THE EARTH'S MOTION OF ROTATION.

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THE EARTH'S MOTION OF ROTATION

INCLUDING THE THEORY OF

PRECESSION AND NUTATION.

BY

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PREFACE.

In offering to the student a treatment of the Problem of the Earth's rotation somewhat different from that which has been usually given in elementary text-books, a few words of explanation are necessary.

The FIRST PART consists of an application of the method of the variation of elements to the general problem of rotation. That the formulæ for calculating these elements are identical in the motions both of translation and rotation, appeared so remarkable, that it might be well to present the latter in a form easily accessible. As far as I am aware, an elementary investigation of these formulæ has not yet been given: in attempting to supply this, I have adopted a method somewhat similar to that which I have given for the corresponding equations of the motion of translation, in an *Elementary Treatise on the Planetary Theory.* The striking analogy, thus developed, between the solutions of problems, in appearance so dissimilar, may, I hope, lead to a more complete study of Lagrange's beautiful theory of the variation of arbitrary constants.

In the SECOND PART the general rotation-formulæ are applied to the particular case of the Earth. These formulæ afford a simple and accurate proof of the important theorem of the Stability of the Earth's axis and of the motion about it, so far as these depend upon the attractions of distant bodies. In this I have followed M. de Pontécoulant. The remaining pages are devoted to a consideration of the motion of the Earth's axis in space. In this I have obtained the formulæ for calculating Precession and Nutation, first, by an application of the general method, and afterwards, by an independent process; but I have not carried the approximation further than has been usual in elementary text-books.

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1, DEAN'S YARD,
WESTMINSTER,
September, 1867.

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THE EARTH'S MOTION OF ROTATION.

PART I.

GENERAL THEORY OF ROTATION.

1. In that part of Physical Astronomy which usually goes by the name of the Planetary Theory we are concerned with the motions of translation only of the planets in space: we now propose to consider their motions of rotation. The principles of the conservation of the motions of translation and rotation permit us to consider these separately, and to treat the latter as if the centre of gravity were a fixed point. We shall adopt a method perfectly rigorous, and free from all assumptions, with the single exception of the hypothesis, already required in the Planetary Theory, that the attracting bodies are so distant that their action may be supposed the same as it would be if their whole masses were condensed into their centres of gravity. Thus we shall obtain, for the determination of the motion, formulæ applicable to the case of any planet or other rigid body: an interesting application of these will then be afforded by the special circumstances which occur in the *Earth's Motion of Rotation.*

2. If the planets were exactly spherical in shape, it is clear that the attractions of the Sun, Moon, and of the other planets could produce no effect upon their rotation, since they would all pass through the centre of gravity. But

although this is not the case, yet the deviation from exact sphericity being very small, the motion will differ only slightly from what it would be if these disturbing forces did not exist. We shall, therefore, by neglecting them obtain first an approximate solution of the problem, and then by the method of the variation of parameters deduce from it accurate results. Since, however, the motion of a rigid body about a fixed point under the action of no forces is discussed in works on Rigid Dynamics, we shall here consider it only so far as is necessary for the purpose of obtaining results which will be required in the sequel.

Undisturbed Motion.

3. Let ω_1, ω_2, ω_3 be the angular velocities of a planet about the principal axes at its centre of gravity; A, B, C the moments of inertia about these axes: then Euler's equations give

$$A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = 0,$$

$$B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 = 0,$$

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = 0.$$

Multiplying these equations by ω_1, ω_2, ω_3 respectively, adding, and integrating, we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = h,$$

where h is the constant of integration.

Again, multiplying by $A\omega_1$, $B\omega_2$, $C\omega_3$, adding, and integrating, we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = k^2,$$

where k^2 is the constant of integration.

From these two equations we obtain

$$\omega_1^2 = \frac{k^2 - Bh + (B - C)\,C\omega_3^2}{(A - B)\,A},$$

$$\omega_2^2 = \frac{-k^2 + Ah + (C - A)\,C\omega_3^2}{(A - B)\,B}.$$

Substituting these values, the third equation of motion becomes

$$C\frac{d\omega_3}{dt} = \frac{\{k^2 - Bh + (B - C)\,C\omega_3^2\}^{\frac{1}{2}}\{-k^2 + Ah + (C - A)\,C\omega_3^2\}^{\frac{1}{2}}}{\sqrt{(AB)}},$$

whence $t + l$

$$= C\sqrt{(AB)}\int\frac{d\omega_3}{\{k^2 - Bh + (B - C)\,C\omega_3^2\}^{\frac{1}{2}}\{-k^2 + Ah + (C - A)\,C\omega_3^2\}^{\frac{1}{2}}},$$

where l is the constant of integration.

This integral cannot in the general case be found; we may however approximate: thus t is known in terms of ω_3, and consequently ω_3 in terms of t; and then from above, ω_1, ω_2 are also known.

4. With respect to the constants introduced by the integration, we may remark that h represents the *vis viva* (Routh's *Rigid Dynamics*, Art. 194), and k the area conserved on the invariable plane. To prove the latter point, the areas conserved on the principal planes being $A\omega_1$, $B\omega_2$, $C\omega_3$ (Routh's *R. D.*, Art 179), and the direction cosines of the invariable plane with reference to the principal axes

$$\frac{A\omega_1}{k},\ \frac{B\omega_2}{k},\ \frac{C\omega_3}{k}$$

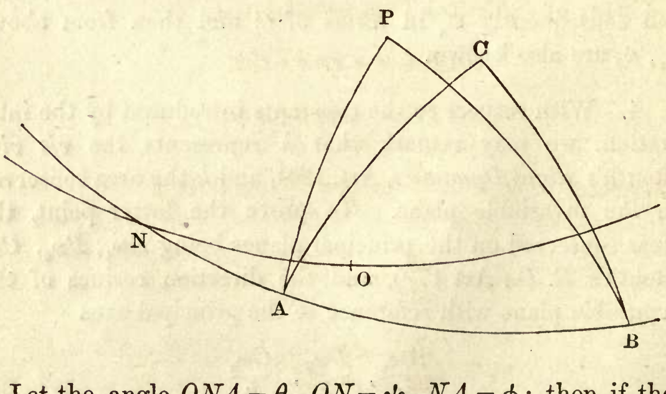
(Routh's *R. D.*, Art. 125), the area conserved on the invariable plane

$$=\frac{A^2\omega_1^2}{k}+\frac{B^2\omega_2^2}{k}+\frac{C^2\omega_3^2}{k}$$

$$=\frac{A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2}{k}=k \text{ (Art. 3).}$$

5. When ω_1, ω_2, ω_3 are known at any time, the resultant angular velocity of the planet is known, and also the position of the instantaneous axis of rotation with reference to the principal axes. It remains to shew how the position of these axes in space may be determined.

Suppose a sphere described with its centre at the centre of gravity of the planet and its radius of any magnitude: take as a plane of reference any fixed plane passing through the centre of gravity, and let it cut the sphere in the great circle ON; also let the principal plane of xy cut the sphere in the great circle NAB, N being the node of this plane upon the fixed plane, and A, B, C the points where the sphere is cut by the principal axes of x, y, z. Take P the pole of ON, and join PA, PB, CA, CB by arcs of great circles.



Let the angle $ONA=\theta$, $ON=\psi$, $NA=\phi$: then if the angles θ, ϕ, ψ be known, the position of the planet will be determined with reference to the fixed plane.

Now we may consider the planet to be moving with angular velocities ω_1, ω_2, ω_3 about the principal axes; or with angular velocities $\frac{d\psi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, the first about a normal to the fixed plane, the second about the line of nodes, the third about the principal axis of z. We shall adopt the usual convention with respect to signs, and consider positive those angular velocities which tend to turn the planet round the axes of x, y, z from y to z, z to x, x to y, respectively.

Thus, resolving about the principal axes, we have

$$\omega_1 = -\frac{d\theta}{dt}\cos\phi - \frac{d\psi}{dt}\cos PA,$$

$$\omega_2 = \quad \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\cos PB,$$

$$\omega_3 = \quad \frac{d\phi}{dt} - \frac{d\psi}{dt}\cos\theta.$$

Now, by Spherical Trigonometry,

$$\cos PA = -\sin\theta\sin\phi,$$

$$\cos PB = -\sin\theta\cos\phi;$$

therefore $$\omega_1 = -\frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi,$$

$$\omega_2 = \quad \frac{d\theta}{dt}\sin\phi + \frac{d\psi}{dt}\sin\theta\cos\phi,$$

$$\omega_3 = \quad \frac{d\phi}{dt} - \frac{d\psi}{dt}\cos\theta.$$

Hence also,

$$\left.\begin{aligned}\frac{d\psi}{dt}\sin\theta &= \omega_1 \sin\phi + \omega_2 \cos\phi,\\ \frac{d\theta}{dt} &= -\omega_1 \cos\phi + \omega_2 \sin\phi,\\ \frac{d\phi}{dt} - \frac{d\psi}{dt}\cos\theta &= \omega_3,\end{aligned}\right\} \text{......(A)*.}$$

By substituting the values of ω_1, ω_2, ω_3 obtained as above (Art 3), and then integrating these equations, θ, ϕ, and ψ would be determined, and thus the position of the principal axes at any time would be known. The integration, however, cann in general be effected; so that we are obliged to have recourse to a special hypothesis with regard to the position of the fixed plane of reference. If we take for this purpose the invariable plane, the process becomes much simplified.

6. Let then θ_1, ϕ_1, ψ_1 denote relatively to the invariable plane the same angles which relatively to the original plane of reference have been denoted by θ, ϕ, ψ. Then, the direction cosines of the invariable plane with reference to the principal axes being

$$\frac{A\omega_1}{k},\quad \frac{B\omega_2}{k},\quad \frac{C\omega_3}{k}$$

respectively, we have (see figure of preceding Article)

$$\frac{A\omega_1}{k} = \cos PA = -\sin\theta_1 \sin\phi_1,$$

$$\frac{B\omega_2}{k} = \cos PB = -\sin\theta_1 \cos\phi_1,$$

$$\frac{C\omega_3}{k} = \cos\theta_1;$$

* There is much disagreement between writers as to the measurement of the angles employed in these kinematical equations; the above, however, agrees with La Place, Poisson, and Pontécoulant.

therefore

$$\tan\phi_1 = \frac{A\omega_1}{B\omega_2} = \sqrt{\left\{\frac{A}{B}\cdot\frac{k^2 - Bh + (B-C)\,C\omega_3^2}{-k^2 + Ah + (C-A)\,C\omega_3^2}\right\}} \text{ by Art. 3,}$$

$$\cos\theta_1 = \frac{C\omega_3}{k}.$$

These equations give θ_1 and ϕ_1: to obtain ψ_1, substitute in the first of the equations (A) of Art. 5; thus

$$\frac{d\psi_1}{dt}\sin^2\theta_1 = \omega_1 \sin\theta_1 \sin\phi_1 + \omega_2 \sin\theta_1 \cos\phi_1;$$

therefore $$\frac{d\psi_1}{dt}(k^2 - C^2\omega_3^2) = -(A\omega_1^2 + B\omega_2^2)\,k$$

$$= -(h - C\omega_3^2)\,k;$$

therefore $$\frac{d\psi_1}{dt} = -\frac{h - C\omega_3^2}{k^2 - C^2\omega_3^2}\,k;$$

combining this with the result of Art. 3, and integrating, we have

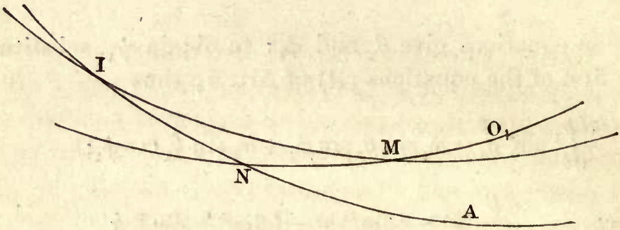
$$\psi_1 + g = -kC\sqrt{(AB)}\times$$

$$\int\frac{(h - C\omega_3^2)\,d\omega_3}{(k^2 - C^2\omega_3^2)\{h^2 - Bh + (B-C)C\omega_3^2\}^{\frac{1}{2}}\{-k^2 + Ah + (C-A)\,C\omega_3^2\}^{\frac{1}{2}}},$$

where g is the constant of integration.

Since ω_3 is known in terms of t from Art. 3, these equations give θ_1, ϕ_1, ψ_1; so that the position of the principal axes is known at any time with reference to the invariable plane. Since, however, when the disturbing forces are taken into account, this plane ceases to be absolutely invariable, it will be convenient to be able to refer the motion to some other plane which does remain fixed, and which may be taken as a plane of reference: this we can now do by Spherical Trigonometry.

7. Let the surface of a sphere of any radius, with its centre at the centre of gravity of the planet, be cut by the fixed plane of reference, the invariable plane, and the principal plane of xy, in the great circles OMN, MI, INA respectively.



As before let $ON = \psi$, $NA = \phi$, the angle $ONA = \theta$; also take M as the origin from which ψ_1 is measured, and let $MI = \psi_1$, $IA = \phi_1$, the angle $MIN = \theta_1$: let OM (the longitude of the node of the invariable plane) $= \alpha$, and IMN (its inclination to the plane of reference) $= \gamma$.

Then the sides of the spherical triangle IMN are

$$\psi - \alpha, \quad \psi_1, \quad \phi_1 - \phi;$$

and the angles respectively opposite to these,

$$\theta_1, \quad \pi - \theta, \quad \gamma.$$

Hence, by the formulæ of Spherical Trigonometry, we have

$$\cos\theta = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1\cos\psi_1,$$

$$\sin(\phi_1 - \phi)\sin\theta = \sin\gamma\sin\psi_1,$$

$$\sin(\psi - \alpha)\sin\theta = \sin\theta_1\sin\psi_1,$$

which determine θ, ϕ, ψ, when θ_1, ϕ_1, ψ_1, α, γ are known.

Disturbed Motion.

8. Having now shewn how to determine the position and velocity of rotation of the planet on the hypothesis that no forces act upon it, we proceed to a rigorous treatment of the problem. We shall employ the principle of the variation of parameters, and suppose the results already obtained to represent the true solution, the arbitrary constants or elements being no longer constants, but variable quantities, which it will be our object to determine. We shall speak of the forces which produce this variation as *disturbing forces.*

The elements which have been already employed in the undisturbed motion are six in number, viz. $h, k, l, g, \alpha, \gamma$: in considering these as variable we shall arrive at the very remarkable result that the equations for calculating their variations are precisely the same as the corresponding equations for the motion of translation in the Planetary Theory.

DEF. The plane of which the direction cosines are

$$\frac{A\omega_1}{k}, \quad \frac{B\omega_2}{k}, \quad \frac{C\omega_3}{k},$$

the area conserved upon which has been shewn (Art. 4) to be equal to k, will in future be termed the *plane of maximum areas,* on account of the property which it possesses, that k is a maximum*; since it can no longer be considered invariable.

* See Routh's *Rigid Dynamics,* Art. 174.

9. *To find an expression for the sum of the moments of the disturbing forces about any line through the centre of gravity of the planet.*

We shall suppose the disturbing body (which may be the Sun, Moon, or another planet), so distant that it may be considered to attract as if condensed into its centre of gravity.

Let m be the mass of the disturbed, m' of the disturbing body, ρ_1 the distance of the centre of gravity of the latter from an element δm_1 of the former: also let $V_1 = \frac{m'}{\rho_1}$. Then, if σ denote the length of the arc of any curve measured from some fixed point to the element δm_1, the disturbing force on this element in direction of the tangent to the curve, and tending to increase σ, will be

$$\delta m_1 \frac{dV_1}{d\sigma}.$$

If we suppose this arbitrary curve so drawn that its tangent at the point where the element is situated is perpendicular to the axis about which the moments are to be taken, and denote by p the distance of δm_1 from this axis, the moment of the force will be

$$\delta m_1 . p \frac{dV_1}{d\sigma}.$$

Let the small arc $\delta\sigma$ subtend an angle $\delta\chi$ at the nearest point of the axis; then $\delta\sigma = p\delta\chi$, and the moment becomes

$$\delta m_1 \frac{dV_1}{d\chi}.$$

Similarly, if V_2 refer to an element δm_2, the moment of the disturbing force on this element will be

$$\delta m_2 \frac{dV_2}{d\chi},$$

$\delta\chi$ being the same as for the element δm_1 since these elements are supposed rigidly connected.

Hence the sum of the moments of the disturbing forces on all the elements of the planet

$$= \delta m_1 \frac{dV_1}{d\chi} + \delta m_2 \frac{dV_2}{d\chi} + \dots$$

$$= \frac{d}{d\chi} \Sigma (\delta m_1 . V_1);$$

or, if we write V' for $\Sigma (\delta m_1 . V_1)$, the sum of the moments will be

$$\frac{dV'}{d\chi},$$

where

$$V' = m' \Sigma \frac{\delta m}{\rho}.$$

COR. If there are several disturbing bodies m', m'', &c. and V', V'', &c. are the functions corresponding, the sum of the moments of the forces due to their action will still be $\frac{dV}{d\chi}$, where $V = V' + V'' + \dots$

The result of this Article may be thus enunciated:—*Suppose a small arbitrary rotation given to the planet about any axis through an angle* $\delta\chi$; *then, supposing* V *expressed in terms of* χ *and quantities which do not vary in this hypothetical motion, the sum of the moments of the disturbing forces about this axis will be expressed by the partial differential coefficient* $\frac{dV}{d\chi}$.

The function V is thus clearly analogous to the disturbing function R of the Planetary Theory.

10. We may express V in various ways which will be found useful:

(i) As a function of θ, ϕ, ψ. Let x, y, z be the coordinates of an element δm of the disturbed planet, the fixed plane of reference being that of xy, and the axis of x the line from which ψ and α are measured; let x', y', z' be the co-ordinates of the centre of gravity of the disturbing body referred to the same axes; and let x_1, y_1, z_1 be the co-ordinates of δm referred to the principal axes of the planet. Then

$$V = m'\Sigma\frac{\delta m}{\rho}, \text{ (Art. 9)},$$

$$= m'\Sigma\frac{\delta m}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}.$$

Now if λ, μ, ν be the angles which the fixed axis of x makes with the principal axes of x, y, z, we have

$$x = x_1\cos\lambda + y_1\cos\mu + z_1\cos\nu;$$

and by Spherical Trigonometry (see fig. to Art. 5),

$$\cos\lambda = \cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta,$$

$$\cos\mu = -\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta,$$

$$\cos\nu = \sin\psi\sin\theta.$$

Thus x may be expressed in terms of θ, ϕ, ψ and of the co-ordinates of δm referred to the principal axes. Similarly, y and z may be expressed in terms of θ, ϕ, ψ. If the values of x, y, z so obtained be now substituted in the expression for V, it will become a function of θ, ϕ, ψ; and, in so far as it depends upon the disturbed planet, of quantities independent of the time.

(ii) As a function of $\theta_1, \phi_1, \psi_1, \alpha, \gamma$. By Art. 7, we have,

$$\theta = f(\theta_1, \psi_1, \gamma),$$

$$\phi - \phi_1 = f(\theta_1, \psi_1, \gamma),$$

$$\psi - \alpha = f(\theta_1, \psi_1, \gamma),$$

the symbol f denoting a different function in each case. If then we suppose V to have been expressed as a function of θ, ϕ, ψ by (i), these equations will enable us to express it as a function of θ_1, ϕ_1, ψ_1, α, γ.

(iii) As a function of t and the elements. Collecting together the results of Arts. 6 and 3, and making $C\omega_3 = s$, we may write

$$\theta_1 = f(k, s),$$
$$\phi_1 = f(h, k, s),$$
$$\psi_1 + g = f(h, k, s),$$
$$t + l = f(h, k, s).$$

Supposing V to have been expressed as a function of θ_1, ϕ_1, ψ_1, α, γ, by (ii), the first three of these equations will enable us to express it as a function of s, h, k, g, α, γ: then, if s be eliminated by means of the fourth equation, it will become a function of $t + l$, h, k, g, α, γ; that is, of t and the elements.

11. We now proceed to obtain equations for calculating the values of the elements at any time, commencing with h, the element of *vis viva*.

Let T denote the *vis viva* due to the rotation of the planet, $\delta m_1 \frac{dV_1}{d\sigma}$ the resolved part of the disturbing force on an element δm_1 of its mass in the direction of motion of the element; then the equation of *vis viva* gives

$$\frac{dT}{dt} = 2\Sigma\left(\delta m_1 \frac{dV_1}{d\sigma}\frac{d\sigma}{dt}\right)$$
$$= 2\Sigma\left(\delta m_1 \frac{d(V_1)}{dt}\right),$$

where in $\frac{d(V_1)}{dt}$, the differential coefficient is taken only in so far as t is involved through the co-ordinates of the element δm_1 of the disturbed planet. This equation may be written

$$\frac{dT}{dt} = 2\frac{d}{dt}\Sigma(\delta m_1 . V_1)$$

$$= 2\frac{d(V)}{dt},$$

if, as in Art. 9, we write $V = \Sigma(\delta m_1 . V_1)$. Now from the result of (iii) in the preceding Article we notice that $t + l$ always occurs in V as one quantity; therefore

$$\frac{d(V)}{dt} = \frac{dV}{dl};$$

also in the undisturbed motion $T = h$; hence our equation becomes

$$\frac{dh}{dt} = 2\frac{dV}{dl};$$

from which h may be calculated.

12. This equation may also be written

$$\frac{dh}{dt} = 2\frac{d(V)}{dt},$$

under which form it is easily seen to be identical with the corresponding equation

$$\frac{d}{dt}\left(-\frac{\mu}{a}\right) = 2\frac{d(R)}{dt},$$

of the Planetary Theory. (See *Planetary Theory*, Art. 26.) In fact it appears that the element of *vis viva* will be given by a similar equation in all cases of motion, whether of translation or rotation, when the disturbing bodies attract according to any law expressed by a function of the distance.

Equations of Motion.

13. Let k_1, k_2, k_3 denote the areas conserved about three rectangular axes of x, y, z, originating in the centre of gravity of the disturbed planet, and moving with angular velocities β_1, β_2, β_3 about their instantaneous positions; also let L, M, N be the moments of the disturbing forces about these axes: then (Routh's *Rigid Dynamics*, Art. 120) we have the equations of motion

$$\frac{dk_1}{dt} - k_2\beta_3 + k_3\beta_2 = L,$$

$$\frac{dk_2}{dt} - k_3\beta_1 + k_1\beta_3 = M,$$

$$\frac{dk_3}{dt} - k_1\beta_2 + k_2\beta_1 = N.$$

Now let k_3 denote the area conserved on the plane of maximum areas: then from the undisturbed motion (Art. 4) we have

$$k_3 = k;$$

and since the areas conserved upon all planes perpendicular to the plane of maximum areas are zero*, we must also have

$$k_1 = 0, \quad k_2 = 0,$$

always; and therefore

$$\frac{dk_1}{dt} = 0, \quad \frac{dk_2}{dt} = 0:$$

* This follows from the fact that the area conserved upon any plane is equal to the projection upon it of the area conserved upon the plane of maximum areas. (Routh's *Rigid Dynamics*, Art. 174.)

hence our equations become

$$k\beta_2 = L,$$

$$-k\beta_1 = M,$$

$$\frac{dk}{dt} = N.$$

We may express β_1 and β_2 in terms of $\frac{d\alpha}{dt}$ and $\frac{d\gamma}{dt}$, α and γ being respectively the longitude of the node and the inclination of the plane of maximum areas to the fixed plane of reference. Since no assumption has yet been made with regard to the position of the axis of x, it will be convenient to suppose it to coincide with the line of nodes. Employing the equations of Art. 5, and making $\omega_1 = \beta_1$, $\omega_2 = \beta_2$, $\theta = \gamma$, $\phi = 0$, $\psi = \alpha$, we have

$$\frac{d\alpha}{dt} \sin\gamma = \beta_2,$$

$$\frac{d\gamma}{dt} = -\beta_1,$$

and the equations of motion become

$$k\frac{d\alpha}{dt} \sin\gamma = L,$$

$$k\frac{d\gamma}{dt} = M,$$

$$\frac{dk}{dt} = N.$$

14. With respect to the last of these equations, we may remark that it is of the same form as it would be if the plane of maximum areas were still invariable. Now on referring to Art. 18 of the *Planetary Theory* it will be seen that the equations for the motion of translation, when referred to axes in the plane of the orbit, take the same forms

as if this plane were at rest, and moreover, that the second equation there obtained is identical with the third of the preceding Article*. This coincidence is not accidental, but arises from the fact that in the motion of translation, the plane of maximum areas *is* the plane of the orbit. In fact, the equations obtained in the preceding Article for the motion of rotation are equally applicable to the motion of translation, and we shall see that the resulting formulæ for the calculation of the elements involved are identical in the two motions.

15. It now only remains to find expressions for L, M, N in terms of the differential coefficients of the function V, in order to obtain from the equations of Art. 13 formulæ for calculating the elements α, γ and k. We shall suppose V expressed as a function of θ_1, ϕ_1, ψ_1, α, γ (Art. 10 (ii)).

To find L:—Suppose a small arbitrary rotation given to the planet about the axis of x, that is, about the line of nodes of the plane of maximum areas: then we may represent the change in position of the planet by supposing γ alone to vary, θ_1, ϕ_1, ψ_1, α remaining constant†. Thus, the tendency of L being to diminish γ, we have

$$L = -\frac{dV}{d\gamma},$$

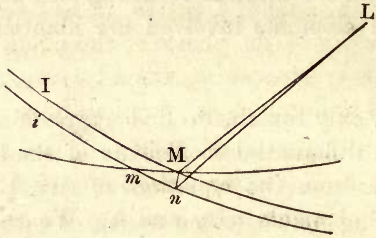
the differential coefficient being partial. Hence by the first equation of motion,

$$\frac{d\alpha}{dt} = -\frac{1}{k\sin\gamma}\frac{dV}{d\gamma}.$$

* The quantity here denoted by k corresponds with what in the Planetary Theory is denoted by h.

† This is equivalent to supposing the plane of maximum areas fixed in the body during the rotation, a supposition perfectly allowable, since the rotation given to the planet is hypothetical, and has nothing to do with its actual

To find M:—Suppose a small arbitrary rotation given to the planet through an angle $\delta\chi$ about the axis of y, that is, about a line through L the centre of gravity of the planet, in the plane of maximum areas perpendicular to LM its line of nodes: let the effect of the rotation be to change the position of this plane from MI to mi: draw Mn perpendicular to mi.



Then $Mn = \delta\chi$, $mn = -\delta\psi_1$, $Mm = \delta\alpha$, the angle $Mmn = \gamma$: and we have $\delta\chi = \delta\alpha \sin\gamma$, $\delta\psi_1 = -\delta\alpha\cos\gamma$;

therefore $$\frac{d\alpha}{d\chi} = \frac{1}{\sin\gamma}, \quad \frac{d\psi_1}{d\chi} = -\cos\gamma\frac{d\alpha}{d\chi} = -\cot\gamma.$$

And $$M = \frac{dV}{d\chi} = \frac{dV}{d\alpha}\frac{d\alpha}{d\chi} + \frac{dV}{d\psi_1}\frac{d\psi_1}{d\chi},$$

since we may suppose α and ψ_1 alone to vary with χ; therefore

$$M = \frac{1}{\sin\gamma}\frac{dV}{d\alpha} - \cot\gamma\frac{dV}{d\psi_1}.$$

Now (Art. 10 (iii)) $\psi_1 + g = f(h, k, s)$,

and θ_1, ϕ_1 do not involve g; therefore

$$\frac{dV}{d\psi_1} = -\frac{dV}{dg}:$$

motion. Any other supposition might be adopted with regard to the motion of this plane, but this is the most convenient, and will be retained throughout this Article.

hence $$M = \frac{1}{\sin\gamma}\frac{dV}{d\alpha} + \cot\gamma\frac{dV}{dg};$$

and the second equation of motion gives

$$\frac{d\gamma}{dt} = \frac{1}{k\sin\gamma}\frac{dV}{d\alpha} + \frac{\cos\gamma}{h\sin\gamma}\frac{dV}{dg}.$$

To find N:—Suppose a small arbitrary rotation given to the planet about the axis of z, that is, about a normal through the centre of gravity of the planet to the plane of maximum areas: then we may suppose ψ_1 alone to vary, and since the tendency of N is to diminish ψ_1, we shall have

$$N = -\frac{dV}{d\psi_1} = \frac{dV}{dg},$$

as in the preceding Article; then the third equation gives

$$\frac{dk}{dt} = \frac{dV}{dg}.$$

16. The equation for calculating γ may also be simply obtained as follows: if we refer the motion to the fixed plane of reference, and suppose V expressed as a function of θ, ϕ, ψ, since $k\cos\gamma$ is the area conserved on this plane, we shall have

$$\frac{d}{dt}(k\cos\gamma) = -\frac{dV}{d\psi}.$$

Now (Art. 10 (ii)), $\psi - \alpha = f(\theta_1, \psi_1, \gamma)$, and θ, ϕ do not involve α; therefore

$$\frac{dV}{d\psi} = \frac{dV}{d\alpha},$$

the latter differential coefficient supposing V expressed as a function either of θ_1, ϕ_1, ψ_1, α, γ, or simply of t and the elements. Hence

$$\frac{d}{dt}(k\cos\gamma) = -\frac{dV}{d\alpha};$$

therefore $$k \sin\gamma \frac{d\gamma}{dt} = \frac{dV}{d\alpha} + \cos\gamma \frac{dk}{dt}$$

$$= \frac{dV}{d\alpha} + \cos\gamma \frac{dV}{dg};$$

therefore $$\frac{d\gamma}{dt} = \frac{1}{k \sin\gamma} \frac{dV}{d\alpha} + \frac{\cos\gamma}{k \sin\gamma} \frac{dV}{dg}.$$

The method of this Article is identical with the second method given in Art. 34 of the *Planetary Theory* for the determination of $\frac{di}{dt}$; we reserve the comparison of the results to a subsequent Article (Art. 22).

17. The following proposition will be useful in finding $\frac{dl}{dt}$.

To shew that $\frac{\mathrm{dl}}{\mathrm{dk}} = 2\frac{\mathrm{d}\psi_1}{\mathrm{dh}}$, *and* $\frac{\mathrm{dl}}{\mathrm{ds}} = -2\frac{\mathrm{d}\phi_1}{\mathrm{dh}}$, *the variables being connected together by the equations of Art.* 10, (iii).

Let $$\left.\begin{aligned} u &= k^2 - Bh + \frac{B-C}{C} s^2 \\ v &= -k^2 + Ah + \frac{C-A}{C} s^2 \end{aligned}\right\} \quad \text{............. (1),}$$

where $s = C\omega_3$. Then from Arts. 3 and 6, we have

$$t + l = \sqrt{(AB)} \int \frac{ds}{\sqrt{(uv)}} \quad \text{.................. (2),}$$

$$\psi_1 + g = -\sqrt{(AB)} \int \frac{k}{k^2 - s^2} \cdot \frac{h - \frac{s^2}{C}}{\sqrt{(uv)}} ds \quad \text{......(3),}$$

$$\tan^2\phi_1 = \frac{Au}{Bv} \quad \text{................................ (4).}$$

From equation (2), by differentiating under the integral sign,

$$\frac{dl}{dk} = -\frac{1}{2}\sqrt{(AB)}\int \frac{u\frac{dv}{dk} + v\frac{du}{dk}}{(uv)^{\frac{3}{2}}}\,ds$$

$$= \sqrt{(AB)}\int \frac{k\,(u-v)}{(uv)^{\frac{3}{2}}}\,ds, \text{ by equations (1).}$$

From equation (3), in like manner, we have

$$\frac{d\psi_1}{dh} = -\sqrt{(AB)}\int \frac{k}{k^2 - s^2}\cdot\frac{uv - \frac{1}{2}\left(h - \frac{s^2}{C}\right)\left(u\frac{dv}{dh} + v\frac{du}{dh}\right)}{(uv)^{\frac{3}{2}}}\,ds$$

$$= -\frac{1}{2}\sqrt{(AB)}\int \frac{k}{(uv)^{\frac{3}{2}}}\cdot\frac{2uv - \left(h - \frac{s^2}{C}\right)(Au - Bv)}{k^2 - s^2}\,ds.$$

To simplify this expression, we find from equations (1),

$$h - \frac{s^2}{C} = \frac{u+v}{A-B},$$

$$k^2 - s^2 = \frac{Au + Bv}{A - B}.$$

Substituting these values, and reducing,

$$\frac{d\psi_1}{dh} = \frac{1}{2}\sqrt{(AB)}\int \frac{k\,(u-v)}{(uv)^{\frac{3}{2}}}\,ds$$

$$= \frac{1}{2}\,\frac{dl}{dk}, \text{ from above;}$$

therefore $$\frac{dl}{dk} = 2\,\frac{d\psi_1}{dh}.$$

Again, from equation (2), we have

$$\frac{dl}{ds} = \sqrt{\left(\frac{AB}{uv}\right)}.$$

Differentiating equation (4), and substituting for $\frac{du}{dh}$ and $\frac{dv}{dh}$ from equations (1),

$$2 \tan \phi_1 \sec^2 \phi_1 \frac{d\phi_1}{dh} = \frac{A}{B} \cdot \frac{-Bv - Au}{v^2};$$

also from equation (4),

$$\tan \phi_1 \sec^2 \phi_1 = \sqrt{\left(\frac{Au}{Bv}\right)} \cdot \frac{Au + Bv}{Bv};$$

hence, by substitution,

$$\frac{d\phi_1}{dh} = -\frac{1}{2}\sqrt{\left(\frac{AB}{uv}\right)};$$

therefore $$\frac{dl}{ds} = -2\frac{d\phi_1}{dh}.$$

18. The following relations between partial differential coefficients of the function V, as well as the results of the preceding Article, will be required in finding $\frac{dl}{dt}$.

In Art. 10 we have shewn that, in order to express V as a function of t and the elements, we may first express it as a function of θ_1, ϕ_1, ψ_1, α, γ, by case (ii), and then eliminate θ_1, ϕ_1, ψ_1 by means of the equations in (iii). The first three of these equations are

$$\theta_1 = f(k, s),$$
$$\phi_1 = f(h, k, s),$$
$$\psi_1 + g = f(h, k, s).$$

If, then, we substitute for θ_1, ϕ_1, ψ_1 from these equations, V will become a function of h, k, s, g, α, γ; and we shall have

$$\left(\frac{dV}{dh}\right)=\frac{dV}{d\theta_1}\frac{d\theta_1}{dh}+\frac{dV}{d\phi_1}\frac{d\phi_1}{dh}+\frac{dV}{d\psi_1}\frac{d\psi_1}{dh}$$

$$=\frac{dV}{d\phi_1}\frac{d\phi_1}{dh}+\frac{dV}{d\psi_1}\frac{d\psi_1}{dh},$$

since θ_1 does not involve h. We have used the brackets to distinguish the partial differential coefficient with respect to h, of V expressed as here supposed, from its partial differential coefficient when expressed as a function of t and the elements. In order to pass from the expression of V as a function of h, k, s, g, α, γ to its expression as a function of t and the elements, we must solve the fourth equation of Art. 10 (iii), with respect to s; thus we shall have

$$s=f(h,\ k,\ t+l),$$

and then

$$\frac{dV}{dh}=\left(\frac{dV}{dh}\right)+\frac{dV}{ds}\frac{ds}{dh};$$

or, writing for $\left(\frac{dV}{dh}\right)$ the value obtained above,

$$\frac{dV}{dh}=\frac{dV}{ds}\frac{ds}{dh}+\frac{dV}{d\phi_1}\frac{d\phi_1}{dh}+\frac{dV}{d\psi}\frac{d\psi_1}{dh},$$

a relation which will be required in the following Article.

Again, differentiating the equation

$$s=f(h,\ k,\ t+l)$$

on the hypothesis that h and l *alone* vary, we have

$$0=\frac{ds}{dh}+\frac{ds}{dl}\frac{dl}{dh},$$

where $\frac{dl}{dh}$ is the same as would be obtained by direct differentiation of the last equation of Art. 10 (iii). Hence

$$\frac{dV}{ds}\frac{ds}{dh} = -\frac{dV}{ds}\frac{ds}{dl}\frac{dl}{dh}$$

$$= -\frac{dV}{dl}\frac{dl}{dh},$$

since l is introduced into V only through s. This relation also will be required in the following Article.

19. We are now in a position to find $\frac{dl}{dt}$, and shall for this purpose refer the motion to the principal plane of xy. We must first, however, obtain an expression for the sum of the moments of the disturbing forces about the axis of z. Consider, then, V as a function of θ_1, ϕ_1, ψ_1, α, γ, and let a small arbitrary displacement be given to the planet about the axis of z. We may express this by supposing θ_1, ψ_1, α, γ to remain constant while ϕ_1 alone varies. Since, then, with the usual convention with respect to signs, the tendency of the moment we are considering is to increase ϕ_1, it will be expressed by $\frac{dV}{d\phi_1}$. Thus, from Euler's equations of motion,

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = \frac{dV}{d\phi_1};$$

whence, writing as before, s for $C\omega_3$,

$$\frac{ds}{dt} = (A - B)\,\omega_1\omega_2 + \frac{dV}{d\phi_1}.$$

From the results obtained in the case of undisturbed motion, we have (Art. 10 (iii))

$$t + l = f(h, k, s).$$

If we differentiate this equation, first considering the elements variable, and then considering them invariable, and equate the results, substituting the above value of $\frac{ds}{dt}$, all terms not depending upon the disturbing force will disappear: thus

$$\frac{dl}{dt}=\frac{dl}{dh}\frac{dh}{dt}+\frac{dl}{dk}\frac{dk}{dt}+\frac{dl}{ds}\frac{dV}{d\phi_1};$$

or, substituting the values of $\frac{dh}{dt}$ and $\frac{dk}{dt}$ from Arts. 11 and 15,

$$\frac{dl}{dt}=2\frac{dV}{dl}\frac{dl}{dh}+\frac{dV}{dg}\frac{dl}{dk}+\frac{dV}{d\phi_1}\frac{dl}{ds}:$$

but by Art. 17,

$$\frac{dl}{dk}=2\frac{d\psi_1}{dh}, \quad \frac{dl}{ds}=-2\frac{d\phi_1}{dh},$$

also, as in Art. 15,

$$\frac{dV}{dg}=-\frac{dV}{d\psi_1},$$

and by Art. 18,

$$\frac{dV}{dl}\frac{dl}{dh}=-\frac{dV}{ds}\frac{ds}{dh};$$

therefore $$\frac{dl}{dt}=-2\left(\frac{dV}{ds}\frac{ds}{dh}+\frac{dV}{d\psi_1}\frac{d\psi_1}{dh}+\frac{dV}{d\phi_1}\frac{d\phi_1}{dh}\right)$$

$$=-2\frac{dV}{dh} \text{ (Art. 18)},$$

from which l may be calculated.

20. To obtain the formula for g, we might of course proceed in the same manner, differentiating the third equation of Art. 10 (iii); but the process would be complicated by the fact that ψ_1 is measured from a moving point and on

a moving plane, and consequently that the form of $\frac{d\psi_1}{dt}$ is not the same in the disturbed as in the undisturbed motion. We shall therefore employ a different method: availing ourselves of the formulæ already obtained for calculating five out of the six elements, we shall obtain the sixth by direct substitution, in the same manner as that in which the epoch was obtained in Art. 36 of the *Planetary Theory*.

By Art. 10 (iii), we may write

$$V = f(t+l, g, h, k, \alpha, \gamma).$$

Differentiating, the elements being considered variable,

$$\frac{dV}{dt} = \frac{dV}{d(t+l)}\left(1 + \frac{dl}{dt}\right) + \frac{dV}{dg}\frac{dg}{dt} + \frac{dV}{dh}\frac{dh}{dt} + \frac{dV}{dk}\frac{dk}{dt} + \frac{dV}{d\alpha}\frac{d\alpha}{dt} + \frac{dV}{d\gamma}\frac{d\gamma}{dt}:$$

differentiating as if the elements were invariable,

$$\frac{dV}{dt} = \frac{dV}{d(t+l)}.$$

Equating the two values of $\frac{dV}{dt}$, we have

$$\frac{dV}{dl}\frac{dl}{dt} + \frac{dV}{dg}\frac{dg}{dt} + \frac{dV}{dh}\frac{dh}{dt} + \frac{dV}{dk}\frac{dk}{dt} + \frac{dV}{d\alpha}\frac{d\alpha}{dt} + \frac{dV}{d\gamma}\frac{d\gamma}{dt} = 0.$$

Substituting in this equation the values of $\frac{dl}{dt}$, $\frac{dh}{dt}$, &c., obtained in the preceding Articles, we find after reduction

$$\frac{dg}{dt} = -\frac{dV}{dk} - \frac{\cos\gamma}{k\sin\gamma}\frac{dV}{d\gamma},$$

from which g may be calculated.

21. We will here recapitulate the formulæ which have been obtained for calculating the elements of the motion:

(i) $$\frac{dh}{dt} = 2\frac{dV}{dl},$$

(ii) $$\frac{dl}{dt} = -2\frac{dV}{dh},$$

(iii) $$\frac{dk}{dt} = \frac{dV}{dg},$$

(iv) $$\frac{dg}{dt} = -\frac{dV}{dk} - \frac{\cos\gamma}{k\sin\gamma}\frac{dV}{d\gamma},$$

(v) $$\frac{d\alpha}{dt} = -\frac{1}{k\sin\gamma}\frac{dV}{d\gamma},$$

(vi) $$\frac{d\gamma}{dt} = \frac{1}{k\sin\gamma}\frac{dV}{d\alpha} + \frac{\cos\gamma}{k\sin\gamma}\frac{dV}{dg}.$$

22. These six equations agree with those obtained by Pontécoulant in his *Théorie Analytique du Système du Monde* (Tome II. p. 189). In Art. 12 we have shewn that the first of them can be expressed in a form in which it is seen to be identical with the formula from which the mean distance is calculated in the Planetary Theory: it will be instructive to compare the remaining equations with those of the motion of translation. In order to do this, we shall first express the latter in terms of elements having a signification analogous to that of the above. Now in the motion of rotation h is the element of *vis viva*, and this in the motion of translation is equal to $-\frac{\mu}{a}$: k is the area conserved on the plane of maximum areas, which plane in the motion of translation becomes that of the orbit, and the area conserved upon it is equal to $\sqrt{\mu a(1-e)^2}$: l is the element added to t

in the integration, and this in the motion of translation is $\frac{1}{n}(\epsilon - \varpi)$: g is the element added to ψ_1, and this corresponds to ϖ in the motion of translation, only we must remember that g is measured wholly on the plane of maximum areas from its node, while ϖ is measured on the fixed plane of reference as far as the node, and thence on the plane of the orbit; so that $\varpi - \Omega$ will correspond to g: lastly, α is the longitude of the node of the plane of maximum areas, and γ its inclination to the plane of reference; in the motion of translation the same quantities defining the position of the plane of the orbit are denoted by Ω and i. In order, then, to compare the rotation formulæ with those of translation, we proceed to replace the elements a, e, ϵ, ϖ of the Planetary Theory by four new elements h, k, l, g, connected with the former by the relations

$$h = -\frac{\mu}{a},$$

$$k^2 = \mu a(1 - e^2),$$

$$l = \frac{1}{n}(\epsilon - \varpi),$$

$$g = \varpi - \Omega.$$

Let R' express the form which the function R takes when the new elements are substituted for the old: then we have

$$\frac{dR}{da} = \frac{dR'}{dh}\frac{dh}{da} + \frac{dR'}{dk}\frac{dk}{da} + \frac{dR'}{dl}\frac{dl}{da}\,*$$

$$= \frac{\mu}{a^2}\frac{dR'}{dh} + \frac{1}{2}na\sqrt{1-e^2}\frac{dR'}{dk} + \frac{3(\epsilon - \varpi)}{2na}\frac{dR'}{dl},$$

* In $\frac{dR}{da}$, which occurs only in the formula for the epoch, the differential coefficient is supposed to be taken with respect to a only in so far as a

$$\frac{dR}{de} = \frac{dR'}{dk}\,\frac{dk}{de} = -\frac{na^2 e}{\sqrt{1-e^2}}\,\frac{dR'}{dk},$$

$$\frac{dR}{d\epsilon} = \frac{dR'}{dl}\,\frac{dl}{d\epsilon} = \frac{1}{n}\,\frac{dR'}{dl},$$

$$\frac{dR}{d\varpi} = \frac{dR'}{dl}\,\frac{dl}{d\varpi} + \frac{dR'}{dg}\,\frac{dg}{d\varpi}$$

$$= -\frac{1}{n}\,\frac{dR'}{dl} + \frac{dR'}{dg},$$

$$\frac{dR}{d\Omega} = \frac{dR'}{d\Omega} + \frac{dR'}{dg}\,\frac{dg}{d\Omega}$$

$$= \frac{dR'}{d\Omega} - \frac{dR'}{dg}.$$

Now if we differentiate the above expressions for h, k, l, g, and then substitute the values of $\frac{da}{dt}$, $\frac{de}{dt}$, &c. obtained in the Planetary Theory, and those obtained here for $\frac{dR}{da}$, $\frac{dR}{de}$, &c., they will reduce to

$$\frac{dh}{dt} = 2\frac{dR'}{dl},$$

$$\frac{dl}{dt} = -2\frac{dR'}{dh},$$

$$\frac{dk}{dt} = \frac{dR'}{dg},$$

$$\frac{dg}{dt} = -\frac{dR'}{dh} - \frac{\cos i}{k\sin i}\,\frac{dR'}{di},$$

occurs explicitly in R: if we suppose a to vary also as contained implicitly in n, this differential coefficient will include the term proportional to the time, which may therefore with this understanding be omitted. In Art. 37 of the *Planetary Theory* this term was removed by a different transformation.

$$\frac{d\Omega}{dt}=\frac{1}{k\sin i}\frac{dR'}{di},$$

$$\frac{di}{dt}=-\frac{1}{k\sin i}\frac{dR'}{d\Omega}+\frac{\cos i}{k\sin i}\frac{dR'}{dg}.$$

On comparing these with the rotation formulæ, it will be seen that they are identical, with the exception only that the sign of α differs from that of Ω; and this is accounted for by the fact that α and Ω are measured in opposite directions. Thus, by employing elements having a like signification in the two motions of translation and rotation, we have arrived at the very remarkable result that *the complete solution of the problem of Planetary perturbation, whether in the motion of translation or of rotation, is expressed by the above simple formulæ.*

PART II.

APPLICATION OF PRECEDING RESULTS.

PRECESSION AND NUTATION.

23. THE formulæ obtained in the *first part* are sufficient completely to determine the motion of a planet or other rigid body about its centre of gravity. They are perfectly rigorous, subject only to the hypothesis that the disturbing bodies may be supposed to attract as if condensed into their respective centres of gravity; a hypothesis admissible if these bodies are either very distant or nearly spherical in form. But though the formulæ are exact, they can be integrated only by approximation. We propose, therefore, in the *second part* to restrict ourselves to the particular case of the Earth, taking advantage of such of the results of observation as may be required to enable us to approximate. In order to treat the problem fully we shall consider, first, the motion of the axis of rotation in the Earth itself, with the velocity of rotation about it; secondly, the motion of this axis in space. The first is of special interest, since any change in the position of the axis in the Earth, were such change possible, would affect the permanence of terrestrial latitudes; any change in the velocity of rotation would affect the length of the day. The second is of great importance to astronomers, since it establishes the fact that the first point of Aries, or vernal equinox, to which they are accustomed to refer celestial longitudes, is not a fixed point.

Stability of the axis of rotation in the Earth and of the velocity about it.

24. We proceed then, first, to consider the motion of the axis of rotation within the Earth, and shall be able to shew that, in so far as it depends upon the attractions of other bodies, this axis can never separate appreciably from the axis of figure, and that the velocity about it must always remain appreciably constant.

We have from Art. 3

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = h,$$
$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = k^2.$$

Eliminating ω_3 from these equations,

$$A(C-A)\omega_1^2 + B(C-B)\omega_2^2 = Ch - k^2.$$

Now if C be either the greatest or least principal moment, both the terms of the left-hand member of this equation must always retain the same sign; and if we take C about the Earth's axis of figure it will be the greatest and this sign will be positive. We may therefore write

$$Ch - k^2 = (C-A)(C-B)e^2:$$

thus $A(C-A)\omega_1^2 + B(C-B)\omega_2^2 = (C-A)(C-B)e^2.$

It follows from this that

ω_1 can never be greater than $e\sqrt{\dfrac{C-B}{A}}$,

ω_2 can never be greater than $e\sqrt{\dfrac{C-A}{B}}$.

If we neglect the disturbing force, h, k and therefore e are constant, and the value of e will be found by substituting the values of ω_1, ω_2 as determined by observation at any given epoch. Now the most delicate observations have hitherto shewn no appreciable separation of the axis of rotation from the axis of figure: hence ω_1, ω_2 are at present inappreciable; thus e is so, and we conclude that, independently of the disturbing force, ω_1, ω_2 must always remain inappreciable.

If in the equations of Art. 6, we write $\omega_1 = 0$, $\omega_2 = 0$, we have also $\theta_1 = 0$; so that, if the disturbing force be neglected, we may consider the plane of maximum areas as coincident with the Earth's equator.

25. Let us now examine the effect of the disturbing force upon the angular velocities ω_1, ω_2. By the preceding Article

$$(C-A)(C-B)e^2 = Ch - k^2.$$

Differentiating, and substituting the expressions for $\frac{dh}{dt}$ and $\frac{dk}{dt}$ obtained in the *first part* (Art. 21),

$$(C-A)(C-B)e\frac{de}{dt} = C\frac{dV}{dl} - k\frac{dV}{dg}.$$

If the approximation be carried to the first power of the disturbing force, we may, in calculating the second member of this equation, suppose $\theta_1 = 0$: thus the equations of Art. 7 give

$$\theta = \gamma, \quad \phi = \phi_1 - \psi_1, \quad \psi = \alpha.$$

Also, if n be the mean angular velocity, we may to the same order of approximation suppose $\omega_3 = n$: thus, from the equations (A) of Art. 5, ψ and θ are constant and

$$\frac{d\phi}{dt} = n;$$

whence by integration

$$\phi = nt + c,$$

c being the value of ϕ at the epoch from which t is measured.

Again, by Art. 10 (iii),

$$\psi_1 + g = f(h, k, s),$$
$$t + l = f(h, k, s).$$

Hence $$\frac{dV}{dl} = \frac{d(V)}{dt},$$

$$\frac{dV}{dg} = -\frac{dV}{d\psi_1} = \frac{dV}{d\phi} = \frac{1}{n}\frac{d(V)}{dt}.$$

Now (Art. 3) $$k^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$$
$$= C^2n^2$$

to the same order of approximation: therefore

$$k = Cn.$$

Hence $$C\frac{dV}{dl} - k\frac{dV}{dg} = C\frac{d(V)}{dt} - \frac{k}{n}\frac{d(V)}{dt} = 0;$$

and therefore $$\frac{de}{dt} = 0:$$

whence it follows that to the first order of the disturbing force e continues absolutely constant.

When the approximation is carried to the order of the square of the disturbing force*, it is found that e^2 can contain no term proportional to the time, nor any inequality, either secular or periodic, raised to the first order in the process of integration. Hence we may consider e^2 as practically constant, and thus conclude, as in the preceding Article, that ω_1, ω_2 will always remain inappreciable.

* See Pentécoulant's *Système du Monde*, tome II. p. 218.

It follows, as in the preceding Article, that to the same degree of approximation the plane of maximum areas may be regarded as coincident with the Earth's equator.

26. We have now to examine the effect of the disturbing force upon the velocity of rotation. Since ω_1, ω_2 are inappreciable, this velocity will be equal to ω_3; thus, as in the preceding Article, we have

$$\omega_3 = \frac{k}{C};$$

therefore
$$\frac{d\omega_3}{dt} = \frac{1}{C}\frac{dk}{dt} = \frac{1}{C}\frac{dV}{dg} \text{ (Art. 21)}$$
$$= \frac{1}{Cn}\frac{d(V)}{dt}.$$

Now considering V as a function of θ, ϕ, ψ, if our approximation be carried to the first power of the disturbing force, we have

$$\theta = \gamma, \quad \phi = nt + c, \quad \psi = \alpha:$$

so that V will involve t only under the form $nt + c$. It follows that $\frac{d(V)}{dt}$ will consist of a series of terms of the form

$$P \begin{matrix}\cos \\ \sin\end{matrix} (pnt + q),$$

where p is some positive integer. And this term will give rise to inequalities whose period is $\frac{2\pi}{pn}$, that is, a day or fraction of a day. Now the most delicate observations have hitherto failed to detect any such inequality: hence we must altogether reject such terms, and conclude that ω_3 will always remain sensibly constant.

27. The preceding Articles prove the stability of the axis of rotation in the Earth and of the velocity about it, so

far as the motion depends upon the attractions of distant bodies; but there is another disturbing influence to be found in the friction produced by the tides. The effect of this (see Thomson's and Tait's *Natural Philosophy*) would be to retard the Earth's velocity; but whether to an appreciable extent is yet uncertain. The fact, discovered by Professor Adams, that a portion of the acceleration of the Moon's mean motion is yet unaccounted for, led Delaunay to suggest that this portion might be apparent only, and really due to the retarding effect of tidal friction upon the Earth's velocity. But other possible explanations have been given, and to pursue the subject further would be beyond the scope of the present treatise.

Precession and Nutation.

28. We have now to consider the motion of the Earth's axis in space, resulting in the phenomena of Precession and Nutation. We have seen that, to the first order of the disturbing force, the plane of maximum areas may be considered as coincident with the Earth's equator, and that consequently to this order of approximation we may write

$$\theta=\gamma, \quad \psi=\alpha.$$

Hence the motion of the equator may be calculated by formulæ (v) and (vi) of Art. 21. Now it may be shewn as in Art. 25 that

$$\frac{dV}{dg}=\frac{1}{n}\frac{d(V)}{dt},$$

and may therefore be neglected, since (as has been proved in Art. 26) it can give rise only to terms whose period is a day or fraction of a day, which are wholly insensible. If, then, we write ψ and θ for α and γ respectively, these formulæ become

$$\frac{d\psi}{dt} = -\frac{1}{k\sin\theta}\frac{dV}{d\theta},$$

$$\frac{d\theta}{dt} = \frac{1}{k\sin\theta}\frac{dV}{d\psi};$$

or, since to the same order of approximation $k = C\omega_3 = Cn$,

$$\frac{d\psi}{dt} = -\frac{1}{Cn\sin\theta}\frac{dV}{d\theta},$$

$$\frac{d\theta}{dt} = \frac{1}{Cn\sin\theta}\frac{dV}{d\psi}.$$

29. These elegant equations, due to Poisson, give ψ and θ, and thus determine the motion of the Earth's axis in space. Before we can integrate them it will be necessary to obtain an expression for V.

Let then m be the mass of the Earth, m' that of the disturbing body which we shall suppose to be the Sun; r' the distance between the centres of gravity of these bodies, r the distance from the Earth's centre of gravity of an element δm of its mass, ρ the distance of the same element from the centre of gravity of the Sun: also let the inclination of r to r' be denoted by v. Then by Art. 9,

$$V = m'\Sigma\frac{\delta m}{\rho}$$

$$= m'\Sigma\frac{\delta m}{(r'^2 - 2rr'\cos v + r^2)^{\frac{1}{2}}}$$

$$= \frac{m'}{r'}\Sigma\left\{\delta m\left(1 - \frac{2r}{r'}\cos v + \frac{r^2}{r'^2}\right)^{-\frac{1}{2}}\right\}.$$

Now since r' is very large in comparison of r, we shall neglect all powers of $\frac{r}{r'}$ above the second: thus

$$V = \frac{m'}{r'} \Sigma \left\{ \delta m \left(1 + \frac{r}{r'} \cos v - \frac{1}{2} \frac{r^2}{r'^2} + \frac{3}{2} \frac{r^2}{r'^2} \cos^2 v \right) \right\}$$

$$= \frac{m'}{r'} \Sigma (\delta m) + \frac{m'}{r'^2} \Sigma (\delta m \,.\, r \cos v) - \frac{m'}{2r'^3} \Sigma (\delta m \,.\, r^2)$$

$$+ \frac{3m'}{2r'^3} \Sigma (\delta m \,.\, r^2 \cos^2 v)$$

$$= \frac{m'}{r'} \Sigma (\delta m) + \frac{m'}{r'^2} \Sigma (\delta m \,.\, r \cos v) + \frac{m'}{r'^3} \Sigma (\delta m \,.\, r^2)$$

$$- \frac{3m'}{2r'^3} \Sigma (\delta m \,.\, r^2 \sin^2 v).$$

Of these terms $\frac{m'}{r'} \Sigma (\delta m)$ and $\frac{m'}{r'^3} \Sigma (\delta m . r^2)$ may be omitted, since they cannot contain the angles θ, ϕ or ψ, and would not be affected by any arbitrary rotation such as is supposed in Art. 9: also since r is measured from the Earth's centre of gravity, $\Sigma (\delta m \,.\, r \cos v) = 0$. Hence we may write

$$V = - \frac{3m'}{2r'^3} \Sigma (\delta m \,.\, r^2 \sin^2 v)$$

$$= - \frac{3m'}{2r'^3} Q,$$

if Q denote the moment of inertia of the Earth about the line joining its centre of gravity with that of the Sun. We see, then, that the effective portion of the disturbing function is proportional to this moment of inertia.

30. We have hitherto made no assumption respecting the Earth's figure. We learn from pendulum observations and geodetic measurements that it is approximately an

oblate spheroid; the moments of inertia about all axes in the plane of the equator are therefore nearly equal, and we may take $B = A$. If, then, δ be the Sun's declination,

$$Q = A\cos^2\delta + C\sin^2\delta:$$

also if n' denote the Sun's mean motion, we have approximately

$$\frac{2\pi}{n'} = \frac{2\pi r'^{\frac{3}{2}}}{\sqrt{m+m'}};$$

therefore $$\frac{m+m'}{r'^3} = n'^2,$$

or, neglecting the Earth's mass in comparison with that of the Sun,

$$\frac{m'}{r'^3} = n'^2.$$

Hence, by substitution,

$$V = -\frac{3n'^2}{2}(A\cos^2\delta + C\sin^2\delta).$$

31. By Art. 9, the moment of the Sun's disturbing force about an equatorial diameter of the Earth perpendicular to the line joining the centres of gravity of the two bodies, and tending to increase δ

$$= \frac{dV}{d\delta} = -\frac{3n'^2}{2}(C-A)\sin 2\delta.$$

It is clear that the moment about any axis perpendicular to this is zero, since an hypothetical rotation about any such axis could produce no change in δ. If, then, the Sun's disturbing force be reduced to a parallel force through the Earth's centre of gravity, and a couple, the tendency of the latter will be to give to the equator a rotation towards the ecliptic about an equatorial diameter perpendicular to the line joining the centres of gravity of the two bodies. The moment of this disturbing couple is

$$\frac{3n'^2}{2}(C-A)\sin 2\delta.$$

32. We shall now return to the equations of Art. 28, and deduce from them the motion of the Earth's axis in space. The results obtained will be of sufficient accuracy for our present purpose if we suppose the ecliptic, to which plane the motion will be referred, to remain fixed.

Let ψ denote the longitude of the first point of Aries measured from some fixed point on the ecliptic, λ the Sun's longitude measured from the same point, θ the obliquity of the ecliptic. Then from the spherical triangle formed by the intersection of the equator, the ecliptic, and a declination circle through the Sun, we have

$$\sin\delta = \sin(\psi-\lambda)\sin\theta;$$

therefore

$$\cos\delta\frac{d\delta}{d\theta} = \sin(\psi-\lambda)\cos\theta,$$

$$\cos\delta\frac{d\delta}{d\psi} = \cos(\psi-\lambda)\sin\theta.$$

If $\psi-\lambda=l$, then l will be the Sun's longitude measured from the first point of Aries in the opposite direction to that in which ψ and λ are measured: thus

$$\cos\delta\frac{d\delta}{d\theta} = \sin l\cos\theta, \quad \cos\delta\frac{d\delta}{d\psi} = \cos l\sin\theta.$$

Hence $$\frac{dV}{d\theta} = \frac{dV}{d\delta}\frac{d\delta}{d\theta} = -3n'^2(C-A)\sin\delta\sin l\cos\theta,$$

$$\frac{dV}{d\psi} = \frac{dV}{d\delta}\frac{d\delta}{d\psi} = -3n'^2(C-A)\sin\delta\cos l\sin\theta.$$

Substituting these values in the equations of Art. 28,

$$\frac{d\psi}{dt} = \frac{3n'^2}{n} \cdot \frac{C-A}{C} \sin\delta \sin l \cot\theta$$

$$= \frac{3n'^2}{n} \cdot \frac{C-A}{C} \sin^2 l \cos\theta$$

$$= \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \cos\theta\,(1-\cos 2l).$$

$$\frac{d\theta}{dt} = -\frac{3n'^2}{n} \cdot \frac{C-A}{C} \sin\delta \cos l$$

$$= -\frac{3n'^2}{n} \cdot \frac{C-A}{C} \sin l \cos l \sin\theta$$

$$= -\frac{3n'^2}{2n} \cdot \frac{C-A}{C} \sin\theta \sin 2l.$$

Let I denote the mean value of the obliquity, then since both $\frac{n'}{n}$ and $\frac{C-A}{C}$ are very small quantities, we may in the second members of these equations write

$$\theta = I, \quad l = n't:$$

thus
$$\frac{d\psi}{dt} = \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \cos I\,(1-\cos 2n't),$$

$$\frac{d\theta}{dt} = -\frac{3n'^2}{2n} \cdot \frac{C-A}{C} \sin I \sin 2n't.$$

By integration,

$$\psi = \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \,.\, \cos I \,.\, t - \frac{3n'}{4n} \cdot \frac{C-A}{C} \,.\, \cos I \,.\, \sin 2n't,$$

$$\theta = I + \frac{3n'}{4n} \cdot \frac{C-A}{C} \,.\, \sin I \,.\, \cos 2n't,$$

ψ being measured from the position of the first point of Aries at the epoch from which the time is reckoned.

33. These formulæ determine the motion of the Earth's axis in space. Although the method by which they have been obtained has the advantage of connecting the two problems of translation and rotation, it may be worth while to arrive at them by an independent process.

Since the Earth is nearly an oblate spheroid, we shall, as before, take C for the greatest principal moment, and suppose $B = A$, so that all axes in the plane of the equator are principal axes. If, then, taking the plane of the equator for that of xy, we suppose the axes of x and y to revolve with an angular velocity θ_3 about the axis of z, we have the equations of motion*

$$A\frac{d\omega_1}{dt} - A\omega_2\theta_3 + C\omega_2\omega_3 = L,$$

$$A\frac{d\omega_2}{dt} + A\omega_1\theta_3 - C\omega_1\omega_3 = M,$$

$$C\frac{d\omega_3}{dt} = N.$$

Now if we take for the axis of x the projection of the Sun's radius vector on the plane of the equator, we have

$$L = 0, \quad N = 0,$$

$$M = -\frac{3n'^2}{2}(C - A)\sin 2\delta, \text{ (Art. 31)};$$

and since θ_3 is very small and occurs only in connexion with ω_1 and ω_2 which are also very small (Arts. 24 and 25), we may write

$$\theta_3 = n' \text{ its mean value.}$$

* See Routh's *Rigid Dynamics*, Art. 107: the equations there given reduce to the above if θ_3 is written for $\omega_3 + \frac{d\chi}{dt}$.

Thus the third equation gives

$$C\frac{d\omega_3}{dt}=0,$$

whence $\omega_3 = \text{const.} = n$ suppose;

and the first two become

$$A\frac{d\omega_1}{dt}+(Cn-An')\,\omega_2=0,$$

$$A\frac{d\omega_2}{dt}-(Cn-An')\,\omega_1=-\frac{3n'^2}{2}(C-A)\sin 2\delta:$$

or, denoting $\dfrac{Cn-An'}{A}$ by p,

$$\frac{d\omega_1}{dt}+p\omega_2=0,$$

$$\frac{d\omega_2}{dt}-p\omega_1=-\frac{3n'^2}{2}\cdot\frac{C-A}{A}\sin 2\delta.$$

Now $\dfrac{C-A}{A}$ is a very small fraction; n', the Sun's mean motion, is very small in comparison of n; and δ varies very slowly; we shall, therefore, in integrating these equations, take no account of its variation. Thus, eliminating ω_1,

$$\frac{d^2\omega_2}{dt^2}+p^2\omega_2=0,$$

the integral of which is

$$\omega_2=A\cos(pt-B),$$

where A and B are constants of integration. This term is independent of the disturbing force; but having for its period $\dfrac{2\pi}{p}$, which does not differ much from a day, it must be rejected, since, as we have already remarked in Art. 26, such terms are altogether insensible. Hence we conclude that

$$\omega_2 = 0;$$

and by substitution in the second equation,

$$\begin{aligned}\omega_1 &= \frac{3n'^2}{2p} \cdot \frac{C-A}{A} \sin 2\delta \\ &= \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \left(1 - \frac{An'}{Cn}\right)^{-1} \sin 2\delta \\ &= \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \sin 2\delta,\end{aligned}$$

powers of n' above the second being neglected. We see, then, that the effect of the Sun's attraction is to cause the Earth's axis to travel in a plane perpendicular to that of the disturbing couple which it produces, and with an angular velocity proportional to its moment. This velocity, though so small as to be insensible to the most delicate observations, yet leads to values of ψ and θ which can by no means be neglected.

34. We proceed to determine the motion of the axes in space. Substituting in the equations of Art. 5, we have

$$\frac{d\psi}{dt} \sin\theta = \frac{3n'^2}{n} \cdot \frac{C-A}{C} \cdot \sin\delta \cos\delta \sin\phi,$$

$$\frac{d\theta}{dt} = -\frac{3n'^2}{n} \cdot \frac{C-A}{C} \cdot \sin\delta \cos\delta \cos\phi.$$

Now from the spherical triangle formed by the intersection of the equator, the ecliptic and a declination circle through the Sun, it is easily seen that

$$\begin{aligned}\sin\phi &= \tan\delta \cot\theta, \\ \sin\delta &= \sin l \sin\theta, \\ \cos l &= \cos\phi \cos\delta.\end{aligned}$$

By means of these equations we obtain

$$\frac{d\psi}{dt} = \frac{3n'^2}{n} \cdot \frac{C-A}{C} \cdot \cos\theta \sin^2 l,$$

$$\frac{d\theta}{dt} = -\frac{3n'^2}{2n} \cdot \frac{C-A}{C} \cdot \sin\theta \sin 2l;$$

whence, as in Art. 32,

$$\psi = \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \cos I \,.\, t - \frac{3n'}{4n} \cdot \frac{C-A}{C} \cdot \cos I \sin 2n't,$$

$$\theta = I + \frac{3n'}{4n} \cdot \frac{C-A}{C} \sin I \cos 2n't.$$

35. It appears from these formulæ that the motion of the Earth's axis is of two kinds; partly secular, partly periodic. It is convenient to consider these separately. The former affects the equinoxes alone, and, being proportional to the time, indicates uniform motion; this motion is one of regression, since ψ has been measured in a direction contrary to that of the apparent motion of the Sun. Considered with reference to the apparent diurnal motion of the stars, the effect is to place the equinoxes in advance of the position they would occupy if fixed: hence it obtained the name of the *Solar Precession of the Equinoxes.* The latter furnishes corrections both on ψ and θ, which go through all their changes in half a year: it is called the *Solar Nutation of the Earth's axis;* the correction on ψ forming the *Nutation in Longitude,* that on θ the *Nutation in Latitude.*

36. We will now examine the effect produced by the Moon's action on the motion of the Earth's axis. Since the investigations which have been given of the effect of the Sun's disturbing force contain nothing to restrict their generality except the special assumptions made with regard to

small quantities, we shall first consider what modifications are required in order that they may be applied in the case of the Moon.

Let ψ', θ', n'', I' denote relatively to the plane of the Moon's orbit the same quantities which relatively to the ecliptic have been denoted by ψ, θ, n', I; also let m, m'' be the masses of the Earth and Moon: then, as in the case of the Sun, we have approximately,

$$\frac{2\pi}{n''} = \frac{2\pi r''^{\frac{3}{2}}}{\sqrt{m+m''}},$$

but we cannot in this case neglect m, as it is in fact much larger than m'': retaining it, we have

$$\frac{m''}{r'^3} = \frac{m''n''^2}{m+m''} = \frac{n''^2}{1+\lambda},$$

if $m = \lambda m''$. Also n'' is small, though not so small as n' in comparison of n. Hence to determine the motion of the Earth's axis with reference to the plane of the Moon's orbit, we have

$$\psi' = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I't$$

$$- \frac{3n''}{4n(1+\lambda)} \cdot \frac{C-A}{C} \cdot I' \sin 2n''t,$$

$$\theta' = I' + \frac{3n''}{4n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \sin I' \cos 2n''t.$$

The periodical terms in these equations go through all their values in half a month, and are so small that they are usually neglected. Thus we may consider the inclination of the plane of the Earth's orbit to that of the Moon as constant and equal to its mean value, and the precession as uniform and given by

$$\psi' = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I' . t;$$

whence, the velocity of precession

$$\frac{d\psi'}{dt} = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I'.$$

These results, deduced from the corresponding formulæ in the case of the Sun, suppose the plane of the Moon's orbit fixed. Now the line of nodes moves too rapidly to allow of this hypothesis, but the only effect of considering its motion would be to add to $\frac{d\psi'}{dt}$ a term depending upon its velocity and not upon the disturbing force of the Moon upon the Earth. Since our object is to trace the effects of this force only, such terms must be omitted.

37. It now remains to determine the motion of the Earth's axis with reference to the ecliptic.

Let θ be the obliquity of the ecliptic, ψ the longitude of the equinox, α the longitude of the node of the Moon's orbit measured from the same origin as ψ, i the inclination of the Moon's orbit to the ecliptic, I' its inclination to the equator. Then supposing ψ' measured from the node of the Moon's orbit on the ecliptic, we have by Spherical Trigonometry (as in Art. 7)

$$\sin(\psi - \alpha) \sin\theta = \sin I' \sin\psi' \quad \text{.................. (1)},$$

$$\cos\theta = \cos i \cos I' - \sin i \sin I' \cos\psi' \quad \text{........... (2)},$$

$$\cos I' = \cos\theta \cos i + \sin\theta \sin i \cos(\psi - \alpha) \quad \text{..... (3)}.$$

We must now differentiate these equations in order to express $\frac{d\psi}{dt}$ and $\frac{d\theta}{dt}$ in terms of $\frac{d\psi'}{dt}$. In so doing we shall omit the terms involving $\frac{d\alpha}{dt}$ and $\frac{di}{dt}$, since these quantities depend not upon the disturbing force, but upon the motion of the Moon's orbit. Thus

$$\cos(\psi - \alpha)\sin\theta\frac{d\psi}{dt} + \sin(\psi - \alpha)\cos\theta\frac{d\theta}{dt}$$

$$= \sin I' \cos\psi' \frac{d\psi'}{dt},$$

$$\sin\theta\frac{d\theta}{dt} = -\sin i \sin I' \sin\psi' \frac{d\psi'}{dt}.$$

Writing Ω for $\psi - \alpha$, we obtain

$$\cos\Omega\sin\theta\frac{d\psi}{dt} = (\sin I'\cos\psi'$$

$$+ \sin i \sin I' \sin\psi' \cot\theta \sin\Omega)\frac{d\psi'}{dt}$$

$$= (\sin I'\cos\psi' + \sin i\cos\theta\sin^2\Omega)\frac{d\psi'}{dt} \text{ ...... by (1),}$$

$$= \left(\frac{\cos i\cos I' - \cos\theta}{\sin i}\right.$$

$$\left. + \sin i\cos\theta - \sin i\cos\theta\cos^2\Omega\right)\frac{d\psi'}{dt} \text{ ...... by (2),}$$

$$= \left(\frac{\cos i\cos I' - \cos\theta\cos^2 i}{\sin i} - \sin i\cos\theta\cos^2\Omega\right)\frac{d\psi'}{dt}$$

$$= (\cos i\sin\theta\cos\Omega - \sin i\cos\theta\cos^2\Omega)\frac{d\psi'}{dt}, \text{ ...... by (3);}$$

therefore $$\frac{d\psi}{dt} = (\cos i - \sin i\cot\theta\cos\Omega)\frac{d\psi'}{dt}.$$

Also $$\frac{d\theta}{dt} = -\frac{\sin i\sin I'\sin\psi'}{\sin\theta}\frac{d\psi'}{dt}$$

$$= -\sin i\sin\Omega\frac{d\psi'}{dt} \text{ .... by (1).}$$

We must now substitute the value of $\frac{d\psi'}{dt}$: thus

$$\frac{d\psi}{dt} = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I' (\cos i - \sin i \cot\theta \cos\Omega),$$

$$\frac{d\theta}{dt} = -\frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I' \sin i \sin\Omega.$$

It remains to eliminate I'. Now

$$\cos I' (\cos i - \sin i \cot\theta \cos\Omega)$$

$$= (\cos\theta \cos i + \sin\theta \sin i \cos\Omega)(\cos i - \sin i \cot\theta \cos\Omega)$$

$$= \cos\theta \cos^2 i - \frac{1}{2}\frac{\cos 2\theta}{\sin\theta} \sin 2i \cos\Omega - \cos\theta \sin^2 i \cos^2\Omega$$

$$= \cos\theta \left(\cos^2 i - \frac{1}{2}\sin^2 i\right) - \frac{1}{2}\frac{\cos 2\theta}{\sin\theta} \sin 2i \cos\Omega$$
$$- \frac{1}{2}\cos\theta \sin^2 i \cos 2\Omega.$$

Also $\cos I' \sin i \sin\Omega$

$$= (\cos\theta \cos i + \sin\theta \sin i \cos\Omega) \sin i \sin\Omega$$

$$= \frac{1}{2}\cos\theta \sin 2i \sin\Omega + \frac{1}{2}\sin\theta \sin^2 i \sin 2\Omega.$$

Hence $$\frac{d\psi}{dt} = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos\theta \, (\cos^2 i$$

$$- \frac{1}{2}\sin^2 i - \cot 2\theta \sin 2i \cos\Omega - \frac{1}{2}\sin^2 i \cos 2\Omega),$$

$$\frac{d\theta}{dt} = -\frac{3n''^2}{4n(1+\lambda)} \cdot \frac{C-A}{C} \cdot (\cos\theta \sin 2i \sin\Omega$$
$$+ \sin\theta \sin^2 i \sin 2\Omega).$$

Let ν be the mean retrograde velocity of the Moon's nodes, Ω_0 the value of Ω at the epoch, then we may write $\Omega = \nu t + \Omega_0$; also for θ we may write its mean value I; and since i is a very small angle we may neglect $\sin^2 i$ when multiplied by a periodical term. Thus

$$\frac{d\psi}{dt} = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I \{\cos^2 i - \frac{1}{2}\sin^2 i$$

$$- \cot 2I \sin 2i \cos(\nu t + \Omega_0)\},$$

$$\frac{d\theta}{dt} = -\frac{3n''^2}{4n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I \sin 2i \sin(\nu t + \Omega_0).$$

The integrals of these equations are

$$\psi = \frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I \{(\cos^2 i - \frac{1}{2}\sin^2 i)\, t$$

$$- \frac{1}{\nu} \cot 2I \sin 2i \sin(\nu t + \Omega)\} + \text{const.},$$

$$\theta = I + \frac{3n''^2}{4n(1+\lambda)} \cdot \frac{C-A}{C\nu} \cos I \sin 2i \cos(\nu t + \Omega_0).$$

38. These equations determine the motion of the Earth's axis due to the attraction of the Moon. They are similar to the equations expressing the Sun's action, and the remarks made in Art. 35 might, *mutatis mutandis*, be repeated here. The Lunar Precession is

$$\frac{3n''^2}{2n(1+\lambda)} \cdot \frac{C-A}{C} \cdot \cos I \left(\cos^2 i - \frac{1}{2}\sin^2 i\right) t.$$

If we add to this the Solar Precession (Art. 32), we find for the whole permanent effect of the Sun and Moon upon the equinoxes

$$\frac{3n}{2} \cdot \frac{C-A}{C} \cdot \cos I \left\{\frac{n''^2}{n^2(1+\lambda)} \left(\cos^2 i - \frac{1}{2}\sin^2 i\right) + \frac{n'^2}{n^2}\right\} t.$$

This is called the *Luni-solar Precession*, to distinguish it from the Precession due to the secular motion of the ecliptic in consequence of the attractions of the Planets, and which is called *Planetary Precession*.

Adding together the effects of Lunar and Solar Nutation, we find for the whole Nutation in Longitude

$$-\frac{3}{2}\cdot\frac{C-A}{C}\cdot\cos I\left\{\frac{n''^2}{n\nu(1+\lambda)}\cot 2I\sin 2i\sin\Omega\right.$$
$$\left.+\frac{n'}{2n}\sin 2\odot\right\},$$

where Ω and $\odot$ denote respectively the mean longitude of the ascending node of the Moon's orbit and of the Sun.

Similarly, the whole Nutation in Latitude is

$$\frac{3}{4}\cdot\frac{C-A}{C}\cdot\cos I\left\{\frac{n''^2}{n\nu(1+\lambda)}\sin 2i\cos\Omega+\frac{n'}{n}\cos 2\odot\right\}.$$

In each of these expressions the first term, due to the action of the Moon, is the most important, since n'' is larger than both n' and ν.

39. If we consider only the motion of the Earth's axis due to Precession, it appears from the preceding formulæ that it maintains a constant inclination to the pole of the ecliptic, and describes a right circular cone about it with uniform velocity. An axis possessing this motion exactly we shall term the *mean axis* of the Earth.

The motion of the axis due to Lunar Nutation can now be exhibited as follows:—

$$\text{Let } x=-\frac{3}{4}\cdot\frac{n''^2}{n\nu(1+\lambda)}\cdot\frac{C-A}{C}\cdot\cos 2I\sin 2i\sin\Omega,$$
$$y=\frac{3}{4}\cdot\frac{n''^2}{n\nu(1+\lambda)}\cdot\frac{C-A}{C}\cdot\cos I\sin 2i\cos\Omega:$$

then, taking unity as the radius of the celestial sphere, x and y are the small *linear* spaces traversed by the intersection of the Earth's axis with the circumference in two directions at right angles. Eliminating Ω from these equations, we have

$$\frac{x^2}{\cos^2 2I}+\frac{y^2}{\cos^2 I}=\left(\frac{3}{4}\,\frac{n''^2}{n\nu(1+\lambda)}\cdot\frac{C-A}{C}\cdot\sin 2i\right)^2.$$

Hence, in consequence of Lunar Nutation, the extremity of the axis may be considered to move in an ellipse whose semi-axes are in the ratio of $\cos 2I$ to $\cos I$. The centre of this ellipse is at the point of intersection of this mean axis with the celestial sphere, and its plane a tangent to the sphere at that point. By supposing the mean axis to describe the cone uniformly, while the true axis describes this ellipse about it, the real motion will be represented. This conception is due to Bradley, who arrived at it by observation.

40. The annual value of the Luni-solar precession (see Art. 38)

$$= \frac{3n}{2n'} \cdot \frac{C-A}{C} \cos I \left\{ \frac{n''^2}{n^2(1+\lambda)} \left(\cos^2 i - \frac{1}{2} \sin^2 i \right) + \frac{n'^2}{n^2} \right\} n'.$$

Observation gives about 50″.1 as the numerical value of this expression. Hence, by substituting the known values of n, n', ν, I, i, we have a relation between $\frac{C-A}{C}$ and λ, by means of which either may be determined when the other is known.

41. We have taken no account in our calculations of the Precession caused by the attractions of the planets on the Earth, since it is too trifling to be appreciable.

THE END.

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