two new functions A' and B', one dimension lower respectively than A and B, and of which f is still a common factor.

By operating in a similar manner upon A' and B', we find two functions A'' and B'', two dimensions lower respectively than A and B, and containing f as a common factor.

By a continuation of this process we must eventually reduce A and B to depend upon functions of two dimensions, and having f as a common factor.

Hence these, upon rejecting all monomial factors, must be the factor f, and must therefore be identical.

And thus the identity of the two results at any stage of the operation indicates that the *H. C. F.* is obtained.

Ex. 1. Let $A \equiv 6x^3 - 7x^2 - 9x - 2$, $B \equiv 2x^3 + 3x^2 - 11x - 6$.

OPERATION.

A	$6x^3 - 7x^2 - 9x - 2$	B	$2x^3 + 3x^2 - 11x - 6$
$3B$ . . .	$6x^3 + 9x^2 - 33x - 18$	$3A$ . . .	$18x^3 - 21x^2 - 27x - 6$
$3B - A$ . . .	$16x^2 - 24x - 16$	$3A - B$	$16x^3 - 24x^2 - 16x$
Reject factor 8,	$2x^2 - 3x - 2$	Reject factor $8x$,	$2x^2 - 3x - 2$

The results being identical shows that the highest common factor is $2x^2 - 3x - 2$.

64. In the preceding example we notice:

(1) That the presence of the variable is unnecessary, and the operation may be carried out upon the coefficients alone.

(2) That as we reject monomial factors wherever they occur, all monomial factors should be removed before beginning the operation, and that any of these that are common to both functions should be set aside and be multiplied by the final result to give the complete *H. C. F.*

Ex. 1. Let $A \equiv 3x^3 - 10x^2 + 9x - 2$, $B \equiv 2x^3 - 7x^2 + 2x + 8$.

A	$3 - 10 + 9 - 2$
$2A$	$6 - 20 + 18 - 4$
$3B$	$6 - 21 + 6 + 24$
A'	$1 + 12 - 28$
$19A'$	$19 + 228 - 532$
$14B'$	$196 - 658 + 532$
A''	$215 - 430$
$\div 215$	$1 - 2$

B	$2 - 7 + 2 + 8$
$4A$	$12 - 40 + 36 - 8$
B'	$14 - 47 + 38$
$14A'$	$14 + 168 - 392$
B''	$215 - 430$
$\div 215$	$1 - 2$

$\therefore x - 2$ is the *H. C. F.*

Ex. 2. Let $A \equiv 3x^4 - 6x^3 - 6x - 3$, $B \equiv 6x^4 + 12x^3 + 12 - 6$.

3 being a common monomial factor, we set it aside and write

A	$1 - 2 + 0 - 2 - 1$
B	$1 + 2 + 0 + 2 - 1$
A'	$4 + 0 + 4$
$\div 4$	$1 + 0 + 1$

B	$1 + 2 + 0 + 2 - 1$
$\frac{1}{4}A'$	$1 + 0 + 1$
B'	$2 - 1 + 2 - 1$
$\frac{1}{2}A'$	$2 + 0 + 2$
B''	$1 + 0 + 1$

$\therefore 3(x^2 + 1)$ is the *H. C. F.*

Ex. **3**. Let $A = 3x^4 - 5x^3 + x^2 + 4x + 1$, $B = 3x^2 - 11x - 4$.

A	$3 - 5 + 1 + 4 + 1$
B	$3 - 11 - 4$
A'	$6 + 5 + 4 + 1$
$2B$	$6 - 22 - 8$
A''	$27 + 12 + 1$
$9B'''$	$27 + 9$
A'''	$3 + 1$

B	$3 - 11 - 4$
$4A''$	$108 + 48 + 4$
B''	$111 + 37$
$\div 37$, B'''	$3 + 1$

$\therefore\ 3x + 1$ is the *H. C. F.*

It may be remarked that the functions must be complete in form before beginning the operation.

65. When two integral functions of the same variable have a common linear factor, the corresponding equations have a common root; and if the functions have a common quadratic factor, the corresponding equations have two common roots, etc.

Ex. To find the relation between the constants in order that the equations $x^2 + ax + b = 0$ and $x^2 + a_1x + b_1 = 0$ may have a common root.

The functions $x^2 + ax + b$ and $x^2 + a_1x + b_1$ must have a common linear factor.

A	$1 + a + b$
b_1A	$b_1 + b_1a + b_1b$
bB	$b + ba_1 + b_1b$
A'	$(b - b_1) + (ba_1 - b_1a)$

B	$1 + a_1 + b_1$
A	$1 + a + b$
B'	$(a - a_1) + (b - b_1)$

These results must be multiples of the same linear factor. Hence reducing the first term to 1 in each gives —

$$1 + \frac{ba_1 - b_1a}{b - b_1} = 1 + \frac{b - b_1}{a - a_1}.$$

$$\therefore\ (b - b_1)^2 = (a - a_1)(ba_1 - b_1a);$$

which is the required relation.

EXERCISE IV. a.

1. Find the *H. C. F.* of each of the following—

i. $x^4 - 4x^3 + 2x^2 + 4x + 1$ and $x^4 - 6x^2 + 1$.

ii. $a^4 - 2a^3 + 6a - 9$ and $3a^4 - 2a^3 - 8a^2 + 6a - 3$.

iii. $10x^3 + x^2 - 9x + 24$ and $20x^4 - 17x^2 + 48x - 3$.

iv. $5x^2(12x^3 + 4x^2 + 7x - 3)$ and $10x(24x^3 - 52x^2 + 14x - 1)$.

v. $x^4 - px^3 + \overline{q-1} \cdot x^2 + px - q$ and $x^4 - qx^3 + \overline{p-1} \cdot x^2 + qx - p$.

vi. $\dfrac{a^4}{b^4} + 1 + \dfrac{b^4}{a^4}$ and $\dfrac{a^3}{b^3} + \dfrac{b^3}{a^3}$.

2. Find the relation between a and b when $x^2 + ax + 10 = 0$ and $x^2 + bx - 10 = 0$ have a common root.

3. Find the value of c when $x^2 - 3x + 2$ and $x^2 + cx + 3$ have a common linear factor.

4. Find the condition that $ax^2 + bx + c$ and $px^2 + qx + r$ may have a common linear factor.

5. Find the condition that

$$x^3 + ax^2 + bx + c = 0$$

and

$$x^3 + a_1x^2 + b_1x + c_1 = 0$$

may have a common root.

6. Find the value of a when $x^2 - x - 6$ and $x^2 + x(3 - a) - 3a$ have a common linear factor.

7. If $x^4 + x + a$ and $x^4 - x + b$ have a common linear factor, show that $(a - b)^4 = -8(b + a)$.

66. An expression which contains two or more given expressions as factors is a common multiple of the given expressions, and that common multiple which is of the lowest possible dimensions is the *lowest common multiple* or the *least common multiple* of the given expressions, the latter term being more particularly applicable to numbers. The contraction *L. C. M.* is used for either.

If *acef* and *abde* be two expressions in which the individual letters represent linear factors, their *L. C. M.* is *abcdef*; and we see that in order to find the *L. C. M.* of two expressions or quantities we take the factors that are common to both, as a and e, and the factors which are peculiar to each, as c and f from the first, and b and d from the second, and form the continued product of all these factors.

Evidently a similar process applies to the case of more than two expressions.

Ex. **1.** To find the *L. C. M.* of $x^2-x(a+b)+ab$, $x^2-ax-x+a$, and $x^2-bx-x+b$.

The expressions factored become —

$$(x-a)(x-b),\ (x-a)(x-1), \text{ and } (x-b)(x-1);$$

and the *L. C. M.* is $(x-a)(x-b)(x-1)$; and this is evenly divisible by each of the given expressions.

Ex. **2.** To find the *L. C. M.* of

$$x^2-ax-bx+ab \text{ and } x^2-2ax+a^2.$$

The expressions factored are $(x-a)(x-b)$ and $(x-a)(x-a)$.

Here $x-a$ is a common factor, $x-b$ is peculiar to the first, and the second $x-a$ to the second.

$\therefore$ the *L. C. M.* is $(x-a)^2(x-b)$.

Ex. **3.** To find the *L. C. M.* of

$$(x-y+z)(x+y-z),\ (y-z+x)(y+z-x), \text{ and } (z-x+y)(z+x-y),$$

or $$x^2-(y-z)^2,\ y^2-(z-x)^2, \text{ and } z^2-(x-y)^2.$$

The *L. C. M.* is $$(x-y+z)(y-z+x)(z-x+y),$$

or $$-\Sigma x^3+\Sigma x^2y-2xyz.$$

67. Theorem. The product of any two quantities or expressions is equal to the product of their *H. C. F.* and their *L. C. M.*

Let A and B denote the expressions, and let f be their *H. C. F.* Then $A = pf$ and $B = qf$, where p and q are the factors peculiar to A and to B respectively. Their *L. C. M.* is pqf.

But $A \cdot B \equiv pqf^2 = pqf \cdot f = L.\,C.\,M. \times H.\,C.\,F.$

Hence we may find the *L. C. M.* of two expressions by dividing their product by their *H. C. F.*, and conversely we may find the *H. C. F.* of two expressions by dividing their product by their *L. C. M.*

APPLICATION TO NUMBERS.

68. The foregoing principles apply to integral numbers in the same manner as to algebraic expressions.

The *H. C. F.*, or, as it is here called, the *G. C. M.* of the numbers, is the greatest number which divides each evenly.

Ex. 1. To find the *G. C. M.* of 3824 and 4160. The difference between any multiples of the two numbers must contain their *G. C. M.* (Art. 62).

Hence, from 4160 subtract the greatest possible multiple of 3824, which in this case is the number itself, and we have 336. We have now to find the *G. C. M.* of 336 and 3824. Taking 11 times 336 from 3824 leaves 128, and we are to find the *G. C. M.* of 128 and 336. By continuing this process we finally arrive at the factor required.

The whole operation appears as follows:

A 3824		B 4160	
$A - 11B'$ . . 128	. . A'	$B - A$ . . 336	. . B'
$A' - B''$ . . 48	. . A''	$B' - 2A'$ . . 80	. . B''
$A'' - B'''$ . . 16	. . A'''	$B'' - A''$ . . 32	. . B'''
		$B''' - A'''$ . . 16	

The identical results show that 16 is the *G. C. M.*

This process may be much shortened by leaving out the letters of reference, and by writing only remainders in the operation.

Ex. 2. To find the *G. C. M.* of 10395 and 20592.

10395	20592	1	quotients.
198	10197	1	
99	99	51	
		1	

And 99 is the *G. C. M.*

Explanation. — 20592 ÷ 10395 gives quotient 1 and remainder 10197. 10395 ÷ 10197 gives quotient 1 and remainder 198. 10197 ÷ 198 gives quotient 51 and remainder 99. And lastly 198 ÷ 99 gives quotient 1 and remainder 99, or it is a case of even division. Hence 99 is the common factor, and hence the *G. C. M.*

The quotients are important in the subject of continued fractions.

69. Two numbers whose *G. C. M.* is 1 are prime to one another; and a number which is prime to every number, except unity, smaller than itself is a **prime** number, or simply a *prime*.

The following are the primes less than 100, and a table of all primes below 1000 is given at the end of this work.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

All numbers not primes are **composite** numbers.

Every composite number can be exhibited as the product of prime factors. This is called the *composition* of the number.

Let N denote a composite number.

Divide N by 2, and the resulting quotient by 2, and so on until a quotient is found which is not divisible by 2. Call this quotient N'. Divide N' by 3, and the quotient by 3, and so on, until a quotient, N'', is found which is not divisible by 3. Divide N'' by 5, etc., and continue the operation by 7, 11, etc., using only primes as divisors.

Then if 2 has been used a times as a divisor, 3 b times, 5 c times, etc.,

$$N \equiv 2^a \cdot 3^b \cdot 5^c \cdots$$

And this is the composition of the number N.

Ex. 1. The composition of 8640 is $2^6 \cdot 3^3 \cdot 5$; and 8640 is said to be decomposed into its prime factors.

Ex. 2. To find the *G. C. M.* of 8640 and 1720.

Composition of 8640 is $2^6 \cdot 3^3 \cdot 5$.
Composition of 1720 is $2^3 \cdot 5 \cdot 43$.
And the *G. C. M.* is $2^3 \cdot 5 = 40$.

70. To find the *L. C. M.* of two or more numbers, we may decompose the numbers into their prime factors, and take the highest power of each factor involved, and form their continued product.

Ex. To find the *L. C. M.* of 8640, 1280, and 1560.

$$8640 \equiv 2^6 \cdot 3^3 \cdot 5, \quad 1280 \equiv 2^8 \cdot 5, \quad 1560 \equiv 2^3 \cdot 3 \cdot 5 \cdot 13.$$

And the *L. C. M.* is $2^8 \cdot 3^3 \cdot 5 \cdot 13$, or 449280.

The operation may also be conveniently carried out as follows:

40	8640	1280	1560	40 is the *G. C. M.* of the three numbers.
8	216	32	39	8 is the *G. C. M.* of 216 and 32, and is prime to 39.
3	27	4	...	
	9	...	13	3 is the *G. C. M.* of 27 and 39, and is prime to 4.

The quotients 9, 4, 13 are prime to each other.

Hence $40 \times 8 \times 3 \times 9 \times 4 \times 13$, or 449280 is the *L. C. M.*

EXERCISE IV. b.

1. Find the *L. C. M.* of the following —

i. $12x - 36,\ x^2 - 9,\ x^2 - 5x + 6$.

ii. $x^2 - (a + b)x + ab,\ x^2 - (b + c)x + bc,\ x^2 - (c + a)x + ca$.

iii. $1 + p + p^2,\ 1 - p + p^2,\ 1 + p^2 + p^4$.

2. If $ax^2 + bx + c$ and $cx^2 + bx + a$ have a linear common factor, show that $a \pm b + c = 0$.

3. Find the compositions of the numbers —

i. 72. ii. 180. iii. 824. iv. 1048. v. 25200.

4. Find the *G. C. M.* of 144, 840, 5040.

5. Find the *L. C. M.* of the 9 digits.

6. What is the least multiplier that will make 720 a complete square? That will make 1440 a complete cube?

7. What is the lowest factor that will make $x^3 - 4x^2 + 5x - 2$ a complete square?

8. What is the least multiplier that will make 144 a multiple of 64? That will make $x^3 - 5x^2 + 5x - 1$ a multiple of $x^2 - 4x + 3$?

CHAPTER V.

FRACTIONS. — SYMBOLS ∞ AND 0.

71. The expression 3/4 denotes that 3 is to be divided by 4. As numbers, 3 cannot be divided by 4; and hence we indicate this impossible arithmetical operation symbolically by writing it in a form of division, and we call the whole symbol a **fraction**. We then discover the laws of transformation of these symbols, and these laws form the working rules for fractions.

In this relation the dividend is called the *numerator* and the divisor the *denominator*.

The expression a/b is an algebraic fraction, in which a and b stand for any numbers or expressions; and the transformations of this symbol must apply to arithmetical fractions as particular cases.

RULES OF TRANSFORMATION.

72. The fractional form $\dfrac{a+b+c+\cdots}{D}$, with the line written beneath $a+b+c+\cdots$ (Art. 10), shows that $a+b+c+\cdots$ is to be taken in its totality to form the numerator.

(1) Let $Q=\dfrac{a+b}{D}$, Q denoting the fraction as a whole.

Then $QD=a+b$ (Art. 37); and if we take d such a multiplier that $Dd=1$, and therefore $d=\dfrac{1}{D}$, we have

$$QDd = ad + bd, \text{ or } Q = \frac{a}{D} + \frac{b}{D}.$$

Hence the denominator of a fraction is distributive throughout the terms of the numerator.

And conversely, the algebraic sum of any number of fractions with the same denominator is that fraction whose numerator is the algebraic sum of the numerators, and whose denominator is the same denominator.

(2) Let $Q = \frac{N}{D}$. Then $QD = N$; and p being any multiplier,

$$QDp = Np.$$

$$\therefore Q = \frac{Np}{Dp} = \frac{N}{D}.$$

Hence, multiplying both numerator and denominator by the same multiplier does not alter the value of the fraction.

(3) For p write $\frac{1}{q}$, and we have from (2)

$$Q = \frac{\frac{N}{q}}{\frac{D}{q}} = \frac{N}{D}.$$

And hence dividing both numerator and denominator by the same divisor does not alter the value of the fraction.

(4) Let $Q = \frac{N}{D}$, and $Q' = \frac{N'}{D'}$.

Then $QD = N$, and $Q'D' = N'$.

$$\therefore QQ'DD' = NN', \text{ and } QQ' = \frac{NN'}{DD'}.$$

Hence, to multiply two fractions together, we multiply together the numerators for a new numerator and the denominators for a new denominator.

(5) From (4), $\frac{QD}{Q'D'} = \frac{N}{N'}$, and multiplying both fractions by $\frac{D'}{D}$ gives $\frac{Q}{Q'} = \frac{N}{D} \cdot \frac{D'}{N'}$.

Hence, to divide one fraction by another we multiply the dividend by the inverted form of the divisor.

(6) Let $Q = \frac{N}{D}$; and since $p = \frac{p}{1}$,

$$pQ = \frac{pN}{D} = \frac{pN \cdot \frac{1}{p}}{D \cdot \frac{1}{p}} = \frac{N}{\frac{D}{p}}.$$

Therefore, we multiply a fraction by a given quantity when we multiply the numerator by that quantity, or when we divide the denominator by that quantity.

Also, by writing $\frac{1}{q}$ for p, we see that we divide a fraction by a given quantity when we divide the numerator by that quantity, or when we multiply the denominator by that quantity.

Cor. If two fractions have equal denominators the greater fraction has the greater numerator, and conversely.

And, if two fractions have equal numerators, the greater fraction has the smaller denominator, and conversely.

73. The operations with fractions are as general in character as those with whole numbers, and beyond the rules of transformation as now established no general directions can be laid down. The principal operation on fractions in themselves is the simplification of their forms. Facility in this is the result of practice.

Ex. 1. $\dfrac{3x^2+x-2}{2x^2-x-3} \equiv \dfrac{(3x-2)(x+1)}{(2x-3)(x+1)} \equiv \dfrac{3x-2}{2x-3}.$

Ex. 2. $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}} \equiv \dfrac{(x+1)^2+(x-1)^2}{(x+1)^2-(x-1)^2} \equiv \dfrac{x^2+1}{2x}.$

Ex. 3. $\dfrac{x^3-1}{x^2+1} \div \dfrac{x+1}{3x^3+2x} \equiv \dfrac{(x^3-1)(3x^3+2x)}{(x^2+1)(x+1)}$

$$\equiv 3x^3-3x^2+2x-\frac{x(5x^2-x+4)}{(x^2+1)(x+1)}.$$

74. From Art. 72 (1), to add fractions we bring them to have the same denominator. We then add the numerators, and place the sum over the common denominator.

Subtraction being addition with changed signs follows the same rule.

When several fractions are to be added the *L. C. M.* of all the denominators is the simplest common denominator.

Ex. 1. Simplify

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}.$$

The *L. C. M.* of the denominators is $(a-b)(b-c)(c-a)$; and the numerators become respectively —

$$a(c-b),\ b(a-c),\ \text{and}\ c(b-a).$$

And the sum is $\dfrac{a(c-b)+c(b-a)+b(a-c)}{(a-b)(b-c)(c-a)}$.

And as the numerator vanishes upon distribution, and the denominator does not, the sum of the three fractions is zero.

Ex. **2.** Simplify

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

The *L. C. M.* of the denominators is

$$(a-b)(b-c)(c-a)(x-a)(x-b)(x-c);$$

and the sum of the new numerators is

$$-a(b-c)(x-b)(x-c)-b(c-a)(x-c)(x-a)$$
$$-c(a-b)(x-a)(x-b).$$

This latter expression factors into

$$x(a-b)(b-c)(c-a).$$

Therefore the simplified sum of the fractions is

$$\frac{x}{(x-a)(x-b)(x-c)}.$$

EXERCISE V. a.

1. Simplify the following—

i. $\dfrac{x^2+\frac{5}{2}xy+y^2}{x^2+\frac{3}{2}xy-y^2}$.

ii. $\dfrac{3x^2+10xy+3y^2}{3x^2+8xy-3y^2}$.

iii. $\dfrac{n(x^2+y^2)+(n^2+1)xy}{n(x^2-y^2)+(n^2-1)xy}$.

iv. $\left(\dfrac{a}{x}-\dfrac{x}{a}\right)\left(\dfrac{a}{x}+\dfrac{x}{a}\right)\left(1-\dfrac{x-a}{x+a}\right)$.

v. $\dfrac{a^3+b^3+c^3-3abc}{a^2+b^2+c^2-ab-bc-ca}$.

vi. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

vii. $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}-\dfrac{x+c}{(x-a)(x-b)}$.

viii. $\dfrac{x^2}{(x-y)(x-z)}+\dfrac{y^2}{(y-x)(y-z)}+\dfrac{z^2}{(z-x)(z-y)}$.

ix. $\Sigma\left\{\dfrac{a^3}{(a-b)(a-c)}\right\}$.

x. $\Sigma\left\{\dfrac{1}{\left(\dfrac{a}{b}-1\right)\left(\dfrac{a}{c}-1\right)}\right\}$.

xi. $\Sigma\left\{\dfrac{y+z}{(x-y)(x-z)}\right\}$.

xii. $\Sigma\left\{\dfrac{(1+ab)(1+ac)}{(a-b)(a-c)}\right\}$.

xiii. $\Sigma\left\{\dfrac{bc(a+x)}{(a-b)(a-c)}\right\}$, where x is not varied.

xiv. $\Sigma\left\{a^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)\right\}\div\Sigma\left\{a\left(\dfrac{1}{b}-\dfrac{1}{c}\right)\right\}$.

xv. $\Sigma\left\{\dfrac{b^2+c^2-a^2}{(a-b)(a-c)}\right\}$.

xvi. $\Sigma\left\{\dfrac{b^2+bc+c^2}{(a-b)(a-c)}\right\}$.

2. If $a=\dfrac{x}{y+z}$, $b=\dfrac{y}{z+x}$, $c=\dfrac{z}{x+y}$, show that

$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

3. Find the value of $\dfrac{x+y-1}{x-y+1}$

when $x=\dfrac{a+1}{ab+1}$ and $y=\dfrac{ab+a}{ab+1}$.

4. If $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}+\dfrac{r^2}{c^2}=1$, show that

$$\frac{p(x-p)}{a^2}+\frac{q(y-q)}{b^2}+\frac{r(z-r)}{c^2}=\frac{px}{a^2}+\frac{qy}{b^2}+\frac{rz}{c^2}-1.$$

5. If $\dfrac{1+x}{1-x}=\dfrac{b}{a}\cdot\dfrac{1+x+x^2}{1-x+x^2}$, show that $x^3=\dfrac{b-a}{b+a}$.

6. Simplify $\left\{\dfrac{fs+rc}{fc-rs}-\dfrac{s}{c}\right\}\div\left\{1+\dfrac{fs+rc}{fc-rs}\cdot\dfrac{s}{c}\right\}$ when $s^2+c^2=1$.

75. Let $x=\dfrac{A}{a-b}$, where A, a, b are all positive finite quantities and b is not greater than a.

If b is $< a$, x has some positive finite value, and the nearer b approaches to a in value the greater the value of x becomes. By making the difference between a and b small enough we can make the value of x as great as we please.

Thus, let A be 10. If $a-b$ is 1, x is 10; if $a-b$ is 0.1, x is 100; if $a-b$ is 0.00001, x is 1000000; etc.

When $a-b$ is made smaller than any conceivable quantity, x becomes greater than any conceivable quantity.

In this case b is said to approach *infinitely* near to a in value; the difference between a and b is then denoted by 0, and the value of x is denoted by ∞, read *infinity.*

Thus we say that $\infty=\frac{A}{0}=\frac{\text{any finite quantity}}{0}$. And assuming that these symbols obey the formal laws of quantitative symbols,

$$\infty \times 0 = \text{any finite quantity.}$$

But 1 is a finite quantity, and $\frac{1}{0}=\infty$;

therefore $\frac{1}{0}\times 0$, or $\frac{0}{0}=$ any finite quantity.

The expressions $\infty \times 0$ and $\frac{0}{0}$ are mere symbols having no particular value except through their history; that is, through a knowledge of the source whence they have come. The expression $\frac{0}{0}$, however, which occurs quite often, does not necessarily mean zero, but may mean any finite quantity whatever.

Also, since $\infty \times 0 = A$, $0=\frac{A}{\infty}$, and hence any finite quantity divided by infinity gives zero as a quotient.

Moreover, we are not justified in writing $\infty-\infty=0$, or $\infty \div \infty = 1$, for ∞ does not mean any definite quantity, but merely a quantity so great as to be undefinable and inconceivable.

76. Special Roots. (1) When an integral function contains the variable as a monomial factor in the first degree, one of the roots is zero.

Thus $x^3 + 3ax^2 + 2a^2x = 0 = x(x + 2a)(x + a)$;

which is satisfied by $x = 0$, since the whole then vanishes.

The roots are accordingly, $0, \ -2a, \ -a$.

If the variable is of two dimensions in the monomial factor, two of the roots are zero.

(2) In the linear equation $x = x + a$, to which equations are sometimes reducible, we transfer x, and obtain $x - x = a$.

Now we are not justified in saying that $x - x = 0$, and therefore $a = 0$, for a is a *given* quantity whose value is not at our disposal; we must endeavor to find some value for x that will satisfy the equation.

But $$x - x = x(1 - 1) = a.$$

$$\therefore\ x = \frac{a}{1-1} = \infty \text{ by Art. 75.}$$

Whence $x = \infty$ is a symbolic root of the equation. The meaning of the solution is that a being finite, the larger x is, the more nearly is the equation $x = x + a$ satisfied, but that it cannot be completely satisfied by any finite value of x. We shall return to this again in Art. 81.

77. Let $x^2 + ax + b = x^2 + px + q$, where a, b, p, q are all finite quantities.

If x is finite, x^2 on one side cancels x^2 on the other, and we have a single finite value of x, namely,

$$x = \frac{q-b}{a-p}.$$

(1) If $q = b$, and $a = p$, $x = \frac{0}{0} =$ any finite quantity by Art. 75, and the solution is *indefinite.*

(2) If $q - b$ is not zero, and $a = p$, $x = \infty$.

(3) If $q = b$ and $a - p$ is not zero, $x = 0$.

Again, as the equation is quadratic it must have two roots, Art. 59 (2).

Dividing throughout by x^2 gives

$$1 + \frac{a}{x} + \frac{b}{x^2} = 1 + \frac{p}{x} + \frac{q}{x^2};$$

and the larger x becomes the more nearly does this become an identity.

Hence $x = \infty$ is a solution.

As the equation may be written

$$x^2(1-1) + x(a-p) + (b-q) = 0,$$

we infer that if the coefficient of the square term, in a quadratic equation, becomes zero, one solution of the equation is $x = \infty$.

Or more generally, if, in any integral equation, the coefficient of the highest power of the variable becomes zero, one root of the equation is ∞.

EXERCISE V. b.

1. Given $\frac{9x+8}{5}+\frac{4}{3}=3x+4$, to find x.

2. Given $\frac{x+3}{x+2}+\frac{2x}{3x}=\frac{5}{3}$, to find x.

3. Given

$$\frac{a}{(x-a)(x-c)}-\frac{c}{(a-c)(a-x)}+\frac{x}{(c-a)(c-x)}=\frac{1}{a-c},$$

to find x. What is the value of x when $c=0$? when $a=0$?

4. Is $\left(1+\frac{x}{a}\right)\left(\frac{1}{x}+\frac{a}{x}\right)=\left(1+\frac{1}{a}\right)\left(1+\frac{a}{x}\right)$ an equation or an identity? What value has x?

5. What expression substituted for x will make

$$\frac{3x-1}{a-1}+\frac{x+2}{a+2}=1 \text{ an identity?}$$

6. From $\frac{x}{x-a}+\frac{x}{x-b}-\frac{2(x^2-ab)}{(x-a)(x-b)}=0$, find a general value of x; and also the particular value when $a+b=0$.

7. Given $\frac{(x-1)(x+2)}{(x-2)(x+1)}=\frac{x+1}{x-1}$, to find x.

8. Given $\frac{(x-1)(x+2)(x-3)}{(x+1)(x-2)(x+3)}+1=0$, to find all the values of x.

9. Given $\frac{3x^3-x^2-6x+2}{x-1}-\frac{4x^3+x^2-8x-2}{x}-\frac{2}{x(1-x)}=0$, to find all the values of x.

10. If $\frac{a}{x}+\frac{x}{b}=\frac{b}{x}+\frac{x}{a}$, what relation holds between a and b?

11. Two numbers differ by 10, and one-half the less is greater by 1 than one-sixth the greater. Find the numbers.

12. One body moves about a circuit in a days, and another in b days, and they start from the same point. How many days will elapse between two conjunctions?

13. The sun moves in the ecliptic $0^\circ.9856$ per day, and the moon moves $13^\circ.1690$ per day. Find the days elapsing between two new moons.

14. Find a number such that if a be added to it and b be subtracted from it, the difference of the squares of the results shall be the number.

What relation must hold between a and b that the number may be (1) zero, (2) infinity?

15. Given $\frac{x+3}{x+1}-\frac{x+4}{x+2}+\frac{x-6}{x-4}=\frac{x^2-2x-15}{x^2-9}$, to find x.

16. Given $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}$, to find all the values of x.

17. Given $\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}=\frac{3a}{x(a^4+a^2x^2+x^4)}$, to find x.

18. Divide \$64 among A, B, and C, so that A may have 3 times as much as B, and C have $\frac{1}{3}$ as much as A and B together.

19. A person spends \$2 and then borrows as much as he has left. He again spends \$2 and borrows as much as he has left; etc. After his fourth spending he has nothing left. How much had he at first?

20. A person spends \$$a$, and borrows as much as he has left; then spends \$$a$, and borrows as much as he has left, etc., for n times, when he has nothing left.

Show that he had at first $\frac{2^n-1}{2^{n-1}}a$ dollars.

21. In a naval battle the number of ships taken was 7 more, and the number burnt 2 less, than the number sunk; 15 escaped, and the fleet consisted of 8 times the number sunk. How many ships were in the fleet?

78. The following properties of equal fractions are of special importance:

I. Let $$\frac{a}{b}=\frac{c}{d},$$

where a, b, c, d denote any quantities or expressions satisfying the indicated relation.

(1) Multiplying by $\frac{b}{c}$ gives $$\frac{a}{c}=\frac{b}{d}.$$

(2) Adding 1 to each member $$\frac{a+b}{b}=\frac{c+d}{d}.$$

(3) Subtracting 1 from each member $$\frac{a-b}{b}=\frac{c-d}{d}.$$

(4) Dividing (2) by (3) $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Relations (1), (2), (3), (4) are all direct consequences of the original relations.

The number of such derived relations is unlimited; those given are of most importance.

It will be noticed that these expressions have that kind of correlative symmetry by which we may interchange a and b if we interchange c and d, or we may interchange a and c if we interchange b and d, but in general we cannot interchange a with c or b with d.

II. Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots=Q, \text{ say.}$$

Then $$a=bQ,\ c=dQ,\ e=fQ, \text{ etc.}$$

And $$Q(lb+md+nf+\cdots)=la+mc+ne+\cdots$$

(5) $$\therefore\ Q=\frac{a}{b}=\cdots=\frac{la+mc+ne+\cdots}{lb+md+nf+\cdots}.$$

By giving particular values to $l, m, n \cdots$ an indefinite number of special relations may be obtained.

The relations of I. and II. are frequently employed with great advantage, the letters a, b, c, etc., being general symbols denoting any quantities or algebraic expressions.

Ex. 1. To find x from the equation $\left(\frac{a+x}{a-x}\right)^2 = 1 + \frac{cx}{ab}$.

Expanding, etc., $$\frac{a^2+2ax+x^2}{a^2-2ax+x^2} = \frac{ab+cx}{ab}.$$

Applying (4) of I., $$\frac{a^2+x^2}{2ax} = \frac{2ab+cx}{cx}.$$

Subtracting denominator from numerator for a new numerator,

$$\frac{(a-x)^2}{2ax} = \frac{2ab}{cx}.$$

$$\therefore\ cx(a-x)^2 = 4a^2bx.$$

Whence $x = 0$, $(76 \cdot 1)$, and $x = a - 2a\sqrt{\left(\frac{b}{c}\right)}$.

Ex. 2. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{(a^2+b^2)(a+b)}{(c^2+d^2)(c+d)} = \frac{a^3}{c^3}$.

Exercises of this kind may be solved in several ways: as (1) by transforming one expression into the other; (2) by using the first relation to show that the second is an identity, etc.

(1) $\frac{a^2}{b^2} = \frac{c^2}{d^2}$. $\therefore \frac{a^2+b^2}{b^2} = \frac{c^2+d^2}{d^2}$, and $\frac{a^2+b^2}{c^2+d^2} = \frac{b^2}{d^2} = \frac{a^2}{c^2}$,

by relations (1) and (2), I.

Again, $$\frac{a+b}{b} = \frac{c+d}{d}, \text{ and } \frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}.$$

$\therefore$ by multiplication, $\frac{a^2+b^2}{c^2+d^2} \cdot \frac{a+b}{c+d} = \frac{a^2}{c^2} \cdot \frac{a}{c} = \frac{a^3}{c^3}$. Q. E. D.

(2) Let $\frac{a}{b}=\frac{c}{d}=p$; then $a=bp,\ c=dp$.

Substitute for a and c in the second expression, and it becomes

$$\frac{b^3(p^2+1)(p+1)}{d^3(p^2+1)(p+1)}=\frac{b^3p^3}{d^3p^3},\ \text{an identity.}$$

Ex. 3. If $\frac{x}{2y-z}=\frac{y}{2z-x}=\frac{z}{2x-y}$, each fraction $=1$.

By (5), $\frac{x}{2y-z}=\frac{x+y+z}{2y-z+2z-x+2x-y}=\frac{x+y+z}{x+y+z}=1.$

Ex. 4. If $\frac{a^2l}{x}=\frac{b^2m}{y}=\frac{c^2n}{z}$, and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, to find the values of x, y, and z in terms of the remaining letters.

The first relation gives $\frac{al}{\frac{x}{a}}=\frac{bm}{\frac{y}{b}}=\frac{cn}{\frac{z}{c}}$;

whence, squaring each fraction and employing (5),

$$\frac{a^4l^2}{x^2}=\frac{a^2l^2+b^2m^2+c^2n^2}{1}.$$

Whence $$x=\frac{a^2l}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}};$$

with symmetrical expressions for y and z.

Question 4 furnishes an example of *collateral* symmetry between the two sets of letters a, b, c and l, m, n. Thus if we change a to b and b to c, we must, at the same time, change l to m and m to n. But we are never supposed to make an interchange between letters from different sets. In like manner we may have collateral symmetry amongst three or even more sets of different letters.

EXERCISE V. c.

1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots$ Show—

i. $\frac{(a-c)^2+(b-d)^2}{(a-c)^2-(b-d)^2}=\frac{a^2+b^2}{a^2-b^2}$.

ii. $\frac{a^3+a^2b+ab^2+b^3}{c^3+c^2d+cd^2+d^3}=\frac{b^3}{d^3}$.

iii. $\frac{a}{b}=\frac{\sqrt{(l^2a^2+m^2c^2+n^2e^2)}}{\sqrt{(l^2b^2+m^2d^2+n^2f^2)}}$.

2. If $\frac{a}{b}=\frac{c}{d}$ and $\frac{A}{B}=\frac{C}{D}$, then $\frac{a\sqrt{A}+b\sqrt{B}}{c\sqrt{C}+d\sqrt{D}}=\frac{a\sqrt{A}-b\sqrt{B}}{c\sqrt{C}-d\sqrt{D}}$.

3. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

4. Under the conditions of Ex. 3, show that

$$\sqrt{(ab)}+\sqrt{(bc)}+\sqrt{(cd)}=\sqrt{\{(a+b+c)(b+c+d)\}}.$$

5. If
$$\frac{x}{a(y+z)}=\frac{y}{b(z+x)}=\frac{z}{c(x+y)},$$
then
$$\frac{x}{a}(y-z)+\frac{y}{b}(z-x)+\frac{z}{c}(x-y)=0.$$

6. Given that $ax^2+by^2+2z=0$, and $\frac{ax}{l}=\frac{by}{m}=\frac{1}{n}=\frac{z}{p}$,

show that
$$\frac{l^2}{a}+\frac{m^2}{b}+2np=0.$$

7. If $\frac{x^2-yz}{x(1-yz)}=\frac{y^2-zx}{y(1-zx)}$, each fraction $=x+y+z$.

8. If $\frac{a+b}{a-b}=\frac{b+c}{2(b-c)}=\frac{c+a}{3(c-a)}$, $8a+9b+5c=0$.

9. If $\frac{y+z}{b-c}=\frac{z+x}{c-a}=\frac{x+y}{a-b}$, then each fraction is

$$\sqrt{\left\{\frac{x^2+y^2+z^2}{(b-c)^2+(c-a)^2+(a-b)^2}\right\}}.$$

10. If $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$, and $a+b+c=2S$,

then $$S=R(\sin A+\sin B+\sin C).$$

CHAPTER VI.

Ratio, Proportion, Variation, or Generalized Proportion.

79. The **ratio** of a to b is the quotient arising from dividing a by b, where a and b denote any numerical quantities. If the division is even, the ratio is an integer, and is expressible; if uneven, the ratio is a fraction and can only be indicated.

In this relation a and b are called the *terms* of the ratio, a being the *antecedent* and b the *consequent*.

The ratio is commonly symbolized as $a : b$.

If $a > b$, the ratio is one of greater inequality.

If $a = b$, it is one of equality; and if $a < b$, it is one of less inequality.

When two ratios are multiplied together, after the manner of fractions, they are said to be *compounded*.

Thus $ac : bd$ is compounded of $a : b$ and $c : d$.

When a ratio is compounded with itself, the terms are squared, and the result is the *duplicate* ratio of the original. Thus $a^2 : b^2$ is the duplicate of $a : b$.

Similarly, $a^3 : b^3$ is the *triplicate* of $a : b$; and $\sqrt{a^3} : \sqrt{b^3}$ is, in physics, sometimes called the *sesquiplicate* ratio of $a : b$.

80. As a ratio is virtually a fraction, all the laws of transformation for fractions apply to ratios.

The ratio of one quantity to another does not depend upon any absolute magnitudes of the quantities (for there is no absolute magnitude), but upon the relative magnitudes of the quantities.

It is thus that the ratio of one quantity to another expresses the true relation of magnitude or greatness existing between the quantities.

The ratio $\frac{a}{x}:\frac{b}{x}$ is the same as $a:b$, whatever x may be. But if x is very small as compared with a and b, both terms become very great; and if x is very great as compared with a and b, both terms are very small; but the relation of greatness existing between the terms remains the same.

If $x=0$, both $\frac{a}{x}$ and $\frac{b}{x}$ become infinite; so that $\infty:\infty$ may be any ratio whatever.

If $x=\infty$, both $\frac{a}{x}$ and $\frac{b}{x}$ becomes zero; so also $0:0$ may be any ratio whatever. (Compare Art. 75.)

81. Theorem. The addition of the same quantity to both terms brings the ratio nearer to unity, or to a ratio of equality.

Let $a:b$ be the ratio, and let x be added to each term, making $a+x:b+x$.

Then $$\frac{a}{b}-\frac{a+x}{b+x}\equiv\frac{x(a-b)}{b(b+x)}=Q\text{, say,}$$

and $$\frac{a+x}{b+x}-1\equiv\frac{a-b}{b+x}=Q_1\text{, say.}$$

All the letters denoting positive quantities, if $a:b>1$, $a>b$ and Q and Q_1 are both $+$.

$$\therefore\ a+x:b+x<a:b, \text{and} >1;$$

or $\quad a+x:b+x$ lies between $a:b$ and 1.

If $a:b<1$, $a<b$, and Q and Q_1 are both $-$.

$$\therefore\ a+x:b+x>a:b, \text{and} <1;$$

or $\quad a+x:b+x$ lies between $a:b$ and 1,

which proves the theorem.

Cor. 1. By writing $-x$ for x, we see that to subtract the same quantity from both terms of a ratio of inequality removes the ratio further from 1, provided the subtraction leaves both terms positive.

Cor. 2. Q_1 decreases as x increases, and by making x great enough, we may make Q_1 as small as we please. That is, by adding the same quantity to both terms of a ratio we may bring the ratio as near 1 as we please.

We have here another proof of Art. 76, for if $x+a=x+b$, we have $x+a:x+b=1$, and whatever values a and b may have, provided they are finite, the statement is satisfied by $x=\infty$, since that value for x makes the ratio to differ from 1 by a quantity less than any assignable quantity.

PROPORTION.

82. Four quantities are *proportional*, or are *in proportion*, or *form a proportion*, when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus a, b, c, d are proportional when

$$a:b=c:d.$$

This may be expressed as $\frac{a}{b} = \frac{c}{d}$, and a proportion, being thus an equality of two fractions, is best dealt with after the manner of fractions.

The proportion is evidently subject to all the transformations of Art. 78, I.

83. In the proportion $a : b = c : d$, a and d are the *extremes* and b and c the *means*.

But since the same proportion may be written $b : a = d : c$, the extremes and means are capable of exchanging places.

Writing the proportion $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$, or $\frac{b}{a} = \frac{d}{c}$, etc., which all express the same relation, we may represent the *form* generally by $\frac{a \mid c}{b \mid d}$, where the letters are written in the four corners formed by the two crossed lines.

In this form a and d, as also b and c, are *opposites* of the proportion, as standing in opposite corners, and we can make the general statement

The terms of a proportion may be written in any order, provided the opposites are unchanged.

By cross-multiplication $ad = bc$. That is, when four quantities are in proportion, the product of each pair of opposites is the same; and conversely, if two equal quantities be each divided into any two factors, these factors form a proportion, of which the factors of the same quantity are a pair of opposites.

84. If $a : b = c : d$, d is a fourth proportional to a, b, and c.

If $a : b = b : c$, b is a mean proportional, or a geometric mean between a and c. In this case $b = \sqrt{(ac)}$.

If $a : b = b : c = c : d =$ etc., the statement is a continued proportion.

Ex. 1. If $a + b : a = a - b : b$, to find $a : b$.

Here $\qquad ba + b^2 = a^2 - ab$; whence $a - b = b\sqrt{2}$,

and $\qquad a = b(1 + \sqrt{2})$, or $a : b = 1 + \sqrt{2}$.

Ex. 2. If $a^2y^2 - b^2x^2 = a^2b^2$, to find approximately the ratio $x : y$ when x becomes indefinitely great, and a and b are finite constants.

Evidently y becomes indefinitely great with x. Dividing by x^2 the relation gives
$$a^2 \cdot \frac{y^2}{x^2} - b^2 = \frac{a^2b^2}{x^2}.$$

When x approaches ∞, $\frac{a^2b^2}{x^2}$ approaches 0, and $\frac{y^2}{x^2}$ approaches $\frac{b^2}{a^2}$.

And when $x = \infty$, $\qquad \frac{x}{y} = \pm \frac{a}{b}$, or $x : y = \pm\, a : b$.

Ex. 3. Two numbers are in the ratio $p : q$; what must be added to each that the ratio of the new numbers may be $P : Q$?

Let mp, mq be the numbers, and add x to each.

Then $\qquad mp + x : mq + x = P : Q$.

whence
$$x = \frac{m(pQ - qP)}{P - Q}.$$

Cor. When $P : Q = 1$, $P = Q$, and the value of x becomes infinite.

Ex. 4. To find 4 numbers in continued proportion such that their sum may be 65.

Let a, b, c, d be the numbers.

Then
$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = z, \text{ say.}$$

Hence $a = dz^3$, $b = dz^2$, $c = dz$, and their sum is

$$d(z^3 + z^2 + z + 1) = 65.$$

$\therefore$ $d(z^2 + 1)(z + 1) = 65$, and since we can make z what we please, the problem is indefinite, *i.e.* it admits of any number of solutions.

If $z = 2$, the numbers are $\frac{104}{3}$, $\frac{52}{3}$, $\frac{26}{3}$, and $\frac{13}{3}$.

If $z = \frac{2}{3}$, the numbers are 8, 12, 18, and 27.

EXERCISE VI. a.

1. For what value of x will the ratio $5 + x : 8 + x$ become $5:8$, $6:8$, $7:8$, $8:8$, $9:8$?

2. In a city A a man assessed for \$10,000 pays \$72 tax, and in a city B a man assessed for \$720 pays \$4.50 tax. Compare the rate of taxation in A to that in B.

3. The ratio $a - 1 : b - 1$ is α, and that of $a + 1 : b + 1$ is β. Find the ratio $a : b$ in terms of α and β.

4. Two men can do in 4 days what 3 boys can do in 5 days. Compare a man's working ability with that of a boy.

5. Find the number for which the cube root of its square is to the square root of its cube as m to n.

6. Given $3x^2 + 10xy + 3y^2 : 3y^2 + 8xy - 3x^2 = 2x : y$, to find the ratio $y : x$.

7. If $a : b = b : c = c : d$, show that $a^2 : b^2 = a : c$, and $a^3 : b^3 = a : d$.

8. If $\sqrt{(p + q)} : \sqrt{(p - q)} = m : n$, find $p : q$, and also the duplicate ratio of $p - q : q$.

9. If $l^2 - m^2 + n^2 = 0$ and $l + m - n = 0$, find $l : m$.

10. Find the ratio of $(x + h)^{-1} - x^{-1} : h$ when h approaches zero.

GENERALIZED PROPORTION, OR VARIATION.

85. Let x be a variable, and let y be connected with x by a constant multiplier m, so that $y = mx$. When x changes its value, becoming x_1, say, y also changes value, becoming y_1, so that $y_1 = mx_1$.

Dividing one equation by the other,

$$\frac{y}{y_1} = \frac{mx}{mx_1} = \frac{x}{x_1}.$$

Whence $$y : y_1 = x : x_1,$$

i.e. any two values of x and the corresponding values of y are in proportion.

Also, if x takes a series of values, x_1, x_2, x_3, etc., and the corresponding values of y be y_1, y_2, y_3, etc.,

$$x_1 : y_1 = x_2 : y_2 = x_3 : y_3 = \text{etc.}$$

The foregoing relations are indicated by saying that y varies as x, or x varies as y, since the relation is mutual, and they are symbolically expressed by writing $y \propto x$, or $x \propto y$.

Hence to say that y varies as x, is to say that one is a constant multiple of the other, or that any two values of x and the corresponding values of y are in proportion.

86. If $y = n \cdot \frac{1}{z}$, y varies as the inverse or reciprocal of z; or y varies *inversely* as z.

If $y = n \cdot \frac{x}{z}$, y varies directly as x and inversely as z.

If $y = nxz$, y varies conjointly as x and z.

Ex. 1. If $x \propto yz$, and y varies inversely as z^2, and if $z = 2$ when $x = 10$, it is required to express x in terms of z.

We have $\qquad x = myz$, and $y \propto \frac{1}{z^2}$. $\quad \therefore x = \frac{n}{z}$.

And $\qquad 10 = \frac{n}{2}$, or $n = 20$.

$$\therefore x = \frac{20}{z}.$$

Ex. **2**. The velocity of a body falling from rest varies as the square root of the space passed over, and when the body has fallen 16 feet its velocity is 32 feet. Find the relation between space and velocity.

$$v = m\sqrt{s}, \text{ where } v = \text{velocity and } s = \text{space}.$$

$$\therefore v = 32 \text{ and } s = 16 \text{ gives } 32 = 4m.$$

$$\therefore m = 8, \text{ and } v = 8\sqrt{s}, \text{ or } v^2 = 64s;$$

which shows that the velocity varies as the square root of the space fallen through.

Ex. **3**. The radius of the earth is r, and the attraction upon a body without varies inversely as the square of the body's distance from the centre. The number of beats made per day varies as the square root of the earth's attraction upon the pendulum. How much will a clock, with a second's pendulum, lose daily if taken to a distance r_1 from the earth's centre, r_1 being greater than r.

Let $n =$ the number of seconds in a day $= 86400$, and let $g =$ the earth's attraction at the surface.

Then $\qquad g \propto \frac{1}{r^2}$, and $n \propto \sqrt{g}$.

$$\therefore n \propto \frac{1}{r}, \text{ and we write } n = \frac{m}{r}, \ m \text{ being constant}.$$

Also, if n_1 be the number of beats per day made by the pendulum in its new position,

$$n_1 = \frac{m}{r_1} = n \cdot \frac{r}{r_1}, \text{ by substituting for } m.$$

And the clock loses $n - n_1$ seconds daily,

$$= n\left(1 - \frac{r}{r_1}\right) = n \cdot \frac{r_1 - r}{r_1} \text{ seconds}.$$

If $r = 3960$ and $r_1 = 3961$, the loss is 21.81 sec.

EXERCISE VI. b.

1. The space passed over by a body falling from rest varies as the square of the time, and a body is found to fall 196 feet in $3\frac{1}{2}$ seconds. Find the relation between the space and the time.

2. If $x \propto y$ and $y = 3\frac{5}{8}$ when $x = 6\frac{1}{2}$, find the value of y when $x = \frac{5}{8}$.

3. y varies inversely as x^2, and z varies directly as x^2. When $x = 2$, $y + z = 340$; and when $x = 1$, $y - z = 1275$. For what value of x is y equal to z?

4. $z \propto u - v$, $u \propto x$, and $v \propto x^2$. When $x = 2$, $z = 48$; and when $x = 5$, $z = 30$. For what value of x is $z = 0$?

5. If $xy \propto x^2 + y^2$, and $x = 3$ when $y = 4$, find the relation connecting x and y.

6. The area of a rectangle is the product of two adjacent sides; if the area is 24 when the sum of the sides is 10, find the sides of the rectangle.

7. If $x + y \propto x - y$, then $x^2 + y^2 \propto xy$.

8. If $x \propto y$, show that $x^2 + y^2 \propto xy$.

9. A watch loses $2\frac{1}{2}$ minutes per day. It is set right on March 15th at 1 P.M.; what is the correct time when the watch shows 9 A.M. on April 20th?

10. The volume of a gas varies directly as its absolute temperature, and inversely as its tension. 1000 cc. of gas at 240° and tension 800 mm. has its temperature raised to 300° and its tension lowered to 600 mm. What volume has the gas then?

11. The attraction at the surface of a planet varies directly as the planet's mass and inversely as the square of its radius. The earth's radius being 3960 miles, and the moon's 1120, and the mass of the earth being 75 times that of the moon, compare the attractions at their surfaces.

12. The length of a pendulum varies inversely as the square of the number of beats it makes per minute, and a pendulum 39.2 in. long beats seconds. When a seconds pendulum loses 30 sec. per day, how much too long is the pendulum?

13. When one body revolves about another by the law of gravitation, the square of the time varies as the cube of the distance. The moon is 240,000 miles from the earth, and makes her circuit in 27 days. In what time would she complete her circuit if she were 10,000 miles distant?

CHAPTER VII.

INDICES AND SURDS.

87. The index law is the result of the convention that when p is a positive integer, $a \cdot a \cdot a \cdots$ to p factors shall be denoted by a^p. And by this law $a^p \cdot a^q = a^{p+q}$, p and q both being positive integers.

Now, if algebra is to be consistent with itself we can have only one index law, whatever p may denote, and instead of making a new convention we must interpret in conformity with this index law the cases in which p is not positive and integral.

The interpretation of p zero, or negative and integral, is given in Art. 38. We deal here with p fractional.

(1) If $p = q = \frac{1}{2}$, $a^p \cdot a^q = a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^1 = a$.

Therefore, $a^{\frac{1}{2}}$ is the same as $\sqrt{a}$, the meaning of which is fully given in Arts. 47 and 48.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$.

And $a^{\frac{1}{3}}$ means that a is to be separated into three identically equal factors, and that one of these is to be taken.

By an obvious extension,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{to } n \text{ terms}} = a;$$

and $a^{\frac{1}{n}}$ tells us to separate a into n identically equal factors, and take one of these factors.

This factor is called the nth root of a, and is often written $\sqrt[n]{a}$, the letter or figure being written in the symbol $\sqrt{\ }$ in all cases where n is not 2. Thus $\sqrt[3]{a}$ denotes the cube root of a, etc.

(2) $a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots$ to m factors becomes $(a^{\frac{1}{n}})^m$ when we consider $(a^{\frac{1}{n}})$ as a single quantitative symbol; and it becomes $a^{\frac{m}{n}}$ by the index law.

Therefore, $(a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$; and either of these expressions denotes that the nth root of a is to be raised to the mth power; or that the mth power of a is to have its nth root taken.

Thus $\qquad (64)^{\frac{2}{3}} = (64^{\frac{1}{3}})^2$, or $(64^2)^{\frac{1}{3}} = 16.$

Also, writing p for $\frac{1}{n}$, and $\frac{1}{q}$ for m, gives —

$$(a^p)^{\frac{1}{q}} = a^{\frac{p}{q}}; \text{ or } (a^p)^{\frac{1}{q}} = a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p.$$

(3) Making $p = \frac{1}{r}$ in (2) gives —

$$(a^{\frac{1}{r}})^{\frac{1}{q}} = a^{\frac{1}{qr}} = (a^{\frac{1}{q}})^{\frac{1}{r}},$$

or $$\sqrt[q]{\sqrt[r]{a}} = \sqrt[qr]{a} = \sqrt[r]{\sqrt[q]{a}}.$$

Thus to get the 6th root of a quantity we may first take the square root and then the cube root of the result, or we may first take the cube root and then the square root of the result.

Ex. 1. To simplify $\left(\frac{x^m}{x^n}\right)^{m+n} \cdot \left(\frac{x^n}{x^l}\right)^{n+l} \cdot \left(\frac{x^l}{x^m}\right)^{l+m}$.

This becomes $(x^{m-n})^{m+n} \cdot (x^{n-l})^{n+l} \cdot (x^{l-m})^{l+m}$

$$= x^{m^2-n^2} \cdot x^{n^2-l^2} \cdot x^{l^2-m^2} = x^{m^2-n^2+n^2-l^2+l^2-m^2} = x^0 = 1.$$

Ex. 2. To simplify $\dfrac{2^3 \cdot 8^{2-n} \cdot 16^{n-1}}{3^{2+n} \cdot 9^n \cdot 27^{1-n}}$.

This becomes

$$2^3 \cdot 2^{6-3n} \cdot 2^{4n-4} \cdot 3^{-2+n} \cdot 3^{-2n} \cdot 3^{-3+3n} = 2^{5+n} \cdot 3^{-5} = \frac{2^{n+5}}{3^5}.$$

EXERCISE VII. a.

1. Simplify $(a^{\frac{m}{n}} \cdot a^{1-\frac{m}{n}} \cdot a^{\frac{3}{2}}) \div (a^{\frac{3}{2}} \cdot a \cdot a^{-1})$.

2. Simplify $\sqrt{z^{\frac{2}{3}}} \cdot \sqrt[3]{z^{\frac{3}{2}}} \cdot z^{\frac{n}{2}} \cdot z^{-\frac{1}{2}(1+n)}$.

3. Simplify $(x^2)^{\frac{n}{2}} \cdot (x^{\frac{1}{2}})^{n-1} \div (x^n)^{\frac{1}{2}} \cdot (x^{2n})^{\frac{1}{n}}$.

4. Simplify $a^{-n+1} \cdot a^{-\frac{3}{2}} \cdot a^{\frac{1}{2}} \div a^{-n} \cdot a^2 \cdot a^{\frac{3}{2}}$.

5. Simplify $(x^{p+q} \div x^q)^p \div (x^q \div x^{q-p})^{p-q}$.

6. Simplify $\sqrt{\{(a^{-2m} \div b^{-2n})^{\frac{p}{m}}\}^{\frac{q}{n}}}$.

7. Write the square of $a^{\sqrt{2}} - a^{-\sqrt{2}}$.

8. Multiply $a^{\frac{5}{3}} - a^{\frac{5}{6}}b^{\frac{3}{4}} + b^{\frac{3}{2}}$ by $a^{\frac{5}{6}} + b^{\frac{3}{4}}$.

9. Express the relation $x = \sqrt[3]{\left\{-\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}}$, so as to be free from irrationals.

10. Express the relation $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a^2 - x^2}} = \sqrt{\left(\dfrac{a+x}{a-x}\right)}$, so as to be free from irrationals.

11. Find x when $a^x \cdot a^{\frac{1}{x}} \div a^{x-\frac{1}{2}} = a$.

12. Find n when $2^n \cdot 2^{n-1} \cdot 2^{n+1} = 2^{1-n} \cdot 2^{1+n} \cdot 2^{1-n}$.

13. Find n when $9^{\frac{n}{3}} \cdot 3^{n-1} = 6^n \cdot 2^{-n} \cdot 3^{n+1}$.

14. Find the relation between m and n when $a^{m^n} = (a^m)^n$.

15. Find n when $4^n \cdot 2^{2n-1} \cdot 8^{1-n} = 4^2 \cdot 2^{-n} \cdot 16^{\frac{n}{3}}$.

16. Divide $x^3 - 3x^2 + 2 - x^{-1} + x^{-3}$ by $x - 3 - 2x^{-1}$, giving quotient and remainder.

17. If $fx \equiv \frac{1}{2}(a^x + a^{-x})$, and $\phi x \equiv \frac{1}{2}(a^x - a^{-x})$, show that—

i. $(fx)^2 - (\phi x)^2 = 1$.

ii. $2(fx)^2 = 1 + f(2x)$.

iii. $2(\phi x)^2 = f\,2x - 1$.

iv. $\phi(2x) = 2(\phi x) \cdot (fx)$.

18. If $fx \equiv \frac{1}{2}(a^{ix} + a^{-ix})$ and $\phi x \equiv \frac{i}{2}(a^{ix} - a^{-ix})$, show that—

i. $(fx)^2 + (\phi x)^2 = 1$.

ii. $\phi(-x) = -(\phi x)$.

iii. $fx = f(-x)$.

SURDS.

88. A **surd** is the incommensurable root of a commensurable number (Chrystal).

Thus $\sqrt{2}$, $\sqrt[3]{5}$, $3^{\frac{1}{4}}$, etc., are symbolic expressions for surds. The arithmetical extraction of the roots indicated would approximate to the numerical values of the surds.

The expression $\sqrt{4}$ is an integer under a surd form.

Many known incommensurables are not surds, and some of them are not, as far as we know, due to any finite combination of surds. As examples, we have 3.1415926 ⋯, which is the ratio of the circumference of a circle to its diameter, and which is usually denoted by the greek letter π; and 2.7182818 ⋯, which is the base of the natural logarithms, and which is usually denoted by ϵ or e.

An expression such as $\sqrt{2 + \sqrt{2}}$ is a surd expression, but is not a surd according to definition.

89. Let z and n be any real positive quantities, n being an integer.

Then x^n will pass through all values from 0 to $+\infty$, if x passes through all values from 0 to $+\infty$. Therefore for some positive value of x, $x^n = z$, and $x = \sqrt[n]{z}$.

That is, every quantity between 0 and $+\infty$ has a positive real nth root. This is the arithmetical root, and is the one with which we are principally concerned.

If n is even, $n = 2m$; and $z^{\frac{1}{n}} = z^{\frac{1}{2m}} = (z^{\frac{1}{m}})^{\frac{1}{2}}$. And since a square root has two signs —

$\therefore$ every even root of any quantity has two values differing only in sign.

90. Let x be negative and n be an odd positive integer. Then x^n is negative, and passes from 0 to $-\infty$, while x passes from 0 to $-\infty$.

Therefore, for some value of x, $x^n = -z$, and $x = \sqrt[n]{-z}$.

That is, every real negative quantity has a real negative nth root, when n is odd.

Thus, every positive real quantity has two real square roots, fourth roots, sixth roots, etc., and one real cube root, fifth root, etc. And every real negative quantity has one real cube root, fifth root, etc., and no real square root, fourth root, etc.

91. Let a be any real positive quantity, and let r be its arithmetical cube root.

Then, since $\omega^3 = \omega^6 = 1$, $r^3 = \omega^3 r^3 = \omega^6 r^3 = a$. And taking the cube root, r, ωr, and $\omega^2 r$ are each cube roots of a. And the three cube roots of a are $\sqrt[3]{a}$, $\omega\sqrt[3]{a}$, $\omega^2\sqrt[3]{a}$, where $\sqrt[3]{a}$ is the arithmetical cube root.

Thus the three cube roots of 27 are 3, 3ω, $3\omega^2$. Hence every real quantity has one arithmetical cube root, and two complex cube roots.

92. Surds which are reducible to the same surd factor are *similar;* otherwise they are dissimilar.

To reduce a surd to its surd factor we proceed as follows:

Decompose the number into its prime factors, and then (1) for a quadratic surd take out the largest square factor possible. The remaining factor is the surd factor.

Ex. 1. $\sqrt{(1350)} \equiv \sqrt{(2 \cdot 3^3 \cdot 5^2)} \equiv 3 \cdot 5\sqrt{(2 \cdot 3)} \equiv 15\sqrt{6}$, and 6 is the surd factor.

(2) For a cubic surd take out the largest cube factor possible. The remaining factor is the surd factor.

Ex. 2. $\sqrt[3]{(9720)} \equiv \sqrt[3]{(2^3 \cdot 3^5 \cdot 5)} \equiv 2 \cdot 3\sqrt[3]{(3^2 \cdot 5)}$

$= 6\sqrt[3]{(45)}$, and 45 is the surd factor.

93. Surds of the same order are added by reducing them to their surd factors, and adding the coefficients of similar surds.

Ex.
$$3\sqrt{2} + \sqrt{18} + 2\sqrt{12} - \sqrt{48}$$
$$\equiv 3\sqrt{2} + 3\sqrt{2} + 4\sqrt{3} - 4\sqrt{3} \equiv 6\sqrt{2}.$$

With dissimilar surds, or with surds of different orders, the addition and subtraction can only be indicated by connecting them with the proper signs.

94. When a fraction contains a surd expression as denominator, it becomes necessary, for the sake of ease in calculation, to so transform the fraction as to make its denominator rational. This is effected by multiplying both parts of the fraction by some expression which will make the denominator rational. Such a multiplier is called a **rationalizing** factor of the denominator.

Ex. 1. $\frac{1}{\sqrt{2}} \equiv \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \equiv \frac{\sqrt{2}}{2}$, and the denominator is rational, and $\sqrt{2}$ is the rationalizing factor.

Ex. 2. $\frac{2}{\sqrt[3]{2} \cdot \sqrt{3}} \equiv \frac{2\sqrt[3]{2^2} \cdot \sqrt{3}}{\sqrt[3]{2^3} \cdot \sqrt{3^2}} \equiv \frac{2\sqrt[3]{4}\sqrt{3}}{6} \equiv \frac{\sqrt[3]{4}\sqrt{3}}{3}$, and the denominator is rational, and $\sqrt[3]{2^2}\sqrt{3}$ is the rationalizing factor.

95. Since $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$, which is rational, the expression $\sqrt{a}-\sqrt{b}$ is a rationalizing factor for $\sqrt{a}+\sqrt{b}$; and reciprocally $\sqrt{a}+\sqrt{b}$ is a rationalizing factor for $\sqrt{a}-\sqrt{b}$.

Ex. 1. To rationalize the denominator of $\frac{3\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

Multiplying by $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ gives $\frac{3\sqrt{6}+6}{1}$, or $3(\sqrt{6}-2)$.

Ex. 2. To rationalize the denominator of $\frac{1+2\sqrt{2}}{1+\sqrt{2}}$.

Multiplying by $\frac{1-\sqrt{2}}{1-\sqrt{2}}$ gives $\frac{-3+\sqrt{2}}{-1}$, or $3-\sqrt{2}$.

In the present case it is better to change the order of the denominator, writing it $\sqrt{2}+1$, since that gives a + quantity in the denominator of the result, and does not affect the value of the fraction.

Ex. 3. To rationalize the denominator in $\frac{1}{1+\sqrt{2}+\sqrt{3}}$.

This may be effected at two steps.

Multiply by $\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$ and we get $\frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2})^2-3}$, or $\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$. And this multiplied by $\frac{\sqrt{2}}{\sqrt{2}}$ gives $\frac{\sqrt{2}+2-\sqrt{6}}{4}$.

Ex. **4.** To rationalize the denominator in $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$.

Multiplying by $\dfrac{\sqrt{2}-\sqrt{3}-\sqrt{5}}{\sqrt{2}-\sqrt{3}-\sqrt{5}}$ gives $\dfrac{-1-\sqrt{10}-\sqrt{15}}{-2\sqrt{6}}$. And multiplying by $\dfrac{\sqrt{6}}{\sqrt{6}}$ gives $\dfrac{-\sqrt{6}-\sqrt{60}-\sqrt{90}}{-12}$,

or $$\frac{\sqrt{6}+2\sqrt{15}+3\sqrt{10}}{12}.$$

A trinomial quadratic denominator may also be rationalized by a single multiplication, as we proceed to show.

96. To find a rationalizing factor for $\sqrt{a}+\sqrt{b}+\sqrt{c}$, where the surds are all dissimilar.

Let $$\sqrt{a}=p,\ \sqrt{b}=q,\ \sqrt{c}=r.$$

Then $$(p+q+r)(p+q-r)\equiv p^2+q^2-r^2+2pq;$$

and $$(p^2+q^2-r^2+2pq)(p^2+q^2-r^2-2pq)$$
$$\equiv(p^2+q^2-r^2)^2-4p^2q^2.$$

In this final product p, q, and r appear only in even powers, and the product is accordingly rational.

The rationalizing factor is the co-factor of $(p+q+r)$; that is, $(p+q-r)(p^2+q^2-r^2-2pq)$.

This factor reduces to $\Sigma p^3-\Sigma p^2q+2pqr$; or restoring $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$, to

$$\Sigma(a-b-c)\sqrt{a}+2\sqrt{(abc)};$$

and this is the rationalizing factor.

The rationalized expression is $\Sigma a^2-2\Sigma ab$, and the fraction $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$ becomes

$$\frac{\Sigma(a-b-c)\sqrt{a}+2\sqrt{(abc)}}{\Sigma a^2-2\Sigma ab}.$$

Ex. To rationalize the denominator of $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.

$$a=1,\ b=2,\ c=3.$$

$$\Sigma(a-b-c)\sqrt{a}+2\sqrt{(abc)}\equiv -2\sqrt{2}-4+2\sqrt{6},$$

and $$\Sigma a^2-2\,\Sigma ab\equiv -8.$$

$\therefore$ the fraction becomes $\dfrac{\sqrt{2}+2-\sqrt{6}}{4}$.

In using the form $\Sigma(a-b-c)\sqrt{a}+2\sqrt{(abc)}$, if any of the terms are negative, the proper signs must be attached to the parts involving the roots only.

Thus for $\sqrt{2}-\sqrt{3}+1$ the rationalizing factor is $-2\sqrt{2}-4-2\sqrt{6}$.

For $\sqrt{7}-\sqrt{3}-\sqrt{2}$ the factor is

$$2\sqrt{7}+6\sqrt{3}+8\sqrt{2}+2\sqrt{42}.$$

The rationalized result $\Sigma a^2-2\,\Sigma ab$, not involving any roots, is not affected by the signs of the terms, and remains the same for all signs.

97. The expression $\Sigma(a-b-c)\sqrt{a}+2\sqrt{abc}$ factors into $(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}-\sqrt{c})$.

This result, although not convenient in practice, is interesting as showing the constitution of a rationalizing factor.

For the three factors are each derived from the expression to be rationalized, $\sqrt{a}+\sqrt{b}+\sqrt{c}$, by keeping one term unchanged and giving to the other two all possible variations of sign. So that of the four trinomial expressions $\sqrt{a}+\sqrt{b}+\sqrt{c}$, $\sqrt{a}+\sqrt{b}-\sqrt{c}$, $\sqrt{a}-\sqrt{b}+\sqrt{c}$, and $\sqrt{a}-\sqrt{b}-\sqrt{c}$, the product of any three is a rationalizing factor of the fourth.

This principle admits of great extension.

98. To find a rationalizing factor for $\sqrt[3]{a}+\sqrt[3]{b}$.

Let $$a=p^3, \text{ and } b=q^3.$$

Then $$(p+q)(p^2-pq+q^2)\equiv p^3+q^3=a+b.$$

$\therefore$ p^2-pq+q^2, or $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$ is a rationalizing factor for $\sqrt[3]{a}+\sqrt[3]{b}$.

Similarly, $\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}$ is a rationalizing factor for $\sqrt[3]{a}-\sqrt[3]{b}$.

Ex. **1.** To rationalize $\sqrt{a}+\sqrt[3]{b}$.

$$(\sqrt{a}+\sqrt[3]{b})(\sqrt{a}-\sqrt[3]{b})\equiv a-\sqrt[3]{b^2}\equiv\sqrt[3]{a^3}-\sqrt[3]{b^2};$$

and this being the difference between two cube roots, the rationalizing factor for it is —

$$(a^3)^{\frac{2}{3}}+(a^3)^{\frac{1}{3}}(b^2)^{\frac{1}{3}}+(b^2)^{\frac{2}{3}},$$

or $$a^2+ab^{\frac{2}{3}}+b^{\frac{4}{3}},$$

and the factor required is —

$$(\sqrt{a}-\sqrt[3]{b})(a^2+a\sqrt[3]{b^2}+b\sqrt[3]{b}).$$

The rationalized expression is a^3-b^2.

EXERCISE VII. b.

1. Simplify —

i. $(3+\sqrt{5})(2-\sqrt{5})$.

ii. $\left(5+\frac{\sqrt{3}}{2}\right)\left(-5+\frac{\sqrt{3}}{2}\right)$.

iii. $(4\sqrt{\tfrac{1}{3}}+5\sqrt{\tfrac{1}{2}})(\sqrt{\tfrac{1}{3}}+2\sqrt{\tfrac{1}{2}})$.

iv. $\sqrt[3]{\{\sqrt[4]{8a^3}\}}$.

v. $5\sqrt{2}(3\sqrt{4+6\sqrt{2}})$.

vi. $\sqrt{48ab^2}+b\sqrt{75a}$.

vii. $2\sqrt{8}-7\sqrt{18}+5\sqrt{72}-\sqrt{50}$.

viii. $\sqrt{18a^5b^2}+\sqrt{50a^3b^4}$.

2. Rationalize the denominators of —

i. $\dfrac{3}{\sqrt{8a}}$. ii. $\dfrac{\sqrt{3}-1}{2-\sqrt{3}}$. iii. $\dfrac{3\sqrt{2}-4}{3-2\sqrt{2}}$.

iv. $\dfrac{\sqrt{2}(\sqrt{5}-1)}{2\sqrt{5}+\sqrt{5}}$.

vi. $\dfrac{5+2\sqrt{2}-3\sqrt{3}-\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}$.

v. $\dfrac{3+2\sqrt{2}-3\sqrt{3}-2\sqrt{6}}{1+\sqrt{2}-\sqrt{3}-\sqrt{6}}$.

vii. $\dfrac{1}{1-\sqrt{2}-\sqrt{3}}$.

3. Which is the greater, $\dfrac{3\sqrt{2}-1}{\sqrt{3}}$ or $\dfrac{2\sqrt{3}-1}{\sqrt{2}}$?

4. Which is the greater, $\dfrac{\sqrt{2}-1}{\sqrt{2}}$ or $\dfrac{\sqrt{3}-1}{\sqrt{3}}$, and what is their difference?

5. Rationalize the denominator of $\dfrac{\sqrt[3]{4}}{\sqrt[3]{2}+\sqrt[3]{3}}$.

6. Rationalize the denominator of $\dfrac{3\sqrt{2}}{\sqrt{2}+\sqrt[3]{2}}$.

99. The following theorems are important in relation to quadratic surds.

(1) The product of two dissimilar surds cannot be rational.

For, if p and q be their surd factors, p and q do not contain the same factors, and hence pq is not made up of square factors only.

Therefore $\sqrt{pq}$ is not rational.

(2) A quadratic surd is not the sum or difference of a rational quantity and a surd.

For, if $\sqrt{a}=b\pm\sqrt{c}$, $a-b^2-c=\pm 2b\sqrt{c}$, by squaring. But $a-b^2-c$ is rational, and therefore $\sqrt{c}$ is rational, which is contrary to hypothesis.

Cor. The sum or difference of two quadratic surds is not rational.

(3) A surd is not the sum or difference of two dissimilar surds.

For, if $\sqrt{a}=\sqrt{b}+\sqrt{c}$, $a-b-c=2\sqrt{bc}$, by squaring. But by (1), $\sqrt{bc}$ is not rational, while $a-b-c$ is rational. $\therefore$ etc.

(4) If $x+\sqrt{y}=a+\sqrt{b}$, where x and a are rational, then $x=a$ and $y=b$.

For, $x-a=\sqrt{b}-\sqrt{y}$; and if $x-a$ has any value, it is rational, and is equal to the difference of two surds, which is impossible by (2), *Cor.*

$$\therefore\ x-a=0,\ \text{or}\ x=a\ \text{and}\ b=y.$$

It may be well to remark here that with an equation, such as $x=a+\sqrt{b}$, we do not mean to assert that x has not an exact finite value, but merely that its value is not that of a quadratic surd; or, in other words, the *square* of x is not rational. So also, if $x-a=\sqrt{b}-\sqrt{y}$, where b and y are not equal, $x-a$ has a real finite value, but this value is not rational, and its square is not rational, *i.e.* $x-a$ is not a surd. Hence, if x and a be both rational, $x-a$ cannot have any value, since such a value must of necessity be rational.

100. The results of Art. 99 enable us to separate a surd expression, such as $a+\sqrt{b}$ or $\sqrt{a}+\sqrt{b}$, into two identical factors, *i.e.* to find its square root.

Let one factor be $\sqrt{x}+\sqrt{y}$.

Then $\quad a+\sqrt{b}=(\sqrt{x}+\sqrt{y})^2=x+y+2\sqrt{xy}.$

$\therefore$ Art. 99 (4), $\quad x+y=a$; and $4xy=b$.

But $\quad x-y\equiv\sqrt{\{(x+y)^2-4xy\}}=\sqrt{\{a^2-b\}}.$

$$\therefore\ x\equiv\tfrac{1}{2}\{\overline{x+y}+\overline{x-y}\}=\tfrac{1}{2}\{a+\sqrt{(a^2-b)}\},$$

$$y\equiv\tfrac{1}{2}\{\overline{x+y}-\overline{x-y}\}=\tfrac{1}{2}\{a-\sqrt{(a^2-b)}\}.$$

$$\therefore\ \sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}(a+\sqrt{a^2-b})}+\sqrt{\tfrac{1}{2}(a-\sqrt{a^2-b})},$$

which is the required factor.

This factor, upon being squared, will reproduce the expression $a+\sqrt{b}$; but the case of practical utility is that in which a^2-b is a complete square. Denote it by c^2.

Then $$\sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}(a+c)}+\sqrt{\tfrac{1}{2}(a-c)}.$$

Ex. **1.** To find the square root of $3+2\sqrt{2}$.

Here $a=3$ and $b=8$, and $a^2-b=c^2=1$, a complete square, whose root is 1.

$$\therefore\ \sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}\cdot 4}+\sqrt{\tfrac{1}{2}\cdot 2}=\sqrt{2}+1.$$

Ex. **2.** To find $\sqrt{\{n^2+n-2n\sqrt{n}\}}$.

Here $$a=n^2+n,\ b=4n^3,\ c=n^2-n.$$

$$\tfrac{1}{2}(a+c)=n^2,\ \tfrac{1}{2}(a-c)=n.$$

$$\therefore\ \sqrt{\{n^2+n-2n\sqrt{n}\}}=n-\sqrt{n}.$$

The sign $-$ before $\sqrt{n}$ in the result is indicated by the same sign before $2n\sqrt{n}$ in the original.

101. An expression of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ can have its root found when the expression is a complete square.

Assume $\sqrt{\{a+\sqrt{b}+\sqrt{c}+\sqrt{d}\}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

Squaring,

$$a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{yz}+2\sqrt{zx}+2\sqrt{xy}.$$

$$\therefore\ a=x+y+z,\ b=4yz,\ c=4zx,\ d=4xy.$$

But $$x^2=\frac{4xy\cdot 4xz}{4\cdot 4yz}=\frac{d\cdot c}{4b},\ \text{and}\ \sqrt{x}=\sqrt[4]{\left(\frac{dc}{4b}\right)}.$$

Similarly, $$\sqrt{y}=\sqrt[4]{\left(\frac{cb}{4d}\right)},\ \sqrt{z}=\sqrt[4]{\left(\frac{bd}{4c}\right)}.$$

$$\therefore \text{ required root} = \sqrt[4]{\left(\frac{cd}{4b}\right)} \pm \sqrt[4]{\left(\frac{db}{4c}\right)} \pm \sqrt[4]{\left(\frac{bc}{4d}\right)}.$$

Ex. Root of $35 + 4\sqrt{15} - 6\sqrt{10} - 12\sqrt{6}$.

Here $\qquad b = 240,\ c = 360,\ d = 864.$

$$\therefore \sqrt[4]{\left(\frac{360 \times 864}{4 \times 240}\right)} \pm \sqrt[4]{\left(\frac{864 \times 240}{4 \times 360}\right)} \pm \sqrt[4]{\left(\frac{240 \times 360}{4 \times 864}\right)}$$

$$= \sqrt{18} \pm \sqrt{12} \pm \sqrt{5}$$

$$= 2\sqrt{3} \pm 3\sqrt{2} \pm \sqrt{5}.$$

The signs are then to be determined. It is readily seen that they must be

$$2\sqrt{3} - 3\sqrt{2} + \sqrt{5}$$

in order to give those of the original.

Moreover, $\qquad (2\sqrt{3})^2 + (3\sqrt{2})^2 + (\sqrt{5})^2 = 35.$

The original expression should be reproduced by squaring this, and that is the only absolute test that we have the true root.

EXERCISE VII. c.

Find the square roots of the following perfect squares from 1 to 11 inclusive.

1. $8 + 2\sqrt{15}$. **2.** $3 + 4i$. **3.** $4 + 2\sqrt{3}$. **4.** $1 + 2u\sqrt{1 - u^2}$.

5. $-2i$. **6.** $2x^2 + 2(x - y)\sqrt{x^2 - y^2} - 2xy$.

7. $9 - 4\sqrt{2} + 4\sqrt{3} - 2\sqrt{6}$. **10.** $\dfrac{4\sqrt{10} + 8}{4\sqrt{10} - 8}$.

8. $25 + 10\sqrt{6}$.

9. $\frac{3}{2} + \sqrt{2}$. **11.** $a^2 - 2 + a\sqrt{a^2 - 4}$.

12. Simplify $\dfrac{1}{\sqrt{(16 + 2\sqrt{63})}} + \dfrac{1}{\sqrt{(16 - 2\sqrt{63})}}$.

13. Simplify $\sqrt{(3 + \sqrt{9 - p^2})} + \sqrt{(3 - \sqrt{9 - p^2})}$.

14. If $a^2d = bc$, then $\sqrt{(a + \sqrt{b} + \sqrt{c} + \sqrt{d})}$ can be put in the form $(\sqrt{x} + \sqrt{y})(\sqrt{X} + \sqrt{Y})$.

CHAPTER VIII.

CONCRETE QUANTITY. — GEOMETRICAL INTERPRETATIONS. — THE GRAPH.

102. A concrete number depends upon a concrete unit which gives to the number its name and character. Thus 8 dollars is 8 of the concrete units called a dollar. So a hours is a times the concrete unit known as an hour.

The symbolism of algebra applies to these abstract numbers, as coefficients of concrete units, but the interpretation is affected by the nature of the concrete unit.

Operations with concrete quantities are in general subject to the two following laws:

(1) To multiply 2 hours by 3, or 3 hours by 2, gives 6 hours; but to multiply 2 hours by 3 hours has no meaning.

Hence, concrete units have no product, *i.e.* they do not admit of being multiplied together.

Hence, also, concrete units have no powers and no roots.

(2) Again, 3 hours and 4 hours make 7 hours, and 5 minutes and 8 minutes make 13 minutes, but 3 hours and 4 minutes do not make 7 hours or 7 minutes.

Hence, concrete quantities are added only when of the same name, and then, by adding the coefficients of the concrete unit.

An important apparent exception to the foregoing laws occurs in units of length; this will be fully dealt with in connection with geometrical interpretations.

Any other apparent exceptions are easily explained.

103. In many cases of the interpretation of concrete results negative quantity has a special significance.

If an idea which can be denoted by a quantitative symbol has an opposite so related to it that one of these ideas tends to destroy the other or to render its effects nugatory, these two ideas can be algebraically and properly represented only by the opposite signs of algebra.

If a man buys an article for b dollars, and sells it for s dollars, his gain is expressed by $s-b$ dollars. So long as $s>b$, this expression is $+$, and gives the man's gain.

But if $s<b$, the expression is $-$. It denotes that whatever his gain is *now*, it is something exactly opposite in character to what it was before. And as he now sells for less than he buys for, he loses. In other words, a negative gain means loss.

Thus *gain* and *loss* are ideas which have that kind of oppositeness which is expressed by oppositeness in sign. If a man gains $+a$ dollars, he is so much the wealthier; if he gains $-a$ dollars, he is so much the poorer.

Whether gain or loss is to be considered positive must be a matter of convenience, but only opposite signs can denote the opposite ideas.

104. Among the ideas which possess this oppositeness of character are the following:

(1) *To receive and to give out;* and hence, to buy and to sell, to gain and to lose, to save and to spend, etc.

(2) *To move in any direction and in the opposite direction;* and hence, measures or distances in any direction and in the opposite direction, as east and west, north and south, up and down, above and below, before and behind, etc.

(3) *Ideas involving time past and time to come;* as the past and the future, to be older than and to be younger than, since and before, etc.

(4) *To exceed and to fall short of;* as, to be greater than and to be less than, etc.

Ex. **1**. A man goes 20 miles north, then 15 miles south, then 8 miles south, then 12 miles north, and lastly 18 miles south. Where is he with respect to his starting-point?

Denoting north by $+$, south becomes $-$.

The traveller has gone $+20-15-8+12-18$ miles, or -9 miles.

That is, he is 9 miles from his starting-point, and the sign $-$ shows that he is south of it.

Ex. **2**. A man invests $10,000. On $\frac{1}{4}$ of his investment he gains 25%; on $\frac{2}{5}$ he loses 20%; and on the remaining portion he gains 2%. What per cent does he gain upon the whole.

Let gain be $+$, and let a denote the amount invested. His gain % is

$$\left(\frac{a}{4}\cdot\frac{25}{100}-\frac{2a}{5}\cdot\frac{20}{100}+\frac{7a}{20}\cdot\frac{2}{100}\right)\div\frac{a}{100}=-1\tfrac{1}{20}\%.$$

Therefore, his gain is $-1\frac{1}{20}\%$, or he loses $1\frac{1}{20}\%$.

Ex. **3**. A goes a miles an hour and B b miles an hour along the same road, and A is m miles in advance of B. When and where will they be together?

Let t be the time in hours, and p be the distance in miles, from B's present position to the point of meeting.

Then $$t=\frac{m}{b-a}, \text{ and } p=\frac{bm}{b-a}.$$

1. Suppose a, b, and m to be all positive.

As a and b may have any values,—

(a) Let $a=b$. Then t and p are both ∞.

Now when $a=b$, A and B are travelling at the same rate in the same direction, and the values of t and p tell us that they will be together after an infinite time, and at an infinite distance from the present position of B, *i.e.* that they will *never* be together. It must be noticed that this is the only way in which the symbols of algebra can answer the question proposed with these conditions.

(b) Let $a<b$. Then as t and p are both positive and finite, the men will be together at some time in the future, and at some distance in the positive direction measured from B's present position.

(c) Let $a>b$. Then t and p are both negative.

This tells us that A and B will be together at some time in the past, *i.e.* they have already been together; for as t denotes time to come, $-t$ must denote time past, and their point of meeting is at some distance in the negative direction from B's present position.

2. Let a be negative. Then A is coming backwards to meet B, and they will meet at some time in the future, and at some place in the positive direction from B's position.

Other variations of signs must be left to the explanation of the reader.

Ex. 4. A and B both have cash in hand and both owe debts. B's cash is 10 times his debt. If B pays A's debt, his cash will be $2\frac{4}{5}$ that of A's, and if A pays B's debt, his cash will be $\frac{3}{20}$ of B's. When all debts are paid, both together have $3400. How much has A after his debts are paid?

Let A's cash $= a$, and B's cash $= b$. Since debt is opposite to cash capital, B's debt is $-\frac{b}{10}$ capital.

Then A's debt $= 3400 + \frac{b}{10} - a - b$ capital.

After paying B's debt A has $a - \frac{b}{10}$ left, and this is equal to $\frac{3}{20} b$. $\therefore a = \frac{1}{4} b$.

After paying A's debt B has $b - \left(3400 + \frac{b}{10} - a - b\right)$ left, and this is equal to $2\frac{4}{5} a$.

$$\therefore \frac{14}{5} \cdot \frac{1}{4} b = b - 3400 - \frac{b}{10} + \frac{b}{4} + b.$$

Whence $\qquad b = 4000 =$ B's capital,

and $\qquad -400 =$ B's debt, as capital.

$\therefore$ B is worth 3600 when his debts are paid, and A is worth -200 when his debts are paid.

That is, A owes 200 dollars more than he is worth.

105. The examples of the preceding article illustrate the fact that a literal algebraic solution of a concrete problem is not a solution of a particular problem, although put in a particular form, but of every problem belonging to a group of which the particular one may be taken as a representative. This group includes all problems derivable from the particular one by (1) varying the magnitudes of the numerical quantities concerned, and (2) changing ideas which admit of it into their opposites, provided always that such changes do not render the problem unintelligible.

When such a literal result is interpreted in language as general as possible, it becomes a *rule;* but it seldom happens that any rule of arithmetic or geometry can be as broad and representative as the literal expression from which it is derived, and of which it professes to be the interpretation.

EXERCISE VIII. a.

1. A man buys m articles at b dollars per article and sells them at s dollars per article; what is his gain per cent?

Interpret the result (i.) when he receives the articles as a gift; (ii.) when he gives the goods away; (iii.) when $b > s$.

Show that a person in dealing may gain any finite percentage, but that he cannot lose more than 100 per cent.

2. One carriage wheel is m, and the other n feet in circumference. How far has the carriage gone when one wheel has made r revolutions more than the other.

Interpret when $m = n$.

3. A has a dollars more than B, but if B gives A b dollars, A will have twice as much as B. How much has each?

Interpret when (i.) a is negative; (ii.) b is negative; (iii.) is there any arithmetical interpretation for both a and b negative?

4. A's age exceeds B's by n years, and is as much below m as B's is above p. Find their ages.

Interpret when $n > m + p$.

5. The hands of a clock go around in the same direction in a and b hours respectively. If they start from the same point together, when will they be again together?

Adapt your result to the case where the hands move in opposite directions.

6. A, B, C, D are four equidistant fixed points in line; find a point O, in the line, for which

$$3\,AO + 5\,BO + 2\,CO + 6\,DO = 0.$$

On which side (the A or D side) of the middle is the point O?

7. Two circles have their radii r and r_1, and their centres lie on a fixed horizontal line. If $r > r_1$ and d is the distance between centres, how are the circles relatively situated when —

i. $d = r + r_1$, iii. $d = -r + r_1$,

ii. $d = r - r_1$, iv. $d = -r - r_1$?

8. A can run a feet in b seconds, and B can run b feet in a seconds. Express the ratio of A's speed to B's.

9. Two ropes are in length as $4:5$, and 6 feet being cut from each, the remainders are as $3:4$. Find their original lengths.

10. The lengths of two ropes are as $a:b$, and c feet being cut from each they are as $a_1 : b_1$. Find their original lengths.

Interpret when c is negative.

11. A river flows 4 miles an hour; a boat going down the river passes a certain point in 20 seconds, and in going up it takes 30 seconds. Find the speed of the boat in still water. Also the length of the boat in feet.

12. A man walks from A to B in h hours. If he had walked a miles an hour faster he would have been b hours less on the road. Find the distance from A to B, and the rate of walking.

13. Change the wording of 12 to suit the case where a and b both change signs.

GEOMETRICAL INTERPRETATIONS.

106. As we have seen in Art. 21, the substitution of numerical quantity for the quantitative symbol gives an arithmetical theorem for an algebraic expression, and especially for an identity.

So, also, an expression written in the symbolism of algebra may admit of a geometric interpretation, when the quantitative symbol stands for some elementary geometric idea. The mode of interpretation must depend upon the character of this idea.

By employing the quantitative symbol to stand for different geometric ideas, mathematicians have developed different geometric algebras, requiring different modes of interpretation, and some of which, not being derived from arithmetic, are not subject to all the formal laws deduced in Chapter I. It need scarcely be said that these latter algebras are not generalized arithmetic, and are not generally applicable to numbers.

Only two modes of interpretation concern us here, and these are such as belong to algebra as already defined and developed.

The distinction between these is as follows:

I. The quantitative symbol stands for a given portion of a straight line, or a line-segment, as a geometric figure. The algebra then becomes a kind of *symbolic geometry,* and is subject to certain restrictions arising from the nature of geometry.

II. The quantitative symbol stands for a number, *i.e.* the number of times a particular line-segment, taken as a unit of measure, is contained in a given line-segment.

This becomes a matter of concrete quantity, admitting of geometric interpretation, and being subject to certain geometric limitations.

I. Symbolic Geometry.

107. The primary ideas in geometry are length and direction, or, mechanically stated, transference and rotation.

Length is denoted by a quantitative symbol, which in this connection will be called a **line-symbol**. Thus a

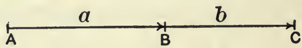
denotes a given line-segment. But it does more than this; it denotes transference from one end of the segment to the other in only one direction. Then $-a$ denotes the same amount of transference in the opposite direction; and thus $a-a=0$ represents a neutralized effect, and is equivalent to no transference, and consequently to no line-segment.

Thus a and $-a$ denote the same segment measured in opposite directions, and we have thus a simple and intelligible interpretation of the signs $+$ and $-$ as applied to line-segments.

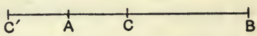
Such a segment is called a *directed* segment, as its direction, being one of two opposites, is determined by algebraic sign.

108. $a+b$ is a segment as long as a and b together; or it represents transference over length a followed by transference over length b in the same direction.

Thus, if $a=AB$, $b=BC$, $a+b=AB+BC=AC$.



Then $a-b$, which is $a+(-b)$ represents transference over length a followed by transference over length b in the opposite direction. If a is longer than b, let $a=AB$, and $b=CB$. Then $-b=BC$, and $a-b=AB+BC$, or transference from A to B, followed by transference from B to C. This is equivalent to transference from A to C, and is positive, being the remainder when the shorter segment b is cut off from the longer a.



If b is longer than a, let $b=C'B$.

Then $a-b=AB+BC'=AC'$, which is negative.

Similarly, na, when n is numerical, denotes a segment n times as long as that denoted by a.

109. The area of a rectangle is denoted, in the symbolism of algebra, by the product form of the line-symbols which denote two adjacent sides of the rectangle.

Thus ab means the area of the rectangle whose adjacent sides are denoted by a and b.

Then a^2 denotes the area of the square whose side is a.

If either a or b is negative, the area ab is negative, and is subtractive from any other area concerned. Hence an area is often spoken of in connection with the symbolism of algebra as a directed area. A square, however, is essentially positive.

110. The volume of a rectangular parallelopiped or cuboid is symbolically expressed as the continued product of the three line-segments which denote any three conterminous edges, known as *direction* edges, of the cuboid.

Thus abc is the volume of the cuboid having a, b, and c as direction edges.

Then a^3, which is the same as aaa, is the volume of the cube whose edge is a.

We thus see the meaning of the terms *square* and *cube* as introduced from geometry into arithmetical algebra.

111. The evidence of the legitimacy of the conventions of the three preceding articles, or rather the proof of the necessity of 109 and 110 as following from the fundamental convention of 107, is a matter for geometry rather than for algebra.

In the elements of geometry it is also shown that, with these conventions in regard to the geometric meanings of the algebraic forms, we have

(1) $a + b = b + a$.

(2) $ab = ba$, and $\therefore\ abc = acb = bac =$ etc.

(3) $a(b + c) = ab + ac$, etc.

Hence the symbols a, b, c are subject to the same formal laws of transformation whether we consider them as line-symbols or as quantitative symbols. And thus every algebraic identity of proper form may be interpreted either as a theorem in numbers, *i.e.* in arithmetic, or as a theorem in lines, areas, and volumes, *i.e.* in geometry.

EXERCISE VIII. b.

Interpret the following identities as geometric theorems —

1. $(a - b)^2 + 2\,ab = a^2 + b^2$.

$(a - b)^2$ is the square on the difference of two line-segments; ab is the rectangle on the segments; and $a^2 + b^2$ is the sum of the squares on the segments, therefore

The square on the difference of two segments and twice the rectangle on the segments are together equal in area to the sum of the squares on the segments.

2. $a(a + b) = a^2 + ab$.

3. $(a + b)(a - b) = a^2 - b^2$.

4. $(a + b)^2 = a^2 + b^2 + 2\,ab$.

5. $(a + b)^2 - (a - b)^2 = 4\,ab$.

6. $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

7. $ab(a + b) = a^2b + ab^2$.

8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

9. $(a + b)^3 = a^3 + b^3 + 3\,ab(a + b)$.

10. If $a : b = c : d$, then $ad = bc$.

11. If $a : b = b : c$, then $b^2 = ac$.

12. If $a : b = b : c = c : d$, then $a : d = a^3 : b^3$.

112. Certain restrictions must be imposed upon algebraic expressions if they are to be interpretable as *real* geometric relations. These, besides the conditions which render an expression arithmetically interpretable, are two in number; namely,

(1) In the line-symbols the expression must be of not more than three dimensions.

This is due to the fact that there are but three dimensions in space, the subject-matter of geometry, and that by our convention each line-symbol in a product represents one of these dimensions.

(2) The expression must be homogeneous in the line-symbols. For the adding of one species of magnitude to another species, as a line to an area, or an area to a volume, is not an intelligible operation.

113. Every homogeneous expression of one dimension in line-symbols denotes a finite line-segment, or is *linear*.

Every homogeneous expression of two dimensions in line-symbols denotes an area; such areas being squares, as a^2, or rectangles, as ab, or areas made up of these.

Every homogeneous expression of three dimensions in line-symbols denotes a volume; such volumes being cubes, as a^3, or cuboids on square bases, as a^2b, or cuboids with three unequal edges, as abc, or volumes made up of these.

114. An expression which represents a geometric relation must always represent a geometric relation however it is transformed, provided it is interpretable. And hence a homogeneous expression cannot be made non-homogeneous by any legitimate transformation.

This fact is useful in many ways.

If at any stage in the transformations of algebraic expressions a homogeneous expression becomes non-homogeneous, or *vice versa,* some error in work is to be looked for.

Non-homogeneous expressions are frequently made homogeneous in *form* by the introduction of a *unit-variable,* on account of the resulting advantages in the after work.

Thus $x^2 + 3x - 2 = 0$ may be written $x^2 + 3xy - 2y^2 = 0$, where y is a variable in form only, and is to be replaced by 1 after all the necessary transformations are made.

115. In applying the symbolism of algebra to develop metrical relations in geometry, a sufficient knowledge of descriptive geometry is required, and in addition to the relations already laid down in 113 and previous articles, the following are necessary:

1. If a, c denote the sides of a right-angled triangle and b denote the hypothenuse, $b^2 = a^2 + c^2$.

2. Similar triangles have their homologous sides proportional.

Ex. **1**. $\sqrt{ab}$ is linear, and denotes the side of the square whose area is equal to that of the rectangle whose sides are a and b.

Ex. **2**. $\sqrt[3]{abc}$ is linear, and denotes the edge of the cube whose volume is equal to that of the cuboid of which a, b, c denote direction edges.

Ex. **3**. $\sqrt{a^2bc}$ is an area. For it is $a \cdot \sqrt{bc}$, and $\sqrt{bc}$ is linear.

Ex. **4**. a^2bc, being of 4 dimensions, has no geometrical interpretation.

Ex. **5.** $ab + bc$ is the sum of two areas, and is therefore an area; but $ab + c$ has no geometrical meaning, not being homogeneous.

Ex. **6.** The base of an isosceles triangle is b and its altitude is a, to find the perpendicular, p, from a basal vertex to a side.

Let ABC be the triangle with B as vertex, D the foot of the altitude, and AE the required perpendicular.

Then $ab = AE \cdot BC$, since each expresses double the area of the triangle.

But BDC being right-angled at D,

$$BC^2 = BD^2 + DC^2 = a^2 + \tfrac{1}{4}b^2.$$

$$\therefore\ ab = p\sqrt{(a^2 + \tfrac{1}{4}b^2)}.$$

Whence $$p = \frac{2\,ab}{\sqrt{(4\,a^2 + b^2)}}.$$

Ex. **7.** If any given area be divided respectively by the areas of the squares on the two sides of a right-angled triangle, the sum of the quotients is equal to that obtained by dividing the given area by the area of the square on the perpendicular from the right angle to the hypothenuse.

Let p be the perpendicular.

Then $ac = pb =$ twice the area of the triangle.

$$\therefore\ a^2c^2 = p^2b^2 = p^2(a^2 + c^2) \qquad \text{(Art. 115, 1)}$$

Whence $$\frac{1}{a^2} + \frac{1}{c^2} = \frac{1}{p^2};$$

and multiplying by any area, u^2 say,

$$\frac{u^2}{a^2} + \frac{u^2}{c^2} = \frac{u^2}{p^2};$$

which interpreted gives the theorem.

EXERCISE VIII. c.

1. AA' is the diagonal of a square, and is trisected at the points C and D. Find the area of the square when the segment CD is a.

2. The sides of a rectangle are as $m:n$ and the diagonal is d—

i. Find the sides.

ii. Find the area.

iii. Find the perpendicular from a vertex to the diagonal.

iv. Find the distance between the feet of the two perpendiculars upon the same diagonal.

v. Show that the rectangle on the diagonal and the line-segment between the feet of the perpendiculars on the diagonal, is equal in area to the rectangle on the sum and difference of the sides.

3. The sides of a rectangle are a and b; to find—

i. The perpendicular upon a diagonal.

ii. The distance between the feet of the perpendiculars upon the same diagonal.

iii. Show that the volume of the cuboid, whose direction edges are the two sides and the line-segment of ii., is equal to that of the cuboid, whose direction edges are the sum and difference of the sides and the line-segment of i.

4. The side of an isosceles triangle is n times the altitude, and the base is $2b$; to find the area. What does the result become when $n = 1$? Explain.

5. The base of an isosceles triangle is one-half the side, and the perpendicular upon the base is $\frac{1}{2}\sqrt{15}$. Find the area.

6. An upright tree is broken over by the wind, and the top touches the ground at 36 feet from the base. Find the length of the whole tree when the remaining upright part is 15 feet.

7. How large a circular disc can be cut from a triangular piece of paper whose edges are 13, 14, and 15 inches respectively?

8. In a right-angled triangle the median to the hypothenuse is r times one of the sides; find the ratio of the sides to one another.

9. In an isosceles triangle where b is the base and s the side, the area is expressed by $\frac{bs}{\sqrt{5}}$; find the ratio of the side to the base.

10. If the side of a square be increased by $\frac{1}{n}$th of itself, where n is a large number, by what part of itself is the area increased?

11. If the edge of a cube be increased by $\frac{1}{n}$th part of itself, where n is a large number, by what part of itself is the volume increased?

12. Find the ratio of the diagonal of a square to —

i. The side.

ii. The join of a vertex with the middle of a side.

13. Find the ratio of the diagonal of a cube to its edge.

14. Compare the area of an equilateral triangle on the side of a square to one on the diagonal.

15. A boat making 10 miles an hour in still water steers directly across a stream flowing 4 miles an hour. Compare the real velocity of the boat with —

i. Its velocity across the stream.

ii. Its velocity down the stream.

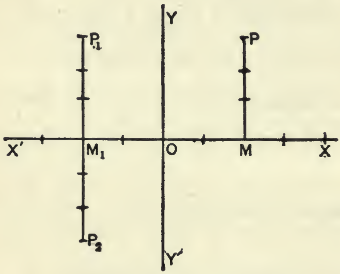
16. A boat goes a certain distance down a stream in t seconds, and requires t_1 seconds to return. Compare the velocity of the boat with that of the stream.

Interpret when $t = t_1$.

17. A street 60 ft. wide has a house 20 ft. high on one side, and a house 30 ft. high on the other. How long a ladder is required, and where must its foot be placed, that it may just reach to the top of each house?

II. Geometry as Concrete Quantity.—The Graph.

116. Take any point O, and through it draw the two lines XX', YY' at right angles to one another.



These lines are lines of reference, and are called *axes*, and O is the origin.

To distinguish the axes, XX' is called the x-axis, and YY' the y-axis.

Measures are made along the x-axis *from* O to the right or left, those to the right being positive and those to the left being negative. Also measures are made *from* the x-axis parallel to the y-axis, upwards or downwards, those taken upwards being positive and those downwards negative. We have thus two sets of measures which represent the two dimensions of the plane, and by means of the convention of signs stated above we may represent any point in the plane.

Let P be any point, and PM be perpendicular to OM.

Then the measures which determine the position of P relatively to the origin and axis are OM and MP, and if these are known, the position of P is known.

Usually, and for the sake of uniformity, the measure OM is called the x of the point P, and the measure MP is called the y of the point P. If, then, the x and y of a point are given, the point can be laid down.

Thus, let the x of P be 2 and its y be 3.

To get the point, we measure OM to the right equal to 2 units from any adopted scale, and then measure MP upwards parallel to OY and equal to 3 units from the same scale. The point thus found is an ocular representation of the given point, and is called the **graph** of the given point.

If x were -2, and y 3, we would take OM_1 to the left and get the point P_1; and if the point had its $x=-2$ and its $y=-3$, we would also take M_1P_2 downward and get the point P_2. Thus any point is completely determined by its x and y with their proper signs.

117. Now let $y=fx$ be any integral function of x. For every value of x we have a corresponding value of y, and if x varies continuously, *i.e.* by infinitely small gradations, y also varies continuously.

Let, then, a number of corresponding values of x and y be found by giving to x any convenient arbitrary values, and finding the resulting value of y for each. These values form the x's and y's for a set of points whose graphs all lie upon the graph of the function fx, this graph being a line or curve passing through all the points.

If all possible values of x could be considered, the points would be infinite in number, and would exactly mark out the graph of the function; but as we cannot practically take every value of x, we take a set of values, usually integral, as being most convenient, and thus get a set of points. We then connect these, as well as possible, by a line or curve, as may be required.

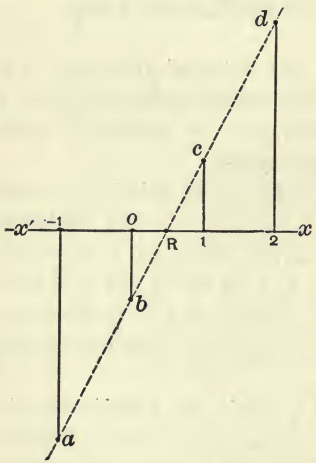
The theoretical graph is an exact geometrical picture of the function, and in itself and in its relation to the axis represents every property of the function. The practical graph is a more or less close approximation to this.

Ex. 1. Let $y = 2x - 1.$

Take $x = -1, \quad 0, +1, +2, \cdots$

Then $y = -3, \ -1, +1, +3, \cdots$

Lay down the graphs of the points whose x's and y's are given, as at $a, b, c, d, \cdots$ in the diagram.



It is readily seen that these points lie in line, and that the graph, which is denoted by the dotted line, is a straight line. Hence the reason for calling $2x-1$ a linear function of x, and $2x - 1 = 0$ a linear equation.

It can be readily shown that the graph of every function of the form $ax + b$ is a straight line.

The **root**. At the point R, where the graph ad cuts the x-axis, we have $y = 0$, and $\therefore \ 2x - 1 = 0$. The x of this point is $0R$, and as this is the value of x, which makes the function zero, $0R$ measures the value of the root. It is readily seen that $0R = \frac{1}{2}$.

Thus the cutting of the x-axis by the graph denotes a *real* root, and the distance from the origin to the point of intersection measures the value of the root, + if to the right, and $-$ if to the left. In the present example the root is $+$.

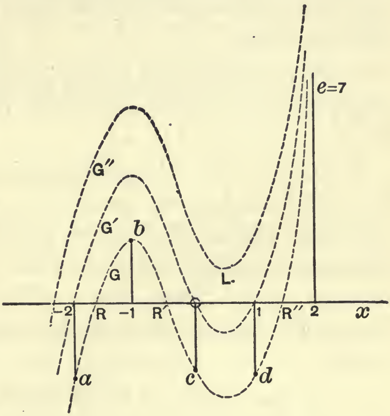
As the line *ad must* cut the x-axis either at a finite point or at ∞, a linear equation has one root only; this root must be real, and may be finite or infinite in value.

Ex. 2. Let $\quad y = x^3 + x^2 - 2x - 1 = fx.$

Take $\quad x = -2 \; -1 \quad 0 \; +1 \; +2.$

Then $\quad y = -1 \; +1 \; -1 \; -1 \; +7.$

The graph is given at G in the diagram, being a curve through the points $a, b, c, d, e \cdots$



1. The graph cuts the x-axis at three points, R, R', and R''.

Hence there are three real roots to the equation

$x^3 + x^2 - 2x - 1 = 0.$

Two of these, $0R$ and $0R'$, are negative, and the third, $0R''$, is positive. The limits of the roots are, $0R > -2$ and < -1, $0R' > -1$ and < 0, and $0R'' > 1$ and < 2.

2. If we move the graph bodily upwards through 1 unit, we bring it to the position G', and increase every value of y by 1 unit. But we may increase the value of y by a unit, by adding a unit to the independent term of the function.

Hence, to increase the independent term of the function is equivalent to moving the graph upwards, and to decrease it is equivalent to moving the graph downwards.

G' is thus the graph of

$$y = x^3 + x^2 - 2x = x(x-1)(x+2),$$

in which a has come to -2, c to 0, and d to 1.

The roots are now -2, 0, and 1.

The two points R' and R'' have come nearer together, and the two R and R' have gone further apart; that is, the roots $0R'$ and $0R''$ have approached one another in value, while the roots $0R'$ and $0R$ have become farther separated in value.

3. Add another unit to the independent term, and the graph is moved into the position G'', which is the graph of $x^3 + x^2 - 2x + 1$.

As the loop L no longer cuts the x-axis, the points R' and R'' have become imaginary, while the point R is still real. Hence the two roots $0R'$ and $0R''$ of G have become imaginary in G'', while the third root, $0R$, remains real. Thus, then, $x^3 + x^2 - 2x + 1$ has two imaginary factors, *i.e.* complex quantities, and one real factor.

4. In the motion of the graph from the position G' to that of G'' there was an intermediate position in which the loop L just touched the x-axis. The points R' and R'' were then coincident, and the roots $0R'$ and $0R''$ were identical.

We thus see that in passing from real to imaginary two roots approach one another in value, become equal, and then become imaginary; and since two roots must always be thus involved together, the roots must become imaginary in pairs; or, more concisely, imaginary roots exist in pairs.

5. The least consideration will show that similar changes take place when the graph is lowered by subtracting from the independent term of the function, the difference being that R and R' will then become imaginary, while R'' remains real.

The mode of representing a function by a graph is due to Descartes, and its invention is one of the great milestones in the progress of mathematics. The graph is largely employed by statisticians, by engineers, by physicists, by chemists, and many others who are able to employ mathematical methods intelligently; and its systematic discussion is the subject-matter of coördinate geometry.

EXERCISE VIII. d.

1. Construct the graphs of $2y + 3x = 6$, and of $3y - 2x = 6$.

2. Construct the graph of $x - y = 0$. How is it situated with respect to the axes?

3. In the graph of $ax + by + c = 0$, what is the effect of —

i. Increasing the independent term?

ii. Increasing the coefficient of x?

iii. Increasing the coefficient of y?

4. Draw the graphs of $x^3 - 4x + 2 = y$; of $x^3 + 3x^2 - x - 1 = y$; and of $x^3 - 4x = y$. How are these graphs situated in relation to one another?

5. What integer added to the independent term of $x^4 - 2x^3 - x^2 + 2x - 1$ will make all the roots imaginary? Will make all the roots real?

6. Explain from the graph why a cubic must have one real root.

CHAPTER IX.

THE QUADRATIC.

118. The most general type of a quadratic function of one variable is

$$ax^2 + bx + c,$$

and the corresponding equation is

$$ax^2 + bx + c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

In the equation we may divide through by a; then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0;$$

and writing p for $\frac{b}{a}$, and q for $\frac{c}{a}$, the equation becomes

$$x^2 + px + q = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

which is the quadratic reduced to its *simplest form*. The roots of this are, by Art. 59,

$$x_1 = \tfrac{1}{2}(-p + \sqrt{p^2 - 4q}), \text{ and } x_2 = \tfrac{1}{2}(-p - \sqrt{p^2 - 4q}).$$

On account of the double sign of the root-symbol, $\sqrt{\ }$ (Art. 48), both values are included in the one expression

$$x = \tfrac{1}{2}(-p \pm \sqrt{p^2 - 4q}),$$

and this is the solution of (B).

In this solution write $\frac{b}{a}$ for p, and $\frac{c}{a}$ for q, and reduce, and we obtain

$$x = \frac{1}{2a}\{-b \pm \sqrt{b^2 - 4ac}\},$$

and this is the solution of (A)

The forms of these solutions should be so mastered that for any quadratic equation, in either of the forms (A) or (B), the solution may be written down at once.

Ex. 1. The roots of $3x^2 + 2x - 4 = 0$ are

$$x = \tfrac{1}{6}(-2 \pm \sqrt{4 + 48}) = \tfrac{1}{3}(-1 \pm \sqrt{13}).$$

Ex. 2. The roots of $2x^2 - 3x + 2 = 0$ are

$$x = \tfrac{1}{4}(3 \pm \sqrt{9 - 16}) = \tfrac{1}{4}(3 \pm i\sqrt{7}).$$

119. The double root, or double solution of the quadratic, is frequently of the highest importance as giving an unexpected answer to a problem, and through this answer giving us a clearer idea of the nature of the problem.

It is only when a problem admits, in spirit, of a double answer, that it involves the solution of a quadratic.

A few examples will make this plain.

Ex. 1. A man buys a horse and sells him for \$24, thus losing as much per cent as the horse cost in dollars. To find the cost.

Let $x =$ the cost. Then $\frac{x}{100} \cdot x = x - 24.$

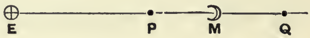
Or $x^2 - 100x + 2400 = 0.$

Whence $x = 60$ or $40.$

This solution shows the problem to be to a certain extent indefinite, since there is no way of determining whether the cost of the horse was \$40 or \$60.

Ex. **2.** The attraction of a planet varies directly as its mass and inversely as the square of the distance from its centre. The earth's mass is 75 times that of the moon, and their distance apart is 240000 miles. To find a point, in the line joining them, where their attractions are equal.

Let P be the point, and let $EP = x$.



Attraction of $E = \frac{75}{x^2}$; and of $M = \frac{1}{(240000 - x)^2}$; and these are to be equal. This gives

$$74\,x^2 - 150 \times 240000\,x + 75 \cdot (240000)^2 = 0.$$

Whence $\qquad x = 215160$ or 271330 miles.

The smaller of these numbers evidently gives EP; the larger, being greater than 240000, gives a second point, Q, beyond the moon, and not contemplated in the problem. Our judgment tells us that there is a second point.

Cor. In the foregoing question let the masses of the moon and earth be the same. Then we have

$$\frac{1}{x^2} = \frac{1}{(240000 - x)^2} \text{ or } x^2(1 - 1) - 480000\,x + (240000)^2 = 0.$$

By Art. 77, $\qquad x = \infty$ or 120000.

That is, one point, P, is half way between the earth and moon, and the other is infinitely distant.

EXERCISE IX. a.

1. Solve the quadratics.

i. $x^2 + x \cdot \frac{1}{2}(b - c) - \frac{1}{4}bc = 0.$

ii. $x^2 - ax + \frac{1}{4}(a^2 - b^2) = 0.$

iii. $abx^2 - (a^2 + b^2)x + ab = 0.$

iv. $3\,x^2 - 2\,x + 1 = 0.$

v. $(a^2 - b^2)x^2 - 2\,ax + 1 = 0.$

vi. $x^2 - x - \frac{1}{4} = 0.$

2. Find the relation between a and b in the equation $(a + x)(b - x) + abx - 1 = 0$, when—

i. The sum of the roots is zero.

ii. The sum of the reciprocals of the roots is zero.

iii. The sum of the reciprocals of the roots is infinite.

3. If the equation $ax^2 + bx + c = 0$ has α and β as its roots, find the equation which has $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ as its roots.

4. Show that the roots of $ax^2 + bx + a = 0$ are reciprocals of one another.

5. The area of a right-angled triangle is a^2 and the difference between the two sides is d; to find the sides. Explain the double solution, and draw figures to represent it, when $a^2 = 4$ and $d = 2$.

6. $ABCD$ is a square. P is a point on AB produced, and Q is on AD, and PCQ is a right angle. Determine BP so that the triangle PCQ shall have a given area, a^2. Explain the double solution.

7. In Ex. 6, AQ is equal to BP; determine BP when the triangle PCQ has a given area, a^2. Explain the double solution.

8. AB and CD are two straight lines intersecting at right angles in O. AC is of a given length, l. Find AO when the triangle ACO has a given area, a^2. Explain the quadruple solution.

9. Find the area of the triangle of Ex. 8, when $AO = 2\,CO$.

10. Find CO, of Ex. 8, when $AO^2 = l \cdot CO$.

120. The rational part, $-\frac{b}{2a}$, in the solution of (A) is the same for each root, the difference in the roots being due to the part $\sqrt{b^2 - 4ac}$.

As this part may be rational, irrational, or imaginary, both roots are alike rational, irrational, or imaginary.

(1) When $\sqrt{b^2 - 4ac}$ is real, the roots are real and different.

This occurs when a and c have unlike signs, or when they have like signs and $b^2 > 4ac$.

Ex. 1. The roots of $x^2 - 2x - 2$ are $1 \pm \sqrt{3}$.

Ex. 2. The roots of $x^2 - 3x + 1$ are $3 \pm \sqrt{5}$.

(2) When $\sqrt{b^2 - 4ac} = 0$, the roots are real and equal. This occurs when $b^2 = 4ac$, in which case the function is a complete square.

Ex. 3. The roots of $x^2 - 4x + 4$ are 2 ± 0.

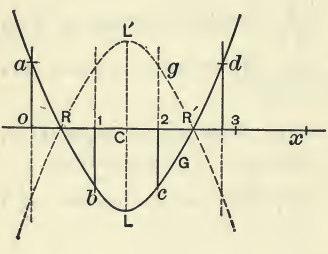
(3) When $\sqrt{b^2 - 4ac}$ is imaginary, the roots are complex numbers, unless b is zero, when they are imaginaries. This occurs when $b^2 < 4ac$.

Ex. 4. The roots of $x^2 - 2x + 2$ are $1 \pm i$.

(4) When $b = 0$, the roots are $\pm \dfrac{1}{2a}\sqrt{-4ac}$, and differ in sign only; but they may be rational, irrational, or imaginary.

(5) If the roots are real, and a is +, they will have the same sign when $b > \sqrt{b^2 - 4ac}$; that is, when c is +. The sign of the roots will be the opposite to that of b.

This takes place in Ex. 2, the roots being real, and a and c being both +.



121. The Graph. The graph, G, of $x^2 - 3x + 1$ is given in the margin. The roots $0R$ and $0R'$ are both positive (Art. 120, 5).

Let $x^2 + px + q = 0$ be the quadratic in form (B), and let x_1 and x_2 be the roots.

Then,

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2 = 0$$

is the equation; and comparing with the former, we have $x_1 + x_2 = -p$, and $x_1x_2 = q$. That is —

The sum of the roots is the coefficient of linear x with changed sign, and the product of the roots is the independent term.

If we put for p and q their values in terms of a, b, and c, we get $x_1 + x_2 = -\frac{c}{a}$, and $x_1x_2 = \frac{b}{a}$.

Since $x_1 + x_2$ is independent of q, the sum of the roots is not affected by changing the value of q. Hence if we move the graph upwards by adding to q (Art. 117, Ex. 2, 2) until L comes to C, the roots become equal and their sum is unchanged. Hence $0C = \frac{1}{2}(0R + 0R')$.

122. Minimum and Maximum.

The y of any point of the graph expresses the value of the function $x^2 - 3x + 1$ for the corresponding value of x; thus for $x = 0$ the value of the function is $0a$, for $x = 01$ the value is $1b$, and for $x = 0C$ the value of the function is CL.

The function has then a least value CL, called its **minimum**, but it has no greatest value.

If we change the signs of the function throughout, we do not affect the roots in any way, but we change the sign of every value of y, and we thus reverse the graph, putting it into the position g.

The function now has a greatest value CL', its **maximum**, but it has no least value.

Hence a quadratic function with the coefficient of x^2 positive has a minimum value but no maximum; and with the coefficient of x^2 negative, it has a maximum, and no minimum.

123. To find the minimum or maximum solution.

It appears, from Art. 121, that when $x = 0C$, the value of the function is either a minimum or a maximum, according as the coefficient of x^2 is positive or negative.

But $0C = \frac{1}{2}(0R + 0R') =$ one-half the sum of the roots $= -\frac{1}{2}p$. Hence the required solution is obtained by substituting for x one-half the coefficient of linear x with changed sign.

Ex. **1**. The minimum value of $x^2 - 3x + 1$ is $(\frac{3}{2})^2 - 3(\frac{3}{2}) + 1$, or $-\frac{5}{4} = CL$.

Ex. **2**. To divide a number into two parts such that their product may be a minimum or a maximum, and to find its value.

Let a be the number, and x one of the parts.

Then $x(a - x)$ is to be a minimum or a maximum.

But the function $ax - x^2$ has a maximum solution (Art. 122). The value is $a \cdot \frac{a}{2} - \left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$, and x is $\frac{a}{2}$, or the number is halved.

Ex. **3**. Two trains A and B are on two roads crossing at right angles and approaching the crossing. A is a miles from the crossing and goes α miles an hour; B is b miles from the crossing and goes β miles an hour. When will they be nearest together, and how far apart will they then be?

Let x be the time in hours. Then—

$a - \alpha x$ is A's distance from the crossing at the end of x hours, and $b - \beta x$ is B's distance.

Their distance apart is $\sqrt{(a-\alpha x)^2 + (b-\beta x)^2}$, and this is to be a minimum. But its square will also be a minimum.

$$\therefore\ (a - \alpha x)^2 + (b - \beta x)^2$$

or

$$x^2(\alpha^2 + \beta^2) - 2x(a\alpha + b\beta) + a^2 + b^2$$

is to be a minimum.

The value of x is $$\frac{a\alpha + b\beta}{\alpha^2 + \beta^2}.$$

If this value be substituted for x, the function reduces to

$$\frac{(b\alpha - a\beta)^2}{\alpha^2 + \beta^2},$$

which is the square of the least distance.

124. We arrive at the results of Art. 123, without using the graph, as follows:

Let $$x^2 + px + q = y.$$

Then $$x = \tfrac{1}{2}(-p \pm \sqrt{p^2 - 4q + 4y}).$$

Now, whatever be the value of $p^2 - 4q$, the expression $\sqrt{p^2 - 4q + 4y}$ cannot be made imaginary by increasing the value of y, while it may be made so by sufficiently diminishing the value of y. If, then, the roots are to be real, y has a minimum value, and this minimum is reached just as the expression $p^2 - 4q + 4y$ is passing from + to −; *i.e.* when the expression is zero.

This gives $x = -\tfrac{1}{2}p$ for the minimum solution; and the value of y is found either by substituting this value of x in the function, or by putting $p^2 - 4q + 4y$ equal to zero and solving for y.

Hence $y = -\tfrac{1}{4}(p^2 - 4q)$; *i.e.* one-fourth of the quantity under the sign $\sqrt{\ }$, in the solution of the equation $x^2 + px + q = 0$, with its sign changed.

Next let $$-x^2 + px + q = y.$$

Then $$x = \tfrac{1}{2}(p \pm \sqrt{p^2 + 4q - 4y}).$$

The part $\sqrt{p^2 + 4q - 4y}$ may be made imaginary by increasing the value of y, but not by diminishing it. Hence the function now admits of a maximum value, but not of a minimum; and the maximum solution as before is given by $x = \frac{1}{2}p$, and the value of the maximum is $y = \frac{1}{4}(p^2 + 4q)$.

As the value of x, which gives a maximum or a minimum solution, does not involve the irrational part of the root, the solution is independent of the nature of the roots, as to whether they are real or imaginary.

125. By studying the graph, we see that for real roots with x^2 positive, the function has a negative value for all values of x lying between the roots, and positive for values lying beyond the roots; and for x^2 negative, the value of the function is positive for all values of x lying between the roots, and negative for all values of x lying beyond the roots.

Ex. For what values of x is $3x^2 - 2x - 1$ positive?

The roots of $3x^2 - 2x - 1 = 0$ are 1 and $-\frac{1}{3}$; and the expression is positive for every value of $x > 1$ and $< -\frac{1}{3}$.

And the expression is negative for every value of $x < 1$ and $> -\frac{1}{3}$.

EXERCISE IX. b.

1. Construct the graphs of —

i. $x^2 - x - 1$. iii. $2 + x - x^2$.

ii. $x^2 + x + 1$. iv. $2 - x - x^2$.

2. Construct the graph of $4x^2 + 4x + 1$.

3. Construct the graph of $\sqrt{4 - x^2}$.

Here we put $y = \sqrt{4 - x^2}$, and hence $y^2 = 4 - x^2$; and y has thus two values differing in sign only for every value of x.

4. Construct the graph of $\sqrt{4x}$.

5. Construct the graph of $x(1 \pm 2)$.

6. Construct the graph of $x^3 + x + 1$.

7. Construct the graph of $x^3 - x^2 - x + 1$.

8. Construct the graph of $\dfrac{x^2 - x + 1}{x + 1}$.

9. Find the maximum or minimum value of the following functions:

i. $x^2 + x - 1$.

ii. $3x^2 - 2x - 1$.

iii. $3x - x^2 + 2$.

iv. $x^2 - 3x$.

10. Find the numerical quantity which exceeds its square by the greatest possible quantity.

11. Divide a number into two parts such that the sum of the squares of the parts may be a minimum.

12. Find the number which when added to its reciprocal gives the smallest sum.

13. Divide a number a into two parts such that the square of one part added to n times the square of the other may be the least possible, and find the sum.

14. Divide a number into two parts such that the difference between the sum of the squares upon the parts and the product of the parts may be a minimum.

15. Divide 20 into two parts such that their product may be 120.

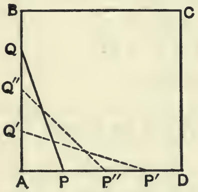
The result is $x = 10 \pm 2i\sqrt{5}$. The factors $2(5 + i\sqrt{5})$ and $2(5 - i\sqrt{5})$ have 20 as their sum and 120 as their product, and thus algebraically the problem is solved. But the complex numbers tell us, in the only way in which algebra can do so, that the question is arithmetically absurd or impossible. We are shown why this is so in Ex. 2 of Art. 123.

16. The sum of a quantity and three times its reciprocal is $\sqrt{3}$; is the quantity real or imaginary?

17. Show that $\dfrac{x^2 - 1}{x^2 - x + 1}$ cannot be greater than $\frac{2}{3}\sqrt{3}$ if x is real.

18. If k is a value of x which makes $\frac{x^2+x-1}{x^2-x+1}$ equal to 2, show that k is a complex number.

19. $ABCD$ is a square; on AD a point P is taken, and on AB a point Q, so that $AP = BQ$. Find AP when the area of the triangle QAP is a given quantity, a^2.



Denote the side of the square by s, and let $AP = BQ = x$. Then $AQ = s - x$, and the area of the triangle APQ is

$$\tfrac{1}{2}x(s-x) = a^2.$$

Thence, $x = \frac{1}{2}(s \pm \sqrt{s^2 - 8a^2})$.

(1) There are two solutions, and therefore two positions for P. This is seen in the diagram, in the triangle $AP'Q'$.

(2) a^2 has a maximum, that is when $8a^2 = s^2$, or the area of the triangle is one-eighth that of the square.

(3) When the triangle has its maximum, the two solutions become one, and $x = \frac{1}{2}s$, as is seen in the triangle $AP''Q''$.

20. In Ex. 19, Q is taken on AB produced, so that $BQ = AP$. Examine the case and show, (1) that there are two solutions for a given area of triangle, (2) that the triangle has a minimum, and (3) that for the minimum $x = -\frac{1}{2}s$, and $a^2 = -\frac{1}{8}s^2$, and explain these negative quantities.

21. Examine Ex. 19, when BQ is so taken that the rectangle contained by AP and BQ is a constant, c^2.

22. In Ex. 6, of IX. a, has the triangle PQC a maximum or a minimum, and what is its value?

23. In Ex. 7, of IX. a, has the triangle PQC a maximum or a minimum, and what is its value?

24. Find the maximum value of the triangle AOC in Ex. 8 of IX. a.

25. A rectangular field is to contain an acre of ground, and a path from one corner to the middle of an opposite side is to be as short as possible. What must be the form and dimensions of the field?

26. An isosceles triangle has its equal sides given, to find the third side when the area is a maximum.

27. Along a road already fenced a rectangular plot of 1 acre is to be inclosed. What must be its form that the cost of fencing the remaining three sides may be the least possible?

28. Two towns, A and B, are on opposite sides of a river 6 miles wide, and B is 10 miles below A. A person can walk along the shore twice as fast as he can row across. At what point must he leave the shore so as to get from A to B in the shortest time?

29. In the equation $\frac{x^2}{a^2}+\frac{b^2}{q^2}\left(1-\frac{xp}{a^2}\right)^2=1$, find the relation between p, q, a, and b when the quadratic in x has equal roots.

30. What are limits between which $x^2-5x+5\frac{1}{4}$ is negative?

31. If α and β denote the roots of $x^2+px+q=0$, find in terms of p and q the value of —

i. $\alpha+\beta$.	iii. $\alpha^2+\beta^2$.	v. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.
ii. $\alpha\beta$.	iv. $\frac{1}{\alpha}+\frac{1}{\beta}$.	vi. $\alpha^3+\beta^3$.

32. If the height of the thermometer is expressed by the function $x^2-2x-29\frac{1}{4}$, where x denotes the number of days counted from a fixed time, for how many days will the thermometer be below zero?

126. Every equation of the form

$$x^{2n}+px^n+q=0$$

can be solved as a quadratic, and be put under the form

$$x^n=\tfrac{1}{2}(-p\pm\sqrt{p^2-4q}).$$

For, put $x^n = y$, and the equation becomes

$$y^2 + py + q = 0;$$

whence the solution follows.

Ex. 1. $x^8 - 3x^4 - 208 = 0.$

Hence $x^4 = \frac{1}{2}(3 \pm 29) = 16$, or -13.

$\therefore x^2 = +4, \ -4, \ +i\sqrt{13}, \ -i\sqrt{13}.$

$x = +2, \ -2, \ +2i, \ -2i, \ \pm\sqrt{i\sqrt{13}}, \ \pm\sqrt{-i\sqrt{13}};$

which gives the 8 roots.

Ex. 2. $x^{\frac{3}{2}} - 7x^{\frac{3}{4}} - 8 = 0.$

Let $y = x^{\frac{3}{4}}$, then $x^{\frac{3}{2}} = y^2$.

$\therefore y^2 - 7y - 8 = 0$, and $y = -1$, or 8.

$\therefore x^{\frac{3}{4}} = -1$, or 8, and $x^3 = 1$, or 4096.

$\therefore x = \sqrt[3]{1}$, or 16.

127. Every equation of the form

$$(fx)^2 + p(fx) + q = 0$$

can be solved as a quadratic, and exhibited in the form

$$fx = \tfrac{1}{2}(-p \pm \sqrt{p^2 - 4q}).$$

Whence, if $fx = 0$ is solvable, the equation can be completely solved.

Ex. 1. Given $(x^2 - x - 1)^2 + 4(x^2 - x) - 6 = 0.$

This can be put into the form —

$$(x^2 - x - 1)^2 + 4(x^2 - x - 1) - 2 = 0.$$

$\therefore x^2 - x - 1 = \frac{1}{2}(-4 \pm \sqrt{24}) = -2 \pm \sqrt{6}.$

Then $x^2 - x + 1 \mp \sqrt{6} = 0.$

Whence $x = \frac{1}{2}(1 \pm \sqrt{\pm 4\sqrt{6} - 3}).$

On account of the double square root, we have by permutation of signs 4 values for x, in all, as we should have, since x rises to the 4th power in the expanded equation.

Ex. 2. $$x^2 - 2x + \sqrt{(x^2 - 2x + 6)} = 6.$$

Add 6 to each side; then

$$x^2 - 2x + 6 + \sqrt{(x^2 - 2x + 6)} = 12.$$

$$\therefore \sqrt{(x^2 - 2x + 6)} = -4, \text{ or } +3.$$

And $$x^2 - 2x + 6 = 16, \text{ or } 9.$$

Whence $$x = 3, \ -1, \ 1 \pm \sqrt{11}.$$

IRRATIONAL EQUATIONS.

128. An equation which involves the variable under a root sign is called an *irrational* equation.

These may always be freed from irrationality, and presented as rational equations; but the rationalizing of them introduces certain uncertainties of solution which it seems impossible to avoid.

A few examples will make this clear.

Ex. 1. Given $\sqrt{a} + \sqrt{x} = \sqrt{ax}$.

This is readily reduced by dividing throughout by $\sqrt{x}$, or by treating the equation as having $\sqrt{x}$ as its variable. Then

$$\sqrt{x}(\sqrt{a} - 1) = \sqrt{a},$$

and $$x = \frac{a}{(\sqrt{a} - 1)^2},$$

and the solution is exact.

Ex. 2. Given $\sqrt{a + x} + \sqrt{a - x} = 2\sqrt{x}$.

As it is always profitable to reduce the number of terms containing $\sqrt{x}$, where possible, we may divide throughout by $\sqrt{x}$, and obtain —

$$\sqrt{\left(\frac{a}{x} + 1\right)} + \sqrt{\left(\frac{a}{x} - 1\right)} = 2.$$

Let $\frac{a}{x} = z$, and square; then

$$2z + 2\sqrt{z^2 - 1} = 4.$$

Divide by 2, transpose z, and square, and

$$z^2 - 1 = 4 - 4z + z^2.$$

$$\therefore z = \infty \text{ (Art. 77), and } z = \tfrac{5}{4}.$$

Whence $\qquad x = 0$, and $\frac{4a}{5}$.

The root $\frac{4a}{5}$ satisfies the irrational equation; but the root $x = 0$ does not satisfy the equation, and although obtained by the legitimate transformations of algebra, is not really a root of the given equation, as the test of a root is that it shall render the equation an identity when substituted for the variable. The root $x = 0$, however, satisfies the equation $\sqrt{a + x} - \sqrt{a - x} = 2\sqrt{x}$, which differs from the other in a single sign.

The probable explanation of this peculiarity is that owing to the disappearance of certain negative signs, in rationalizing the equations by squaring, both these equations reduce to the same rational form; and as far as this form is concerned, there is nothing by which we can know from which of the two irrational equations it has come.

Hence there is no reason why the roots obtained should not satisfy one of the irrational equations as well as the other. But it is quite evident that both roots cannot satisfy both irrationals.

Ex. 3. $\qquad \sqrt{x} + \sqrt{a - \sqrt{ax + x^2}} = \sqrt{a}.$

Transposing $\sqrt{x}$, squaring, and cancelling a,

$$\sqrt{ax + x^2} = 2\sqrt{ax} - x.$$

$\sqrt{x}$ is a divisor of this. Hence (Art. 76) $\sqrt{x}=0$, and $x=0$, and

$$\sqrt{a+x}=2\sqrt{a}-\sqrt{x}.$$

Squaring, $$a+x=4a+x-4\sqrt{ax}.$$

$$\therefore\ x=\infty \text{ (Art. 81, } Cor \text{ 2), and } 3a=4\sqrt{ax}.$$

Whence $$x=\tfrac{9}{16}a.$$

Of these roots, 0 and $\frac{9}{16}a$ satisfy the given irrational equation, while $x=\infty$ satisfies the equation

$$\sqrt{x}-\sqrt{a+\sqrt{ax+x^2}}=\sqrt{a},$$

in which the signs before two root-symbols are changed.

Ex. 4. Given $$3x+\sqrt{30x-71}=5.$$

Transposing $3x$ and squaring, we obtain

$$x=4, \text{ or } 2\tfrac{2}{3}.$$

Neither of these roots satisfies the given irrational equation, they being roots of

$$3x-\sqrt{30x-71}=5.$$

Whether the given equation has an expressible root or not, it cannot be found by the usual methods of solution.

EXERCISE IX. c.

1. Solve the following —

i. $a+x+\sqrt{2ax+x^2}=b$. iii. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$.

ii. $a+x+\sqrt{a^2+x^2}=b$. iv. $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$.

v. $\sqrt[m]{a+x}=\sqrt[2m]{\{x^2+5ab+b^2\}}$.

2. Find x, when $\dfrac{1-ax}{1+ax}\cdot\sqrt{\left(\dfrac{1+bx}{1-bx}\right)}=1$.

3. Find x, when $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}=\dfrac{1+\sqrt{1+x}}{1-\sqrt{1-x}}$.

4. Find x, when $1+\sqrt{\left(1-\frac{a}{x}\right)}=\sqrt{\left(1+\frac{a}{x}\right)}$.

5. Find x, when $a+x+\sqrt{2ax+x^2}=b^2\{a+x-\sqrt{2ax+x^2}\}$.

6. Solve the equation $6x-4\sqrt{6x+1}=x^2-2x-4$.

7. Solve the equation $x(\sqrt{x}+1)^2=8(x+\sqrt{x})+240$.

8. Solve the equation $a+x\sqrt{1+a^2}=a\sqrt{1-x^2}+x\sqrt{1-a^2}$.

9. Solve the equation $9x+8+2x\sqrt{9x+4}=15x^2+4$.

10. Solve the equation $x+\sqrt{(x^2-ax+b^2)}=\frac{x^2}{a}+b$.

CHAPTER X.

Indeterminate and Simultaneous Equations of the First Degree. — Simultaneous Quadratics.

129. When a positive integral equation contains a single variable, the value or values of that variable may be found, theoretically at least, in terms of the constants. But if the function contains two variables, the value of either will contain not only constants, but the other variable, and thus this value will not be constant, but variable, and therefore arbitrary.

Thus if $3x+2y=6$, $x=2-\frac{2}{3}y$, and y may take as many values as we please, and to every value of y will correspond a single value of x; and, conversely, to every value of x will correspond a single value of y. Such equations are accordingly called *Indeterminate*, and as we have seen in Art. **117**, they have a graph which is a straight line.

The study of Indeterminate equations is practically the study of their graphs, and as a consequence Indeterminate equations in relation to their graphs form the subject-matter of the great body of higher geometry known as *Analytic Geometry.*

130. Linear Indeterminate equations are considered here only under the restrictions that the corresponding values of x and y shall be positive integers.

The subject is best discussed in examples.

Ex. 1. To find positive integral values for x and y which shall satisfy the equation $7x + 11y = 103$.

This gives $x = \frac{103 - 11y}{7}$, which is to be integral.

But $\frac{103 - 11y}{7} = 14 - y + \frac{5 - 4y}{7}$; and as y is to be integral, $14 - y$ is integral, and therefore $\frac{5 - 4y}{7}$ is integral.

Also, as the product of integers is integral, $2 \times \frac{5 - 4y}{7}$, or $\frac{10 - 8y}{7}$ is integral; *i.e.* $\frac{3 - y}{7}$ is integral.

The purpose in multiplying by 2 is to make the coefficient of y greater by 1 than a multiple of 7, so that after casting out all integers the coefficient of y may be 1.

Now, put $\frac{3 - y}{7} = p$. Then $y = 3 + 7p$; and putting this value for y in the original equation gives $x = 10 - 11p$.

Therefore $\left.\begin{aligned} x &= 10 - 11p \\ y &= 3 + 7p \end{aligned}\right\}$ is the *general* solution.

The particular solutions are got by giving to p any allowable integral values, provided such values do not make x or y negative.

We readily see that in the present question p can have only one value, zero; and $x = 10$, $y = 3$, is the only solution.

Ex. 2. Can $1 be paid in 9-cent pieces and 7-cent pieces, and if so, how?

Let $x =$ the number of 9-cent pieces, and $y =$ the number of 7-cent pieces.

Then $9x + 7y = 100$ is our equation.

The solution gives $y = 4 - 9p$ and $x = 8 + 7p$.

When
$$\begin{aligned} p &= 0, \quad -1, \\ x &= 8, \quad\ \ 1, \\ y &= 4, \quad\ 13. \end{aligned}$$

There are thus two solutions, one by 8 9-cent pieces with 4 7-cent pieces; the other, by 1 9-cent piece with 13 7-cent pieces.

131. It will be noticed that in the general solution the coefficient of p in the value of x is the coefficient of y in the equation, and the coefficient of p in the value of y is that of x in the equation, one of the signs being changed.

Hence in the equation $ax - by = c$, p will have the same sign in the values of both x and y, and the number of solutions will be unlimited.

Ex. To find solutions of $7x - 5y = 23$, we easily obtain

$$x = 4 + 5p,\ y = 1 + 7p.$$

Therefore when $\quad p = 0,\ 1,\ 2,\ 3,\ 4 \cdots$

$$x = 4,\ 9,\ 14,\ 19,\ 24 \cdots$$

$$y = 1,\ 8,\ 15,\ 22,\ 29 \cdots$$

132. In the equation $ax \pm by = c$ there can be no solution in positive integers if a and b have a common factor which is not also a factor of c.

For, let $\quad a = mf$, and $b = nf$.

Then $\quad ax \pm by = mfx \pm nfy = c$;

and $\quad mx \pm ny = \dfrac{c}{f}$.

But $\dfrac{c}{f}$ is a fraction by hypothesis, and is not the sum or difference of two integers.

133. The following problem is nearly related to one of the three preceding articles.

Ex. To find an integral number which when divided by 3 leaves 1, by 5 leaves 4, and by 7 leaves 2.

If x denotes the number, x is evidently of any one of the forms

$3m+1$, $5n+4$, or $7p+2$, and these are to represent the same number; we have

$$3m+1=5n+4=7p+2.$$

$$\therefore m=\frac{5n+3}{3}=\text{an integ., and } \frac{n}{3}=\text{an integ.}=m,$$

and $$n=3m.$$

Again, $$15m+4=7p+2, \text{ or } m=\frac{7p-2}{15}=\text{an integ.}$$

$$\therefore \frac{p-11}{15}=\text{an integ.}=m, \text{ and } p=15m+11.$$

Hence $x=7p+2=105m+79$, which is the general solution. If $m=0$, we have 79 as the lowest number satisfying the conditions.

The following method of solution is also convenient. Let x be the number.

Then $\frac{x-1}{3}$, $\frac{x-4}{5}$, and $\frac{x-2}{7}$ are all integers. Put $\frac{x-1}{3}=p$, and $x=3p+1$. Substitute this value of x in the second fraction, and $\frac{3p-3}{5}$ is an integer, or $\frac{p-1}{5}$ is an integ. $=q$.

$$\therefore p=5q+1, \text{ and } x=15q+4.$$

Substitute this new value of x in the third fraction, and

$$\frac{15q+2}{7} \text{ is an integ., or } \frac{q+2}{7} \text{ is an integ.}=r.$$

Then $q=7r-2$, and $x=105r-26$, which is the general solution. $r=1$ gives 79 as the lowest number satisfying the conditions.

EXERCISE X. a.

1. Find positive integral solutions to the following —

i. $3x+7y=101$. iii. $45x-13y=38$.

ii. $13x+17y=200$. iv. $x-11y=48$.

2. Find multiples of 23 and 15 which differ by 1; which differ by 2; by 3.

3. How can I measure off a length of 4 feet by means of two measures, one 7 inches long and the other 13 inches long?

4. I have nothing but 4-pound and 7-pound weights. How can I weigh exactly 45 pounds?

5. A company of soldiers when arranged 4 abreast lacks 1 man, when 5 abreast it lacks 2 men, when 6 abreast it has 3 too many, and when 7 abreast it forms a complete block. How many men, at least, are in the company?

6. A wall 27 ft. 9 in. long is to be panelled with two widths of boards, 8 in. and 5 in. wide. How many of each kind must be used so that —

i. The narrow boards may exceed the wide by the least number possible?

ii. The wide may exceed the narrow by the least number possible?

iii. The whole number of boards may be the least?

LINEAR SIMULTANEOUS EQUATIONS.

134. The equations $ax+by+c=0$ and $a_1x+b_1y+c_1=0$ are both indeterminate, and being linear, both have straight lines as their graphs (Art. 117, Ex. 1).

These straight lines have some common point, their point of intersection, and at this point the corresponding values of x and y must be such as to satisfy both equations; and as the graphs have only one common point, there is only one such set of values.

When two equations are given, and it is required to find corresponding values of x and y that shall satisfy both, the equations are called **simultaneous**, and these particular values of x and y form the *solution* to the set of two equations.

Ex. The two equations $2x+3y=9$ and $3x+2y=11$ are satisfied by the values $x=3$, $y=1$, and by no other values.

135. Problem. To solve a set of two simultaneous equations with two variables. The methods will be explained through examples.

Let $4x - 3y = 26$ and $3x + 5y = 5$ be the equations.

First method. — By addition and subtraction.

Add 5 times the first equation to 3 times the second, and the coefficients of y, being equal with opposite signs, disappear, and we have left

$$29x = 145; \text{ whence } x = 5.$$

Again, to get rid of x, we subtract 4 times the second equation from 3 times the first, and obtain

$$-29y = 58; \text{ and hence } y = -2.$$

Second method. — By substitution.

The first equation gives $x = \frac{26 + 3y}{4}$; and substituting this for x in the second gives

$$\frac{78 + 9y}{4} + 5y = 5, \text{ or } 78 + 29y = 20.$$

Whence $$29y = -58, \text{ and } y = -2.$$

Next substitute -2 for y in either of the equations, and we get the value of x.

Third method. — By comparison.

The values of x found from the two equations are $\frac{26 + 3y}{4}$ and $\frac{5 - 5y}{3}$; and as these must be equal, we have

$$78 + 9y = 20 - 20y, \text{ or } y = -2.$$

Similarly, $$y = \frac{4x - 26}{3} = \frac{5 - 3x}{5}.$$

$$\therefore\ 20x - 130 = 15 - 9x, \text{ or } x = 5.$$

Fourth method. — By an arbitrary multiplier.

Multiply one of the equations, the first, by an arbitrary multiplier m, and add to the other.

We have $(4m+3)x-(3m-5)y=26m+5.$

As m is arbitrary, we may give to it such a value as will make either of the brackets zero.

If $m=\frac{5}{3}$, $(\frac{20}{3}+3)x=26\times\frac{5}{3}+5$, or $x=5$.

If $m=-\frac{3}{4}$, we similarly obtain $y=-2$.

EXERCISE X. b.

1. $7(x-5)=y-2$, $4y-3=\frac{1}{3}(x+10)$, to find x and y.

2. $(x+5)(y+7)=(x-1)(y-9)+104$, and $2x+10=3y+1$, to find x and y.

3. $x-a=c(y-b)$, $a(x-a)+b(y-b)+abc=0$, to find x and y.

4. $\frac{x}{a}+\frac{y}{b}=1=\frac{x-a}{b}+\frac{y-b}{a}$, to find x and y.

5. $2s=n(a+z)$, $d(n-1)=z-a$, to find a relation not containing n.

6. A and B have \$500 between them. A gains from B $\frac{1}{3}$ of B's money and \$50. B then gains from A $\frac{1}{4}$ of A's original money and \$50, and they then have the same amount. What had they to begin with?

7. A fraction is such that if 2 be added to its numerator it becomes $\frac{1}{2}$, and if 1 be added to its denominator it becomes $\frac{1}{3}$. Find the fraction.

8. A number of two digits has the sum of the digits 12, and if 6 times the first digit be subtracted from the number, the digits are exchanged. Find the number.

9. There are two kinds of coin such that a and b pieces respectively are equal to \$1. How many pieces of each kind must be taken so that the value of c pieces together may be one dollar?

10. A farm was taxed at 30 cents an acre, and the tenant being allowed 10 % off his rent found the allowance to amount to 15 dollars more than the taxes. The next year the taxes were doubled, and the farmer was allowed 15 % off his rent, which just paid his taxes. What was the rent of the farm, and how many acres did it contain?

SET OF THREE LINEAR EQUATIONS.

136. Denote the three equations by A, B, and C, and let the variables be x, y, z.

Find from A the value of x in terms of y, z, and the constants, and substitute this value for x in equations B and C. We are then said to have *eliminated* x between A and B, and also between A and C; and we have two new equations, D and E, which contain only y and z as variables.

Thus in eliminating one variable we reduce the number of our equations by one.

Now eliminate y between D and E, and we are left with a single equation F, which is just sufficient to determine z in terms of the constants. Hence we readily find y and x.

Ex. Let A be $x + 3y - 2z = 1,$

B be $3x - 2y + z = 5,$

C be $2x + 4y - 3z = 1.$

From A we have $x = 1 - 3y + 2z.$

Substituting this value for x gives —

in B, $3 - 9y + 6z - 2y + z = 5$, or $-11y + 7z = 2$ (D)

in C, $2 - 6y + 4z + 4y - 3z = 1$, or $-2y + z = -1$ (E)

And eliminating y between D and E gives

$$3z = 15, \text{ or } z = 5.$$

Thence E gives $y = \frac{z+1}{2} = 3$, and $x = 1 - 9 + 10 = 2$.

$$\therefore\ x = 2,\ y = 3,\ z = 5 \text{ is the solution.}$$

This method of eliminating x and y is always sufficient, but it may not always be the most convenient, and any method of elimination will answer if carried out in proper form. Thus whatever process of elimination be employed, the *same* letter must be eliminated between one of the equations and each of the others. The following method is very convenient and gives the same results as the last.

Subtract B from $3A$. This gives $11y - 7z = -2$ . . . (D)

Subtract C from $2A$. This gives $2y - z = 1$ . . . . (E)

Subtract $11(E)$ from $2(D)$, and $-3z = -15$, or $z = 5$.

137. From the preceding article it appears that when we eliminate a variable from a system of equations we lose one equation; and conversely, that by combining one of the equations with each of the others of the set we can eliminate one variable from the set.

Hence the two following results:

(1) That n equations are just sufficient to determine n variables; and conversely, that n variables can be determined from a system of n equations, provided the equations be *independent, i.e.* such that any one cannot be derived from the others.

(2) That with n variables, and $n-1$ equations, the final result will be a single equation containing two variables, and be thus indeterminate. And hence a system of $n-1$ equations with n variables is indeterminate.

Thus the two equations with three variables,

$$3x + 2y - 6z = 4, \text{ and } 4x + y + z = 10,$$

give, by eliminating x, the single indeterminate equation

$$5y - 27z = 36.$$

If this be solved for positive integers, we may find one or more systems of positive integral values for x, y, and z, which will satisfy the two given equations.

138. Let there be n variables and $n+1$ equations; an important case requiring consideration.

As the $n+1$ equations are sufficient to determine $n+1$ variables, we make up this number of variables by taking one of the constants, provided it be literal, and considering it as a variable. After eliminating the n true variables, we have a single equation in which this pseudo-variable is the only one occurring; or, in other words, we have a single equation expressing a necessary relation amongst the constants. This equation is called the **Eliminant** of the system.

Ex. Given $3x + 2y = a$, $2x - 4y = b$, and $x + 5y = c$, to find the eliminant.

Eliminating x gives

$$13y = 3c - a, \text{ and } 14y = 2c - b.$$

And now eliminating y, we have

$$14a - 13b - 16c = 0,$$

as the necessary relation between a, b, and c that the three equations may be *compatible*.

If a, b, and c are numbers which do not satisfy the eliminant, the equations are *incompatible*, and cannot be satisfied by any one set of corresponding values of x and y.

On the other hand, if a, b, and c are numbers which do satisfy the eliminant, one of the equations is derivable from the other two, and thus expresses no relation but what is already given by the other two. The equations are then not independent, and one of them is *redundant.*

139. An important system is one of homogeneous equations, in which the number of variables is greater by one than the number of equations.

Let $a_1x + b_1y + c_1z = 0 = a_2x + b_2y + c_2z$ be such a system.

Dividing through by y, these take the form

$$a_1\frac{x}{y} + c_1\frac{z}{y} + b_1 = 0 = a_2\frac{x}{y} + c_2\frac{z}{y} + b_2,$$

which is a system of two equations, with the two variables $x:y$ and $z:y$.

Solving, we get $$\frac{x}{y} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}.$$

$$\therefore \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}, \text{ by symmetry.}$$

Or $$x : y : z = b_1c_2 - b_2c_1 : c_1a_2 - c_2a_1 : a_1b_2 - a_2b_1.$$

And the variables are any quantities proportional to the denominators of the fractions.

Ex. Let $2x + 6y - 3z = 0 = 4x - 3y + z.$

Then $x : y : z = 3 : 14 : 30,$

and the numbers 3, 14, and 30, or any multiples of these, will satisfy the equations.

If in these equations we make $z =$ a constant, both x and y take fixed values, and we have the common case of two equations with two variables.

140. Problem. To solve a system of three linear equations with three variables by arbitrary multipliers.

Let the system be

$$a_1x + a_2y + a_3z = d_1,$$
$$b_1x + b_2y + b_3z = d_2,$$
$$c_1x + c_2y + c_3z = d_3,$$

Multiply the equations by the arbitrary multipliers l, m, and n, respectively, and add; then

$$(la_1+mb_1+nc_1)x+(la_2+mb_2+nc_2)y+(la_3+mb_3+nc_3)z$$
$$= ld_1 + md_2 + nd_3.$$

As l, m, n are arbitrary, we may so take their values as to make any two of the brackets zero.

Thus to eliminate y and z we must have

$$la_2 + mb_2 + nc_2 = 0 = la_3 + mb_3 + nc_3.$$

The solution of this is, by the preceding article,

$$\frac{l}{b_2c_3 - b_3c_2} = \frac{m}{c_2a_3 - c_3a_2} = \frac{n}{a_2b_3 - a_3b_2}.$$

And l, m, n are any quantities proportional to the denominators. Naturally we take as the multipliers the denominators themselves.

The reader will find, upon trial, that these multipliers cause the coefficients of y and z each to become zero.

We notice that the multiplier for any equation does not contain any coefficient from that equation or any coefficient of the variable to be determined. Thus the multiplier for the first equation is $b_2c_3 - b_3c_2$, and does not contain a suffix 1, or an a, etc. A little observation on the forms of these multipliers is better than any description.

Ex. **1.** Given
$$x + 2y + 3z = 9,$$
$$2x + y + z = 14,$$
$$3x + 2y + 5z = 3.$$

To eliminate y and z the multipliers are
$$l = 3,\ m = -4,\ n = -1.$$
$$\therefore\ 3x - 8x - 3x = 27 - 56 - 3,$$
or
$$-8x = -32, \text{ and } x = 4.$$

To eliminate x and y the multipliers are
$$l = 1,\ m = 4,\ n = -3.$$
$$\therefore\ 3z + 4z - 15z = 9 + 56 - 9,$$
or
$$-8z = 56, \text{ and } z = -7.$$

Ex. **2.** Given
$$ax + y + z = 0,$$
$$x + ay + z = 1,$$
$$x + y + az = -1.$$

The multipliers for eliminating y and z are $a^2 - 1$, $1 - a$, and $1 - a$.

Thence, $\{a(a^2 - 1) + 2(1 - a)\}x = 0$, and $x = 0$.

Similarly, we find $y = \dfrac{1}{a - 1}$, $z = \dfrac{1}{1 - a}$.

141. With four or more equations, multipliers may also be found which will eliminate all the variables but one, but these multipliers are too complex for convenient use. In the chapter on Determinants it is shown that all sets of linear equations are solvable upon the same general principle.

A set of four equations may be dealt with as follows:

Ex. Let them be
$$\overline{x + y} + z + u = 4 \quad \ldots\ldots \quad (A)$$
$$\overline{x + 2y} + 3z + 4u = 10 \quad \ldots\ldots \quad (B)$$
$$\overline{2x + y} + 3z + u = 7 \quad \ldots\ldots \quad (C)$$
$$\overline{3x + 3y} + z + 2u = 10 \quad \ldots\ldots \quad (D)$$

Take the first three equations, and consider the x and y part as forming a single term.

The multipliers for eliminating z and u are -9, 2, 1; and these give $5x + 4y = 9$.

Now take the last three equations; the multipliers are 5, -2, -9; and these give $26x + 19y = 54$.

From these two new equations we find $x = 5$, $y = -4$. The values of u and z are then readily found to be $z = -1$, $u = 4$.

EXERCISE X. c.

1. Solve the set, $2x + 4y + 5z = 49$, $3x + 5y + 6z = 64$, $4x + 3y + 4z = 55$.

2. Solve the set, $2x - 3y + z = 2$, $x + y - 2z = 1$, $3x + 2y - 3z = 5$.

3. Solve the set, $x - y - 2z = 3$, $2x + y - 3z = 11$, $3x - 2y + z = 4$.

4. Solve the set, $ax + by - az = b(a+b)$, $bx - ay + z = b(b-a)$, $x + 2y - 2z = 4x - b$.

5. Solve the set, $x + y + z = 0$, $(a+b)x + (b+c)y + (c+a)z = 0$, $abx + bcy + caz = 1$.

6. Solve the set, $\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = 7\frac{3}{5}$, $\frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = 10\frac{1}{6}$, $\frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = 16\frac{1}{10}$.

7. If $2x + 3y = a$, $x - y = b$, $x + 2y = c$, find the eliminant.

8. Find the eliminant of $ax + y = 1$, $bx + 3y = 6$, $cx + 5y = 10$.

9. Find the eliminant of $3x + 2y + a = 0$, $x - 3y + b = 0$, $2x + y - c = 0$.

10. Solve the set, $3x - 2y + 5z = 14$, $2x + y - 8z = 10$, $8x - 3y + 2z = 38$; and explain the cause of any difficulties.

11. If $a_1x + b_1y + c_1z = a_2x + b_2y + c_2z = a_3x + b_3y + c_3z = 0$, show that $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$.

12. When $x+y-z-u=2x-2y+z+u=3x-y+3z-u=0$, find four numbers having the ratios $x:y:z:u$.

13. What does Ex. 12 reduce to when $u=1$?

14. Solve the set, $a+2b+3c+4d=29$, $4a+b+2c+3d=23$, $3a+4b+c+2d=25$, $2a+3b+4c+d=23$.

15. Solve the set, $x+by-az=\frac{b}{a}$, $ax+y-\frac{1}{a}z=a^2$, $\frac{1}{a}x+ay-z=1$.

16. Solve the set, $a^2x+ay+z=-a^3$, $b^2x+by+z=-b^3$, $c^2x+cy+z=-c^3$; and reduce the values of the variables to lowest terms.

17. Find the eliminant of $ax+by+cz=bx+cy+az=cx+ay+bz=0$.

18. Solve the set, $\frac{1}{x}\left(\frac{2}{z}+\frac{1}{yz}\right)+\frac{1}{y}\left(\frac{3}{x}+\frac{1}{z}\right)=\frac{1}{y}\left(\frac{2}{x}-\frac{6}{xz}\right)+\frac{1}{z}\left(\frac{3}{y}+\frac{1}{x}\right)=\frac{1}{z}\left(\frac{2}{y}-\frac{7}{xy}\right)+\frac{1}{x}\left(\frac{3}{z}+\frac{1}{y}\right)=0$.

19. Given $x(y+z)=a^2$, $y(z+x)=b^2$, $z(x+y)=c^2$, and $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}$, to find the relation connecting a, b, c.

(Find the values of x, y, and z from the first three equations, and substitute these in the fourth.)

SIMULTANEOUS QUADRATICS.

142. A system consisting of one quadratic and one linear can always be solved.

The most general type of a quadratic equation of two variables may be written

$$ax^2+by^2+hxy+gx+fy+c=0.$$

And any linear of two variables may be written

$$x+py+r=0.$$

If we substitute for x from the linear into the quadratic, we have

$$a(py+r)^2 + by^2 - hy(py+r) - g(py+r) + fy + c = 0,$$

a quadratic from which to determine y.

Ex. To find two numbers such that the sum of their squares and their product is a and the sum of the numbers is b.

We have, $x^2 + y^2 + xy = a$, and $x + y = b$, where x and y denote the numbers.

Substituting for x from the linear into the quadratic,

$$(b-y)^2 + y^2 + (b-y)y = a,$$

or $$y^2 - by = a - b^2.$$

Whence $$y = \tfrac{1}{2}(b \pm \sqrt{4a - 3b^2}),$$

and $$x = \tfrac{1}{2}(b \mp \sqrt{4a - 3b^2}).$$

If $4a < 3b^2$, the numbers are complex.

The equation $x^2 + y^2 + xy = a$ and $x + y = b$ are symmetrical in x and y; and whenever this is the case, the values of x and y must be interchangeable, so that having the two values of y, we have also the two values of x.

Thus if $a = 19$, and $b = 5$, we have $y = 3$ or 2, and $x = 2$ or 3.

143. A system of two quadratics with two variables does not in general admit of being solved as a quadratic, since substituting the value of a variable from one of the equations into the other will in general give rise to an equation of four dimensions.

Thus the system $x^2 + y = a$, and $y^2 + x = b$ gives, by substituting for y, $x^4 - 2ax^2 + x = b - a^2$, a quartic equation.

There are, however, many cases in which a sufficient relation exists between the forms of the equations, to make a solution possible without going beyond the quadratic.

No general list of such can be given, and no very general rules of procedure can be laid down for such cases when they occur. Practice and observation are the only keys to success.

The following are given by way of illustration:

144. When two quadratics have a common linear factor in the portions involving the variables, they can be solved.

For let A be the common linear factor, and let C and C' be the independent terms.

Then the equations are of the forms $AB=C$ and $AB'=C'$, and B and B' must be linear factors.

Dividing the first equation by the second, we have

$$\frac{B}{B'}=\frac{C}{C'}, \text{ or } B=B'\cdot\frac{C}{C'}.$$

And as $\frac{C}{C'}$ is a constant, one variable is linearly expressible in terms of the other. And hence by substitution we obtain a quadratic for finding one of the variables.

Ex. Given $$3x^2-4y^2+4xy=-21,$$

$$12x^2+2y^2-11xy=-3.$$

The first equation is $(3x-2y)(x+2y)=-21,$

and the second is $(3x-2y)(4x-y)=-3.$

Dividing, $$\frac{x+2y}{4x-y}=7, \text{ and } \therefore\ y=3x.$$

Substituting in the first equation,

$$21x^2=21; \text{ and } x=\pm 1.$$

Thence $$y=\pm 3.$$

145. When the equations are homogeneous in the parts involving the variables, they can often be readily solved by putting $y = ux$, and then dividing one equation by the other. This gives a quadratic for finding u, and hence a known linear relation between the variables.

Since u will, in general, have two values, we will get two quadratics to determine x, and hence x will have in all four values, as it should have. So also y will have four values.

Ex. Given
$$x^2 + xy + 4y^2 = 6,$$
$$3x^2 + 8y^2 = 14.$$

Let $y = ux$, and divide equation by equation; then

$$\frac{x^2 + ux^2 + 4u^2x^2}{3x^2 + 8u^2x^2} = \frac{3}{7},$$

or
$$4u^2 + 7u = 2, \text{ and } u = -2, \text{ or } \tfrac{1}{4}.$$

$\therefore y = -2x$, or $\frac{1}{4}x$; and substituting these values in one of the equations, the second by preference, as being the simpler,

$$3x^2 + 32x^2 = 14, \text{ and } 3x^2 + \tfrac{1}{2}x^2 = 14.$$

$$\therefore x = \pm \tfrac{1}{5}\sqrt{10}, \ = \pm 2,$$

and
$$y = \mp \tfrac{2}{5}\sqrt{10}, \ = \mp \tfrac{1}{2}.$$

146. When two variables are involved symmetrically, it frequently simplifies the solution to assume two new variables whose sum shall be one of the original variables, and their difference the other.

Ex. Given
$$x^2 + y^2 + x + y = 8,$$
$$x + y + xy = 5.$$

The variables being symmetrically involved, assume $x = u + v$, and $y = u - v$.

Then
$$u^2 + v^2 + \ \ u = 4,$$

and
$$u^2 - v^2 + 2u = 5.$$

Adding, $2u^2 + 3u = 9$; and $u = \frac{3}{2}$, or -3.

Substituting these values for u in one of the new equations, the first, we get

$$\tfrac{9}{4} + v^2 + \tfrac{3}{2} = 4, \text{ and } 9 + v^2 - 3 = 4.$$

$$\therefore v = \pm \tfrac{1}{2}, = \pm i\sqrt{2}.$$

And $$x = 2 \text{ or } 1, = -3 \pm i\sqrt{2}.$$

And y being symmetrical with x has the same values; then

$$x = 2,\ y = 1;\ x = 1,\ y = 2;\ x = -3 + i\sqrt{2};$$
$$y = -3 - i\sqrt{2};\ x = -3 - i\sqrt{2};\ y = -3 + i\sqrt{2};$$

are the four sets of corresponding values of x and y.

The present equations may be otherwise solved as follows:

Add twice the second equation to the first, and it becomes

$$(x + y)^2 + 3(x + y) = 18,$$

whence $$x + y = 3, \text{ or } -6.$$

Then from the second, $xy = 2$, or 11,

and $$(x - y)^2 \equiv (x + y)^2 - 4xy = 1, \text{ or } -8.$$

$$\therefore x - y = \pm 1, \text{ or } 2i\sqrt{2}.$$

Whence $$x \equiv \tfrac{1}{2}(x + y + x - y) = 2, \text{ or } -3 + i\sqrt{2};$$
$$y \equiv \tfrac{1}{2}(x + y - \overline{x - y}) = 1, \text{ or } -3 - i\sqrt{2};$$

and the values of x and y being interchangeable give the four values as before.

147. Various devices are employed to obtain solutions of simultaneous quadratics and other simultaneous equations. These cannot be given in detail, but will be illustrated in the following examples.

Ex. 1. Given $x^5 + y^5 = 275,\ x + y = 5.$

$$(x + y)^5 = x^5 + 5xy(2x^2y + 2xy^2 + x^3 + y^3) + y^5 = 3125.$$

Subtracting $x^5 + y^5 = 275$ leaves

$$5\,xy\,(x^3 + 2\,x^2y + 2\,xy^2 + y^3) = 2850.$$

$$\therefore\ x^3 + 2\,x^2y + 2\,xy^2 + y^3 = \frac{570}{xy}.$$

But $$(x + y)^3 = x^3 + 3\,x^2y + 3\,x^2y^2 + y^3 = 125.$$

$$\therefore\ xy(x + y) = 5\,xy = 125 - \frac{570}{xy}.$$

Whence $$(xy)^2 - 25\,xy = -114,$$

and $$xy = 19,\ \text{or}\ 6.$$

Then having xy and $x + y$, we readily find

$$x = 2,\ \text{or}\ 3,\ \text{or}\ \tfrac{1}{2}(5 \pm i\sqrt{51});$$

$$y = 3,\ \text{or}\ 2,\ \text{or}\ \tfrac{1}{2}(5 \mp i\sqrt{51}).$$

Ex. 2. Given $$x + \sqrt{x^2 - y^2} = \frac{8}{y}(\sqrt{x + y} + \sqrt{x - y}),$$

and $$(x + y)^{\frac{3}{2}} - (x - y)^{\frac{3}{2}} = 26.$$

Put $$x + y = 2\,s^2,\ \text{and}\ x - y = 2\,t^2.$$

This reduces the equations to

$$(s + t)^2(s^2 - t^2) = 8\,(s + t)\sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and $$2\,(s^3 - t^3)\sqrt{2} = 26 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Divide (a) by $s + t$, and $s + t = 0$, or $s = -t$.

Multiplying out the quotient,

$$s^3 - t^3 + st\,(s - t) = 8\sqrt{2}.$$

And substituting $s^3 - t^3$ from (b),

$$st\,(s - t) = \tfrac{3}{2}\sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Dividing (b) by (c),

$$\frac{s^2 + st + t^2}{st} = \frac{13}{3}.$$

Whence $$\frac{(s + t)^2}{(s - t)^2} = \frac{16}{4},\ \text{and}\ \frac{s + t}{s - t} = \pm 2.$$

$$\therefore\ s = 3\,t.$$

From this we readily obtain

$$s = \tfrac{3}{2}\sqrt{2},\ t = \tfrac{1}{2}\sqrt{2},\ x = 5,\ y = 4.$$

Ex. **3**. Given $x(y+z) = a,\ y(z+x) = b,\ z(x+y) = c.$

Adding the first and second and subtracting the third,

$$xy + xz + yz + yx - zx - zy = 2xy = a + b - c.$$

Similarly, $2yz = b + c - a,\ 2zx = c + a - b.$

Multiplying together two of these new equations and dividing by the third,

$$\frac{2xy \cdot 2yz}{2zx} = 2y^2 = \frac{(a+b-c)(b+c-a)}{c+a-b}.$$

$$\therefore\ y = \sqrt{\left\{\frac{(a+b-c)(b+c-a)}{2(c+a-b)}\right\}},$$

with symmetrical expressions for z and x.

Ex. **4**. Given $x^2 - yz = a,\ y^2 - zx = b,\ z^2 - xy = c.$

Then, $(x^2 - yz)^2 - (y^2 - zx)(z^2 - xy) = a^2 - bc,$

i.e. $x(x^3 + y^3 + z^3 - 3xyz) = a^2 - bc.$

$$\therefore\ x^3 + y^3 + z^3 - 3xyz = \frac{a^2 - bc}{x} = \frac{b^2 - ca}{y} = \frac{c^2 - ab}{z},$$

since the left-hand expression is symmetrical in x, y, and z.

Thence each fraction $= \sqrt{\left\{\dfrac{(a^2-bc)^2 - (b^2-ca)(c^2-ab)}{a}\right\}}.$

$$\therefore\ x = \sqrt{\left\{\frac{a(a^2-bc)^2}{(a^2-bc)^2 - (b^2-ca)(c^2-ab)}\right\}},$$

$$= \frac{a^2 - bc}{\sqrt{(a^3 + b^3 + c^3 - 3abc)}},$$

with symmetrical expressions for y and z.

EXERCISE X. d.

1. Given $x : y = 3 : 2$, and $(2 - x)^2 + (1 - y)^2 = 25$, to find all the values of x and y.

2. Given $x + y = a$, and $xy(x^2 + y^2) = b$.

3. Given $xy = 750$, and $x : y = 10 : 3$.

4. Given $x + y = xy = x^2 - y^2$; that is, to find the two quantities for which the sum, product, and difference of squares may be the same.

5. Given $(x - y)(x^2 - y^2) = 160$; $(x + y)(x^2 + y^2) = 580$.

6. Given $x + y + xy = 34$; $x^2 + y^2 - (x + y) = 42$.

7. Given $x^2 + y^2 + x + y = 330$; $x^2 - y^2 + x - y = 150$.

8. Given $4x^2 + y^2 + 4x + 2y = 6$; $2xy = 1$.

9. Given $x + y = 18$; $x^4 + y^4 = 14096$.

10. Given $x + y = 5$; $(x^2 + y^2)(x^3 + y^3) = 455$.

11. Given $\frac{5}{x}\sqrt{x + y} + \frac{5}{y}\sqrt{x + y} = \frac{32}{3}$.

$$\frac{3}{y}\sqrt{x - y} + \frac{3}{x}\sqrt{x - y} = \frac{4}{5}.$$

12. Given $\frac{y}{x} - \frac{x}{x + y} = \frac{x^2 - y^2}{y}$; $\frac{x}{y} - \frac{x + y}{x} = \frac{y}{x}$.

13. Given $x - \sqrt{x^2 - y^2} = x(x + \sqrt{x^2 - y^2})$;

$$x\sqrt{1 - y} = y\sqrt{1 + x}.$$

14. Given $xy = a$; $yz = b$; $zx = c$.

15. Given that the sides of a right-angled triangle are in geometrical progression, and the area is a^2, to find the sides.

16. Find three numbers such that the product of each into the sum of the other two may be the numbers 48, 84, and 90, respectively.

17. Given $x^2y^3z^4 = a$, $x^3y^4z^2 = b$, $x^4y^2z^3 = c$, to find x, y, and z.

18. Given $y^2 = 4ax$, $x - p = -2(a + x)$, $y - q = \frac{y^3}{4a^2x}(a + x)$ to find the relation between a, p, and q.

19. Given $x^2 + xy + y^2 = 14x$, $x^4 + x^2y^2 + y^4 = 84x^2$, to find x and y. (Divide one equation by the other.)

20. $x^3 + y^3 + xy(x + y) = 65$, $(x^2 + y^2)x^2y^2 = 468$, to find x and y. (Put $x + y = u$ and $xy = v$.)

21. $x + y + z = 13$, $x^2 + y^2 + z^2 = 61$, $xy + xz = 2\,yz$, to find x, y, and z.

22. The sum of the two sides of a right-angled triangle is 51, and the hypothenuse is greater than the longer side by 3. Find the sides.

23. The sum of the three sides of a right-angled triangle is 60, and the sides are in A. P. To determine the triangle.

CHAPTER XI.

REMAINDER THEOREM. — TRANSFORMATION OF FUNCTIONS.

149. When we divide x^3+ax^2+bx+c by $x-p$, we get the quotient $x^2+(a+p)x+p^2+ap+b$, and the remainder p^3+ap^2+bp+c.

We notice here that the remainder can be obtained from the dividend by simply putting p for x; or, in other words, the remainder is the same function of p as the dividend is of x.

To prove that this is always the case.

Let fx be any integral function of x, and let it be divided by $x-p$. Then if Q denotes the quotient, and R the remainder, we have

$$fx=(x-p)Q+R.$$

As $x-p$ is of one dimension in x, R is independent of x, and is not affected by any change in x.

Change x to p; *i.e.* put p for x, and $x-p=0$, and

$$fp=R,$$

which proves the

Theorem. If an integral function of x be divided by $x-p$, the remainder is the same function of p.

Ex. 1. The remainder when x^8-5x^4+6x-2 is divided by $x-1$ is $1^8-5\cdot 1^4+6\cdot 1-2\equiv 0$; and $x-1$ is a factor of the given expression.

Ex. 2. The remainder when $x^7 - 6x^5 + 3x - 2$ is divided by $x + 3$ is

$$(-3)^7 - 6(-3)^5 + 3(-3) - 2 \equiv -740.$$

Ex. 3. To find the result of substituting 6 for x in the function

$$x^5 - 3x^4 + x^3 - 2x - 1.$$

To substitute 6 for x is to find the remainder when the function is divided by $x - 6$.

Therefore,

$$\begin{array}{r} 1 - 3 + \ 1 - \ \ 2 - \ \ 1 \,\underline{|\ 6} \\ + 6 + 18 + 114 + 672 \\ \hline 1 + 3 + 19 + 112 + 671 \end{array}$$

And $R = 671$ is the result.

Hence to substitute a for x in an integral function of x is equivalent to dividing the function by $x - a$ and taking the remainder.

EXERCISE XI. a.

1. Find the value of $x^{10} - 3x^7 + x^4 - 5x + 6$ when $x = 4$, when $x = -4$, when $x = 1$.

2. Find the value of $x^3 - 3.6x^2 + 4.32x - 1.728$ when $x = 1.2$.

3. What is the remainder when $x^5 - 6x^4 + 5x^3 - 4x^2 + 3x - 2$ is divided by $x + 5$?

4. Find the remainder when $(a + b + c)(ab + bc + ca) - abc$ is divided by $a + b$.

5. Find the remainder when $x^3 - 7x + 10$ is divided by $x - 1$, by $x - 2$, by $x + 3$.

What relation does $x^3 - 7x + 6$ hold to the three divisors?

6. Find the result of substituting 1.71 for x in the function $x^3 - 5$.

What, approximately, is the relation between 1.71 and 5?

7. Find the result of substituting 1.27 for x in the function $x^4 - x^2 - 1$.

What relation does 1.27 hold to the function?

8. Find the remainder when $x^5 + 2x^3 - 3x^2 + x + 1$ is divided by $x^2 - x + 1$.

The function may be written

$$x(x^4 + 2x^2 + 1) - 3x^2 + 1,$$

and the divisor gives $x^2 = x - 1$.

$$\therefore\ x\{(x-1)^2 + 2(x-1) + 1\} - 3(x-1) + 1 = R$$

$$\equiv x(x^2) - 3x + 4 \equiv x(x-1) - 3x + 4$$

$$\equiv -3x + 3.$$

We thus substitute $x - 1$ for x^2, wherever x^2 occurs, and continue the reduction until only linear x remains.

9. What is the value of $x^{10} - x^4 + x - 1$ when $x^3 + 2x - 1 = 0$?

150. Divide $x^3 - 3x^2 + 2x - 1$ by $x - 1$; we get a quotient $x^2 - 2x + 0$, and a remainder -1.

Divide $x^2 - 2x + 0$ by $x - 1$; we get a quotient $x - 1$, and a remainder -1.

Hence $x^3 - 3x^2 + 2x - 1 \equiv (x-1)(x^2 - 2x) - 1$

$$\equiv (x-1)(x-1)(x-1) + (x-1)(-1) - 1$$

$$\equiv (x-1)^3 - (x-1) - 1.$$

We have thus expressed the function of x,

$$x^3 - 3x^2 + 2x - 1,$$

as a function of $(x - 1)$, viz.,

$$(x-1)^3 - (x-1) - 1;$$

and we notice that the function is simplified in form as it lacks the square term.

This result tells us that to substitute any value for x in $x^3 - 3x^2 + 2x - 1$ is equivalent to substituting a number less by unity in the function

$$y^3 - y - 1.$$

The foregoing transformation may also be effected as follows:

Let $x - 1 = y$; then $x = y + 1$, and putting $y + 1$ for x in the function, and expanding, we obtain

$$y^3 - y - 1;$$

or, since $y = x - 1$, $(x-1)^3 - (x-1) - 1$.

151. Let it be required to express $x^3 - 4x^2 - 3x + 6$ as a function of $x - 2$.

The transformed function will take the form

$$(x-2)^3 + R_2(x-2)^2 + R_1(x-2) + R,$$

where the coefficients R, R_1, R_2 are to be found.

We have the identity

$$x^3 - 4x^2 - 3x + 6 \equiv (x-2)^3 + R_2(x-2)^2 + R_1(x-2) + R.$$

Writing 2 for x gives $R = -8$.

But to substitute 2 for x is equivalent to dividing by $x - 2$ and taking the remainder; so that if we divide the given function by $x - 2$ the remainder gives R.

Hence rejecting -8 from each member, and dividing throughout by $x - 2$, we have from the remainder

$$R_1 = -7.$$

The next similar operation gives

$$R_2 = 2.$$

Therefore the transformed function is

$$(x-2)^3+2(x-2)^2-7(x-2)-8.$$

The whole operation carried out in the form of division is as follows:

$$\begin{array}{rrrrl}
 & & & \underline{\quad 2} & \\
1 & -4 & -3 & +6 & \\
 & 2 & -4 & -14 & \\
\hline
1 & -2 & -7 & -8 & = R \\
 & +2 & +0 & & \\
\hline
1 & -0 & -7 & = \ldots\ldots & R_1 \\
 & 2 & & & \\
\hline
1 & 2 & = \ldots\ldots\ldots & & R_2 \\
1 & & & &
\end{array}$$

We here divide by $x-2$, getting the remainder $-8=R$. We then cut this off, and divide the quotient x^2-2x-7 by $x-2$, and get the remainder $-7=R_1$. Cutting this off, we divide again by $x-2$, etc., until the whole is completed.

Ex. 1. To express $x^4-12x^3+36x^2-x-77$ as a function of $x-3$.

$$\begin{array}{rrrrr}
 & & & & \underline{\quad 3} \\
1 & -12 & +36 & -1 & -77 \\
 & 3 & -27 & +27 & +78 \\
\hline
1 & -9 & +9 & +26 & +1 \\
 & 3 & -18 & -27 & \\
\hline
1 & -6 & -9 & -1 & \\
 & 3 & -9 & & \\
\hline
1 & -3 & -18 & & \\
 & 3 & & & \\
\hline
1 & 0 & & &
\end{array}$$

And the required function is

$$(x-3)^4-18(x-3)^2-(x-3)+1.$$

Ex. 2. To express $m^3 - 3m^2n + 2mn^2 - 3n^3$ as a function of $m - n$ in place of m.

Both parts being homogeneous, this is equivalent to expressing $\left(\frac{m}{n}\right)^3 - 3\left(\frac{m}{n}\right)^2 + 2\left(\frac{m}{n}\right) - 3$ as a function of $\frac{m}{n} - 1$.

The coefficients are readily found to be 1, 0, -1, -3, and the function becomes

$$(m-n)^3 - n^2(m-n) - 3n^3.$$

EXERCISE XI. b.

1. Express $x^3 - 3x^2 + 2x + 1$ as a function of $x + 1$.

2. Express $3a^5 - a^3 + 4a^2 + 5a - 8$ as a function of b, where $b = a - 2$.

3. Express $y^5 - 5y^4 + 10y^3 - 10y^2 + 5y - 2$ as a function of x, when $x = y - 1$.

4. Express $x^5 + 15x^4 + 90x^3 + 270x^2 + 404x + 241$ as a function of $x + 3$.

5. Express $a^5 + 2a^4b - 2a^3b^2 - 12a^2b^3 - 15ab^4 - 5b^5$ in terms of $a + b$ and b.

6. Express $x^3 - 7x + 6$ as a function of $x - 1$.

What relation does $x - 1$ hold to the function?
What relation does 1 hold to the corresponding equation?

7. If $x^2 + ax + b$ be developed as a function of $x - z$, for what value of z will the function take the form

$$(x-z)^2 + A(x-z)?$$

152. If we have to express fx as a function of $x - (a + b + c \cdots)$, we may carry out the operation in successive parts by putting $y = x - a$, and transforming to a function of y; then putting $z = y - b$, and transforming to a function of z, etc., until all the quantities $a, b, c \cdots$ are taken in. Where b, c, etc., are decimals, this is generally the most convenient method, as the whole operation is very compact.

Ex. To express $x^3 - 2x^2 + 3x - 4$ as a function of $x - 1.23$.

In the following operation only the algebraical sums are put down in each column.

1 −2	+3	−4	1.23
−1	+2	−2	
0	2		
1			
1.2	2.24	−1.552	
1.4	2.52		
1.6			
1.63	2.5689	−1.474933	
1.66	2.6187		
1.69			

Hence, $x^3 - 2x^2 + 3x - 4 \equiv (x-1)^3 + (x-1)^2 + 2(x-1) - 2$

$$\equiv (x-1.2)^3 + 1.6(x-1.2)^2 + 2.52(x-1.2) - 1.552$$

$$\equiv (x-1.23)^3 + 1.69(x-1.23)^2 + 2.6187(x-1.23) - 1.474933.$$

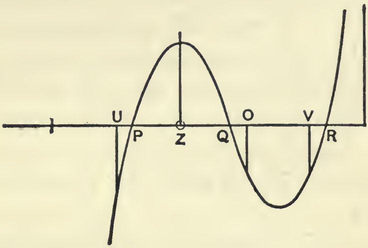
The work may be still more condensed by leaving out the decimal points, and remembering that when beginning with a new quotient figure the product must be written one place further to the right in the first column, two places in the second column, three in the third, etc.

153. Let $fx \equiv x^3 + x^2 - 2x - 1 = y.$

When $x = -2, -1, 0, 1, 2, \cdots$

$$y = -1, +1, -1, -1, +7, \cdots$$

The graph is given in the figure, with O as origin, from which values of x are measured.



Transform to a function of z where

$$z = x + 1,$$

and we have

$$z^3 - 2z^2 - z + 1 = y.$$

And when $\quad z = -1, 0, 1, 2, 3, \cdots$

$$y = -1, +1, -1, -1, +7 \cdots$$

By comparing results for the same values of y, we perceive that z is measured from an origin Z, one unit to the left of the origin for x.

Of the equation $x^3 + x^2 - 2x - 1 = 0$, two roots, OP and OQ, are negative, and one root, OR, is positive. Of the equation $z^3 - 2z^2 - z + 1 = 0$, one root, ZP, is negative, and two roots, ZQ and ZR, are positive.

Similarly, by transforming to a function of u where $u = x + 2 = z + 1$, the origin is moved to U, and the resulting equation, $u^3 - 5u^2 + 6u - 1 = 0$, has all its roots positive.

Hence the transforming of fx to a function of $x - a$ is equivalent to moving the origin with respect to the graph, through a units along the x-axis, to the right if a is positive, and to the left if a is negative.

Now take the equation $x^3 - 5x^2 + 6x - 1 = 0$, for which the origin is at U, and all the roots are accordingly positive.

For some value of a, the origin will be moved from U to P, and UP represents this value of a. But UP also represents one of the roots.

Therefore the value of a which transfers the origin from U to P is one of the roots of the equation.

Similarly, the values of a which transfer the origin from U to Q, and from U to R, are the two other roots.

154. As the roots of the equation $x^3 - 5x^2 + 6x - 1 = 0$ are real and incommensurable, they may be approximated to. Let us then endeavor to find the value of UR.

For this purpose we first transfer the origin through three units from U to V. Then, having carefully drawn the graph, we estimate the distance VR in tenths of a unit, as nearly as we can, and we move the origin onwards through this estimated distance.

To know whether our estimated distance is too great or not, we have the following test:

As long as the origin lies between V and R, the graph at that point is below the x-axis, and the independent term is negative; but when the origin passes R, the graph rises above the x-axis, and the independent term is positive.

Hence a change in the sign of the independent term indicates that we have caused the origin to pass R.

Now transforming $x^3 - 5x^2 + 6x - 1 = 0$ to a function of $x - 3$ gives $y^3 + 4y^2 + 3y - 1$, where $y = x - 3$.

Next transform to a function of $y - 0.2$, and we get $z^3 + 4.6z^2 + 4.72z - 0.232$, where $z = y - 0.2$; and the independent term being $-$ shows that our new origin is still to the left of R.

The student is advised to try the effect upon the independent term of transforming to $y-0.3$.

Again transforming the last equation to a function of $z-0.04$, we obtain $u^3+4.72\,u^2+5.0928\,u-0.035776$, where $u=z-0.04$.

We have thus transferred the origin in all through the distance 3.24, and this is to two decimals a correct approximation to the root UR. A repetition of the same process will furnish as close an approximation as may be desired.

The work is carried out practically in the following condensed form, where only algebraic sums are written, and decimal points are not employed:

		3.24698
1−5	+6	−1
−2	0	−1...
+1	3..	−0232...
4.	384	−0035776...
42	472..	−0005047984
44	49056	
46.	50928..	
464	5121336	
468	5149728	
472.		
4726		
4732		
4738		

It will be noticed that the independent term is being successively reduced in value; and as this term gives the distance from the origin to the graph, measured parallel to the y-axis, this reduction shows that the origin is approaching the point R.

We have said that the first decimal figure must be estimated, and then tried. So, to a certain extent, must the remaining figures. But the values of the others are readily found. Thus, after finding the 2 of the quotient, and finishing the transformation, we add a cipher to 0232 of the last column, and divide by 472 of the second column; this gives 4 for the next figure. Similarly, 357760 divided by 50928 gives 6 for the next, and so on.

In fact, after obtaining the three decimals 246, and completing the transformations thus far, we may safely obtain three more decimals by simple division. In this way 9, 8 are obtained by dividing 5047984 by 514972.

Ex. To find the root UQ. The various transformations give as coefficients —

for $x - 1$,	1,	-2,	-1,	$+1$;
for $x - 1.5$,	1,	-0.5,	-2.25,	$+0.125$;
for $x - 1.55$,	1,	-0.35,	-2.2925,	$+0.011375$;

and the approximate root is 1.554 ⋯

It will be noticed that the independent term for this root is +, as the graph lies above the x-axis to the left of Q, or between P and Q.

155. The preceding methods offer an elegant means of extracting roots of numbers.

Ex. To approximate to the cube root of 12. Let $x^3 - 12 = 0$, and solve this as a cubic equation. This equation has but one real root, and that is the arithmetic cube root of 12.

1	0	0	-12	2.2894
	2	4	-4	
	4	12..	$-1.352\ldots$	
	6	1324	-0.147648	
	62	1452..		
	64	150544		
	66	155952		
	668			
	676		$\therefore \sqrt[3]{12} = 2.2894 \cdots$	

EXERCISE XI. c.

1. Determine the integral values between which the real roots of the following equations lie —

i. $x^3 - 3x^2 + 2x - 2$. iii. $x^4 - 4x^2 + 3x - 4$.

ii. $x^3 + 3x^2 + 2x + 2$. iv. $x^4 + 2x^3 - 4x - 2$

2. Find to 3 decimals the greatest positive root of

$$x^4 - 2x^3 - 3x^2 + 6x - 1 = 0.$$

3. Transform $x^3 - 3x^2 + 3x - 4 = 0$ to an equation in $(x - 1)$, and thence find the roots of the given equation.

4. Transform $x^4 - 4x^3 + 2x^2 + 4x + 6 = 0$ to an equation in $(x - 1)$, and thence find the roots of the given equation.

5. Show that if $x^3 - px^2 + qx + r = 0$ be transformed to an equation in $\left(x - \frac{p}{3}\right)$, the equation will assume the form $z^3 + Qz + R = 0$, where the square term is wanting.

6. Remove the second term from $x^3 - 6x^2 + 12x + 9 = 0$, and thence solve the equation.

7. In the equation $x^3 - 2x^2 - x - 6 = 0$, move the origin 3 units to the right, and thence find the roots.

8. Find to 5 decimals the cube root of 3.1416.

9. A gallon contains 277.273 cu. in. Find the length in inches of the edge of a cubical box that shall hold just 10 gallons.

10. Find the fifth root of 100, to 3 decimals.

CHAPTER XII.

THE PROGRESSIONS. — INTEREST AND ANNUITIES.

156. A series is a succession of terms which follow some fixed law, by means of which any term, after some fixed term, usually not far removed from the beginning, may be obtained from the preceding terms and from constants.

Thus $1 + 3 + 5 + 7 + \cdots$ is a series in which each term is got from the preceding one by adding 2. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots$ is a series in which each term is one-half the preceding term.

The doctrine of series is a very extensive and important one, and has given rise to a distinct calculus, that of Finite Differences; but two series, the simplest of their species, are of such common application as to be treated of in elementary algebra, and even in arithmetic, under the name of the *Progressions.*

A series in which each term differs from the preceding one by a constant, as in the first of the foregoing examples, is an Arithmetic Series, or an Arithmetic Progression, and is symbolized as an A. P.; and a series in which each term is a constant multiple of the preceding one, as in the second example, is a Geometric Series, or a Geometric Progression, contracted to G. P.

A third kind, the nature of which will be explained in the proper place, is called a Harmonic Series, or a Harmonic Progression, contracted to H. P.

ARITHMETIC SERIES.

157. The quantities normally occurring here are: a, the first term of the series; d, the common difference; n, any given number of terms; and s, the sum of n terms.

If fn is such a function of n that the substituting of any integral number for n gives that numbered term in the series, fn is called the nth term of the series, and is all-important, not only in an A. P., but in all series, as expressing the law of the series.

Evidently, to know the form of fn is to know the series, since its consecutive terms are given by the substitution of 1, 2, 3, ⋯ etc., for n.

The consecutive terms of an A. P. are

$$a,\ a+d,\ a+2d,\ a+3d, \text{ etc.},$$

and it is readily seen that the nth term is $a+(n-1)d$. It being often convenient to denote this general term by a single letter, z, we have

$$z=a+(n-1)d \quad . \quad . \quad . \quad . \quad . \quad (A)$$

158. An A. P., like any other series of numbers, is not necessarily limited in extent, but may be continued at pleasure in either direction.

When we consider any portion containing n consecutive terms of this unlimited series, we call the first term a, and the nth, or last term, z. And thus any two unequal numbers may be any two terms of an A. P.

Ex. To find the A. P. whose 5th term is 12, and whose 11th term is 24.

There will evidently be 7 terms in this portion, and hence

$$24 = 12 + 6d.$$

$\therefore\ d = 2$; and the first term of the series is $12 - 4d$ or 4. Hence the nth term is $4 + 2(n - 1)$ or $2 + 2n$. And the series is

$$4 + 6 + 8 + 10 + \cdots$$

159. It will be noticed that the nth term of an A. P. is a linear function of n.

Now any function of n taken as the nth term will give rise to some series, and if it is a positive integral function of n higher than linear, the series will be of the same species as an A. P. but of a higher order. So that the A. P. is the simplest series of its species.

Thus if the nth term be $\frac{1}{2}(n^2 + n)$ or $\frac{1}{2}n(n + 1)$, the series is

$$1, 3, 6, 10, 15 \cdots$$

It is worthy of notice that the differences of the terms in this series form an A. P.

Similarly, if the nth term be of the form $n^3 + n^2$, or a function of three dimensions in n, the differences of the differences of the terms of the series will be an A. P.

160. As n denotes the number of a term in a series, it must be integral; and hence the presentation of a non-integral value for n indicates some absurdity or impossibility.

Ex. Is 100 a term of the series whose first term is 3 and whose difference is 4?

As the nth term is $3 + (n - 1)4$ or $4n - 1$, the equation $100 = 4n - 1$ will give an integral value for n if 100 is a term of the series.

It does not give such a value, and therefore 100 is not a term of the series.

It is readily seen that 99 is the 25th term.

161. S being the sum of n terms of an A. P., we have

$$S = a + (a + d) + (a + 2d) + \cdots (a + \overline{n - 1} \cdot d).$$

Also, by reversing the order,

$$S = (a + \overline{n-1} \cdot d) + (a + \overline{n-2} \cdot d) + (a + \overline{n-3} \cdot d) + \cdots + a.$$

Adding,

$$2S = (2a + \overline{n-1} \cdot d) + (2a + \overline{n-1} \cdot d) + \cdots n \text{ terms}$$
$$= n(2a + \overline{n-1} \cdot d).$$

$$\therefore\ S = \frac{n}{2}(2a + \overline{n-1} \cdot d) \quad . \quad . \quad . \quad . \quad . \qquad (B)$$

The following method of investigation for the sum of an A. P. is important.

We have
$$\begin{aligned} 1^2 &= 0^2 &&+ 2 \cdot 0 &&+ 1, \\ 2^2 &= 1^2 &&+ 2 \cdot 1 &&+ 1, \\ 3^2 &= 2^2 &&+ 2 \cdot 2 &&+ 1, \\ &\ldots \\ n^2 &= (n-1)^2 &&+ 2(n-1) &&+ 1, \\ (n+1)^2 &= n^2 &&+ 2n &&+ 1. \end{aligned}$$

$\therefore$ by addition,

$$2(1 + 2 + 3 + \cdots n) = (n+1)^2 - (n+1);$$

whence $S = \dfrac{n(n+1)}{2}$ = the sum of the first n natural numbers.

Then, the terms of an A. P. are

$$a,\ a + d,\ a + 2d,\ \cdots\ a + (n-1)d;$$

and summing these gives

$$S = na + (1 + 2 + 3 + \cdots \overline{n-1})d$$
$$= na + \tfrac{1}{2}n(n-1)d.$$

And this is the same as (B).

Ex. The sum of the first n odd numbers is

$$\frac{n}{2}(2 + \overline{n-1} \cdot 2), \text{ or } n^2.$$

162. Upon multiplying out,

$$S = n^2 \cdot \frac{d}{2} + n \cdot \frac{2a - d}{2}.$$

Hence the sum of n terms of an A. P. is a quadratic function of n, with no independent term.

And every quadratic in n, without the independent term, is the sum of n terms of some A. P. The independent term, if present, would appear as an extraneous term which might or might not follow the law of the series.

Ex. To find the A. P. of which $2n^2 - 3n$ expresses the sum of n terms.

Let $n = 1$; the sum of 1 term is $-1 = a$.
Let $n = 2$; the sum of 2 terms is $2 = 2a + d$.
$\therefore$ $d = 4$, and the nth term $= 4n - 5$;
which gives the series.

Otherwise, the sum of n terms, or S_n, is $2n^2 - 3n$; and the sum of $n - 1$ terms, or S_{n-1}, is $2(n-1)^2 - 3(n-1)$.

But S_n is got by adding the nth term to S_{n-1}.

$\therefore$ nth term $= S_n - S_{n-1} = 2n^2 - 3n - 2(n-1)^2 + 3(n-1) = 4n - 5$.

163. Any positive integral function of n, of a higher degree than the second, and lacking the independent term, expresses the sum of n terms of some series of the same species as the A. P. but of a higher order.

Ex. Let $2n^3 - 3n^2 + n$ be the sum of n terms.

Then $S_n - S_{n-1} = 2n^3 - 3n^2 + n - 2(n-1)^3 + 3(n-1)^2 - (n-1)$

$$= 6n^2 - 12n + 6,$$

which is the nth term. And the series is

$$0 + 6 + 24 + 54 + \cdots,$$

a series whose differences form an A. P. Compare Art. 159.

164. In problems where n is to be found from conditions involving the sum of n terms, n may have two values. If both be integral, both will satisfy the conditions, but non-integral values of n must be rejected as being inapplicable to the case.

Ex. How many terms of the series whose nth term is $27 - 2n$ will make 144?

The sum of n terms is $\frac{n}{2}(2a + \overline{n-1} \cdot d)$, and this is to be 144.

We readily find $a = 25$, and $d = -2$.

$$\therefore \frac{n}{2}(52 - 2n) = 144.$$

Whence $n = 18$, or 8.

And the sum of 18 terms = sum of 8 terms = 144.

165. When three quantities form three consecutive terms of an A. P., the middle one is called an arithmetic mean between the other two.

Let A be an arithmetic mean between a and b. Then $A - a = b - A$; whence

$$A = \tfrac{1}{2}(a + b).$$

Or the arithmetic mean between two quantities is one-half their sum.

The following miscellaneous examples illustrate the subject of arithmetic progression.

Ex. 1. The pth term of an A.P. is P and the qth term is Q, to find the nth term, and the sum of n terms.

From the pth to the qth term the difference is added $q-p$ times.

$$\therefore\ d=\frac{Q-P}{q-p}.$$

Also, from the first to the pth term the difference is added $p-1$ times.

$$\therefore\ a=P-(p-1)\cdot\frac{Q-P}{q-p}=\frac{P(q-1)-Q(p-1)}{q-p}.$$

Hence the nth term is $\dfrac{Pq-Qp+n(Q-P)}{q-p}$.

Also,
$$S=\frac{n}{2}(a+n\text{th term})$$
$$=\frac{n}{2}\left\{\frac{2(Pq-Qp)+(n+1)(Q-P)}{q-p}\right\}.$$

Ex. 2. In the A.P.'s $6+7\frac{1}{2}+9+\cdots$ and $-3-1+1\cdots$ to determine —

(1) If there be a common term, and if so, its value.

(2) If there be a common number of terms for which the sum in each series is the same, and if so, to find the sum.

(1) $6+(n-1)\frac{3}{2}=-3+(n-1)2$ gives an integral value for n if there is a common term.

This gives $n=19$, and the 19th term is common.

Its value is $6+18\times\frac{3}{2}$ or 33.

(2) $S=\dfrac{n}{2}\left(12+\overline{n-1}\cdot\dfrac{3}{2}\right)=\dfrac{n}{2}(-6+\overline{n-1}\cdot 2)$ gives the condition for a common sum, if n is integral.

This gives $n = 37$, and the sum of the first 37 terms is the same in each series.

The sum is $\frac{37}{2}(12 + 36 \times \frac{3}{2}) = 1221.$

EXERCISE XII. a.

1. Find the nth terms in the A.P.s, two of whose terms are given as follows —

i. 1st term $= m$, 2d $= p$.

ii. 1st $= a + \overline{n-1} \cdot b$, 2d $= a$.

iii. 1st $= 3$, 3d $= 12$.

iv. 1st $= 0$, 10th $= 50$.

v. 3d $= 8$, 8th $= b$.

vi. 10th $= 4$, 4th $= 20$.

vii. $(n-1)$th$=a$, $(n+1)$th$=b$.

2. Sum the following A. P.s —

i. $1 + 1\frac{1}{2} + \cdots$ to 12 terms.

ii. $\frac{1}{2} + \frac{2}{3} + \cdots$ to 10 terms.

iii. $0 + 3 + \cdots$ to 7 terms.

iv. $\dfrac{1}{a} + \dfrac{1}{a+b} + \cdots$ to n terms.

v. $103 + 97 + \cdots$ to 24 terms.

vi. $2 + 4 + \cdots$ to n terms.

3. Sum 100 terms of the A.P. whose 3d term is 5, and 10th term 75.

4. Sum to n terms the series whose nth term is $\frac{1}{2}(n-1)$.

5. Find the sum of all the multiples of 7 lying between 200 and 400.

6. If a, b, c are in A. P., so also are $a^2(b+c)$, $b^2(c+a)$, and $c^2(a+b)$.

7. Find the A. P. for which $s = \frac{1}{4}(3n^2 - 2n)$.

8. Find the A. P. for which $s = 72$, $a = 17$, $d = -2$.

9. The A. P.s whose sums are $29n - 2n^2$ and $\frac{1}{4}(17n + 3n^2)$ have a common term. Find it.

10. One hundred apples are placed in line at two feet apart, and a basket is placed at an extreme apple. How far will a person travel who takes the apples one by one to the basket?

11. n apples are placed in line d feet apart, and a basket is placed in the same line, m feet from the first apple. How far will a person travel who takes the apples one by one to the basket?

What is the difference between m positive and m negative?

12. A debt of $1000 is paid in 20 annual instalments of $50 with simple interest at 6% on all, due at the time of each payment. How much money is paid in discharging the debt?

13. A person receives $1.50 daily. He spends 15 cents the first day, 18 the second, 21 the third, and so on.

i. When will he be worth the most, and how much will it be?

ii. When will he be worth nothing?

iii. When will he be worth exactly $21.30?

iv. Will he at any time have exactly $10?

14. A and B start from the same place at the same time. A goes westward 10 miles the first day, 8 miles the 2d, etc., in A. P. B goes eastward 3 miles the 1st day, 4 miles the 2d, etc., in A.P.

i. Where and when will they be together?

ii. When will they be 70 miles apart?

15. Find the sum of $1 + 2 - 3 + 4 - 5 + 6 - + \cdots$ to 120 terms.

16. Two sides of an equilateral triangle are each divided into 100 equal parts, and corresponding points of division are joined. Find the total length of all these joins.

Into how many equal parts must the sides be divided, so that the sum of the joins may be m times the side of the triangle?

17. $100 is deposited annually in a bank for 20 years to be left at simple interest at 6%. What is the accumulated sum at the last payment?

18. The side of an isosceles triangle is a and the base is b, and the altitude is an arithmetic mean between the side and base. Show that $a : b = 1 + \sqrt{7} : 3$.

19. Insert 4 terms of an A. P. between a and b.

20. Find the series for which $S = \frac{1}{6} n(n+1)(n+2)$.

21. Find the series for which $S = \frac{1}{6} n(n+1)(2n+1)$.

22. Find the series for which $S = \{\frac{1}{2} n(n+1)\}^2$.

23. The sums of two A. P.s to n terms each, are $n^2 + pn$ and $3n^2 - 2n$. For what value of p will they have a common nth term?

Show that they cannot have a common term unless p is a multiple of 4.

24. Two sets of n lines each are drawn parallel to adjacent sides of a parallelogram. Find the whole number of parallelograms thus formed.

25. The natural numbers are divided into groups as follows —

$$(1)(2, 3)(4, 5, 6)(7, 8, 9, 10) \cdots$$

Find the sum of the numbers in the nth group.

GEOMETRIC PROGRESSION.

166. The quantities with which we have normally to deal in Geometric Progression are

a, the first term; r, the common ratio; n, the number of terms; z, the last or nth term; and S, the sum of n terms.

By definition, the terms of a Geometric Progression are $a, ar, ar^2, ar^3 \cdots$, and it is readily seen that the nth term is ar^{n-1}.

$$\therefore z = ar^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Again, $$S = a + ar + ar^2 + \cdots ar^{n-1},$$

and $$rS = \quad ar + ar^2 + \cdots ar^{n-1} + ar^n.$$

$$\therefore S(1-r) = a(1-r^n),$$

and $$S = a \cdot \frac{1-r^n}{1-r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

Relation (B) may also be obtained as follows:

By division,

$$\frac{a}{1-r} = a + ar + ar^2 + \cdots ar^{n-1} + \frac{ar^n}{1-r}.$$

$$\therefore\ a + ar + ar^2 + \cdots ar^{n-1} = \frac{a}{1-r} - \frac{ar^n}{1-r} = a \cdot \frac{1-r^n}{1-r}.$$

Ex. The population of a city increases at the rate of 5% per annum, and it is now 20000. What will it be 10 years hence?

It will evidently be the 11th term of the G. P. whose first term is 20000, and whose ratio is 1.05.

$\therefore$ The population will be $20000\,(1.05)^{10} = 32578$ to the nearest integer.

167. The finding of r or n in a geometric progression cannot usually be conveniently done without employing logarithms. Thus in the preceding example the labor of raising 1.05 to the 10th power is very great. And in the case of n, on account of its appearing as an exponent, the common operations of arithmetic do not suffice for finding it.

Ex. 1. In a G. P. the first term is $\frac{1}{3}$, and the second term is $\frac{1}{2}$; to find the nth term and the sum of n terms.

Since $a = \frac{1}{3}$ and $ar = \frac{1}{2}$. $\therefore\ r = \frac{1}{2} \div \frac{1}{3} = \frac{3}{2}$.

Then $z = n\text{th term} = \frac{1}{3}\left(\frac{3}{2}\right)^{n-1} = \frac{3^{n-2}}{2^{n-1}}$.

And $S = a \cdot \frac{r^n - 1}{r - 1} = \frac{1}{3}\left\{\left(\frac{3}{2}\right)^n - 1\right\} \div \left(\frac{3}{2} - 1\right) = \frac{2}{3}\left\{\left(\frac{3}{2}\right)^n - 1\right\}$.

Ex. 2. How many terms of the series $1 + 2 + 4 + 8 + \cdots$ will make 127?

Here $$127 = \frac{r^n - 1}{r - 1} = \frac{2^n - 1}{1} = 2^n - 1.$$

$$\therefore\ 2^n = 128, \text{ and } n \text{ is evidently } 7.$$

168. When $r < 1$, r^n diminishes as n increases, and by taking n great enough r^n may be made as small as we please. At the limit when $n = \infty$, $r^n = 0$, and (B) becomes

$$S = \frac{a}{1-r}.$$

This is the **limit** towards which the sum of n terms of the series approaches as n is continually increased; and by taking n sufficiently great we can make the sum of n terms approach this value as near as we please.

For convenience, this limit is called the *sum of the series to infinity,* although it is more properly spoken of as the *limit* of the series as *n approaches infinity.*

Ex. **1.** To find the limit of 0.333 ⋯ *ad infinitum.*

This is $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

And its limit is $\frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) = \frac{3}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{1}{3}.$

Ex. **2.** To find the value of the circulating decimal $0.2\dot{4}\dot{1}$.

This is $\frac{2}{10} + \frac{41}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) = \frac{2}{10} + \frac{41}{10^3} \cdot \frac{10^2}{99}$

$$= \frac{2}{10} + \frac{41}{990} = \frac{2 \times 99 + 41}{990} = \frac{241 - 2}{990}.$$

Hence the rule — Subtract the non-repeating part from the whole, and write as denominator as many 9's as there are digits in the repeating part, followed by as many ciphers as there are digits in the non-repeating part.

EXERCISE XII. b.

1. Find the sum of n terms of the series —

i. $1 + 3 + 9 + \cdots$ ii. $1 - \frac{2}{3} + \frac{4}{9} - + \cdots$

iii. $\frac{3}{10} + \frac{9}{100} + \cdots$ What is this when $n = \infty$?

iv. $\sqrt{2} + (2 - \sqrt{2}) + \cdots$ when $n = \infty$.

2. Sum to ∞ the series $1 + \dfrac{n}{n+1} + \dfrac{n^2}{(n+1)^2} + \cdots$

What are the results when $n = 1$? $= 2$? $= 3$?

3. Find the nth term of the G.P. whose first term is $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$, and the second is $\dfrac{1}{2-\sqrt{2}}$.

4. A cask of wine contains 30 gallons. 5 gallons are drawn off and the cask is filled up with water. After this has been done 5 times, how many gallons of the original wine are in the cask?

5. A circle is inscribed in an equilateral triangle; a second circle touching the first and two sides; a third touching the second and the same two sides, etc. If the side be s, find the radius of the nth circle so described. Also, find the total area of all the circles continued to ∞.

6. A square is inscribed in an acute-angled triangle having one side of the square on the base of the triangle. A second square is inscribed similarly in the triangle above; a third square above that; etc. to ∞. If the base of the triangle be b and its altitude a, find the total area of all the squares.

7. How many terms of the G.P. $1 + 2 + 4 + \cdots$ will make 1023?

8. Find the sum of n terms of the series whose nth term is $na + a^n$.

9. If S, a, r, z be taken in their common acceptation, show that

$$r = \frac{S-a}{S-z}.$$

10. A country whose annual production is now 5 millions, increases at the rate of 5 % per annum. What will it be 5 years hence?

169. When three quantities are three consecutive terms of a G. P., the middle quantity, is a *geometric mean* between the other two. It is also called a mean proportional. Art. 84.

Let G be a geometric mean between a and b.

Then $\frac{G}{a}=\frac{b}{G}$; and hence $G=\sqrt{ab}$.

Problem. To insert n terms betwen two extremes, a and b, so that the whole may be a G. P.

Let the series be

$$a+t_1+t_2+\cdots t_{n-1}+t_n+b.$$

Then $$\frac{t_1}{a}=\frac{t_2}{t_1}=\cdots\frac{t_{n-1}}{t_{n-2}}=\frac{t_n}{t_{n-1}}=\frac{b}{t}=r;$$

and $$\frac{b}{a}=\frac{t_1}{a}\cdot\frac{t_2}{t_1}\cdots\frac{t_{n-1}}{t_{n-2}}\cdot\frac{t_n}{t_{n-1}}\cdot\frac{b}{t_n}=r^{n+1}.$$

$$\therefore\ r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}};\ t_1=ar=(a^nb)^{\frac{1}{n+1}};$$

$$t_2=(a^{n-1}b^2)^{\frac{1}{n+1}};\ \text{and generally,}\ t_m=(a^{n-m+1}b^m)^{\frac{1}{n+1}}.$$

EXERCISE XII. c.

1. Insert 8 terms between 1 and 512, to form a G. P.

2. A body weighs a grams in one scale pan of a balance and b grams in the other. Show that its true weight is $\sqrt{ab}$.

3. Three circles each touch the same two lines, and one circle touches both the others. Show that the radii of the circles form a G. P.

4. If a, b, c, d be in G. P., prove that

$$(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2.$$

5. A right-angled non-isosceles triangle has a perpendicular drawn from the right-angled vertex to the hypothenuse. The larger of the resulting triangles is treated in the same manner; and so on. Show that the perpendiculars so drawn form a G. P.

6. AB is the diameter of a circle, and CT is a tangent at any point C on the circle, and AT is perpendicular to CT. Prove that AC is a geometric mean between AB and AT.

HARMONIC SERIES.

170. A number of terms form a Harmonic Series when their reciprocals form an Arithmetic Series.

Thus $\quad 1, 2, 3, 4, 5,$ etc., are in A. P.

And $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5},$ etc., are in H.P.

Let $\quad a, b, c$ be three terms in H. P.

Then $\quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P., and $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}.$

$$\therefore \quad \frac{a}{c} = \frac{a-b}{b-c}.$$

That is, three quantities are in H. P. when the first is to the third as the difference between the first and second is to the difference between the second and third.

The term Harmonic is derived from the property that a string of a musical instrument stopped at lengths corresponding to the terms of an H. P., sounds the harmonics in music.

In algebra itself Harmonic Progression does not play any important part; it is in geometry that it has its principal applications.

No method of summing an H. P. is known.

EXERCISE XII. d.

1. Find the Harmonic Mean between a and b.

2. If A, G, H denote the Arithmetic, Geometric, and Harmonic Means between two quantities, show that $G = \sqrt{AH}$.

3. A, P, B, Q are four points in line, and C is half-way between A and B. If AP, AB, AQ are in H. P., then CP, CB, CQ are in G. P.

4. If 1, c, a are in A. P. and 1, c, b, in G. P., can c, a, b be in H. P.?

5. If x, y, z be in H. P., a, x, b in A. P., and a, z, b in G. P., show that

$$y = 2(a+b)\left\{\left(\frac{a}{b}\right)^{\frac{1}{4}} + \left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}^{-2}.$$

6. Three numbers are in G. P. If the first two be each increased by 1, the result is in A. P.; and if 2 be then added to the third, the result will be in H. P. Find the numbers.

INTEREST AND ANNUITIES.

171. Let P denote the principal, r the rate per unit, t the time in years, and A the sum of the principal and interest at the end of t years.

Then $Pr =$ the interest for 1 year.

(1) If the interest is simple, this interest remains the same for every year, and in t years becomes Prt. And adding the principal gives

$$A = P(1 + rt),$$

which is the relation connecting the quantities in simple interest.

(2) In a case of compound interest, the amount at the end of the first year becomes the principal for the second

year; the amount at the end of the second year becomes the principal for the third year, etc.

$$\text{Amount at end of 1st year} = P(1+r).$$
$$\text{Amount at end of 2d year} = P(1+r)(1+r) = P(1+r)^2.$$
$$\text{Amount at end of 3d year} = P(1+r)^3.$$

Similarly,

$$\text{Amount at end of } t \text{ years} = P(1+r)^t,$$

or

$$A = P(1+r)^t,$$

which expresses the relation connecting the quantities in compound interest.

172. The present value of a sum of money, payable at some fixed future date, is that sum which put at interest will amount to the given sum at the given date.

Ex. **1.** What is the present value of a sum, S, payable t years hence, money being worth r per unit.

Let V be the value.

Then $V(1+r)^t = S. \quad \therefore \; V = \dfrac{S}{(1+r)^t}.$

The result is given for compound interest, as in all such cases compound interest is the only kind practically considered.

Ex. **2.** If the sum S pays yearly interest at rate r, and money is worth rate r_1, we have

$$S(1+r)^t = \text{amount of } S \text{ in } t \text{ years, rate } r,$$

and

$$V(1+r_1)^t = \text{amount of } V \text{ in } t \text{ years, rate } r_1,$$

and as these must be equal,

$$V = S\left(\frac{1+r}{1+r_1}\right)^t.$$

Ex. **3**. A loan of $5000 pays interest annually at 4% for 4 years, and is to be then paid in full. What is its present value, reckoning money at 6%?

$$V = 5000\left(\frac{1.04}{1.06}\right)^4 = \$4633.23\cdots$$

OF ANNUITIES.

173. An annuity is a fixed payment of money made at stated and equidistant intervals.

If the payments continue for a definite time, it is an *annuity certain*, or a fixed annuity; if they continue only during a person's life, it is a *life annuity;* and if they continue for all time, it is a *perpetuity*.

Annuities may pay annually, or semi-annually, or quarterly, or at any other stated times; but as the principles are the same in dealing with all of these, we shall, unless otherwise stated, consider the payments as being made annually.

Problem. To find the present value of a fixed annuity.

Let P be the annual payment, r the rate per unit of interest, t the number of years the annuity is to run, V its present value, and let R stand for $1 + r$.

Let us suppose that the annuity is paid into a bank, and left there for t years from the time of purchase, to accumulate at compound interest.

The 1st payment draws interest for $t-1$ years, and amounts to PR^{t-1}.

The second payment, similarly, amounts to PR^{t-2}.

The 3d payment amounts to PR^{t-3}.

etc. etc.

The last payment is PR^{t-t} or P.

Therefore the whole amount is

$$P(1+R+R^2+\cdots R^{t-1})=P\cdot\frac{R^t-1}{R-1}.$$

Now if the purchase money were deposited in the same way, it should, in t years, amount to the same sum. But V dollars in t years amounts to $V(1+r)^t$

$$\therefore\ VR^t=P\cdot\frac{R^t-1}{R-1}.$$

Whence
$$V=P\cdot\frac{1-R^{-t}}{r}.$$

Ex. What is the present value of $100 paid annually for 6 years at 10% compound interest?

Here $R^t=(1.1)^6=1.77156$; and $R^{-t}=0.56447.$

Then $V=100\times\frac{1-0.56447}{0.1}=\$435.53.$

Cor. When $t=\infty$, the annuity becomes a perpetuity, and for its present value

$$V=\frac{P}{r}.$$

174. When an annuity does not begin to pay until after the lapse of a number of years, it is said to be *deferred*, or to be in *reversion*.

Problem. To find the present value of an annuity in reversion.

Let p be the number of years the annuity is deferred, and let t be the number of years through which its payments run.

The amount of the annuity at the end of $p+t$ years is

$$P\cdot\frac{1-R^t}{1-R};$$

and the amount of V for the same time is

$$VR^{p+t},$$

and these must be equal.

$$\therefore\ V=\frac{P}{R^{p+t}}\cdot\frac{R^t-1}{R-1}=P\left\{\frac{R^{-p}}{r}-\frac{R^{-(p+t)}}{r}\right\}.$$

Cor. When $t=\infty$, we have as the present value of a deferred perpetuity

$$V=P\left\{\frac{R^{-p}}{r}\right\}=\frac{P}{rR^p}.$$

Ex. A young man, at the age of 19, will come into a property at 23 that will pay him \$1000 yearly during his life. If his life probability at 23 is 40 years, how much is his annuity now worth, money being at 6%?

Here $P=1000,\ p=4,\ t=40,\ r=0.06,\ R=1.06.$

And $V=\frac{1000}{0.06}\left\{\frac{1}{(1.06)^4}-\frac{1}{(1.06)^{44}}\right\}.$

This cannot be conveniently worked out without the use of either logarithms, or tables of the powers of 1.06.

The value is \$11918, to the nearest dollar.

175. An annuity which has not been paid for a number of years is said to be *foreborne.* The present value of a foreborne annuity is the cash value of all due, together with the present value of the annuity as continued into the future.

To find the present value of a foreborne annuity. Let the annuity be foreborne for q years.

Its cash value is $P\cdot\frac{1-R^q}{1-R}$ or $P\cdot\frac{R^q-1}{r}.$

And its value for the future is $P\cdot\dfrac{1-R^{-t}}{r}$, t being the number of years it is to continue.

$$\therefore\ V=P\left\{\frac{R^{q}-R^{-t}}{r}\right\}.$$

Cor. If $t=\infty$, we have for the present value of a foreborne perpetuity

$$V=P\cdot\frac{R^{q}}{r}.$$

176. The following problem is of special importance.

Problem. A corporation borrows A dollars, which is to be paid in t equal annual instalments, each instalment to cover all interest due at the time of payment. To find the value of each instalment.

A part of the instalment goes to pay interest, and the remainder goes to reduce the debt.

Let $a, b, c, \cdots t$, be the parts applied in successive years to the reduction of the debt, and let p be one of the annual instalments.

Then,

1st payment $=p=a+Ar$.
Reduced debt $=A-a$.
2d payment $=p=b+(A-a)r$; whence $b=aR$.
Reduced debt $=A-a-b=A-a-aR$.
3d payment $=p=c+(A-a-aR)r$; whence $c=aR^2$.
Reduced debt $=A-a-b-c=A-a-aR-aR^2$.
. . . .
tth payment $=p=t+\{A-a-aR-aR^2\cdots-aR^{t-2}\}r$.
Reduced debt $=A-\{a+aR+aR^2+\cdots aR^{t-1}\}$.

But after the tth payment the debt must be nothing.

$$\therefore\ A = a(1 + R + R^2 + \cdots R^{t-1}) = a\frac{R^t - 1}{R - 1}.$$

But $a = p - Ar$, and eliminating a between these gives

$$p = A \cdot \frac{rR^t}{R^t - 1},$$

which is the value of the annual payment.

EXERCISE XII. e.

1. A mortgage for $1200 pays $400 annually for 3 years without interest. What is its cash value when drawn, money being reckoned at 6% compound interest?

2. Find the present value of the mortgage, of Ex. 1, if it pays interest at 4%, while money is worth 5%.

3. An annual annuity of $1000 is foreborne for 5 years, and is to run 8 years in all. What is it now worth, money being reckoned at 4% interest?

4. A man borrows $500 on a mortgage and wishes to pay principal and interest in 5 equal instalments. What is the amount of each instalment, calculating interest at 6%?

5. A corporation borrows $30000 at 4% interest, and is to repay principal and interest in 30 equal instalments. What is the value of an instalment?

CHAPTER XIII.

PERMUTATIONS, COMBINATIONS, BINOMIAL THEOREM.

177. If from n different objects we form groups each containing r objects, such that no two groups contain the same assemblage of objects, each group is called a **Combination**, and the possible number of such groups is the number of combinations of n things r together.

This number is symbolized by ${}^nC_r$, and read 'n objects combined by r's.'

Thus ${}^4C_3$, taking letters as objects, is 4, and the several groups or combinations are

abc, *abd*, *acd*, and *bcd*.

Similarly ${}^5C_2$ has for its groups *ab, ac, ad, ae, bc, bd, be, cd, ce,* and *de*; or 10 in all.

The combination *abc* is the same as *acb*, the same as *bac*, etc., since all have the same assemblage of letters.

If, however, we take relative position into consideration, *abc* and *acb* are not the same, since, although they contain the same letters, the letters are differently arranged. Distinguishing different groups in this way, each group is called a **Permutation**, and the possible number of such groups is the number of permutations of n things r together.

This number is symbolized by ${}^nP_r$.

The combination *abc* gives 6 permutations: *abc, acb, bac, bca, cab,* and *cba*; and as each combination may be treated similarly, the number of permutations of 4 letters 3 together is 24; or ${}^4P_3 = 24$.

PERMUTATIONS.

178. Problem. To find the number of permutations of n things r together, n being greater than r.

If we have r boxes, $A, B, C, \cdots$, etc., into each of which one of the n letters, $a, b, c \cdots$ is to be put, the number of ways in which the distribution can be effected is the number of permutations of n things r together.

In filling box A, we may choose any one of the n letters, and we have therefore n choices.

Having filled A, we have $n-1$ choices in filling B, and any one of these $n-1$ choices may be combined with the n choices in filling A.

Hence in filling A and B we have $n(n-1)$ choices.

Similarly, in filling $A, B,$ and C, we have $n(n-1)(n-2)$ choices, and so on through the r boxes.

$$\therefore\ {}^nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

179. Multiply the value found for ${}^nP_r$ by the unit fraction

$$\frac{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdots(n-r-1)(n-r)},$$

and we obtain

$${}^nP_r = \frac{1\cdot 2\cdot 3\cdots(n-2)(n-1)n}{1\cdot 2\cdot 3\cdots(n-r)}.$$

The continued product of m consecutive natural numbers, beginning at 1, is called *factorial m*, and is symbolized as $m!$, or $\underline{|m}$.

Hence $${}^nP_r = \frac{n!}{(n-r)!}.$$

Cor. When $r = n$, we have

$${}^nP_n = n! \text{ (Art. 178)} = \frac{n!}{0!} \text{ (Art. 179)},$$

and hence $0!$ must be interpreted as meaning 1.

Ex. 1. The number of permutations of 12 things 5 together is

$${}^{12}P_5 = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95040.$$

Ex. 2. The number of permutations of n things 3 together is $14\frac{2}{3}$ times the number of permutations of $n-2$ things 2 together, to find n.

Here $${}^nP_3 = 14\tfrac{2}{3} \cdot {}^{n-2}P_2,$$

or $$\frac{n(n-1)(n-2)}{(n-2)(n-3)} = \frac{44}{3} = \frac{n^2-n}{n-3}.$$

Whence $$n = 12, \text{ or } 3\tfrac{2}{3}.$$

As n must be integral, its value is 12, and $3\frac{2}{3}$ must be rejected as being inapplicable to the nature of the problem.

Nevertheless, $(3\frac{2}{3} \cdot 2\frac{2}{3} \cdot 1\frac{2}{3}) \div (1\frac{2}{3} \cdot 0\frac{2}{3}) = \frac{44}{3}$.

180. In the permutations ${}^nP_r$ to find how many contain a particular object or letter, as a.

Putting a aside, we form a group of $r-1$ from $n-1$ objects, and this can be done in

$$(n-1)(n-2)\cdots(n-r+1) \text{ ways.}$$

In each of these groups a can have r positions; namely, from preceding all the other letters to following them all.

Hence the number of permutations containing a is

$$r(n-1)(n-2)\cdots(n-r+1).$$

Similarly, the number of permutations containing two particular letters, as a and b, is

$$r(r-1)(n-2)\cdots(n-r+1).$$

Containing 3 particular letters together, it is

$$r(r-1)(r-2)(n-3)\cdots(n-r+1).$$

etc. etc.

Ex. How many numbers can be made from 5 figures, 1, 2, 3, 4, 5, three at a time; and how many of these will contain 1? How many contain 1 and 2?

1st. $^nP_r = 5\cdot 4\cdot 3 = 60.$

2d. $r(n-1)(n-2) = 3\cdot 4\cdot 3 = 36.$

3d. $r(r-1)(n-2) = 3\cdot 2\cdot 3 = 18.$

181. To find the number of permutations of n things, all together, when u of the things are alike.

Denote the number by $^nP(u)$.

If the u things were all different, they would in themselves give rise to $u!$ permutations, each of which combined with each of the permutations of $^nP(u)$ would give ^nP_n.

$$\therefore\ ^nP(u)\times u! = {}^nP_n.$$

Or
$$^nP(u) = \frac{n!}{u!}.$$

Similarly, if $^nP(u)(v)$ denotes the number of permutations all together, when u articles are alike of one kind, and v articles are alike of another kind,

$$^nP(u)(v) = \frac{n!}{u!\,v!}, \text{ etc.}$$

Ex. How many permutations can be made from the letters in *Mississippi?*

Here there are 11 letters, of which 4 are i's, 4 are s's, and 2 are p's.

$$\therefore\ {}^nP(u)(v)(w) = \frac{11\,!}{4\,!\,4\,!\,2\,!} = 34650.$$

If the permutations were to be such as not to have repeated letters, we have only 4 different letters, and the number is

$${}^nP_n = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

EXERCISE XIII. a.

1. Find the values of —

i. ${}^3P_2$. ii. ${}^3P_3$. iii. ${}^7P_4$. iv. ${}^{n-l}P_m$.

2. Given ${}^nP_4 = 3\,{}^{n-1}P_4$, to find ${}^nP_6$.

3. Given ${}^nP_3 = \frac{15}{7}\,{}^{n-2}P_3$, to find n.

4. How many permutations can be made from the letters in *College?* in *Oporto?* in *Amsterdam?*

In each of these how many permutations would have letters repeated?

5. A person writes at random 3 of the figures 1, 2, 3, 4, 5, 6. What is the probability that they will be consecutive and in ascending order?

6. With four different consonants and a vowel, how many words of 3 letters can be made having a vowel in each.

7. The figures from 1 to 9 are written down, and a person erases 3 figures at random. What is the chance that the figures erased may be consecutive?

8. Six points are taken on a circle. In how many different ways may they be joined by twos?

COMBINATIONS.

182. As a combination has no reference to arrangement, each combination of r articles can give rise to $r!$ permutations r together.

Hence $${}^nP_r = r! \times {}^nC_r.$$

$$\therefore\ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!},$$

or, by reduction, $${}^nC_r = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots \text{ to } r \text{ factors.}$$

Cor. Since ${}^nC_r$ is necessarily an integer, it follows that the continued product of any r consecutive integers is divisible by factorial r.

183. Substituting $n-r$ for r gives

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,r!} = {}^nC_r.$$

Or the number of combinations of n things r together is the same as the number $n-r$ together.

This is quite self-evident, for every time we take an assemblage of r things out of n things we leave an assemblage of $n-r$ things, and the numbers of the two assemblages must necessarily be equal.

${}^nC_r$ and ${}^nC_{n-r}$ are *complementary* combinations.

184. Since ${}^nC_r = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots$, the number of combinations will increase with r as long as the last fractional factor is greater than 1. But this factor is

$\frac{n-r+1}{r}$; and while this is >1, the number of combinations increases.

$$\therefore\ n-r+1>r, \text{ or } r<\frac{n+1}{2}.$$

And r is to be the integer nearest to but less than $\frac{1}{2}(n+1)$. Therefore

If n is even, the value of r which makes ${}^nC_r$ greatest is $r=\frac{1}{2}n$; and if n is odd, the value of r is $\frac{1}{2}(n-1)$, or $\frac{1}{2}(n+1)$, the latter value giving a unit-factor.

Ex. $${}^{12}C_4=\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}={}^{12}C_8=495.$$

$${}^{12}C_6=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}=924.$$

$${}^{11}C_5=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}={}^{11}C_6.$$

185. In the combinations ${}^nC_r$, to find the number of times any particular object, as a, will be present.

If we form ${}^{n-1}C_{r-1}$ from all the letters except a, taken $r-1$ together, we can place a with each of these groups, and we then have all the combinations of n letters r together containing a.

Thus a occurs ${}^{n-1}C_{r-1}$ times.

Similarly, ab occurs ${}^{n-2}C_{r-2}$ times, etc.

Ex. Out of a guard of 12 men 5 are drafted for duty each night. Relatively, how often will A be on duty? How often will A and B be together on duty? How often will A be on duty without B?

As ${}^{12}C_5=792$, this is the total number of different drafts.

1. A is on duty ${}^{11}C_4=330$ times out of 792.
2. A and B are together ${}^{10}C_3=120$ times out of 792.
3. $\therefore$ A is present without B, 210 times out of 792.

EXERCISE XIII. b.

1. If $a = {}^nC_3$, and $b = {}^nP_2$, find the relation between a and b.

2. If $a = {}^nC_r$, and $b = {}^{n-1}P_{r-1}$, find the relation between a and b.

3. At an election there are 10 candidates, of which 4 are to be elected. If a man may vote for 1 or more up to 4, how many different votes can he cast?

4. If ${}^nC_{n-1} = {}^{m+1}C_{m-1}$, then $m(m+1) = 2n$.

5. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

6. Prove that $n\{{}^nC_r + {}^{n-1}C_{r-1}\} = (n+r){}^nC_r$.

7. 3 black and 2 white balls are put into a bag; what is the chance of drawing 2 black balls at a single drawing of 2 balls?

THE BINOMIAL THEOREM.

186. The expansion of $(1+x)^n$, with n a positive integer, is the simplest form of the binomial theorem, or binomial formula. The theorem is then generalized and adapted to any value of n whatever. The simplest case is first established.

I. n A POSITIVE INTEGER.

The number of terms in Σa with n letters is ${}^nC_1$; the number of terms in Σab is ${}^nC_2$; and generally the number in $\Sigma ab \cdots r$ is ${}^nC_r$.

These statements are self-evident.

Now $(x+a)(x+b)(x+c)\cdots$ to n factors is

$$x^n + \Sigma a \cdot x^{n-1} + \Sigma ab \cdot x^{n-2} + \cdots + \Sigma abc \cdots r \cdot x^{n-r} + \cdots abc \cdots n.$$

And making $a = b = c = \cdots = 1$, we obtain

$$(x+1)^n = x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \cdots + {}^nC_n.$$

Or, since $\quad {}^nC_n = 1,\ {}^nC_{n-1} = {}^nC_1,\ {}^nC_{n-2} = {}^nC_2$, etc.

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \cdots {}^nC_rx^r + \cdots \qquad (A)$$

Also writing the factor values of ${}^nC_1$, ${}^nC_2$, etc.,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2$$
$$+ \cdots \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}\cdot x^r + \cdots \qquad (B)$$

(A) and (B) are common forms of the binomial theorem; and

$$\frac{n(n-1)\cdots(n-r+1)}{r!}x^r,$$

the $(r+1)$th term from the beginning, is called the general term.

Knowing the particular value of each coefficient, these coefficients are often denoted by a single letter with subscript numbers, and the theorem then becomes

$$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots c_rx^r \cdots \qquad (C)$$

where $\qquad c_0 \equiv 1,\ c_1 \equiv n,\ c_2 \equiv \dfrac{n(n-1)}{2!}$, etc.

Ex. 1. $\quad (a+x)^n = a^n\left(1+\dfrac{x}{a}\right)^n = a^n\left(1 + c_1\dfrac{x}{a} + c_2\dfrac{x^2}{a^2} + \cdots\right)$

$$= a^n + c_1a^{n-1}x + c_2a^{n-2}x^2 + \cdots c_ra^{n-r}x^r + \cdots$$

and thus the expansion of any binomial depends upon that of $(1+x)$.

Ex. 2. $\quad (1-x)^n = 1 + c_1(-x) + c_2(-x)^2 + c_3(-x)^3 + \cdots$

$$= 1 - c_1x + c_2x^2 - c_3x^3 + - \cdots$$

the signs being alternately $+$ and $-$.

187. Since $(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$, making $x = 1$ gives

$$2^n = 1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n.$$

$\therefore$ The total number of combinations of n things taken 1 at a time, 2 at a time, and so on to n at a time, is $2^n - 1$.

Also, the sum of the coefficients of the expansion of $(1+x)^n$ is 2^n.

Also, since $(1-x)^n = 1 - c_1x + c_2x^2 - c_3x^3 + \cdots$, by making $x = 1$ we have

$$0 = 1 + c_2 + c_4 + \cdots - (c_1 + c_3 + c_5 + \cdots).$$

Or, the sum of the odd coefficients in the expansion of $(1+x)^n$ is equal to the sum of the even coefficients.

Ex. To find the sum of the squares of the coefficients of $(1+x)^n$.

$$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n;$$

$$(x+1)^n = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + c_3x^{n-3} + \cdots c_n.$$

The coefficient of x^n from the product of the right-hand members is

$$c_0^2 + c_1^2 + c_2^2 + \cdots c_n^2.$$

But the coefficient from the product of the left-hand members is the coefficient of x^n from the expansion of $(1+x)^{2n}$;

that is, $${}^{2n}C_n \text{ or } \frac{(2n)!}{n!\,n!}.$$

$$\therefore c_0^2 + c_1^2 + c_2^2 + \cdots c_n^2 = \frac{(2n)!}{(n!)^2}.$$

Cor. 1. This last expression on the right is the number of permutations of $2n$ articles, when half of them are alike of one kind, and the other half alike of another kind.

Cor. 2. The coefficient of x^{n-2} from the right product is

$$c_0c_2 + c_1c_3 + c_2c_4 + \cdots c_{n-2}c_n.$$

And from the left it is $\dfrac{(2n)!}{(n-2)!\,(n+2)!}$.

The student is left to generalize this.

EXERCISE XIII. c.

1. Write the general, or $(r+1)$th, term in the expansions —

i. $(a+x)^n$. ii. $(a-x)^n$. iii. $(1+x)^{2n}$.

2. Show that

$$\frac{(a+x)^n}{n!} = \frac{a^n}{n!}\cdot\frac{x^0}{0!} + \frac{a^{n-1}}{(n-1)!}\cdot\frac{x^1}{1!} + \frac{a^{n-2}}{(n-2)!}\cdot\frac{x^2}{2!} + \cdots + \frac{a^0}{0!}\cdot\frac{x^n}{n!}.$$

3. Find the 10th term in the expansion of $(1+x)^{12}$.

4. What is the factor which changes the $(r+1)$th term into the $(r+2)$th term?

5. Find the value of r that the factor of question 4 may be the last one greater than unity.

6. Find the greatest coefficient in the expansion of $(2+3x)^5$.

7. Show that the 5th term has the greatest coefficient in the expansion of $(3+2x)^{12}$.

8. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

9. In the expansion of $\left(x+\frac{1}{x}\right)^n$ show that the coefficient of x^r is

$$\frac{n!}{\left(\overline{\tfrac{1}{2}n-r}\right)!\left(\overline{\tfrac{1}{2}n+r}\right)!}.$$

II. n A NEGATIVE INTEGER.

188. Let the expression $\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}$ be denoted by ${}^nH_r$.

Then $${}^nH_r + {}^{n+1}H_{r-1} = \frac{n(n+1)\cdots(n+r-1)}{r!}$$

$$+\frac{(n+1)(n+2)\cdots(n+r-1)}{(r-1)!} = \frac{(n+1)(n+2)\cdots(n+r)}{r!}$$

$$= {}^{n+1}H_r \quad \dots\dots \quad (D)$$

Now we know from division that

$$(1-x)^{-2}=\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots nx^{n-1}+\cdots$$

$$=1+\frac{2}{1}x+\frac{2\cdot 3}{1\cdot 2}x^2+\frac{2\cdot 3\cdot 4}{1\cdot 2\cdot 3}x^3+\cdots\frac{2\cdot 3\cdot 4\cdots n}{(n-1)!}x^{n-1}+\cdots$$

$$=1+{}^2H_1x+{}^2H_2x^2+{}^2H_3x^3+\cdots{}^2H_{n-1}x^{n-1}+\cdots$$

Therefore let us assume that

$$\frac{1}{(1-x)^n}=1+{}^nH_1x+{}^nH_2x^2+{}^nH_3x^3+\cdots+{}^nH_rx^r+\cdots \quad (E)$$

Divide both sides by $1-x$; the left-hand member becomes $\dfrac{1}{(1-x)^{n+1}}$, and the right as follows:

$$\begin{array}{l|l} & 1 \\ \hline 1+{}^nH_1 \quad +{}^nH_2 \quad +{}^nH_3 \quad +\cdots & \\ \quad +1 \qquad +{}^{n+1}H_1+{}^{n+1}H_2+\cdots & \\ \hline 1+{}^{n+1}H_1+{}^{n+1}H_2+{}^{n+1}H_3+\cdots & \end{array}$$

because ${}^nH_r+{}^{n+1}H_{r-1}={}^{n+1}H_r$, from (D).

$$\therefore \frac{1}{(1-x)^{n+1}}=1+{}^{n+1}H_1x+{}^{n+1}H_2x^2+{}^{n+1}H_3x^3$$
$$+\cdots+{}^{n+1}H_rx^r+\cdots$$

Hence if the expansion is true for any value of n, it is true for the next greater value. But it is true for $n=2$, and therefore for $n=3$, for $n=4$, etc.; that is, it is generally true. And the expansion, (E), is true.

Now $(1-x)^n=1-nx+\dfrac{n(n-1)}{1\cdot 2}-\dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\cdots$

Change n to $-n$, and this becomes

$$(1-x)^{-n}$$
$$=1-(-n)x+\frac{-n(-n-1)}{1\cdot 2}x^2-\frac{-n(-n-1)(-n-2)}{1\cdot 2\cdot 3}x^3+\cdots$$
$$=1+nx+\frac{n(n+1)}{1\cdot 2}x^2+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3+\cdots$$
$$=1+{}^nH_1x+{}^nH_2x^2+{}^nH_3x^3+\cdots, \text{ agreeing with } (E).$$

And we see that the general form of the binomial theorem holds good for all integral values of n, positive or negative.

With n positive, the series terminates when $\frac{n-r+1}{r\,!}$ becomes zero; *i.e.* when $r=n+1$. Or the series contains $n+1$ terms.

With n negative, however, the series is infinite, as $n(n+1)(n+2)\cdots$ cannot become zero by extending the number of factors.

189. To interpret ${}^nH_r$.

$$\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\cdots+a^rx^r+\cdots$$
$$\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\cdots+b^rx^r+\cdots$$
$$\frac{1}{1-cx}=1+cx+c^2x^2+c^3x^3+\cdots+c^rx^r+\cdots$$

By multiplying n such equations together, the coefficient of x^r, on the right, is the sum of all the homogeneous terms of r dimensions that can be made out of n letters. But if we make $a=b=c=\cdots=1$, the left-hand product becomes $(1-x)^{-n}$, and the coefficient of x^r is ${}^nH_r$.

$\therefore\ {}^nH_r =$ the *number* of homogeneous terms of r dimensions which can be formed from n letters, and their powers.

Thus, if $n = 4$ and $r = 2$, ${}^nH_r \equiv {}^4H_2 = \frac{4 \cdot 5}{1 \cdot 2} = 10$; that is, there are 10 homogeneous terms; namely, a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, and cd.

It is well to notice that if we denote ${}^nH_1$ by h_1, ${}^nH_2$ by h_2, etc.,

$$(1+x)^n = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots \text{ signs all } +.$$

$$(1-x)^{-n} = 1 + h_1x + h_2x^2 + h_3x^3 + \cdots \text{ signs all } +.$$

$$(1-x)^n = 1 - c_1x + c_2x^2 - c_3x^3 + - \cdots \text{ signs alternate.}$$

$$(1+x)^{-n} = 1 - h_1x + h_2x^2 - h_3x^3 + - \cdots \text{ signs alternate.}$$

EXERCISE XIII. d.

1. Expand $(1-x)^{-4}$, and show that the coefficients are the sums of the coefficients of $(1-x)^{-3}$.

2. Expand $\dfrac{1+x}{1+x+x^2}$ in ascending powers of x, and find the coefficient of x^n in the expansion.

3. Find the coefficient of x^n in the expansion of $\left(\dfrac{1+x}{1-x}\right)^n$.

4. Prove that $(1+x)^n = 2^n \left\{1 - h_1 \cdot \dfrac{1-x}{1+x} + h_2\left(\dfrac{1-x}{1+x}\right)^2 - + \cdots\right\}$.

5. Find the coefficient of x^{100} in $\dfrac{3-5x}{(1-x)^2}$.

6. Find the coefficient of x^n in the expansion of

$$(1 - 2x + 3x^2 - 4x^3 + - \cdots)^{-n}.$$

7. Show that if n is a positive integer, $(5 + 2\sqrt{6})^n$ is odd in its integral part.

Since $5^2-(2\sqrt{6})^2=1$ and $5+2\sqrt{6}>1$, $\therefore$ $5-2\sqrt{6}<1$, and is accordingly a proper fraction.

$\therefore$ $5^n-c_1 5^{n-1}\cdot 2\sqrt{6}+c_2 5^{n-2}2^2\cdot 6-+\cdots=f'=$ proper fraction.

$5^n+c_1 5^{n-1}\cdot 2\sqrt{6}+c_2 5^{n-2}2^2\cdot 6++\cdots=I+f'=$ an integral part + a proper fraction.

$\therefore$ $2\{5^n+c_2\cdot 5^{n-2}2^2\cdot 6+\cdots\}=I+f+f'=$ an integer.

$\therefore$ $f+f'$ must $=1$; and as $I+f+f'$ must be even, I must be odd.

8. Show that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

9. By the Binomial theorem find $99^{\frac{1}{4}}$.

10. Prove that $c_0-2c_1+3c_2-+\cdots+(-1)^n(n+1)c_n=0$.

11. Show that ${}^nH_r={}^{n+r-1}C_r$.

III. n A FRACTION.

190. With n fractional there are certain difficulties in the Binomial theorem which we cannot here explain; and no very satisfactory proof of the theorem with n fractional can be given without involving higher considerations than occur in this work.

Several methods, however, will furnish proofs which are morally sufficient.

The following is Euler's.

$(1+x)^n$ is a function of n; denote it by fn.

Then $(1+x)^m=fm$, and $(1+x)^{m+n}=f(m+n)$.

But $(1+x)^m\cdot(1+x)^n=(1+x)^{m+n}$ by the index law.

$$\therefore\ fm\cdot fn=f(m+n).$$

Similarly, $fm\cdot fn\cdot fp=f(m+n+p)$;

and generally,

$$fm\cdot fn\cdot fp\cdots k\text{ factors}=f(m+n+p\cdots k\text{ terms}).$$

Now let $$m = n = p = \cdots = \frac{h}{k}.$$

Then $$\left\{ f\left(\frac{h}{k}\right) \right\}^k = fh,$$

and $$f\left(\frac{h}{k}\right) = (fh)^{\frac{1}{k}} = (1+x)^{\frac{h}{k}}.$$

But, $$fn = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \cdots$$

and $$\therefore\ f\left(\frac{h}{k}\right) = 1 + \frac{h}{k} x + \frac{\frac{h}{k}\left(\frac{h}{k} - 1\right)}{1 \cdot 2} x^2 + \cdots$$

Or $$(1+x)^{\frac{h}{k}} = 1 + \frac{h}{k} x + \frac{\frac{h}{k}\left(\frac{h}{k} - 1\right)}{1 \cdot 2} x^2 + \cdots$$

And the form of the Binomial theorem holds good for n fractional.

Cor. If we make $k = -1$, $f(-h) = (fh)^{-1}$.

But

$$f(-h) = 1 - hx + \frac{-h(-h-1)}{1 \cdot 2} x^2 + \frac{-h(-h-1)(-h-2)}{1 \cdot 2 \cdot 3} x^3 + \cdots$$

and $$(fh)^{-1} = \frac{1}{fh} = \frac{1}{(1+x)^h}.$$

$$\therefore\ \frac{1}{(1+x)^h} = (1+x)^{-h}$$

$$= 1 - hx + \frac{h(h+1)}{1 \cdot 2} x^2 - \frac{h(h+1)(h+2)}{1 \cdot 2 \cdot 3} x^3 + \cdots$$

which proves the theorem for negative indices.

EXERCISE XIII. e.

1. To find $\sqrt{1-x}$. This is $(1-x)^{\frac{1}{2}}$,

and $(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2 - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}x^3 + \cdots$

$$= 1 - \tfrac{1}{2}x - \frac{1}{2\cdot 4}x^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 - \cdots$$

2. Write the general term, $(r+1)$th, of Ex. 1.

3. Expand $(1+x)^{\frac{3}{2}}$ in ascending powers of x.

4. Find the approximate value of $a(1-x)\sqrt{1+x^2}$ when x^5 is so small as to be rejected.

5. Expand $(1+x)^{\frac{1}{x}}$, and find the result when $x=0$.

6. Expand $\left(1+\frac{1}{x}\right)^x$, and find the result when $x=\infty$.

7. Find the value of \$1 compounded every moment for t years at r% per annum.

8. By expanding $(1+x)^{\frac{1}{n}}$ and making $x=3$ and $n=2$, show that 2 is the limit of the series

$$1 + \frac{3}{2} - \frac{1}{1\cdot 2}\left(\frac{3}{2}\right)^2 + \frac{1\cdot 3}{1\cdot 2\cdot 3}\left(\frac{3}{2}\right)^3 - + \cdots$$

9. Find the limit of the series to infinity —

$$1 + \tfrac{1}{2}\cdot 2 - \tfrac{1}{8}\cdot 2^2 + \tfrac{3}{48}\cdot 2^3 - \tfrac{15}{384}\cdot 2^4 + - \cdots$$

10. If $e \equiv \left(1+\frac{1}{x}\right)^x$ where $x=\infty$, show that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots \frac{y^n}{n!} + \cdots$$

CHAPTER XIV.

INEQUALITIES.

191. When two unequal expressions are compared, particularly with the purpose of showing that the expressions are not equal, the whole is called a *non-equation* or *inequality*.

An inequality employs the signs $>$ and $<$ between its members, and sometimes the signs $\neq$, read *not equal to*, $\ngtr$, read *not greater than*, and $\nless$, read *not less than*.

It usually happens that some values of the variables will change an inequality to an equality, *i.e.* an identity.

The working rules for inequalities being in some respects different from those for equations, must be here established.

I. Let $$a > b,$$

and let all the quantitative symbols denote positive quantities.

1. Let $a = b + \beta$, and add p to both sides.

$\therefore\ a + p = b + p + \beta$, or $a + p > b + p$.

2. Subtract p from both sides, and

$$a - p = b - p + \beta, \text{ or } a - p > b - p.$$

Hence the same quantity may be added to or subtracted from both members of an inequality; and hence a term may be transposed from one member to the other by changing the sign of the transposed term, without affecting the character of the inequality.

3. Subtract both members from p.

Then $p - a = p - b - \beta$; or $p - a < p - b$, and the character of the inequality is changed.

Therefore, if both members be subtracted from the same quantity, the character of the inequality is reversed.

4. $ma = mb + m\beta$; or $ma > mb$.

5. $\frac{a}{m} = \frac{b}{m} + \frac{\beta}{m}$; or $\frac{a}{m} > \frac{b}{m}$.

Hence, if both members be multiplied or be divided by the same quantity, the character of the inequality is unchanged.

6. $\frac{m}{a} = \frac{m}{b+\beta} = \frac{m}{b} - \frac{m\beta}{b(b+\beta)}$; or $\frac{m}{a} < \frac{m}{b}$.

Hence, dividing the same quantity by both members changes the character of the inequality.

7. To multiply or divide both sides by a negative quantity is equivalent to exchanging the members, and therefore it reverses the character of the inequality.

II. Let $a > b$ and $c > d$.

Put $a = b + \beta$, and $c = d + \delta$.

8. $a + c = b + d + \beta + \delta$; or $a + c > b + d$.

9. $a - c = b - d + \beta - \delta$; from which we cannot infer whether $a - c > b - d$ or $< b - d$.

If $\beta > \delta$, $a - c > b - d$; but if $\beta < \delta$, $a - c < b - d$.

Hence, inequalities of the same character may have corresponding members added; but they do not in general admit of being subtracted.

192. Inequalities are usually referred to certain standard forms, or determined by fixed relations.

(1) For all values of x and y, except equality,

$$x^2 + y^2 > 2xy.$$

Proof. $(x - y)^2$ is essentially positive, and $\therefore > 0$.

$$\therefore\ x^2 + y^2 - 2xy > 0,$$

and $$x^2 + y^2 > 2xy \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Ex. 1. The sum of a number and its reciprocal is greater than 2.

$$x + \frac{1}{x} > 2,$$

if $$x^2 + 1^2 > 2x \cdot 1.$$

And this latter is true (A). $\therefore$ the former is.

Ex. 2. To show that $1 + a^2 + a^4 > \frac{3}{2}(a + a^3)$.

If a is negative, this is evidently true, since the left-hand member is essentially positive.

Let a be positive.

To prove that $\quad 2 + 2a^2 + 2a^4 > 3a + 3a^3.$

$$(a - 1)(a^3 - 1) = a^4 - a - a^3 + 1;\ \text{and is} + \text{when } a \text{ is } +.$$

$$\therefore\ 1 + a^4 > a + a^3.$$

Also $$1 + a^2 > 2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Ex. } 1)$$

and $$a^2 + a^4 > 2a^3.$$

$\therefore$ Adding, $$2 + 2a^2 + 2a^4 > 3a + 3a^3.$$

193. (2) $(x^n - y^n)(x^m - y^m) > 0$, if m and n are both odd or both even positive integers.

Proof. If x and y have the same sign or opposite signs, both factors have the same sign, and the product is positive.

Ex. 1. $x^6 + y^6 \gtrless x^5y + xy^5$ according as

$$x^6 - x^5y + y^6 - xy^5 \gtrless 0,$$

as $$(x^5 - y^5)(x - y) \gtrless 0.$$

But $$(x^5 - y^5)(x - y) > 0.$$

$$\therefore\ x^6 + y^6 > x^5y + xy^5.$$

EXERCISE XIV

1. If $a, b, c \cdots a$ be any unequal quantities forming a cycle, show that $\Sigma a^2 > \Sigma ab$.

2. Show that $a^2 + 3b^2 > 2b(a + b)$.

3. Show that $a^3b + ab^3 > 2a^2b^2$.

4. Show that $(a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2$.

5. With three letters, $\Sigma a^2 b > 6abc$.

6. $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} > 6$, unless $a = b = c$.

7. Which is the greater —

i. $(a^2 + b^2)(c^2 + d^2)$ or $(ac + bd)^2$?

ii. $m^2 + m$ or $m^3 + 1$?

iii. $\sqrt{\left(\frac{a^2}{b}\right)} + \sqrt{\left(\frac{b^2}{a}\right)}$ or $\sqrt{a} + \sqrt{b}$?

8. If x is real, $x^2 - 8x + 22 \not< 6$.

9. Under what circumstances is $x + \frac{3}{x} >$ or < 4?

10. An isosceles triangle is greater in area than a scalene triangle with the same perimeter and base.

CHAPTER XV.

Undetermined Coefficients and their Applications.

194. Theorem. If an integral function of x of n dimensions is satisfied by more than n different quantities, it is satisfied by all quantities, or its coefficients are severally zero.

Let $\quad fx = ax^n + bx^{n-1} + cx^{n-2} + \cdots sx + t = 0$

be satisfied by the n values, α, β, $\gamma \cdots \tau$.

Then $\quad fx = a(x-\alpha)(x-\beta)(x-\gamma)\cdots(x-\tau) = 0.$

Now, if it is satisfied by an $(n+1)$th value z,

$$fz = a(z-\alpha)(z-\beta)(z-\gamma)\cdots(z-\tau) = 0.$$

But z is different from α, and β, and γ, etc., so that none of the binomial factors are zero.

$\therefore a = 0$. And rejecting ax^n, we can show in like manner that $b = 0$; thence that $c = 0$, etc. And the coefficients being severally zero, the function is satisfied by all values for x, since it is zero identically.

195. Let

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + T = ax^n + bx^{n-1} + cx^{n-2} + \cdots + t.$$

Then

$$(A-a)x^n + (B-b)x^{n-1} + (C-c)x^{n-2} + \cdots + (T-t) = 0.$$

And if this equation is to be true independently of the value of x, that is, for all values of x, we must have

$$A = a,\ B = b,\ C = c \cdots T = t.$$

And this establishes the principle of **undetermined coefficients** for functions of finite dimensions.

The statement of the principle is, that if a positive integral function of x of finite dimensions be zero for all values of x, the coefficients of the several powers of x are each equal to zero.

The extension to functions of infinite dimensions will be established hereafter.

We shall now consider applications of this prolific principle.

I. PARTIAL FRACTIONS.

196. The sum of the fractions $\frac{2}{1-x}$ and $\frac{1}{1+x}$ is $\frac{3+x}{1-x^2}$; and with reference to this latter fraction, the parts which make it up by addition are called its *partial* fractions. It is often necessary to separate a fraction into its partials, it being understood that the denominators of the partials shall be linear whenever practicable, but at any rate be less complex than that of the original.

Ex. 1. To separate $\frac{3+x}{1-x^2}$ into its partials.

Since the denominator is $(1-x)(1+x)$,

assume $$\frac{3+x}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x};$$

where A and B are coefficients to be determined.

Then, $3+x=A(1+x)+B(1-x).$

And as this is to be true for all values of x, we apply the principle of undetermined coefficients, which gives

$$3=A+B, \text{ and } 1=A-B.$$

Whence $A=2$, and $B=1$;

and $$\frac{3+x}{1-x^2}=\frac{2}{1-x}+\frac{1}{1+x}.$$

Ex. **2**. To separate $\frac{x^2}{(x-1)(x-2)(x-3)}$ into its partials.

Assume $$\frac{x^2}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

Then

$$x^2=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2).$$

We might now equate coefficients; but the following method is simpler.

Since this equation is to hold for all values of x,

Make $x=1$; then $1=2A$, and $A=\frac{1}{2}$.
Make $x=2$; then $4=-B$, and $B=-4$.
Make $x=3$; then $9=2C$, and $C=\frac{9}{2}$.

$$\therefore \frac{x^2}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}.$$

Ex. **3**. To separate $\frac{x^2-x+1}{(x-1)^2(x+2)}$ into its partials.

In forming this fraction by addition there may have been a fraction of the form $\frac{a}{x-1}$ and another of the form $\frac{b}{(x-1)^2}$, and in our assumption we make provision for these. Therefore we assume

$$\frac{x^2-x+1}{(x-1)^2(x+2)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+2}.$$

Then $x^2-x+1=A(x+2)+B(x-1)(x+2)+C(x-1)^2.$

Let $x=1$; then $1=3A$, and $A=\frac{1}{3}$.

Substitute $\frac{1}{3}$ for A, and

$$x^2 - \tfrac{4}{3}x + \tfrac{1}{3} = B(x-1)(x+2) + C(x-1)^2.$$

Let $\qquad x = -2$; then $7 = 9C$, and $C = \frac{7}{9}$.

Substitute $\frac{7}{9}$ for C, and equate the coefficients of x^2, which gives

$$1 = B + \tfrac{7}{9}, \text{ or } B = \tfrac{2}{9}.$$

$$\therefore \frac{x^2 - x + 1}{(x-1)^2(x+2)} = \frac{1}{3(x-1)^2} + \frac{2}{9(x-1)} + \frac{7}{9(x+2)}.$$

Ex. 4. To separate $\dfrac{3x^2 + x - 1}{x^3 - 1}$ into partials.

The denominator is $(x-1)(x^2+x+1)$, and the quadratic factor is not separable into real factors.

But a proper fraction with a quadratic factor in its denominator may have a linear factor in its numerator. We make provision for this by assuming

$$\frac{3x^2 + x - 1}{x^3 - 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}.$$

Then, $\qquad 3x^2 + x - 1 = A(x^2 + x + 1) + (Bx + C)(x - 1),$

whence we readily find $\qquad A = 1,\ B = 2,\ C = 2.$

$$\therefore \frac{3x^2 + x - 1}{x^3 - 1} = \frac{1}{x-1} + \frac{2x + 2}{x^2 + x + 1}.$$

For a fuller discussion of this subject the student is referred to works on Higher Algebra, and to the Calculus.

EXERCISE XV. a.

1. Separate into partial fractions the following —

i. $\dfrac{x+2}{(x-1)(x-2)}$.

ii. $\dfrac{x+1}{x^2 - 5x + 6}$.

iii. $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$.

iv. $\dfrac{ax+b}{(a-x)(b-x)b}$.

v. $\dfrac{7x}{(2x+3)(x+2)^2}$.

vi. $\dfrac{ax}{a^2 - x^2}$.

197. We shall now extend the principle of undetermined coefficients to the case of an integral function of x of infinite dimensions.

Theorem. In a positive integral function of x of infinite dimensions, and arranged in ascending powers, any term may be made greater than the sum of all that follow by making x sufficiently small.

Let $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \cdots$ be the function, and let cx^2 be the term chosen.

Also let k be greater than any coefficient following c.

Then $\quad kx^3(1 + x + x^2 + \cdots) > dx^3 + ex^4 + fx^5 + \cdots$

Or $\quad kx^3 \cdot \dfrac{1}{1-x} > dx^3 + ex^4 + fx^5 + \cdots$

But $\quad cx^2 > kx^3 \cdot \dfrac{1}{1-x}$ if $c > \dfrac{kx}{1-x}$.

And since $\dfrac{x}{1-x} = 0$ when $x = 0$, and c and k are constants, $\dfrac{kx}{1-x}$ can be made less than c by taking x sufficiently small.

$$\therefore\ cx^2 \text{ can be made } > dx^3 + ex^4 + fx^5 + \cdots$$

Now let $A + Bx + Cx^2 + \cdots = a + bx + cx^2 + \cdots$ be true for all values of x. Then

$$A - a + (B - b)x + (C - c)x^2 + \cdots = 0$$

is true for all values of x.

But when x is sufficiently small, $A - a$ is greater than all that follows, and its sign controls that of the series; but the whole series is zero; therefore $A - a = 0$.

$$\therefore\ A = a.$$

And by striking out A and a as being equal, we prove in like manner that $B=b$; thence $C=c$, etc.

II. Expansion of Functions.

198. If a function of x which has but one value for each value of x be expanded in powers of x, it must take the form

$$a+bx+cx^2+dx^3+\cdots$$

where every exponent is a positive integer.

For if there be a term of the form gx^{-t}, this term will become infinite when $x=0$, and therefore the independent term a must be infinite, and the expansion is impossible.

Again, if there be a term of the form $hx^{\frac{m}{n}}$, this term has n values, and therefore the expansion has at least n values for each value of x, which is contrary to the hypothesis.

Ex. 1. To expand $\dfrac{1+x-3x^2}{1-x-x^2+x^3}$.

Assume $$\frac{1+x-3x^2}{1-x-x^2+x^3}=1+ax+bx^2+cx^3+dx^4+\cdots$$

Then
$$1+x-3x^2=\begin{array}{r|r|r|r|l} 1+a & x+b & x^2+c & x^3+d & x^4+\cdots \\ -1 & -a & -b & -c & \\ & -1 & -a & -b & \\ & & +1 & +a & \end{array}$$

And equating coefficients,

$$1=a-1,\ -3=b-a-1,\ 0=c-b-a+1,\ 0=d-c-b+a, \text{ etc.,}$$

whence $$a=2,\ b=0,\ c=1,\ d=-1,\ e=0, \text{ etc.}$$

And the expansion is

$$1+2x+0x^2+x^3-x^4+0x^5\cdots$$

Ex. 2. To expand $\sqrt{1+x}$.

Assume $\sqrt{1+x} = 1 + ax + bx^2 + cx^3 + dx^4 + \cdots$

$$\text{Then}\quad 1 + x = 1 + 2ax + \left.\begin{matrix}2b\\a^2\end{matrix}\right| x^2 + \left.\begin{matrix}2c\\2ab\end{matrix}\right| x^3 + \left.\begin{matrix}2d\\2ac\\b^2\end{matrix}\right| x^4 + \cdots$$

whence $\quad a = \frac{1}{2},\ b = -\frac{1}{8},\ c = \frac{1}{16},\ d = -\frac{5}{128}$, etc.

$$\therefore \sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 \cdots$$

$$= 1 + \tfrac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$$

EXERCISE XV. b.

1. Expand $\sqrt{(1 + 2x + 3x^2 + 4x^3 + \cdots nx^{n-1} + \cdots)}$.

2. Expand $\sqrt{(1 + x + x^2)}$ to x^4.

3. Expand $\sqrt{\left(\frac{1-x}{1+x}\right)}$ to x^4.

4. Expand $\sqrt[3]{(1+x)}$ to x^4.

5. Given $(1+x)^n = \frac{1}{(1+x)^{-n}} = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots$, to find the coefficients of the expansion of $(1+x)^{-n}$ in terms of c_1, c_2, c_3, etc., up to the fourth.

6. If $y = a_1x + a_2x^2 + a_3x^3 + \cdots$, find x in terms of y.

Assume $x = Ay + By^2 + Cy^3 + \cdots$; write for y, in this assumption, the value given, and equate coefficients.

7. If $y = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, find x in terms of z, where $z = y - 1$.

8. If $x = y - 2y^2 + y^3$, develop y as a function of x.

9. If $(a + bx + cx^2 + \cdots)^2 = (a + 2bx + 2^2cx^2 + \cdots)$, find the values of a, b, c, etc.

III. Summation of Series.

199. It will be noticed in Article 163 that the expression for the nth term, as a function of n, is one dimension lower than the expression for the corresponding sum, and this can be shown to be true for all series of that species.

For suppose $S_n = an^p + bn^{p-1} + cn^{p-2} + \cdots$

Then $S_{n-1} = a(n-1)^p + b(n-1)^{p-1} + c(n-1)^{p-2} + \cdots$

And the difference, $S^n - S^{n-1}$, is the nth term; and on expanding n^p disappears.

The coefficient of n^{p-1} is $b - ap - b$, or ap, which cannot be zero unless a or p is zero, both of which suppositions are contrary to the assumption.

$\therefore$ $S^n - S^{n-1}$ is of the $(p-1)$th dimension.

Ex. 1. To find the sum of the series of squares of the natural numbers, viz., $1^2 + 2^2 + 3^2 + \cdots n^2$.

Since the nth term is of two dimensions, assume

$$S_n = an^3 + bn^2 + cn.$$

Then $$S_{n-1} = a(n-1)^3 + b(n-1)^2 + c(n-1).$$

$$\therefore\ S_n - S_{n-1} = 3an^2 - (3a - 2b)n + a - b + c = n\text{th term}$$
$$= n^2.$$

And equating coefficients,

$$a = \tfrac{1}{3},\ b = \tfrac{1}{2},\ c = \tfrac{1}{6}.$$

$$\therefore\ S_n = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = \frac{n}{6}(n+1)(2n+1).$$

Ex. 2. To find the nth term, and the sum of n terms, of the series

$$1 + 4 + 8 + 14 + 23 + 36 + \cdots$$

Taking first differences, we have

$$3 + 4 + 6 + 9 + 13 + \cdots$$

and for second differences,

$$1 + 2 + 3 + 4 + \cdots, \text{ an A. P.}$$

Now as the nth term of an A. P. is linear, the nth term of the first difference is quadratic, and of the original series is cubic.

Therefore, assume the nth term $= a + bn + cn^2 + dn^3$.

$$\begin{aligned} \text{When } n = 1,\quad 1 &= a + b + c + d. \\ n = 2,\quad 4 &= a + 2b + 4c + 8d. \\ n = 3,\quad 8 &= a + 3b + 9c + 27d. \\ n = 4,\quad 14 &= a + 4b + 16c + 64d. \end{aligned} \qquad \therefore \begin{aligned} 3 &= b + 3c + 7d. \\ 4 &= b + 5c + 19d. \\ 6 &= b + 7c + 37d. \end{aligned}$$

$$\text{Thence}\quad \begin{aligned} 1 &= 2c + 12d, \\ 2 &= 2c + 18d. \end{aligned} \qquad 1 = 6d. \quad \therefore d = \tfrac{1}{6}.$$

Thence $\quad c = -\frac{1}{2},\ b = \frac{20}{6},\ a = -2.$

And the nth term $= \frac{1}{6}(n^3 - 3n^2 + 20n - 12)$.

Next, for the sum, assume

$$S_n = an + bn^2 + cn^3 + dn^4.$$

$$S_{n-1} = a(n-1) + b(n-1)^2 + c(n-1)^3 + d(n-1)^4.$$

Then $\quad \frac{1}{6}(n^3 - 3n^2 + 20n - 12) = S_n - S_{n-1} =$ the nth term

$$= a - b + c - d + (2b - 3c + 4d)n + (3c - 6d)n^2 + 4dn^3.$$

And equating coefficients, we get

$$d = \tfrac{1}{24},\ c = -\tfrac{2}{24},\ b = \tfrac{35}{24}, \text{ and } a = -\tfrac{10}{24}.$$

$$\therefore\ S_n = \frac{n\{n^3 - 2n^2 + 35n - 10\}}{24}.$$

EXERCISE XV. c.

1. If the nth term of the series $1 + 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5 + \cdots$ is of the form $1 + ax + bx^2$, find the nth term.

2. Develop $\dfrac{1+2x-x^2}{(1-x)(1+x)^2}$, and show what two series it is the sum of.

(Separate into partial fractions and develop each.)

3. Sum to n terms the series whose nth term is $1-n+n^2$.

4. Sum the series whose nth term is $\frac{1}{2}(n^3+n)$.

5. Sum to n terms, $1\cdot 2+3\cdot 4+5\cdot 6+\cdots$

6. Sum to n terms, $1\frac{1}{2}+2+1\frac{1}{2}+0-2\frac{1}{2}\cdots$

7. Sum to n terms, $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots$

8. Find the series whose nth term is the sum of the natural numbers from 1 to n.

IV. MISCELLANEOUS.

200. The following are miscellaneous applications of the principle of undetermined coefficients to problems which fall under none of the previous heads.

Ex. 1. To find the condition under which ax^2+bx+c shall be a complete square.

Assume $$ax^2+bx+c=(px+q)^2.$$

Then expanding $(px+q)^2$, and equating coefficients,

$$a=p^2,\ \ b=2pq,\ \ c=q^2.$$

But $$(2pq)^2=4p^2q^2.$$

$$\therefore\ b^2=4ac$$

is the required condition.

Ex. 2. To find the condition that the equation $x^3+ax^2+bx+c=0$ may have two equal roots.

This is the condition that x^3+ax^2+bx+c may have a square factor.

Assume $$x^3+ax^2+bx+c=\left(x+\frac{c}{p^2}\right)(x+p)^2.$$

Expanding, and equating coefficients,

$$a = 2p + \frac{c}{p^2}, \quad b = p^2 + \frac{2c}{p},$$

from which we must determine p.

$$\text{1st} \quad \begin{cases} 2p^2 + \dfrac{c}{p} = ap, \\ p^2 + \dfrac{2c}{p} = b; \end{cases} \quad \text{whence} \quad \begin{cases} ap^2 - 2bp + 3c = 0, \\ 3p^2 - 2ap + b = 0. \end{cases}$$

Eliminating linear p, $\qquad p^2 = \dfrac{3ac - b^2}{a^2 - 3b}.$

Eliminating square p, $\qquad p = \dfrac{9c - ab}{2(a^2 - 3b)}.$

$$\therefore\ 4(3ac - b^2)(a^2 - 3b) = (9c - ab)^2$$

is the required relation.

Cor. For the equation $x^3 + bx + c = 0$, we get by making $a = 0$, $4b^3 = 27c^2$, as the condition.

Ex. **3**. To find the condition that

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c \quad \dots\dots \quad (A)$$

may be the product of two factors, rational in x and y; and to find the factors.

Assume

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = \left(ax + \frac{by}{p} + s\right)\left(x + py + \frac{c}{s}\right),$$

and equate coefficients of xy and x, and we obtain

$$ap^2 - 2hp + b = 0, \text{ and } s^2 - 2gs + ac = 0.$$

Whence $\qquad p = \dfrac{1}{a}(h + \sqrt{h^2 - ab}),\ s = g + \sqrt{g^2 - ac}.$

Denote $\sqrt{h^2 - ab}$ by H, and $\sqrt{g^2 - ac}$ by G, and the factors become

$$\left\{ax + \frac{aby}{h + H} + g + G\right\}\left\{x + \frac{(h + H)y}{a} + \frac{c}{g + G}\right\},$$

or $\qquad \dfrac{1}{a}\{ax + (h - H)y + g + G\}\{ax + (h + H)y + g - G\}.$

As these factors do not contain f, we equate the coefficients of linear y from the function and from its factored equivalent, and obtain

$$2f = \frac{1}{a}\{(g+G)(h+H)+(g-G)(h-H)\} = \frac{2}{a}(gh+GH).$$

And putting for G and H their values, and reducing, we obtain

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad . \; . \; . \; . \; . \; . \quad (B)$$

which expresses the necessary condition.

This very important function, B, is called the **discriminant** of the function A.

Ex. **4**. To find a number such that if 1 be added to it the sum shall be a square, and if 1 be subtracted from it the difference shall be a square.

Let x denote the number.

Then $x+1$, and $x-1$, and consequently x^2-1, are all to be squares.

Assume $\quad x^2 - 1 = (x-p)^2 = x^2 - 2px + p^2.$

Then $$x = \frac{1+p^2}{2p},$$

and $$x + 1 = \frac{(1+p)^2}{2p},$$

which will be a square if $2p$ is a square.

Let $\quad 2p = s^2$; then $p = \frac{s^2}{2}$, and $x = \frac{4+s^4}{4s^2}$;

where s may be any quantity whatever.

When $$s = \tfrac{1}{3},\ \tfrac{1}{2},\ 1,\ 2,\ 3,\ 4 \cdots$$

$$x = \tfrac{325}{36},\ \tfrac{65}{16},\ \tfrac{5}{4},\ \tfrac{5}{4},\ \tfrac{85}{36},\ \tfrac{65}{16} \cdots$$

and the problem is thus indeterminate.

EXERCISE XV. d.

1. Find the condition that $x^2 - abx + bc$ may be a complete square.

2. If $x^3 + ax^2 + bx + c$ is a complete cube, show that $27c = a^3$, and $3b = a^2$.

3. Find the condition that $ax^2 + bx + c$ and $a_1x^2 + b_1x + c_1$ may have a common linear factor.

4. Determine λ so that the equation $\frac{2A}{x+a} + \frac{\lambda}{x} + \frac{2B}{x-a} = 0$ may have equal roots in x.

5. Show that $1 + \frac{2x}{1} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$ is the square of $1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, up to x^5.

6. Find the value of m that $y - mx - 3 = 0$ may be compatible with $y - x - 1 = 0$ and $y - 2x - 2 = 0$.

7. Find the value of c in order that $2x^2 + y^2 - 4xy + 6y + c$ may be put into rational factors in x and y, and find the factors.

8. Show that $ax^2 + 2hxy + by^2$ can always be rationally factored, whatever be the values of a, h, and b; and find the factors.

9. Find a formula for numbers which put for x make $x^2 + b$ a complete square.

10. Determine the fraction of form $\frac{a + bx}{1 + cx + dx^2}$, which expands into $1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + 29x^6 + \cdots$

11. Find the relation between a and b that $(x - a)^2 + (x - b)$ may be a square.

12. Express $x^4 - 4x^3 + x^2 + 2x$ in the form $(x^2 + ax + b)^2 - (x + c)^2$; and thence show how the corresponding equation can be solved.

13. Put $x^2 + xy - 2y^2 + 2x + 7y - 3$ into factors.

14. Find the value of m that $2x^2 - 3xy + 2x - y + m$ may be put into rational factors in x and y.

15. If $ax^2 + by^2 + 2hxy + 2gx + 2fy + c$ is expressible in factors rational in x and y, show that h has two values for given values of a, b, f, g, and c, unless $f^2 = bc$, or $g^2 = ac$.

CHAPTER XVI.

Elementary Continued Fractions.

201. Take any proper fraction, preferably in its lowest terms, as $\frac{11}{25}$.

Then,

$$\frac{11}{25}=\frac{1}{\frac{25}{11}}=\frac{1}{2+\frac{3}{11}}=\frac{1}{2+\frac{1}{\frac{11}{3}}}=\frac{1}{2+\frac{1}{3+\frac{2}{3}}}=\frac{1}{2+\frac{1}{3+\frac{1}{1+\frac{1}{2}}}}.$$

This expanded result is a **continued fraction**, and is often written

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{1}+\frac{1}{2}.$$

Or, for the purpose of saving space, it is more generally written

$$\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2}.$$

From the nature of the expansion, it is evident that every fraction can be expressed as a finite continued fraction.

Ex. $$\frac{40}{111}=\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{4}.$$

202. Problem. To express any vulgar fraction as a continued fraction.

Proceed as in finding the *G. C. M.* of the numerator and denominator of the given fraction, and write the quotients in order as denominators of the continued fraction, the numerators being 1.

Ex. 1. To change $\frac{56}{103}$ to a continued fraction.

$$\begin{array}{r|r|l} 56 & 103 & 1 \\ 9 & 47 & 1 \\ 1 & 2 & 5 \\ & & 4 \\ & & 2 \end{array}\Bigg\} \text{quotients.} \quad \therefore \frac{56}{103}=\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{4+}\,\frac{1}{2}.$$

Ex. 2. To change 3.1416 to a continued fraction.

This is $3+\frac{1416}{10000}$, and we change $\frac{1416}{10000}$ to a continued fraction.

$$\begin{array}{r|r|l} 1416 & 10000 & 7 \\ 8 & 88 & 16 \\ & & 11 \end{array} \qquad \therefore \frac{1416}{10000}=\frac{1}{7+}\,\frac{1}{16+}\,\frac{1}{11},$$

$$\text{and } 3.1416 = 3+\frac{1}{7+}\,\frac{1}{16+}\,\frac{1}{11}.$$

Ex. 3. To express $\dfrac{x^2}{x^2+x+1}$ as a continued fraction.

$$\frac{x^2+x+1}{x^2}=1+\frac{x+1}{x^2};\quad \frac{x^2}{x+1}=x+\frac{-x}{x+1};\quad \frac{x+1}{-x}=-1-\frac{1}{x}.$$

$$\therefore \frac{x^2}{x^2+x+1}=\frac{1}{1+}\,\frac{1}{x+}\,\frac{1}{(-1)+}\,\frac{1}{(-x)}.$$

203. The quotients obtained by the process of finding the *G. C. M.* are called **partial quotients**.

Let the partial quotients be denoted by a_1, a_2, a_3, etc., all the quantities being affected with the sign $+$.

Then the continued fraction is

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots+\frac{1}{a_n},$$

which in its totality is equal to the fraction from which it is derived, and which we shall denote by x.

$\frac{1}{a_1}$ is the 1st *convergent,* which denote by v_1;

$\frac{1}{a_1 +}\frac{1}{a_2}$ is the 2d *convergent,* which denote by v_2;

$\frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3}$ is the 3d *convergent,* v_3;

etc. etc. etc.

Then

$\frac{1}{a_1} > x$, since its denominator a_1 is too small.

v_2, or $\frac{1}{a_1 +}\frac{1}{a_2} < x$, since its denominator is too great.

v_3, or $\frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3} > x$; v_4 or $\frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}\frac{1}{a_4} < x$;

etc. ⋯

Thus, all the odd convergents are greater than x, and all the even convergents are less than x; so that the value of x lies between that of any two consecutive convergents, until we reach the last convergent, which is the value of x itself.

ILLUSTRATION. — The convergents to $\frac{11}{25}$ (Art. 201) are

$$v_1 = \tfrac{1}{2},\ v_2 = \tfrac{3}{7},\ v_3 = \tfrac{4}{9},\ v_4 = \tfrac{11}{25} = x.$$

Now $\frac{11}{25} = x = 0.44.$

$v_1 = 0.50$, and is too great by 0.06.

$v_2 = 0.42857 \cdots$ and is too small by $0.02142 \cdots$

$v_3 = 0.4444 \cdots$ and is too great by $0.0044 \cdots$

Thus the consecutive convergents, while being alternately too great and too small, approximate more and more nearly to the value of x; and hence the names **convergent** and **converging fractions.**

We thus see that one obvious application of continued fractions is to find a fraction, with few figures, which shall be a close approximation in value, to a given fraction with so many figures as to be unwieldy.

Thus $\frac{6}{7}$ and $\frac{19}{22}$ are close approximations to $\frac{3817}{4421}$.

204. The first convergent $v_1 = \frac{1}{a_1}$, and the second, $v_2 = \cfrac{1}{a_1 + \cfrac{1}{a_2}}$; so that to get the second convergent from the first we write $a_1 + \frac{1}{a_2}$ for a_1 in the first.

Similarly, to get v_3 from v_2 we write $a_2 + \frac{1}{a_3}$ for a_2 in the value of v_2; and generally, to get v_{n+1} from v_n, we write $a_n + \frac{1}{a_{n+1}}$ for a_n in the value of v_n.

Now let $v_1 = \frac{P_1}{Q_1}$, $v_2 = \frac{P_2}{Q_2}$, $v_3 = \frac{P_3}{Q_3}$, etc.

Then $$v_1 = \frac{P_1}{Q_1} = \frac{1}{a_1};\ v_2 = \frac{P_2}{Q_2} = \frac{a_2}{a_2 a_1 + 1};$$

$$v_3 = \frac{P_3}{Q_3} = \frac{a_3 a_2 + 1}{a_3(a_2 a_1 + 1) + a_1} = \frac{a_3 P_2 + P_1}{a_3 Q_2 + Q_1};$$

$$v_4 = \frac{P_4}{Q_4} = \frac{\left(a_3 + \frac{1}{a_4}\right) P_2 + P_1}{\left(a_3 + \frac{1}{a_4}\right) Q_2 + Q_1} = \frac{a_4 P_3 + P_2}{a_4 Q_3 + Q_2};\ \text{etc.}$$

We here see that the forms for v_3 and v_4 are exactly alike, the only difference being that the subscripts are each increased by unity.

To prove that this is always the case.

Assume $v_n = \frac{P_n}{Q_n} = \frac{a_n P_{n-1} + P_{n-2}}{a_n Q_{n-1} + Q_{n-2}}$.

Then $$v_{n+1} = \frac{P_{n+1}}{Q_{n+1}} = \frac{\left(a_n + \frac{1}{a_{n+1}}\right) P_{n-1} + P_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) Q_{n-1} + Q_{n-2}}$$

$$= \frac{a_{n+1}(a_n P_{n-1} + P_{n-2}) + P_{n-1}}{a_{n+1}(a_n Q_{n-1} + Q_{n-2}) + Q_{n-1}} = \frac{a_{n+1} P_n + P_{n-1}}{a_{n+1} Q_n + Q_{n-1}}.$$

Which shows that the form is true for v_{n+1} if it is true for v_n. But it is true for v_3 and v_4, and therefore for v_5, v_6, etc.; *i.e.* it is generally true.

205. The result of the preceding article furnishes a convenient means of finding all the consecutive convergents, when we have any two consecutive ones, and the partial quotients.

Taking $\frac{P_{n-1}}{Q_{n-1}}$, $\frac{P_n}{Q_n}$, a_{n+1}, we get the $(n+1)$th convergent as follows:

Multiply P_n by a_{n+1} and add P_{n-1}, for P_{n+1}; and multiply Q_n by a_{n+1} and add Q_{n-1}, for Q_{n+1}.

This operation gives $\frac{P_{n+1}}{Q_{n+1}} = \frac{a_{n+1} P_n + P_{n-1}}{a_{n+1} Q_n + Q_{n-1}}$; which is correct.

For convenience we assume a fictitious convergent, $v_0 = \frac{0}{1}$, and carry out the operation as in the following example:

Ex. Let 2, 1, 3, 1, 2 be partial quotients.

$$\frac{0}{1} \quad \frac{1}{2} \quad \begin{array}{cccc} 1 & 4 & 5 & 14 \\ \hline 1 & 3 & 1 & 2 \\ \hline 3 & 11 & 14 & 39 \end{array}$$

The partial quotients after the first are written between the lines, and the parts of the corresponding convergents are written above and below the lines.

Thus the convergents are

$$v_1 = \tfrac{1}{2},\ v_2 = \tfrac{1}{3},\ v_3 = \tfrac{4}{11},\ v_4 = \tfrac{5}{14}, \text{ and } v_5 = \tfrac{14}{39}.$$

206. Taking the convergents of the preceding example,

$$v_1 - v_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{2 \times 3}; \quad v_2 - v_3 = \frac{1}{3} - \frac{4}{11} = \frac{-1}{3 \times 11};$$

$$v_3 - v_4 = \frac{4}{11} - \frac{5}{14} = \frac{1}{11 \times 14}; \text{ etc.}$$

Thus the difference between two consecutive convergents is the fraction whose numerator is 1, and whose denominator is the product of the denominators of the two convergents. And this difference, taken in regular order, is alternately positive and negative.

To prove that this is always the case.

$$v_n - v_{n-1} = \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{P_n Q_{n-1} - Q_n P_{n-1}}{Q_n Q_{n-1}}.$$

But, Art. 204,

$$P_n Q_{n-1} - Q_n P_{n-1} = (a_n P_{n-1} + P_{n-2}) Q_{n-1} - (a_n Q_{n-1} + Q_{n-2}) P_{n-1}$$
$$= -(P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}).$$

Similarly,

$$-(P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}) = P_{n-2} Q_{n-3} - Q_{n-2} P_{n-3} = \text{etc.},$$

so that $P_n Q_{n-1} - Q_n P_{n-1}$ evidently has, with the exception of sign, the same value for all values of n.

But when $n = 2$, $P_2Q_1 - Q_2P_1 = a_2a_1 - (a_1a_2 + 1) = -1$; and for $n = 3$, $P_3Q_2 - Q_3P_2 = +1$.

Hence $$v_n - v_{n-1} = \frac{(-1)^{n-1}}{Q_nQ_{n-1}}.$$

Which proves the theorem. And as the difference between any two consecutive convergents is a fraction with unity as numerator, every convergent is in its lowest term, *i.e.* its parts are prime to one another.

207. As the value of x lies between those of two consecutive convergents, $\frac{P_n}{Q_n}$ differs from x in value by less than $\frac{1}{Q_nQ_{n+1}}$, *i.e.* by less than $\frac{1}{Q_n^2}$.

Thus $\frac{4}{11}$, a convergent to $\frac{14}{39}$, differs from the latter fraction by less than $\frac{1}{121}$; and $\frac{5}{14}$, another convergent to the same, differs from it by less than $\frac{1}{196}$.

208. $$\frac{P_n}{Q_n} = \frac{a_nP_{n-1} + P_{n-2}}{a_nQ_{n-1} + Q_{n-2}} = \frac{P_{n-1} + \frac{1}{a_n} \cdot P_{n-2}}{Q_{n-1} + \frac{1}{a_n} \cdot Q_{n-2}}.$$

Now if a_n is relatively large, the quantities $\frac{1}{a_n} \cdot P_{n-2}$, and $\frac{1}{a_n} \cdot Q_{n-2}$ are relatively small, and the whole fraction differs but little from $\frac{P_{n-1}}{Q_{n-1}}$.

That is, if a_n is relatively large, the difference between v_{n-1} and v_n is relatively small, and hence v_{n-1} is a close approximation to x.

Hence, the last convergent preceding a large partial quotient is a close approximation to the value of the fraction.

Thus, if the partial quotients be

$$1,\ 2,\ 1,\ 3,\ 15,\ 2,$$

the convergents are

$$v_1 = 1,\ v_2 = \tfrac{2}{3},\ v_3 = \tfrac{3}{4},\ v_4 = \tfrac{11}{15},\ v_5 = \tfrac{168}{229},\ v_6 = x = \tfrac{347}{473};$$

and v_4, preceding the quotient 15, differs from x by less than

$$\frac{1}{15 \times 229} \text{ or } \frac{1}{3435}.$$

EXERCISE XVI. a.

1. Find all the convergents to $\frac{190}{251}$.

2. If $\dfrac{P}{Q}$ be the convergent preceding $\dfrac{a}{b}$, show that Pb differs from Qa by unity.

Thence form a rule for finding multiples of two numbers prime to one another, so that such multiples shall differ by a given integer.

3. Find multiples of 23 and 31 that shall differ by 6.

4. Find an improper fraction to express 3.1416 to a near approximation.

5. Express 1.4142 by a vulgar fraction, each of whose parts are less than 100.

209. A continued fraction may be non-terminating; *i.e.* its partial quotients may be an endless series of numbers.

The convergents approximate in the same way whether the fraction is terminating or not, but no convergent, however high its order, can express exactly the quantity denoted by the continued fraction. Such a fraction has in general an incommensurable for its value.

If the partial quotients exist in recurring periods, like the figures in a circulating decimal, the fraction is a *periodic continued fraction,* and every such fraction is the development of a square root or quadratic surd.

An infinite continued fraction, in which the partial quotients are not periodic, may be the expansion of a cubic or higher form of surd expression, but, in general, the equivalent surd expression cannot be found.

Ex. 1. Let the partial quotients be 2, 2, 2 ⋯, and let x be the equivalent fraction.

Then $$x = \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots = \frac{1}{2+x}.$$

$$\therefore\ x^2 + 2x = 1, \text{ and } x = \sqrt{2} - 1.$$

Ex. 2. Let the partial quotients be 1, 2, 3, 1, 2, 3 ⋯

Then $$x = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + x}}} = \frac{7 + 2x}{10 + 3x}.$$

$$\therefore\ 3x^2 + 8x = 7, \text{ and } x = \tfrac{1}{3}(\sqrt{37} - 4).$$

And the method applies to all, however great the periodic part may be.

210. To expand the square root of a non-square number into a periodic fraction.

We give the method of operation by means of examples.

Ex. 1. To develop $\sqrt{7}$.

Since 2 is the highest integer in the root of 7, we subtract 2 from $\sqrt{7}$, and throughout the operation no number greater than 2 is to be thus subtracted.

$$7 = 2 + \sqrt{7} - 2 = 2 + \frac{3}{\sqrt{7} + 2};$$

multiplying $$\frac{\sqrt{7} - 2}{1} \text{ by } \frac{\sqrt{7} + 2}{\sqrt{7} + 2}.$$

$$\frac{\sqrt{7}+2}{3}=1+\frac{\sqrt{7}-1}{3}=1+\frac{2}{\sqrt{7}+1};$$

adding 1 to 2 to get an integral quotient, 1, and subtracting 1 from $\sqrt{7}$ so as to keep the whole unchanged.

$$\frac{\sqrt{7}+1}{2}=1+\frac{\sqrt{7}-1}{2}=1+\frac{3}{\sqrt{7}+1};$$

$$\frac{\sqrt{7}+1}{3}=1+\frac{\sqrt{7}-2}{3}=1+\frac{1}{\sqrt{7}+2};$$

$$\frac{\sqrt{7}+2}{1}=4+\sqrt{7}-2, \text{ etc.}$$

The partial quotients obtained are 2, 1, 1, 1, 4; and as we have now to begin again with $\sqrt{7}-2$, the same quotients will constantly recur. We notice then that when a quotient which is double the first one appears, the period is complete.

$$\therefore\ \sqrt{7}=2+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\cdots$$

Ex. 2. $$\sqrt{13}=3+\sqrt{13}-3=3+\frac{4}{\sqrt{13}+3};$$

$$\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{3}{\sqrt{13}+1};$$

$$\frac{\sqrt{13}+1}{3}=1+\frac{\sqrt{13}-2}{3}=1+\frac{3}{\sqrt{13}+2};$$

$$\frac{\sqrt{13}+2}{3}=1+\frac{\sqrt{13}-1}{3}=1+\frac{4}{\sqrt{13}+1};$$

$$\frac{\sqrt{13}+1}{4}=1+\frac{\sqrt{13}-3}{4}=1+\frac{1}{\sqrt{13}+3};$$

$$\sqrt{13}+3=6+\sqrt{13}-3, \text{ etc.}$$

$$\therefore\ \sqrt{13}=3+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\cdots$$

The convergents to the fractional part are

$$v_1 = 1,\ v_2 = \tfrac{1}{2},\ v_3 = \tfrac{2}{3},\ v_4 = \tfrac{3}{5},\ \text{etc.}$$

$\therefore$ the convergents to $\sqrt{13}$ are

$$4,\ 3\tfrac{1}{2},\ 3\tfrac{2}{3},\ 3\tfrac{3}{5},\ 3\tfrac{20}{33},\ \text{etc.}$$

EXERCISE XVI. b.

1. Find the value of $1 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3}\cdots$

2. Find the value of $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1}\cdots$

3. Show that the C. F. whose partial quotients are 1, -2, 3, -4, forming a period, has 3 for its total value, and find the first 10 convergents.

4. Find the value of $2 + \frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\cdots$

5. Expand $\sqrt{2}$ and $\sqrt{6}$ into periodic C. F.'s.

6. Expand $\sqrt{17}$ and $\sqrt{19}$ into periodic C. F.'s.

CHAPTER XVII.

LOGARITHMS AND EXPONENTIALS.

211. In the expression $a^x = b$, x is called the **logarithm** of b to the base a; and this relation is otherwise indicated by writing $x = \log_a b$.

The base, a, being some fixed positive number, to every value of b there is a corresponding value of x.

If these corresponding values be tabulated in opposing columns, the b-column is a column of *numbers*, and the x-column is a column of *logarithms*, and the whole forms a table of logarithms to the base a.

As will be shown hereafter, the general properties of logarithms are the same for all bases, and any positive number, commensurable or incommensurable, may be taken as a base; but certain numbers offer special advantages as bases in working with logarithms, and in calculating them.

As a consequence logarithms are, in practice, taken to one of two bases; namely, 10, as being the base of our numerical system; and a certain incommensurable, usually denoted by e, and called the *Napierian* or *natural* base.

Logarithms to the base 10 are decimal or common logarithms, and those to the base e are Napierian or natural logarithms.

GENERAL PROPERTIES OF LOGARITHMS.

212. Let $a^x = b$, and $a^y = c$.

Then $x = \log_a b$, and $y = \log_a c$.

(1) $bc = a^x \cdot a^y = a^{x+y}$,

and $\log_a(bc) = x + y$,

or $\log_a(bc) = \log_a b + \log_a c$.

That is, the logarithm of the product of two numbers is the sum of the logarithms of the numbers.

(2) $\frac{b}{c} = \frac{a^x}{a^y} = a^{x-y}$.

$$\therefore \log_a\left(\frac{b}{c}\right) = x - y,$$

or $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$.

That is, the logarithm of the quotient of two numbers is the logarithm of the dividend diminished by the logarithm of the divisor.

(3) $(a^x)^n = b^n = a^{nx}$.

$$\therefore \log_a(b^n) = nx = n \log_a b.$$

That is, the logarithm of the nth power of a number is n times the logarithm of the number.

(4) Writing $\frac{1}{n}$ for n,

$$\log_a(b^{\frac{1}{n}}) = \frac{1}{n} \log_a b,$$

or the logarithm of the nth root of a number is one-nth of the logarithm of the number.

These four relations form the working rules of logarithms in their applications to quantity.

213. The results of the preceding article show that multiplication in numbers corresponds to addition in logarithms; division in numbers, to subtraction in logarithms; the raising of a number to a power, to the multiplication of a logarithm by a number; and the extracting of the root of a number, to the division of a logarithm by a number.

There are in arithmetic, as confined to numbers, no known operations which correspond to the multiplication or division of one logarithm by another, and hence to the raising of a logarithm to a power, or to the extraction of its root.

Such operations upon logarithms can correspond only to some hyper-arithmetical processes. Thus logarithms not only facilitate the more difficult arithmetical operations; they also, by an extension of processes, give rise to a sort of transcendental arithmetic.

EXPONENTIAL EQUATIONS.

214. An exponential equation is one in which the variable appears as an exponent.

Thus $a^x = b$, with x variable, is an exponential equation.

The method of solution is obvious; for taking the logarithms of both members,

$$\log(a^x) = x \log a = \log b.$$

$$\therefore\ x = \frac{\log b}{\log a}.$$

And the operation which gives x is the transcendental one which corresponds to the division of one logarithm by another.

Ex. 1. Given $a^x + a^{-x} = 2b$; to find x.

Multiply by a^x,

$$a^{2x} - 2ba^x + 1 = 0.$$

Whence $$a^x = b \pm \sqrt{b^2 - 1}, \qquad \text{(Art. 126)}$$

and $$x = \frac{\log\{b \pm \sqrt{b^2-1}\}}{\log a}.$$

Ex. 2. To express the logarithm of $\dfrac{\sqrt{3}(1.05)^6}{2^{\frac{2}{5}}(216)^{\frac{1}{5}}}$, in terms of the logarithms of 2, 3, 5, and 7.

The given expression reduces to $(3^{\frac{59}{10}} \cdot 7^6) \div (2^{13} \cdot 5^6)$; and hence its logarithm is

$$\tfrac{59}{10}\log 3 + 6\log 7 - 13\log 2 - 6\log 5.$$

EXERCISE XVII. a.

1. If 3 be taken as a base, what are the logarithms of 9, of 81, of 729, of $\frac{1}{27}$, of $\frac{1}{243}$?

2. If 6 be the base, show that $\frac{3}{2}$ is the logarithm of 14.697.

3. If $\frac{1}{2}$ be the base, of what numbers are 1, 2, 0, $-\dfrac{1}{3}$, $-\dfrac{1}{n}$ the logarithms?

4. Prove that, with any positive base, 1 is the logarithm of the base, 0 is the logarithm of 1, and $-\infty$ is the logarithm of 0.

5. If $\log_b a = n \log_a b$, show that $\log_b a = \dfrac{n \cdot \log b}{\log a}$, where the logarithm without a suffix is taken to any base.

6. Solve the exponential equations —

i. $20^x = 100$.

ii. $(2^3)^x \cdot (3^2)^x = 4 \cdot 9$.

iii. $3^{2x} \cdot 5^{3x-4} = 7^{x-1} 11^{2-x}$.

iv. $x^y = y^x$, and $x^3 = y^2$.

v. $2a^{4x} + a^{2x} = a^{6x}$.

vi. $a^1 \cdot a^2 \cdot a^3 \cdots a^x = n$.

7. Express the logarithm of $(8\sqrt{3}\cdot\sqrt[3]{12})\div(\sqrt{2}\cdot\sqrt[3]{15})$ in terms of the logarithms of 2, 3, and 5.

8. Given $\log 2 = 0.30103$; find $\log 64$, $\log 256$, $\log\sqrt{128}$, $\log\frac{1}{2}$, $\log 25$. (Base = 10.)

9. Express in terms of $\log 2$ and $\log 3$ the logarithms of 6, 18, 72, $\frac{1}{12}$, 0.25, $0.041\dot{6}$.

10. How many terms of the G. P. $1+\frac{2}{3}+\frac{4}{9}+\cdots$ will make $\frac{18915}{6561}$?

11. How long will it take a sum of money to double at 5% compound interest?

OF THE TABLE OF LOGARITHMS.

215. In $a^x = b$, if b is greater than a and less than a^2, x is >1 and <2; *i.e.* $x = 1 +$ a proper fraction.

If b is $>a^2$ and $<a^3$, $x = 2 +$ a proper fraction; etc.

Thus a logarithm consists of an integral part, called the **characteristic**, and a fractional part, called the **mantissa.** Either of these may, however, become zero.

Taking 10 as a base, every integral power of 10 consists of 1 followed by ciphers only, and the logarithm of such power is an integer, or characteristic, being the index of the power.

Thus $\log 100 = 2$, $\log 1000 = 3$, etc.

For any number between 100 and 1000 the logarithm is $2 +$ a decimal; for a number between 1000 and 10000, it is $3 +$ a decimal; etc.

Hence one convenience of decimal logarithms is that we know the characteristic at sight, and it is not necessary to tabulate it.

The following rule gives the characteristic for decimal logarithms:

Call the units' place of the number zero, and count to the significant figure farthest upon the left; the number of this figure is the characteristic, positive if counted leftward, and negative if counted rightward.

Thus the characteristic of the logarithm of 0.000074 is -5, of 386.50 it is 2, and of 430070 it is 5.

216. The **Mantissa**. Let n be a number, and let c and m be the characteristic and mantissa of its logarithm.

Then $$\log n = c + m.$$

To divide n by 10^x we subtract $\log 10^x$ from $\log n$. But $\log 10^x = x$; and dividing a number by an integral power of 10 has no effect other than moving the decimal point.

Therefore $$\log(n \div 10^x) = (c - x) + m,$$

and since x is integral, the mantissa is unchanged.

Hence the mantissa of a logarithm to base 10 does not depend upon the position of the decimal point, but only upon the arrangement of figures in the number; so that the same arrangement always corresponds to the same mantissa, and *vice versa*.

The characteristic, on the other hand, is determined wholly by the position of the decimal point.

Thus the logarithms of 0.0024, 0.24, 240, 24000, etc., all have the same mantissa, while the characteristics are -3, -1, 2, and 4 respectively.

217. As the logarithms of integral numbers are mostly incommensurable, the approximation to their mantissæ is carried to 4, 5, 6, 7, etc. decimal places, thus giving rise to tables of 4-place, 5-place, 6-place, or 7-place logarithms.

Portions of a table of 5-place logarithms. A, from number 1780 to 1889; B, from number 5700 to 5769; and C, from number 7320 to 7429.

N.	0	1	2	3	4	5	6	7	8	9	D.
					A.						
178	25042	066	091	115	139	164	188	212	237	261	
9	285	310	334	358	382	406	431	455	479	503	24
180	527	551	575	600	624	648	672	696	720	744	
1	768	792	816	840	864	888	912	935	959	983	
2	26007	031	055	079	102	126	150	174	198	221	
3	245	269	293	316	340	364	387	411	435	458	
4	482	505	529	553	576	600	623	647	670	694	
5	717	741	764	788	811	834	858	881	905	928	
6	951	975	998	021	045	068	091	114	138	161	23
7	27184	207	231	253	277	300	323	346	370	393	
8	416	439	462	485	508	531	554	577	600	623	
....	....	....	....	....	....	....	....	....	...	....	
					B.						
570	75587	595	603	610	618	626	633	641	648	656	
1	663	671	679	686	694	702	709	717	724	732	
2	740	747	755	762	770	778	785	793	800	808	
3	815	823	831	838	846	853	861	868	876	884	8
4	891	899	906	914	921	929	937	944	952	959	
5	967	974	982	989	997	005	012	020	027	035	
6	76042	050	057	065	072	080	087	095	103	110	
....	....	....	....	....	....	....	....	....	....	....	
					C.						
732	.86451	457	463	469	475	481	487	493	499	504	
3	510	516	522	528	534	540	546	552	558	564	
4	570	576	581	587	593	599	605	611	617	623	
5	629	635	641	646	652	658	664	670	676	682	
6	688	694	700	705	711	717	723	729	735	741	
7	747	753	759	764	770	776	782	788	794	800	6
8	806	812	817	823	829	835	841	847	853	859	
9	864	870	876	882	888	894	900	906	911	917	
740	923	929	935	941	947	953	958	964	970	976	
1	982	988	994	999	005	011	017	023	029	035	
2	87040	046	052	058	064	070	075	081	087	093	
....	....	....	....	....	....	....	....	....	....	....	
P.	24	2	5	7	10	12	14	17	19	22	
	23	2	5	7	9	11	14	16	18	21	
	8	1	2	2	3	4	5	6	6	7	
	6	1	1	2	2	3	4	4	5	5	

The larger tables are mostly 7-place, but 5-place logarithms are sufficiently accurate for the majority of arithmetical applications.

We give on page 261, and merely for purposes of illustration, portions of a 5-place table, in which, as is usual, only mantissæ are registered.

218. The *working* a table of logarithms consists in two operations inverse to one another; namely,

(a) to find the mantissa corresponding to a given arrangement of figures in a number, and

(b) to find the arrangement corresponding to a given mantissa.

(a)

A complete 5-place table gives the mantissæ for every arrangement of 4 figures from 1000 to 9999; the three right-hand figures being taken from column **N**, and the fourth from the horizontal line at the top of the table.

Thus, for 1854 the mantissa is 26811; for 1864 it is 27045, the last three figures 045 being in distinctive type to show that the first two figures of the mantissa are to be taken from the first column and the line below, being 27 instead of 26.

Ciphers occurring before or after an arrangement do not affect the mantissa.

Thus, 23, 2300, 23000, .023, etc., have the same mantissa.

Ex. **1.** To find the mantissa of 18347.

Mantissa for	18340	is	26340.
Mantissa for	18350	is	26364.
Difference	10		24.

Thus each unit between 18340 and 18350 adds $\frac{24}{10}$ to the mantissa, and hence 7 adds $7 \times \frac{24}{10}$, or 17 nearly.

$$\therefore\ 26340 + 17 = 26357 \text{ is the mantissa required.}$$

The column marked **D** (differences) and the row at the bottom marked **P** (proportional parts) are intended to facilitate this operation.

Thus, for 1834 we find **D** to be 24, and in line with 24, in the row **P**, we have 17 in the column having 7 at the top. This quantity, 17, is to be added to the mantissa of 1834 to give the mantissa of 18347.

(*b*)

Ex. **2**. To find the arrangement corresponding to the mantissa 26845.

The tabular mantissa next below this is 26834, and the corresponding arrangement is 1855.

The excess of 26845 is 11, and **D** being 23, we find in line with 23, in row **P**, that 11 is in the column having 5 at the top. Then 5 is to be attached to 1855, giving 18555 as the arrangement corresponding to 26845.

219. It must be remembered that the mantissa is *always positive,* while the characteristic is negative for numbers less than 1, zero for numbers from 1 to 10, and positive for numbers above 10.

To mark the negative characteristic the minus sign is written above the characteristic instead of before it.

Ex. **1**. To find the value of $(1.8471)^7$.

$$\begin{aligned} \log 1.8471 &= 0.26649 \\ &\underline{7} \\ \log (1.8471)^7 &= 1.86543 \\ \therefore\ (1.8471)^7 &= 73.355 \cdots \end{aligned}$$

Ex. **2**. To find the value of $(18.71)^{\frac{1}{5}}$.

$$\log 18.71 = 1.27207$$

Divide by 5. $\log (18.71)^{\frac{1}{5}} = 0.25441$

$$\therefore\ (18.71)^{\frac{1}{5}} = 1.7964 \cdots$$

Ex. **3.** To find the value of $(0.185)^7$.

$$\log (0.185) = \bar{1}.26717$$
$$\underline{\qquad\qquad 7}$$
$$\therefore\ \log (0.185)^7 = \bar{6}.87019$$
$$\therefore\ (0.185)^7 = 0.0000074162 \cdots$$

Ex. **4.** To find the value of $(0.001836)^{\frac{5}{11}}$.

$$\log (0.001836) = \bar{3}.26387$$
$$\log (0.001836)^5 = \overline{14}.31935$$
$$= \overline{22} + 8.31935$$
$$\therefore\ \log (0.001836)^{\frac{5}{11}} = \bar{2}.75631$$

and $$(0.001836)^{\frac{5}{11}} = 0.057056 \cdots$$

Notice that to divide the negative characteristic, $\overline{14}$, by 11, we make it evenly divisible by subtracting 8 from it and adding 8 to the mantissa, so as to keep the whole unchanged.

EXERCISE XVII. b.

(All the exercises here given can be worked by means of the portions of logarithmic table given.)

1. Find the continued product of 1.783, 1.791, and 1.799.

2. Find the value of $(18.43 \times 18.65 \times 1.876 \times 5736) \div (1854 \times 186.6 \times 5766)$.

3. Find the value of $(0.1866)^{\frac{4}{7}} \times (7.365)^{\frac{1}{5}}$.

4. Find the value of $(1.8337)^{3.3037}$.

5. Given that $17.80 \times 17.977 = 320$, to find the logarithm of 5.

6. To what power must 74 be raised to give 57 ?

NAPIERIAN BASE, AND EXPONENTIAL SERIES.

220. Definition. The quantity which we have denoted by e, and called the Napierian base in Art. 211, is the limiting value of $(1+n)^{\frac{1}{n}}$ as n approaches the value zero.

By the Binomial theorem

$$(1+n)^{\frac{1}{n}} = 1 + \frac{1}{n}\cdot n + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\cdot 2}n^2 + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1\cdot 2\cdot 3}n^3 + \cdots$$

$$= 1 + 1 + \frac{1(1-n)}{1\cdot 2} + \frac{1(1-n)(1-2n)}{1\cdot 2\cdot 3} + \cdots$$

$$= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots,$$

when $n = 0$.

$$\therefore\ e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

By adding a sufficient number of terms, we find for the approximate value of e,

$$e = 2.7182818\cdots \text{ to 7 decimals.}$$

This peculiar incommensurable quantity is one of the most important constants in mathematics.

221. From the definition of e in Art. 220,

$$e^x = \text{the limit of } \{(1+n)^{\frac{1}{n}}\}^x, \text{ as } n \text{ approaches zero.}$$

But

$$\{(1+n)^{\frac{1}{n}}\}^x = (1+n)^{\frac{x}{n}}$$

$$= 1 + \frac{x}{n} \cdot n + \frac{\frac{x}{n}\left(\frac{x}{n}-1\right)}{1 \cdot 2} n^2 + \frac{\frac{x}{n}\left(\frac{x}{n}-1\right)\left(\frac{x}{n}-2\right)}{1 \cdot 2 \cdot 3} n^3 + \cdots$$

$$= 1 + x + \frac{x(x-n)}{1 \cdot 2} + \frac{x(x-n)(x-2n)}{1 \cdot 2 \cdot 3} + \cdots$$

and when $n = 0$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad . \quad . \quad (B)$$

Any power of e is got in the form of a series, by writing the index of the power in the place of x in the series for e^x. Thus,

$$\frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots$$

and $$\sqrt{e} = e^{\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{2!\,2^2} + \frac{1}{3!\,2^3} + \frac{1}{4!\,2^4} + \cdots$$

222. Let $a = e^c$. Then $c = \log_e a$; or, denoting, in future, Napierian logarithms by the single italic l followed or not by a point, $c = l \cdot a$.

And $$a^x = e^{cx} = 1 + cx + \frac{c^2x^2}{2!} + \frac{c^3x^3}{3!} + \cdots$$

or $$a^x = 1 + xl \cdot a + \frac{x^2(l \cdot a)^2}{2!} + \frac{x^3(l \cdot a)^3}{3!} + \cdots \quad (C)$$

This last series is called the **exponential series**; and it expresses any power (x) of a given number (a) in terms of the exponent and the Napierian logarithm of the number.

Cor. Making $x=1$,

$$a=1+l\cdot a+\frac{(l\cdot a)^2}{2\,!}+\frac{(l\cdot a)^3}{3\,!}+\cdots \qquad (D)$$

and this series which expresses a number (a) in terms of its Napierian logarithm is sometimes called the *anti-logarithmic* series.

EXERCISE XVII. c.

1. Show that $e=$ limit of $\left(1+\frac{1}{n}\right)^n$ as n approaches ∞.

2. Prove that $(e^2-1)^2 \div 8\,e^2=\frac{2^0}{2\,!}+\frac{2^2}{4\,!}+\frac{2^4}{6\,!}+\cdots$

3. If x is positive, then $e > x^{\frac{1}{x}}$.

4. The series $\frac{1}{1\cdot 2}+\frac{1\cdot 3}{1\cdot 2\cdot 3\cdot 4}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\cdots$ is the expansion of $(\sqrt{e}-1)$.

5. Find the sum of $e+e^{-1}$.

6. Show that $\frac{1}{2}(e^{ix}+e^{-ix})=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\cdots$

7. Show that $\frac{1}{2i}(e^{ix}-e^{-ix})=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+-\cdots$

8. Expand $\frac{x}{e^x-1}$.

Assume the expansion to be $\quad a+bx+cx^2+\cdots$

Then $\qquad x=(a+bx+cx^2+\cdots)\left(x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\cdots\right)$

Distribute and equate coefficients of like powers of x, and find a, b, c, etc.

9. Show that $\frac{1^2}{1\,!}+\frac{2^2}{2\,!}+\frac{3^2}{3\,!}+\frac{4^2}{4\,!}+\cdots=2\,e$.

10. Find the value of $\frac{1\cdot 2}{1\,!}+\frac{2\cdot 3}{2\,!}+\frac{3\cdot 4}{3\,!}+\frac{4\cdot 5}{4\,!}+\cdots$

LOGARITHMIC SERIES.

223. In the exponential series (C, Art. 222) the Napierian logarithm of a is the coefficient of linear x in the expansion of a^x. And as x is arbitrary it follows from the principle of undetermined coefficients that if we expand a^x in ascending powers of x, by *any means*, the coefficient of linear x in the expansion will still be the Napierian logarithm of a.

$$\text{But}\quad a^x = (1 + \overline{a-1})^x$$
$$= 1 + x(a-1) + \frac{x(x-1)}{1\cdot 2}(a-1)^2$$
$$+ \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}(a-1)^3 + \cdots$$

by the Binomial Theorem.

And picking out the terms which form the coefficient of linear x, we have

$$l\cdot a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - + \cdots \qquad (E)$$

which gives $l\cdot a$ in terms of the number less by unity. Writing $1+x$ for a,

$$l\cdot(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + - \cdots \qquad (F)$$

which is the **logarithmic series.**

224. Writing 1 for x in F gives a series for $l\cdot 2$; but this series is so slowly convergent (see limits of a series, Chap. XVIII.) as to be of no practical utility in computations.

We transform the logarithmic series as follows:

In (F) write $-x$ for x, and we get

$$l\cdot(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\cdots;$$

and subtracting this latter series from (F),

$$l\cdot\frac{1+x}{1-x}=2(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\cdots) \qquad (G)$$

Now make $x=\dfrac{1}{2z-1}$; then $\dfrac{1+x}{1-x}=\dfrac{z}{z-1}$, and this reduces (G) to

$$l\cdot z=l(z-1)$$

$$+2\left\{\frac{1}{2z-1}+\frac{1}{3}\cdot\frac{1}{(2z-1)^3}+\frac{1}{5}\cdot\frac{1}{(2z-1)^5}+\cdots\right\} \qquad (H)$$

This makes $l\cdot z$ depend upon $l(z-1)$, and a function of z which makes up the difference between the two logarithms.

Ex. **1.** Let $z=2$. Then since $l\cdot 1=0$,

$$l\cdot 2=2\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\cdots\right\}$$

$$=0.69315\cdots \text{ to five places.}$$

Ex. **2.** Let $z=5$. Then since $l\cdot 4=2\,l\cdot 2$,

$$l\cdot 5=2\,l\cdot 2+2\left\{\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\frac{1}{5}\cdot\frac{1}{9^5}+\cdots\right\}$$

$$=1.60944\cdots \text{ to five places.}$$

Ex. **3.** $l\cdot 10=l\cdot 2+l\cdot 5=2.30259$ to five places.

225. The series now obtained furnishes a practical method for computing logarithms to the base e.

Now let $a^x = b$, and take the Napierian logarithms of both members of the equation; then

$$l \cdot a^x = xl \cdot a = l \cdot b,$$

and $$x = \log_a b.$$

$$\therefore \log_a b = \frac{1}{l \cdot a} \cdot l \cdot b.$$

Hence $\frac{1}{l \cdot a}$ is a multiplier which changes $l \cdot b$ into $\log_a b$. And a being given, $l \cdot a$ is also given and constant, being the Napierian logarithm of the given base.

The multiplier, $(l \cdot a)^{-1}$, is called the **modulus** of the system of logarithms having a as base.

The modulus for base 10, or $(l \cdot 10)^{-1}$, is $0.43429448 \cdots$ to 8 decimals.

Thus the Napierian logarithm of any number is changed into the decimal logarithm of the same number by being multiplied by $0.43429448 \cdots$; and the decimal logarithm of any number is changed into the Napierian logarithm, by being divided by $0.43429448 \cdots$ or by being multiplied by the reciprocal of this quantity, namely,

$$2.30258509 \cdots, \text{ which is } l \cdot 10.$$

The logarithms of any two systems are thus connected by a constant multiplier, the modulus of one system with respect to the other.

Napierian logarithms are most convenient in analysis, and decimal logarithms in practical applications, and the change from one system to the other is easily effected.

EXERCISE XVII. d.

1. Find x from the equation $a(b+1) = l\cdot(a+x)$.

2. Show that $x = e^{l\cdot x} = a^{\frac{lx}{la}} = a^{\log_a x}$.

3. If $y = e^{\frac{1}{1-l\cdot z}}$, $z = e^{\frac{1}{1-l\cdot x}}$, then $x = e^{\frac{1}{1-l\cdot y}}$.

4. Prove that $l\cdot a - l\cdot x = \dfrac{a-x}{x} - \dfrac{1}{2}\left(\dfrac{a-x}{x}\right)^2 + \dfrac{1}{3}\left(\dfrac{a-x}{x}\right)^3 - + \cdots$

5. Prove that

$$n\{(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots\}$$
$$= \frac{x^n-1}{x^n} - \frac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2 + \frac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3 - + \cdots$$

6. Show that $l\cdot\{(1+x)^{\frac{1+x}{2}}\cdot(1-x)^{\frac{1-x}{2}}\} = \dfrac{x^2}{1\cdot 2} + \dfrac{x^4}{3\cdot 4} + \dfrac{x^6}{5\cdot 6} + \cdots$

7. Prove that the ratio $\log a : \log b$, where a and b are given numbers, is the same for all systems.

8. The modulus which changes from base 10 to base 3 is $\log 10 : \log 3$, taken in any system.

9. Show that $\log_{10} 9 = 1 - m\left(\dfrac{1}{10} - \dfrac{1}{2}\cdot\dfrac{1}{10^2} + \dfrac{1}{3}\cdot\dfrac{1}{10^3} - + \cdots\right)$ where m is the modulus to base 10.

10. Show that the logarithm of a number cannot be developed in terms of the number itself.

226. The exponential and the logarithmic series can be obtained by other methods besides the ones already employed. We give some of these as examples and exercises.

Ex. 1. Assume $a^x = 1 + b_1x + b_2x^2 + b_3x^3 + \cdots$

Then $$a^{2x} = 1 + b_1(2x) + b_2(2x)^2 + b_3(2x)^3 + \cdots \quad (1)$$

But $$a^{2x}=(a^x)^2=1+2\,b_1x+2\,b_2\left|\begin{matrix}x^2+2\,b_3\\ \end{matrix}\right|x^3+\cdots \qquad (2)$$
$$\begin{array}{l|l|l} 1+2\,b_1x+2\,b_2 & x^2+2\,b_3 & x^3+\cdots \\ \,b_1^2 & 2\,b_1b_2 & \end{array}$$

Equate coefficients of like powers of x in the two expressions for a^{2x}.

Ex. **2.** In Ex. 1, b_1 is indeterminate; what does it mean?

Ex. **3.** How do we know that the expansion of a^x must begin with 1?

Ex. **4.** If $a=e^{b_1}$, show from the result of Ex. 1 that

$$e=1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\cdots$$

Ex. **5.** Assume $a^x=1+b_1x+b_2x^2+\cdots b_nx^n+\cdots$

Then $a^y=1+b_1y+b_2y^2+\cdots b_ny^n+\cdots$

And $a^x-a^y=b_1(x-y)+b_2(x^2-y^2)+\cdots b_n(x^n-y^n)+\cdots$ (3)

But $a^x-a^y=a^y(a^{x-y}-1)$

$$=a^y\{b_1(x-y)+b_2(x-y)^2+\cdots b_n(x-y)^n+\cdots\} \qquad (4)$$

Make (3) = (4) as they are equivalents; divide throughout by $x-y$, which is a factor; put $y=x$; and equate coefficients of x^n.

Then $$b_{n+1}=\frac{b_n}{n+1}\cdot b_1.$$

From this relation obtain the coefficients in terms of b_1, which is indeterminate.

Ex. **6.** In Ex. 5, why is it necessary to divide by $x-y$ before making $y=x$?

Ex. **7.** If $e^x=b,\ x=l\cdot b.$

But $$b-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

Assume $$x=A(b-1)+B(b-1)^2+C(b-1)^3+\cdots \qquad (5)$$

In (5) put the value of $b-1$ taken from the preceding line, and equate coefficients of like powers of x. Then find A, B, C, etc.

This gives $l.\ b$ in terms of $b-1$.

Ex. **8**. Starting from $a^x = b$, proceed as in Ex. 7 to find $\log_a b$ in terms of $b' - 1$. What is the meaning of the indeterminate coefficient in the result?

Ex. **9**. Assume $\log(1 + x) = ax + bx^2 + cx^3 + \cdots$

Then n being an arbitrary quantity,

$$n \log(1 + x) = n(ax + bx^2 + cx^3 + \cdots) \quad . \quad . \quad . \quad (6)$$

But $$n \log(1 + x) = \log(1 + x)^n$$

$$= \log(1 + \overline{{}^nC_1x + {}^nC_2x^2 + \cdots})$$

$$= a({}^nC_1x + {}^nC_2x^2 + \cdots) + b({}^nC_1x + {}^nC_2x^2 + \cdots)^2 + \cdots \quad . \quad (7)$$

by taking the expression ${}^nC_1x + {}^nC_2x^2 + \cdots$ as a variable.

(6) and (7) are equivalents. Make $n = 0$, and equate coefficients of like powers of x.

The result contains the indeterminate a. What is this, and what is the effect of making $a = 1$?

Ex. **10**. In the assumption of Ex. 8, how do we know that the first term of the expansion must contain x?

CHAPTER XVIII.

OF SERIES.

227. Series are too varied in character to be rigidly classified, but the greater number of them have a relation to the geometric series, or to the arithmetic series, or to both.

In series related to the geometric, any term is connected with one or more of the preceding terms by constant multipliers, or by multipliers which vary with the number of the term in the series.

Thus in $$1 + 3 + 7 + 15 + 31 + 63 + \cdots$$

$$63 = 3 \times 31 - 2 \times 15,\ 31 = 3 \times 15 - 2 \times 7, \text{ etc.},$$

with constant multipliers, 3 and -2.

In $$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

the nth term is got from the $(n-1)$th term by multiplying by $\dfrac{x}{n}$, and the multiplier is a function of the position of the term in the series.

228. The nth term of a series is such a function of n that after some particular term, usually near the beginning of the series, any term is got by writing the number of the term for n in the function of n.

Usually, however, when the series contains a variable, x, in ascending powers, the absolute term is not considered in the counting, so that the term counted as the nth is the $(n+1)$th from the beginning. This usage makes the nth term contain x^n; and in purely numerical series a unit variable is frequently introduced for this and other purposes.

Thus $(2^{n+1}-1)x^n$ is the nth term of the series

$$1+3x+7x^2+15x^3+31x^4+\cdots$$

RECURRING SERIES.

229. A *recurring* series is generally the expanded form of a proper fraction, and is analogous to the circulating decimal in arithmetic.

Thus $$\frac{1}{1-x}=1+x+x^2+x^3+\cdots, \text{ a geometric series.}$$

$$\frac{1+2x}{1-3x+2x^2}=1+5x+13x^2+29x^3+61x^4+\cdots,$$

a recurring series of the second order.

$$\frac{1-x}{1+x-2x^2-x^3}=1-2x+4x^2-7x^3+13x^4-23x^5+\cdots,$$

a recurring series of the third order.

etc. etc. etc.

In this relation the fraction which by its expansion produces the series, is called the **Generating Function** (G) of the series; and the denominator of the fraction is the **Scale of Relation** (R).

When the R is binomial and linear, the series is geometric, and is of the first order; and generally, the order of the recurring series is the same as the dimensions of the R.

230. Problem. Given a recurring series and its order, to find its R and its G.

Take the recurring series of the 2d order given above, viz.:

$$1 + 5x + 13x^2 + 29x^3 + 61x^4 + \cdots$$

Assume $\dfrac{N}{1 + ax + bx^2} = 1 + 5x + 13x^2 + 29x^3 + 61x^4 + \cdots$

Multiply by the denominator. Then

$$N = 1 + \begin{array}{c|c|c|c|c} 5 & x + 13 & x^2 + 29 & x^3 + 61 & x^4 + \cdot \\ a & 5a & 13a & 29a & \\ & b & 5b & 13b & \end{array}$$

But N cannot be higher than linear in x; and therefore all the coefficients after linear x must vanish.

That is,

$$13 + 5a + b = 0;$$
$$29 + 13a + 5b = 0;$$
$$61 + 29a + 13b = 0, \text{ etc.}$$

The first two equations give $a = -3$, $b = 2$, and these values satisfy the third equation.

This compatibility shows that the series is of the second order and the R is quadratic.

If the third equation were not satisfied, the R would be of higher dimensions, and the series would be of a higher order.

The R is $1 - 3x + 2x^2$.

By putting -3 for a in the terms up to the linear inclusive, we get $N = 1 + 2x$.

$$\therefore \text{ G} = \frac{1 + 2x}{1 - 3x + 2x^2}.$$

231. We see from the foregoing article, that if z be the order of the series, it requires z terms to find the R, and z terms to find N. Hence the number of terms required to determine completely a recurring series of any order is twice the number of the order; and if z be the order, $z-1$ terms at most, counting from the beginning, may not follow the law of the rest of the series.

232. Problem. To find the nth term of a recurring series.

Taking the series of Art. 230, its R factors into $(1-x)(1-2x)$, and going to partial fractions,

$$\frac{1+2x}{1-3x+2x^2}=\frac{4}{1-2x}-\frac{3}{1-x}.$$

Then expanding these partials, we get the equivalent geometric series:

$$\frac{4}{1-2x}=4(1+2x+2^2x^2+\cdots 2^nx^n+\cdots).$$

$$\frac{3}{1-x}=3(1+x+x^2+\cdots x^n+\cdots).$$

And confining ourselves to the coefficient of x^n,
$(2^{n+2}-3)x^n=$ the nth term.

This shows that the terms of a recurring series are, in general, the algebraic sums of the corresponding terms of two or more geometric series.

And in finding the nth term we need not write out the geomctric series; for if

$$\frac{A}{1-\alpha x},\quad \frac{B}{1-\beta x},\quad \frac{C}{1-\gamma x},\quad \text{etc.,}$$

be the partial fractions,

$$(A\alpha^n + B\beta^n + C\gamma^n)x^n$$

is the nth term.

233. Our ability to find the nth term depends upon our ability to factor the R. If the R rises only to a quadratic, or is separable into factors none of which are higher than quadratic, it is *possible* to find the nth term; but when the linear factors are irrational, the operation may be laborious.

We give one example.

Ex. The series $1 - 4x + 19x^2 - 17x^3 + 265x^4 - \cdots$

has $1 + 4x + x^2$ as its R, and $N = 1 - x$.

$$1 + 4x + x^2 = \{1 + (2 + \sqrt{3})x\}\{1 + (2 - \sqrt{3})x\}.$$

Assuming $\dfrac{1 - x}{1 + 4x + x^2} = \dfrac{A}{1 + (2 + \sqrt{3})x} + \dfrac{B}{1 + (2 - \sqrt{3})x}$,

we obtain $A = \frac{1}{2}(1 + \sqrt{3}),\ B = \frac{1}{2}(1 - \sqrt{3})$.

Thence the nth term is

$$(-)^n \tfrac{1}{2}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n + \sqrt{}[(2 + \sqrt{3})^n - (2 - \sqrt{3})^n]\}x^n.$$

EXERCISE XVIII. a.

1. Find the R and the G of $1 + 2x + 3x^2 + 4x^3 + \cdots$, it being of the second order.

2. Find the R and G of $1 + 3x + 4x^2 + 5x^3 + \cdots$, a recurring series of the second order.

3. Find the next two terms of the series of Ex. 2.

4. Develop the series whose G is $\dfrac{1 + 2x - x^2}{(1 - x)(1 + x)^2}$.

5. The terms $1 + x - 2x^2 + 3x^3$ are the first four terms of a recurring series of the second order, and also of one of the third order. Find the G's of the series, and the 5th term in each.

6. The first four terms of a recurring series of the third order are $1 - x + 2x^2 - 2x^3$. Find an expression for the nth term, and thence find the 99th term.

7. Find the nth term of the series of Ex. 4.

8. Find the nth term of the series of the second order,

$$1 + 5x + 19x^2 + 65x^3 + \cdots$$

9. If there be n terms given, N may contain any number of terms from 1 to $\frac{n}{2} - 1$ if n is even, and from 1 to $\frac{n-1}{2}$ if n is odd.

10. If n terms be given, they may belong to a series of any order from $n - 1$ to $\frac{n}{2} + 1$ if n is even, and from $n - 1$ to $\frac{n+1}{2}$ if n is odd.

DIFFERENCE SERIES.

234. These have an alliance with arithmetic series. Take the series

	$2 + 3 + 6 + 12 + 22 + 37 + \cdots$
1st differences	$1 + 3 + 6 + 10 + 15 +$
2d "	$2 + 3 + 4 + 5 +$
3d "	$1 + 1 + 1 +$
4th "	$0 + 0 +$

By subtracting each term from the following, we obtain a set of series similar to the first, but of successively lower orders, called the series of 1st differences or Δ_1-series, the series of 2d differences or Δ_2-series, etc.

In the example given, the Δ_2-series is arithmetic, and the Δ_4-series vanishes; and for any true difference series some Δ-series is arithmetic, and the second one thereafter vanishes.

Thus in the series of cubes 1, 8, 27, 64, 125, etc., the Δ_2-series is arithmetic. In 1^5, 2^5, 3^5, etc., the Δ_4-series is arithmetic, etc.

Evidently if any general relation exists between the original series and its Δ_1-series, a similar relation must exist between each two consecutive Δ-series.

235. Let $u_0 + u_1 + u_2 + \cdots$ be a difference series in which the suffix serves the purpose of the exponent of a variable.

Then,

u-series	$u_0 + u_1 + u_2 + u_3 + u_4 + \cdots$
Δ_1-series	$\Delta_1 \quad \Delta_1' \quad \Delta_1'' \quad \Delta_1'''$
Δ_2-series	$\Delta_2 \quad \Delta_2' \quad \Delta_2'' \quad \ldots$
Δ_3-series	$\Delta_3 \quad \Delta_3'$
Δ_4-series	Δ_4

Now, $u_1 = u_0 + \Delta_1,\ \Delta_1' = \Delta_1 + \Delta_2,$

$$\Delta_2' = \Delta_2 + \Delta_3,\ \Delta_3' = \Delta_3 + \Delta_4, \text{ etc.}$$

Again, $u_2 = u_1 + \Delta_1' = u_0 + 2\Delta_1 + \Delta_2.$

$$\therefore\ \Delta_1'' = \Delta_1 + 2\Delta_2 + \Delta_3.$$

$$\Delta_2'' = \Delta_2 + 2\Delta_3 + \Delta_4.$$

Again, $u_3 = u_2 + \Delta_1'' = u_0 + 3\Delta_1 + 3\Delta_2 + \Delta_3.$

$$\therefore\ \Delta_1''' = \Delta_1 + 3\Delta_2 + 3\Delta_3 + \Delta_4.$$

Again, $u_4 = u_3 + \Delta_1''' = u_0 + 4\Delta_1 + 6\Delta_2 + 4\Delta_3 + \Delta_4.$

And obviously, from the mode of formation of the terms,

$$u_n = u_0 + {}^nC_1\Delta_1 + {}^nC_2\Delta_2 + {}^nC_3\Delta_3 + \cdots$$

which is an expression for the nth term of a difference series.

Ex. To find the nth term of $2 + 3 + 6 + 12 + 22 + \cdots$

$$u_0 = 2,\ \Delta_1 = 1,\ \Delta_2 = 2,\ \Delta_3 = 1,\ \Delta_4 = 0.$$

$$\therefore\ u_n = 2 + n + n(n-1) + \tfrac{1}{6}n(n-1)(n-2)$$
$$= 2 + \tfrac{1}{6}n(n+1)(n+2).$$

INTERPOLATION.

236. Take the difference series whose first 5 terms are

7, 2, 1, 4, 11.

The expression for the nth term is

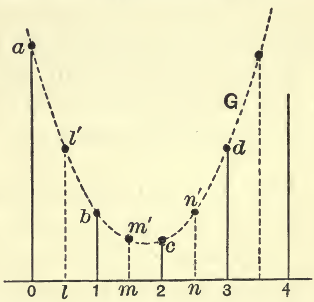
$$u_n = 7 + n(2n - 7).$$

Regarding n as a variable of the function whose value is denoted by u_n, we draw, as in the figure, the graph (G) of the function, in which n takes the place of x, and u_n of y

Then $0a = u_0 = 7,\quad 1b = u_1 = 2,$

$2c = u_2 = 1,\quad 3d = u_3 = 4$, etc.

And the points a, b, c, d, etc., represent, by their ordinates, $0a$, $1b$, etc., the terms of the series.



Hence we may define a series as a set of *point values* of a function of a variable, corresponding to equidistant values of the variable; the equidistant values being generally regarded in all series as the consecutive integers from zero upwards, or the numbers of the successive term.

Now $n = \frac{1}{2}$ gives the point value ll' corresponding to the middle point of $0\,1$; $n = \frac{3}{2}$ gives mm' corresponding to the middle point of $1\,2$; etc. And the points 0, l, 1, m, 2, etc., being equidistant, we have a, l', b, m', c, etc., as terms of a new series, such that every alternate term, counting from the first, belongs to the original series.

We are then said to have **interpolated** single mean terms in the original series.

Our unit on the x-axis being arbitrary, we may make $0l$ the unit by writing $\frac{1}{2}n$ for n in the function, and leaving u_n unchanged.

This gives
$$u_n = 7 + \frac{n}{2}(n - 7)$$

for the new series; and the series itself is

$$7, 4, 2, 1, 1, 2, 4, 7, 11 \cdots$$

So that $ll' = 4$, $mm' = 1$, $nn' = 2$, etc.

In a similar manner by writing $\frac{n}{3}$ for n in the nth term of the original series, we obtain the nth term of a series in which two mean terms are interpolated between each two consecutive terms of the given series, etc.

In like manner, if any real value whatever be given to n, the resulting value of u_n is the ordinate corresponding to that particular value of n.

237. The least consideration will show that interpolations can be made *accurately* whenever the nth term can be accurately expressed, and that the last condition is satisfied for any series in which an order of differences becomes zero. Also, that if no order of differences is zero, the nth term can be expressed only approximately, and the interpolated terms will be only approximately correct.

As an illustration of the latter statement consider a table, such as that of logarithms, for example.

The logarithmic series is a function of a variable n, and the tabulated logarithms are the point values of this function corresponding to consecutive equidistant values of the variable, as 30, 31, 32, etc., say.

These logarithms do not form a proper difference series, and the nth term cannot be exactly expressed.

Thus

	Δ_1	Δ_2
$\log 30 = 1.47712$		
	1424	
$\log 31 = 1.49136$		-45
	1379	
$\log 32 = 1.50515$		-43
	1336	
$\log 33 = 1.51851$		

But Δ_2 is small as compared with Δ_1, and *nearly* constant, so that

$$u_n = 1.47712 + 1424\,n - \tfrac{45}{2}\,n(n-1)$$

is approximately true for small values of n, as from 0 to 1; *i.e.* the result will be practically correct for the logarithm of any number lying between 30 and 31.

$$\text{Thus}\ \log 30.3 = 1.47712 + \tfrac{3}{10} \cdot 1424 + \tfrac{45}{2} \cdot \tfrac{3}{10} \cdot \tfrac{7}{10}$$
$$= 1.47712 + 427 + 5$$
$$= 1.41844,$$

which is true to the last figure.

This example shows that in a case like the present proportional parts are not always sufficient.

EXERCISE XVIII. b.

1. Find the orders of differences of the difference series, 50, 52, 50, 45, 38, 30, etc.

2. Find the expression for the nth term of Ex. 1.

3. Find the nth term of the difference series of which the first four terms are $1 + 7 + 11 + 13$.

4. Find the nth term of $1\frac{1}{3}$, 2, 3, $4\frac{5}{6}$, $8 \cdots$

5. Interpolate mean terms in the series 4, 1, 2, 7.

6. If a, b, c be three consecutive terms of a difference series, and m and n be mean terms between a, b and b, c, respectively, show that $m = \frac{1}{8}(3a + 6b - c)$, and $n = \frac{1}{8}(3c + 6b - a)$.

7. If in Ex. 6, m, n be two interpolated mean terms between a and b, and p, q be two between b and c, show that, upon the supposition that $\Delta_3 = 0$, $m = \frac{1}{9}(5a + 5b - c)$, $n = \frac{1}{9}(2a + 8b - c)$, $p = \frac{1}{9}(2c + 8b - a)$, and $q = \frac{1}{9}(5c + 5b - a)$.

8. The expectation of life at 10 years of age is 48.8, at 20 it is 41.5, at 30 it is 34.3, and at 40 it is 27.6. What is it at 15? at 25?

9. At 9 o'clock the distance of a star from the moon is 42′, at 10 it is 19′, and at 11 it is $-3'$. How were they situated at 10 h. 52 m.?

10. Given $\sin 24° = 0.40674$, $\sin 25° = 0.42262$, $\sin 26° = 0.43837$, $\sin 27° = 0.45399$; find $\sin 24° 25'$.

SUMMATION OF SERIES.

238. The *Sum of a Series* is a somewhat indefinite expression, as the following statements will show.

(1) If the series be numerical, the *sum* of its first n terms is intelligible, whether a general expression for such a sum can be found or not.

Thus the sum of n terms of the series $1 + 2 + 3 + \cdots$ is $\frac{1}{2}n(n+1)$, for all values of n; and the sum of any given number of terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ may be found, although no general expression for the sum of n terms has ever been obtained.

(2) In many numerical series the *sum of the series* to infinity may be given as a finite expression; but as we cannot properly speak of summing an infinite number of terms, this expression is more correctly spoken of as the **limit** of the series, *i.e.* the value towards which the sum of the terms approaches, as more and more of the terms are included in the summation, and to which the sum may be made to approach as near as we please.

Thus 2 is the limit of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ ad inf.

(3) If a series contains a variable in ascending or descending powers, what is called the sum is in reality the Generating Function of the series.

Thus e^x is the G. of $1 + x + \frac{x^2}{2!} + \cdots$ ad inf., and cannot be spoken of as the sum of the series without extending the meaning of the word *sum* quite beyond that usually given to it.

Similarly, $\frac{r^n - 1}{r - 1}$ is frequently spoken of as the sum of $1 + r + r^2 \cdots + r^{n-1}$, because, when developed by division, it produces the series. But it is evidently a generating function rather than a sum.

Thus in reference to series, the word *sum* applies properly to a finite number of terms of a *numerical* series.

The word *limit* applies to an infinite numerical series; and the term *generating function* to a series, finite or infinite, containing ascending or descending powers of a variable.

We shall not, however, always apply these distinctions rigidly.

SERIES TO n TERMS.

I. Generating Functions.

239. As a particular case take the recurring series of the second order:

$$\frac{1+x}{1-2x+x^2} \text{ is the G of}$$

$$1+3x+5x^2+\cdots(2n+1)x^n+\cdots \text{ad inf.}$$

Assume

$$\frac{N_n}{1-2x+x^2}=1+3x+5x^2+\cdots(2n+1)x^n.$$

Then $N_n =$

$$\begin{array}{l|l|l|l|l}
1+3 & x+5 & x^2+\cdots(2n+1) & x^n & \\
\;\;-2 & \;-6 & \;\;-2(2n-1) & \;\;-2(2n+1) & x^{n+1} \\
 & \;+1 & \;\;\;(2n-3) & \;\;\;(2n-1) & \;\;+(2n+1)x^{n+2}
\end{array}$$

$$=1+x-\{2n+3-(2n+1)x\}x^{n+1}.$$

$$\therefore\ G_n=\frac{1+x-\{2n+3-(2n+1)x\}x^{n+1}}{1-2x+x^2}$$

is the generating function required; and this fraction, by division, gives the series to n terms, and no more.

The variable x may take any value except 1 (as for this value the G_n becomes indeterminate), and the G_n becomes the sum of n terms of a numerical series.

Thus, putting $x=2$, we have

$$S_n=3+(2n-1)2^{n+1},$$

as the sum of the first n terms of the series

$$1+6+20+56+\cdots(2n+1)2^n.$$

Similarly, for $x = \frac{1}{2}$ we get $\dfrac{3 \cdot 2^{n+1} - (2n+5)}{2^n}$ as the sum of n terms of the series

$$1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \cdots \frac{2n+1}{2^n}.$$

240. Now take the general case, and let

$$\frac{N_n}{1 + px + qx^2} = a_0 + a_1 x + \cdots a_n x^n.$$

$$\therefore\ N_n = \begin{array}{r|r|r|r|l} a_0 + a_1 & x + \cdots a_n & x^n + \cdots & x^{n+1} \cdots & x^{n+2} \\ pa_0 & pa_{n-1} & + pa_n & \cdots & \\ & qa_{n-2} & + qa_{n-1} & + qa^n & \end{array}$$

And from the property of the R, that every column with three terms is zero,

$$N_n = a_0 + (a_1 + pa_0)x + \{pa_n + qa_{n-1} + qa_n x\} x^{n+1}.$$

Also $$a_{n+1} + pa_n + qa_{n-1} = 0.$$

$$\therefore\ N_n = a_0 + (a_1 + pa_0)x - (a_{n+1} - qa_n x)x^{n+1}.$$

And the required G is

$$\frac{a_0 + (a_1 + pa_0)x - (a_{n+1} - qa_n x)x^{n+1}}{1 + px + qx^2}.$$

In a similar manner the G_n can be found for a recurring series of any order.

241. To find the sum of n terms of the series whose nth term is $\dfrac{1}{n(n+1)(n+2)}$.

This series is allied to a recurring series, but the scale of relation is not of finite dimensions.

We have $\dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{2n} - \dfrac{2}{2(n+1)} + \dfrac{1}{2(n+2)}.$

And by putting $n = 1, 2, 3$, etc., we express the given series as the sum of three series, viz.:

$$S_n = \frac{1}{2}\left\{\begin{matrix} 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \frac{1}{n} \\ -\frac{2}{2} - \frac{2}{3} - \cdots - \frac{2}{n} - \frac{2}{n+1} \\ + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \end{matrix}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right\}.$$

Cor. If $n = \infty$, we obtain as the limit of the series to ∞,

$$S = \tfrac{1}{4}.$$

Series of the form of the foregoing can always be summed when the numerator of the nth term is constant, and the denominator has its factors of the form

$$(n+k)(n+2k)(n+3k)\cdots \text{ etc.}$$

II. Difference Series.

242. Let $u_0 + u_1 + u_2 + \cdots u_n$ be a given difference series, and let $U_0 + U_1 + U_2 + \cdots$ be the series of which $u_0 + u_1 + u_2 + \cdots$ is the first series of differences.

Then $U_0 = 0$, $U_1 = u_0$, $U_2 = u_0 + u_1$, $U_3 = u_0 + u_1 + u_2$, and generally $U_n = u_0 + u_1 \cdots + u_{n-1} = S_{n-1}$.

But the nth term of the U-series is given by

$$U_n = U_0 + {}^nC_1u_0 + {}^nC_2\Delta_1 + {}^nC_3\Delta_2 + \cdots \qquad \text{(Art. 235)}$$

and $\because\ U_0 = 0$,

$$S_{n-1} = nu_0 + {}^nC_2\Delta_1 + {}^nC_3\Delta_2 + \cdots$$

which gives the sum of $n-1$ terms counting from the 2d term, or of n terms counting from the 1st.

Hence for the sum of n terms from the beginning of the series,

$$S_n = nu_0 + {}^nC_2\Delta_1 + {}^nC_3\Delta_2 + \cdots$$

Ex. The sum of the 5th powers of the first n natural numbers is —

$$n + \tfrac{31}{2}n(n-1) + 30\,n(n-1)(n-2) + \tfrac{65}{4}n(n-1)(n-2)(n-3)$$
$$+ 3n(n-1)(n-2)(n-3)(n-4)$$
$$+ \frac{u}{6}(n-1)(n-2)(n-3)(n-4)(n-5);$$

which reduces to

$$\tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1).$$

LIMIT OF A SERIES.

243. The limit of a series is either finite or infinite. When the limit is finite, it is often called the sum of the series to infinity, and the series is said to be *convergent*. In general, series which are not convergent are classed together as *divergent*, and cannot be said to have any sum.

The limit of a converging series may be rational, or incommensurable; but in either case the rational value, or a sufficiently close approximation to the incommensurable, may be employed in place of the series in computations.

Such is the case with logarithms, with e, with trigonometrical functions, etc.

To know whether a series has a sum or not, we must determine whether it is convergent or not.

CONVERGENCY OF SERIES.

When a series contains a variable, its convergency or divergency is usually dependent upon the numerical value assumed by the variable.

Thus x being positive, the series $1+x+x^2+\cdots$ is convergent only when $x<1$.

Some series of this kind, however, and especially such as have increasing factorials in the denominators of their terms, are convergent for all numerical values of the variable.

244. It is shown under geometric series that the series $1+x+x^2+x^3+\cdots$ ad inf. has $\frac{1}{1-x}$ as its limit when x is positive and less than 1, and hence that under these conditions the series is convergent.

Now let $u_0+u_1+u_2+u_3+\cdots$ be an infinite series. Then

$$S=u_0\left\{1+\frac{u_1}{u_0}+\frac{u_2}{u_1}\cdot\frac{u_1}{u_0}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}\cdot\frac{u_1}{u_0}+\cdots\right\}.$$

And if each ratio $\frac{u_1}{u_0}$, $\frac{u_2}{u_1}$, $\frac{u_3}{u_2}$, etc., be $<x$,

$$S<u_0\{1+x+x^2+x^3+\cdots\};$$

which is convergent if $x<1$.

Therefore the series $u_0+u_1+u_2+\cdots$ is convergent if, after some finite term, $\frac{u_{n+1}}{u_n}$ is less than a quantity which is less than 1 for all values of n.

Ex. 1. The Binomial series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots$$

where n is negative or fractional, is infinite in extent.

To show that it is convergent if x be < 1.

$$\frac{u_{n+1}}{u_n} = \frac{n(n-1)\cdots(n-r)}{(r+1)!}x^{r+1} \cdot \frac{r!}{n(n-1)\cdots(n-r+1)}\frac{1}{x^r}$$

$$= \left(\frac{n}{1+r} - \frac{r}{1+r}\right)x.$$

But $\frac{n}{1+r} = 0$, and $\frac{r}{1+r} = 1$, when $r = \infty$. And the whole is < 1 if $x < 1$, which proves that the series is convergent if $x < 1$.

Ex. 2. The series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is convergent for all numerical values of x.

$\frac{u_{n+1}}{u_n} = \frac{x}{n+1}$; and for all finite values of x, this is < 1, when n is great enough, and is zero when $n = \infty$.

245. The series $u_0 - u_1 + u_2 - u_3 + - \cdots$, with alternating signs, is convergent when each term is greater than the following one.

For $$S = u_0 - (u_1 - u_2) - (u_3 - u_4) - \cdots$$

And as $u_1 > u_2$, $u_3 > u_4$, etc., every bracket is positive, and

$$S < u_0.$$

Again, $$S = (u_0 - u_1) + (u_2 - u_3) + (u_4 - u_5) + \cdots$$

And every bracket being positive, $S > u_0 - u_1$.

$\therefore$ S lies between u_0 and $u_0 - u_1$, and is finite.

Ex. The logarithmic series

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + - \cdots$$

is convergent when $x < 1$.

For if x be < 1, the condition is evidently satisfied.

246. When the sum of a few of the first terms of a series is a close approximation to its limit, the series is *rapidly* convergent.

If the ratio $u_{n+1} : u_n$ approximates to zero as n approaches ∞, the series is rapidly convergent. But even when this ratio approaches 1 as n approaches ∞, we are not justified in saying that the series is not convergent, as it may even then be slowly convergent, and may require further examination.

247. Theorem. The series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \frac{1}{n^p} + \cdots$ is convergent if $p > 1$.

Separate the terms of the given series, after the first, into groups of 2, 2^2, 2^3, etc., terms; then,

$$\frac{2}{2^p} > \frac{1}{2^p} + \frac{1}{3^p};\ \frac{4}{4^p} > \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p};\ \frac{8}{8^p} > \frac{1}{8^p} + \cdots \text{etc.}$$

$\therefore$ If S be the limit of the given series,

$$S < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots$$

i.e.
$$< 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots$$

But this latter series is convergent if $\frac{1}{2^{p-1}} < 1$; that is, if $p > 1$

$$\therefore\ 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \text{ is convergent if } p > 1.$$

248. The series $u_0 + u_1 + u_2 \cdots + u_n + \cdots$ is convergent if $u_n = \frac{1}{n^p}$, and $p > 1$.

Then, $$\frac{u_n}{u_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{n^2} + \frac{A}{n^3} + \cdots$$

And $$n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = p + \frac{p-1}{n} + \frac{A}{n^2} + \cdots$$

Every term of the right-hand member except the first vanishes when $n = \infty$. Hence the series

$$u_0 + u_1 + u_2 + \cdots u_n + \cdots$$

is convergent if the function $n\left\{\frac{u_n}{u_{n+1}} - 1\right\} > 1$, when $n = \infty$.

Ex. To examine the series whose nth term is $\frac{1+n}{1+n^3}$.

$$\frac{u_{n+1}}{u_n} = \frac{1+(n+1)}{1+(n+1)^3} \cdot \frac{1+n^3}{1+n} = \frac{n^4 + 2n^3 + n + 2}{n^4 + 4n^3 + \cdots + 2} = 1, \text{ when } n = \infty,$$

and this test is not sufficient.

Again, $$n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \frac{2n^4 + 6n^3 \cdots}{n^4 + 2n^3 + n + 2} = 2, \text{ when } n = \infty.$$

$\therefore$ the series is convergent.

The tests here given are the most useful tests of convergency. Readers who wish to make themselves acquainted with other tests will find such in larger Algebras, where special attention is given to the subject, or better, in works on Finite Differences.

249. When the sum of n terms of a series can be found, and is of such a form as to become finite when n

is infinite, the series is convergent, and the finite expression found by making $n = \infty$ is the limit of the series.

This is quite self-evident.

Ex. The sum of n terms of $\frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots$ is $\frac{n}{3(n+3)}$, and when $n = \infty$, the limit of the series is $\frac{1}{3}$.

EXERCISE XVIII. c.

1. Find the G_n of $1 + 2x + 3x^2 + \cdots (n+1)x^n$.

2. Sum n terms of the series $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$

3. Find the G_n of $1 + 3x + 6x^2 + \cdots \frac{1}{2}(n)(n+1)x^{n-1}$.

4. Sum n terms of the series $1 + 2 \cdot 2 + 3 \; 2^2 + 4 \cdot 2^3 + \cdots$

5. Find the G_n of $1 - 2x + 3x^2 - 4x^3 + - \cdots$

6. Sum n terms of the series $1 - 2 + 3 - 4 + 5 - 6 + \cdots$

7. Sum n terms of $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$

8. Sum n terms of $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$

9. Sum n terms of $\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \cdots$

10. Show that $S_n\left\{\frac{n}{(n+1)(n+2)}\right\} = S_n\left(\frac{1}{n+1}\right) - \frac{n}{n+2}$, where S_n is the sum of n terms from the beginning.

11. Find the limit of $\frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots n^2)$, when $n = \infty$.

12. Find the limit of $\frac{1}{n^4}(1^3 + 2^3 + 3^3 + \cdots n^3)$, when $n = \infty$.

13. Find the limit of $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\cdots$ ad inf.

14. Is the series $1+\frac{x}{2}+\frac{1\cdot 3\cdot x^2}{2\cdot 4}+\frac{1\cdot 3\cdot 5x^3}{2\cdot 4\cdot 6}+\cdots$ convergent?

15. Under what condition is $1+2^2x+3^2x^2+4^2x^3+\cdots$ convergent?

16. Show that $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots$ is convergent if $x<1$; thence find to four decimals the approximation to the value of

$$16\left\{\frac{1}{5}-\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}-+\cdots\right\}-4\left\{\frac{1}{239}-\frac{1}{3\,(239)^3}+-\cdots\right\}.$$

17. Find the limit of the series in Ex. 2.

18. Find the limit of $1-\frac{2}{2}+\frac{3}{2^2}-\frac{4}{2^3}+\cdots$

CHAPTER XIX.

DETERMINANTS.

250. Two figures, as 3 and 4, are in *order* when the less precedes the greater, and they form an *inversion* when the greater precedes the less.

Thus 1 2 3 4 are in order; 1 3 2 4 has 1 inversion, 3 2; 1 4 2 3 has 2 inversions, 4 2 and 4 3; and 4 1 3 2 has 4 inversions, 4 1, 4 2, 4 3, and 3 2.

For illustration, take any arrangement of figures,

3 1 4 2 5 6 8.

This contains 3 inversions. Interchange any two consecutive figures, as 2 and 5; the arrangement becomes 3 1 4 5 2 6 8, and contains 4 inversions. Or interchange 4 and 2, and the new arrangement has 2 inversions.

Thus the interchange of any two consecutive figures increases or decreases the number of inversions by one.

This is readily seen to be always the case; for if the figures be in order before the interchange, they form an inversion afterwards, and *vice versa*, while their relations to the figures which precede or follow them are unchanged.

251. Starting with any arrangement, as

3 1 4 2 5 6 8,

let us interchange two figures which are not consecutive, as 1 and 6. To do this, we must move 1 through 4 places to the right, and then move 6 through 3 places to the left; or we must move 1 through 3 places to the right, and 6 through 4 places to the left. In either case we make 7 consecutive changes in all; and the number of inversions is thus increased or decreased by an *odd* number. The new arrangement, 3 6 4 2 5 1 8, has 10 inversions, or 7 more than the original.

Similarly, if the orders of any two figures be denoted by m and n, where $m<n$, to interchange m and n requires us to move m through $n-m$ places to the right, and then to move n through $n-m-1$ places to the left; or, to make $2(n-m)-1$ consecutive interchanges in all. And this being an odd number gives the important

Theorem. — *To interchange any two numbers in an arrangement increases or decreases the number of inversions by an odd number.*

252. Consider the four-dimensional term $a_1 b_2 c_3 d_4$, composed of four letters with attached suffixes, and in which both the letters and suffixes are in order.

Keeping the letters in order, let us permute the four suffixes in every possible way, as $a_1 b_3 c_2 d_4$, $a_2 b_1 c_4 d_3$, etc. As we can permute the four suffixes in ${}^4P_4$ or 24 ways, we shall have 24 terms in all, of which no two have the same suffixes attached to the same letters, or are wholly alike.

The term $a_1 b_2 c_3 d_4$, being the one from which the others are derived, is called the *principal* or *leading* term; but as the whole set of terms may be derived from any one of them, any term may be taken as a principal term.

Let us take as our principal term that one having no inversions, and calling this positive, let us agree that in forming the other terms every inversion is to be accompanied by a change of sign. Then a term with one inversion in its suffixes, as $a_1 b_3 c_2 d_4$, is negative; a term with two inversions, as $a_1 b_4 c_2 d_3$, is positive; and generally, *a term is + or − according as its suffixes contain an even or an odd number of inversions.*

These considerations apply to a leading term of any number of letters, and its derived terms.

253. A **Determinant** is the algebraic sum of all the terms that can be derived from a leading term, by permuting the suffixes without changing the letters, each term being taken with its proper sign.

A letter with its attached suffix is an **element** of the determinant, and as each letter takes in turn each suffix, if there are n letters, there are also n suffixes and n^2 elements.

A determinant with n letters and n suffixes is of the nth order, and contains $n!$ terms.

The determinant of the second order is $a_1b_2 - a_2b_1$, and of the third order it is

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

254. A letter with its attached suffix, standing as an element of a determinant, is symbolic, and may be replaced by any quantitative symbol whatever.

But it is only through this symbolic and symmetrical notation that we are enabled to discuss with any facility the general properties of determinants. Moreover, owing to their unwieldiness when written at length, it becomes necessary to employ some symbolic or contracted form for the whole expression.

The symbols $\Sigma \pm a_1b_2c_3 \cdots$ and $|a_1b_2c_3 \cdots|$, where the leading term is written after $\Sigma \pm$ or between straight-line brackets, are both employed. But the *working* form known as a **matrix** is made by writing the elements in a square between parallel vertical lines, in such a manner that all the same letters stand in the same column, and all the same suffixes are situated in the same row.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

The determinant of the 4th order is written as a matrix in the margin.

255. The diagonal of the matrix from the upper left-hand corner to the lower right-hand corner, namely, $a_1\, b_2\, c_3\, d_4$, is the *principal* or *leading diagonal*, as giving the principal term of the determinant.

The sign of the matrix as a whole depends upon that of its principal diagonal.

The matrix being a symbolic form for a determinant must be capable of being expanded so as to give the determinant, and with a matrix having symbolic elements this expansion can be effected by permuting the suffixes of the leading diagonal according to the definition of Art. 253. Hence two matrices containing the same symbolic elements can differ only in sign, and the signs of two such matrices will be the same or opposite according as the number of inversions in their principal diagonals differ by an even or by an odd number, Art. 252.

256. Consider the matrices

$$(1)\ \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},\quad (2)\ \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix},\quad (3)\ \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

(1) is the standard matrix of the 4th order, the suffixes being in order in the columns, and the letters being in order in the rows. Its principal diagonal is $a_1\, b_2\, c_3\, d_4$.

(2) differs from (1) in having the rows of (1) for its columns and the columns of (1) for its rows, the letters and suffixes still being in order. The principal diagonal of (2) is $a_1\, b_2\, c_3\, d_4$, and being the same as that of (1), the expansions give the same determinant.

Therefore, a matrix is not changed in value by changing its rows to columns and its columns to rows, provided the letters and suffixes maintain the same order. Hence *whatever is true for a matrix with respect to its columns, is true also with respect to its rows*, and *vice versa*.

(3) differs from (1) in having its 2d and 3d rows interchanged. This introduced one inversion into its principal diagonal, and hence changes the sign of the matrix. And it is readily seen from the principles of Art. 252, that the interchange of any two rows, or of any two columns in a matrix changes the sign of the matrix, since it increases or decreases the number of inversions in the principal diagonal by an odd number.

257. Theorem. *If two columns or two rows of a matrix be identical, the value of the matrix is zero.*

For, by interchanging the identical columns or the identical rows, the matrix changes sign. But the columns being identical leaves the matrix unchanged. The only quantity or expression which remains unchanged when you change its sign is zero. Hence the value of the determinant is zero.

Thus the matrix with two a-columns, as in the margin, expands into $a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1$, which is identically zero.

$$\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix}$$

258. Expansion of the Matrix. Let us take a matrix of the third order to begin with.

As every term in the determinant contains each letter once and each suffix once, the terms which contain a_1 cannot contain any other a or any other letter with suffix 1. Hence the coefficient of a_1 is the sum of all the terms that can be made from the remaining letters and the remaining suffixes. But this, for a determinant of the 3d order, is the expression denoted by the matrix $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.

Similarly, the coefficient of a_2 is $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, and of a_3 it is $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$.

Hence the expansion takes the form

$$\pm a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \pm a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \pm a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

But taking the principal diagonals, $a_1 b_2 c_3$ is $+$, $a_2 b_1 c_3$ is $-$, and $a_3 b_1 c_2$ is $+$. And the expansion is

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \qquad (A)$$

and the matrix of the third order is made to depend upon matrices of the second order.

Similarly, the expansion of the matrix of the 4th order is

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix},$$

which makes it depend upon matrices of the 3d order.

So also a matrix of the nth order may be expanded to depend upon matrices of the $(n-1)$th order; and these again upon those of the $(n-2)$th order; and so on.

259. Reducing the matrices of (A), the determinant of the third order becomes —

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

Comparing this with the matrix here written, in which the first two rows are repeated in order below the matrix, we see that the three terms $a_1\,b_2\,c_3$, $a_2\,b_3\,c_1$, and $a_3\,b_1\,c_2$, read in the direction of the principal diagonal, are $+$, and the three, $a_3\,b_2\,c_1$, $a_1\,b_3\,c_2$, and $a_2\,b_1\,c_3$, read in the direction of the other diagonal, are $-$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}$$

This is the **rule of Sarrus** for expanding a matrix of the third order; in practice the portion without the matrix is not written, the operation being carried on mentally.

Ex. **1.** $$\begin{vmatrix} 2 & 4 & 7 \\ 1 & 2 & 1 \\ 3 & 5 & 6 \end{vmatrix} = 2\cdot2\cdot6 + 4\cdot1\cdot3 + 7\cdot1\cdot5 - 3\cdot2\cdot7 - 2\cdot5\cdot1 - 1\cdot4\cdot6 = -5.$$

Ex. **2.** $$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1\cdot5\cdot9 + 2\cdot6\cdot7 + 3\cdot4\cdot8 - 3\cdot5\cdot7 - 6\cdot8\cdot1 - 9\cdot2\cdot4 = 0.$$

Ex. **3.** $$\begin{vmatrix} a & 1 & a \\ a & a & 1 \\ 1 & a & a \end{vmatrix} = a^3 + a^3 + 1 - a^2 - a^2 - a^2 = 2\,a^3 - 3\,a^2 + 1.$$

EXERCISE XIX. a.

1. Find the value of the following matrices of the third order —

i. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ ii. $\begin{vmatrix} 7 & 1 & 6 \\ 1 & -2 & 4 \\ 3 & -5 & -1 \end{vmatrix}$ iii. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ iv. $\begin{vmatrix} x & 1 & 1 \\ 0 & x & 1 \\ 0 & 0 & x \end{vmatrix}$

v. $\begin{vmatrix} 1 & x & y \\ 1 & x^2 & y^2 \\ 1 & x^3 & y^3 \end{vmatrix}$ vi. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ vii. $\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$ viii. $\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$

2. Expand $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 3 & 2 \end{vmatrix}$. (Expand to depend upon matrices of the third order, and then expand these.)

3. Show that $\begin{vmatrix} x & x & 1 & 1 \\ x & 1 & x & 1 \\ x & 1 & 1 & x \\ x & 1 & 1 & 1 \end{vmatrix} = x(1-x)^3.$

260. We see from the preceding article that as soon as a matrix is reduced to depend upon matrices of the third order, we can write out its expansion.

We turn our attention now to the investigation of those properties of the matrix which enable us to expand or reduce it more readily.

OPERATIONS ON THE MATRIX.

Let **D** denote a determinant, of any order, with symbolic elements, and let A_1 be the coefficient of a_1. Then (Art. 258) A_1, which is called a *first minor* of **D**, is a determinant of the next order lower than **D**, and contains no a and no letter with suffix 1. Similarly, let A_2 be the coefficient of a_2, A_3 of a_3, etc.

Then (Art. 258)

$$\mathbf{D} = a_1A_1 - a_2A_2 + a_3A_3 - + \cdots (-)^{n-1} a_nA_n.$$

261. To multiply the matrix by any quantity, m.

$$m\mathbf{D} = ma_1A_1 - ma_2A_2 + ma_3A_3 - + \cdots$$

But this is the expansion of the matrix in which ma_1 is written for a_1, ma_2 for a_2, etc., throughout the a-column.

Therefore, *to multiply a matrix by* m *we multiply every element of a column or of a row by* m.

Also, as multiplying by a fraction with unit numerator is equivalent to dividing by the denominator,

Therefore, *to divide a matrix by* m *we divide every element of a column or of a row by* m.

Ex. 1. $m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{etc.}$

Ex. 2. $\begin{vmatrix} \frac{1}{2} & 1 & 1 \\ 1 & \frac{1}{3} & 1 \\ 1 & 1 & \frac{1}{4} \end{vmatrix} = \frac{1}{12}\begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 4 \\ 2 & 3 & 1 \end{vmatrix} = 1\frac{11}{12}.$

Ex. 3. $\begin{vmatrix} 8 & 4 & 2 \\ 12 & 4 & 3 \\ 4 & 6 & 5 \end{vmatrix} = 8\begin{vmatrix} 2 & 2 & 2 \\ 3 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 16\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 2 & 5 \end{vmatrix} = -64.$

262. Let each element of a column be the algebraic sum of two quantities.

Thus, let $a_1 = p_1 + q_1,\ a_2 = p_2 + q_2,$ etc.

Then $\mathbf{D} = (p_1+q_1)A_1 - (p_2+q_2)A_2 + (p_3+q_3)A_3 - + \cdots$

$= p_1A_1 - p_2A_2 + p_3A_3 - + \cdots$

$+ \{q_1A_1 - q_2A_2 + q_3A_3 - + \cdots\}$

= the matrix with p written for a, + the matrix with q written for a.

And the matrix thus becomes the sum of two matrices of the same order.

Ex. 1. $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1+1 & 3 & 1 \\ 4+1 & 2 & 1 \\ 2+1 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 4 & 2 & 1 \\ 2 & 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 4 & 2 & 1 \\ 2 & 4 & 1 \end{vmatrix} = 4,$

since the second matrix, having two columns alike, vanishes.

$$\text{Ex. 2.}\quad \begin{vmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 0+2 & 3 & 1 \\ 3+2 & 2 & 1 \\ 1+2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 4,$$

as the second matrix will vanish after dividing by 2.

263. Let a_1 be changed to $a_1 + nb_1$, a_2 to $a_2 + nb_2$, a_3 to $a_3 + nb_3$, etc., throughout the a-column; and let the value of the new matrix be noted by $\mathbf{D}'$.

Then

$$\mathbf{D}' = (a_1 + nb_1)A_1 - (a_2 + nb_2)A_2 + (a_3 + nb_3)A_3 - + \cdots$$
$$= a_1A_1 - a_2A_2 + a_3A_3 - + \cdots$$
$$+ n\{b_1A_1 - b_2A_2 + b_3A_3 - + \cdots\}$$
$$= \mathbf{D} + n \text{ times the matrix with } b \text{ put for } a.$$

But the matrix with b written for a has two b-columns, and therefore vanishes.

Hence $\mathbf{D}' = \mathbf{D}$, and we have the following important

Theorem. *The value of a matrix is not changed by increasing or diminishing any column by a multiple of any other column, or any row by a multiple of any other row.*

In the examples which follow we shall denote the columns from left to right by C_1, C_2, C_3, etc., and the rows from above downwards by R_1, R_2, R_3, etc. Then $R_2 + R_1$ indicates that we are to add the second row to the first row, and $C_2 - nC_1$ denotes that we are to subtract n times the first column from the second column; etc.

$$\text{Ex. 1.}\quad \begin{vmatrix} 1 & 5 & 3 & 6 \\ 3 & 1 & 4 & 1 \\ 4 & 6 & 1 & 7 \\ 2 & 2 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 3 & 6 \\ 0 & 14 & 5 & 17 \\ 0 & 2 & -9 & 3 \\ 0 & 8 & 1 & 10 \end{vmatrix} = 2\begin{vmatrix} 7 & 5 & 17 \\ 1 & -9 & 3 \\ 4 & 1 & 10 \end{vmatrix} = -24.$$

Here we write $3R_1 - R_2$ for R_2; $R_3 - 2R_4$ for R_3; and $2R_1 - R_4$ for R_4. The C_1 of the new matrix being all ciphers except 1, the matrix is at once reduced to one of the third order.

Ex. 2. $\begin{vmatrix} 1 & a & b & c \\ 1 & a^2 & b^2 & c^2 \\ 1 & a^3 & b^3 & c^3 \\ 1 & a^4 & b^4 & c^4 \end{vmatrix}$.

1. Divide by abc.
2. $R_1,\ R_2 - R_1,\ R_3 - R_2,\ R_4 - R_3$.
3. Divide by $(1-a)(1-b)(1-c)$.
4. $C_1,\ C_1 - C_2,\ C_2 - C_3$.
5. Divide by $(a-b)(b-c)$, and reduce.

Result, $-abc(1-a)(1-b)(1-c)(a-b)(b-c)(c-a)$.

Ex. 2 may also be reduced as follows —

1. If a, b, or $c = 0$, the matrix vanishes. $\therefore a, b, c$ are monomial factors.

2. If a, or b, or $c = 1$, two columns are alike, and the matrix vanishes. $\therefore a-1,\ b-1,\ c-1$ are binomial factors.

3. If $a = b$, or $b = c$, or $c = a$, two columns are alike, and the matrix vanishes. $\therefore a-b,\ b-c,\ c-a$ are binomial factors.

Since the expansion cannot have any term higher than of 9 dimensions, these are all the literal factors. And the expansion is readily seen to be $abc(a-1)(b-1)(c-1)(a-b)(b-c)(c-a)$.

EXERCISE XIX. b.

1. Evaluate the following determinants:

i. $\begin{vmatrix} 3 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \end{vmatrix}$. ii $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{vmatrix}$. iii $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$.

2. If $a_1 | \cdots | - b_1 | \cdots | + c_1 | \cdots | - d_1 | \cdots |$ be the expansion of a 4th-order determinant, fill in the brackets.

3. If a matrix be rotated through ninety degrees, the rows become columns, and the columns rows, but they are not disposed in the original order. How does this affect the sign?

4. If the rotated matrix of 3 be turned over in the plane, so that the 4th column may become the 1st, the 3d, the 2d, etc., how does this affect the sign? How does the matrix now compare with the original?

5. Show that to keep the suffixes in order and permute the letters is equivalent to keeping the letters in order and permuting the suffixes.

6. If p_1 and p_2 denote the diagonals of a matrix of the nth order, show that the sum of the inversions in p_1 and p_2 is $\frac{1}{2}n(n-1)$.

7. A matrix of the nth order has both diagonals of the same sign when n or $n-1$ is a multiple of 4; and of opposite signs in other cases.

DETERMINANTS IN THEIR RELATION TO EQUATIONS.

264. Take the set of linear equations, with symbolic coefficients —

$$a_1x + b_1y + c_1z = g_1,$$

$$a_2x + b_2y + c_2z = g_2,$$

$$a_3x + b_3y + c_3z = g_3.$$

Let A_1 be the first minor of $|a_1\, b_2\, c_3|$ with respect to a_1, A_2 with respect to a_2, etc.

Multiply the 1st equation by A_1, the 2d by A_2, the 3d by A_3, and add; then —

$$(a_1A_1 - a_2A_2 + a_3A_3)x + (b_1A_1 - b_2A_2 + b_3A_3)y$$
$$+ (c_1A_1 - c_2A_2 + c_3A_3)z = g_1A_1 - g_2A_2 + g_3A_3.$$

But the coefficient of x is $|a_1\, b_2\, c_3|$.

The coefficient of y is $|a_1\, b_2\, c_3|$ with b put for a, and it therefore vanishes.

The coefficient of z is $|a_1\, b_2\, c_3|$ with c for a, and it vanishes.

The independent term is $|g_1\, b_2\, c_3|$, *i.e.* $|a_1\, b_2\, c_3|$ with g put for a.

$$\therefore\ x = \frac{|g_1\, b_2\, c_3|}{|a_1\, b_2\, c_3|}.$$

Similarly, $$y = \frac{|a_1\, g_2\, c_3|}{|a_1\, b_2\, c_3|},\ \text{and}\ z = \frac{|a_1\, b_2\, g_3|}{|a_1\, b_2\, c_3|}.$$

Thus, in the solution of the set of 3 linear equations in 3 variables, each variable appears as a fraction whose parts are matrices. The denominator is common to all, and is the matrix formed by taking the coefficients of the variables in order as elements of the matrix.

The numerators are the denominators in which one column is replaced by the independent terms of the equations, the coefficients of x being replaced in finding the value of x, those of y in finding y, and those of z in finding z.

Ex. To solve the set
$$\begin{aligned} 2x + 3y + 4z &= 16, \\ x + 4y + 2z &= 13, \\ 3x + y + z &= 7. \end{aligned}$$

Here
$$x = \frac{\begin{vmatrix} 16 & 3 & 4 \\ 13 & 4 & 2 \\ 7 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 2 & 16 & 4 \\ 1 & 13 & 2 \\ 3 & 7 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 2 & 3 & 16 \\ 1 & 4 & 13 \\ 3 & 1 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix}},$$

whence $$x = 1,\ y = 2,\ z = 2.$$

265. From the nature of the investigation of Art. 264, it is evident that the same method of solution applies to the case of n linear equations in n variables.

So that in the case of any number of linear equations in the same number of variables, we can, at sight, write down the values of the variables in the forms of fractions whose parts are matrices. The rest of the solution consists in the evaluation of the matrices.

266. Take the set of four homogeneous linear equations in four variables —

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1u = 0 \\ a_2x + b_2y + c_2z + d_2u = 0 \\ a_3x + b_3y + c_3z + d_3u = 0 \\ a_4x + b_4y + c_4z + d_4u = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (A)$$

Divide the last three equations throughout by u, and solve for the ratios $\frac{x}{u}$, $\frac{y}{u}$, and $\frac{z}{u}$.

Then, $\frac{x}{u} = -\frac{|\,b_2\,c_3\,d_4\,|}{|\,a_2\,b_3\,c_4\,|}$, $\frac{y}{u} = \frac{|\,a_2\,c_3\,d_4\,|}{|\,a_2\,b_3\,c_4\,|}$, $\frac{z}{u} = -\frac{|\,a_2\,b_3\,d_4\,|}{|\,a_2\,b_3\,c_4\,|}$.

$$\therefore \frac{x}{|\,b_2\;c_3\;d_4\,|} = \frac{-y}{|\,a_2\;c_3\;d_4\,|} = \frac{z}{|\,a_2\;b_3\;d_4\,|} = \frac{-u}{|\,a_2\;b_3\;c_4\,|} \quad (B)$$

which gives the ratios $x : y : z : u$.

Ex. Given $3x + 2y - z = 4x - 6y + z = 0$, to find the ratios $x : y : z$.

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ -6 & 1 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 2 \\ 4 & -6 \end{vmatrix}}, \text{ or } \frac{x}{4} = \frac{y}{7} = \frac{z}{26}.$$

$$\therefore x : y : z = 4 : 7 : 26.$$

267. Since the denominators, in (B) of Art. 266, are proportional to the numerators, write these denominators for x, y, z, and u in the first equation of (A) of the same article, then —

$$a_1\,|\,b_2\;c_3\;d_4\,| - b_1\,|\,a_2\;c_3\;d_4\,| + c_1\,|\,a_2\;b_3\;d_4\,| - d_1\,|\,a_2\;b_3\;c_4\,| = 0.$$

But this expression is the expansion of the matrix

$$|\,a_1\ b_2\ c_3\ d_4\,|.$$

Hence the *Eliminant* (Art. 138) of a set of homogeneous linear equations, having the same number of equations as of variables, is the determinant formed from the coefficients taken in order.

The vanishing of the eliminant indicates that the set of equations is consistent, or that the equations are compatible.

Ex. **1.** Given the equations

$$2x + 3y - 1 = 3x - 2y + 4 = x + 6y - m = 0,$$

to find the value of m that the equations may be compatible.

Writing a unit-variable, z, in the independent term, we have

$$\begin{cases} 2x + 3y - z = 0 \\ 3x - 2y + 4z = 0 \\ x + 6y - mz = 0 \end{cases} \qquad \text{whence} \begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 4 \\ 1 & 6 & -m \end{vmatrix} = 0,$$

from which $m = 4\tfrac{4}{13}$.

Ex. **2.** Given $al + bm : bm + cn : cn + al = x : y : z$, to find the ratios $a : b : c$ and $l : m : n$.

We have $\dfrac{al + bm}{x} = \dfrac{bm + cn}{y} = \dfrac{cn + al}{z} = u$, say.

Then,

$$\begin{aligned} al + bm \phantom{{}+cn} - xu &= 0 \\ bm + cn - yu &= 0 \\ al \phantom{{}+bm} + cn - zu &= 0 \end{aligned}$$

And treating a, b, c, and u as variables,

$$\frac{a}{\begin{vmatrix} m & 0 & x \\ m & n & y \\ 0 & n & z \end{vmatrix}} = \frac{-b}{\begin{vmatrix} l & 0 & x \\ 0 & n & y \\ l & n & z \end{vmatrix}} = \frac{c}{\begin{vmatrix} l & m & x \\ 0 & m & y \\ l & 0 & z \end{vmatrix}} \qquad \text{(Art. 266)}$$

Whence $a : b : c = mn(x-y+z) : nl(y-z+x) : lm(z-x+y)$.

Similarly, $l : m : n = bc(x-y+z) : ca(y-z+x) : ab(z-x+y)$.

EXERCISE XIX. c.

1. Express the condition that $y = m_1x + h_1$, $y = m_2x + h_2$, and $y = m_3x + h_3$ may be satisfied for the same values of x and y.

How would you interpret this fact with respect to the graphs of the given functions?

2. Express the condition that $Ax_1 + By_1 + C = Ax_2 + By_2 + C = Ax_3 + By_3 + C = 0$ may be true, A, B, and C being considered as variables.

3. If 3 sheep, 1 cow, and 4 horses are worth \$318, 4 sheep, 3 cows, and 1 horse are worth \$137, and 6 sheep, 2 cows, and 5 horses are worth \$420, what is the value of 2 sheep, 5 cows, and 3 horses?

4. If $a + \frac{1}{l}(hm + gn) = b + \frac{1}{m}(hl + fn) = c + \frac{1}{n}(gl + fm) = \lambda$, eliminate l, m, and n, and find a cubic equation for determining λ.

5. Find the condition that $x^3 + ax = b$ and $x^3 + a_1x = b_1$ may have a common root.

6. By means of $Ax + By + Cz = 0$ and $Px + Qy + Rz = 0$ eliminate x, y, and z from the statement $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

7. Determine m, n, and p from the set

$$u_0 - 4m + 6u_1 - 4n + u_2 = m - 4u_1 + 6n - 4u_2 + p$$
$$= u_1 - 4n + 6u_2 - 4p + u_3 = 0.$$

(If u_0, u_1, u_2, u_3, be four terms of a difference series, m, n, and p are intermediate equidistant terms.)

8. Find the eliminant of $a^2x^{-2} + b^2y^{-2} + c^2z^{-2} = alx^{-2} + bmy^{-2} + cnz^{-2} = al'x^{-2} + bm'y^{-2} + cn'z^{-2} = 0$.

9. Show that $\begin{vmatrix} a+b-c & c & c \\ a & b+c-a & a \\ b & b & c+a-b \end{vmatrix} \equiv \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

10. Solve the equation in x, $\begin{vmatrix} a_1 + b_1x & c_1 & d_1 \\ a_2 + b_2x & c_2 & d_2 \\ a_3 + b_3x & c_3 & d_3 \end{vmatrix} = 0.$

MISCELLANEOUS EXERCISES.

1. If $x+y=a,\ y+z=b,\ z+x=c,$ and $a^2+b^2+c^2=0,$ then
$$xy+yz+zx=\tfrac{1}{2}(ab+bc+ca).$$

2. Extract the fifth root of 78502725751.

3. Expand $\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}$ into an ascending series.

4. Given $\sqrt{\left(\dfrac{x+1}{x-1}\right)}+\sqrt{\left(\dfrac{x-1}{x+1}\right)}=a,$ to find x.

5. Given $x+by-az=\dfrac{b}{a},\ ax+y-\dfrac{z}{a}=a^2,\ \dfrac{x}{a}+ay-z=1,$ to solve the set.

6. Find all the positive integral solutions of $17x+31y=100$.

7. Expand $\dfrac{1+x-x^2}{1-x+x^2}$ in ascending powers of x.

8. Divide 100 into two parts such that twice the square of one added to three times the square of the other may be a minimum.

9. Determine if $\dfrac{x^2-x+1}{x^2+x+1}$ has a maximum or a minimum value, or both for real values of x.

10. Factor $3x^2-10x-25$ and thence show for what values of x the expression is positive.

11. Show that $\left(\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\right)\left(\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}\right)\not<9.$

12. Find the value of $\sqrt{x^2+ax}-x$ when $x=\infty$.

13. Find the maximum or minimum value of $\dfrac{(a+x)(b+x)}{x^2}.$

14. Write the nth term of $1-2x+5x^2-8x^3+11x^4\cdots$

15. Solve the set, $x^2+y^2+x+y=14,\ 2(x^2+y^2)+3xy=29.$

16. A person distributes 50 cents among beggars, giving 7 cents to some, and 12 cents to others. How many were there?

17. A and B play four games, each having the same sum at starting. In the first game A wins $\frac{1}{3}$ of B's money. In the second B wins \$20. In the third A wins $\frac{1}{4}$ of what B has; and in the fourth B wins \$5, and has then $\frac{1}{2}$ as much as A. What did each start with?

18. Find three numbers in A. P. whose sum is 21, while the first and second terms are together equal to $\frac{3}{4}$ of the second and third.

19. Sum to n terms the series whose nth term is $n^2 - n + 1$.

20. A falling body descends $\frac{1}{2}f$ feet the first second, $\frac{3}{2}f$ feet the second second, $\frac{5}{2}f$ feet the third, and so on. How far will it fall in the nth second? How far in n seconds?

21. Solve $2x^3 - x^2 - 2x + 1 = 0$ by putting it under the form $x^4 - (x^2 + px + q)^2 = 0$.

22. If $2n - 10$ is the nth term, how many terms will make $n^2 - 3b$?

23. Find three numbers in G. P. whose sum is 13, and the sum of whose squares is 91.

24. The sums of n terms of two A. P. s are $n(n+1)$ and $\frac{n}{2}\left(\frac{n}{2} - 1\right)$. Determine if they have a common term.

For what same numbered term is the term in one series 8 times that in the other?

25. A sum of \$1000 is compounded annually for 4 years and amounts to \$1464.10. What is the rate?

26. Divide 1 into 5 parts in A. P. so that the sum of the squares of the parts may be $\frac{9}{40}$.

27. Factor $2x^2 - 21xy - 11y^2 - x + 34y - 3$.

28. If $a^2 + b^2 : a + c = a^2 - b^2 : a - c$, then $a : b = b : c$; and $a - 2b + c = (a - b)^2/a = (b - c)^2/c$.

29. If $z \propto (x + y)$, and $y \propto x^2$, and if when $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$, express z in terms of x.

30. Find n when ${}^{n}P_3 : {}^{n+2}P_3 = 1 : 5$.

31. At a game of cards 3 are dealt to each person, and each can hold 425 times as many different hands as there are cards in the pack. How many cards are in the pack?

32. If the number of combinations of n things 10 together be the same as the number 5 together, what is the number 2 together?

33. The sums of n terms of two A. P. s are as $11-5n$ to $11+3n$. Find the ratio of their sixth term.

34. How many different guards of 6 soldiers can be formed from a company of 30? And relatively,

i. How often will A be on duty?

ii. How often will A and B be on duty?

iii. How often will A be with B, without C?

iv. How often will A be without B or C?

35. At what time after T o'clock will the hands make an angle of a° with one another?

36. A, B, and C start from the same point at the same moment to travel around an island 34 miles in circumference. A goes 13, B, 7, and C, 4 miles an hour. When and where will they first be together again?

37. Find the expansion of $\left(1 - \frac{x-1}{x}\right)^x$ when $x = 0$.

38. In a G. P., $s = 73.5682$, $r = 1.1$, $a = 3$, to find the number of terms.

39. Expand $\sqrt{11}$ as a periodic C. F.

40. Find values of x and y so that $314x \sim 451y = 13$.

41. Distribute $\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)\left(\sqrt{\frac{b}{c}}-\sqrt{\frac{c}{b}}\right)\left(\sqrt{\frac{c}{a}}-\sqrt{\frac{a}{c}}\right)$.

42. Divide $\quad a^3(b-c)+b^3(c-a)+c^3(a-b)$

by $\quad a^2(b-c)+b^2(c-a)+c^2(a-b)$.

43. Express $x^5 - 1$ in terms of u when $u = x - 1$.

44. If $s = a + \frac{1}{a}$, prove that $a^4 + \frac{1}{a^4} = s^2(s^2 - 4) + 2$.

45. Put $4x^4 - 4x^3 + 5x^2 + 3$ under the form $A^2 - B^2$.

46. Factorize $2x^2 + 6y^2 - 8xy + xz - 3yz - 2x + 2y - z$.

47. Expand $\dfrac{4}{3 - x(1 - x)}$ in ascending powers of x.

48. If $\dfrac{x}{y + z} = a$, $\dfrac{y}{z + x} = b$, $\dfrac{z}{x + y} = c$, show that

$$\frac{x^2}{a(1 - bc)} = \frac{y^2}{b(1 - ca)} = \frac{z^2}{c(1 - ab)}.$$

49. From each of two towns 45 miles apart a traveller sets out, at 9 o'clock, towards the other town, and the rate of one is 5 miles more than $\frac{3}{4}$ that of the other. They meet at 12:45. Where?

50. If $y = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots$ and $z = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 \cdots$, show that $x = 2(1 - yz) + \frac{8}{3}(1 - yz)^2 + \frac{40}{9}(1 - yz)^3 + \cdots$

51. P is any point on the bisector of the angle AOB, and A, P, B are in line. Show that the sum of the reciprocals of OA and OB is constant.

52. A gives to B and C as much as each one has. B then gives to A and C $\frac{1}{2}$ as much as they have, and then C gives to A and B $\frac{1}{3}$ of what they then have, when A has 72 cents, B 104 cents, and C 28 cents. What had they at first?

53. $xy = c(x + y)$, $yz = a(y + z)$, $zx = b(z + x)$, to solve in x, y, and z.

54. If $x(\sqrt{B} - \sqrt{b}) = h\sqrt{b}$, find the value of $x(B - b) + Bh$ in terms of B, b, and h.

55. If a carriage wheel, $16\frac{1}{2}$ feet in circumference, took one second longer to revolve, the rate of the carriage would be $1\frac{7}{8}$ miles less per hour. How fast does the carriage go? Explain the double solution.

56. Prove by inequalities that the triangle of greatest area, with given base and perimeter, is isosceles.

57. If $xy + yz + zx = 0$, and $xyz = a$, find y and z in terms of a and x.

58. Resolve $\dfrac{x^2 + 6x - 2}{(x^2 - x + 1)(x - 1)^2}$ into partial fractions.

59. By the expansion of $(1 + \overline{1 - x}^2)^n$ show that

$$c_1(1 + c_1 + c_2 + \cdots) = 2c_1 + 4c_2 + 6c_3 + \cdots$$

60. By the expansion of $(1 + \overline{x - 1})^n$ show that

i. $c_0 - c_1 + c_2 - c_3 \cdots c_n = 0.$

ii. $c_1 - 2c_2 + 3c_3 - 4c_4 \cdots = 0.$

61. If P and Q are the pth and qth terms of an A. P., what is the $(p + q)$th term?

62. Divide a number into two parts such that the product of the parts shall be equal to the difference of the squares of the parts.

63. Three numbers are in A. P. If 1 be taken from each of the first two, the three terms will be in G. P.; and if the last term be increased by the first, they will be in H. P. Find the numbers.

64. Find the square root of $2x + 2\sqrt{\{y(2 - y) + x^2 - 1\}}$.

65. How many sums can be made of a farthing, a penny, a sixpence, a shilling, a half-crown, a crown, a half-sovereign, and a sovereign?

66. A stream runs 2 miles an hour, and a rower who rows 4 miles an hour wishes to go directly across it. At what angle up the stream must he direct his boat?

67. Find the approximate value of

$$\{\sqrt{1 + x} + \sqrt[3]{(1 + x)^2}\} \div \{1 + x + \sqrt{1 + x}\}$$

when first powers of x only are retained.

68. Expand $\dfrac{2-3x+x^3}{1-x-3x^2+5x^3-2x^4}$ to the term containing x^6.

69. Find the fraction which expanded gives

$$2-3x+x^2+2x^4-3x^5+x^6+2x^8-\cdots$$

70. Show that $\dfrac{1+x+x^2+x^3+\cdots}{1-x+x^2-x^3+\cdots}=\dfrac{1+x}{1-x}$.

71. Express $x^5-5x^4y+9x^3y^2-7x^2y^3+2xy^4-y^5$ as a function of y and z, when $z=x-y$.

72. A rectangle is inscribed in a triangle, and has a side coincident with the base of the triangle. Show that the area of the rectangle is a maximum when its altitude is one-half that of the triangle.

73. A cubic foot of lead weighs 715 pounds. If 10 pounds of lead is formed into a cubic block, how long is the edge?

74. What three linear expressions divided into $x^3-7x+10$ will each give a remainder 4?

75. Express $x^3+3x^2y-5xy^2-2y^3$ as a function of y and z, when $z=x+y$.

76. If $x^2-2x-1=3$, find the values of $x^7-5x^5+3x^2-x+1$ as a linear function of x.

77. If $Bac=b\Delta$, $Cab=c\Delta$, find the value of

$$\frac{a^2BC}{2(B\sqrt{1-C^2}+C\sqrt{1-B^2})}$$

in terms of abc and Δ.

78. Put $x^3+x^2(a+b)-x(a^2-ab)-a^2b$ into linear factors.

79. If $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}=\dfrac{b+x}{b-x}+\dfrac{b-x}{b+x}$, prove that $a=b$.

80. If $\dfrac{a}{lx(ny-mz)}=\dfrac{b}{my(lz-nx)}=\dfrac{c}{nz(mx-ly)}$, then

$$\frac{a}{lx}(l-x)+\frac{b}{my}(m-y)+\frac{c}{nz}(n-c)=0.$$

81. Two circles, whose diameters are D and d, are made to overlap until their common chord is c. Find the distance between the centres, and explain the quadruple solution.

82. Two perpendiculars from a point to adjacent sides of a square are 6 and 7, and the line to the vertex not related to those sides is 5. Find the side of the square, and explain the double solution.

83. The wheel of a barrow is 20 in. in diameter, and its axle is 1 in. out of the centre. If the wheel turns uniformly at the rate of 1 revolution per second, find the greatest and least velocity of the barrow in feet per second.

84. A V-shaped trough has an angle of 60°, and is 6 feet long. A sphere 12 inches in diameter is placed in it and rolls throughout its length. How many revolutions does it make?

85. If $a + x\sqrt{1 + a^2} = a\sqrt{1 - x^2} + x\sqrt{1 - a^2}$, find x.

86. From a log 20 in. diameter and 20 ft. long, the largest beam is to be cut so as to be twice as wide as thick. How many cubic feet will it contain?

87. Given $x^2 + \dfrac{4}{x^2} = 12 + 6\left(x + \dfrac{2}{x}\right)$, to find x.

88. Given $n^2(x^2 + y^2) = c^2(p - lx - my)^2$, to find n in terms of c, when the terms of two dimensions in x and y form a square.

89. To go through a rectangular field along a diagonal is 10 rods shorter than going around, and one side of the field is twice the other. Find its area.

90. Divide a into two parts, such that the difference of their squares divided by the sum of their squares is a max. or a min.

91. Given $x(\sqrt{x} + 1)^2 = 102(x + \sqrt{x}) - 2576$, to find x.

92. If two chords of a circle intersect at right angles, the sum of the squares on the segments is equal to the square on the diameter.

93. Find the nth term of the A. P. whose sum to n terms is $(pn^2 - qn)/(p-q)$.

94. A pair of wheels, whose diameters are d and d', are rigidly fixed upon an axle, a feet apart. How large a circle will the smaller wheel describe when the set rolls on a plane?

95. $ABCD$ is a square with side s. $AE = BF = CG = DH = ns$, E being on AB, F, or BC, etc. Determine n so that the square formed by AF, BG, CH, and DE may have a given area a^2.

Explain the double solution, and thence find the maximum and minimum values of a^2.

96. A farmer bought some sheep for \$72. If he had bought 6 more for the same money, he would have paid \$1 less for each.

How many did he buy? Explain the double solution, and change the wording of the question accordingly.

97. A G. P. and an A. P. each has its first term $= a$, and the sums of the first three terms are equal.

Find r in terms of a and d, and show that both values of r satisfy the conditions.

98. If a, b, c are all positive, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ngtr \frac{a^2+b^2+c^2}{abc}$ and $\nless \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{abc}}$.

99. If $u = 1 + x + x^2 + \cdots$ and $v = 1 - x + x^2 - + \cdots$, $\frac{1}{u}+\frac{1}{v} = 2$.

100. Show that $1-(ll'+mm'+nn')^2 = (mn'-m'n)^2 + (nl'-l'n)^2 + (lm'-l'm)^2$, if $l^2+m^2+n^2 = l'^2+m'^2+n'^2 = 1$.

101. X and Y are towns 20 miles apart. A person goes a certain number of miles in a certain direction, and then changing his course through a right angle arrives at Y after travelling 1 mile further upon the second course than the first. How long was he in going from X to Y at 4 miles an hour?

102. On the rectangle $ABCD$, with sides $AB = a$ and $BC = b$, P, Q, R, S are taken, so that $AP = na = CR$, and $BQ = nb = DS$. Show that area of Rectangle : area of Parallelogram $PQRS = 1 : 2n^2 - 2n + 1$; and explain the result when $n = 1$. When n is negative.

Has the parallelogram any maximum? any minimum?

103. If $n = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \cdots$, prove that

$$\frac{n^2+1}{n} = 2\left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right\}.$$

104. A ladder 20 feet long leans against an upright wall, and has its foot 3 feet from the wall. If a person pulls the foot outward, compare the rates with which the foot and the top begin to move.

105. A room 18 by 24 is to be so carpeted as to have its floor two-thirds covered, and the width of the uncovered part is to be uniform around the room. Find the size of the carpet.

106. In a rectangular garden, 40 by 60 feet, a flag pole 50 feet high is placed at 6 feet from a longer side of the garden and 15 feet from an adjacent side. How far is the top of the pole from each corner of the garden?

107. If $ab^2 - bc^2 = a$, $bc^2 - ca^2 = b$, $ca^2 - ab^2 = c$, show that

$$(ab + bc + ca)(a^2 + b^2 + c^2) = -(a^4 + b^4 + c^4).$$

108. Find all the values of x from $\left(x + \frac{1}{x}\right)^2 + 1 = \frac{4}{x^2}$.

109. Two wheels, in the same plane, are 3 feet and 1 foot in diameter, and their centres are 4 feet apart.

Find the length of the belt which envelops the wheels and crosses between them.

110. Show that $1 + x + x^2 + x^3 = -\frac{1}{x}\left(1 + \frac{1}{x} + \frac{1}{x^2} + \cdots\right)$.

111. If $\left(\frac{x + yz}{y + zx}\right)^2 = \frac{1 - y^2}{1 - x^2}$, then each fraction $= 1$.

112. Express as a fraction $1 - 2x(1 - x)(1 + x^2 + x^4 + \cdots) + x\left\{1 + \frac{x}{3} - \frac{2}{1 \cdot 2}\left(\frac{x}{3}\right)^2 + \frac{2 \cdot 5}{1 \cdot 2 \cdot 3}\left(\frac{x}{3}\right)^3 - + \cdots\right\}$.

113. Given

$$\frac{a}{(x-a)(x-c)} - \frac{c}{(a-c)(a-x)} + \frac{x}{(c-x)(c-a)} = \frac{1}{a-c},$$

to find x; and also the ratio $(c+a)^2 : (cx+a^2)$.

114. Determine the isosceles triangle in which the altitude is equal to one-third the whole perimeter.

115. The radius of a circle being r, find the angle between two radii when the triangle formed by them and the chord through their extremities is a maximum.

116. Through a point P to draw a line so as to form with two given intersecting lines the minimum triangle, show that there are two minima, and explain.

117. A man walks from A to B. If he had walked m miles an hour faster he would have been h hours less on the road, and if he had walked m_1 miles an hour slower he would have been h_1 hours longer.

Find the distance from A to B, and the rate of walking.

118. If $x = rc$, $y = rs$, $r = as$, and $s^2 + c^2 = 1$, eliminate c, s, and r.

119. How many sets of positive integers satisfy the system $2x + y - z = 4$, $2y + z - x = 33$?

120. A rectangular garden, sides a and b, is to be bordered with a walk of uniform width which shall occupy one-half the plot. Find the width of the walk; and explain the double solution.

121. If $f(x) \equiv 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, find $f(i) + f(-i)$, where $i = \sqrt{-1}$.

122. If s, p, q, be the sum, product, and quotient of two numbers, $p = s^2(q - 2q^2 + 3q^3 - 4q^4 + \cdots)$.

123. Two wheels, A and B, geared together should move, as nearly as practicable, with relative velocities of 1401 and 1945.

The number of teeth in a wheel being limited to not more than 120, find the numbers to be employed.

After 100 revolutions of A how much will B be in advance of, or behind, its true place?

124. The resistance to sliding a stone on the ground varies as the weight, and the weight varies as the cube of the diameter. The power of a running stream to move a stone varies conjointly as the square of the diameter and the square of the velocity of the stream. Show that if the velocity of a stream be doubled it can move a stone 2^6 times as heavy as before.

125. If a_1, a_2, a_3, a_4 be perpendiculars from the vertices of a square to any line, and p be the perpendicular from the centre of the square to the same line, show that $\Sigma a^2 = 4p^2 + s^2$, where s is the side of the square.

126. If b_1, b_2, b_3, b_4 be line-segments from any point to the vertices of a square, and p be the line-segment to the centre, show that $\Sigma b^2 = 4p^2 + 2s^2$.

127. The natural numbers are grouped as follows:

$$(1)\ (2,\ 4)\ (3,\ 5,\ 7)\ (6,\ 8,\ 10,\ 12)\ (9,\ 11,\ 13,\ 15,\ 17)\ \cdots$$

Show that the sum of the numbers in the nth group is

$$\tfrac{1}{2}n\{n^2 + \tfrac{3}{2} + (-)^n \tfrac{1}{2}\}.$$

128. If $xy = 1$, show that $x^n + y^n = \begin{vmatrix} (x+y)^n & {}^nC_1 & {}^nC_2 & \cdots \\ (x+y)^{n-2} & 1 & {}^{n-2}C_1 & \cdots \\ (x+y)^{n-4} & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}.$

TABLE OF PRIME NUMBERS UNDER 1000.

The hundreds are found in the top row, and the two remaining figures in the body of the table.

0	1	2	3	4	5	6	7	8	9
1	01	11	07	01	03	01	01	09	07
2	03	23	11	09	09	07	09	11	11
3	07	27	13	19	21	13	19	21	19
5	09	29	17	21	23	17	27	23	29
7	13	33	31	31	41	19	33	27	37
11	27	39	37	33	47	31	39	29	41
13	31	41	47	39	57	41	43	39	47
17	37	51	49	43	63	43	51	53	53
19	39	57	53	49	69	47	57	57	67
23	49	63	59	57	71	53	61	59	71
29	51	69	67	61	77	59	69	63	77
31	57	71	73	63	87	61	73	77	83
37	63	77	79	67	93	73	87	81	91
41	67	81	83	79	99	77	97	83	97
43	73	83	89	87		83		87	
47	79	93	97	91		91			
53	81			99					
59	91								
61	93								
67	97								
71	99								
73									
79									
83									
89									
97									

ANSWERS TO EXERCISES.

I.

a. **2.** i. -3. ii. -183. iii. $30521/415800$. **3.** i. $a-b$. ii. $a-b+b^2-ab^2+ab^3$. **4.** i. $a^2-9b^2+4c^2+4ac$. ii. $2(m^2-1)a^2+2(n^2-1)b^2$. **10.** 4th. **11.** pth.

b. **4.** i. 12. ii. $4\frac{1}{2}$. **5.** $1-x$. **6.** -5. **7.** 800. **8.** $2a+3b$. **9.** 48, 24, 36. **10.** \$1100. **11.** \$120. **12.** 120. **13.** 67.

II.

b. **6.** i. 0. ii. $12abc\Sigma a$. iii. 0. iv. abc. **10.** $\Sigma a^3+3\Sigma a^2b+6abc$. **11.** $\Sigma a^3b+\Sigma a^2b^2+3\Sigma a^2bc$. **12.** $\Sigma a^3+3\Sigma a^2b+6\Sigma abc$.

c. **1.** $x^4-10x^3+35x^2-50x+24$. **2.** $x^4+2x^3-13x^2-14x+24$. **3.** $24x^4+154x^3+269x^2+154x+24$. **4.** $x^3+x^2\Sigma a+x(2\Sigma ab-\Sigma a^2)+(\Sigma a^2b-\Sigma a^3-2abc)$. **5.** $\Sigma a^5+5\Sigma a^4b+10\Sigma a^3b^2+20\Sigma a^3bc+30\Sigma a^2b^2c$. **6.** $\Sigma a^4+4\Sigma a^3b+6\Sigma a^2b^2+12\Sigma a^2bc+24abcd$.

d. **1.** 26, -46. **2.** x^2-x. **6.** 2.71828. **7.** $x^4+x^3-x^2+2x$.

e. **1.** $1+2x^2+3x^4+14x^6+25x^8$. **2.** 1. **3.** x^2-2x-1. **4.** $mc+2nab+pa^3$. **5.** $-\frac{1}{4}b(8a+1)$. **6.** $1+x$. **7.** 1. **8.** x^2-1. **10.** $1^2+c_1{}^2+c_2{}^2+\cdots$. **11.** $c_2+c_1c_3+c_2c_4+c_3c_5+\cdots$. **12.** $m-\frac{1}{2}m^2+\frac{1}{3}m^3-+\cdots$. **13.** $A=\frac{1}{a},\ B=-\frac{b}{a^3},\ C=\frac{2b^2-ac}{a^5}$, etc. **14.** 3.1416. **15.** 3.1416.

f. **1.** x^3+2x^2+3x+4. **2.** $3a^3+a^2-1$. **3.** $1-2x+2x^2$. **4.** $1-2a^{-1}+2a^{-2}-2a^{-3}\cdots$. **5.** $Q=x^2-2x+6$, $R=-18x^2+18x-7$. **6.** $1+(3a-a^2+a^3+5a^4)(1+a^4+a^8+\cdots)$. **8.** $1-x+x^2-x^3+-\cdots$. **9.** $x+2x^2+3x^3+4x^4+\cdots$, and $x^{-1}+2x^{-2}+3x^{-3}+\cdots$. **10.** $x-1$.

11. $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3$. **13.** $a^2(b-c)-a(b^2-c^2)+bc(b-c)$, or $-(a-b)(b-c)(c-a)$. **14.** x^2+y^2. **15.** $x(1+2x)/(1-x+x^2)$. **16.** b^2c. **17.** $a+2-2a^{-2}-a^{-3}$. **18.** $x^4+x^3+3x^2+2x-2x^{-1}-3x^{-2}-x^{-3}-x^{-4}$. **19.** $4x^3+x^2+x+1$. **21.** $A=a^{-2}$, $B=a^{-3}$, $C=0$, $D=-a^{-5}$, $fx=-x^4(a-x)/a^5(a^2-ax+x^2)$.

III.

a. **1.** $(x-1)(x+1)(x-2)(x+2)$. **2.** $(2a+3b+c)(a-b-c)$. **3.** $(x^2-px+1)(x^2-qx-1)$. **4.** $(x+y+a)(x+y+b)$. **5.** $(a+x-1)(a+x-2)$. **6.** $169xy$. **7.** $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$. **8.** $(3x-1)(4x-1)$ and $3(x-2)(2x-3)$. **9.** $(x-1)(x+1)(x^2-px+q)$.

b. **1.** i. $(b+c)(c+a)(a+b)$. ii. $(b-c)(c-a)(a-b)$. iii. $(b-c)(c-a)(a-b)$. iv. $(a-b)(b-c)(c-a)(a+b+c)$. v. vanishes. vi. $(a-b)(b-c)(c-a)(\Sigma a^2+\Sigma ab)$. vii. $(a-b+1)(b-c+1)(c-a+1)$. viii. $-2abc$. ix. vanishes. **2.** $(a^2-bc)(b^2-ca)(c^2-ab)$. **5.** $(x-1)(x-3)(x+5)(x+7)$. **6.** $(x+2)(x+6)(x^2+8x+10)$. **7.** $(x+\sqrt{3})(x-\sqrt{3})(x+\sqrt{2})(x-\sqrt{2})$. **8.** $24abc$. **9.** $(a+b-c)(b+c-a)(c+a-b)$. **10.** $(a-b)(b-c)(c-d)(d-a)$. **11.** abc. **12.** $(a-b)(b-c)$. **13.** $(x-a+b)(x-b+c)(x-c+a)$.

c. **1.** $a+b-1$. **2.** $(x+2y-2)^2$. **3.** $(b-c)^2\sqrt{a}$. **5.** $x^2-2ax+a^2+b^2$. **7.** i. $\{x-\frac{1}{2}(3+\sqrt{21})\}\{x-\frac{1}{2}(3-\sqrt{21})\}$. ii. $(x-1+2i)(x-1-2i)$. iii. $(2x-1+\sqrt{3})(2x-1-\sqrt{3})$. iv. $5(x+\frac{3}{10}+\frac{1}{10}\sqrt{129})(x+\frac{3}{10}-\frac{1}{10}\sqrt{129})$. v. $a\left(x+\frac{1}{2a}+\frac{1}{2a}\sqrt{1-4a^2}\right)\left(x+\frac{1}{2a}-\frac{1}{2a}\sqrt{1-4a^2}\right)$. vi. $(x-1)(px-1)$. vii. $(\overline{a-b}\cdot x+1)(\overline{a+b}\cdot x+1)$.

IV.

a. **1.** i. x^2-2x-1. ii. x^2-3. iii. $2x^2-3x+3$. iv. $5x$. v. x^2-1. vi. $a^2b^{-2}-1+b^2a^{-2}$. **2.** $a^2-b^2=40$. **3.** $c=-4$, $=-3\frac{1}{2}$. **4.** $(cp-ar)^2=(bp-aq)(cq-br)$. **5.** $(BC-AQ)(CQ-BR)=(C^2-AR)^2$, where $A\equiv a-a_1$, $B\equiv b-b_1$, $C\equiv c-c_1$, $Q\equiv ca_1-ac_1$, $R\equiv cb_1-bc_1$. **6.** $a=3$, $=-2$.

b. **1.** i. $12(x-2)(x-3)(x+3)$. ii. $(x-a)(x-b)(x-c)$. iii. $1+p^2+p^4$. **3.** i. $2^3 3^2$. ii. $2^2 3^2 5$. iii. $2^3 103$. iv. $2^3 131$. v. $2^4 3^2 5^2 7$. **4.** $2^2 3$. **5.** 2520. **6.** 5, 150. **7.** $x-2$. **8.** 4, $x-3$.

V.

a. **1.** i. $(2x+y)/(2x-y)$. ii. $(3x+y)/(3x-y)$. iii. $(nx+y)/(nx-y)$. iv. $2(a-x)(a^2+x^2)/ax^2$. v. $a+b+c$. vi. $1/(2x^2-1)$. vii. 0. viii. 1. ix. $a+b+c$. x. 1. xi. 0. xii. -1. xiii. x. xiv. $a+b+c$. xv. -2. xvi. 0. **3.** a. **6.** r/f.

b. **1.** $-\frac{8}{9}$. **2.** ∞. **3.** ∞, $(c^2+2ac-a^2)/2c$; $-\infty$; $\frac{c}{2}$. **5.** $(a^2+2)/(4a+5)$. **6.** ∞, $2ab/(a+b)$; ∞. **7.** ∞. **8.** 0, $\sqrt{5}$. **9.** 0, 1, $\frac{1}{2}(1\pm\sqrt{17})$. **10.** $a=b$. **11.** 8, 18. **12.** $ab/(a-b)$. **13.** $29 \cdot 5 \cdots$. **14.** $x=(b^2-a^2)/(2a+2b-1)$. **15.** ∞, 1. **16.** ∞, 4. **17.** $3/2a^2$. **18.** 36, 12, 16. **19.** \$3.75. **21.** 32.

VI.

a. **1.** 0, 4, 16, ∞, -32. **2.** 144 : 125. **3.** $(2\alpha\beta-\alpha-\beta)/(\alpha+\beta-2)$. **4.** 15 : 8. **5.** $(n/m)^{\frac{6}{5}}$. **6.** 1, -3, $\frac{2}{3}$. **8.** $(m^2+n^2):(m^2-n^2)$, $\{2n^2:(m^2-n^2)\}^2$. **9.** 0, -1. **10.** $-1:x^2$.

b. **1.** $s=16t^2$. **2.** 145/416. **3.** $x=\pm 4$. **4.** $x=0, 6$. **5.** $(x-y)^2=\frac{1}{12}xy$. **6.** 6, 4. **9.** 10 hrs. 29.74 min. **10.** $1666\frac{2}{3}$. **11.** ⊕ : ☽ $=75\times(28)^2:(99)^2$. **12.** 0.027 in. **13.** 5 hrs. 31 m. nearly.

VII.

a. **1.** a. **2.** $\sqrt[3]{z}$. **3.** $x^{\frac{2n-5}{2}}$. **4.** $a^{-\frac{7}{2}}$. **5.** x^{pq}. **6.** $(b^n/a^m)^{\frac{pq}{mn}}$. **7.** $a^{2\sqrt{2}}+a^{-2\sqrt{2}}-2$. **8.** $a^{\frac{5}{2}}+b^{\frac{9}{4}}$. **9.** $x^6+rx^3+\frac{1}{27}q^3=0$. **10.** $(\overline{a+x}^2-2a)^2=4(a^2-x^2)$. **11.** 2. **12.** $n=\frac{3}{4}$. **13.** $n=-6$. **14.** $m^{n-1}=n$. **15.** $n=3$. **16.** $x^2+2+8x^{-1}+27x^{-2}$; $R=97x^{-2}+55x^{-3}$.

b. **1.** i. $1-\sqrt{5}$. ii. $-24\frac{1}{4}$. iii. $\frac{1}{6}(38+13\sqrt{6})$. iv. $\sqrt[4]{2a}$. v. $30\sqrt{2+3\sqrt{2}}$. vi. $9b\sqrt{3a}$. vii. $8\sqrt{2}$. viii. $ab(3a+5b)\sqrt{2a}$. **2.** i. $3\sqrt{2a}/4a$. ii. $1+\sqrt{3}$. iii. $\sqrt{2}$. iv. $\frac{1}{20}(3\sqrt{5}-5)\sqrt{10+2\sqrt{5}}$.

v. $1+\sqrt{2}$. vi. $2-\sqrt{3}$. vii. $\frac{1}{4}(2-\sqrt{2}-\sqrt{6})$. **3.** The former. **4.** The latter, by $\frac{1}{6}(3\sqrt{2}-2\sqrt{3})$. **5.** $\frac{1}{5}(\sqrt[3]{36}-2\sqrt[3]{3}+2\sqrt[3]{2})$. **6.** $\frac{3}{2}\sqrt{2}(\sqrt{2}-\sqrt[3]{2})(2+\sqrt[3]{4}+\sqrt[3]{2})$.

c. **1.** $\sqrt{5}+\sqrt{3}$. **2.** $2+i$. **3.** $1+\sqrt{3}$. **4.** $u+\sqrt{(1-u^2)}$. **5.** $1-i$. **6.** $x-y+\sqrt{(x^2-y^2)}$. **7.** $2+\sqrt{3}-\sqrt{2}$. **8.** $\sqrt{10}+\sqrt{15}$. **9.** $1+\frac{1}{2}\sqrt{2}$. **10.** $\frac{1}{3}(\sqrt{15}+\sqrt{6})$. **11.** $(a+\sqrt{a^2-4})/\sqrt{2}$. **12.** 3. **13.** $\sqrt{(6+2p)}$.

VIII.

a. **1.** $100(s-b)/b$. **2.** $mnr/(m-n)$. **3.** $A=2a+3b$, $B=a+3b$. **4.** A's $=\frac{1}{2}(m+n+p)$, B's $=\frac{1}{2}(m-n+p)$. **5.** $ab/(a-b)$. **6.** $AO=1\frac{11}{16}a$, where a is the distance between consecutive points. **8.** a^2/b^2. **9.** 24 and 30. **10.** $ac(b_1-a_1)/(ab_1-a_1b)$, $bc(b_1-a_1)/(ab_1-a_1b)$. **11.** 20 miles an hour and 704 ft. long. **12.** dist. $=ha(h-b)/b$; rate $=a(h-b)/b$.

c. **1.** $\frac{9}{2}a^2$. **2.** i. $dm/\sqrt{m^2+n^2}$, $dn/\sqrt{m^2+n^2}$. ii. $d^2mn/(m^2+n^2)$. iii. $dmn/(m^2+n^2)$. iv. $d(m^2-n^2)/(m^2+n^2)$. **3.** i. $ab/\sqrt{a^2+b^2}$. ii. $(a^2-b^2)/\sqrt{a^2+b^2}$. **4.** $b^2/\sqrt{n^2-1}$. **5.** $\frac{1}{4}\sqrt{15}$. **6.** 54. **7.** 4 in. rad. **8.** $\sqrt{(4r^2-1)}$. **9.** $\frac{1}{2}\sqrt{5}$. **10.** $\frac{2}{n}$ths nearly. **11.** $\frac{3}{n}$ths nearly. **12.** i. $\sqrt{2}$. ii. $\frac{2}{5}\sqrt{10}$. **13.** $\sqrt{3}$. **14.** $1:2$. **15.** i. $\sqrt{29}:5$. ii. $\sqrt{29}:2$. **16.** $t+t_1:t-t_1$. **17.** ladder $=39\frac{1}{6}$ ft. nearly; position, $34\frac{1}{6}$ ft. from wall.

d. **5.** 3, 1.

IX.

a. **1.** i. $-\frac{1}{2}b$, $\frac{1}{2}c$. ii. $\frac{1}{2}(a-b)$, $\frac{1}{2}(a+b)$. iii. a/b, b/a. iv. $\frac{1}{3}+\frac{1}{3}i\sqrt{2}$, $\frac{1}{3}-\frac{1}{3}i\sqrt{2}$. v. $1/(a-b)$, $1/(a+b)$. vi. $\frac{1}{2}-\frac{1}{2}\sqrt{2}$, $\frac{1}{2}+\frac{1}{2}\sqrt{2}$. **2.** i. $b=a/(1+a)$. ii. $b=a/(1+a)$. iii. $ab=1$. **3.** $cx^2+bx+a=0$. **5.** $\frac{1}{2}(\sqrt{d^2+8a^2}\pm d)$. **6.** $BP=\sqrt{(2a^2-s^2)}$. **7.** $BP=\frac{1}{2}(s\pm\sqrt{5s^2-8a^2})$. **8.** $AO=\sqrt{\{\frac{1}{2}(l^2\pm\sqrt{l^4-16a^4})\}}$. **9.** $l^2=5a^2$. **10.** $CO=\frac{1}{2}l(\sqrt{5}-1)$.

b. **9.** i. min. $-\frac{5}{4}$. ii. min. $-\frac{4}{3}$. iii. max. $4\frac{1}{4}$. iv. min. $-2\frac{1}{4}$. **10.** $\frac{1}{2}$. **12.** 1. **13.** $a/(1+n)$, $na/(1+n)$, $na^2/(1+n)$. **14.** $\frac{1}{2}a$, $\frac{1}{2}a$. **16.** imag. **22.** min. $\frac{1}{2}s^2$. **23.** max. $\frac{5}{8}$ths of the square. **24.** $\frac{1}{4}l^2$. **25.** $4\sqrt{5}$, $8\sqrt{5}$. **26.** $s\sqrt{2}$. **27.** $4\sqrt{5}$, $8\sqrt{5}$.

28. $10 - 2\sqrt{3}$ from A. **29.** $p^2/a^2 + q^2/b^2 = 1$. **30.** $1\frac{1}{2}$, $3\frac{1}{2}$. **31.** i. $-p$. ii. q. iii. $p^2 - 2q$. iv. $-p/q$. v. $(p^2 - 2q)/q^2$ vi. $3pq - p^3$. **32.** 11 days.

c. **1.** i. $(b-a)^2/2b$, ∞. ii. ∞, $b(b-2a)/2(b-a)$. iii. ∞, $(b-a)^2/(2a-b)$. iv. 0, $\frac{4}{5}a$. v. ∞, $(5ab + b^2 - a^2)/2a$. **2.** 0, $\sqrt{\{(2a-b)/a^2b\}}$. **3.** 0, $\frac{1}{2}\sqrt{3}$. **4.** $\frac{2}{3}a\sqrt{3}$. **5.** $\frac{1}{2}a(b^{\frac{1}{2}} - b^{-\frac{1}{2}})^2$. **6.** 0, 4. **7.** 16, 25. **8.** 0, $-2a(\sqrt{1+a^2} - \sqrt{1-a^2})/\{a^2 + 2 - 2\sqrt{1-a^4}\}$. **9.** $\frac{4}{3}$, $-\frac{1}{3}$. **10.** 0, a, $\frac{1}{2}(a \pm \sqrt{5a^2 - 8ab})$.

X.

a. **1.** i. $x = 29 + 7p$, $y = 2 - 3p$. ii. $x = 1 + 17p$, $y = 11 - 13p$. iii. $x = 13p + 2$, $y = 45p + 4$. iv. $x = 48 + 11p$, $y = p$. **2.** 2 and 3, 4 and 6, 6 and 9. **3.** 5 7-in. and 1 13-in. **4.** 6 4-lbs. and 3 7-lbs. **5.** 63; general formula $420p - 357$. **6.** i. 21 wide, 33 narrow. ii. 26 wide, 25 narrow. iii. 41 wide, 1 narrow.

b. **1.** $x = 5$, $y = 2$. **2.** $x = 3$, $y = 5$. **3.** $x = a - abc^2/(b + ac)$, $y = b^2/(b + ac)$. **4.** $x = a^2/(a - b)$, $y = b^2/(b - a)$. **5.** $2sd = (z + a)(z - a + d)$. **6.** $A = 200$, $B = 300$. **7.** $\frac{5}{14}$. **8.** 93. **9.** $a(b-c)/(b-a)$ of the first, $b(a-c)/(a-b)$ of the second. **10.** 150 acres, rent = \$600.

c. **1.** $x = 7$, $y = 5$, $z = 3$. **2.** $x = 2$, $y = z = 1$. **3.** $x = 3$, $y = 2$, $z = -1$. **4.** $x = y = b$, $z = 0$. **5.** $x = (b-a)/(a-b)(b-c)(c-a)$, with symmetrical expressions for y and z. **6.** $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$. **7.** $3a = b + 5c$. **8.** $5b = 3c$. **9.** $7a + b + 11c = 0$. **12.** $x : y : z : u = 1 : 3 : 1 : 3$. **13.** $x = z = \frac{1}{3}$, $y = 1$. **14.** $a = 1$, $b = 3$, $c = 2$, $d = 4$. **15.** $x = a$, $y = a^{-1}$, $z = 1$. **16.** $x = -\Sigma a$, $y = \Sigma ab$, $z = -abc$. **17.** $3abc = \Sigma a^3$. **18.** $x = 3$, $y = 1$, $z = -2$. **19.** $\Sigma a^4 + 2\Sigma a^2b^2 = 8a^2b^2$.

d. **1.** $x = 6$, $-2\frac{4}{13}$, $y = 4$, $-1\frac{7}{13}$. **2.** $x = \frac{1}{2}(a + \sqrt[4]{a^4 - 8b})$, $y = \frac{1}{2}(a - \sqrt[4]{a^4 - 8b})$. **3.** $x = \pm 50$, $y = \pm 15$. **4.** $x = \frac{1}{2}(3 \pm \sqrt{5})$, $y = \frac{1}{2}(1 \pm \sqrt{5})$. **5.** $x = 7$, $y = 3$. **6.** $x = 4$, $\frac{1}{2}(-11 + i\sqrt{59})$, $y = 6$, $\frac{1}{2}(-11 - i\sqrt{59})$. **7.** $x = 15$, -16, $y = 9$, -10. **8.** $x = \frac{1}{2}$, $\frac{1}{2}(\sqrt{3} - 2)$, $y = 1$, $\sqrt{3} - 2$. **9.** $x = 10$, $y = 8$. **10.** $x = 3$, $y = 2$. **11.** $x = 1.786\cdots$, $y = 1.731\cdots$. **12.** $x = -\frac{1}{9}$, $y = -\frac{1}{18}$. **13.** $x = \frac{1}{4}$, $y = \frac{1}{5}$. **14.** $x = \sqrt{(ac/b)}$, $y = \sqrt{(ab/c)}$, $z = \sqrt{(bc/a)}$.

15. $2a/\sqrt[4]{2+2\sqrt{5}}$, $a\sqrt[4]{2+2\sqrt{5}}$, $\frac{1}{2}a\sqrt[4]{(2+2\sqrt{5})^3}$. **16.** $x=3$, $y=7$, $z=9$. **17.** $x=\sqrt[27]{bc^{10}a^{-8}}$, $y=\sqrt[27]{ab^{10}c^{-8}}$, $z=\sqrt[27]{ca^{10}b^{-8}}$. **18.** $27aq^2=4(p-2a)^3$. **19.** $x=2$, $y=4$. **20.** $x=2$, $y=3$. **21.** $x=4$, $y=6$, $z=3$. **22.** 15, 36. **23.** 15, 20, 25.

XI.

a. **1.** 999666, 1098010, 0. **2.** 0. **3.** -7617. **4.** 0. **5.** 4 in each case. **6.** 0.000211. **7.** $-0.0114\cdots$. **9.** $12x^2-31x+11$.

b. **1.** $(x+1)^3-6(x+1)^2+11(x+1)-5$. **2.** $3b^5+30b^4+119b^3+238b^2+249b+106$. **3.** x^5-1. **4.** $(x+3)^5-(x+3)+1$.

5. $(a+b)^5-3b(a+b)^4-4b^3(a+b)^2+b^5$.

6. $(x-1)^3+3(x-1)^2-4(x-1)$.

c. **1.** i. 2 and 3. ii. -2 and -3. iii. -3 and -2, 1 and 2. iv. 0 and -1, 1 and 2. **2.** 2.2284. **3.** $1+\sqrt[3]{3}$. **4.** $1+\sqrt{(2+i\sqrt{5})}$. **6.** $2-\sqrt[3]{17}$. **7.** 3, $\frac{1}{2}(-1+i\sqrt{7})$. **8.** $1.46460\cdots$. **9.** 14.0487 in. **10.** $2.5119\cdots$.

XII.

a. **1.** i. $m+(n-1)(p-m)$. ii. $a+b(n-1)(2-n)$. iii. $\frac{3}{2}(3n-1)$. iv. $\frac{50}{9}(n-1)$. v. $\frac{1}{5}(64-3b+n\cdot\overline{b-8})$. vi. $\frac{1}{3}(92-8n)$. vii. $\frac{1}{2}(a+b)$. **2.** i. 45. ii. $12\frac{1}{2}$. iii. 63. iv. $n(2a+3b-nb)/2a(a+b)$. v. 816. vi. $n(n+1)$. **3.** 48000. **4.** $\frac{1}{4}n(n-1)$. **5.** 8729. **7.** $n\text{th}=\frac{1}{4}(6n-5)$. **8.** 6 or 12 terms. **9.** 11. **10.** 19800. **12.** \$1630. **13.** i. 46 days. ii. 91 days. iii. 20 or 71 days. iv. no. **14.** i. 27 days. ii. 7 or 20 days. **15.** 62. **16.** $49\frac{1}{2}s$, $n=2m+1$. **17.** \$3140. **19.** diff. $=\frac{1}{5}(b-a)$. **20.** $n\text{th}=\frac{1}{2}n(n+1)$. **21.** $n\text{th}=n^2$. **22.** $n\text{th}=n^3$. **23.** $p=4(n-1)$. **24.** $\frac{1}{4}(n+1)^2(n+2)^2$. **25.** $\frac{1}{2}n(n^2+1)$.

b. **1.** i. $\frac{1}{2}(3^n-1)$. ii. $\frac{3}{5}\{1-(-\frac{2}{3})^n\}$. iii. $\frac{3}{7}\{1-(\frac{3}{10})^n\}$, $\frac{3}{7}$. iv. $\sqrt{2}+1$. **2.** $n+1$. **3.** $\frac{1}{2}\{(\sqrt{2}-1)/\sqrt{2}\}^{n-3}$. **4.** $12.056\cdots$ gals. **5.** $s\sqrt{3}/2\cdot 3^n$, $\frac{3}{32}\pi s^2$. **6.** $a^2b/(2a+b)$. **7.** 10. **8.** $\frac{1}{2}an(n+1)+(a^{n+1}-a)/(a-1)$. **10.** 6381407.

c. **1.** 2, 4, 8, 16, 32, 64, 128, 256.

d. **1.** $2ab/(a+b)$. **6.** 1, 2, 4.

e. **1.** \$1069.20. **2.** \$1156. **3.** \$8192. **4.** \$118.75. **5.** \$1734.90.

XIII.

a. **1.** i. 6. ii. 6. iii. 840. iv. $(n-l)!/(n-l-m)!$. **2.** 720. **3.** 10. **4.** 1260, 120, 90720. **5.** 1 in 30. **6.** 36. **7.** 1 in 72. **8.** 60.

b. **1.** $6a=(n-2)b$. **2.** $nb=a(r)!$. **3.** 385. **7.** 3 in 10.

c. **1.** i. $\{n(n-1)\cdots(n-r+1)a^{n-r}x^r\}/r!$. ii. i $\times(-1)^r$. iii. $\{2n(2n-1)\cdots(2n-r+1)x^r\}/r!$. **3.** ${}^{12}C_9x^9$. **4.** $\{(n-r)x\}/\{a(r+1)\}$. **5.** $\dfrac{nx-a}{a+x}>r>\dfrac{nx-a}{a+x}-1$. **6.** 1080.

d. **2.** 1, 0, -1, according as n is of form $3m$, $3m+1$, or $3m+2$. **3.** $c_0h_0+c_1h_1+c_2h_2+\cdots+c_nh_n$. **5.** -197. **6.** $\dfrac{(-)^n(3n-1)!}{n!(2n-1)!}$. **9.** $3\left\{1+\dfrac{1\cdot2}{4\cdot9}-\dfrac{3}{4\cdot8}\left(\dfrac{2}{9}\right)^2+\dfrac{3\cdot7}{4\cdot8\cdot12}\left(\dfrac{2}{9}\right)^3-+\cdots\right\}$.

e. **2.** $1\cdot3\cdot5\cdots(2r-3)x^r/2\cdot4\cdot6\cdots2r$. **3.** $1+\dfrac{3}{2}x+\dfrac{3}{2\cdot4}x^2-\dfrac{3}{2\cdot4\cdot6}x^3+\cdots$. **4.** $a(1-x+\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{8}x^4)$. **9.** $\sqrt{3}$.

XIV.

7. i. The first. ii. The second. iii. The first. **9.** Less than 4 for all values of x from 1 to 3.

XV.

a. i. $\dfrac{4}{x-2}-\dfrac{3}{x-1}$. ii. $\dfrac{4}{x-3}-\dfrac{3}{x-2}$.

iii. $\dfrac{1}{2(x-1)}-\dfrac{4}{x-2}+\dfrac{7}{2(x-3)}$.

iv. $\dfrac{a+1}{(a-b)(b-x)}-\dfrac{a^2+b}{b(a-b)(a-x)}$

v. $\dfrac{21}{x+2}-\dfrac{42}{2x+3}+\dfrac{14}{(x+2)^2}$. vi. $\dfrac{a}{2(a-x)}-\dfrac{a}{2(a+x)}$.

b. **1.** $1+x+x^2+x^3+\cdots$. **2.** $1+\frac{1}{2}x+\frac{3}{8}x^2-\frac{3}{16}x^3+\frac{3}{128}x^4\cdots$. **3.** $1-x+\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{3}{4}x^4\cdots$. **4.** $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4\cdots$. **5.** $(1+x)^{-n}=1-c_1x+(c_1^2-c_2)x^2-(c_1^3-2c_1c_2+c_3)x^3+(c_1^4-3c_1^2c_2$

$+2\,c_1c_3+c_2^2-c_4)x^4\cdots$. **6.** $x=1/a_1\cdot y-a_2/a_1^3\cdot y^2+(2\,a_2^2-a_3a_1)/a_1^5\cdot y^3\cdots$. **7.** $x=z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4\cdots$. **8.** $y=x+2\,x^2+7\,x^3+\cdots$. **9.** $a=1,\ b=b,\ c=b^2/2\,!,\ d=b^3/3\,!$, etc.

c. **1.** $(n+1)x^{n-1}$. **2.** $(2-x+2\,x^2-\cdots)-(1-2\,x+3\,x^2-4\,x^3\cdots)$. **3.** $\frac{1}{3}n(n^2+2)$. **4.** $\frac{1}{8}n(n+1)(n^2+n+2)$. **5.** $\frac{1}{3}n(n+1)(4\,n-1)$. **6.** $\frac{n}{12}(11-2\,n)(1+n)$. **7.** $\frac{1}{4}n(n+1)(n+2)(n+3)$.

d. **1.** $a^2b=4\,c$. **3.** $(ca'-c'a)^2=(ab'-a'b)(bc'-b'c)$. **4.** $\lambda=2\sqrt{AB}-(A+B)$. **6.** 3. **9.** $(b-p^2)/2\,p$. **10.** $(1+2\,x)/(1-x-x^2)$. **11.** $a-b=\frac{1}{4}$. **12.** $(x^2-2\,x-1)^2-(x+1)^2$. **13.** $(x+2\,y-1)(x-y+3)$. **14.** $m=\frac{4}{9}$.

XVI.

a. **1.** $1,\ \frac{3}{4},\ \frac{25}{33},\ \frac{28}{37},\ \frac{81}{107},\ \frac{190}{251}$. **3.** 558, 552. **4.** $\frac{22}{7},\ \frac{355}{113}$. **5.** $\frac{99}{70}$.

b. **1.** $\frac{1}{2}(\sqrt{15}-1)$. **2.** $\frac{1}{5}(2\sqrt{39}-9)$. **4.** $\sqrt{6}$. **5.** $1+\frac{1}{2+}\cdots$, $2+\frac{1}{2+}\frac{1}{4+}\cdots$. **6.** $4+\frac{1}{8+}\cdots$, $4+\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8+}\cdots$.

XVII.

a. **1.** 2, 4, 6, -3, -5. **3.** $\frac{1}{2},\ \frac{1}{4},\ 1,\ 2^{\frac{1}{3}},\ 2^{\frac{1}{n}}$. **6.** i. $1.537\cdots$. ii. $0.8379\cdots$. iii. 1.242. iv. $x=2\frac{1}{4},\ y=3\frac{3}{8}$. v. $\log(\sqrt{2}+1)/2\log a$. vi. $\frac{1}{2}\sqrt{(1+8\log n/\log a)}-\frac{1}{2}$. **7.** $\frac{19}{6}\log 2+\frac{1}{2}\log 3-\frac{1}{3}\log 5$. **8.** 1.80618, 2.40824, 1.05361, $\bar{1}.69897$, 1.39794. **9.** $\log 2+\log 3$, $\log 2+2\log 3$, $3\log 2+2\log 3$, $-(2\log 2+\log 3)$, $-2\log 2$, $-(3\log 2+\log 3)$. **10.** 8. **11.** $t=\log 2/\log 1.05$.

b. **1.** 5.7449. **2.** 0.0018542. **3.** 0.57122. **4.** $7.4123\cdots$. **5.** 0.69897. **6.** 0.93936.

c. **10.** $3\,e$.

d. **1.** $x=e^{a(b+1)}-a$.

XVIII.

a. **1.** $R=1-2\,x+x^2,\ G=1/R$. **2.** $R=1-2\,x+x^2$. $G=(1+x-x^2)/R$. **3.** $6\,x^4+7\,x^5$. **4.** $1+x-x^2+3\,x^3-3\,x^4+5\,x^5-5\,x^6\cdots$. **5.** $(3+8\,x)/(3+5\,x+x^2)$, $1/(1-x+3\,x^2-8\,x^3)$,

$-4\frac{1}{3}x^4$, $17x^4$. **6.** $\frac{1}{4}\{1+(-)^n(2n+3)\}$, -50. **7.** $\frac{1}{2}\{1+(-)^n(1-2n)\}$. **8.** $(3^{n+1}-2^{n+1})x^n$.

b. **2.** $50+\frac{n}{6}(n^2-15n+26)$. **3.** $1+7n-n^2$. **4.** $\frac{1}{12}(n^3-4n^2+13n+6)$. **5.** 4, 2, 1, 1, 2, 4, 7. **8.** 45.14, 37.89. **9.** $-0'.13$. **10.** 0.41337.

c. **1.** $\{1-(n+2)x^{n+1}+(n+1)x^{n+2}\}/(1-x)^2$. **2.** $(2^{n+2}-n-3)/2^n$. **3.** $\{1-\frac{1}{2}(n+1)(n+2)x^n+n(n+2)x^{n+1}-\frac{1}{2}n(n+1)x^{n+2}\}/(1-x)^3$. **4.** $1+n\cdot 2^{n+1}$. **5.** $\{1+(-)^n(\overline{n+2}\,x^{n+1}+\overline{n+1}\,x^{n+2})\}/(1+x)^2$. **6.** $\frac{1}{4}\{1+(-)^n(2n+3)\}$. **7.** $n/(n+1)$. **8.** $n/(2n+1)$. **9.** $n/3(n+1)+n/6(n+2)+n/9(n+3)$. **11.** $\frac{1}{3}$. **12.** $\frac{1}{4}$. **13.** $\frac{11}{18}$. **14.** conv. if $x<1$. **15.** $x<1$. **17.** 4. **18.** $\frac{4}{9}$.

XIX.

a. **1.** i. -18. ii. 173. iii. $abc+2fgh-af^2-bg^2-ch^2$. iv. x^3. v. $xy(y-x)(1-x)(1-y)$. vi. $3abc-\Sigma a^3$. vii. -4. viii. $c(ae-c^2)+b(cd-be)+d(bc-ad)$. **2.** 0.

b. **1.** i. 0. ii. -160. iii. 1.

c. **3.** \$301. **4.** $\lambda^3-\lambda^2\Sigma a+\lambda(\Sigma ab-\Sigma f^2)-(abc+2fgh-af^2-bg^2-ch^2)=0$, where abc and fgh are collateral systems of letters. **5.** $(a-a_1)^2(ab_1-a_1b)=(b-b_1)^3$. **6.** $mn(BR-CQ)=nl(CP-AR)=lm(AQ-BP)$. **7.** $m=\frac{1}{16}(5u_0+15u_1+u_3-5u_2)$, $n=\frac{1}{16}(9u_1+9u_2-u_0-u_3)$, $p=\frac{1}{16}(5u_3+15u_2+u_0-5u_1)$. **8.** $abc\,|amn'|$. **10.** $x=-|a_1c_2d_3|/|b_1c_2d_3|$.

MISCELLANEOUS.

2. 151. **3.** $-2(1+2x^2+2x^4+2x^6+\cdots)$. **4.** $x=a/\sqrt{a^2-4}$. **5.** $x=a$, $y=1/a$, $z=1$. **6.** none. **7.** $1+2(x-x^3-x^4+x^6+x^7\cdots)$. **8.** 60 and 40. **9.** max. 3, min. $\frac{1}{3}$. **10.** x must not lie between $-\frac{5}{3}$ and 5. **12.** $\frac{1}{2}a$. **13.** min. $-(a-b)^2/4ab$. **14.** $(-)^n(3n-1)x^n$. **15.** $x=3$, $y=1$. **16.** 5. **17.** \$120. **18.** 5, 7, 9. **19.** $\frac{1}{3}n(n^2+2)$. **20.** $\frac{1}{2}f(2n-1)$, $\frac{1}{2}n^2f$. **21.** 1, -1, $\frac{1}{2}$. **22.** $\frac{1}{3}b$. **23.** 1, 3, 9. **24.** no, 5th. **25.** 0.1 per unit. **26.** $\frac{2}{20}$, $\frac{3}{20}$, $\frac{4}{20}$, $\frac{5}{20}$, $\frac{6}{20}$. **27.** $(x-11y+1)(2x+y-3)$. **29.** $z=\frac{3}{10}x+\frac{2}{5}x^2$. **30.** 4.

31. 52. **32**. 105. **33**. -1. **34**. 593775; i. 118755; ii. 20475; iii. 17550; iv. 80730. **35**. $\frac{120}{11}(30T \pm a°)$ seconds after hour T. **36**. 34 hrs., at starting point. **37**. Put $1-z$ for x, expand, and then make $z=1$. **38**. 13. **39**. $3+\frac{1}{3+}\frac{1}{6+}\cdots$. **40**. 1027, 715. **41**. $\{ab(b-a)+bc(c-b)+ca(a-c)\}/abc$. **42**. $a+b+c$. **43**. $u^5+5u^4+10u^3+10u^2+5u$. **45**. $(2x^2-x+2)^2-(2x-1)^2$. **46**. $(2x-2y+z)(x-3y-1)$. **47**. $\frac{4}{3}(1+\frac{1}{3}x-\frac{2}{9}x^2-\frac{5}{27}x^3+\cdots)$. **49**. 15 miles from one town. **52**. A, 120; B, 60; C, 24. **53**. $x=2abc/(ab-bc+ca)$, with sym. expressions for y and z. **54**. $h(B+b+\sqrt{Bb})$. **55**. $5\frac{5}{8}$ miles. **57**. $y=\frac{1}{2}(\sqrt{a^2-4ax^3}-a)/x^2$, $z=\frac{1}{2}(-\sqrt{a^2-4ax^3}-a)/x^2$. **58**. $5/(x-1)^2+3/(x-1)-(3x+4)/(x^2-x+1)$. **61**. $(qQ-pP)/(q-p)$. **62**. greater part $=\frac{1}{2}a(\sqrt{5}-1)$. **63**. 2, 3, 4. **64**. $\sqrt{x-y+1}+\sqrt{x+y-1}$. **65**. 255. **67**. $1-\frac{1}{6}x$. **68**. $2-x+5x^2-7x^3+17x^4-31x^5+65x^6\cdots$. **69**. $(2-x)/(1+x)(1+x^2)$. **71**. $z^5-z^3y^2-y^5$. **73**. 2.8912 in. **74**. $x-1, x-2, x+3$. **75**. $z^3-8zy^2+5y^3$. **76**. $437x+557$. **78**. $(x+b)\{x+\frac{1}{2}a(1-\sqrt{5})\}\cdot\{x+\frac{1}{2}a(1+\sqrt{5})\}$. **81**. $\frac{1}{2}(\sqrt{D^2-c^2}+\sqrt{d^2-c^2})$. **83**. $22\pi/12$, and $18\pi/12$. **84**. 3.82. **85**. 0, $a(\sqrt{1-a^2}-\sqrt{1+a^2})/(1+\frac{1}{2}a^2-\sqrt{1-a^4})$. **86**. $22\frac{2}{9}$ cu. ft. **87**. $4\pm\sqrt{14}$, $-1\pm i$. **88**. $n^2=c^2(l^2+m^2)$. **89**. $25(7+3\sqrt{5})$ sq. rds. **90**. $x=\frac{1}{2}a$, a min. **91**. $\sqrt{x}=7, -8, \frac{1}{2}(\sqrt{185}-1)$. **93**. $(2pn-p-q)/(p-q)$. **94**. rad.$=d'\sqrt{(a^2+\frac{1}{4}\overline{d-d'}^2)}/(d-d')$. **95**. $n=(s^2\pm a\sqrt{2s^2-a^2})/(s^2-a^2)$. **96**. 18, or -24. **101**. $\frac{1}{4}\sqrt{799}$. **104**. $\sqrt{391}:3$. **105**. 14.234 by 20.234. **106**. $\sqrt{2761}$, $\sqrt{4561}$, $\sqrt{3881}$, $\sqrt{5681}$. **108**. $x=(-3\pm\sqrt{21})^{\frac{1}{2}}/\sqrt{2}$. **109**. $4\sqrt{3}+\frac{8}{3}\pi$. **112**. $(1-x+x\cdot\overline{1+x^3}^{\frac{4}{3}})/(1+x)$. **113**. $x=(c^2+2ac-a^2)/2c$; ratio is 2. **114**. side : base $=13:10$. **115**. 90°. **117**. rate $=mm_1(h+h_1)/(mh_1-m_1h)$; dist. $=mm_1hh_1(m+m_1)(h+h_1)/(mh_1-m_1h)^2$. **118**. $x^2+y^2=ay$. **119**. 12 solutions, $1+3\rho$, $12-\rho$, $10+5\rho$. **120**. $\frac{1}{4}(a+b\pm\sqrt{a^2+b^2})$. **121**. $2\left(1-\frac{1}{2!}+\frac{1}{4!}-+\cdots\right)$. **123**. 85 and 118; $+\frac{1}{328}$th, nearly.

INDEX.

The Numbers refer to the Articles.