ELEMENTARY ALGEBRA

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ELEMENTARY ALGEBRA

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"As regards the method of teaching algebra, I would make it, in the earlier stages, as much a generalized arithmetic as possible. Results obtained by algebra would be verified by arithmetical instances; and the use of a formula would be indicated as including any number of instances. Elaborate (and to my mind wearisome) processes, useful for solving artificial combinations of difficulties, would be at least deferred. With a comparative beginner, progress towards new ideas or new stages of old ideas can, I think, best be made by the simplest instances, and it is on this account that I would build algebra entirely on arithmetical foundations so far as concerns the teaching of beginners."

—*Professor Forsyth, M.A., D.Sc., F.R.S., Cambridge.*

"It is assumed that pupils will be required throughout the course to solve numerous problems which involve putting questions into equations. Some of these problems should be chosen from mensuration, from physics, and from commercial life. The use of graphical methods and illustrations, particularly in connection with the solution of equations, is also expected."

—*Extract from the Report of the American Mathematical Society*, 1903.

PREFACE

The aim of this book is to introduce the young student to the study of algebra, in particular to those portions of algebra that are indispensable to the study of geometry, mensuration, physics, and chemistry as pursued in secondary schools. The book is an outgrowth of classroom experience, lays stress on fundamental principles, and illustrates these principles so that the beginner may not "regard algebra as a very arbitrary affair, involving the application of a number of fanciful rules to the letters of the alphabet."

As far as the authors know, this is the first beginners' book that graphically illustrates the fundamental rules, fundamental laws and facts, and incidentally brings out in bold relief the essential connections of arithmetic, algebra, and concrete geometry. Whoever wishes to obtain a clear and sound knowledge of the fundamental operations of algebra must have recourse to arithmetic and to geometric illustrations, since learning is, at bottom, largely a process of visualizing.

Every point that we have found to give trouble to the young learner is dealt with in a way that will bring into play the perceptive powers of the student. Professor Minchin well says: "Effective teaching requires a great deal more than a bare recitation of facts, even if these are duly set forth in logical order. The probable difficulties which the intelligent student will naturally and necessarily encounter in some statement of fact or theory, — these things our authors seldom or never notice, and yet a recognition and anticipation of them by the author would often be of priceless value to the student." Few of our pupils in secondary schools have a clear conception of why having like signs in the multiplication of two numbers produces a plus result. This is one place where a text-book should come to the assistance of the student.

The first ten chapters furnish an easy introduction to the study of algebra. There are a great many simple problems, the typical solutions are natural, and there is no obvious striving to arrive at hasty generalizations. The last ten chapters demand more maturity on the part of the learner. It is a good practice to fix principles in the mind by means of easy examples, and then to move forward to difficult ones. The examples here are of medium grade, neither too easy nor too hard, and have been used in teaching classes in the Ball High School. Occasionally an example is inserted that will provoke serious thought.

This book leaves out the Euclidean method of finding the Highest Common Factor because it is not a practical topic. It also omits certain theorems in the Theory of Exponents, Surds, and Imaginary Numbers and the Theory of Limits because they are too difficult for the pupils of secondary schools.

A large number of the examples are new; the others are mainly from examination papers.

THE AUTHORS.

GALVESTON, TEXAS.
April 7, 1912.

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ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION

THE LANGUAGE OF ALGEBRA. THE EQUATION

1. Algebra, like arithmetic, deals with number. Algebra, however, goes farther than arithmetic, and enables one to solve with readiness problems which cannot be solved by arithmetic.

Algebra makes use of the symbols of arithmetic. Thus, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, are used in algebra to represent numbers in exactly the same way that they are used in arithmetic.

In algebra the sign of addition is $+$; the sign of subtraction is $-$; the sign of multiplication is $\times$, or a period; the sign of division is $\div$, /, or the fractional notation.

In arithmetic it is customary to use abbreviations for concrete quantities. Thus, instead of the word *cent* or *cents*, the letter c. is used. For example, 54 cents is written 54¢. Instead of the word *acre* or *acres*, the letter A. is used. Thus, 5 acres is written 5 A. Instead of the word *ton* or *tons*, the letter T. is used. Thus, 10 tons is written 10 T. In the metric system of measures the linear unit is the meter. The abbreviation for *meter* is m. Thus, 12 meters is written 12 m. A meter is, approximately, 39.37 inches. Hence, 12 m. is equivalent to 12×39.37 inches.

2. In algebra, symbols are carried farther than in arithmetic. In algebra, a letter of the alphabet may stand for any number whatever. When a number is written before a letter, it indicates multiplication. Thus, $4a$ means 4 times a. If a stands for 5, then $4a$ stands for 4 times 5. If a stands for 90, then $4a$ stands for 4 times 90.

Letters written in succession indicate multiplication. Thus, ab means a times b. abc means the continued product of the numbers a, b, and c, that is, a times b, and this result times c. If $a=4$, $b=7$, and $c=8$, then $abc = 4 \cdot 7 \cdot 8$, or $4 \times 7 \times 8$.

3. Each of the letters in an indicated product is called **a factor** of the product. Any one factor is called a **coefficient** of the number expressed by the remaining factors. Thus, in the indicated product, $9x$, 9 is the coefficient of x. In the indicated product, $4ab$, 4 is the coefficient of ab, and $4a$ is the coefficient of b. When no coefficient is written, 1 is understood. In the expression $x+3$, 1 is the coefficient of x.

4. Algebra uses a language of its own. The sum of two numbers, 7 and 4, is written $7+4$. The sum of two numbers, x and y, is written $x+y$. The difference of two numbers, 18 and 7, is written $18-7$. The difference of two numbers, a and b, is written $a-b$. The quotient of 56 divided by 11 is written $\frac{56}{11}$. The quotient of a divided by b is written $\frac{a}{b}$.

a^2 is a short method of writing a times a, or aa.

a^3 is a short method of writing a times a times a, or aaa.

a^4 is a short method of writing a times a times a times a, or $aaaa$.

The number indicating how many times a number or letter is taken as a factor is called an **exponent**. Thus, in a^4, 4 is the exponent.

It is important that the student distinguish carefully between coefficient and exponent. $4a$ means 4 times a. If $a = 10$, $4a = 40$, and $a^4 = 10^4 = 10{,}000$.

EXERCISE I

1. If $x = 7$, find the value of $2x$, $7x$, $11x$, $19x$, $25x$.
2. If $x = 11$, find the value of $8x$, $12x$, $15x$, $24x$, $32x$.
3. If $x = 8$, find the value of $5x$, $9x$, $7x$, $14x$.
4. If $a = 9$, find the value of $7a$, $16a$, $20a$, $30a$.
5. If $a = 4$, find the value of $2a$, a^2, $3a$, a^3, $5a$.
6. If $b = 12$, find the value of b^2, $3b^2$, b^3, $4b^2$, $7b^2$.
7. If $x = 5$, find the value of x^2, $x^2 + x$, $2x^2 + 3x$, $x^2 - x$.
8. If $a = 4$, $b = 3$, find the value of $a + b$, $a - b$, $2a + b$, $3a - b$.
9. If $a = 4$, $b = 7$, find the value of ab, $5ab$, $\frac{7a}{b}$, $\frac{a^2}{b}$, $\frac{a^3}{b}$.
10. If $a = 12$, $b = 3$, find the value of ab^2, $\frac{a^2}{b}$, $\frac{3a}{4b}$, $\frac{9a}{8b}$.
11. If $a = 6$, $b = 1$, find the value of $a^2 + b^2$, $a^2 - b^2$, $a^2 - ab$.
12. If $a = 7$, $c = 3$, find the value of $2ac$, $a^2 + ac$, $a^2 - ac$, ac^2.

13. If $x = 8$, $y = 4$, find the value of $\frac{x}{y}$, $\frac{x^2}{y}$, $\frac{x}{y^2}$, $\frac{x^2}{4}$.

14. If $x = 10$, $z = 6$, find the value of $\frac{3x}{z}$, $\frac{2x^2}{z}$, $\frac{z^2}{x}$, $\frac{z^3}{36}$.

5. In arithmetic a statement in symbols that one quantity equals another quantity is called an **equation**. (The algebraic meaning of equation will be given later.) For example, $5x + 5 = 83$ is an equation.

The part of an equation to the left of the sign of equality is called the **first member** of the equation, the part of the equation to the right of the sign of equality is called the **second member** of the equation.

EXERCISE 2

Find the value of x in each of the following equations:

1. $6x=24$.	5. $18x=90$.	9. $17x=187$.	13. $4x=2$.
2. $7x=56$.	6. $5x=120$.	10. $14x=210$.	14. $6x=3$.
3. $9x=63$.	7. $4x=192$.	11. $13x=78$.	15. $8x=52$.
4. $15x=45$.	8. $11x=143$.	12. $15x=5$.	16. $12x=78$.

EXERCISE 3

1. How many are 5 apples and 8 apples? How many are $5a$ and $8a$?

2. How many are 7 horses and 8 horses? How many are $7h$ and $8h$?

3. How many are 11 cents and 7 cents? How many are $11c$ and $7c$?

4. How many are 15 dollars and 8 dollars? How many are $15d$ and $8d$?

5. How many are 13 acres and 9 acres? How many are $13a$ and $9a$?

6. How many are 17 yards and 8 yards? How many are $17y$ and $8y$?

7. How many are 19 boys and 11 boys? How many are $19b$ and $11b$?

8. $3x + 5x + 8x = ?$

9. $7x + 5x + 4x = ?$

10. $9x + 8x + 6x = ?$

11. $8x + 7x + 5x = ?$

12. $14b + 5b + 6b = ?$

13. $8c + 9c + 10c = ?$

14. $12z + 8z + 13z = ?$

15. What is the sum of $4ab$ and $2ab$?

16. What is the sum of $12bc$ and $9bc$?

17. What is the sum of $5c^2$ and $3c^2$?

18. A man had x dollars and earned y dollars. How many dollars did he then have?

19. Take $4x$ from $10x$.

20. Take $9x$ from $16x$.

21. Take $13x$ from $22x$.

22. Take $7y$ from $13y$.

23. Take $15y$ from $27y$.

24. Take $19y$ from $25y$.

25. Take $2ab$ from $5ab$.

26. Take $7ab$ from $14ab$.

27. Take $16ab$ from $23ab$.

28. Diminish $15x$ by $3x$.

29. By how much does 12 exceed 7?

30. By how much does $12x$ exceed $7x$?

31. By how much does $24x$ exceed $16x$?

32. By how much does $30b$ exceed $21b$?

33. By how much does 40 exceed 10?

34. By how much does 40 exceed x?

35. The sum of two numbers is 27, and one of the numbers is 17. Find the other number.

36. The sum of two numbers is $11a$, and one of the numbers is $8a$. Find the other number.

37. The sum of two numbers is $100c$, and one of the numbers is $79c$. Find the other number.

38. The sum of two numbers is 35, and one of the numbers is 24. Find the other number.

39. The sum of two numbers is 35, and one of the numbers is x. Find the other number.

40. A man had a head of cattle and bought b head. How many head of cattle did he then have?

EXERCISE 4

1. The price of a bushel of wheat is b cents. Find the price of 13 bushels.

2. If the cost of an acre of land is x dollars, find the cost of 5 acres.

3. A horse sells for z dollars. Find the selling price of 9 such horses.

4. A city lot is sold for y dollars. Find the value of 10 such lots.

5. One side of a square is n yards. Find the perimeter of the square.

6. The dimensions of a rectangle are $2n$ and n. Find the perimeter of the rectangle.

7. A rectangle is 48 yards long and a yards wide. Find the area of the rectangle.

8. A rectangle is 10 rods long and b rods wide. Find its area.

9. The side of a square is b. Find the area of the square.

10. Six houses sold for $18x$ dollars. Find the average selling price of a house.

11. Five sheep sold for $10y$ dollars. Find the selling price of each sheep.

12. The area of a rectangle is $20x$ square rods, and one side is 10 rods long. Find the other dimension of the rectangle.

13. The area of a rectangle is $80a$ square yards, and one side is 40 yards. Find the other dimension.

14. Eight bushels of barley sold for $24d$ cents. Find the selling price of a bushel of barley.

15. A train runs $90x$ miles in 15 hours. Find its average rate per hour.

16. A steamboat runs $40h$ miles in 20 hours. What is its average rate per hour?

17. A man divides $5x$ acres of land into 10 equal parts. Find the number of acres in each part.

18. A man bequeathes $12x$ dollars to be divided equally among his three sons. How many dollars does each son receive?

EXERCISE 5

Translate into ordinary English the following symbols:

1. $18 + 5$. *Ans.* Eighteen increased by five, or the sum of eighteen and five.

2. $19 - 8$.

3. 17×7.

4. $x + 2x$.

5. $a + 4a$.

6. $5a + 6a$.

7. $5x - x$.

8. $7x - 3x$.

9. $14x - 11x$.

10. $a + b$.

11. $2a + b$.

12. $5a + 2b$.

13. $a - b$.

14. $3a - b$.

15. $2a^2$.

16. $a^2 + a$.

17. $b^2 - b$.

18. $\frac{x}{y}$.

19. $\frac{x+y}{y}$.

20. $\frac{3x^2}{a}$.

21. $\frac{b}{c^2}$.

22. $3a^2 - 4a$.

23. $11x^2y$.

EXERCISE 6

Translate into algebraic symbols, using x in each instance involving one number, and x and y whenever two numbers are involved :

1. The sum of a number and twice the same number. *Ans.* $x + 2x$.

2. Five times a number and the number.

3. Three times a number increased by twice the number.

4. Seven times a number increased by four times the number.

5. Eleven times a number increased by five times the number.

6. The difference between five times a number and two times the number.

7. Ten times a number diminished by seven times the same number.

8. Thirteen times a number diminished by six times the same number.

9. One number diminished by another number.

10. One number increased by another number.

11. Three times one number increased by another number.

12. The sum of six times a number and seven times another number.

13. Eight times a number diminished by five times another number.

14. Fifteen times one number diminished by eight times another number.

15. The difference between four times one number and five times another number.

16. The product of two numbers. Seven times the product of two numbers.

17. The square of a number. The square of a number times a second number.

18. The cube of a number. Eight times the cube of a number.

19. The fourth part of a number. Six times the fourth part of a number.

20. The fourth power of a number. Five times the fourth power of a number.

21. The quotient of one number divided by another number.

22. Seven times one number divided by another number.

23. Nine times one number divided by another number.

24. The square of one number divided by another number.

25. The sum of two numbers divided by the difference of the same numbers.

26. The difference of two numbers divided by the sum of the same numbers.

27. The cube of one number divided by the square of another number.

28. The sum of the squares of two numbers.

29. The difference of the squares of two numbers.

30. The square of a number diminished by the number.

Example **1.** The area of a rectangle is 2160 square yards, and its length is 72 yards. Find its width.

ARITHMETIC SOLUTION. 72 times the number of yards in the width equals 2160, the number of square yards in the area of the rectangle.

Hence, 72 times the number of yards = 2160.

Therefore, the number of yards $= \frac{2160}{72} = 30$.

Hence, the width of the rectangle is 30 yards.

ALGEBRAIC SOLUTION. Let x = the number of yards in the width of the rectangle.

Then $72x$ square yards = the area of the rectangle.

Hence, $72x = 2160$.

Dividing by 72, $x = 30$.

EXERCISE 7

1. The area of a rectangle whose length is 14 rods is 182 square rods. Find its width.

2. The area of a parallelogram is 1600 square yards, and its base is 64 yards. Find its altitude.

3. The area of a parallelogram is 342 square inches, and its altitude is 18 inches. Find its base.

4. If 75 acres of land is sold for $2775, find the average price per acre.

5. If 24 horses are sold for $2088, find the selling price of one horse.

6. A dealer sold a number of cattle at $39 per head, and received in payment $702. How many head of cattle did he sell?

7. Seven times the distance by rail from New York to Albany equals eleven times the distance by rail between New York City and Philadelphia. The distance from New York City to Albany is 143 miles. Find the distance from New York City to Philadelphia.

8. Three times the distance by rail from New York City to Hartford, Conn., equals twice the distance from New York City to Dover, Del. Hartford is 110 miles from New York City. Find the distance from Dover to New York City.

9. Omaha, Neb., is five times as far from New York City as Concord, N.H., is from New York City. Omaha is 1400 miles from New York City. Find the distance from Concord to New York City.

10. In the year 1901 Alabama produced 150 ounces of gold valued at $3100. Find the value of one ounce of gold.

11. In 1901 Montana produced 13,131,700 ounces of silver valued at $7,879,020. Find the value of one ounce of silver.

12. In the same year Colorado produced 18,437,800 ounces of silver valued at \$11,062,680. Find the value of one ounce of silver.

13. A boat travels at the rate of 9 miles per hour. At this rate how long will it take to travel 567 miles?

Example 1. Divide \$1 between two boys, giving one boy four times as much as the other boy.

SOLUTION. Let $x =$ the number of cents one boy receives.

Then, $4x =$ the number of cents the other boy receives.

$x + 4x =$ the number of cents both boys receive.

Hence, $x + 4x = 100.$

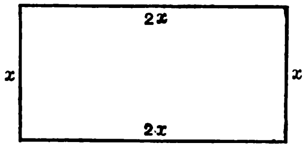
Therefore, $5x = 100.$

Dividing by 5, $x = 20.$

Multiplying by 4, $4x = 80.$ *Ans.* 80¢, 20¢.

Check. 20 cents + 80 cents = 100 cents.

Example 2. The perimeter of a rectangular garden is 150 yards, and its length is twice its breadth. Find the dimensions of the garden.



SOLUTION. Let $x =$ the number of yards in the breadth of the garden.

Then, $2x =$ the number of yards in the length of the garden.

$x + 2x + x + 2x =$ the number of yards in the perimeter of the garden.

Hence, $x + 2x + x + 2x = 150.$

Therefore, $6x = 150.$

Dividing by 6, $x = 25.$

Multiplying by 2, $2x = 50.$

The length is 50 yards; the width is 25 yards.

Check. $25 + 50 + 25 + 50 = 150.$

EXERCISE 8

1. The cost of an arithmetic and an algebra is \$1. If the algebra costs three times as much as the arithmetic, find the cost of each.

2. Two men enter into a partnership contributing \$3600. If one man contributes twice as much as the other, find each person's share in the partnership.

3. The sum of the ages of father and son is 52 years. If the father is three times as old as his son, find the age of each.

4. The sum of two angles is 90 degrees. If one angle is five times the other angle, how many degrees are in each angle?

5. A farmer plants 80 acres of cotton and wheat. If he plants three times as many acres in cotton as in wheat, how many acres does he plant in each?

6. A farmer plants 210 acres in corn and in oats. If the number of acres of corn is six times the number of acres of oats, find the number of acres of each.

7. A house and a barn cost \$3450. If the house cost twice as much as the barn, find the cost of each.

8. A house and lot cost \$3520. If the house cost three times as much as the lot, find the cost of each.

9. The number of national banks and private banks in North Dakota, in 1903, was 84. There were eleven times as many national banks as private banks. Find the number of national banks and private banks in North Dakota in 1903.

10. The sum of two angles, A and B, is 180 degrees. If angle A is four times as large as angle B, how many degrees are in each angle?

11. The perimeter of a rectangle is 245 yards, and the rectangle is four times as long as wide. Find its dimensions.

12. The perimeter of a rectangular farm is 372 rods. If the farm is twice as long as wide, find its dimensions.

Example **1.** A man bought a horse and carriage. The carriage cost three times as much as the horse. If the carriage cost \$240 more than the horse, what did each cost?

SOLUTION. Let $x =$ the number of dollars the horse cost.

Then, $3x =$ the number of dollars the carriage cost.

Hence, $3x - x = 240.$

Therefore, $2x = 240.$

Dividing by 2, $x = 120.$

Multiplying by 3, $3x = 360.$

Check. $\$360 - \$120 = \$240.$

\$120, cost of the horse.

\$360, cost of the carriage.

EXERCISE 9

1. A and B enter into a partnership. A contributes $5460 more than B. If A's interest in the partnership is four times B's, find each man's capital in the partnership.

2. Thomas Smith buys a lot on which he builds a house costing three times as much as the lot. If the house costs $1900 more than the lot, find the cost of each.

3. A dictionary and stand are bought for a schoolroom. The dictionary costs five times as much as the stand, and the cost of the dictionary exceeds the cost of the stand by $9.60. Find the cost of each.

4. A horse costs three times as much as a wagon. If the cost of the horse is $340 more than the cost of the wagon, what does each cost?

5. Nine times a number exceeds four times the same number by 495. What is the number?

6. The price of a suit of clothes is five times the price of a hat. If the suit of clothes cost $14.40 more than the hat, how much does each cost?

7. A farmer plants 205 acres more in corn than in barley. If he plants six times as many acres in corn as in barley, how many acres does he plant in each?

8. A farmer raises 460 bushels more of potatoes than of beans. If the number of bushels of potatoes is twenty-four times the number of bushels of beans, find how many bushels of each are raised.

Example 1. Divide $100 between A and B, giving A $5 more than B.

SOLUTION. Let $x =$ number of dollars B receives.

Then $x + 5 =$ number of dollars A receives.

Therefore, $2x + 5 =$ number of dollars both receive.

Hence, $2x + 5 = 100.$

Subtracting 5, $2x = 95.$

$$x = 47.50.$$

$$x + 5 = 52.50.$$

A's money is \$52.50.

B's money is \$47.50.

Check. \$47.50 + \$52.50 = \$100.

EXERCISE 10

1. Divide 25 cents between two boys, giving one boy 3 cents more than the other boy.

2. Divide 50 cents between two boys, giving one boy 4 cents more than the other boy.

3. The sum of two numbers is 150, and one of the numbers is 10 more than the other. Find the numbers.

4. At an election two candidates, Mr. Davis and Mr. Myers, received 2955 votes. Mr. Davis received 465 votes more than Mr. Myers. Find the number of votes each received.

5. In an election for governor of a state there were 290,119 votes cast. The successful candidate received 2553 more votes than the other candidate. Find the number of votes each received.

6. In an election two candidates received 167,729 votes. The successful candidate received 7295 more votes

than the other candidate. Find the number of votes each received.

7. A rectangular field is 99 yards longer than it is wide, and the sum of its two dimensions is 1065 yards. Find its dimensions.

8. Divide a line 36 inches long into two parts so that one part may be 5 inches longer than the other part.

9. A man buys a horse and a wagon, paying for the horse $45 more than for the wagon. If he pays $140 for both, find the cost of each.

10. In the years 1901 and 1902 there were produced in the United States 7,635,397 ounces of gold. The yield in the former year was 24,397 ounces more than in the latter year. Find the number of ounces of gold produced in each of these years.

11. The quantity of gold coined by the mints of the world during the years 1900 and 1901 is estimated at 29,172,590 ounces. If the coinage of 1900 was 5,168,416 ounces more than that of 1901, find the number of ounces coined in each of these years.

12. The receipts of the United States government from customs and internal revenue for the year 1903 is estimated at $515,289,705.98. The receipts from customs exceeded those from internal revenue by $53,669,357.64. Find the receipts from each source.

13. The combined areas of Connecticut and Delaware are 7992 square miles. Connecticut contains 3232 square miles more than Delaware. Find the area of each.

14. The combined areas of New Hampshire and New Jersey are 17,550 square miles, and the area of New Hampshire is 1204 square miles more than that of New Jersey. Find the area of each.

15. The Indian population in Indian Territory and Oklahoma in 1902 was 99,456. There were 72,714 more Indians in Indian Territory than in Oklahoma. Find the Indian population of each.

Example 1. Seven eighths of the water area of New York State is 5278 square miles. Find the area of the water surface of New York State.

Let $x =$ the number of square miles in the area.

Then, $\frac{7x}{8} = 5278.$

Dividing by 7, $\frac{x}{8} = 754.$

Multiplying by 8, $\frac{8x}{8} = 6032.$

Therefore, $x = 6032.$ *Ans.* 6032 square miles.

Check. $\frac{7}{8}$ of 6032 square miles = 5278 square miles.

EXERCISE II

1. If $\frac{3}{5}$ of the cost of a geography is 75 cents, find the cost of the geography.

2. If $\frac{7}{8}$ of the price of an acre of land is \$42, find the price of an acre of land.

3. If $\frac{3}{4}$ of the quantity of coal consumed by a family during a winter was $4\frac{1}{2}$ tons, find the number of tons consumed that winter.

4. Seven twenty-fifths of the number of Indians in Utah in 1902 was 609. Find the Indian population of Utah that year.

5. Seven ninths of the area of South Dakota is 60,340 square miles. Find the area of South Dakota.

6. Two thirds of the number of post offices in the United States in 1903 was 49,354. Find the number of post offices in the United States in that year.

7. Four ninths of the number of bales of cotton produced in the United States in 1903 was 4,767,804. Find the number of bales of cotton produced that year.

8. Find the average weight of a bale of cotton, if $\frac{11}{21}$ of the average weight of a bale is 253 pounds.

9. Five eighths of the world's production of sugar for 1903 was 6,134,445 tons. Find the world's production of sugar that year.

10. Eight elevenths of the distance by rail from Charleston, S.C., to San Francisco, Cal., is 2448 miles. Find the distance from Charleston to San Francisco.

Example 1. Two men are employed to do a piece of work, one receiving \$3 a day and the other \$2 a day. If the wages of both is \$95, find the number of days the two men worked.

Let $x =$ the number of days the men worked.

Then, $\$3\,x =$ the wages of one man.

$\$2\,x =$ the wages of the other man.

Therefore,

$\$3\,x + \$2\,x =$ the wages of both.

Therefore,

$$3x + 2x = 95.$$

Hence,

$$5x = 95.$$

$$x = 19.$$ *Ans.* 19 days.

Check. $\$3 \times 19 + \$2 \times 19 = \$95.$

EXERCISE 12

1. A father and his son work together, the father getting \$5 a day and the son \$2 a day. If the wages of both is \$112, how many days did they work?

2. A man buys the same number of horses and wagons. The horses cost \$75 apiece and the wagons \$37 apiece. If he spent all together \$1008, how many horses did he buy?

3. A dealer buys two kinds of coal, the same quantity of each, one kind at \$7 a ton and the other kind at \$5 a ton. If the entire purchase costs \$1788, how many tons of each does he buy?

4. I have the same number of nickels and quarters, and have \$1.80 all together. How many of each have I?

5. The arithmetics and the readers for a class cost \$6.75. If an arithmetic costs 25 cents, and a reader 20 cents, how many pupils are in the class?

6. A farmer sold the same number of turkeys and geese, the turkeys at \$2 apiece, and the geese at \$1 apiece. If the turkeys and geese brought \$75, how many of each did he sell?

7. A property owner builds a number of cottages at \$1500 apiece, and an equal number of cottages at \$900

apiece. If he spends all together $14,400, how many cottages of each kind does he build?

8. A clothier buys a number of suits at $5 apiece, and the same number of suits at $6 apiece. If the suits cost $2365, how many of each kind does he buy?

9. A shoe dealer buys a number of boys' shoes at $1 a pair, and an equal number of men's shoes at $3 a pair, paying all together $436. How many pairs of shoes of each kind does he buy?

10. A woman buys a number of yards of ribbon at 18 cents a yard, and the same number of yards of ribbon at 13 cents a yard, paying for both $9.30. How many yards of each kind does she buy?

REVIEW. EXERCISE 13

1. Divide $150 between two persons, giving one $36 more than the other.

2. Divide $210 between A and B, giving A five times as much as B.

3. A bookkeeper receives a salary at the rate of $165 per month. If he is paid his monthly salary in $5 and $10 bills, the same number of each, how many of each does he receive?

4. According to the census of 1900, the population of Alabama is 1,828,697. The male population exceeded the female population by 4831. Find the male and female population of Alabama.

5. The census of 1900 gives Columbus, Ohio, 442 persons more than twice the population of Lawrence, Mass.

The population of both cities that year was 188,119. Find the population of each city.

6. The population of Fall River, Mass., in 1900, was 5530 more than three times the population of Sioux City, Ia. The population of both cities in that year was 137,974. Find the population of each city.

7. The number of banks in New Mexico in 1903 was 35. There were four more state banks than private banks, and the number of national banks was one more than four times the number of private banks. How many banks of each kind were in New Mexico?

8. Four ninths of the number of post offices in the United States in 1902 increased by 256 is 34,000. Find the number of post offices in the United States in 1902.

EXERCISE 14

Perform the indicated operations:

1. $9 + 7 + 4$.
2. $8 + 9 + 6$.
3. $11 + 5 + 9$.
4. $12 - 5 + 3$.
5. $14 - 9 + 8$.
6. $27 - 11 - 9$
7. $a + 4a + 5a$.
8. $2a + 9a + 4a$.
9. $9x + 4x + 8x$.
10. $15y + 3y + 7y$.
11. $3m + 5m + 7m$.
12. $12n + 9n + 4n$.
13. $15m - 8m + 3m$.
14. $16y + 4y - 19y$.
15. $21y - 3y - 9y$.
16. $30x - 21x - 5x$.
17. $25m - 6m - 11m$.
18. $27x + x - 5x$.
19. $39x + x - 20x$.
20. $33x - 11x + 2x$.
21. $44a - 37a + a$.
22. $50a - 32a + 2a$.

23. $60a - 39a + 11a$.
24. $100x - 58x - 32x$.
25. $2x^2 + 3x^2 + 7x^2$.
26. $11x^2 + 7x^2 + 9x^2$.
27. $12x^2 + 9x^2 + 14x^2$.
28. $15x^2 + 4x^2 + x^2$.
29. $12xy + 7xy + 11xy$.
30. $16xy + 8xy - 20xy$.
31. $18ab + 10ab - 13ab$.
32. $40mn + 11mn - 50mn$.

EXERCISE 15

Find the value of the unknown quantity in each of the following equations:

1. $9x + 3x + 4x = 48$.
2. $5x + 9x + x = 60$.
3. $10x - 3x - 5x = 42$.
4. $9x - 4x + x = 48$.
5. $13x - 5x + x = 99$.
6. $11x - 8x + 4x = 56$.
7. $12y - 3y - 2y = 63$.
8. $19y - 8y + 2y = 143$.
9. $x + 5x - 2x = 64$.
10. $4x + 5x - 8x = 11$.
11. $17x + 7x - 12x = 132$.
12. $18x + 7x - 5x = 400$.
13. $24x - 5x + 3x = 352$.
14. $32y - 10y - 8y = 154$.
15. $35y - 18y - 14y = 297$.
16. $42x - 25x - 14x = 240$.
17. $38y - 13y - 12y = 247$.
18. $41y - y - 32y = 128$.
19. $3y + 4y + 6y + 3 = 146$.
20. $2y + 9y - 8y + 5 = 29$.
21. $6y + 4y - y + 8 = 80$.
22. $11y + 3y - 7y + 4 = 74$.
23. $12y + 9y - 17y + 11 = 47$.
24. $9y + 4y - 2y + 4 = 125$.
25. $6y + 5y - 3y + 9 = 57$.
26. $14y - 8y - 4y + 8 = 38$.
27. $25y - 11y - 9y + 7 = 97$.
28. $9x - 2x - 4x + 11 = 59$.

29. $14x - 11x + 2x + 7 = 57.$

30. $23x - 12x + 4x + 5 = 80.$

31. $39x + 4x - 23x + 9 = 109.$

32. $26x + 8x - 11x + 9 = 170.$

33. $54x - 40x + 6x + 10 = 410.$

34. $75y - 60y + y + 4 = 260.$

35. $37y - 13y + 5y + 11 = 98.$

36. $25y - 12y + 7y + 30 = 580.$

37. $32y - 18y + 3y + 11 = 300.$

38. $38y - 9y + 3y + 4 = 100.$

39. $100y - 79y + 4y + 5 = 230.$

40. $94y - 58y - 22y + 4 = 130.$

POSITIVE AND NEGATIVE NUMBERS

6. Hitherto the symbols plus and minus have been used as symbols of operation, plus denoting addition, and minus denoting subtraction. Their meaning will now be extended.

Suppose the mercury in a thermometer stands at 20° above zero. If the temperature falls 10°, the mercury will then stand at 10° above zero. This may be expressed algebraically

$$20° - 10° = 10°.$$

If the mercury in a thermometer is at 20° and the temperature falls 15°, the mercury will then stand at 5° above zero. This statement may be expressed algebraically

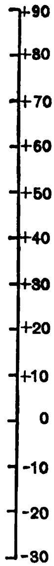
$$20° - 15° = 5°.$$

If the mercury in a thermometer is at 20° above zero and the temperature falls 20°, the mercury will then be at zero. This statement may be expressed algebraically

$$20° - 20° = 0°.$$

If the mercury in a thermometer is at 20° above zero and the temperature falls 30°, the mercury will then be at 10° below zero. This statement may be expressed algebraically

$$20° - 30° = -10°.$$



If the mercury in a thermometer is at 20° above zero and the temperature falls 25°, the mercury will stand at 5° below zero. This statement may be expressed algebraically

$$20° - 25° = -5°.$$

From an arithmetical point of view, 20 − 30 is an impossibility. It will be seen later that 20 − 30 is not an impossibility from an algebraic point of view.

In order to distinguish degrees above zero from degrees below zero the former are marked plus and the latter are marked minus. If no mark is written the plus sign is understood. There are then two styles of marking on a thermometer; one consisting of the natural series of numbers beginning with naught, extending upward, and each marked +, and the other beginning with naught, extending downward, and each marked −. Thus, there is

$$0, +1, +2, +3, +4, \text{etc., and}$$
$$0, -1, -2, -3, -4, \text{etc.}$$

Numbers marked + are called positive numbers, and numbers marked − are called negative numbers.

Suppose a boy has \$20 and owes \$10, how much is he worth? This solution may be indicated as follows:

$$\$20 - \$10 = \$10.$$

Suppose a boy has \$20 and owes \$25, how much is he worth? In this case he owes more money than he has. If he endeavors to settle his indebtedness by paying the \$20 he has, he will still owe \$5. This solution may be indicated as follows:

$$\$20 - \$25 = -\$5.$$

Accordingly what a person owes may be indicated by prefixing the minus sign to the amount he owes. If a person owes \$40, this may be indicated by $-\$40$, and if a person has \$40, this may be indicated by $+\$40$.

If a merchant gains \$50 one day, and loses \$30 the next day, what is his net gain for the two days?

This solution may be indicated as follows:

$$\$50 - \$30 = \$20.$$

If a merchant gains \$40 one day and loses \$55 the next day, what is his net loss for the two days?

This solution may be indicated as follows:

$$\$40 - \$55 = -\$15.$$

Losses may be designated by prefixing the minus sign to the amount lost, and, on the other hand, gains may be designated by prefixing the plus sign to the amount gained.

Similarly, money received may be designated as a positive quantity, and money paid out as a negative quantity. Thus, if $+\$10$ means \$10 received, $-\$10$ means \$10 paid out.

From the above illustrations it may be seen that the signs + and − prefixed to quantities denote opposites. For example, if +5 miles means 5 miles east, then −5 miles means 5 miles west. If +10° means 10° north of any particular point on the earth's surface, then −10° means 10° south of the same place.

Counting forward is regarded as positive, and counting backward is regarded as negative. Beginning with 0 and counting forward, the natural number series is formed. Thus, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, etc.

Counting backward, beginning, say, with 5, there is the following series:

$$-7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

7. Positive numbers are known as **arithmetic numbers.**

Positive and negative numbers are known as **algebraic numbers.**

Plus and minus when used in the sense above indicated are symbols of quality, and have the force of adjectives.

EXERCISE 16

Solve the following problems, indicating in each instance the algebraic solution:

1. A section laborer on a railroad goes in the forenoon 10 miles north, and in the afternoon 4 miles farther north. How far is he, and in what direction, from the starting point? *Ans.* $10 + 4 = +14$.

2. A section laborer goes 10 miles north, and afterwards 8 miles south. How far, and in what direction, is he from his starting point?

3. A person travels on Monday 300 miles north, and

on Tuesday 180 miles south. How far, and in what direction, is he from his starting point?

4. If a man travels on Wednesday 100 miles north and on Thursday 130 miles south, how far, and in what direction, is he from his starting point?

5. If a merchant loses \$40 one day and \$50 the next day, how much does he lose in the two days?

6. A boy has 40 cents and spends 25 cents. How many cents has he then?

7. A merchant's assets are \$1500 and his liabilities are \$2000. How much is he worth?

8. A man travels 150 miles east, and afterwards 200 miles west. How far, and in what direction, is he from his starting place?

9. If a man owes \$50 to one creditor and \$25 to another creditor, how much does he owe the two creditors?

10. A train travels 90 miles south, and afterward 100 miles north. How far, and in what direction, is the train from its starting place?

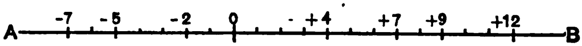
11. A coal dealer buys on Monday 200 tons of coal. He sells on Tuesday 15 tons and on Wednesday 25 tons. How many tons has he remaining?

12. A merchant buys 150 pounds of tea. He sells at one time 30 pounds, and at another time 35 pounds. How many pounds has he remaining?

8. *Example* 1. Represent graphically the following numbers:

$$+4,\ +7,\ +9,\ +12,\ -2,\ -5,\ -7.$$

To do this, take a straight line AB, and mark a point on it 0. Then take any convenient unit, and with a compass mark it as indicated.



EXERCISE 17

Represent graphically the following numbers:

1. 3, 6, 9, 14, -6, -10.
2. 5, 10, 15, 21, -8, -15.
3. 6, 12, 14, 17, 0, -4, -11, -13.
4. 45, -24, 25.
6. 48, 72, -48.
5. 64, -36, -50.
7. 80, -30, -70.
8. 90, -40, -60.

EXERCISE 18

1. Suppose the temperature at midnight is 40° Fahrenheit, and the following forenoon it is 56°. How many degrees does it rise?

2. If the mercury in a thermometer stands at 38° and during the next 24 hours it falls to 24°, how many degrees does it fall?

3. If the mercury in a thermometer stands at 12° one day and at 28° the next day, how many degrees does the mercury rise?

4. If the mercury stands at $-10°$ and rises to $+8°$, how many degrees does it rise?

5. If the mercury stands at $-4°$ and rises to $+20°$, how many degrees does it rise?

6. A ship sails due north from a point 8° south lati-

tude to a point 23° north latitude. How many degrees does it sail?

7. How many degrees are there from a point 18° east longitude to a point in the same latitude 78° west?

8. How many dollars must a man make whose property is valued at $-\$200$ so as to have $+\$300$?

9. A merchant loses on Monday \$40. How many dollars must he gain on Tuesday so as to have a net gain of \$30 for two days?

10. How many years from 430 B.C. to 1904 A.D.?

11. How many years from 150 B.C. to 1876 A.D.?

12. The date of the Second Punic War is 218 B.C. How many years from that date to the present time?

13. A steamer whose rate is 19 miles an hour travels up a river whose rate is 3 miles an hour. Find the rate of the steamer up the river.

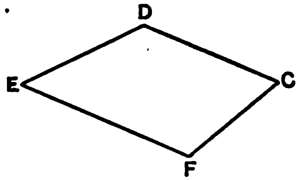
14. The following table gives the highest and lowest temperatures in degrees Fahrenheit recorded at the Weather Bureau Stations named up to December, 1902. Find in each case the range of temperature:

		Highest	Lowest
1.	Augusta, Ga.	105°	3°
2.	Baltimore, Md.	104°	– 7°
3.	Bismarck, N.D.	106°	– 44°
4.	Buffalo, N.Y.	95°	– 14°
5.	Cincinnati, O.	105°	– 17°
6.	Denver, Col.	105°	– 29°
7.	Eagle, Alaska	87°	– 68°
8.	El Paso, Texas	113°	– 5°
9.	Fresno, Cal.	114°	20°
10.	Hannibal, Mo.	108°	– 20°

		Highest	Lowest
11.	Helena, Mont.	103°	– 42°
12.	Galveston, Texas	98°	8°
13.	Kansas City, Mo.	106°	– 22°
14.	La Crosse, Wis.	104°	– 43°
15.	Jacksonville, Fla.	104°	10°
16.	Minneapolis, Minn.	102°	– 33°
17.	Nashville, Tenn.	104°	– 13°
18.	North Platte, Neb.	107°	– 35°
19.	Port Huron, Mich.	99°	– 25°
20.	Yuma, Ariz.	118°	20°

DEFINITIONS

9. Anything which has size or extent is called a **magnitude.** For example, the line *AB*, or the figure *EFCD*.



10. The result of measuring a magnitude is called a **number** or **quantity.** For example, if the line *AB* is measured and found to be 2.53 inches, this result, 2.53, which tells how many times the line *AB* contains another line one inch long, is called a number or quantity.

11. Any collection of symbols representing number is called an **expression.** For example, $x + 2y - 3$ is an expression.

12. The members of an expression connected by the signs plus or minus or both are called the **terms.** For example, in the expression $3x + 5y - z$, the terms are $3x$, $+5y$, $-z$.

13. An expression consisting of one term is called a **monomial.** For example, $6abc$ is a monomial.

14. An expression consisting of more than one term is called a **polynomial.** For example, $x^2 + 5xy - 6y^2$ is a polynomial.

15. A polynomial consisting of two terms is called a **binomial.** For example, $3x - 2y$ is a binomial.

16. A polynomial of three terms is called a **trinomial.** For example, $4a + 5b - 8c$ is a trinomial.

17. The degree of a term with reference to a letter or letters is the exponent of the letter or the sum of the exponents of the letters in the term. For example, the degree of $9x^2y^3$ is 5, that is, the sum of 2 and 3.

The degree of an algebraic expression is the highest degree of any of its terms. For example, in the expression $x^3 + 5x + 9$, the highest degree is 3, and hence this expression is of the third degree.

18. Terms whose literal factors are of the same degree are called **like terms.** For example, $9x^3$, $5x^3$, are like terms. $7x^2y$, $4x^2y$, are also like terms. $4x^2y$ and $5xy^2$ are not like terms, because the factors of these terms have not the same exponents.

19. The product obtained by multiplying a number by itself a number of times is called in arithmetic a **power** of that number. For example, 8 is the cube or third power of 2. 81 is the fourth power of 3. In algebra, power means this much and something more which will be referred to later.

20. The number which multiplied by itself a number of times produces another number is called a **root** of that number. For example, 5 is the square root of 25, since $5 \times 5 = 25$. 6 is the cube root of 216, since $6 \times 6 \times 6 = 216$. 2 is the fifth root of 32, since $2 \times 2 \times 2 \times 2 \times 2 = 32$. Square root of a number is indicated by $\sqrt{\ }$. Cube root by $\sqrt[3]{\ }$. Fourth root by $\sqrt[4]{\ }$.

CHAPTER II

THE FOUR SIMPLE RULES. FUNDAMENTAL LAWS

ADDITION

21. Addition is the process of combining numbers into a single number. The result in addition is called the **sum.**

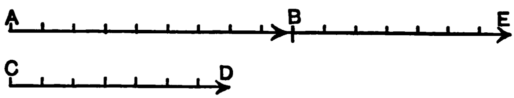
22. To add two arithmetical numbers, count forward from the first number as many units as there are units in the second number. Thus, to add six and five, count forward from six as many units as there are in five. To add three arithmetical numbers, add the first two, and then add this result and the third number.

Example 1. What is the sum of $+9$ and $+7$?

The answer may be stated in symbols as follows:

$$+9+7=+16.$$

The process may be represented graphically as follows:



Let $AB=+9$, and $CD=+7$. To add these two lines, place the beginning point of the second line at the end point of the first line. Then AE will be the sum. Next, count the marks from A to E. The result will be the sum.

Example 2. Add -6 and -5.

This problem and result may be stated in symbols as follows:

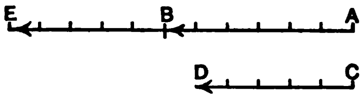
$$-6+(-5)=-11,$$

or

$$-6-5=-11.$$

The solution may be represented graphically as follows:

Let $AB=-6$, and $CD=-5$. To add AB and CD, place the beginning point of CD at the end point of AB, and then count the marks from A to E.



To add two positive numbers, add the numbers regardless of sign, and prefix the plus sign to the result.

To add two negative numbers, add the numbers regardless of sign, and prefix the minus sign to the result.

The value of a number, regardless of its sign, is called the **absolute value** of the number. Hence,

The sum of two numbers affected by the same sign equals the sum of the absolute values of the numbers with the common sign prefixed.

23. Addition of numbers with unlike signs.

Example 1. Suppose a man travels 100 miles north one day, and 80 miles south the next day, how far is he from his starting point?

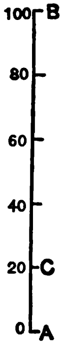
This question may be expressed in symbols.

$$+100 \text{ miles} + (-80 \text{ miles}) = ?$$

or

$$+100 \text{ miles} - 80 \text{ miles} = ?$$

Obviously the man travels 20 miles more in a northern direction than in a southern direction, and hence he is 20 miles north of his starting point. Therefore,



$$+100 \text{ miles} + (-80 \text{ miles}) = +20 \text{ miles}.$$

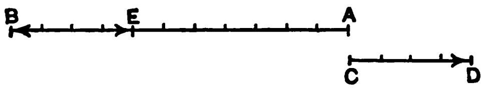
To represent this graphically, let AB represent 100 miles, and BC represent 80 miles. Then $AC = 20$ miles.

Example 2. What is the sum of -11 and $+4$?

SOLUTION. Counting four units to the right from -11 the result is -7. This may be expressed in symbols as follows:

$$-11 + 4 = -7.$$

This may be represented graphically as follows: $AB = -11$, $CD = +4$. To get the sum, place the



beginning point of CD at the end point of AB, then AE will be the sum.

To add two numbers, one positive and the other negative, take the difference of their absolute values and to it prefix the sign of the number greater in absolute value.

EXERCISE 19

Add and represent graphically in each case:

1.	2.	3.	4.	5.	6.	7.
$+4$	$+3$	$+11$	$+14$	$+16$	$+19$	$+25$
$+9$	$+11$	$+7$	$+8$	$+12$	$+6$	$+18$

8.	9.	10.	11.	12.	13.	14.
-12	-9	-6	-4	-17	-21	-29
-8	-15	-13	-35	-22	-28	-24

15.	16.	17.	18.	19.	20.	21.
$+11$	$+13$	$+17$	$+25$	$+32$	$+27$	$+31$
-7	-8	-15	-25	-35	-38	-40

22.	23.	24.	25.	26.	27.	28.
-10	-16	-22	-28	-21	-30	-39
$+4$	$+11$	$+17$	$+26$	$+33$	$+37$	$+41$

24. Addition of three or more numbers.

Example 1. Suppose a man makes \$14 Monday, \$18 Tuesday, loses \$20 Wednesday, and \$25 Thursday, does he gain or lose in the four days, and how much?

SOLUTION. At the end of Tuesday he has made \$32. At the end of Wednesday he has made \$32 − \$20, or \$12. At the end of Thursday he has made \$12 − \$25, or − \$13. In the four days he loses \$13.

This solution may be expressed in symbols.

$$+\$14+\$18-\$20-\$25=-\$13.$$

As far as the total loss is concerned, it does not matter on which of the four days the man lost the \$20 and the \$25, nor on which days he made the \$18 and the \$14. Hence, it may be inferred generally that the result of adding several numbers is the same no matter in what order the numbers may be arranged. From the same illustration it may also be inferred that numbers may be grouped in any manner and the sum will remain the same. Thus, \$14 and \$18 may be added, and − \$20 and − \$25, and then the sum of these two results taken for a final result.

EXERCISE 20

1. $25 - 9 + 7 = ?$
2. $16 - 10 + 4 = ?$
3. $13 - 9 - 7 = ?$
4. $26 - 11 - 5 = ?$
5. $34 - 14 - 10 = ?$
6. $15 - 10 - 8 = ?$
7. $22 - 9 - 14 = ?$
8. $32 - 20 - 9 = ?$
9. $17 - 19 + 5 = ?$
10. $35 - 26 - 14 = ?$
11. $37 - 15 - 26 = ?$
12. $31 - 21 - 14 = ?$
13. $-12 - 5 - 16 = ?$
14. $-22 + 5 + 18 = ?$
15. $-32 - 9 + 50 = ?$
16. $-14 - 7 + 30 = ?$
17. $-24 - 7 + 11 + 19 = ?$
18. $-34 - 7 + 37 - 9 = ?$
19. $-36 - 8 - 32 - 33 - 9 + 56 = ?$
20. $-18 + 24 - 27 + 31 - 9 = ?$

EXERCISE 21

Find the value of :

1. 2^3; 3^2; 2^4; 3^3; 4^2; 4^3; 7^2; 2^7; 7^3; 3^7.
2. 3×2^2; 2×3^2; 3×2^3; 2×3^3; 5×4^3; 4×5^3.
3. $2^2 + 2$; $2^3 + 2$; $2^3 + 2^2$; $3^2 + 3^3$; $4^2 + 4^3$; $5^3 + 5$.
4. $2^4 - 2^3$; $3^3 - 3$; $4^2 - 3^2$; $5^3 - 4^3$; $6^2 - 6$; $7^3 - 7$.
5. $7^2 - 6^2$; $9^2 - 5^2$; $11^3 - 9^3$; $2^4 - 1$; $5^4 - 1$.
6. $3 \times 7^2 - 2 \times 6^2$; $5 \times 8^3 - 8 \times 5^3$; $4 \times 10^3 - 5 \times 11^3$.
7. $3^2 \times 2^2$; $2^3 \times 4^2$; $4^2 \times 3^3$; $5^2 \times 3^2$; $6^2 \times 2^3$.
8. $\frac{2^3}{2^2}$; $\frac{3^3}{3^2}$; $\frac{5^2}{2^3}$; $\frac{6^3}{3^3}$; $\frac{8^3}{4^4}$; $\frac{9^3}{3^5}$.

25. Evaluation of algebraic expressions.

Example 1. If $a = 7$, $b = 3$, $c = 5$, find the value of

$$4a^3 - 6b^2 - 2c^4.$$

SOLUTION. This problem is solved by writing 7 for a, 3 for b, and 5 for c.

Therefore, $4a^3 - 6b^2 - 2c^4 = 4 \times 7^3 - 6 \times 3^2 - 2 \times 5^4$

$$= 1372 - 54 - 1250 = 68.$$

EXERCISE 22

If $a = 4$, $b = 2$, $c = 1$, find the value of:

1. $a + b + c$.
2. $a - b + c$.
3. $a - b - c$.
4. $4a + 2b + 7c$.
5. $6a + 11b + 9c$.
6. $5a - 10b + 2c$.
7. $9a - 4b - 11c$.
8. $19a - 17b - 6c$.
9. $17a - 25b - 13c$.
10. $15a - 11b - 10c$.
11. $a^2 + b^2 + c^2$.
12. $a^2 + ab + b^2$.
13. $a^2 - 2ac + c^2$.
14. $b^2 - 2bc + c^2$.
15. $2a^2 - 3ab + 3c^2$.
16. $ab + ac + bc$.
17. $ab - ac + bc$.
18. $a^2 + b^2 - c^2$.
19. $a^3 - a^2 + a$.
20. $b^3 - b^2 - b$.
21. $a^3 + b^3 + c^3$.
22. $2a^2 + 3b^2 + 5c^2$.

If $a = 10$, $b = 5$, $c = 6$, find the value of:

23. a^2b.
24. ab^2.
25. ac^2.
26. a^2c.
27. a^2b^2.
28. a^2c^2.
29. abc.
30. a^2bc.

31. ab^2c. **35.** ab^2c^2. **39.** ac^3. **43.** $b^3 - bc^2$.

32. abc^2. **36.** $a^2b^2c^2$. **40.** a^3c. **44.** $5a^2 - 5b^2$.

33. a^2b^2c. **37.** a^3b. **41.** $a^2 - 2ab$. **45.** $6c^2 + 7b^2$.

34. a^2bc^2. **38.** ab^3. **42.** $a^3 - a^2b$. **46.** $3a^3 - 10b^3$.

26. Addition of monomials.

Example 1. Add \$5, \$7, −\$8, −\$9.

SOLUTION. First, find the sum of 5 and 7; second, find the sum of −8 and −9; third, find the sum of these two results. $5 + 7 = 12$. $-8 + -9 = -17$. $12 - 17 = -5$. *Ans.* −\$5.

Example 2. Find the sum of $5x$, $7x$, $-8x$, $-9x$.

Here x is merely substituted for dollars. Hence, the sum is $-5x$.

Example 3. Add $5xy$, $7xy$, $-8xy$, $-9xy$.

Here xy takes the place of dollars in the first example. Hence, the sum is $-5xy$.

Like monomials are added by taking the sum of the coefficients of the positive terms, and of the negative terms separately, and then writing the algebraic sum of these two results before the common symbol.

EXERCISE 23

Add:

1. $4a$, $3a$, $-7a$.
2. $7b$, $-4b$, $6b$.
3. $9b$, $-6b$, $-4b$.
4. $11x$, $-8x$, $-4x$.
5. $10c$, $-7c$, $-8c$.
6. $6c$, $-3c$, $8c$.
7. $10c$, $-8c$, $-5c$.
8. $9c$, $-3c$, $+14c$.
9. $-12c$, $9c$, $-19c$.
10. $15a$, $-6a$, $3a$, $-4a$.

11. $12\,xy,\ -4\,xy,\ -5\,xy,\ +3\,xy.$

12. $15\,xy,\ -11\,xy,\ 7\,xy,\ -xy.$

13. $17\,ab,\ -4\,ab,\ 12\,ab,\ -3\,ab.$

14. $20\,ab,\ -14\,ab,\ -7\,ab,\ -ab.$

15. $23\,mn,\ -15\,mn,\ -11\,mn,\ +4\,mn.$

16. $ax,\ -7\,ax,\ -2\,ax,\ -5\,ax.$

17. $-by,\ -5\,by,\ 6\,by,\ 8\,by.$

18. $11\,ab,\ -5\,ab,\ 14\,ab,\ -4\,ab.$

19. $-8\,a^2,\ -12\,a^2,\ 27\,a^2.$

20. $-18\,a^2,\ +9\,a^2,\ +14\,a^2,\ -8\,a^2.$

27. Addition of polynomials.

Example 1. Add a, b, and $-c$.

Since these terms are unlike, the addition can be only indicated. Hence, the answer is $a+b-c$.

Example 2. Add $4\,x^2+2\,xy-y^2$, $6\,xy-x^2-4\,y^2$, $5\,y^2-3\,xy+2\,x^2$, $6\,y^2-x^2-5\,xy$.

Since the sum of a number of terms is the same no matter in what order they are written, the terms can be arranged in any way. The most convenient arrangement is to have the like terms standing in the same vertical column. Rewriting the expressions, placing like terms in the same vertical column, there is the following arrangement:

$$\begin{array}{rrr} 4\,x^2 & +\,2\,xy & -\quad y^2 \\ -\,x^2 & +\,6\,xy & -\,4\,y^2 \\ 2\,x^2 & -\,3\,xy & +\,5\,y^2 \\ -\,x^2 & -\,5\,xy & +\,6\,y^2 \\ \hline 4\,x^2 & & +\,6\,y^2 \end{array}$$

The coefficients of the terms in x^2 are 4, -1, $+2$, -1. The sum of these coefficients is $+4$. Write $4\,x^2$. The coefficients of the terms in xy are 2, 6, -3, -5. The sum of these coefficients is 0. The coefficients of the terms in y^2 are -1, -4, $+5$, $+6$. The sum of these coefficients is $+6$. Write $+6y^2$. The answer is $4x^2+6y^2$.

EXERCISE 24

Add:

1. $a-b,\ a+b.$
2. $2a-b,\ 2a+b.$
3. $3x+3y,\ 6x-3y.$
4. $4x+5y,\ 5x+4y.$
5. $9x+2y,\ 11x-6y.$
6. $-6x+4y,\ 2x-5y.$
7. $-9x+3y,\ 4x+7y.$
8. $a+b+c,\ a+b-c,\ a-b+c.$
9. $a-b-c,\ a+b-c,\ b+c-a.$
10. $2a+3b+5c,\ 3a-2b-c,\ 5a-4b-c.$
11. $3a-3b+6c,\ 6a+3b-2c,\ 4a-5b-4c.$
12. $7a-6b-9c,\ 5a-b-c,\ 8a-3b-4c.$
13. $15a-7b-7c,\ 4a-9b-7c,\ 4a-5b+9c.$
14. $-16a-11b+10c,\ 6a-6b-8c,\ 7a-7b-7c.$
15. $-11a+4b+5c,\ 8a-5b-9c,\ 4a-4b-11c.$
16. $x^2-xy+y^2,\ 3x^2-3xy-7y^2,\ 4x^2-7xy-8y^2.$
17. $3x^2-6xy+3y^2,\ 8x^2-15xy+6y^2,\ 7x^2+7xy+9y^2.$
18. $a^2+a-4,\ 3a^2-3a+9,\ 9a^2-8a-10.$
19. $7x^2+7x-7,\ 9x^2+8x-13,\ -10x^2+5x+8.$
20. $5x^2+4x+4,\ 8x^2-5x+6,\ -10x^2-10x-10.$
21. $6x^2+7x+9,\ 5x^2+11x-8,\ 9x^2-8x-8.$
22. $4a^2-9ab-10b^2,\ 5a^2-10ab+11b^2,\ 8a^2-15b^2,$ $a^2+ab+b^2.$

23. $a^2 + a + 1$, $a^2 - a + 5$, $11 + a - a^2$, $5a^2 - 8$.

24. $x^2 + xy + yz$, $x^2 - xy + 4yz$, $3yz - x^2 - xy$, $5x^2 - 4xy - 6yz$.

25. $a - b$, $b - c$, $c - a$, $a + b + c$.

26. $a^2 + b^2 + c^2$, $a^2 - b^2 - c^2$, $c^2 - a^2 - b^2$, $b^2 - a^2 - c^2$.

27. $4mn - 5my - 6mz$, $9mn + 6my + 3mz$, $8mn - 4mz$, $9my - 4mn$.

28. $5ab - 6ac - 7cd$, $-8ab - 7ac + 11cd$, $4ab + 2ac - 3cd$, $ab + 11ac - cd$.

29. $3a^3 + 2a^2 + 4a$, $7a^3 - 11a^2 - 11a$, $-2a^3 + 14a$, $-8a^3 + 5a^2 + 6a$.

30. $3x^3 - 3xy + y^3$, $5x^3 - 8xy - 5y^3$, $9y^3 - 7x^3 + 4xy$, $2y^3 - 5x^3 - 3xy$.

31. A man owns property worth \$2000. He has a bank account of \$783, and he has owing to him \$285. He owes a note for \$850, and miscellaneous items amounting to \$250. How much is he worth?

32. A passenger agent travels on Monday 420 miles north, on Tuesday 180 miles north, on Wednesday 350 miles south, and on Thursday 275 miles south. How far is he from his starting point?

33. A man walks 15 miles east, then 5 miles west, then 12 miles east, and then 27 miles west. Where is he with reference to his starting point?

34. The mercury in a thermometer rises 10° on Monday, 12° on Tuesday, 7° on Wednesday, falls 17° on Thursday, and falls 6° on Friday. What is the temperature on Friday?

SUBTRACTION

28. In addition two or more numbers are given and their sum is required. In subtraction the sum of two numbers and one of the numbers are given and the other number is sought. The given sum in the subtraction exercise is called the **minuend**, and the given number is called the **subtrahend**. The number sought is called the **remainder**, or **difference**.

Example 1. A man buys a pair of shoes for $3.75, and gives in payment a five-dollar bill. How much change does he receive?

A clerk will solve this problem as follows: Counting from $3.75 to $4, there is 25¢, and from $4 to $5, there is $1. Hence, the change is $1.25.

Example 2. The mercury in a thermometer stands at 10°. How many degrees must it rise so that it may stand at 70°?

To solve this problem, count from 10° to 70°. Since the counting is forward, the remainder will be positive. Hence, the answer is 60°.

This solution may be expressed algebraically

$$70° - (+10°) = +60°.$$

Example 3. The mercury in a thermometer stands at 20°. How many degrees must it fall to stand at 13°?

To solve this problem, count backward from 20° to 13°. The result is a fall of 7°. This solution may be expressed algebraically

$$13° - (+20°) = -7°.$$

To verify the answer, add $-7°$ to $+20°$.

Example 4. The mercury in a thermometer stands at $-10°$. How many degrees must it rise to stand at $+6°$?

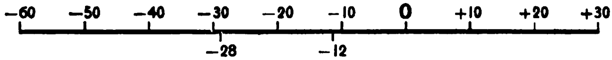
SOLUTION. Count from $-10°$ to $+6°$. Since the counting is forward, the result will be positive. Hence, the answer is 16°.

This process may be stated in symbols as follows:

$$6° - (-10°) = +16°.$$

Example 5. Subtract -28 from -12.

First, write the series of positive and negative numbers as far as necessary.



Next, count from -28 to -12. The result will be the answer. As the counting is forward, the answer will be positive. From 12 to 28 there are sixteen marks. Hence the answer is $+16$.

This process may be represented algebraically as follows:

$$-12 - (-28) = +16.$$

Check. $-28 + 16 = -12$.

EXERCISE 25

Give results and represent graphically the following:

	1.	2.	3.	4.	5.	6.
From	40	45	56	15	12	5
Take	20	15	10	4	−10	−30

	7.	8.	9.	10.	11.	12.
From	−6	−10	−20	−40	−25	20
Take	−16	−20	−10	−24	− 5	40

13. A man has $100. How much must he lose to have $-$100$?

14. A man is worth $-$20$. How much must he earn in order to have $+$80$?

15. A man is worth $-$30$. How much must he earn in order to have $+$60$?

29. *Example* **1.** Subtract $x+7$ from $4x$.

SOLUTION. If x is taken from $4x$, the remainder is $3x$. If a number greater than x is taken from $4x$, the remainder will be as many less as the number is greater. Hence, the remainder, when $x+7$ is taken from $4x$, will be $3x-7$.

This may be represented algebraically

$$4x-(x+7)=4x-x-7=3x-7.$$

Example **2.** Subtract $a-5$ from $7a$.

SOLUTION. If a is taken from $7a$, the remainder will be $6a$. Hence, if $a-5$ is taken from $7a$, the remainder will be 5 greater. Therefore, the remainder is $6a+5$.

This process may be represented algebraically as follows:

$$7a-(a-5)=7a-a+5=6a+5.$$

A little consideration of Examples 1 and 2 will enable one to see that an exercise in subtraction in algebra can be most readily performed by **changing the signs of the terms of the subtrahend and then adding.**

EXERCISE 26

Subtract:

1. $4x+9$ from $10x+12$.

2. $3x-5$ from $5x-4$.

3. $4x-3$ from $7x+7$.

4. $5a+b$ from $7a-2b$.

5. $a - b$ from $a + b$.

6. $2a - 3b$ from $5a - 7b$.

7. $9m - 5n$ from $8m + 6n$.

8. $10x + 9y$ from $9x + 18y$.

9. $xy - yz$ from $6xy - 3yz$.

10. $4xy - 9xz$ from $9xy - 8xz$.

11. $12mn - 15xy$ from $8mn - 4xy$.

12. $5ab - 13bc$ from $7ab - 11bc$.

13. $ad - ac$ from $7ad - 12ac$.

14. $3x - 5y + 4z$ from $6x - 9y - 2z$.

15. $2x - 2y - z$ from $x - 4y - 4z$.

16. $x^2 - 2x - 2$ from $5x^2 - 3x - 5$.

17. $4x^2 - 7x + 5$ from $3x^2 + 2x + 9$.

18. $x^2 - 2xy + y^2$ from $2x^2 + 4xy + 2y^2$.

19. $3y^2 + 3yz + z^2$ from $7y^2 - 3yz + z^2$.

20. $4x^3 - 5x^2 - 10x$ from $x^3 + 9x^2 + x$.

30. Removal of parentheses.

When several terms are taken collectively and subjected to an operation, these terms are inclosed in a parenthesis. For example, $a + (b + c)$ means that the sum of b and c is to be added to a. $a - \{b - c\}$ means that the difference of b and c is to be subtracted from a. $a(b + c)$ means a times the sum of b and c.

31. *Example* 1. Simplify $a + (3a - 5)$.

Here $3a - 5$ is to be added to a. Hence, $a + (3a - 5)$ $= a + 3a - 5 = 4a - 5$. Therefore, if a parenthesis is affected with the plus sign, it may be removed, and each of its terms written with its proper sign.

Example 2. Simplify $4a-(2a-5)$.

Here $2a-5$ is to be taken from $4a$. This is done by changing the sign of each term of $2a-5$ and then combining as in addition. Hence, $4a-(2a-5)=4a-2a+5=2a+5$.

If a parenthesis is preceded by the minus sign, remove the parenthesis, taking care to change the sign of each term within it.

If a parenthesis is preceded by the plus sign, remove the parenthesis, writing each term with its original sign.

EXERCISE 27

Simplify:

1. $3x+2+(5x-4)$.
2. $7x-6+(3x-5)$.
3. $9x-11+(11-4x)$.
4. $6x-12-(10-3x)$.
5. $5x-6-(4x-2)$.
6. $9a+3b-(7b-8a)$.
7. $6a+4b-(2b-4a)$.
8. $x^2+x-(2x-x^2)$.
9. $(2a+3b+5)-(a-4b-2)$.
10. $(3a-5b+6)+(5b+5-4a)$.
11. $(a-b)-(b+c)-(c-a)$.
12. $(2x-4)-(3x+5)-(7-6x)$.
13. $(4n-m)-(3n-2m)+(m+n)$.
14. $(5x-3y)+(7y-3x)-(2x-8y)$.
15. $(5a-2b)-(9b-4a)+(b+5a)$.
16. $(6a+4b)-(5b+2a)+(6b-7a)$.
17. $(x-y)-(x-3y)-(2x-7y)$.
18. $(3x-8)-(5x-7)+(11-4x)$.
19. $(x-4)-(4-x)-(2x-8)$.
20. $(9y-14)+(2y-11)-(10y-25)$.

MULTIPLICATION

32. In arithmetic multiplication of whole numbers is repeated addition. 5 times 7 means $7 + 7 + 7 + 7 + 7$.

In a multiplication problem the multiplicand is taken as addend as many times as there are units in the multiplier.

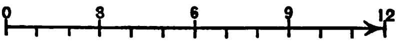
In order to understand the principles underlying the multiplication of algebraic numbers, consider the following illustrations:

(1) A man walks at the rate of 3 miles an hour in an easterly direction. How far will he go in 4 hours?

Obviously, he will go 4 times 3 miles, or 12 miles.

Distance in an easterly direction may be regarded as positive, and time to come, also, as positive. Therefore, this problem may be algebraically stated

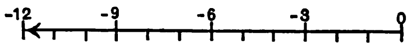
$$(+4) \times (+3) = +12.$$[1]



(2) If a man walks at the rate of 3 miles an hour in a westerly direction, how far will he go in 4 hours?

In this problem the man walks 3 miles to the west each hour. This may be regarded as -3 miles an hour. Hence,

$$(+4) \times (-3) = -12.$$



(3) A man traveling at the rate of 3 miles an hour goes in an easterly direction. Where was he 4 hours ago?

[1] For the symbol × read times.

Obviously, he was 12 miles west of his position now. Hence,

$$(-4)\times(+3)=-12.$$

(4) A man travels in a westerly direction at the rate of 3 miles an hour. Where was he 4 hours ago?

Obviously, he was 12 miles east of his position now. Hence,

$$(-4)\times(-3)=+12.$$

Illustrations such as the above will enable one to justify the following definitions:

To multiply $+7$ by $+5$ means to take $+7$ five times as addend.

To multiply -7 by $+5$ means to take -7 five times as addend.

To multiply $+7$ by -5 means to take $+7$ five times as addend, and then to change the sign of the result.

To multiply -7 by -5 means to take -7 five times as addend, and then to change the sign of the result.

These four results are,

(1) $(+5)\times(+7)=+35.$

(2) $(+5)\times(-7)=-35.$

(3) $(-5)\times(+7)=-35.$

(4) $(-5)\times(-7)=+35.$

Hence, in multiplication **like signs give plus, unlike signs give minus.** This is the *Law of Signs for Multiplication.*

33. Perform the following indicated operations:

$$2 \cdot 3 \cdot 5 \cdot 7$$
$$5 \cdot 3 \cdot 2 \cdot 7$$
$$7 \cdot 3 \cdot 5 \cdot 2$$
$$7 \cdot 5 \cdot 3 \cdot 2$$

Illustrations like the above will enable one to see that the order in which numbers are multiplied is immaterial. This principle is known as the *Commutative Law.*

34. Perform the following indicated operations:

$$(2 \cdot 3) \times (5 \cdot 7)$$
$$(2 \cdot 5) \times (3 \cdot 7)$$
$$(2 \cdot 7) \times (3 \cdot 5)$$
$$(2 \cdot 3 \cdot 5) \times 7.$$

When several numbers are to be multiplied, the numbers may be grouped in any manner, and the product remains the same. This principle is known as the *Associative Law* for multiplication.

EXERCISE 28

Express in symbols and represent graphically:

1. The mercury in a thermometer rises on each of three successive days 6°. How many degrees higher was the mercury in the thermometer at the end of the third day?

2. The mercury in a thermometer falls on each of four successive days 5°. How many degrees did the mercury fall in the four days?

3. The mercury in a thermometer rises 5° on each of three successive days. How many degrees lower was the mercury in the thermometer three days ago than it is now?

4. The mercury in a thermometer falls 5° on each of four successive days. How many degrees higher was the mercury in the thermometer four days ago than it is now?

5. A man earns $3 a day. How much more is he worth to-day than he was worth five days ago?

6. A merchant loses on an average $10 a day. How much more was he worth six days ago than he is worth to-day?

7. A merchant makes $12 a day. How much more is he worth to-day than he was worth five days ago?

8. A man travels due south at the rate of 9 miles an hour. Where is he, with reference to his starting point, five hours hence?

9. Interpret the following symbols:

(*a*) $3 \times \$9$,

(*b*) $(-3) \times \$8$,

(*c*) $7 \times (-\$9)$,

(*d*) $(-6) \times (-\$4)$.

35. *Example* 1. Multiply $\frac{2}{3} a^2$ by 12.

SOLUTION. $\frac{2}{3} a^2 \times 12 = \frac{2}{3} \times 12 \times a^2 = 8 a^2$.

Commutative Law.

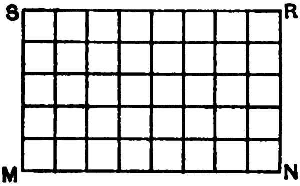
Example 2. Multiply $7\frac{3}{4}$ by 10.

$$10 \times 7\tfrac{3}{4} = (10 \times 7) + (10 \times \tfrac{3}{4}) = 70 + 7\tfrac{1}{2} = 77\tfrac{1}{2}.$$

Example 3. Find the product of 14 by 98.

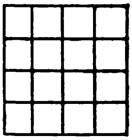
$$14 \times 98 = 14(100 - 2) = 1400 - 28 = 1372.$$

Consider the rectangle *MNRS*. Suppose its length is eight units and its breadth is five units; then its area is 8×5 times the unit of square measure.



In a similar manner, if the dimensions of a rectangle are 17 units by 13 units, the area of this rectangle will be 17×13 times the unit of square measure. And, in general, if a and b be the dimensions of a rectangle, the **number** of units in its area will be ab.

A rectangle may, therefore, graphically represent the product of two numbers.

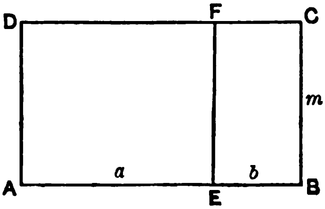


If the number of units in the side of a square be four, the number of units in its area will be 4^2. And, in general, if the side of a square contains a units, its area will contain a^2 units of square measure. For this reason *the second power of a number is called the square of a number.*

Example 4. Multiply $a+b$ by m.

$a+b$	(1) Multiply a by m.
m	(2) Multiply b by m.
$ma+mb$	(3) Write the algebraic sum.

Geometric Proof.



Let $AE = a$,

$EB = b$.

Upon AB construct a rectangle so that $BC = m$. At E erect a perpendicular EF to AB. Then,

$$ABCD = m(a+b),$$

$$AEFD = ma,$$

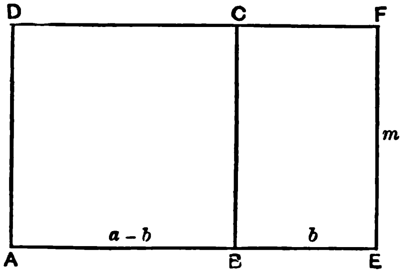
$$EBCF = mb.$$

Also, $$ABCD = AEFD + EBCF.$$

Hence, $$m(a+b) = ma + mb.$$

Example 5. Multiply $a-b$ by m.

$a-b$	(1) Multiply a by m.
m	(2) Multiply $-b$ by m.
$ma-mb$	(3) Write the algebraic sum.



Geometric Proof.

Let $AE = a$.

$EB = b$.

Upon AE construct a rectangle $AEFD$ so that $EF = m$. At B erect a perpendicular to AE. Then,

$$ABCD = m(a-b).$$

$$AEFD = ma.$$

$$BEFC = mb.$$

Also, $$ABCD = AEFD - BEFC.$$

Hence, $$m(a-b) = ma - mb.$$

Notice the steps in the geometric illustration correspond exactly with the steps of the algebraic process of multiplication. Notice further the geometric illustration exemplifies the rule of signs in multiplication, namely, unlike signs give minus.

The product of a polynomial by a monomial equals the algebraic sum of the products of the monomial by each term of the polynomial.

EXERCISE 29

Multiply:

1. $x + 3$ by 10.
2. $2x + 4$ by 8.
3. $5x - 2$ by 3.
4. $9x - 11$ by 7.
5. $6x - 7$ by 9.
6. $5a^2 - 2$ by 11.
7. $9a^2 - 8$ by 12.
8. $5y^2 - 8y$ by 5.
9. $9n^2 - 8n$ by 10.
10. $14m - 15$ by 2.
11. $4x - 4$ by -3.
12. $6x - 7$ by -4.
13. $5x + 8$ by -6.
14. $9y^2 - 12$ by -8.
15. $4\frac{1}{2} - m^2$ by -4.
16. $\frac{3}{4} - \frac{1}{2}n^2$ by -8.
17. $19 - \frac{2}{3}a^2$ by -9.
18. $1\frac{2}{5} - \frac{7}{10}a^3$ by -20.

36. *Example* 1. Multiply 5^3 by 5^4.

$$5^3 = 5 \cdot 5 \cdot 5.$$

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5.$$

Therefore, $\quad 5^3 \cdot 5^4 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^7.$

Example 2. Multiply $2^3 \times 3^2$ by $2^2 \times 3^4$.

$$(2^3 \cdot 3^2) \times (2^2 \cdot 3^4) = 2^3 \cdot 2^2 \cdot 3^2 \cdot 3^4$$

$$= 2^5 \cdot 3^6. \quad \text{Associative Law.}$$

Example 3. Multiply a^2 by a^4.

$$a^2 = a \cdot a.$$

$$a^4 = a \cdot a \cdot a \cdot a.$$

Therefore, $\quad a^2 \cdot a^4 = (a \cdot a) \times (a \cdot a \cdot a \cdot a) = a^6.$

Example 4. Multiply a^2b^2 by a^3b.

$$a^2b^2 = aabb.$$

$$a^3b = aaab.$$

Therefore, $$a^2b^2 \cdot a^3b = aabb \cdot aaab$$
$$= aaaaabbb$$
$$= a^5b^3. \qquad \text{Associative Law.}$$

Example 5. Multiply $4\,a^2y$ by $7\,a^4y^3$.

SOLUTION. $$4\,a^2y \times 7\,a^4y^3 = 4 \times 7 \times a^2y \times a^4y^3$$
$$= 28\,a^6y^4. \qquad \text{Associative Law.}$$

To multiply a power of a letter or number by a power of the same letter or number, write the letter or number, and for its exponent write the sum of the exponents of the multiplier and multiplicand.

This is known as the *Index Law for Multiplication.* Expressed in symbols the index law is,

$$a^m \times a^n = a^{m+n}.$$

NOTE. When no exponent is written, the exponent is 1. The exponent of a in the expression ab^3 is 1.

EXERCISE 30

Perform the following indicated operations and give results in index notation:

1. $3^2 \times 3^4$.	6. $9^4 \times 9^6$.	11. $a^7 \times a^4$.
2. $2^4 \times 2^3$.	7. $10^2 \times 10^7$.	12. $b^8 \times b^5$.
3. $7^3 \times 7^5$.	8. $10^4 \times 10^8$.	13. $b^3 \times b^{10}$.
4. $5^4 \times 5^6$.	9. $2^{12} \times 2^8$.	14. $ab^2 \times a^2b$.
5. $8^2 \times 8^3$.	10. $a^2 \times a^3$.	15. $a^3b^2 \times a^2b^4$.

16. $a^5b \times ab^5$.

17. $a^4b^3 \times a^3b^5$.

18. $a^3 \times a^3$.

19. $a^4 \times a^4$.

20. $a^7 \times a^7$.

21. $4\,xy \times 8\,x$.

22. $7\,ab \times 7\,b$.

23. $6\,a^2b \times -8\,a$.

24. $9\,a^3b^2 \times -8\,ab$.

25. $10\,a^4b^4 \times -3\,a^2b$.

26. $15\,a^3b^5 \times -3\,ab^4$.

27. $30\,a^4b^6 \times -4\,a^4b$.

28. $25\,m^3n \times -mn^3$.

29. $23\,m^4n^2 \times -m^4n$.

30. $12\,m^4n^2 \times -m^4n^3$.

31. $a^n \times a^n$.

32. $x^n \times x$.

33. $x^n \times x^4$.

EXERCISE 31

Perform the following indicated multiplications:

1. $4(3\,x-5\,y-4)$.

2. $7(2\,x-3\,a+8)$.

3. $9(x-4\,a-8)$.

4. $-5(2\,x-3\,y-7)$.

5. $-3(4\,a-2\,b-5)$.

6. $-8(3\,a-7\,b-9)$.

7. $4\,x(x^2-2\,x-3)$.

8. $5\,a(a^2-3\,a-7)$.

9. $-a(a^2-9\,a-6)$.

10. $-b(b^2-ab-10)$.

11. $-8\,x(2\,x^2-3\,xy+y^2)$.

12. $-9\,x(2\,x^3-5\,x^2y-8\,y^2)$.

13. $6(\frac{2}{3}x-\frac{1}{2}y-\frac{5}{6})$.

14. $8(\frac{3}{4}x-\frac{1}{8}y-\frac{3}{8})$.

15. $9(\frac{2}{3}x-\frac{2}{9}y-\frac{5}{9})$.

16. $10(\frac{2}{5}x-\frac{7}{10}y-\frac{1}{2})$.

17. $-12(\frac{5}{6}x+\frac{3}{4}y-\frac{7}{12})$.

18. $-20(\frac{4}{5}a-\frac{3}{10}b-\frac{3}{4})$.

19. $-30(\frac{5}{6}x-\frac{2}{15}y-\frac{1}{2})$.

20. $-50(\frac{1}{5}x-\frac{2}{25}y-\frac{4}{5})$.

21. $-100(\frac{7}{10}x-\frac{9}{10}y-\frac{79}{100})$.

22. $-80(\frac{5}{16}x-\frac{7}{40}y-\frac{41}{80})$.

23. $60(\frac{5}{12}x-\frac{1}{4}x-\frac{37}{60})$.

24. $50(\frac{3}{5}x-\frac{1}{2}x-\frac{23}{50})$.

DIVISION

37. In multiplication two or more numbers are given and their product is sought. In division the product of two numbers and one of the numbers are given, and the other number is sought. Zero as a divisor is excluded.

Division is the inverse of multiplication.

38. Since $(+a)\times(+b) = +ab$, therefore $\frac{+ab}{+b} = +a$.

Since $(+a)\times(-b) = -ab$, therefore $\frac{-ab}{-b} = +a$.

Since $(-a)\times(+b) = -ab$, therefore $\frac{-ab}{+b} = -a$.

Since $(-a)\times(-b) = +ab$, therefore $\frac{+ab}{-b} = -a$.

Hence, the law of signs for division is:

Like signs give +. Unlike signs give −.

39. Divide 6^5 by 6^2.

SOLUTION. $\frac{6^5}{6^2} = \frac{6\cdot 6\cdot 6\cdot 6\cdot 6}{6\cdot 6} = \frac{6\cdot 6\cdot 6}{1\cdot 1} = 6^3$.

Example 2. Divide a^7 by a^3.

SOLUTION. $\frac{a^7}{a^3} = \frac{a\,a\,a\,a\,a\,a\,a}{a\,a\,a} = \frac{a\,a\,a\,a}{1} = a^4$.

Hence, to divide a power of a number by a power of the same number, write the number and for its exponent write the exponent of the dividend minus the exponent of the divisor. This is the *Index Law for Division.* Expressed in symbols this law is, $a^m \div a^n = a^{m-n}$.

EXERCISE 32

Express each of the following indicated quotients as a single power:

1. $\frac{3^8}{3}$.
2. $\frac{4^4}{4}$.
3. $\frac{5^4}{5^2}$.
4. $\frac{7^6}{7^2}$.
5. $\frac{3^7}{3^4}$.
6. $\frac{3^8}{3^3}$.
7. $\frac{5^7}{5^6}$.
8. $\frac{11^5}{11^2}$.
9. $\frac{8^6}{8^4}$.
10. $\frac{6^{10}}{6^4}$.
11. $\frac{3^9}{3^5}$.
12. $\frac{9^7}{9^5}$.
13. $\frac{13^7}{13^3}$.
14. $\frac{17^8}{17^6}$.
15. $\frac{a^2}{a}$.
16. $\frac{a^4}{a}$.
17. $\frac{a^6}{a^2}$.
18. $\frac{a^{10}}{a^3}$.
19. $\frac{b^8}{b^5}$.
20. $\frac{c^7}{c^4}$.
21. $\frac{c^9}{c^7}$.
22. $\frac{a^9}{a^2}$.
23. $\frac{x^{10}}{x^3}$.
24. $\frac{x^{11}}{x^7}$.
25. $\frac{y^7}{y^3}$.
26. $\frac{m^{11}}{m^8}$.
27. $\frac{n^{12}}{n^4}$.
28. $\frac{n^{13}}{n^4}$.

40. *Example* 1. Divide $14 \times 35 \times 36$ by $9 \times 7 \times 5$.

SOLUTION. $$\frac{\overset{2}{\cancel{14}} \times \overset{7}{\cancel{35}} \times \overset{4}{\cancel{36}}}{\underset{1}{\cancel{9}} \times \underset{1}{\cancel{7}} \times \underset{1}{\cancel{5}}} = \frac{2 \times 7 \times 4}{1 \times 1 \times 1} = 56.$$

Example 2. Divide $21\,x^7y^3$ by $3\,x^4y^2$.

SOLUTION. $$\frac{21\,x^7y^3}{3\,x^4y^2} = 7\,x^3y.$$

Here the principle of cancellation is made use of. 3 is contained in 21, 7 times, x^4 is contained in x^7, x^3 times; y^2 is contained in y^3, y times. The answer is $7\,x^3y$.

Example 3. Divide $40\,x^4z^3$ by $8\,xz^3$.

SOLUTION. $$\frac{40\,x^4z^3}{8\,xz^3} = 5\,x^3.$$

8 is contained in 40, 5 times; x is contained in x^4, x^3 times; z^3 is contained in z^3, 1 time. The answer is $5 \times x^3 \times 1$, or simply $5\,x^3$.

EXERCISE 33

Perform the following indicated divisions:

1. $\frac{12\,x}{3}$.
2. $\frac{15\,x}{3}$.
3. $\frac{24\,a}{8}$.
4. $\frac{40\,b}{-5}$.
5. $\frac{-60\,c}{15}$.
6. $\frac{-72\,c^2}{9}$.
7. $\frac{54\,n^3}{-18}$.
8. $\frac{14\,x^2}{7\,x}$.
9. $\frac{18\,x^2}{6\,x}$.
10. $\frac{24\,x^3}{-6\,x^2}$.
11. $\frac{-25\,c^3}{5\,c}$.
12. $\frac{30\,n^3}{5\,n}$.
13. $\frac{-40\,n^4}{4\,n^2}$.
14. $\frac{22\,a^5}{-11\,a^2}$.
15. $\frac{-18\,a^5}{6\,a^3}$.
16. $\frac{27\,m^6}{-9\,m^4}$.
17. $\frac{36\,x^2y}{12\,x}$.
18. $\frac{35\,x^3y}{7\,xy}$.
19. $\frac{42\,x^2y^3}{-6\,x^2y}$.
20. $\frac{45\,a^4b^3}{9\,ab^2}$.
21. $\frac{48\,a^3b^5}{-8\,a^2b^2}$.
22. $\frac{-90\,c^4n^2}{9\,c^2n}$.
23. $\frac{84\,a^4c^5}{-12\,a^2c^2}$.
24. $\frac{75\,a^5c^8}{5\,ac^4}$.
25. $\frac{12\,a^4c^3}{-6\,ac}$.
26. $\frac{27\,a^9c^4}{-3\,a^3c^3}$.
27. $\frac{39\,c^4n^5}{13\,cn^3}$.
28. $\frac{11\,a^5b}{a^2b}$.
29. $\frac{x^n}{x}$.
30. $\frac{x^{2a}}{x^a}$.
31. $\frac{x^{n+1}}{x^{n-1}}$.
32. $\frac{x^{2a}}{x^{a-8}}$.

41. *Example* 1. Divide $18\frac{6}{11}$ by 3.

SOLUTION. $3\overline{)18\frac{6}{11}}$ Divide 18 by 3. Next divide

$\quad 6\frac{2}{11}$ $\frac{6}{11}$ by 3. The required quotient is $6\frac{2}{11}$.

Example 2. Divide $18a - 6b$ by 3.

SOLUTION. $3\overline{)18a - 6b}$
$6a - 2b$

Divide $18a$ by 3. Next divide $-6b$ by 3. Write the quotients. The answer is $6a - 2b$.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and write the quotients in succession.

EXERCISE 34

Divide:

1. $5a + 5$ by 5.
2. $4x + 8$ by 2.
3. $10x + 15$ by 5.
4. $30x - 12$ by 6.
5. $40x - 8$ by 8.
6. $42x - 18$ by 6.
7. $32a - 24b$ by 8.
8. $24a - 72b$ by 12.
9. $60a - 84b$ by 6.
10. $44a^2 - 52b^2$ by 4.
11. $12a^2 - 16b^2$ by 4.
12. $21x^3 - 24x^2 - 12x$ by $3x$.
13. $a^2 + ab - ac$ by a.
14. $a^3 + a^2 - 2a$ by a.
15. $4a^3 + 8a^2 + 4a$ by $4a$.
16. $8x^2y^2 - 12xy$ by $4xy$.
17. $16a^2b^2 - 24ab^3$ by $8ab$.
18. $18x^3y^2 - 27x^2y^3$ by $9x^2y^2$.
19. $24x^4 - 32x^3$ by $8x^2$.
20. $36x^5 - 48x^4$ by $12x^2$.
21. $14a^4b^2 - 21a^3b^3$ by $7a^2b^2$.
22. $22x^5 - 33x^4$ by $11x^3$.
23. $12a^2 - 13a^3$ by a^2.
24. $30x^6 - 40x^4$ by $10x^4$.

CHAPTER III

FACTORS. FRACTIONS

FACTORS

42. When the factors of an integer are sought, it is understood that only the integral factors are required.

The factors of an integer are then its exact integral divisors.

43. An algebraic expression is integral with respect to a letter when that letter does not occur in the denominator of a term. Thus,

$$x^2 + \frac{1}{3}x + \frac{1}{6}, \quad \frac{x^2}{a} + \frac{x}{b} + c,$$

are integral with respect to x.

44. An expression is rational with respect to a letter when it does not contain the square root or any other root of the letter.

45. The factors of a rational and integral expression are the rational and integral expressions which multiplied produce it.

46. *Example* 1. Multiply $x^2 - 2$ by x. The product is $x^3 - 2x$.

Consider next the inverse problem. What are the factors of $x^3 - 2x$?

By inspection it is seen that x is a factor of x^3 and also of $-2x$. Dividing by x and indicating the product of divisor and quotient, we have

$$x^3 - 2x = x(x^2 - 2).$$

Example 2. Factor $5a^2 + 5ab$.

By inspection it is seen that 5 is a factor of each of the terms, and also a is a factor of each of the terms. Hence, $5a$ is a factor of both terms, and hence of the expression $5a^2 + 5ab$. Hence,

$$5a^2 + 5ab = 5a(a + b).$$

$5a$ is called the **monomial factor**, and $a + b$ the **binomial factor**, of $5a^2 + 5ab$.

EXERCISE 35

Factor:

1. $5x + 5$.
2. $7x - 14$.
3. $6a - 12$.
4. $8a + 8b$.
5. $9x - 9y$.
6. $11x - 33$.
7. $12x - 60$.
8. $a^2 + a$.
9. $m^2 + m$.
10. $y^2 - 2y$.
11. $x^2 - 4x$.
12. $2x^2 - x$.
13. $4x^2 - xy$.
14. $5a^2 - 5ab$.
15. $7ax - 14ay$.
16. $9a^2 - 18a$.
17. $4a^3 - 8a$.
18. $6a^3 - 12a^2$.
19. $10x^3 - 10x^2$.
20. $x^2y - xy^2$.
21. $x + \frac{x}{100}$.
22. $x - \frac{x}{50}$.
23. $x - \frac{x}{40}$.
24. $\frac{1}{2}x + \frac{1}{2}$.
25. $\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z$.
26. $\frac{1}{2}bc + \frac{1}{2}bd$.
27. $\frac{1}{2}br + \frac{1}{2}cr + \frac{1}{2}dr + \frac{1}{2}er$.
28. $ah + bh + ch + dh$.
29. $a(m + n) + b(m + n) - c(m + n)$.

FRACTIONS

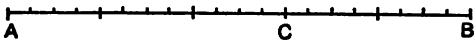
47. A fraction is an indicated quotient.

$\frac{3}{4}$ is a symbol indicating that 3 is to be divided by 4. $\frac{a}{b}$ is a symbol indicating that a is to be divided by b. Such symbols are called **fractions**. The number written above the horizontal line is called the **numerator**, and the number written below the horizontal line is called the **denominator**.

The numerator and the denominator are called the **terms** of a fraction.

48. The most important property of a fraction is that its value remains unchanged when its terms are multiplied by the same number.

This may be shown in a particular instance as follows:



Take a line AB equal to 5 units, and let CB be equal to 2 units. Then, $CB = \frac{2}{5}$ of AB. Next subdivide the unit into, say, 4 equal parts.

Then, $AB = 4 \times 5$ new units, and $CB = 4 \times 2$ new units.

Hence, $$CB = \frac{4 \times 2}{4 \times 5} \text{ of } AB.$$

But $$CB = \tfrac{2}{5} \text{ of } AB.$$

Therefore, $$\tfrac{2}{5} \text{ of } AB = \frac{4 \times 2}{4 \times 5} \text{ of } AB,$$

or $$\frac{2}{5} = \frac{4 \times 2}{4 \times 5}.$$

49. *Example* 1. Multiply $\frac{7}{8}$ by $\frac{3}{4}$.

Let $\qquad x =$ the product.

Then, $\qquad x = \frac{7}{8} \times \frac{3}{4}$.

Multiply these equals by 8 times 4.

Then, $(8 \times 4) \times x = (8 \times 4) \times (\frac{7}{8} \times \frac{3}{4}) = (8 \times \frac{7}{8}) \times (4 \times \frac{3}{4})$.

Therefore, $\qquad (8 \times 4) \times x = 7 \times 3. \qquad$ Associative Law.

Dividing by 8×4, $\qquad x = \frac{7 \times 3}{8 \times 4}$.

Hence, the product of two fractions is a fraction whose numerator is the product of the numerators of the fractions, and whose denominator is the product of the denominators of the fractions.

Example 2. Multiply $\frac{3x^2}{2a}$ by $\frac{a^2}{x}$.

$$\frac{3x^2}{2a} \times \frac{a^2}{x} = \frac{\overset{x}{3x^2} \cdot \overset{a}{a^2}}{2\underset{1}{a} \cdot \underset{1}{x}} = \frac{3ax}{2}.$$

EXERCISE 36

Perform the indicated operations:

1. $\frac{2x}{3} \times 9$.
2. $\frac{4x}{5} \times 15$.
3. $\frac{7x}{8} \times 12$.
4. $\frac{5x}{6} \times 21$.
5. $\frac{4x}{9} \times 24$.
6. $\frac{8x}{11} \times 2\frac{3}{4}$.
7. $\frac{6x}{13} \times 2\frac{1}{6}$.
8. $\frac{9x^2}{14} \times 3\frac{1}{2}$.

9. $\frac{12x^2}{17} \times 5\frac{1}{3}$.

10. $\frac{13a}{19} \times 9\frac{1}{2}$.

11. $\frac{18a^2}{25} \times 5\frac{5}{6}$.

12. $\frac{5a}{2x} \times \frac{8}{15a^2}$.

13. $\frac{7a}{x} \times \frac{x^2}{14a^2}$.

14. $\frac{18a^2}{7y} \times \frac{14y^2}{3a^2}$.

15. $\frac{3x^3}{2y^2} \times \frac{8y^2}{6x^4}$.

16. $\frac{5xy}{2ab} \times \frac{4a^2}{xy^2}$.

17. $\frac{9a^2b}{10x^2} \times \frac{25x^3}{6a^2b^2}$.

18. $\frac{27x^2y^2}{28a^2b} \times \frac{14a^3b}{9x^3y}$.

19. $\frac{30a^4b}{49x^2y} \times \frac{98x^3y^2}{15a^4b^2}$.

20. $\frac{36x^3y^4}{119a^2b^3} \times \frac{133a^3b^2}{54x^2y^4}$.

21. $\frac{26a^4b}{51x^3y^2} \times \frac{68x^4y^2}{39a^4b^2}$.

22. $\frac{38mn}{69ab^2} \times \frac{92a^2b}{57mn^2}$.

50. *Example* 1. Multiply $\frac{3}{4}(x-2)$ by 16.

$$\tfrac{3}{4}(x-2) \times 16 = \tfrac{3}{4} \times 16 \times (x-2) = 12(x-2) = 12x - 24.$$

Example 2. Multiply $-\frac{2x-5}{6}$ by 18.

$$-\frac{2x-5}{6} \times 18 = -\frac{18(2x-5)}{6} = -\frac{3(2x-5)}{1}$$
$$= -3(2x-5) = -6x+15.$$

Check. Take $x = 1$.

$$-\frac{2x-5}{6} \times 18 = -\frac{-3}{6} \times 18 = -(-9).$$
$$-6x+15 = -6+15 = 9.$$

The horizontal line in fractional notation has the force of a parenthesis.

EXERCISE 37

Multiply:

1. $\frac{2}{3}(x-1)$ by 6.
2. $\frac{7}{8}(x-4)$ by 16.
3. $\frac{9}{10}(a-b)$ by 50
4. $\frac{7}{11}(m-n)$ by 33.
5. $\frac{7}{15}(2x-9)$ by 45.
6. $\frac{11}{18}(3x-7)$ by 54.
7. $\frac{x-8}{4}$ by 12.
8. $\frac{3x-10}{20}$ by 100.
9. $-\frac{7x-11}{12}$ by 24.
10. $-\frac{9x+2}{14}$ by 42.
11. $-\frac{9-3x}{4}$ by 48.
12. $-\frac{3x-7}{6}$ by -18.

DIVISION

51. Two numbers are reciprocals when their product is 1. Thus, 9 and $\frac{1}{9}$ are reciprocals. $\frac{2}{7}$ and $\frac{7}{2}$ are reciprocals.

Example 1. Divide $\frac{3}{5}$ by $\frac{7}{9}$.

Let $x =$ the quotient.

Then, $\frac{7}{9}$ of $x = \frac{3}{5}$.

Multiply these equals by $\frac{9}{7}$.

Hence, $\frac{9}{7} \times \frac{7}{9}$ of $x = \frac{3}{5} \times \frac{9}{7}$.

$$\frac{9}{7} \times \frac{7}{9} = 1.$$

Therefore, $x = \frac{3}{5} \times \frac{9}{7} = \frac{27}{35}$.

Hence, **to divide by a fraction, multiply by its reciprocal**

EXERCISE 38

Divide:

1. $\frac{3x}{4}$ by 6.

2. $\frac{9x}{11}$ by 18.

3. $\frac{5x}{12}$ by $2\frac{2}{5}$.

4. $\frac{6x}{17}$ by $\frac{3}{34}$.

5. $\frac{8x}{15}$ by $3\frac{3}{4}$.

6. $\frac{10xy}{21}$ by $1\frac{3}{7}$.

7. $\frac{4a}{b}$ by $\frac{ab}{d}$.

8. $\frac{3a^2}{5}$ by $\frac{2}{3}a$.

9. $\frac{4a^2b}{7x}$ by $\frac{2ab}{7}$.

10. $\frac{6mn^2}{ab^2}$ by $\frac{mn}{a^2b}$.

11. $\frac{3m^2n}{4a^2b}$ by $\frac{7}{8}mn^2$.

12. $\frac{4}{5}ab^3$ by $\frac{4}{7}a^2b^2$.

52. *Example* 1. Express $x+\frac{x}{8}$ as a fraction.

SOLUTION. $x=\frac{8x}{8}$.

Hence, $$x+\frac{x}{8}=\frac{8x}{8}+\frac{x}{8}=\frac{9x}{8}.$$

Example 2. Express as a fraction $x-\frac{x-2}{7}$.

SOLUTION. $x=\frac{7x}{7}$.

Hence,

$$x-\frac{x-2}{7}=\frac{7x}{7}-\frac{x-2}{7}=\frac{7x-(x-2)}{7}=\frac{7x-x+2}{7}=\frac{6x+2}{7}.$$

To reduce a mixed quantity to an equivalent fraction, **multiply the denominator of the fractional part by the integral part, and add the numerator if the fraction is preceded**

by the plus sign; subtract the numerator if the fraction is preceded by the minus sign.

Write the result for the numerator of the required fraction, and for its denominator write the denominator of the given fraction.

EXERCISE 39

Express as a fraction:

1. $x + \frac{x}{2}$.
2. $x + \frac{2x}{3}$.
3. $a + \frac{2a-1}{4}$.
4. $2a + \frac{3a-4}{5}$.
5. $a - \frac{a}{3}$.
6. $a - \frac{a-1}{3}$.
7. $2a - \frac{3a-4}{7}$.
8. $4a - \frac{2a+5}{6}$.
9. $3a - \frac{2a-3}{5}$.
10. $a + \frac{b^2}{a}$.
11. $a - \frac{c^2}{a}$.
12. $3a - \frac{2a^2-b}{a}$.
13. $6a - \frac{4a^2+b}{a}$.
14. $1 + \frac{1}{x-1}$.
15. $2 - \frac{3}{x-2}$.
16. $3 - \frac{4}{x+1}$.
17. $a + \frac{b^2-a^2}{a}$.
18. $x+2+\frac{4}{x-2}$.

53. Simplify the expression $\frac{a-1}{2} - \frac{a-2}{3} + \frac{2a+1}{4}$.

The L. C. M. of 2, 3, 4 is 12.

$$\frac{a-1}{2}=\frac{6a-6}{12}.$$

$$-\frac{a-2}{3}=-\frac{4(a-2)}{12}=-\frac{4a-8}{12}.$$

$$\frac{2a+1}{4}=\frac{6a+3}{12}.$$

Hence,

$$\frac{a-1}{2}-\frac{a-2}{3}+\frac{2a+1}{4}=\frac{6a-6-4a+8+6a+3}{12}=\frac{8a+5}{12}.$$

Check. Take $a=2$, $\frac{2-1}{2}-\frac{2-2}{3}+\frac{4+1}{4}=\frac{8\times 2+5}{12}$.

$$\tfrac{1}{2}-0+1\tfrac{1}{4}=1\tfrac{3}{4}.$$

EXERCISE 40

Simplify:

1. $\frac{a}{2}+\frac{a+3}{4}+\frac{a-4}{3}$.

2. $\frac{c}{3}+\frac{c-1}{4}+\frac{5c-4}{6}$.

3. $\frac{3c-1}{5}-\frac{4c-2}{3}+\frac{11c+1}{15}$.

4. $\frac{2c-4}{6}-\frac{3c-5}{9}+\frac{4c+1}{18}$.

5. $\frac{4a-5}{4}-\frac{2a+7}{3}+\frac{2a-5}{2}$.

6. $\frac{5a-3}{9}-\frac{2a-4}{3}+\frac{2a-5}{2}$.

7. $\frac{11a+2}{12}-\frac{9a+7}{18}+\frac{2a-5}{9}.$

8. $\frac{3a-5}{10}-\frac{2a-8}{15}-\frac{a-2}{6}.$

9. $\frac{5c-2}{10}-\frac{3c+9}{4}+\frac{c-2}{8}.$

10. $\frac{4a-11}{9}-\frac{a-10}{5}-\frac{2a+6}{15}.$

11. $\frac{4m+5}{14}+\frac{3m-2}{28}-\frac{m+4}{7}.$

12. $\frac{7n-3}{21}-\frac{2n-7}{14}-\frac{n-3}{7}.$

13. $\frac{3a-5b}{3}-\frac{2a-b}{5}-\frac{a-8b}{12}.$

14. $\frac{2x-7y}{4}-\frac{x+11y}{9}+\frac{x-2y}{12}.$

15. $\frac{3m-2n}{3}-\frac{m-n}{5}-\frac{12m-7n}{15}.$

16. $\frac{4a-5b}{9}-\frac{2a-3b}{5}-\frac{2a+2b}{45}.$

17. $a-\frac{a+b}{11}+\frac{3b-20a}{33}.$

18. $x-\frac{x-y}{3}-\frac{x-2y}{4}.$

19. $\frac{a^2+2ab+b^2}{4}-\frac{a^2-2ab+b^2}{4}.$

20. $\frac{2a^2+2b^2}{3}-\frac{a^2+b^2}{2}-\frac{a^2+b^2}{6}.$

CHAPTER IV

THE EQUATION. PROBLEMS

54. The following is an equation:

$$3x - 4 = 23.$$

If this equation is solved, it will be found that $x = 9$. For no other value of x will $3x - 4 = 23$.

In the equation $\quad x^2 + 14 = 9x,$

x has two values, 2 and 7; for

$$2^2 + 14 = 9 \cdot 2,$$

and $$7^2 + 14 = 9 \cdot 7.$$

An equation is a statement of equality which is true for a particular value or values of the letter or letters contained therein.

55. $x + 7 = 7 + x$ is not an equation, for in this statement x may have any value and the equality holds good. An expression of this character is called an **identity**.

56. The letter whose value is sought in an equation is called the **unknown quantity** in the equation.

57. The process of finding the value of the unknown quantity in an equation is called **solving the equation.**

58. A value of the unknown quantity in an equation is called a **root** of the equation, and is said to **satisfy the equation.**

59. In solving equations, use is made of certain simple statements assumed to be true. These statements are called **axioms.**

An axiom is a statement whose truth is taken for granted.

AXIOM 1. *Quantities which are equal to the same quantity are equal to each other.*

ILLUSTRATION.

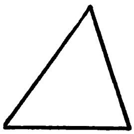


FIG. 1.

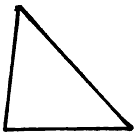


FIG. 2.

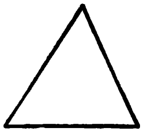


FIG. 3.

Suppose Fig. 1 equals Fig. 3 in area, and suppose Fig. 2 equals Fig. 3 in area, then Fig. 1 equals Fig. 2 in area.

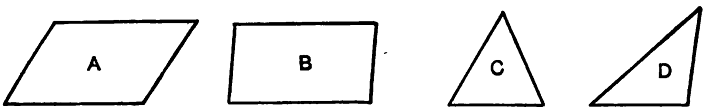
AXIOM 2. *If equals be added to equals, the sums are equal.*

ILLUSTRATION.

Suppose $A = B$ in area,

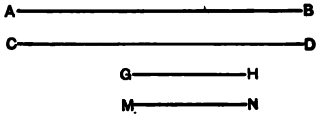
and $C = D$ in area;

then, $A + C = B + D$.



AXIOM 3. *If equals be taken from equals, the remainders are equal.*

ILLUSTRATION.



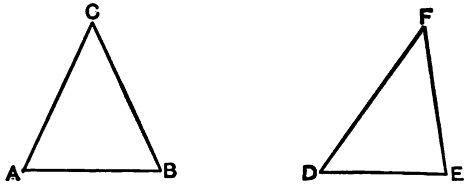
If the line $AB =$ the line CD,

and the line $GH =$ the line MN,

then, $AB - GH = CD - MN$.

AXIOM 4. *If equals be multiplied by the same number, the products are equal.*

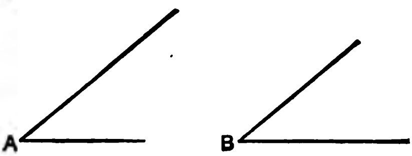
ILLUSTRATION.



If the triangle ABC equals the triangle DEF, then twice the triangle ABC equals twice the triangle DEF, then three times the triangle ABC equals three times the triangle DEF, then four times the triangle ABC equals four times the triangle DEF, etc.

AXIOM 5. *If equals be divided by the same number (zero excluded as a divisor), the quotients are equal.*

ILLUSTRATION.



If angle A equals angle B, then half of angle A equals half of angle B, one third of angle A equals one third of angle B, etc.

60. Solve: $7x - 3 = 4x + 12.$

Subtract $4x$ from each member; then, by Axiom 3,

$$(1)\quad 7x - 4x - 3 = 12.$$

Add 3 to each member; then, by Axiom 2,

$$(2)\quad 7x - 4x = 12 + 3.$$

Uniting like terms, (3) $3x = 15$,

(4) $x = 5$. Axiom 5.

From this solution it appears that a term may be transposed from one member of an equation to the other member by changing its sign. This process is called **transposition**. Accordingly, the above solution may be shortened as follows:

$$7x - 3 = 4x + 12.$$

Transposing, $7x - 4x = 12 + 3.$

Uniting terms, $3x = 15,$

$x = 5.$ Axiom 5.

Check. $7 \cdot 5 - 3 = 4 \cdot 5 + 12.$

EXERCISE 41

Solve:

1. $3x + 4 = x + 14.$
2. $6x + 9 = 4x + 25.$
3. $7x - 3 = x + 21.$
4. $8x - 5 = x + 30.$
5. $10x - 11 = 3x - 4.$
6. $9x - 17 = 4x + 13.$
7. $4x - 11 = 59 - 3x.$
8. $5x + 14 = 42 - 2x.$
9. $3x - 11 = 53 - 3x.$
10. $10x - 15 = 7x + 21.$

11. $12x + 17 = 7x + 37$.
12. $14x + 19 = 8x + 43$.
13. $19x - 17 = 10x - 98$.
14. $16x - 15 = 11x - 10$.
15. $20x + 40 = 13x + 19$.
16. $10x - 25 = 2x + 15$.
17. $21x - 11 = 11x + 79$.
18. $18x + 14 = 3x + 89$.
19. $22x + 13 = 3x + 108$.
20. $30x + 11 = 9x + 221$.

Solve $3x - 7 - 3(7 - x) = 10(x - 4)$.

Removing the parentheses,

$$3x - 7 - 21 + 3x = 10x - 40.$$

Transposing, $3x + 3x - 10x = 7 + 21 - 40$.

Combining, $-4x = -12$.

Dividing by -4, $x = 3$.

Check. $3 \times 3 - 7 - 3(7 - 3) = 10(3 - 4)$.

Hence, $9 - 7 - 12 = -10$.

$$-10 = -10.$$

EXERCISE 42

Solve:

1. $5x + (2x - 3) = 3(x + 3)$.
2. $4x - (x - 4) = 2(x + 4)$.
3. $5x - (7 - 2x) = 2(3x - 1)$.
4. $9x + 3(x + 5) = 5(2x + 5)$.
5. $7x - 3(x - 4) = x + 30$.
6. $2(x + 9) - 3(5 - x) = 4(2x - 3)$.
7. $5(x + 1) - 6(x - 3) = 4(2x - 1)$.
8. $7(x + 1) - 4(x + 2) = 4(x - 2)$.
9. $6(2x - 3) + 2(8 - 3x) = 4(x + 4)$.

10. $9(x-1)-7(x+4)=62-7x$.

11. $4(x-3)-2(2x-5)=2(3x-13)$.

12. $8(3x+5)+2(9-10x)=2(x-5)$.

13. $9(4x-6)-7(5x-1)=4(x-23)$.

14. $10x-50-(6x-67)=3(x+12)$.

15. $7(3x-2)-4(2x-9)=11(5x+2)$.

16. $(3x-4)-(17-4x)=9(x-3)$.

17. $x^2-3x-(x^2+2x-15)=3(3x-23)$.

18. $2x^2+7-(x^2+2x+5)=x^2-(3x-2)$.

19. $2(4x-9)-(3x+10)=4(x-3)-(x+2)$.

20. $3(9-x)+2(3x-8)=4(x+3)-2x-1$.

EXERCISE 43

1. What is the next odd number after 17? What is the next odd number after 99?

2. Given an odd number, how is the next odd number obtained?

3. If n is an odd number, what is the next odd number?

4. What is the next even number after 36? After 88? After 112?

5. If m is an even number, what is the next even number?

6. What is the even number next before m?

7. What is the number which exceeds x by 1? By 10? By 24?

8. 19 exceeds a number by 4. What is the number?

9. x exceeds a number by 4. What is the number?

10. Write three consecutive numbers the least of which is 16.

11. Write three consecutive numbers the least of which is n.

12. Write three consecutive numbers the greatest of which is 14.

13. Write three consecutive numbers the greatest of which is x.

14. Write four consecutive even numbers the least of which is 22.

15. $2x$ is the least of four consecutive even numbers. Write the other three.

16. Name the four consecutive odd numbers the least of which is 35.

17. Name the four consecutive odd numbers the least of which is n.

18. The sum of two numbers is 14, and one of the numbers is 5. Find the other number.

19. The sum of two numbers is 40, and one of the numbers is x. Find the other number.

20. By how much does 24 exceed 12?

21. By how much does 30 exceed 6?

22. By how much does 36 exceed x?

23. By how much does 54 exceed x?

24. By how much does 38 exceed $x - 5$?

25. By how much does 34 exceed $x + 6$?

26. One number exceeds another by 10, and the greater is x. Find the other number.

27. Diminish 25 by 7. Diminish x by 9.

28. How much must be added to 11 to make 29?

29. How much must be added to x to make 50?

30. A boy is n years old. How old will he be in 5 years?

31. A boy is 13 years old. How old will he be in m years? How old was he m years ago?

Example 1. The sum of the ages of a father and his son is 80 years. Twice the son's age exceeds the father's age by 10 years. Find their ages.

Let $x =$ the son's age in years.

Then, $2x - 10 =$ father's age in years.

And, $x + 2x - 10 =$ the sum of their ages.

Therefore, $x + 2x - 10 = 80.$

Transposing, $x + 2x = 80 + 10.$

Combining, $3x = 90.$

$x = 30$, son's age.

$2x - 10 = 50$, father's age.

Check. $50 + 30 = 80.$

ANOTHER SOLUTION.

Let $x =$ son's age in years.

Then, $80 - x =$ father's age in years.

$2x =$ twice the son's age.

Therefore, $2x - (80 - x) = 10.$

Removing parenthesis, $2x - 80 + x = 10.$

Transposing, $2x + x = 10 + 80.$

Combining, $3x = 90.$

$x = 30$, son's age.

$80 - x = 50$, father's age.

Check. $2 \cdot 30 - 50 = 10.$

EXERCISE 44.

1. Divide 90 cents between two boys, giving one of the boys 15 cents more than twice the share of the other boy.

2. The sum of the ages of a father and his son is 94 years, and three times the age of the son exceeds the father's age by 2 years. Find their ages.

3. In a family there are three boys, Henry, Thomas, and John. Henry is 2 years older than Thomas, and 7 years older than John. The sum of their ages is 42 years. Find the age of each.

4. In a certain class in a high school there are 40 pupils. The number of girls exceeds twice the number of boys by 1. Find the number of boys and the number of girls in this class.

5. In an election there were 3521 votes cast for two candidates. The successful candidate received 1066 votes less than twice the number cast for the other candidate. Find the number of votes each received.

6. Divide a line 90 inches long into two parts so that the greater part may be 10 inches longer than three times the smaller part.

7. In 1903 there were in Illinois 1227 banks. There were 18 more national banks than state banks, and the number of private banks was 35 more than the sum of the other two. Find the number of banks of each kind.

8. In an election there were three candidates, C, F, and H. C received 20 votes more than three times the number of votes H received, and F received 119 votes more than C and H together. The total number of votes cast was 935. Find the number of votes each received.

9. The annual salaries of two officials, A and B, amount to $12,500. Twice B's salary exceeds A's salary by $1000. Find A's and B's salary.

10. Three government officials, A, B, and C, receive yearly salaries amounting to $22,500. B receives $500 more than C, and A $500 more than B. Find each person's salary.

11. A basket contains 50 apples and pears. The sum of three times the number of apples and twice the number of pears is 120. How many apples and how many pears are in the basket?

12. A field containing 40 acres is planted in corn and wheat. Four times the number of acres planted in corn added to three times the number of acres planted in wheat make 135. How many acres are planted in each?

13. A jobber buys 17 horses and mules, paying for the horses $80 apiece and for the mules $60 apiece. He pays in all $1220. How many of each does he buy?

14. A man buys two farms containing 130 acres. For one he pays at the rate of $35 an acre, and for the other $40

an acre. If he pays $4800 for the two farms, how many acres does each contain?

15. The perimeter of a rectangle is 260 yards and its length is 30 yards longer than its width. Find its dimensions.

16. The length of a rectangle exceeds its width by 50 yards, and three times the length exceeds five times the width by 70 yards. Find its dimensions.

17. A boy has 55 cents in nickels and dimes, and he has 8 coins in all. How many nickels and how many dimes has he?

18. A girl has $1.70 in dimes and quarters. She has three more dimes than quarters. How many of each has she?

19. A man has $115 in $5 and $10 bills. He has 5 more $5 bills than $10 bills. How many of each has he?

20. The sum of two consecutive odd numbers is 100. Find them.

21. The sum of three consecutive numbers is 99. Find them.

22. The sum of three consecutive even numbers is 108. Find them.

23. Find two consecutive numbers such that five times the greater is 21 more than four times the less.

24. Two numbers differ by 5, and three times the less added to twice the greater is 65. Find them.

25. The sum of four numbers is 80, and each number after the first is 4 greater than the one before it. Find the numbers.

26. A man bought a hat, a pair of shoes, and a suit of clothes for \$29. He paid \$1 more for the hat than for the pair of shoes, and the suit of clothes cost four times as much as the hat. Find the cost of each.

27. Two trains start at the same time from two stations 486 miles apart and travel toward each other at the rates of 24 and 30 miles per hour. In how many hours will they meet?

28. Two stations, A and B, are 380 miles apart. A train leaves A for B at 10 o'clock A.M. and travels at the rate of 18 miles an hour. Two hours later a train leaves B and travels toward A at the rate of 25 miles an hour. When will they meet?

29. A boy has \$4 in quarters, 10-cent, and 5-cent pieces. He has three times as many 10-cent pieces as quarters and three times as many 5-cent pieces as 10-cent pieces. How many of each has he?

30. The length of a garden exceeds its width by 30 yards. If the length is increased by 15 yards, the area of the garden would be increased by 600 square yards. Find the dimensions of the garden.

Example 1. Solve $\frac{x}{2}+\frac{x}{5}=14$.

Multiply both members of the equation by 10, the L. C. M. of 2 and 5.

Therefore, $5x+2x=140.$

Combining, $7x=140.$

$x=20.$

Check. $\frac{20}{2}+\frac{20}{5}=14.$

Example 2. Solve $\frac{3}{4}(x-2)-\frac{2}{3}(x-4)=1\frac{2}{3}$.

SOLUTION. Multiply both members of the equation by 12, the L. C. M. of the denominators.

$$12\times\frac{3}{4}(x-2)=9(x-2)=9x-18. \qquad (1)$$

$$12\times-\frac{2}{3}(x-4)=-8(x-4)=-8x+32. \qquad (2)$$

$$12\times 1\frac{2}{3}=20. \qquad (3)$$

Therefore, $9x-18-8x+32=20.$

Transposing, $9x-8x=20+18-32.$

$$x=6.$$

Check. $\frac{3}{4}(6-2)-\frac{2}{3}(6-4)=1\frac{2}{3}.$

$$3-1\frac{1}{3}=1\frac{2}{3}.$$

A little practice will enable one to perform the indicated operations, (1), (2), (3), mentally.

61. To solve an equation containing fractions:

(1) **Clear of fractions by multiplying both members of the equation by the least common multiple of the denominators of the fractions.**

(2) **Transpose the unknown quantities to the first member of the equation and the known quantities to the second member of the equation.**

(3) **Combine like terms.**

(4) **Divide by the coefficient of the unknown quantity.**

EXERCISE 45

Solve:

1. $\frac{x}{2}+\frac{x}{3}=15.$
2. $\frac{x}{3}+x-1=11.$
3. $\frac{2x}{3}+\frac{x}{2}=14.$
4. $\frac{x}{2}+\frac{x}{5}=21.$

5. $\frac{2x}{3} - \frac{x}{4} = 10.$

6. $\frac{x}{4} + \frac{x}{6} = 15.$

7. $\frac{3x}{4} - \frac{x}{2} = 3.$

8. $\frac{3x}{4} - \frac{2x}{3} = 2.$

9. $\frac{3x}{5} + \frac{x}{10} = 3\frac{1}{2}.$

10. $\frac{4x}{5} - \frac{3x}{10} = 10.$

11. $\frac{4x}{5} + \frac{x}{2} = 6\frac{1}{2}.$

12. $\frac{3x}{8} + \frac{3x}{4} = 4\frac{1}{2}.$

13. $\frac{7x}{8} - \frac{x}{2} = 1\frac{1}{2}.$

14. $\frac{9x}{10} - \frac{x}{5} = 3\frac{1}{2}.$

15. $\frac{x}{3} + \frac{2x}{5} = 2\frac{3}{4}.$

16. $\frac{2x}{3} + \frac{3x}{5} = 6\frac{1}{3}.$

17. $x + \frac{x}{2} - \frac{x}{3} = 7.$

18. $x - \frac{x}{4} - \frac{x}{5} = 5\frac{1}{2}.$

19. $\frac{1}{2}(x+1) + x = 11.$

20. $\frac{1}{3}(x+2) + x = 14.$

21. $\frac{1}{4}(x-1) + \frac{x}{2} = 9\frac{1}{2}.$

22. $\frac{1}{5}(2x-1) + \frac{x}{5} = 3\frac{2}{5}.$

23. $\frac{2}{3}(x-5) - \frac{x}{2} = -2\frac{1}{2}.$

24. $\frac{3}{4}(2x-7) - \frac{x}{3} = 1\frac{3}{4}.$

25. $\frac{1}{6}(3x-2) - \frac{x}{3} = 2\frac{1}{6}.$

26. $\frac{5}{6}(x-1) - \frac{1}{2}(x-4) = 3\frac{5}{6}.$

27. $\frac{3}{7}(2x+4) - \frac{1}{14}(x-1) = 5\frac{5}{7}.$

28. $\frac{3}{8}(3x-4) - \frac{1}{16}(5x-1) = 1\frac{13}{16}.$

29. $\frac{3}{4}(5x-6) - \frac{7}{8}(2x+3) = 8\frac{7}{8}.$

30. $\frac{2}{3}(2x-3) - \frac{1}{4}(x-5) = 3\frac{7}{12}.$

31. $\frac{7}{12}(4x-7) - \frac{x}{3} - 1 = 4\frac{11}{12}.$

32. $\frac{9}{10}(x-3)-\frac{x}{5}-2=1\frac{3}{5}$.

33. $\frac{4}{5}(1+x)-\frac{x}{2}-3\frac{4}{5}=0$.

34. $\frac{7}{8}(3+x)-\frac{1}{3}(x+5)=3\frac{2}{3}$.

35. $x-\frac{x}{3}-\frac{1}{4}(x-2)=7\frac{1}{6}$.

Example 1. If $\frac{1}{3}$ of a field is planted in wheat, $\frac{1}{4}$ in barley, and the remainder, amounting to 25 acres, is planted in corn, how many acres are in the field?

Let $x =$ number of acres in the field.

Then, $\frac{x}{3} =$ number of acres of wheat.

$\frac{x}{4} =$ number of acres of barley.

$25 =$ number of acres of corn.

Therefore, $\frac{x}{3}+\frac{x}{4}+25 =$ number of acres of wheat, corn, and barley.

Hence, $x=\frac{x}{3}+\frac{x}{4}+25.$

Clearing of fractions, $12x=4x+3x+300.$

Combining, $5x=300.$

$x=60.$

Check. $\frac{60}{3}+\frac{60}{4}+25=60.$

EXERCISE 46

1. Find the cost of a book if the sum of $\frac{1}{4}$ of the cost and $\frac{1}{3}$ of the cost is 50 cents.

2. The sum of $\frac{1}{2}$ and $\frac{1}{5}$ of the cost of a horse is \$70. What is the cost of the horse?

3. The sum of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ of a number is 85. Find the number.

4. A man travels from Norfolk, Va., to New York City in three days. The first day he travels $\frac{1}{3}$ of the distance, the second day he travels $\frac{2}{5}$ of the distance, and the third day he travels 120 miles. Find the distance from Norfolk, Va., to New York City.

5. If $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{6}$ of the distance from Pittsburg, Pa., to New York is 407 miles, find how far it is from Pittsburg to New York.

6. After $\frac{3}{5}$ and $\frac{7}{25}$ of a barrel of sugar are sold, 39 pounds remain. How many pounds of sugar did the barrel originally contain?

7. The sum of $\frac{1}{2}$ and $\frac{3}{7}$ of a man's age exceeds $\frac{2}{3}$ of his age by 11 years. How old is he?

8. A man's monthly salary was increased at one time by $\frac{1}{6}$, and at another by \$10. If his salary was then \$150, what was it before it was increased?

9. A corporation decides to increase its capital stock by $\frac{1}{5}$ of itself. If the stock is then \$120,000, how much was it before it was increased?

10. A farmer sold to one merchant $\frac{1}{2}$ of his apple crop, to another $\frac{1}{6}$ of it, to a third $\frac{1}{8}$ of it, and had then remaining 75 barrels. How many barrels did he have at first?

11. A farmer sold $\frac{1}{4}$ of the number of bushels of corn he raised to one merchant, and $\frac{1}{5}$ of the number of bushels

he raised to another merchant. If he sold 75 bushels more to the former than to the latter, how many bushels did he raise?

12. In going a certain journey a man travels $\frac{1}{2}$ the entire distance by rail, $\frac{2}{5}$ by boat, and the remainder, a distance of 8 miles, by carriage. How many miles did he travel in all?

13. If $\frac{4}{5}$ of a certain number is diminished by 10, the remainder is 10 more than $\frac{3}{4}$ of the number. Find the number.

14. If $\frac{2}{3}$ of a man's yearly income increased by \$60 equals $\frac{9}{10}$ of his income diminished by \$164, find his income.

15. In a certain school the number of girls is 40 more than $\frac{1}{2}$ of the number enrolled, and the number of boys is 10 more than $\frac{2}{5}$ of the number enrolled. Find the number enrolled.

Example **1.** The sum of two numbers is 84, and their difference is equal to $\frac{1}{4}$ of the greater number. Find them.

Let $x =$ the smaller number.

Then, $84 - x =$ the greater number.

$(84 - x) - x =$ the difference of the numbers.

$\frac{1}{4}(84 - x) = \frac{1}{4}$ of the greater number.

Therefore, $(84 - x) - x = \frac{1}{4}(84 - x)$.

Multiplying by 4, $336 - 4x - 4x = 84 - x$.

Transposing, $-4x - 4x + x = 84 - 336$.

Combining, $-7x = -252$.

Dividing by -7, $x = 36.$

$84 - x = 48.$

Check. $48 - 36 = \frac{1}{4}$ of 48.

EXERCISE 47

1. Divide 120 into two parts, so that $\frac{1}{2}$ of one part equals $\frac{1}{3}$ of the other part.

2. The difference of two numbers is 30, and $\frac{1}{4}$ of the smaller number equals $\frac{1}{5}$ of the larger number. Find the numbers.

3. Find two numbers differing by 4, such that 5 added to $\frac{1}{3}$ of the smaller number is 7 less than $\frac{1}{2}$ of the larger number.

4. Two farms together contain 180 acres. $\frac{1}{2}$ of the number of acres in the larger farm exceeds $\frac{1}{7}$ of the number of acres in the smaller farm by 45. Find the number of acres in each.

5. Divide 150 into two parts, so that one part is equal to $\frac{2}{3}$ of the other part.

6. Divide 35 cents between two boys, giving one boy $\frac{3}{4}$ as many cents as the other boy.

7. A rectangle is 20 yards longer than it is wide, and $\frac{1}{4}$ of the width added to $\frac{1}{5}$ of the length is equal to 40 yards. Find its dimensions.

8. The length of a rectangle is $2\frac{1}{2}$ times the width, and the difference of the length and width is 750 yards. Find its dimensions.

9. Divide \$110 between A and B so that $\frac{1}{2}$ of A's share may be \$5 more than twice B's share.

10. In 5 years A will be half as old as B was 10 years ago. B is four times as old as A. Find their ages.

11. The sum of the ages of a husband and wife is 70 years. Ten years ago the husband's age was $1\frac{1}{2}$ times the age of his wife. Find their ages.

12. A ticket agent sells 6 single tickets and 5 return tickets for $21.70. A return ticket cost 30 cents less than twice the cost of a single ticket. Find the cost of each.

13. A boy has 21 coins consisting of quarters and dimes, and the value of all the quarters is equal to the value of all the dimes. Find the number of each.

14. A man is hired to work for 35 days on condition that for every day he works he receives $2, and for every day he is absent he pays $1. If he receives all together $43, how many days did he work?

15. A man left his estate to his wife and four children. The wife's share was $\frac{1}{3}$ of the estate, and the children received equal portions. If the wife received $850 more than the share of a child, find the value of the estate.

16. Surrounding a rectangle whose length is twice its width, is a walk 5 feet wide. The area of the walk is 3700 square feet. Find the dimensions of the rectangle.

17. Mary is twice as old as Ann. 10 years ago she was $2\frac{1}{2}$ times as old as Ann. Find their ages.

18. A father is 5 times as old as his son. In 3 years he will be four times as old as his son. Find their ages.

19. B has $35 more than A. If A gave B $15, A would have $\frac{5}{18}$ as much as B. Find A's and B's money.

20. A man buys 5 horses and 12 cows for \$975. A horse costs \$25 more than a cow. Find the price of each.

21. A woman buys a number of pounds of tea at 60 cents a pound, and $\frac{3}{5}$ as many pounds at 75 cents a pound. If the tea costs all together \$10.50, how many pounds of each does she buy?

22. A farmer has two lots planted in wheat, the second lot containing $\frac{3}{4}$ as many acres as the first. The first lot yields 18 bushels to the acre, and the second lot 20 bushels to the acre. The two lots yield 990 bushels. How many acres are in each lot?

23. Divide \$420 among A, B, and C, giving A one half as much as B, and one third as much as C.

Example 1. Solve $\frac{x-7}{3}-\frac{x-2}{4}-1\frac{1}{2}=4\frac{1}{4}-x.$

Multiply both members of the equation by 12.

$$4x-28-3x+6-18=51-12x.$$

Transposing, $\quad 4x-3x+12x=51+28-6+18.$

Combining, $\quad 13x=91.$

$$x=7.$$

Check. $\quad \frac{7-7}{3}-\frac{7-2}{4}-1\frac{1}{2}=4\frac{1}{4}-7.$

$$0-1\frac{1}{4}-1\frac{1}{2}=-2\frac{3}{4}.$$

In this example the multiplication of $-\frac{x-2}{4}$ by 12 may puzzle some beginners. To make this clear, indicate the work as follows:

$$-\frac{x-2}{4}\times 12=-\frac{12(x-2)}{4}=-\frac{3(x-2)}{1}$$
$$=-3(x-2)=-3x+6.$$

EXERCISE 48

Solve:

1. $\frac{x+2}{4}+\frac{x+3}{2}=2\frac{1}{4}.$

2. $\frac{x-3}{3}-2\frac{1}{2}=4\frac{1}{2}-x.$

3. $\frac{2x-1}{4}+\frac{x+5}{2}=4x+\frac{3}{4}.$

4. $\frac{9x-11}{5}-x=\frac{x}{2}-1.$

5. $\frac{7x-8}{6}+\frac{2x+5}{3}=2x-1.$

6. $\frac{5x+4}{7}-\frac{2x-4}{14}=x-3.$

7. $\frac{4x+11}{3}-\frac{2x+15}{5}=x.$

8. $\frac{5x-7}{4}-\frac{x-5}{6}=x-\frac{1}{3}.$

9. $\frac{9x-13}{8}-\frac{3x+1}{16}=\frac{x+19}{8}.$

10. $\frac{6x+2}{5}+\frac{1}{4}(3x-20)=2x-5.$

11. $\frac{4x-1}{10}-\frac{3x-12}{5}=\frac{1}{2}.$

12. $\frac{9x+13}{2}-2(x-3)=7x-1.$

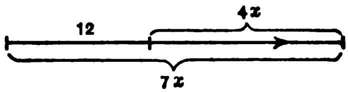
13. $\frac{3x-20}{15}-\frac{1}{4}(x+7)=6x-3\frac{1}{12}.$

14. $\frac{8x-3}{7}+\frac{3x+12}{14}=5x-\frac{7x+2}{2}.$

15. $\frac{2x+5}{9}+\frac{11-x}{2}=\frac{x+7}{18}+2.$

Example 1. A vagrant leaves a town, traveling at the rate of 4 miles an hour. Three hours later an officer follows him, traveling at the rate of 7 miles an hour. In how many hours will the officer overtake the vagrant?

In 3 hours the vagrant travels 12 miles. Hence, the vagrant has a start of 12 miles.



Let $x =$ the number of hours.

Then, $7x =$ the distance traveled by the officer.

$4x =$ the distance traveled by the vagrant in x hours.

$4x + 12 =$ the entire distance traveled by the vagrant.

Hence, $7x = 4x + 12.$

$3x = 12.$

$x = 4.$ *Ans.* 4 hours.

Check. $7 \times 4 = 12 + 4 \times 4.$

EXERCISE 49

1. A freight train leaves a station, traveling at the rate of 20 miles an hour. Three hours later a passenger train leaves the same station, traveling at the rate of 32 miles an hour. In how many hours will the latter overtake the former?

2. A ship leaves a port and travels at the rate of 15 miles an hour. Nine hours later a second ship follows

her at the rate of 20 miles an hour. After how many hours will the ships have gone the same distance?

3. A and B start with the same capital. A saves $500 a year and B $900 a year. Within a year A receives a legacy of $2000. After how many years will B be worth as much as A?

4. Two trains leave at the same time two stations 325 miles apart and travel in opposite directions. If their rates are 32 and 33 miles an hour, in how many hours will they meet?

5. A train leaves a station P at 4 o'clock A.M. and travels toward G at the rate of 28 miles an hour. Four hours later a train leaves G and travels toward P at the rate of 30 miles an hour. When will they meet, if P is 518 miles from G?

EXERCISE 50

Solve:

1. $\frac{x+4}{3}+\frac{x+2}{4}=6\frac{1}{2}.$

2. $\frac{x-1}{4}+\frac{x+7}{2}=x.$

3. $\frac{2x-5}{2}-\frac{x-1}{5}=3\frac{3}{10}.$

4. $\frac{5x-1}{6}-\frac{2x+5}{3}=2x-11.$

5. $\frac{4x+5}{7}+\frac{x+1}{2}=24-x.$

6. $\frac{3x+8}{4}-\frac{2x-1}{5}=\frac{7x-6}{10}.$

7. $\frac{x+11}{8}+\frac{2x+1}{3}=\frac{5x+7}{6}.$

8. $\frac{10x-1}{6}-\frac{x-2}{2}=4x-2.$

9. $\frac{4x-1}{9}+\frac{2x+4}{3}=\frac{3x-3}{2}.$

10. $\frac{7x+1}{5}-\frac{x+2}{3}=\frac{26x-7}{15}.$

11. $\frac{4x+2}{5}-\frac{x-4}{4}-5=x-9.$

12. $2x-\frac{x-3}{4}-\frac{1}{3}(2x-1)-x=0.$

13. $x-\frac{1}{4}(x+1)-\frac{1}{3}(x+6)-4=0.$

14. $5x+\frac{7-x}{7}-\frac{8-17x}{2}+3=0.$

15. $\frac{1}{5}(x-1)+\frac{1}{6}(x+9)-x+12=0.$

16. A train traveling at the rate of 24 miles per hour takes 3 hours less time to go a certain distance than a train traveling 20 miles an hour. Find the distance.

17. A passenger train travels $1\frac{1}{2}$ times as fast as a freight train, and goes in 5 hours 60 miles farther than the freight train goes in $4\frac{1}{2}$ hours. Find their rates.

18. If a train increases its rate by one fourth, it would go 160 miles in one hour less time than it formerly took. Find its rate per hour.

19. The cost of a suit of clothes is $3\frac{1}{2}$ times the cost of a pair of shoes, and the cost of 4 suits of clothes is $7 more than the cost of 12 pairs of shoes. Find the cost of a pair of shoes and of a suit of clothes.

CHAPTER V

PERCENTAGE. INTEREST

62. Computation on the basis of 100 is called **percentage.**

Per cent means "on the hundred," or by the hundred. By rate per cent is meant a rate on a basis of 100.

Thus, if a man buys a house and afterwards sells it, making a profit of \$6 on every \$100 invested in the house, his *rate per cent* of gain is 6, and his *rate of gain* is $\frac{6}{100}$. Rate and ratio are allied terms. A rate may be based on any quantity whatever. When the basis is 100 the rate is called a rate per cent.

Example 1. Find 7% of 250.

$$7 \times \tfrac{250}{100}, \text{ or } \tfrac{7}{100} \times 250 = 17.5.$$

Example 2. 8% of a number is 54. Find the number.

Let $x =$ the number.

Then, $\frac{8}{100}$ of x, or $\frac{8x}{100} = 8\%$ of the number.

Therefore, $\frac{8x}{100} = 54.$

Clearing of fractions, $8x = 5400.$

$$x = 675.$$

Check. $675 \times \frac{8}{100} = 54.$

EXERCISE 51

1. 7% of a man's salary is \$91. Find his salary.

2. An agent charges a commission of 4% for selling land. If his commission is \$225, find the amount of his sales.

3. A certain copper ore contains 6% of copper. If the copper obtained from a piece of this ore weighs 6.75 pounds, find the weight of the piece of ore.

4. 1250 persons in a certain city can neither read nor write. If this number is 4% of the population, find the population.

5. A certain stock yields a dividend at the rate of 5%. If each share of stock yields a dividend of \$4, find the price of a share of stock.

6. 85% of a farm consists of arable land. If the number of acres of arable land in the farm is 77.35, how many acres does the farm contain?

7. 81 pupils in a school failed to be promoted. This number is 18% of the number of pupils in the school. Find the number of pupils in the school.

8. The length of the Orinoco is 60% of the length of the La Plata. The Orinoco is 1500 miles long. How long is the La Plata?

9. 9% of a man's age is 3.78 years. Find the man's age.

10. A man's income from an investment yielding $4\frac{1}{2}$% is \$684. Find the sum invested.

Example 1. The cost of insuring a building valued at \$5470 is \$136.75. Find the rate per cent of insurance.

Let $x =$ the rate per cent.

Then, $5470 \times \frac{x}{100} =$ number of dollars charged for insurance.

$$5470 \times \frac{x}{100} = \frac{547x}{10}.$$

Therefore, $\frac{547x}{10} = 136.75.$

Clearing of fractions, $547x = 1367.5.$

$x = 2.5.$ *Ans.* $2\frac{1}{2}\%$.

Check. $2\frac{1}{2}\%$ of \$5470 = \$136.75.

Example 2. Find what per cent of 19 is 1.52.

Let $x =$ required per cent.

Then, $19 \times \frac{x}{100} = 1.52.$

Clearing of fractions, $19x = 152.$

$x = 8.$ *Ans.* 8%.

Check. 8% of 19 = 1.52.

EXERCISE 52

1. What per cent of 1 mile is 132 feet?

2. What per cent of a square mile is 80 acres?

3. In a school of 350 pupils 280 are promoted. What per cent of the entire number is promoted?

4. When a share of stock sells for \$90 and pays a

dividend of \$4.50, what per cent does money invested in such stock bring?

5. A bankrupt's liabilities amount to \$4800, and his assets amount to \$4000. What per cent of his debts is he able to pay?

6. When a life insurance policy for \$4500 costs the assured \$112.50, find the rate per cent charged by the insurance company.

7. An agent sells a tract of land for \$6500, and charges a commission of \$292.50. Find the rate of commission.

8. The area of Maine is 33,040 square miles. The water surface of Maine is 3145 square miles. What per cent of Maine is covered with water?

9. The total income of a college is \$254,000. The income from tuition fees is \$114,300. What per cent of the income is derived from tuition fees?

10. A clothier sells for \$18.50 a suit of clothes which cost him \$12.50. Find his gain per cent.

Example 1. An organ is sold for \$900 at a profit of $12\frac{1}{2}\%$. Find the cost of the organ.

Let $x =$ the number of dollars in the cost.

$\frac{12\frac{1}{2}x}{100}$ or $\frac{x}{8} =$ the number of dollars profit.

$x + \frac{x}{8} =$ the number of dollars in the selling price.

Therefore, $x + \frac{x}{8} = 900.$

$$8x + x = 7200.$$

$$9x = 7200.$$

$$x = 800.$$

Check. $\$800 + 12\frac{1}{2}\%$ of $\$800 = \$900.$

EXERCISE 53

1. A trader sells a horse for $144 and makes thereby a profit of 80%. Find the cost of the horse.

2. By selling a box of oranges for $2.75 a dealer makes a profit of 25%. How much does the dealer pay for the oranges?

3. A piano is sold for $215, at a profit of $7\frac{1}{2}\%$. Find its cost.

4. A watch is sold for $25.50, at a loss of 15%. Find the cost price of the watch.

5. An agent charges 4% commission for selling land. If the agent remits to his principal $9216, find the amount of his sales.

6. A tree is broken in a storm. The part left standing is 96 feet in height. If 36% of the height of the tree was broken off, how high was the tree?

7. The price of a suit of clothes after receiving a discount of 15% from the marked price was $16.15. What was the marked price?

8. Goods are sold at a discount of 6% for cash. If the cash value of a bill of goods is $741, find the amount of the bill.

9. The value of the cotton crop of the United States in the year 1902–1903 was $480,000,000. This was $6\frac{3}{8}\%$ more than the value of the cotton crop in the United States the previous year. Find the value of the crop in the year 1901–1902.

10. The number of ounces of gold coined by the mints of the world in the year 1902 was 12,002,000. This was 30.1% less than the number of ounces coined in 1901. Find the number of ounces coined in 1901.

SIMPLE INTEREST

63. Money paid for the use of money is called **interest.**

The sum lent is called the **principal.**

The interest of a principal for one year is obtained by multiplying the principal by the rate of interest.

64. To calculate the interest of a sum of money for a number of years, find first the interest of the sum for one year, and then multiply this result by the number of years. Hence,

Interest = principal × rate × time (in years).

Example 1. The interest of $579 at 7% is $54.04. Find the time.

Let $x =$ the number of years.

Then, $\$579 \times \frac{7}{100} \times x =$ the interest, *i.e.* $54.04.

Hence, $$\frac{579 \times 7 \times x}{100} = 54.04.$$

Clearing of fractions,

$$579 \times 7 \times x = 5404.$$

$$x=\frac{5404}{579\times 7}=1\tfrac{1}{3}.$$

Ans. $1\frac{1}{3}$ years, or 1 year 4 months.

Check. $\$579\times .07\times \frac{4}{3}=\$54.04.$

EXERCISE 54

In what time will:

1. \$885 produce \$53.10 at 6% ?
2. \$728 produce \$54.60 interest at 5% ?
3. \$670 produce \$70.35 interest at 7% ?
4. \$1260 produce \$126 interest at 8% ?
5. \$385 produce \$32.34 interest at 7% ?
6. \$2750 produce \$137.50 interest at 3% ?
7. \$3345 produce \$267.60 interest at 6% ?
8. \$783 produce \$34.80 interest at 4% ?
9. \$597 produce \$41.79 interest at 5% ?
10. \$3000 produce \$56 interest at 7% ?

Example 1. At what rate per cent will \$975 produce \$23.40 interest in 7 months 6 days?

Let $x=$ rate % or rate on \$100.

Then, $\frac{x}{100}=$ rate.

7 months 6 days $=216$ days $=\frac{216}{360}$ year $=\frac{3}{5}$ year.

$$\$975\times\frac{x}{100}\times\tfrac{3}{5}=\text{interest, } i.e.\ \$23.40.$$

$$975\times\tfrac{3}{5}\times\frac{x}{100}=\frac{195\times 3\times x}{100}.$$

Therefore, $\frac{195 \times 3 \times x}{100} = 23.40.$

$$195 \times 3 \times x = 2340$$

$$x = \frac{2340}{195 \times 3} = 4. \quad \textit{Ans.}\ 4\%.$$

Check. $\$975 \times .04 \times \frac{3}{5} = \$23.40.$

EXERCISE 55

At what per cent will:

1. \$785 produce \$39.25 interest in 1 year?
2. \$680 produce \$47.60 interest in 1 year?
3. \$960 produce \$86.40 interest in 1 year?
4. \$1250 produce \$25 interest in 4 months?
5. \$4500 produce \$126 interest in 4 months 24 days?
6. \$5100 produce \$306 interest in 9 months?
7. \$548 produce \$6.85 interest in 3 months?
8. \$940 produce \$37.60 interest in 1 year 4 months?
9. \$1800 produce \$9 interest in 1 month 15 days?
10. \$2100 produce \$231 interest in 2 years 9 months?

Example 1. What principal will produce \$18.10 interest in 3 months 25 days at 4%?

Let x = number dollars in the principal.

3 months 25 days = 115 days = $\frac{115}{360}$ year = $\frac{23}{72}$ year.

$$x \times \frac{4}{100} \times \frac{23}{72} = \frac{23x}{1800} = \text{number dollars interest.}$$

Therefore, $\frac{23x}{1800} = 18.10.$

$$x = \frac{1800 \times 18.10}{23} = 1416.52.$$

Ans. $1416.52.

EXERCISE 56

What principal will produce:

1. $30.48 interest in 1 year at 4%?
2. $36.90 interest in 1 year at 6%?
3. $61.84 interest in 1 year at 8%?
4. $35.40 interest in 1 year 6 months at 4%?
5. $102.75 interest in 1 year 6 months at 5%?
6. $118.40 interest in 1 year 4 months at 6%?
7. $97.92 interest in 1 year 6 months at 4%?
8. $52.80 interest in 1 year 2 months 12 days at 5%?
9. $57.42 interest in 1 year 1 month 15 days at 8%?
10. $23.60 interest in 3 months 28 days at 6%?

Example 1. What principal will amount to $792 in 9 months 18 days at 7%?

Let $x =$ number of dollars in principal.

9 months 18 days = 288 days = $\frac{288}{360}$ year = $\frac{4}{5}$ year.

$x \times \frac{7}{100} \times \frac{4}{5} = \frac{28x}{500} =$ number of dollars interest.

$x + \frac{28x}{500} =$ number dollars in the amount.

Therefore, $x + \frac{28x}{500} = 792.$

$$500x + 28x = 396000.$$

$$528x = 396000.$$

$$x = 750. \quad \textit{Ans.}\ \$750.$$

Check. $\$750 \times .07 \times \frac{4}{5} = \$42. \quad \$750 + \$42 = \$792.$

EXERCISE 57

What principal will amount to:

1. $756 in 1 year at 5%?
2. $1605 in 1 year at 7%?
3. $2014 in 1 year at 6%?
4. $523.80 in 1 year at 8%?
5. $483.36 in 1 year 6 months at 4%?
6. $1896.25 in 6 months at 5%?
7. $1684 in 9 months at 7%?
8. $1221 in 1 year 4 months 15 days at 8%?
9. $429.40 in 2 years 1 month at 6%?
10. $626.75 in 1 year 9 months 18 days at 5%?

EXERCISE 58

1. In what time will $100 produce $20 interest at 5%? at 6%? at 8%?

2. In what time will $100 produce $50 interest at 4%? at 6%? at 10%?

3. In what time will $100 amount to $140 at 5% simple interest?

4. In what time will \$100 amount to \$200 at 6%? at 8%? at 7%?

5. At what rate per cent will \$100 amount to \$135 in 7 years?

6. At what rate per cent will \$100 double itself in 20 years? in 25 years? in $12\frac{1}{2}$ years?

7. At what rate per cent will \$100 amount to \$124 in 4 years?

8. Find the time in which a sum of money amounts to $1\frac{1}{2}$ times itself at 5%? at 6%? at 8%?

9. The cost price of an article is \$ C and the profit is r%. Find the selling price.

10. Find the selling price if the cost price of an article is \$ C and it was sold at a loss of r%.

CHAPTER VI

SIMULTANEOUS EQUATIONS

65. In solving problems requiring two answers, it is generally best to use two letters to represent the unknown quantities, and then from the conditions of the problem to form two equations and find from these equations the values of the unknown quantities.

As an illustration take the following problem:

Example 1. The sum of two numbers is 30 and their difference is 6. Find the numbers.

Let $x =$ the greater number.

$y =$ the less number.

Then, $x + y =$ the sum of the two numbers.

$x - y =$ the difference of the two numbers.

Hence, $x + y = 30.$ (1)

$x - y = 6.$ (2)

Adding the first members, and also the second members, of the equations (1) and (2), the result in one case is $2x$ and in the other case 36.

Hence, $2x = 36.$ (3)

$x = 18.$

Substitute in equation (1) 18 for x,

$18 + y = 30.$

Transposing, $y = 12.$

The numbers are 18 and 12.

Check. $18 - 12 = 6.$

66. An equation involving two unknown quantities, such as $x + y = 30$, has no definite solution. For if $x = 1$, then $y = 29$. If $x = 2$, then $y = 28$. If $x = 5$, then $y = 25$.

The equation, $x + y = 30$, states in algebraic language that the sum of two numbers is 30. Nothing more is known about the two numbers until another condition is given. When that other condition is given, then, and not before, can the numbers be determined.

Equations containing two or more letters, the same letters representing the same numbers in each equation, are called **simultaneous equations** (Latin, *simul*, at the same time, and *teneo*, I hold).

Simultaneous equations, then, express different conditions of the same problem and are so called because the same letter has the same value in each of the equations.

Example 2. The difference of two numbers is 15 and the greater number exceeds twice the less by 1. Find the numbers.

Let $x =$ the greater number.

$y =$ the less number.

Then, $x - y = 15. \quad (1)$

$x - 2y = 1. \quad (2)$

Subtract the members of equation (2) from those of equation (1) and get $y = 14. \quad (3)$

Substitute 14 for y in equation (1),

$$x - 14 = 15. \qquad (4)$$

Transposing, $x = 29.$

Check. $29 - 14 = 15.$

$$29 - 2 \times 14 = 1.$$

Example 3. $2x + 5y = 27. \qquad (1)$

$$7x + 3y = 22. \qquad (2)$$

Step 1. Multiply the members of equation (1) by 3 and the members of equation (2) by 5,

$$6x + 15y = 81. \qquad (3)$$

$$35x + 15y = 110. \qquad (4)$$

Step 2. Subtract the members of equation (4) from those of equation (3), and get by axiom 3,

$$-29x = -29. \qquad (5)$$

$$x = 1.$$

Step 3. Substitute in either of the original equations 1 for x. Substituting 1 for x in equation (1), the result is

$$2 + 5y = 27.$$

Transposing, $5y = 25.$

$$y = 5.$$

Check by substituting the values of x and y in equation (2), $7 + 3 \times 5 = 22.$

Example 4. $5x - 4y = -4. \qquad (1)$

$$3x + 2y = 24. \qquad (2)$$

Step 1. Make the coefficients of y in both equations equal in absolute value. This is done by multiplying the members of equation (2) by 2,

$$6x + 4y = 48. \qquad (3)$$

$$5x - 4y = -4. \qquad (1)$$

Step 2. Add the members of (3) and (1),

$$11x = 44. \qquad (4)$$

$$x = 4.$$

Step 3. Substitute in equation (1) 4 for x,

$$20 - 4y = -4. \qquad (5)$$

Transposing, $-4y = -24.$

$$y = 6.$$

Check. $3 \times 4 + 2 \times 6 = 24.$

NOTE. Since equations are not numbers or quantities, it is incorrect to say add, subtract, multiply, or divide equations. The members of equations are quantities, hence they can be added, subtracted, multiplied, or divided.

67. To solve two simultaneous equations:

(1) **Make the coefficients of the same letter in both equations equal in absolute value.**

(2) **Add if the coefficients of this letter in the two equations have unlike signs, subtract if they have like signs.**

(3) **Solve the resulting equation.**

(4) **Substitute the value of the letter thus found in either of the original equations.**

EXERCISE 59

Solve:

1. $x + y = 18,$
   $x - y = 8.$

2. $x + y = 10,$
   $x - y = 2.$

3. $x + y = 15,$
   $x - y = 7.$

4. $x + y = 19,$
   $x - y = 3.$

5. $2x - y = 16,$
   $x + y = 11.$

6. $2x + y = 17,$
   $x - y = 1.$

7. $3x - 2y = 4,$
   $x + 2y = 20.$

8. $4x - y = 18,$
   $x + y = 7.$

9. $x + 3y = 34,$
   $-x + 4y = 43.$

10. $5x - 4y = 9,$
    $x + 4y = 21.$

11. $2x - 3y = 5,$
    $x + y = 5.$

12. $2x + y = 24,$
    $3x - 2y = 22.$

13. $5x - y = 44,$
    $3x + 4y = 31.$

14. $7x - 6y = 0,$
    $x + y = 13.$

15. $3x + 8y = 40,$
    $x - y = 6.$

16. $4x - 9y = 21,$
    $x + 3y = 0.$

17. $5x + 8y = 54,$
    $3x - 2y = 12.$

18. $11x - 4y = 48,$
    $2x + y = 7.$

19. $2x + 3y = 16,$
    $7x + 2y = 39.$

20. $4x + 3y = 6,$
    $9x + 4y = 19.$

21. $5x + 7y = 14,$
    $2x + 5y = -1.$

22. $8x + 9y = -4,$
    $5x + 3y = 8.$

23. $7x + 11y = 21,$
    $3x + 2y = 9.$

24. $8x - 5y = 0,$
    $3x - 2y = -1.$

25. $3x + 7y = 10,$
$5x + 14y = 19.$

26. $9x - 4y = 10,$
$3x - 2y = 2.$

27. $4x - 15y = 23,$
$7x - 5y = 19.$

28. $11x - 16y = 43,$
$5x - 8y = 21.$

29. $13x - 10y = 56,$
$5x - 2y = 16.$

30. $15x - 11y = 89,$
$3x + 2y = 1.$

68. Solve:

$$\frac{x}{2} - 4 = \frac{y}{3} + 3. \qquad (1)$$

$$\frac{x}{y+1} = 2. \qquad (2)$$

Step 1. Clear equation (1) of fractions, and then transpose.

Multiplying by 6,

$$3x - 24 = 2y + 18.$$

Transposing, $3x - 2y = 42.$ (3)

Step 2. Clear equation (2) of fractions and then transpose.

Multiplying by $y + 1$, $x = 2y + 2.$

Transposing, $x - 2y = 2.$ (4)

Step 3. Solve equations (3) and (4),

$$3x - 2y = 42 \qquad (3)$$
$$x - 2y = 2 \qquad (4)$$
$$2x = 40$$
$$x = 20.$$

Substituting in (4), $y = 9.$

Check. $\frac{20}{2} - 4 = \frac{9}{3} + 3.$

EXERCISE 60

Solve:

1. $\frac{x}{2}+\frac{y}{3}=6,$
$2x-y=10.$

2. $\frac{x}{3}-5=\frac{y-4}{4}+2,$
$x=2y-1.$

3. $\frac{x-1}{5}+\frac{y-2}{4}=3,$
$5x=3y.$

4. $\frac{x+3}{4}-\frac{y-1}{5}=1,$
$x=y-2.$

5. $\frac{x}{y+3}=1,$
$3x=4y+4.$

6. $\frac{2x}{y-4}=2,$
$\frac{2y}{x-5}=5.$

7. $3x-\frac{1}{2}y=17,$
$\frac{1}{2}x+2y=19\frac{1}{2}.$

8. $\frac{1}{3}x+3y=16,$
$4x+\frac{1}{2}y=50.$

9. $\frac{x+3y}{x+y}=1+\frac{4}{x+y},$
$x=8y.$

10. $\frac{10x+y}{x+y}=9,$
$x+y=9.$

11. $\frac{x}{y+1}=3.$
$\frac{1}{3}x+\frac{1}{2}y=4.$

12. $\frac{x+17}{y}=1\frac{1}{2},$
$3x=2y-1.$

69. *Example* 1. The sum of the digits of a number of two figures is 11, and if 45 is subtracted from the number, the digits will be interchanged.

Let $x=$ the digit in the tens' place.

$y=$ the digit in the units' place.

Then, $x+y=11.$ (1)

The value of a digit in the tens' place is ten times the value of the same digit in the units' place.

Hence, $10x + y =$ the number whose digits are x and y.

If the digits are interchanged, the resulting number is y tens plus x, *i.e.* $10y + x$.

Therefore, $10x + y - 45 = 10y + x$. (2)

Transposing, $9x - 9y = 45$. (3)

Dividing by 9, $x - y = 5$. (4)

But, $x + y = 11$. (1)

Adding, $2x = 16$.

$$x = 8.$$

$y = 3$. The number is 83.

Check. $83 - 45 = 38$.

Example 2. A bill of $1.15 is paid with 16 coins consisting of dimes and 5-cent pieces. Find the number of each.

Let $x =$ the number of dimes.

$y =$ the number of 5-cent pieces.

Then, $10x =$ the value in cents of all the dimes.

$5y =$ the value in cents of all the 5-cent pieces.

115 cents $=$ the amount of the bill.

Hence,

$$10x + 5y = 115. \quad (1)$$

$$x + y = 16. \quad (2)$$

Dividing (1) by 5,

$$2x + y = 23. \quad (3)$$

Subtracting,

$$x = 7.$$

$$y = 9.$$

Check. 7 dimes = 70 cents.

9 5-cent pieces = 45 cents.

70 cents + 45 cents = \$1.15.

EXERCISE 61

1. The sum of two numbers is 59 and their difference is 11. Find them.

2. The sum of two numbers is 80 and the greater exceeds twice the less by 8. Find them.

3. Find two numbers differing by 10, such that twice the greater exceeds five times the less by 5.

4. Two cords of oak wood and 3 cords of pine wood together cost \$22, and 3 cords of oak wood and 2 cords of pine wood together cost \$23. Find the cost of a cord of each.

5. The daily wages of 5 bricklayers and 4 carpenters amounts to \$37, and the daily wages of 3 bricklayers and 7 carpenters amounts to \$36. Find the daily wages of a bricklayer and a carpenter.

6. The monthly salaries of a bookkeeper and a clerk amount to \$140, and five times the salary of the bookkeeper equals nine times the salary of the clerk. Find the salary of each.

7. The price of 7 arithmetics and 5 grammars is \$6.35, and the price of 4 of those arithmetics and 3 of

the grammars is $3.70. Find the price of an arithmetic and a grammar.

8. The sum of the digits of a number of two figures is 11, and if 27 be subtracted from the number, the digits will be interchanged. Find the number.

9. The sum of the digits of a number of two figures is 9, and if 63 be added to the number, the digits will be interchanged. Find the number.

10. A number is expressed by two digits. The units' digit equals three times the tens' digit. The number exceeds three times the sum of its digits by 2. Find the number.

11. A number expressed by two digits is equal to seven times the sum of its digits. The tens' digit is 4 less than three times the units' digit. Find the number.

12. A man pays a debt of $100 with 17 bills, part $5 bills and part $10 bills. How many of each did he use?

13. A man pays a bill of $85 with 12 coins composed of half-eagles and eagles. How many of each did he use?

14. A donkey and a mule journeyed together, each bearing a heavy burden. "Why dost thou sigh?" said the mule. "Give me one measure of thy burden and mine shall be double thine." "No," said the donkey; "give me one of thine and our burdens shall be equal." How many measures did each carry?

15. A says to B, "Give me $\frac{1}{2}$ of your money and I shall have $110." "No," said B; "but give me $\frac{1}{2}$ of yours, and I shall have $100." How many dollars has each?

16. A has $\frac{3}{5}$ as much money as B. If B gives A \$10, he will then have $\frac{3}{5}$ as much money as A. How much money has each?

17. Divide \$120 between A and B so that five times A's money shall be equal to seven times B's money.

18. A man has \$8000 at interest, part at 4% and part at 5%. If he receives \$350 from both investments, how many dollars has he in each?

19. Two trains start at the same time from two stations 540 miles apart and travel toward each other. One train travels as far in 4 hours as the other does in 5 hours. If they meet in 10 hours, find the rate of each.

20. Find the price per pound of ham and of breakfast bacon when the cost of a pound of ham and half a pound of breakfast bacon is 24 cents, and the cost of 3 pounds of ham and 5 pounds of breakfast bacon is \$1.35.

21. A merchant finds that a mixture of 3 bushels of oats and 7 bushels of corn can be sold without gain or loss at 42 cents per bushel, and a mixture of 2 bushels of oats and 3 bushels of corn can be sold at 41 cents per bushel. Find the price of oats and corn.

22. A merchant mixes two kinds of tea. If he mixes 2 pounds of good tea and 3 pounds of inferior tea, he can sell the mixture at 71 cents per pound. If he mixes 8 pounds of good tea and 7 pounds of inferior tea, he can sell the mixture at 73 cents per pound. Find the price of each kind per pound.

23. A certain fraction is equal to $\frac{8}{5}$, and its denominator exceeds its numerator by 16. Find the numerator and denominator.

24. A has twice as many dollars as half dollars, and B, who has as many coins as A, has twice as many half dollars as dollars. Together they have $18. How many dollars and half dollars has A?

25. A father is three times as old as his son. Six years ago he was four times as old as his son. Find their ages now.

26. Two trains start at the same time from two stations 288 miles apart. They meet in 6 hours. The rate of one train is $\frac{2}{3}$ that of the other. Find their rates.

27. The daily wages of A and B is $5.60. A earns as much in 3 days as B does in 4 days. Find A's and B's daily wages.

28. The sum of two angles is 90°, and $\frac{1}{4}$ of the number of degrees in one angle equals $\frac{1}{5}$ of the number of degrees in the other angle. How many degrees are in each angle?

29. In a right triangle one acute angle is equal to $\frac{2}{3}$ of the other acute angle. Find the number of degrees in each angle of the triangle.

30. One third of the width of a rectangular garden equals $\frac{1}{4}$ of its length. If the length was diminished by 8 yards and the width increased by 2 yards, the garden would then be a square. Find its dimensions.

70. If three simultaneous equations of the first degree, involving three unknown quantities, are given, the values of the unknown quantities may be determined as follows:

(1) Eliminate one of the letters from a pair of the equations.

(2) Eliminate the same letter from another pair of the equations.

(3) Solve the resulting equations.

(4) Substitute the values of the two unknown quantities thus found in any one of the three original equations, and then solve the resulting simple equation to get the value of the remaining letter.

Example 1. Solve the simultaneous equations:

$$4x - y - 2z = 2. \qquad (1)$$

$$3x - 4y + 3z = 13. \qquad (2)$$

$$5x - 3y - 4z = -7. \qquad (3)$$

Step 1. Eliminate z between (1) and (2) by multiplying (1) by 3 and (2) by 2, and adding.

$$\begin{aligned} 12x - 3y - 6z &= 6 \\ 6x - 8y + 6z &= 26 \\ \hline 18x - 11y \qquad &= 32 \qquad (4) \end{aligned}$$

Step 2. Eliminate z between (1) and (3) by multiplying (1) by 2 and subtracting.

$$\begin{aligned} 8x - 2y - 4z &= 4 \\ 5x - 3y - 4z &= -7 \\ \hline 3x + y \qquad &= 11 \qquad (5) \end{aligned}$$

Step 3. Solving (4) and (5), we find

$$x = 3.$$

$$y = 2.$$

Step 4. Substituting these values of x and y in (1), we have

$$12 - 2 - 2z = 2.$$

Transposing,

$$-2z = -8,$$

$$z = 4.$$

Check by substituting these values of x, y, z, in equation (2) or (3).

EXERCISE 62

Solve the simultaneous equations:

1. $x + y + z = 6,$
$3x + 2y + 3z = 16,$
$4x + 3y + 2z = 16.$

2. $x + 2y - z = 3,$
$2x - 3y - 5z = -20,$
$4x - y - z = 6.$

3. $3x - 2y - 5z = -2,$
$4x - 3y - 2z = 11,$
$5x - 4y - 7z = 0.$

4. $5x - 3y - 4z = 8,$
$4x - 3y - 2z = 10,$
$3x - 2y - 3z = 2.$

5. $7x - 4y - 4z = 29,$
$2x - 2y - z = 6,$
$3x + y + z = 26.$

6. $5x - y - z = -1,$
$6x + 4y - 7z = -14,$
$7x - 3y - 3z = -11.$

7. $4x + 5y = 0,$
$3y + 4z = -4,$
$5x + 3y = 13.$

8. $x + y = 5,$
$7x - 4y = 46,$
$9y - 4z = -1.$

9. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 6,$
$2x + 2y + z = 28,$
$5x - 4y - z = -12.$

10. $\frac{x}{3} + y + 2z = 23,$
$3x - \frac{y}{2} - z = -2,$
$4x + y + z = 24.$

11. $2x + 5y = 60,$
$3y - 2z = 26,$
$7x - 3z = 73.$

12. $\frac{1}{5}x + \frac{1}{2}y = 6\frac{1}{2},$
$2y - 3z = 24,$
$4z - x = -18.$

PROBLEMS

Example 1. A number consists of three digits whose sum is 18. If 396 be added to the number, the digits in the hundreds' and the units' places will be interchanged. The units' digit falls short of the sum of the other two by 2. Find the number.

Let $x =$ digit in hundreds' place.

$y =$ digit in tens' place.

$z =$ digit in units' place.

From the first condition,

$$x + y + z = 18. \quad (1)$$

From the second condition,

$$100x + 10y + z + 396 = 100z + 10y + x. \quad (2)$$

From the third condition,

$$z = x + y - 2. \quad (3)$$

Transposing and combining terms in equation (2), we have

$$99x - 99z = -396.$$

Dividing by 99, $x - z = -4. \quad (4)$

Transposing x and y in equation (3), we have

$$-x - y + z = -2. \quad (5)$$

We have then to solve the following system of simultaneous equations:

$$x + y + z = 18. \quad (1)$$

$$x - z = -4. \quad (2)$$

$$-x - y + z = -2. \quad (3)$$

Adding the first and third, $2z = 16.$

$$z = 8.$$

Substituting 8 for z in the second,

$$x = 4.$$

Substituting 4 for x and 8 for z in the first,

$$y = 6.$$

The number is 468.

EXERCISE 63

1. A number consists of three digits whose sum is 18. The first digit is $\frac{1}{8}$ of the number formed by the other two, and the units' digit is also $\frac{1}{8}$ of the number formed by the first two. Find the number.

2. A number consists of three digits whose sum is 12. If 495 be added to the number, the digits will be reversed. The tens' digit is $\frac{1}{3}$ the sum of the other two. Find the number.

3. A number composed of three digits whose sum is 8 is equal to five times the number composed of the tens' and units' digits. Also the tens' digit is 4 less than the sum of the other two. Find the number.

4. Divide 90 into three parts so that twice the first part exceeds the sum of the other two by 21; also three times the second part exceeds the sum of the other two by 26.

5. Divide 80 into three parts so that when the first part is divided by the second the quotient is 2, and when three times the second is divided by the third the quotient is 2 and the remainder is 2.

6. A and B together earn $7 a day, B and C $10 a day, and A and C $9 a day. Find the daily wages of each.

7. A boy has 11 coins consisting of dollars, half dollars, and quarters, and he has \$5 in all. Four times the number of half dollars exceeds the number of quarters by 6. Find the number of each coin.

8. Three numbers are in the ratio of 2 : 3 : 4. The sum of twice the first, three times the second, and four times the third is 174. Find the numbers.

9. A boy buys 3 apples and 2 pears for 13 cents, 2 apples and 3 oranges for 18 cents, and 5 pears and 2 oranges for 18 cents. Find the cost of an apple, a pear, and an orange.

10. A man has \$3420 invested partly in 3 % stock at 72, 5 % stock at 90, and 6 % stock at 120. His total income is \$164. The amount of money invested in the first stock is \$540 less than in both the other stocks. Find the amount of money invested in each stock.

GRAPHS OF FUNCTIONS

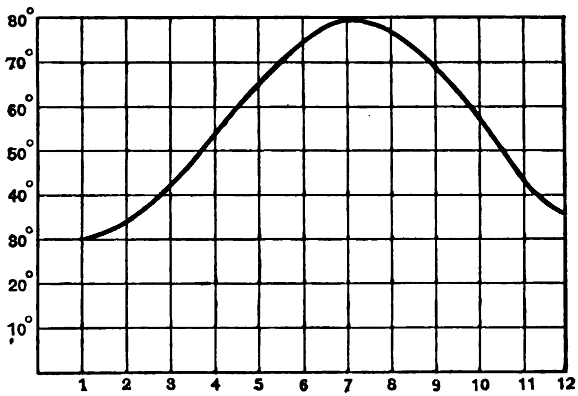
71. Pupils in grammar grades are familiar with the method of locating places on the earth's surface. The position of every point on the globe is given with reference to two imaginary lines called the **equator** and the **prime meridian.** For example, take New Orleans; the latitude of New Orleans is 30° N.; its longitude is 90° W. These two quantities, 90° W. and 30° N., serve to definitely locate the city of New Orleans.

The representation of numerical results by means of two lines of reference serves to convey a clear, accurate, and comprehensive idea of the quantities under consideration. As an illustration, construct a mental representation of the following natural phenomena: The mean monthly temperatures of the city of St. Louis are as follows:

January, 30°.	July, 79°.
February, 35°.	August, 77°.
March, 43°.	September, 69°.
April, 56°.	October, 58°.
May, 66°.	November, 44°.
June, 75°.	December, 36°.

To represent these graphically, take two lines at right angles to each other, one being horizontal and the other vertical. Take any convenient unit and mark the horizontal line 1, 2, 3, 4, etc., to represent the months in succession. Mark in a similar manner the vertical line. From the marks 1, 2, 3, 4, etc., measure upward 30°, 35°, 43°, etc., taking 10° as unit and mark these results with dots. Draw a line through the dots in succession, beginning with the first. Such a line is called a **curve of temperature**, and is also known as a **graph**.

Below is the curve of temperature of St. Louis.



EXERCISE 64

Draw the curve of temperature from the subjoined data for each of the following cities:

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
Albany, N.Y.	23°	24°	32°	46°	59°	69°	72°	70°	63°	51°	39°	29°
Boston, Mass.	27	28	34	45	56	66	71	69	62	52	41	31
Chicago, Ill.	23	27	34	46	56	67	72	71	64	52	38	29
Fort Smith, Ark.	36	42	50	62	68	76	80	78	72	61	49	43
Indianapolis, Ind.	28	32	40	53	63	72	76	74	66	54	41	33
Marquette, Mich.	16	17	23	37	49	59	65	64	57	45	31	23
New Orleans, La.	54	58	62	69	75	80	82	82	78	70	61	56
Raleigh, N.C.	41	44	48	59	68	76	77	76	70	58	50	44
Williston, N.Dak.	4	8	24	43	53	63	68	66	56	43	25	13
St. Paul, Minn.	11	16	28	45	58	67	72	69	60	47	30	19

72. The mean monthly rainfall for the city of Nashville, Tenn., is given in inches as follows:

JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
5.09	5.29	5.2	4.8	3.6	4.3	4.3	3.4	4.1	2.5	3.8	3.7

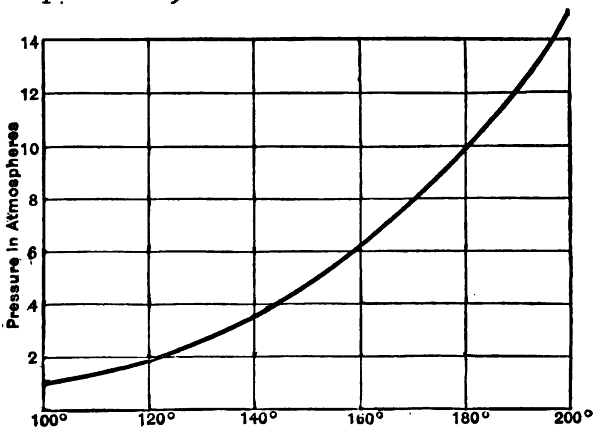
The following is a graphical representation of the rainfall in Nashville, Tenn.

EXERCISE 65

From the subjoined data (expressed in inches) draw the curve for rainfall in the following cities:

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
Baltimore, Md.	3.3	3.5	4.1	3.4	3.8	4	4.7	4.	3.9	3	3	3.2
Charleston, S.C.	4	3.3	3.9	3.6	4	3.6	7.6	7.6	6.6	4.2	3	3.4
Key West, Fla.	2.1	1.6	1.2	1.2	3.2	4	3.8	4.8	7.4	5.2	2.3	1.6
Meridian, Miss.	5.2	5.8	5.5	4.4	4.8	6.2	5.4	4.5	2.8	1.6	3.1	5.3
Neahbay, Wash.	15.3	10.3	10.5	8.5	5.3	4.8	2.3	2.3	7.3	10.4	13.9	16.3
Tatoosh Island, Wash.	12.7	8.5	9.1	7.4	4.6	4	2.1	2.4	6.6	8.5	12.2	14.4
San Antonio, Tex.	1.7	2	2	3	3.2	2.7	2.2	3.8	3.4	1.7	2.1	1.9
Springfield, Mo.	2.5	3.2	3.7	3.8	6	4.5	4.3	3.6	3.8	3.1	3.1	2.6

73. It has been found by experiment that the pressure of aqueous vapor at 100° Centigrade is equal to 1 atmosphere; at 120°, 1.96 atmospheres; at 140°, 3.57 atmospheres; at 160°, 6.1 atmospheres; at 180°, 9.9 atmospheres; at 200°, 15.4 atmospheres. (One atmosphere is 14.7 lb. to the square inch.)



Take the equation $y = 2x - 3$.

$$\text{If } x = 0,\ y = -3;$$
$$\text{if } x = 1,\ y = -1;$$
$$\text{if } x = 2,\ y = 1;$$
$$\text{if } x = -1,\ y = -5;$$
$$\text{if } x = -2,\ y = -7.$$

The value of y in this equation evidently depends upon the value of x.

74. A number or magnitude which may assume in any given process any number of values is called a **variable.** In the equation $y = 2x - 3$, x is a variable and also y is a variable. A number which retains the same value in any given process is called a **constant.** In the above equation -3 is a constant and so also is 2.

75. A variable which depends upon another variable for its value is called a **function** of that variable. In the equation $y = 2x - 3$, y is a function of x. The area of a circle is a function of its radius.

In the above illustration the pressure of aqueous vapor is a function of the temperature.

76. Equations may be graphically represented.

Take two straight lines in a plane, XX' and YY', Fig. 1, at right angles to each other. These lines are called respectively the **x-axis** and the **y-axis.** Their intersection, O, is called the **origin.**

Take any point P in the plane and draw PM perpendicular to XX', and PN perpendicular to YY'. The distance of the point P from the line YY', *i.e.* PN, or its equal OM, is denoted by x, and the distance of the point P from XX', *i.e.* PM, is denoted by y. No other point in the plane is at the same distances from the two axes YY' and XX'.

x is positive if measured in the direction OX, and negative if measured in the direction OX'. y is positive if

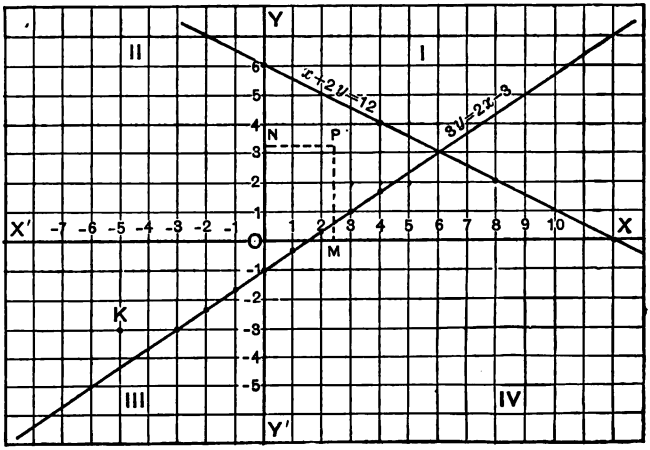


Fig. 1.

measured in the direction OY, and negative if measured in the direction OY'. In other words, distances measured to the right are positive, and distances measured to the left are negative. Distances measured upwards from the x-axis are positive, and distances measured downwards from the x-axis are negative.

77. The four parts into which the plane is divided by the axes are called **quadrants**. These are marked I, II, III, IV, as shown in the figure.

78. To every point in the plane corresponds one value of x and one value of y. These values of x and y are called the **coördinates** of the point. If the coördinates of a point are known, the position of the point is uniquely determined. Suppose, for example, the point $x = -5$, $y = -3$ is sought. In order to mark it, measure 5 units on the x-axis to the left of the origin O. Next, measure 3 units downwards from the point -5 on the x-axis. The point K, Fig. 1, is the point sought. K is obviously 5 units to the left of YY' and 3 units below XX'.

Marking the position of a point when its coördinates are given is called **plotting the point.**

79. Trace the graphs of $3y = 2x - 3$.

In this equation, if $x = 0$, then $y = -1$;

if $x = 1$, then $y = -\frac{1}{3}$;

if $x = 2$, then $y = \frac{1}{3}$;

if $x = 3$, then $y = 1$;

if $x = 4$, then $y = 1\frac{2}{3}$;

if $x = -1$, then $y = -1\frac{2}{3}$;

if $x = -2$, then $y = -2\frac{1}{3}$;

if $x = -3$, then $y = -3$.

For shortness, these results may be tabulated as follows:

$$\begin{array}{lcccccccc} x = & 0, & 1, & 2, & 3, & 4, & -1, & -2, & -3. \\ y = & -1, & -\frac{1}{3}, & \frac{1}{3}, & 1, & 1\frac{2}{3}, & -1\frac{2}{3}, & -2\frac{1}{3}, & -3. \end{array}$$

To trace the curve, measure from the points 0, 1, 2, 3, etc., on the x-axis -1, $-\frac{1}{3}$, $\frac{1}{3}$, 1, etc., respectively, and mark in each case with a dot. Next draw a line through the dots (see Fig. 1).

It will be observed that the line is a straight line. In fact, every equation of the first degree represents a straight line. For this reason first degree equations are often called linear equations.

The coördinates of every point on the line are values of x and y which satisfy the equation.

Draw now the graph of another equation,

$$x + 2y = 12.$$

Notice these two graphs intersect in a point. The coördinates of the point of intersection satisfy both equations and are consequently a solution of the system of linear equations,

$$3y = 2x - 3.$$
$$x + 2y = 12.$$

80. Notation. The point (a, b) means the point whose coördinates are

$$x = a, \quad y = b.$$

EXERCISE 65 (a)

Plot the points:

1. $(2, 3)$; $(4, 1\frac{1}{2})$; $(3\frac{1}{2}, 1\frac{1}{4})$.
2. $(-1, 2)$; $(-2, 4)$; $(-3, -4)$.
3. $(4, -3)$; $(1.5, -5)$; $(1, -2\frac{1}{2})$.
4. $(0, 4)$; $(0, -2)$; $(-2, 0)$; $(-5, 0)$.

Trace the graphs of:

5. $y = x$; $y = x + 1$; $y = x - 1$; $y = x + 2$; $y = x - 2$.
6. $y = 2x$; $y = 2x - 1$; $y = 2x + 1$; $y = 2x - 3$.
7. $y = 4$; $y = -3$; $x = 2$; $x = -3$.
8. $x = 0$; $y = 0$.

9. Draw the graphs of the simultaneous equations in Exercise 59, and determine in each case their point of intersection.

10. Determine graphically the integral values of x and y which satisfy the equation,

$$2x + 3y = 13.$$

11. A square whose side is 9 units is constructed so that two of its sides are parallel to the x-axis and the other two sides are parallel to the y-axis, and the intersection of its diagonals coincides with the origin. Determine the coördinates of its four vertices.

12. Draw a straight line through the points $(1, -3)$ and $(-3, 2)$.

CHAPTER VII

MULTIPLICATION. DIVISION. SYMBOLS OF AGGREGATION

81. *Example* 1. Multiply $x+6$ by $x+4$.

$$\begin{array}{l} x+6 \\ \underline{x+4} \\ x^2+6x \\ \underline{\quad +4x+24} \\ x^2+10x+24 \end{array}$$

(1) Multiply $x+6$ by x.

(2) Multiply $x+6$ by 4.

(3) Add the two results for the required product.

Check. Take $x=1$. Then,

$$(x+6) \text{ times } (x+4) = 7 \times 5 = 35.$$

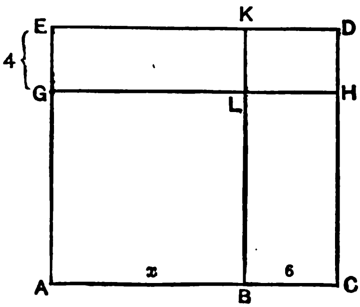
$$x^2+10x+24 = 1+10+24 = 35.$$

Geometric Proof.

Let $AB = x$.

$BC = 6$.

Upon AC construct a rectangle $ACDE$ so that $CD = x+4$. Make $DH = 4$ and draw the lines BK, HG at right angles to AC and CD respectively.



Then,

$ACDE = (x+6)(x+4).$

$ACHG = x(x+6).$

$GHDE = 4(x+6).$

$ABLG = x^2.$

$BCHL = 6x.$

$GLKE = 4x.$

$LHDK = 4 \times 6.$

Also, $ACDE = ACHG + GHDE$

$$= (ABLG + BCHL) + (GLKE + LHDK).$$

Hence,

$$(x+6)(x+4) = x^2 + 6x + 4x + 24 = x^2 + 10x + 24.$$

Example 2. Multiply $x - b$ by $x - a$.

$$\begin{array}{r} x - a \\ x - b \\ \hline x^2 - ax \qquad\quad \\ -bx + ab \\ \hline x^2 - ax - bx + ab \end{array}$$

(1) Multiply $x - a$ by x.

(2) Multiply $x - a$ by $-b$.

(3) Add the two results for the required product.

Geometric Proof.

Let $AB = x$.

$BC = a$.

Upon AB construct a square. Make $LK = b$ and draw the lines KE, CH at right angles to BL and AB respectively.

Then,

$ACDE = (x-a)(x-b)$.

$ACHG = x(x-a)$.

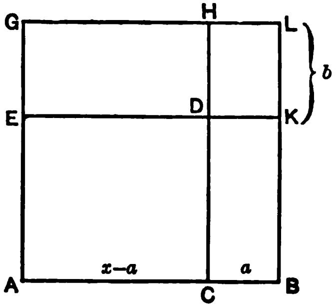
$EDHG = b(x-a)$.

$ABLG = x^2$.

$BCHL = ax$.

$LKEG = bx$.

$LHDK = ab$.



Also, $ACDE = ACHG - EDHG$ (1)

$= (ABLG - BCHL)$

$- (LKEG - LHDK)$ (2)

$= ABLG - BCHL$

$- LKEG + LHDK.$ (3)

Therefore, $(x-a)(x-b)=x^2-ax-bx+ab.$

NOTE. The steps in the geometric proof exactly correspond with the steps in the algebraic process of multiplication. The geometric proof furnishes a justification of the Law of Signs in multiplication. For lines marked (1), (2), (3) represent, respectively, the following lines (4), (5), (6).

$$(x-a)(x-b)=x(x-a)-b(x-a), \quad (4)$$
$$=(x^2-ax)-(bx-ab), \quad (5)$$
$$=x^2-ax-bx+ab. \quad (6)$$

Note the signs of the products x by x, x by $-a$, $-b$ by x, $-b$ by $-a$. Note further the multiplication of $x-a$ by $-b$ is performed by multiplying $x-a$ by b and then changing the signs of the terms of the product. Compare with definition of multiplication by a negative quantity.

EXERCISE 66

Multiply:

1. $x+1$ by $x+2$.
2. $x+3$ by $x+5$.
3. $x+2$ by $x+8$.
4. $x+12$ by $x+9$.
5. $x+6$ by $x+5$.
6. $x+7$ by $x+9$.
7. $x-4$ by $x+2$.
8. $x-8$ by $x+3$.
9. $x-9$ by $x-2$.
10. $x-1$ by $x-5$.
11. $x-3$ by $x-8$.
12. $x-5$ by $x-5$.
13. $2a-4$ by $2a-5$.
14. $3a-2$ by $4a+3$.
15. $5a-6b$ by $6a-5b$.
16. $3a-4c$ by $3a+7c$.
17. $2a-5c$ by $2a+5c$.
18. $8a+5c$ by $8a-5c$.
19. $4m+6n$ by $4m-8n$.
20. $9m-4n$ by $3m-5n$.
21. $7c-2d$ by $8c-3d$.
22. $4b-5d$ by $5b-8d$.
23. $9a-2x$ by $8a-7x$.
24. a^2+b^2 by a^2+2b^2.
25. $4a^2+5b^2$ by $6a^2+3b^2$.
26. $7c^2+4$ by $7c^2-4$.
27. $9x^2+x$ by $8x^2-x$.
28. $7x^2+3x$ by $6x^2-5x$.

29. $11x^2+2x$ by $5x^2-4x$.
30. $6x^2+y^2$ by $6x^2-y^2$.
31. $4a^2-9b^2$ by $4a^2+9b^2$.
32. $3x^2-n^2$ by $2x^2+3n^2$.
33. x^3+1 by $x+1$.
34. a^3-a by $a-1$.
35. x^3+y^3 by x^3-y^3.
36. $2x^3-3y^3$ by $2x^3+3y^3$.
37. x^2+y^2 by x^2+y^2.
38. x^4-1 by x^4+1.
39. x^4+y^4 by x^4-y^4.
40. a^3-a by a^2+1.

82. Multiply $x^3+xy^2-x^2y-y^3$ by $x+y$.

$$\begin{array}{l} x^3-x^2y+xy^2-y^3 \\ \underline{x+y} \\ x^4-x^3y+x^2y^2-xy^3 \\ \underline{\quad +x^3y-x^2y^2+xy^3-y^4} \\ x^4 \qquad\qquad\qquad\qquad -y^4 \end{array}$$

(1) Arrange the terms according to the descending power of the leading letter x, *i.e.* write first the term containing the highest exponent of x, then the term containing the next highest exponent of x, and so on.

(2) Multiply the multiplicand by x.

(3) Multiply the multiplicand by y.

(4) Add the partial products.

EXERCISE 67

Multiply:

1. x^2+x+1 by $x-1$.
2. x^2-x+1 by $x+1$.
3. x^2+xy+y^2 by $x-y$.
4. x^2-xy+y^2 by $x+y$.
5. x^2+x-2 by $x-2$.
6. x^2-2x-3 by $x+3$.
7. $2x^2+5x+2$ by $2x-1$.
8. $3x^2-7x+11$ by $3x-2$.
9. $4x^2-6xy+9y^2$ by $x+2y$.

10. x^4+x^2+1 by x^2-1.

11. x^4-x^2+1 by x^2+1.

12. $a^4+a^2b^2+b^4$ by a^2-b^2.

13. $a^4-a^2b^2+b^4$ by a^2+b^2.

14. a^4+3a^2+2 by a^2+3.

15. x^3+2x^2+5x by $x-5$.

16. $x^2-4xy+4y^2$ by $x-2y$.

17. $x^2-4xy+4y^2$ by $x+2y$.

18. $x^2-6xy+9y^2$ by $x+3y$.

19. $4x^2-12xy+9y^2$ by $2x-3y$.

20. $9x^2-6x-1$ by $3x-2$.

Simplify:

21. $(1-x)(1+x)(1+x^2)$.

22. $(a-b)(a+b)(a^2+b^2)$.

23. $(2x-1)(2x+1)(4x^2+1)$.

24. $(3x+y)(3x-y)(9x^2+y^2)$.

25. $(x^2+y^2+xy-x+y+1)(x-y+1)$.

83. Division of integral expressions.

Example 1. Divide $7x-7x^2+15+x^3$ by $x-3$.

$$
\begin{array}{r|l}
 & x^2-4x-5 \\
x-3 & x^3-7x^2+7x+15 \\
 & \underline{x^3-3x^2} \\
 & -4x^2+7x \\
 & \underline{-4x^2+12x} \\
 & -5x+15 \\
 & \underline{-5x+15}
\end{array}
$$

Step 1. Arrange the terms of the dividend and divisor according to the descending powers of x.

Step 2. Divide the first term of the divisor, x, into the first term of the dividend, x^3, and write the result, x^2, as the first term of the quotient; multiply the divisor by x^2 and subtract the product from the dividend.

Step 3. Take down the next term, $7x$.

Step 4. Divide x, the first term of the divisor, into $-4x^2$, write the result, $-4x$, in the quotient, multiply the divisor by $-4x$ and subtract.

Step 5. Divide x into $-5x$ and proceed as before.

Example 2. Divide $x^3 - 8y^3$ by $x - 2y$.

$$\begin{array}{r|l} & x^2 + 2xy + 4y^2 \\ x - 2y) & x^3 - 8y^3 \\ & \underline{x^3 - 2x^2y} \\ & 2x^2y \\ & \underline{2x^2y - 4xy^2} \\ & 4xy^2 - 8y^3 \\ & \underline{4xy^2 - 8y^3} \end{array}$$

EXERCISE 68

Perform the following indicated divisions:

1. $\dfrac{x^2 - 3x + 2}{x - 1}$.

2. $\dfrac{x^2 - 4x + 4}{x - 2}$.

3. $\dfrac{x^2 - 5x + 6}{x - 3}$.

4. $\dfrac{a^2 - 7ab + 6b^2}{a - b}$.

5. $\dfrac{a^2 - 8ab + 15b^2}{a - 3b}$.

6. $\dfrac{a^2 - 5ab - 14b^2}{a + 2b}$.

7. $\dfrac{c^2 - 7cd - 8d^2}{c - 8d}$.

8. $\dfrac{x^2 + 3xy - 18y^2}{x + 6y}$.

9. $\dfrac{12a^2 - 17ab + 6b^2}{3a - 2b}$.

10. $\dfrac{6a^2 - 23ab + 20b^2}{3a - 4b}$.

11. $\dfrac{20a^2 - 7ab - 6b^2}{5a + 2b}$.

12. $\dfrac{21a^2 + 29ab - 10b^2}{7a - 2b}$.

13. $\dfrac{9a^2 - 4b^2}{3a - 2b}$.

14. $\dfrac{16x^2 - y^2}{4x + y}$.

15. $\frac{25x^2-1}{5x-1}$.

16. $\frac{a^3-a}{a-1}$.

17. $\frac{4x^3-x}{2x+1}$.

18. $\frac{16x^3-20x^2+6x}{4x-3}$.

19. $\frac{6x^3+2x^2-20x}{2x+4}$.

20. $\frac{6x^3-5x^2-21x}{2x+3}$.

21. $\frac{12a^3-9a^2b-30ab^2}{4a+5b}$.

22. $\frac{10m^2n+11mn^2-6n^3}{5m-2n}$.

23. $\frac{x^3-5x+2}{x-2}$.

24. $\frac{x^3-4x-15}{x-3}$.

25. $\frac{a^3-7a+36}{a+4}$.

26. $\frac{a^3-11a-6}{a+3}$.

27. $\frac{x^3-x+60}{x+4}$.

28. $\frac{2x^3-5x-225}{x-5}$.

29. $\frac{4x^3-11x^2+3x+6}{x-2}$.

30. $\frac{a^3-1}{a-1}$.

31. $\frac{x^3-8}{x-2}$.

32. $\frac{x^3+27}{x+3}$.

33. $\frac{a^3+b^3}{a+b}$.

34. $\frac{8x^3-125y^3}{2x-5y}$.

35. $\frac{27a^3-64b^3}{3a-4b}$.

36. $\frac{125x^3+y^3}{5x+y}$.

37. $\frac{216x^3-343y^3}{6x-7y}$.

38. $\frac{a^2-ab-ac+bc}{a-c}$.

39. $\frac{x^3+x^2-x-1}{x+1}$.

40. $\frac{x^3+3x^2-9x-27}{x^2-9}$.

Divide $1 + 8y^3$ by $1 + 2y$.

$$
\begin{array}{r|l}
 & 1 - 2y + 4y^2 \\
1 + 2y & 1 + 8y^3 \\
 & \underline{1 + 2y} \\
 & -2y \\
 & \underline{-2y - 4y^2} \\
 & +4y^2 + 8y^3 \\
 & \underline{4y^2 + 8y^3}
\end{array}
$$

The terms are here arranged according to the ascending powers of y. If the terms were arranged according to the descending powers of y, the quotient would be the same.

EXERCISE 69

Divide:

1. $1 - x^3$ by $1 - x$.
2. $1 + x^3$ by $1 + x$.
3. $1 - x^4$ by $1 - x$.
4. $8 - a^3$ by $2 - a$.
5. $27 + b^3$ by $3 + b$.
6. $16 - a^4$ by $2 - a$.
7. $81 - x^4$ by $3 - x$.
8. $8 + 4x - 2x^3$ by $2 - x$.
9. $21 + 20x - 3x^3$ by $3 - x$.
10. $16 - 20x + x^3$ by $4 - x$.
11. $1 + a^2 + a^4$ by $1 - a + a^2$.
12. $1 + 2a^2 + 9a^4$ by $1 + 2a + 3a^2$.

84. Symbols of aggregation.

Parentheses have already been used to denote that the quantities enclosed by them have to be taken as a whole. For example, $a - 2(b - c)$ implies that twice the difference between b and c is to be taken from a.

There are other symbols to denote that expressions have to be considered as a whole. These are the square brackets [ ], the braces { }, and a horizontal line drawn above an expression. The last is used most frequently in connection with the radical sign. For example, $\sqrt{5}\,a$ means that a is to be multiplied by the square root of 5. $\sqrt{5\,a}$ means the square root of 5 times a. $\sqrt{a}+b$ means b is to be added to the square root of a. $\sqrt{a+b}$ means b is to be added to a and the square root of the sum then taken.

Example 1. Simplify $5\,a-\{2\,a-(a-4)\}$.

Here the quantity within the brace is to be taken from $5\,a$. The quantity within the brace is the remainder obtained by subtracting $a-4$ from $2\,a$.

$$\begin{aligned}\text{Hence, } 5\,a-\{2\,a-(a-4)\} &= 5\,a-\{2\,a-a+4\}\\ &= 5\,a-2\,a+a-4\\ &= 4\,a-4.\end{aligned}$$

In this solution the inner symbol is first removed and then the outer one. The outer symbol might be first removed and then the inner one.

Two quantities are to be subtracted from $5\,a$. One of these is $2\,a$ and the other is $-(a-4)$. Hence, according to the rule for subtraction,

$$\begin{aligned}5\,a-\{2\,a-(a-4)\} &= 5\,a-2\,a+(a-4)\\ &= 5\,a-2\,a+a-4\\ &= 4\,a-4.\end{aligned}$$

The latter method is the shorter.

EXERCISE 70

Simplify by removing symbols of aggregation:

1. $11-(9-3)$.
2. $13-(9-4)-(6-8)$.
3. $15-(11-7)+(7-12)$.
4. $20-(16+5)+(12-3)$.
5. $18-\{19-(6-4)\}$.
6. $27-\{14+(9-3)-(2-7)\}$.
7. $32+\{(8-5)-(15-28)\}$.
8. $2a-b-\{a-(b-2a)\}$.
9. $4a+b-\{2a-(a-b)\}$.
10. $7c+d+\{4c-(5c+d)\}$.
11. $(a+b)-\{(2a+b)-(a-2b)\}$.
12. $x+y-\{3x-y+(x+2y)\}$.
13. $(m+n)+\{(2m+3n)-(3m-4n)\}$.
14. $(4m+n)-\{4m-3(m-n)\}$.
15. $2(m+4n)-\{5m+3(n-m)\}$.
16. $4(x-y)-\{2(y-x)-3(x-2y)\}$.

85. Parentheses written in succession without connecting signs indicate multiplication. Thus,

$$(a+b)(c+d)(m-c)$$

means the continued product of $a+b$, $c+d$, and $m-c$.

$(-a)^2$ means minus a multiplied by minus a.

$(a+b)^2$ means $(a+b)$ times $(a+b)$, or the square of $(a+b)$.

86. In a series of indicated operations involving addition, subtraction, multiplication, and division, the multiplications and divisions are performed first and then the remaining operations.

Example 1. Find the value of

$$7+4-3\times 5+16\div 8.$$

Performing the multiplication and division, we have

$$7+4-15+2.$$

Combining these, the result is -2.

Example 2. If $a=2$, $b=3$, find the value of

$$-a^2+(-b)^2-(-a)(-b)+2\,ab.$$

Substituting 2 for a and 3 for b, the expression becomes

$$-2^2+(-3)^2-(-2)(-3)+2\times 2\times 3=$$
$$-4+9-6+12=11.$$

EXERCISE 71

Perform indicated operations :

1. $(-1)(-2)(-3)(-4)$.
2. $(-2)(-3)(-5)$.
3. $a^2+(-a)^2$.
4. $a^2-(-a)^2$.
5. $(-a)^2-a^2$.
6. $(-a)^3+a^3$.
7. $(-a)^3-a^3$.
8. $(-a^2)(-a)^2$.
9. $(-a)^3(-a)^3$.
10. $(-a)(-a)^2(-a)^3$.
11. $(-a^2)^2$.
12. $(-b^2c)^2$.
13. $(-a^2c^3)^2$.
14. $(-3\,a^3b^2)^3$.
15. $(a-b)^2$.
16. $(a-2\,c)^2$.
17. $(a-b-c)^2$.
18. $(x^2-x-1)^2$.
19. $(a^2-2\,a+1)^2$.
20. $(x^2-y^2)^2$.

If $a = 3$, $b = 2$, find the value of :

21. $(a+b)(a-b)$.

22. $(2a-3b)^2$.

23. $(a-2b)^2$.

24. $5(a+b)^2$.

25. $3(3b-a)^2$.

26. $(a^2+b^2)(a-b)$.

27. $(a^2-b^2)(a-b)$.

28. $\dfrac{a^2+b^2}{a^2-b^2}$.

29. $(ab+b)(ab-b)$.

30. $a^2-(a+b)(a-b)$.

31. $b^2-(a-b)(a+b)$.

32. $a^3-(a-b)^3$.

33. $(a^2+b^2)(a^2-b^2)$.

34. $(a^2+ab)(a^2-ab)$.

35. $ab(a-b)$.

36. $ab(a+b)$.

37. a^3b-ab^3.

38. $ab^2(a^2-b^2)$.

Selecting arbitrary values of a and b, show that:

39. $(a+b)^2 = a^2+2ab+b^2$.

40. $(a-b)^2 = a^2-2ab+b^2$.

41. $a^2-b^2 = (a+b)(a-b)$.

42. $a^3-b^3 = (a-b)(a^2+ab+b^2)$.

43. $a^3+b^3 = (a+b)(a^2-ab+b^2)$.

44. $a^2+b^2 = (a+b)^2-2ab$.

45. $a^2+b^2 = (a-b)^2+2ab$.

46. $(a-b)^3 = a^3-b^3-3ab(a-b)$.

47. $(a+b)^3 = a^3+b^3+3ab(a+b)$.

48. $a^4+a^2b^2+b^4 = (a^2+ab+b^2)(a^2-ab+b^2)$.

49. $a^2+b^2 = \frac{1}{2}(a+b)^2+\frac{1}{2}(a-b)^2$.

50. $ab = \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2$.

IMPORTANT IDENTITIES

87. Multiply $a+b$ by $a+b$.

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad +ab+b^2} \\ a^2+2\,ab+b^2 \end{array}$$

The required product is the sum of $a(a+b)$ and $b(a+b)$.

Hence, $(a+b)^2 = a^2+2\,ab+b^2$.

Hence, **The square of the sum of two numbers is equal to the square of the first number plus twice the product of the first number and the second number plus the square of the second number.** (I)

Geometric Proof.

Let $A'B = a.$

$BC = b.$

Upon AC construct a square $ACDE$, Fig. 1. Make $DG = b$ and draw the lines BK, GH at right angles to AC and CD respectively.

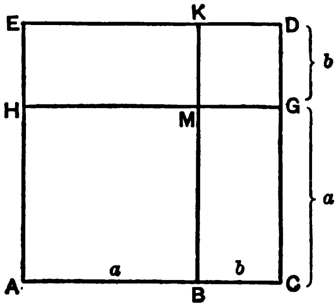


Fig. 1.

Then, $ACDE = (a+b)^2.$

$ACGH = a(a+b).$

$HGDE = b(a+b).$

$ABMH = a^2.$

$MGDK = b^2.$

$HMKE = ab.$

$BCGM = ab.$

Also, $ACDE = ACGH + HGDE$

$= (ABMH + BCGM) + (HMKE + MGDK).$

Hence, $(a+b)^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

Point out the correspondence between the steps in the geometric proof and the steps in the algebraic process of multiplication.

88. Multiply $a-b$ by $a-b$.

$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad -ab+b^2} \\ a^2-2ab+b^2 \end{array}$$

The required product is the difference of $a(a-b)$ and $b(a-b)$.

Hence, **The square of the difference of two numbers is equal to the square of the first number minus twice the product of the first number and the second number plus the square of the second number.** (II)

Geometric Proof.

Let $AB = a$.

$BC = b$.

Describe a square upon AB and also upon AC, Fig. 2. Prolong CH, FH.

Then, $ACHF = (a-b)^2$.

$ABGF = a(a-b)$.

$CBGH = b(a-b)$.

$ABDE = a^2$.

$HGDK = b^2$.

$CBDK = ab$.

$FGDE = ab$.

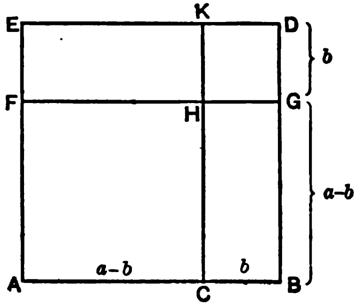


FIG. 2.

Also, $ACHF = ABGF - CBGH$

$= (ABDE - FGDE) - (CBDK - HGDK)$

$= ABDE - FGDE - CBDK + HGDK.$

Hence, $(a-b)^2 = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2.$

NOTE. The steps in the geometric proof exactly correspond with the steps in the algebraic process of multiplying $a - b$ by $a - b$. This proof illustrates the truth of the Law of Signs in multiplication.

89. Multiply $a + b$ by $a - b$.

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2 \qquad\quad -b^2 \end{array}$$

The required product is the difference between $a(a+b)$ and $b(a+b)$.

Hence, **The product of the sum and difference of two numbers is equal to the difference of the squares of the numbers.** (III)

Geometric Proof. Let $AB = a$,

$BC = b$.

Upon AB describe the square $ABKE$, Fig. 3. Make $EG = b$. Draw GH, CD at right angles to AE and AC respectively; prolong EK.

Then, $ACHG = (a+b)(a-b)$.

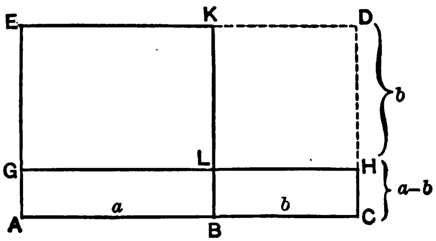


FIG. 3.

$ACDE = a(a+b)$.

$GHDE = b(a+b)$.

$ABKE = a^2$.

$BCDK = ab$.

$GLKE = ab$.

$LHDK = b^2$.

Also, $ACHG = ACDE - GHDE$.

$= (ABKE + BCDK) - (GLKE + LHDK)$

$= ABKE + BCDK - GLKE - LHDK$.

Hence, $(a+b)(a-b) = a^2 + ab - ab - b^2 = a^2 - b^2$.

Example 1. Expand by statement (I) 83^2.

SOLUTION. $83^2 = (80+3)^2 = 80^2 + 2(80 \times 3) + 3^2$

$= 6400 + 480 + 9 = 6889$.

Example 2. Expand by statement (II) 67^2.

$67^2 = (70-3)^2 = 70^2 - 2(3 \times 70) + 3^2$

$= 4900 - 420 + 9 = 4489$.

Example 3. Multiply 43 by 37.

$43 \times 37 = (40+3)(40-3) = 40^2 - 3^2 = 1600 - 9 = 1591$.

EXERCISE 72

Expand by statement (I):

1. 32^2.
2. 43^2.
3. 53^2.
4. 62^2.
5. 74^2.
6. 82^2.
7. 91^2.
8. 93^2.
9. $(x+4)^2$.
10. $(x+10)^2$.
11. $(2x+1)^2$.
12. $(3x+2)^2$.
13. $(x+2a)^2$.
14. $(x+4a)^2$.
15. $(2x+3b)^2$.
16. $(5x+4b)^2$.
17. $(x^2+x)^2$.
18. $(2x^2+x)^2$.
19. $(3x^2+y^2)^2$.
20. $(4a^2+3b^2)^2$.

Expand by statement (II):

21. 18^2.
22. 29^2.
23. 37^2.
24. 48^2.
25. 59^2.
26. 66^2.
27. 78^2.
28. 87^2.
29. 99^2.
30. 97^2.
31. $(x-1)^2$.
32. $(x-3)^2$.
33. $(2x-1)^2$.
34. $(3x-2)^2$.
35. $(5x-2)^2$.
36. $(1-2ab)^2$.
37. $(1-x^2)^2$.
38. $(5-y^2)^2$.
39. $(5x-2x^2)^2$.
40. $(6xy-x)^2$.

Expand by statement (III):

41. 18×22.
42. 28×32.
43. 39×41.
44. 79×81.
45. 88×92.
46. 189×211.
47. 191×209.
48. $(x+a)(x-a)$.
49. $(x+4y)(x-4y)$.
50. $(9x+11)(9x-11)$.
51. $(xy+x)(xy-x)$.
52. $(3x^2+7x)(3x^2-7x)$.

For other important identities, see page 143, examples 42 to 50 inclusive.

CHAPTER VIII

FACTORS. FRACTIONS

EXERCISE 73

Factor:

1. $a^2 + 2a.$	7. $a^2x - b^2x - c^2x.$
2. $x^2 - x.$	8. $x^2 - \frac{x}{8}.$
3. $ax + bx.$	9. $m^2 - \frac{1}{2}m.$
4. $ax^2 + cx^2.$	10. $\frac{1}{2}cm - \frac{1}{2}cn.$
5. $x^3 - x^2 - x.$	11. $3x^3 - 6x^2.$
6. $xy + xz - xu.$	12. $12x^4 - 18x^2.$

90. The square of a sum.

Since $(a + b)^2 = a^2 + 2ab + b^2$, hence, any trinomial expression consisting of the sum of the squares of two quantities plus twice their product can be resolved into two equal factors, each of which is the sum of the two quantities.

Example 1. Factor $9x^2 + 30xy + 25y^2$.

$$9x^2 = (3x)^2.$$

$$25y^2 = (5y)^2.$$

$$30xy = 2(3x \times 5y).$$

Therefore, $9x^2 + 30xy + 25y^2 = (3x + 5y)^2$.

EXERCISE 74

Factor:

1. x^2+2x+1.
2. x^2+4x+4.
3. $x^2+10x+25$.
4. $a^2+14a+49$.
5. $a^2+20a+100$.
6. $4x^2+4xy+y^2$.
7. $9x^2+6xy+y^2$.
8. $16x^2+24xy+9y^2$.
9. $25x^2+20xy+4y^2$.
10. $25x^2+40xy+16y^2$.
11. $49a^2+14ab+b^2$.
12. $64a^2+48ab+9b^2$.
13. $81a^2+36ab+4b^2$.
14. $100a^2+140ab+49b^2$.
15. a^4+2a^2+1.
16. $x^4+2x^2y^2+y^4$.
17. $9x^4+6x^2y^2+y^4$.
18. $16x^4+8x^2+1$.
19. $4+28a^2+49a^4$.
20. x^6+4x^3+4.

91. The square of a difference.

Since $(a-b)^2=a^2-2ab+b^2$, a trinomial consisting of the sum of the squares of two quantities diminished by twice their product can be resolved into two factors, each of which is the difference of the two quantities.

Example 1. Factor $16x^4-24x^2y^2+9y^4$.

$$16x^4=(4x^2)^2.$$

$$9y^4=(3y^2)^2.$$

$$-24x^2y^2=-2(4x^2\times 3y^2).$$

Therefore, $16x^4-24x^2y^2+9y^4=(4x^2-3y^2)^2$.

EXERCISE 75

Factor:

1. $4x^2-4x+1$.
2. $9x^2-6x+1$.
3. $x^2-2xy+y^2$.
4. $16x^2-8xy+y^2$.

5. $25x^2 - 30xy + 9y^2$.
6. $81a^2 - 90ab + 25b^2$.
7. $100a^2 - 60ab + 9b^2$.
8. $144m^2 - 168mn + 49n^2$.
9. $x^4 - 4x^2y^2 + 4y^4$.
10. $x^8 - 6x^4 + 9$.
11. $121a^2 - 132ab + 36b^2$.
12. $169a^4 - 130a^2b^2 + 25b^4$.

92. Difference of two squares.

Since $(a+b)(a-b) = a^2 - b^2$, hence, the difference of the squares of any two quantities equals the product of the sum and the difference of the two quantities.

Example 1. Factor $81x^2 - y^4$.

$$81x^2 = (9x)^2.$$
$$y^4 = (y^2)^2.$$

Hence, $81x^2 - y^4$ is the difference of the squares of the quantities $9x$ and y^2, and consequently

$$81x^2 - y^4 = (9x + y^2)(9x - y^2).$$

EXERCISE 76

Factor:

1. $x^2 - 1$.
2. $a^2 - 4$.
3. $b^2 - 9$.
4. $a^2 - 36$.
5. $a^2 - 64$.
6. $m^2 - 100$.
7. $4x^2 - 25$.
8. $9x^2 - 49$.
9. $16x^2 - 81$.
10. $25x^2 - 36y^2$.
11. $64y^2 - 9x^2$.
12. $100x^2 - 1$.
13. $144y^2 - 25z^2$.
14. $a^2b^2 - 49c^2$.
15. $x^4 - 25y^2$.
16. $b^4 - 121c^2$.
17. $25c^4 - 16d^4$.
18. $169a^4 - b^4$.
19. $400 - x^4$.
20. $900 - a^4$.

21. $1600 - x^2$.

22. $a^2b^2 - 196\,d^2$.

23. $x^2 - \frac{1}{4}$.

24. $4\,x^2 - \frac{1}{9}$.

25. $a^2 - \frac{1}{25}$.

26. $x^2 - \dfrac{1}{a^2}$.

Find the value of:

27. $43^2 - 37^2$.

$$43^2 - 37^2 = (43 + 37)(43 - 37) = 80 \times 6 = 480.$$

28. $67^2 - 53^2$.

29. $82^2 - 18^2$.

30. $79^2 - 21^2$.

31. $94^2 - 86^2$.

32. $75^2 - 5^2$.

33. $85^2 - 5^2$.

34. $95^2 - 5^2$.

35. $73^2 - 27^2$.

36. $91^2 - 9^2$.

93. Trinomials.

Multiply $(x + 8)$ by $(x + 3)$.

$$\begin{array}{r} x + 8 \\ x + 3 \\ \hline x^2 + 8\,x \qquad \\ 3\,x + 24 \\ \hline x^2 + 11\,x + 24 \end{array}$$

Multiply $(x - 8)$ by $(x - 3)$.

$$\begin{array}{r} x - 8 \\ x - 3 \\ \hline x^2 - 8\,x \qquad \\ -\ 3\,x + 24 \\ \hline x^2 - 11\,x + 24 \end{array}$$

Hence, $x^2 + 11\,x + 24 = (x + 8)(x + 3)$.

$x^2 - 11\,x + 24 = (x - 8)(x - 3)$.

The product of two binomials having a common term equals the square of the common term, the algebraic sum of the other two terms into the common term, and the product of the other two terms.

Example 1. Factor $x^2 + 12x + 32$.

The first term of each of the two factors is x, the other terms are the factors of 32 whose sum is 12. By trial these factors are found to be 8 and 4.

Hence, $x^2 + 12x + 32 = (x + 8)(x + 4)$.

Example 2. Factor $x^2 - 10x + 9$.

The first term of each factor is x. The other terms are the factors of 9 whose sum is -10. Since the sum of the factors of 9 is negative and their product is positive, both must be negative. By inspection these factors are found to be -9, and -1.

Hence, $x^2 - 10x + 9 = (x - 9)(x - 1)$.

EXERCISE 77

Factor:

1. $x^2 + 3x + 2$.
2. $x^2 + 5x + 6$.
3. $x^2 + 7x + 12$.
4. $x^2 + 8x + 12$.
5. $x^2 + 13x + 12$.
6. $x^2 + 15x + 56$.
7. $x^2 + 6x + 9$.
8. $x^2 + 8x + 16$.
9. $x^2 + 9x + 20$.
10. $x^2 + 11x + 30$.
11. $x^2 + 11x + 18$.
12. $x^2 + 11x + 28$.
13. $x^2 - 6x + 5$.
14. $x^2 - 7x + 10$.
15. $x^2 - 10x + 25$.
16. $x^2 - 12x + 36$.

17. $x^2 - 13x + 42$.

18. $x^2 - 23x + 42$.

19. $x^2 - 27x + 50$.

20. $x^2 - 28x + 52$.

21. $x^2 - 19x + 60$.

22. $x^2 - 22x + 72$.

23. $a^2 - 25a + 24$.

24. $a^2 - 33a + 90$.

25. $a^2 + 14ab + 24b^2$.

26. $c^2 + 15cd + 36d^2$.

27. $m^2 - 16mn + 48n^2$.

28. $m^2 - 18mn + 72n^2$.

29. $a^2 - 21ab + 90b^2$.

30. $p^2 - 21pq + 98q^2$.

31. $b^2 + 29bc + 204c^2$.

32. $b^2 - 32bc + 240c^2$.

33. $b^2 - 27bc + 182c^2$.

34. $x^2 - 29xy + 210y^2$.

94. Multiply $(x+8)$ by $(x-3)$, also $(x-8)$ by $(x+3)$.

$$\begin{array}{l} x + 8 \\ \underline{x - 3} \\ x^2 + 8x \\ \underline{\quad - 3x - 24} \\ x^2 + 5x - 24 \end{array} \qquad \begin{array}{l} x - 8 \\ \underline{x + 3} \\ x^2 - 8x \\ \underline{\quad + 3x - 24} \\ x^2 - 5x - 24 \end{array}$$

Hence, **the product of two binomials having a common term equals the square of the common term, the algebraic sum of the other two terms multiplied by the common term, and the product of the other two terms.**

Example 1. Factor $x^2 - x - 72$.

The first term of each of the two factors is x. The other terms are the factors of -72 whose algebraic sum is -1. These are found by trial to be -9 and $+8$.

Hence, $\quad x^2 - x - 72 = (x-9)(x+8)$.

EXERCISE 78

Factor:

1. $x^2 + 2x - 15$.
2. $x^2 + 3x - 28$.
3. $x^2 - 3x - 28$.
4. $x^2 + 4x - 5$.
5. $x^2 + 4x - 12$.
6. $x^2 - 4x - 12$.
7. $x^2 + 5x - 36$.
8. $x^2 - 5x - 50$.
9. $x^2 - 6x - 7$.
10. $x^2 + 7x - 18$.
11. $x^2 - 7x - 18$.
12. $x^2 - x - 30$.
13. $x^2 + x - 42$.
14. $x^2 + x - 90$.
15. $x^2 - 11x - 12$.
16. $x^2 + 11x - 12$.
17. $x^2 - 10x - 24$.
18. $x^2 + 4x - 117$.
19. $x^2 - 5x - 84$.
20. $x^2 + 7x - 60$.
21. $x^2 - 10x - 96$.
22. $x^2 - 9x - 90$.
23. $x^2 - 6x - 135$.
24. $a^2 + 5a - 104$.

95. *Example* 1. Factor $12x^2 - 23x + 10$.

$$12x^2 - 23x + 10 = \tfrac{12}{12}(12x^2 - 23x + 10) \qquad (1)$$

$$= \frac{(12x)^2 - 23(12x) + 120}{12} \qquad (2)$$

$$= \frac{(12x - 8)(12x - 15)}{12} \qquad (3)$$

$$= \frac{4(3x - 2) \cdot 3(4x - 5)}{12} \qquad (4)$$

$$= (3x - 2)(4x - 5).$$

Check by multiplying the factors, or by assuming any value for x.

Multiply the terms of the expression by the coefficient of x^2 and divide by the same coefficient. Doing so, we have line (2), which is of the form $s^2 - 23s + 120$, whose factors are $(s-8)$, $(s-15)$. Hence, we have line (3). The remaining steps are obvious.

Example 2. Factor $8x^2 - 10x - 3$.

$$8x^2 - 10x - 3 = \tfrac{8}{8}(8x^2 - 10x - 3)$$
$$= \frac{(8x)^2 - 10(8x) - 24}{8}$$
$$= \frac{(8x-12)(8x+2)}{8}$$
$$= \frac{4(2x-3)\cdot 2(4x+1)}{8}$$
$$= (2x-3)(4x+1).$$

Check. $8(2^2) - 10(2) - 3 = (2\times 2 - 3)(4\times 2 + 1)$
$$32 - 20 - 3 = 1 \times 9.$$

EXERCISE 79

Factor:

1. $3x^2 + 17x + 20$.
2. $2x^2 + 5x + 2$.
3. $2x^2 + 11x + 5$.
4. $2x^2 + 15x + 18$.
5. $8x^2 + 18x + 9$.
6. $6x^2 - 17x + 5$.
7. $12x^2 - 19x + 4$.
8. $10x^2 - 29x + 10$.
9. $6x^2 + 5x - 6$.
10. $12x^2 - x - 6$.
11. $14x^2 + 59x - 18$.
12. $4x^2 + 23x - 35$.
13. $2x^2 + 15x - 8$.
14. $4x^2 - 17x + 4$.
15. $2x^2 + 13x - 7$.
16. $8x^2 - 14x - 15$.

96. Lowest common multiple.

By a **common multiple** of two or more expressions is meant an expression that contains each of them exactly.

The **lowest common multiple** of two or more expressions is the expression of lowest degree that is a multiple of each of them.

Lowest common multiple is denoted by the letters L. C. M.

Example 1. What is the L. C. M. of 12, 18, 50?

Resolve the numbers into their prime factors.

$$12 = 2^2 \times 3.$$

$$18 = 2 \times 3^2.$$

$$50 = 2 \times 5^2.$$

The L. C. M. must contain every one of the factors of the numbers, each factor affected with the highest exponent which it has in any of the numbers. Hence, the L. C. M. of 12, 18, 50 is

$$2^2 \times 3^2 \times 5^2, \text{ or } 900.$$

Example 2. Find the L. C. M. of a^2b^2, a^3b, ab^3x.

Here the different factors are a, b, x. The highest exponent of a is 3, of b is 3, and of x is 1. Hence, the L. C. M. is a^3b^3x.

Example 3. Find the L. C. M. of $6x^3y^2$, $8xy$, $12x^4y^5$.

The L. C. M. of 6, 8, 12, is 24.
The L. C. M. of x^3y^2, xy, x^4y^5, is x^4y^5.
Hence, the required L. C. M. is $24x^4y^5$.

EXERCISE 80

Find the L. C. M. of:

1. ab, a^2c, ab^2c.
2. x^2y, xy^2, x^3.
3. x^2y^2, x^2y^3, x^3y^2.
4. $2x^2y$, $3xy^2$.

5. $9x$, $12xy$, $6x^2y$.
6. $8x^2$, $7y^2$, $14xy$.
7. $8x$, $9x^2$, $12xy$.
8. $10a^2$, $15b^2$, $5ab$.
9. $12x^2$, $18xy$, $24xy^2$.
10. $14a^2b^2$, $21ab^3$, $7a^4$.
11. $8x^3y^3$, $10x^2y^2$, $20y^4$.
12. $4ab^3$, $6a^4b$, $8a^2b^5$.

97. *Example* 1. Find the L. C. M. of x^2-1, x^2-3x+2.

$$x^2-1=(x+1)(x-1).$$
$$x^2-3x+2=(x-2)(x-1).$$

Hence, L. C. M. is $(x-2)(x-1)(x+1)$.

To find the L. C. M. of two or more polynomial expressions, first, factor the expressions; second, take the product of every one of the different factors, each factor being affected with the highest exponent it has in any of the given expressions.

EXERCISE 81

Find the L. C. M. of:

1. x, $x+3$.
2. $3x$, $x+2$.
3. $2x$, $x-1$.
4. $3x$, $x-3$.
5. x^2-1, x^2-x.
6. x^2+2x, x^2+3x+2.
7. x^2+3x, x^2+4x+3.
8. x^2-2x, x^2.
9. x^2-4x, x^2.
10. $x^2-7x+10$, x^2-6x+5.
11. x^2-y^2, $x^2-5xy+4y^2$.
12. x^2-4y^2, $x^2+5xy+6y^2$.
13. $1-x^2$, $x-x^2$.
14. $1-4x^2$, $(1-2x)^2$.
15. $9x^2-1$, $(3x-1)^2$.
16. x^2-16y^2, $x^2-8xy+16y^2$.

FRACTIONS

98. Reduction of mixed quantities to fractions.

Example 1. Express as a fraction $5-\frac{3}{4}$.

$$5=\frac{5\times 4}{4}=\frac{20}{4}.$$

Hence, $$5-\frac{3}{4}=\frac{20}{4}-\frac{3}{4}=\frac{20-3}{4}=\frac{17}{4}.$$

Example 2. Express as a fraction $a-b+\frac{a^2}{a+b}$.

$$a-b=\frac{(a-b)(a+b)}{a+b}.$$

Hence, $$a-b+\frac{a^2}{a+b}=\frac{(a-b)(a+b)}{a+b}+\frac{a^2}{a+b}$$

$$=\frac{(a-b)(a+b)+a^2}{a+b}=\frac{2a^2-b^2}{a+b}.$$

Example 3. Express as a fraction $1-\frac{x+1}{x^2+1}$.

$$1=\frac{x^2+1}{x^2+1}.$$

Hence, $$1-\frac{x+1}{x^2+1}=\frac{x^2+1}{x^2+1}-\frac{x+1}{x^2+1}$$

$$=\frac{x^2+1-x-1}{x^2+1}=\frac{x^2-x}{x^2+1}.$$

To subtract two fractions having the same denominator, subtract their numerators and write the result over the common denominator. Hence, in an exercise like Example 3 above, when a minus sign precedes a fraction, the signs of terms of the numerator must be changed.

EXERCISE 82

Express as a fraction:

1. $1-\dfrac{x-1}{x+1}.$

2. $1-\dfrac{a-b}{a+b}.$

3. $a+\dfrac{a^2}{a+b}.$

4. $x-1-\dfrac{1}{x+1}.$

5. $a-b-\dfrac{b^2}{a-b}.$

6. $x^2-x+1+\dfrac{1}{x+1}.$

7. $m-2n+\dfrac{n^2}{m+2n}.$

8. $x^2-xy+\dfrac{xy^2}{x+y}.$

9. $a-b+\dfrac{a^2+b^2}{a+b}.$

10. $x^2+x+1-\dfrac{x^3}{x-1}.$

11. $4-\dfrac{b+4a}{a+b}.$

12. $-1+\dfrac{x-4y}{x+4y}.$

13. $m-4+\dfrac{12}{m+3}.$

14. $-2+\dfrac{4}{2-a}.$

15. $x-5-\dfrac{25}{x-5}.$

16. $-x+6-\dfrac{36}{6-x}.$

99. Addition and subtraction.

To add two or more fractions, first, reduce the fractions to equivalent fractions having the same denominator; second, add the numerators and write the sum over the common denominator.

To subtract one fraction from another fraction, first, reduce them to the same denomination; second, subtract the numerator of the subtrahend from the numerator of the minuend, and write the result over the common denominator.

Example 1. Simplify $\frac{2}{x}-\frac{x-2}{x+3}$.

The L. C. M. of x and $x+3$ is $x(x+3)$.

$$\frac{2}{x}=\frac{2(x+3)}{x(x+3)}.$$

$$-\frac{x-2}{x+3}=-\frac{x(x-2)}{x(x+3)}.$$

Hence,

$$\frac{2}{x}-\frac{x-2}{x+3}=\frac{2(x+3)}{x(x+3)}-\frac{x(x-2)}{x(x+3)}=\frac{2(x+3)-x(x-2)}{x(x+3)}$$

$$=\frac{2x+6-x^2+2x}{x(x+3)}=\frac{4x-x^2+6}{x(x+3)}.$$

In actual practice the work is shortened as follows:

$$\frac{2}{x}-\frac{x-2}{x+3}=\frac{2x+6-x^2+2x}{x(x+3)}=\frac{4x-x^2+6}{x(x+3)}.$$

x is contained in $x(x+3)$, $x+3$ times; multiply $x+3$ by 2 and write the product, $2x+6$. $x+3$ is contained in $x(x+3)$, x times; multiply x by $x-2$ and change signs of the terms of the product. Then combine like terms.

EXERCISE 83

Simplify:

1. $\frac{x-2}{3}+\frac{x+2}{4}+\frac{1}{6}$.
2. $\frac{2x-1}{4}+\frac{2x-3}{5}+\frac{3}{10}$.
3. $2+\frac{x}{2}+\frac{x-4}{5}$.
4. $1+\frac{x-1}{3}+\frac{x-2}{5}$.
5. $1\frac{1}{2}+\frac{3x+4}{6}-\frac{3x-1}{8}$.
6. $x-\frac{x-1}{2}-\frac{x-2}{3}$.

7. $x-\frac{x}{7}+\frac{x-4}{14}$.

8. $x+\frac{x}{5}-\frac{x+9}{10}$.

9. $\frac{3x}{2}-\frac{2x}{5}-\frac{2x-1}{10}-\frac{x+2}{15}$.

10. $\frac{a}{b}+\frac{4a-b}{5b}-\frac{2a-7b}{3b}$.

11. $\frac{a}{b}-\frac{2a}{3b}+\frac{a-b}{2b}$.

12. $\frac{3-x}{x}-\frac{2-x}{2x}+\frac{5+3x}{3x}$.

13. $\frac{3x-2}{3x}-\frac{2x-4}{4x}+\frac{7x+11}{6x}$.

14. $\frac{9x-5}{5x}-\frac{3x-7}{6x}-\frac{2x+3}{10x}$.

15. $\frac{7a+9}{3a}-\frac{9a+4}{8a}-\frac{24a-12}{24a}$.

16. $\frac{a-b}{a}+\frac{bc-b^2}{ac}$.

17. $\frac{x-3}{x}-\frac{y-4}{y}$.

18. $\frac{3x-5y}{2x}-\frac{9y-7x}{6y}$.

19. $\frac{4x-7y}{3y}+\frac{28x-3y}{12x}$.

20. $\frac{x+a}{x}+\frac{y+b}{y}-\frac{2xy+ay+bx}{xy}$.

21. $\frac{x-2}{y}-\frac{2-y}{x}-\frac{x^2+y^2}{xy}$.

22. $\frac{x-3}{a}-\frac{x+y}{b}-\frac{bx-ax}{ab}$.

23. $\frac{1}{x}-\frac{1}{x+1}$.

24. $\frac{1}{x-1}-\frac{1}{x+1}$.

25. $\frac{x+1}{x-1}-\frac{x-1}{x+1}$.

26. $\frac{1}{3x}-\frac{1}{3x+7}$.

27. $\frac{5}{2x}+\frac{5}{2x-7}$.

28. $\frac{3}{4x}-\frac{2}{4x+5}$.

29. $\frac{7}{x-4}-\frac{7}{x}$.

30. $\frac{9}{x-4}-\frac{9}{x+4}$.

31. $\frac{x}{x-y}-\frac{y}{x}$.

32. $\frac{a}{a+b}-\frac{b}{a}+\frac{ab+b^2}{a(a+b)}$.

33. $\frac{a}{a-b}+\frac{b}{a+b}-\frac{a^2-3b^2}{a^2-b^2}$.

34. $\frac{1}{x}+\frac{1}{x+3}-\frac{3-x}{x^2+3x}$.

100. Multiplication and division.

Example 1. Multiply $1+\frac{1}{x}$ by $\frac{x+1}{3x}$.

(1) Reduce $1+\frac{1}{x}$ to a fraction. $1+\frac{1}{x}=\frac{x+1}{x}$.

(2) Multiply this fraction by $\frac{x+1}{3x}$.

$$\frac{x+1}{x}\times\frac{x+1}{3x}=\frac{(x+1)^2}{3x^2}.$$

Example 2. Divide $\frac{x^2-5x+6}{x^2-x-2}$ by $\frac{x^2-8x+15}{x^2-3x-4}$.

SOLUTION. $\frac{x^2-5x+6}{x^2-x-2}=\frac{(x-2)(x-3)}{(x-2)(x+1)}$.

$$\frac{x^2-8x+15}{x^2-3x-4}=\frac{(x-3)(x-5)}{(x-4)(x+1)}.$$

Next, invert the terms of the divisor and then proceed as in multiplication.

$$\frac{(x-2)(x-3)}{(x-2)(x+1)}\times\frac{(x-4)(x+1)}{(x-3)(x-5)}=\frac{x-4}{x-5}.$$

Remember that factors only can be canceled. Single terms in polynomial expressions can never be canceled.

EXERCISE 84

1. $\frac{3x}{7y}\times\frac{2y}{3x}$.

2. $\frac{4x}{5y}\times\frac{5y^2}{2x^2}$.

3. $\frac{5x}{8y}\div\frac{10x}{3y}$.

4. $\frac{x}{x+y}\times\frac{x+y}{5x^2}$.

5. $\frac{x^2}{x-y}\times\frac{x-y}{3x^2}$.

6. $\frac{1}{x^2-x}\times\frac{x-1}{x}$.

7. $\frac{x-1}{(x-2)^2}\times\frac{x-2}{(x-1)^2}$.

8. $\frac{x^2-8x}{y^2-y}\div\frac{x-8}{y-1}$.

9. $\frac{x^2-1}{x^2+x}\div\frac{x-1}{x+1}$.

10. $\left(1-\frac{1}{x}\right)\times x$.

11. $\left(x-\frac{1}{x}\right)\times x^3$.

12. $\left(x-\frac{x}{10}\right)\times 60$.

13. $\left(x-\frac{x}{12}\right)\times(x-5)$.

14. $\left(a-\frac{a-1}{12}\right)\times\frac{a-2}{2}$.

15. $\left(\frac{x-5}{3}+2\right)\div\frac{x+1}{4}$.

16. $\left(\frac{x-2}{2}-1\right)\times\frac{3}{x-4}$.

17. $\left(1-\frac{y}{x+y}\right)\div\left(1+\frac{y}{x}\right)$.

18. $\left(1-\frac{a-b}{a+b}\right)\times\frac{1}{2}(a+b)$.

19. $\left(1+\frac{b}{a-b}\right)\times\left(1-\frac{b}{a}\right)$.

20. $\frac{1}{2}(x-3)\times\frac{1}{3}$ of $\frac{2}{x+3}$.

21. $\frac{1}{3}(x+2)\times\frac{1}{4}$ of $\frac{24}{5(x+2)}$.

22. $\frac{x^2-7x+12}{(x-4)^2}\div\frac{x-3}{x-4}$.

23. $\frac{a^2-1}{a+2}\times\frac{(a+2)^2}{(a-1)^2}\div\frac{1}{2}$ of $(a+2)$.

24. $\left(\frac{1}{x^2}-\frac{1}{y^2}\right)\div\left(\frac{1}{x}-\frac{1}{y}\right)$.

CHAPTER IX

SIMPLE EQUATIONS. RATIO AND PROPORTION. PROBLEMS

101. If an equation contains fractions, it is generally best to clear of fractions by multiplying the members of the equation by the L. C. M. of the denominators of the fractions.

Example 1. Find the value of x in the equation

$$\frac{3}{x}-\frac{2}{x-1}=\frac{2x-7}{x(x-1)}.$$

The L. C. M. of the denominators is $x(x-1)$.

$$\frac{3}{x}\times\frac{x(x-1)}{1}=3x-3.$$

$$-\frac{2}{x-1}\times\frac{x(x-1)}{1}=-2x.$$

$$\frac{2x-7}{x(x-1)}\times\frac{x(x-1)}{1}=2x-7.$$

Hence, $$3x-3-2x=2x-7.$$

Transposing, $$3x-2x-2x=3-7.$$

$$-x=-4.$$

$$x=4.$$

Check. $$\frac{3}{4}-\frac{2}{4-1}=\frac{8-7}{4\times 3},$$

i.e. $$\tfrac{3}{4}-\tfrac{2}{3}=\tfrac{1}{12}.$$

EXERCISE 85

Solve:

1. $\frac{x}{2}-\frac{x}{3}=\frac{x}{5}-2.$

2. $\frac{x}{3}-\frac{x}{4}=\frac{x}{6}-1.$

3. $\frac{2x}{3}-\frac{x}{5}=\frac{4x+45}{15}.$

4. $\frac{5x+2}{8}-\frac{x+6}{6}=\frac{5x-6}{12}.$

5. $\frac{x+4}{5}+\frac{1}{2}(x+1)=x-2.$

6. $\frac{x+3}{4}+\frac{1}{5}(x+7)=\frac{3x+1}{5}.$

7. $\frac{3x+5}{7}-\frac{2x+5}{5}=\frac{x-10}{7}.$

8. $\frac{4x-2}{13}-\frac{x-4}{26}=1\frac{1}{13}.$

9. $\frac{5x+4}{16}-\frac{x-2}{10}=\frac{x}{4}.$

10. $\frac{11x-12}{3}+\frac{13x-4}{4}=5x-5.$

11. $(x-1)(x-2)-(x-3)(x-4)=x-1.$

12. $(x-5)(2x-1)-(2x-3)(x+1)+8x+4=0.$

13. $(2x-3)(2x-1)-(4x-3)(x+2)=19-15x.$

14. $(x-4)(x-11)+(x+4)(x+7)=2x(x+6)+8.$

15. $(3x-2)^2-(2x-3)^2=5x^2-x-3.$

16. $(x-7)^2-(x-5)^2=x-11.$

17. $(x-9)^2-(x-8)^2=3(x-11).$

18. $(2x+5)^2+(2x+1)^2=(8x+1)(x+3)+15.$

19. $(3x+1)^2+(x+2)^2=10x(x+2)-5.$

20. $(x-6)^2-(x+5)^2=5-21x.$

21. $(x-24)^2-(x-25)^2=26-x.$

22. $\frac{3}{2x}-\frac{5}{3x}=\frac{1}{x}-1\frac{1}{6}.$

23. $\frac{10}{x}+\frac{3}{2x}-1=\frac{4}{x}+\frac{7}{8}.$

24. $\frac{12}{x}-\frac{x-1}{3x}-2=\frac{2}{3x}+1\frac{5}{9}.$

25. $\frac{5}{x}-\frac{3}{2x}+\frac{1}{3x}=\frac{7}{6x}+2\frac{2}{3}.$

26. $\frac{9}{x}-\frac{4}{3x}-\frac{1}{4x}=\frac{7}{x}+\frac{5}{12}.$

27. $\frac{1}{x}-\frac{1}{x+1}=\frac{x-7}{x(x+1)}.$

28. $\frac{2}{x}-\frac{2}{x-2}=\frac{x-8}{x(x-2)}.$

29. $\frac{3}{x}-\frac{5}{x+3}=\frac{21-3x}{x(x+3)}.$

30. $\frac{4}{x}+\frac{x+3}{x-5}=1+\frac{16}{x(x-5)}.$

31. $\frac{5}{x}+\frac{4}{2x-1}=\frac{9x+5}{x(2x-1)}.$

32. $\frac{x+2}{x}+\frac{x+1}{2x+3}=1+\frac{x^2+6x+5}{x(2x+3)}.$

33. $\frac{4-3x}{4}-\frac{7-6x}{8}=\frac{x-5}{x+2}.$

34. $\frac{2x-1}{3}-\frac{x+2}{x+5}=\frac{2x-3}{3}.$

35. $\frac{5x-3}{4}+\frac{x-1}{x-2}=\frac{10x+1}{8}.$

36. $\frac{3x}{5}-\frac{2x-2}{x+1}=1-\frac{11-6x}{10}.$

37. $\dfrac{3x-1}{4}+\dfrac{x+6}{x-5}=\dfrac{6x+10}{8}.$

38. $\dfrac{5x-4}{5}+\dfrac{2x+3}{2x-7}=\dfrac{10x+7}{10}.$

39. $\dfrac{2x-5}{3x+4}=\dfrac{4x-13}{6x+7}.$

40. $\dfrac{4x-5}{2x+7}=\dfrac{6x-5}{3x+13}.$

RATIO

102. The ratio of one quantity to another quantity of the same kind is the quotient of the first quantity divided by the second quantity. Thus, the ratio of 2 feet to 3 feet is the quotient of 2 feet divided by 3 feet, *i.e.* $\frac{2}{3}$.

Magnitudes of the same kind only can be compared. It would be incorrect to say, What is the ratio of 7 feet to $5?

103. The first of the two numbers in a ratio is called the **antecedent**, the second is called the **consequent.**

The symbol for a ratio is the colon between the antecedent and the consequent. The ratio of 3 to 4 is written 3 : 4.

104. The inverse ratio of two numbers is the ratio of the second number to the first number. The inverse ratio of 5 to 3 is 3 : 5.

105. Since a ratio is a quotient, its value remains unchanged when its terms are multiplied or divided by the same number.

EXERCISE 86

The land and water areas of the states named are given in the following table:

	Land Surface Square Miles	Water Surface Square Miles
(*a*) California,	156,200	2,030
(*b*) Delaware,	1,969	411
(*c*) Illinois,	56,000	2,350
(*d*) Louisiana,	45,400	4,227
(*e*) Maryland,	9,875	2,422
(*f*) Minnesota,	79,997	6,338
(*g*) North Carolina,	48,972	3,702
(*h*) Pennsylvania,	44,680	1,249
(*i*) Wisconsin,	55,117	10,688

1. Find the value of the ratio of the land surface of each state to the water surface.

2. Find the value of the ratio in each case of the total surface to the land surface.

3. A farm contains 180 acres. Find the value of the ratio of the area of this farm to 1 square mile.

4. What is the ratio of 1 inch to 1 centimeter?

Example 1. Divide $70 between A and B in the ratio of 2 : 5.

Let $x =$ number of dollars B gets.

Then, $\frac{2}{5}$ of $x =$ number of dollars A gets.

Therefore, $x + \frac{2x}{5} = 70.$

Multiplying by 5, $5x + 2x = 350.$

$$7x = 350.$$

$$x = 50.$$

$\frac{2}{5}$ of $50 = 20.$

A's share, \$20; B's share, \$50.

Check. $\$20 + \$50 = \$70.$

Or, let $x =$ number of dollars A gets.

$70 - x =$ number of dollars B gets.

Then, $$\frac{x}{70 - x} = \frac{2}{5}.$$

Clearing of fractions, $5x = 140 - 2x.$

Transposing, $7x = 140.$

$$x = 20.$$

$$70 - x = 50.$$

EXERCISE 87

1. Divide 200 in the ratio of 3 : 5.

2. Divide 280 in the ratio of 5 : 9.

3. Divide 320 in the inverse ratio of 3 : 2.

4. Two numbers in the ratio of 4 to 7 differ by 39. Find them.

5. Two numbers are in the ratio of 4 to 5. If the consequent is diminished by 18, its ratio to the antecedent will be 3 to 4. Find the numbers.

6. A's money is to B's in the ratio of 2 to 3, and if A gave B \$10, the ratio of A's money to B's would be 1 to 2. Find A's money and B's money.

7. A is 24 years old and B is 36 years old. After how many years will the ratio of A's age to B's age be 5 : 7?

8. A is 50 years old and B is 30 years old. How many years has it been since the ratio of their ages was 2:1?

9. A and B are partners, A's capital in the concern being $2100, and B's being $2800. If they make a profit of $1400, how ought it to be divided?

10. If $\frac{3}{5}$ of A's money is equal to $\frac{7}{12}$ of B's, express in integers the ratio of A's money to B's money.

11. Common solder is an alloy of tin and lead, 7 parts tin to 1 part lead. How many pounds of tin and of lead are in 22 pounds of common solder?

12. Pewter is an alloy of 3 parts tin to 1 part lead. How many pounds of tin and of lead are in 34 pounds of pewter?

13. Dutch brass consists of 5 parts copper to 1 part zinc. How many ounces of each are in 27 ounces of Dutch brass?

14. A 68-acre lot is planted in corn and wheat. If $\frac{2}{3}$ the number of acres of corn equal $\frac{3}{4}$ the number of acres of wheat, how many acres of corn and of wheat are there?

15. There are two numbers in the ratio of 2:3. If the lesser number is diminished by 2 and the greater increased by 3, the ratio of the two results will be 1:2. Find the numbers.

16. The dimensions of the floor of a room are in the ratio of 4:3. If the width is increased by $16\frac{2}{3}\%$ and the length diminished by 2 feet, the dimensions would be equal. Find the dimensions.

PROPORTION

106. Four quantities are in proportion if the ratio of the first to the second equals the ratio of the third to the fourth. The numbers 2, 3, 6, 9, are in proportion, since the ratio $2:3$ equals the ratio $6:9$. The formal statement of this proportion is $2:3=6:9$. This proportion is read, "2 is to 3 as 6 is to 9." This means 2 is as large compared with 3 as 6 is compared with 9. The above proportion may also be written $\frac{2}{3}=\frac{6}{9}$.

107. In a proportion the first and fourth terms are called the **extremes** and the second and third terms are called the **means** of the proportion.

108. Two quantities are directly proportional, or vary directly, if an increase in the one produces a corresponding increase in the other. For example, the cost of a quantity of apples varies with the quantity. The distance traveled at a given rate per hour varies with the time. If a certain distance is traveled in one hour at a given rate, twice the distance would be traveled in two hours, three times the distance in three hours, and so on.

109. Two quantities are inversely proportional or vary inversely if an increase in the one produces a corresponding decrease in the other. For example, the time in which 7 men do a piece of work is to the time in which 4 men can do the same work in the ratio $4:7$.

Also, the time consumed in going a given distance varies inversely as the rate: the greater the rate the less the time.

A third example is the number of articles which can be bought for a given sum of money. The greater the price of the articles, the fewer the number of articles that can be bought for the given sum of money.

110. In a proportion the product of the means equals the product of the extremes.

This may be shown as follows:

Let the proportion be $a : b = c : d$,

or $$\frac{a}{b} = \frac{c}{d}.$$

Multiply by bd, $$ad = bc.$$

Example 1. If 7 head of cattle cost $292, find the cost of 11 head of cattle.

Let $x =$ the number of dollars 11 head cost.

Then, the cost of 11 head : the cost of 7 head = 11 : 7.

Hence, $$x : 292 = 11 : 7,$$

or $$\frac{x}{292} = \frac{11}{7}.$$

Multiply by 292, $$x = \frac{11 \times 292}{7} = 462.$$ *Ans.* $462.

EXERCISE 88

1. A train goes 244 miles in 8 hours. At the same rate, how far will it go in 10 hours?

2. If 8 horses cost $364, at this rate find the cost of 7 horses.

3. The earth revolves on its axis 15° in 1 hour. Through how many degrees does it revolve in 19 minutes?

4. When a dozen eggs bring 18 cents, how much will 42 eggs bring at the same rate?

5. If 12 horses plow a field of 50 acres in 10 days, how many acres will 15 horses plow in the same time?

6. A pole 15 feet high casts a shadow 12 feet long. Find the length of the shadow cast by a flagstaff 70 feet high.

7. The dimensions of a triangular tract of land are 784 yards, 629 yards, 842 yards. A map of this tract is drawn to a scale of 1 inch to 125 yards. Find the dimensions of the tract on this map.

8. Two towns are 40.5 miles distant from each other. On a map these towns are $2\frac{1}{4}$ inches apart. Find the scale of the map.

9. On a certain map $1\frac{1}{2}$ inches corresponds to 21 miles. To how many miles will $3\frac{1}{2}$ inches correspond on this map?

10. The height of Mount Everest is about 5.5 miles. On a globe this height is represented by $\frac{1}{8}$ of an inch. Find the diameter of this globe. The diameter of the earth is 7920 miles. (*b*) Find the length of an arc of 1° on a great circle of the moon. The diameter of the moon is 2160 miles.

11. Suppose the distance from the earth to the sun is represented by a line 10 inches long. Find how many inches will represent the distance of each of the other planets from the sun.

	Distance from the Sun in Miles
Mercury	36 million
Venus	67.2 million
Earth	92.9 million
Mars	141.5 million
Jupiter	483.3 million
Saturn	886 million
Uranus	1782 million
Neptune	2792 million

12. A certain map of Great Britain is constructed on the scale 1 : 887,000, *i.e.* 1 inch represents 887,000 inches. On this map Oxford is marked 4.26 inches from Greenwich; Cambridge, 3.65 inches from Greenwich; Liverpool, 5.62 inches from Birmingham; and Glasgow, 3.04 inches from Edinburgh. Find, correct to $\frac{1}{10}$ of a mile, the distances between these cities.

13. A stone is dropped from an elevation and strikes the earth $3\frac{1}{2}$ seconds later. Find the velocity; given, that the velocity at the end of each successive second is proportional to the time, and the velocity at the end of the first second is 32 feet.

14. If 83% of a number is 124.5, find 115% of the number.

15. If a ball 1 inch in diameter represents the earth, how large a ball would represent the sun? The sun's diameter is 866,000 miles.

Example 1. If 6 horses eat a quantity of corn in 14 days, how long would it take 8 horses to eat a quantity equally large?

Let $x =$ the number of days.

It will take 8 horses less time to eat the corn.

Hence, $x : 14 = 6 : 8,$

or $$\frac{x}{14} = \frac{6}{8}.$$

Therefore, $x = \frac{14 \times 6}{8} = 10\frac{1}{2}$. *Ans.* $10\frac{1}{2}$ days.

EXERCISE 89

1. If the interest on $850 for 16 months is $68, what principal will produce the same interest in 12 months?

2. If $940 yields $28.20 interest in 219 days, what sum will yield the same interest in 146 days?

3. A block of marble whose specific gravity is 2.7 contains 3 cubic feet. Find the volume of a piece of iron just as heavy. The specific gravity of iron is 7.2.

4. How long would it take 18 men to pave a street which 15 men pave in 15 days?

5. Find a number whose ratio to 12 is equal to the inverse ratio of 4 to 5.

6. If $5\frac{3}{4}$ tons of coal can be bought for a certain sum of money, how many tons can be bought for the same sum of money if the coal rises 15% in price?

7. If $21\frac{1}{4}$ yards of carpet can be bought for a certain sum of money, how many yards can be bought for the same sum of money, when the price of carpet falls 15%?

8. A wheel $3\frac{1}{2}$ feet in diameter makes 480 revolutions in going 1 mile. How many revolutions will a wheel 4 feet 8 inches in diameter make in going 1 mile?

9. The volume of a certain quantity of gas is 25 cubic inches when the pressure is 16 pounds to the square inch. Find the volume when the pressure is 20 pounds to the square inch. (The volume varies inversely as the pressure.)

10. 82,000 tons of rock are transported at the rate of 90 cents per ton. How many tons could be transported for the same money, if the rate was $1.20 per ton?

PROBLEMS

Example 1. Two numbers differ by 3 and the difference of their squares is 111. Find them.

Let $x =$ the less number.

Then, $x + 3 =$ the greater number.

Hence, $(x + 3)^2 - x^2 = 111,$

or $x^2 + 6x + 9 - x^2 = 111.$

$$6x = 102.$$

$$x = 17.$$

$$x + 3 = 20.$$

Check. $20^2 - 17^2 = 111.$

EXERCISE 90

1. The difference of the squares of two consecutive numbers is 71. Find the numbers.

2. The difference of the squares of two consecutive odd numbers is 120. Find the numbers.

3. Find two numbers differing by 5, the difference of whose squares is 275.

4. Find two numbers differing by 6, the square of whose sum exceeds 4 times the square of the less number by 516.

5. Two numbers differ by 4, and the square of the greater number exceeds the product of the two numbers by 76. Find them.

Example 1. A rectangular garden is 10 yards longer than it is wide. If its length is diminished by 5 yards and its width by 4 yards, its area is diminished by 380 square yards. Find its dimensions.

Let $x =$ the number of yards in its width.

Then, $x + 10 =$ the number of yards in its length.

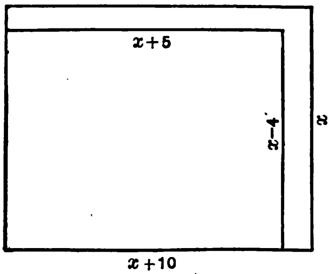
$x(x + 10) =$ the number of square yards in its area.

$x - 4 =$ the number of yards in width after it is diminished.

$x + 5 =$ the number of yards in length after it is diminished.

$(x - 4)(x + 5) =$ the number of square yards in the area.

Therefore, $x(x + 10) - 380 = (x - 4)(x + 5).$



$$x^2 + 10x - 380 = x^2 + x - 20.$$

$$9x = 360.$$

$$x = 40.$$

Width, 40 yards.

$$x + 10 = 50.$$

Length, 50 yards.

Check.

$$50 \times 40 - 380 = 36 \times 45.$$

EXERCISE 91

1. A rectangle whose length exceeds its width by 5 yards is increased in area by 190 square yards, if its length is increased by 5 yards and its width by 3 yards. Find its dimensions.

2. The length of a rectangle exceeds its width by 10 yards. If its dimensions are each diminished by 5 yards, its area is diminished by 325 square yards. Find its dimensions.

3. The length of a rectangle exceeds its width by 4 rods. If the length is increased by 3 rods and the width diminished by 2 rods, the area is diminished by 2 square rods. Find its dimensions.

4. There is a square plot of ground such that if one side be increased by 12 yards and the other diminished by 9 yards, the area of the resulting rectangle will equal the square. Find the side of the square.

5. The length of a rectangle is twice its breadth. If each side is diminished by 2 feet, its area is diminished by 68 square feet. Find its dimensions.

Example 1. A man invested $1280 partly in 4% stock at 80 and partly in 5% at 90. His income from both is $68. How much did he invest in each stock?

Let $x =$ number of dollars in 4% stock.

$1280 - x =$ number of dollars in 5% stock.

$\frac{4}{80}$ of $x =$ number of dollars' income from 4% stock.

$\frac{5}{90}$ of $(1280 - x) =$ number of dollars' income from 5% stock.

Therefore, $\frac{x}{20}+\frac{5}{90}(1280-x)=68.$

Multiplying by 180,

$$9x+12800-10x=12240.$$

$$-x=-560.$$

$$x=560, \$560 \text{ in } 4\% \text{ stock.}$$

$$1280-x=720, \$720 \text{ in } 5\% \text{ stock.}$$

Check. $\frac{4}{80}$ of $560+\frac{5}{90}$ of $720=68.$

EXERCISE 92

1. A man has \$50,000 invested, part at 4% and part at 5%. His income from both investments is \$2140. How much has he invested at each rate?

2. A man invests \$5000, partly at 6% and partly at 5%, and his income from both is \$290. What sum is invested at each rate?

3. A man invests \$1980, part in 3% stock at 72 and part in 4% stock at 84. His dividend from the two stocks is \$90. How much does he invest in each stock?

4. If I invest $\frac{1}{2}$ my capital at 5% interest, $\frac{1}{3}$ at 4% and the rest at 7%, my annual income is \$600. Find my capital.

5. By investing a sum of money at 5% interest \$30 more is made annually than would have been made if the sum was invested in 4% stock at 90. Find the sum.

Mixtures

Example 1. A piece of bronze weighing 15 pounds contains 80% of copper and 20% of tin. With how many pounds of copper must this be melted so as to give an alloy containing 85% of copper?

Let $x =$ number of pounds of copper.

80% of 15 = 12, number of pounds of copper in the bronze.

$12 + x =$ number of pounds of copper in the alloy.

$15 + x =$ number of pounds in the alloy.

Hence, $12 + x = 85\%$ of $(15 + x)$,

or $12 + x = \frac{85}{100}(15 + x)$.

$$1200 + 100x = 1275 + 85x.$$

$$15x = 75.$$

$x = 5$. *Ans.* 5 pounds.

Check. $12 + 5 = 85\%$ of $(15 + 5)$.

EXERCISE 93

1. A piece of bronze weighing 20 pounds contains 70% of copper and 30% of tin. With how many pounds of copper must this be melted to make an alloy containing 76% of copper? With how many of tin to make an alloy containing 44% tin?

2. An alloy weighing 10 pounds contains 60% of lead and 40% of tin. How much lead must be added to it so that the new alloy may contain 80% of lead? How much tin so that the alloy may contain 64% of tin?

3. An alloy contains 60% of copper and 40% of zinc. A second alloy contains 30% of copper and 70% of zinc. How many pounds of the latter must be melted with 5 pounds of the former to make an alloy half copper and half zinc?

4. A certain mixture contains by volume 90% of alcohol and 10% of water. How many gallons of water must be added to 10 gallons of the mixture to make a mixture containing 72% of alcohol? How many gallons of water to make a mixture containing 81% of alcohol?

5. The specific gravity of a mixture of milk and water is 1.024. The specific gravity of the milk is 1.03. How many parts of water and how many parts of milk are in the mixture?

6. The specific gravity of Dead Sea water is 1.24. How many pounds of fresh water must be added to 3 pounds of Dead Sea water to make the specific gravity of the mixture 1.048?

7. A quart measure is filled with ice and butter. The specific gravity of ice is .92, and of butter .94. If the specific gravity of the ice and butter is .938, determine what part by weight of the quart measure is ice and what butter.

8. How much corn at 65 cents per bushel should be mixed with 11 bushels of oats at 48 cents per bushel to make a mixture worth 54 cents per bushel?

9. How many pounds of tea at 63 cents per pound should be mixed with 10 pounds of tea at 40 cents per pound to make a mixture worth 53 cents per pound?

10. Gold coins consist of an alloy of gold and copper. The specific gravity of gold is 19.26, of copper, 8.95, and of the alloy is 18.229. In 10 ounces of the alloy, how many ounces are gold and how many are copper?

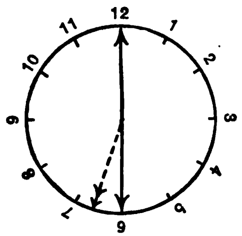
11. Silver coins consist of an alloy of silver and copper.

The specific gravity of silver is 10.47 and of silver coins is 10.318. How many ounces of silver and of copper are in a quantity of silver coins weighing 10 ounces?

Clocks, Circular Motion

Example 1. At what time between 6 and 7 o'clock are the hands of a clock together?

Solution. They will be together when the minute hand gains on the hour hand the 30 minute spaces the hour hand is ahead.



Let $x =$ number of minute spaces the minute hand moves.

Then, $\frac{x}{12} =$ number of minute spaces the hour hand moves.

Hence, $x = 30 + \frac{x}{12}$.

$x = 32\frac{8}{11}$. *Ans.* $32\frac{8}{11}$ minutes past 6 o'clock.

Example 2. At what time between 4 and 5 o'clock are the hands of a clock at right angles?

Solution. The hands are at right angles when they point to positions 15 minute spaces apart. At 4 o'clock they point 20 minute spaces apart. Hence, when the minute hand gains (20 – 15) minute spaces, they are at right angles. Also, when the minute hand gains (20 + 15) minute spaces, they are at right angles. Hence, as in Example 1 above,

$$x = 5 + \frac{x}{12}.$$

$$x = 35 + \frac{x}{12}.$$

Solving these equations, the times are found.

EXERCISE 94

1. At what time are the hands of a clock together (*a*) between 2 and 3 o'clock? (*b*) between 4 and 5 o'clock? (*c*) between 7 and 8 o'clock? (*d*) between 9 and 10 o'clock?

2. At what time are the hands of a clock at right angles (*a*) between 1 and 2 o'clock? (*b*) between 3 and 4 o'clock? (*c*) between 5 and 6 o'clock? (*d*) between 11 and 12 o'clock?

3. At what time do the hands of a clock point in opposite directions (*a*) between 1 and 2 o'clock? (*b*) between 3 and 4 o'clock? (*c*) between 6 and 7 o'clock? (*d*) between 9 and 10 o'clock?

4. At what time after 4 o'clock is the hour hand for the first time $\frac{3}{20}$ of a revolution ahead of the minute hand? The minute hand $\frac{2}{5}$ of a revolution ahead of the hour hand?

5. A and B travel around a circular track in the same direction in 10 minutes and 7 minutes respectively. They start together. After how many minutes will B have made one more circuit than A? If they travel in opposite directions, when will they meet?

HINT. Find the number of revolutions each makes in x minutes. In the former the difference is 1. In the latter the sum is 1.

6. A and B travel around a circular track in 15 and 12 minutes respectively. If they start at the same time and travel in the same direction, when will B be $\frac{2}{3}$ of a revolution ahead of A? If they travel in opposite directions, when will they together have made two circuits?

Example 1. The sum of the digits of a number of two figures of which the tens' digit is the greater is 11. If the number be divided by the difference of its digits, the quotient is 24 and the remainder 2. Find the number.

Let $x =$ the tens' digit.

$y =$ the units' digit.

x tens $+ y$, *i.e.* $10x + y =$ the number.

Hence, $$x + y = 11. \qquad (1)$$

$$\frac{10x + y - 2}{x - y} = 24. \quad \text{Why?} \qquad (2)$$

Clearing (2) of fractions,

$$10x + y - 2 = 24x - 24y.$$

Transposing, $-14x + 25y = 2.$

Multiplying (1) by 14,

$$14x + 14y = 154.$$

Adding, $$39y = 156.$$

$$y = 4.$$

$$x = 7. \quad \textit{Ans. } 74.$$

Check. $$\frac{74}{7 - 4} = 24\tfrac{2}{3}.$$

EXERCISE 95

1. A number consists of two digits. If the number formed by inverting the digits be divided by the difference of the digits, the quotient is 23. If 27 be taken from the number, the digits will be interchanged. Find the number.

2. A number consists of two digits, of which the first exceeds the second by 5. If the number be divided by the sum of the tens' digit and twice the units' digit, the quotient is 6 and the remainder is 6. Find the number.

3. A number consists of two digits, of which the first exceeds the second by 3, and if the digits be reversed, a number is formed which is $\frac{4}{7}$ of the former. Find it.

4. Two numbers differ by 14, and the greater divided by the difference between the less number and unity gives $2\frac{1}{2}$ for quotient. Find the numbers.

5. A certain fraction is equal to $\frac{1}{4}$ when its denominator is diminished by 1, and equal to unity when its numerator is increased by 22. Find it.

6. A certain fraction is equal to $\frac{4}{5}$ when its terms are each diminished by 1, and equal to $\frac{5}{6}$ when its terms are each increased by 3. Find it.

7. Divide 100 into two parts so that the greater divided by the less gives 2 for quotient and 1 for remainder.

8. Two numbers are in the ratio of 4 to 9. If the numbers are each diminished by 9, the ratio of the remainders will be 3 to 8. Find them.

Example 1. A boat is rowed down a stream at the rate of 7 miles an hour, and up the stream at the rate of 4 miles an hour. Find the rate of rowing in still water and the rate of the stream.

Let $x =$ rate of rowing per hour in still water.

$y =$ rate of the stream.

Then, $x + y =$ rate of rowing per hour down stream.

$x - y =$ rate of rowing per hour up stream.

Therefore, $x + y = 7.$

$x - y = 4.$

Adding, $2x = 11.$

$x = 5\frac{1}{2}.$

Subtracting, $2y = 3.$

$y = 1\frac{1}{2}.$

The rate of rowing in still water is equal to half the sum of the rates with the stream and against the stream, and the rate of the stream equals half the difference of the same two numbers.

Example 2. A boat is rowed a distance of 10 miles down a stream and back in 5 hours. The rates of rowing down and back are in the ratio of 3 : 2. Find the rate in still water and the rate of the stream.

SOLUTION. The simplest solution of the problem is an indirect one. Find first the time of rowing down and of rowing up.

Since the rates down and up are in the ratio of 3 : 2, the times down and up will be in the inverse ratio of 3 : 2, *i.e.* 2 : 3.

Let $x =$ time down.

$y =$ time up.

Then, $x + y = 5. \quad (1)$

$$\frac{x}{y} = \frac{2}{3}. \quad (2)$$

Clearing (2) of fractions,

$$3x = 2y.$$

$$3x - 2y = 0.$$

Multiplying (1) by 2,

$$2x + 2y = 10.$$

Adding,

$$5x = 10.$$

$$x = 2.$$

$$y = 3.$$

The time of rowing down is 2 hours. Hence, the rate in miles per hour down is $\frac{10}{2}$, *i.e.* 5. The time of rowing up is 3 hours, and hence the rate in miles per hour of rowing up is $\frac{10}{3} = 3\frac{1}{3}$.

$$\tfrac{1}{2}(5 - 3\tfrac{1}{3}) = \text{rate in miles of the stream.}$$

EXERCISE 96

1. A boat is rowed 15 miles down a stream and back in $7\frac{1}{2}$ hours. The rate down is twice the rate up. Find the rate of rowing in still water and the rate of the stream.

2. A steamboat goes 20 miles down a stream and back in $3\frac{5}{9}$ hours. It goes 5 miles with the stream in the same time it goes 3 miles against the stream. Find the rate of the stream and the rate of the boat in still water.

3. How far can a person ride in a carriage which goes 8 miles an hour so as to return at the rate of 3 miles an hour and be gone 2 hours 45 minutes?

4. A person travels a certain distance in 3 hours. If he increases his rate by $\frac{1}{5}$, he would go 3 miles farther in the same time. Find his rate.

5. A person rides on a bicycle a certain distance in 2 hours. If he diminishes his rate by $\frac{1}{4}$, it would take him $2\frac{1}{2}$ hours to go a distance of 1 mile less. Find the distance and the rate.

Miscellaneous Problems

EXERCISE 97

1. A father is 4 times as old as his son, and 4 years ago he was 6 times as old as his son. Find their ages.

2. A mother is 3 times as old as her daughter, and 5 years ago she was 4 times as old as her daughter. Find their ages.

3. A says to B, "Give me $10 and I shall have twice as much money as you." "No," says B, "give me $10 and I shall have as much money as you." How much money has each?

4. The price of a suit of clothes after giving a discount of 10% from the marked price, is $13.50. Find the marked price.

5. By investing half his capital at 5% and half at 4%, a person derives an annual income of $405 from his investment. Find his capital.

6. The difference of the squares of two consecutive odd numbers is 48. Find the numbers.

7. $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a certain number together make 156. Find it.

8. $\frac{1}{4}$ of a number exceeds $\frac{1}{7}$ of it by $10\frac{1}{2}$. Find the number.

9. The perimeter of a rectangle is 54 yards, and its length exceeds its breadth by 7 yards. Find its dimensions.

10. The length of a rectangle exceeds its breadth by 10 yards. If the length is diminished by 2 yards and

the breadth increased by 3 yards, the area is increased 44 square yards. Find the dimensions of the rectangle.

11. The length and breadth of a rectangle are respectively 14 yards longer and 10 yards shorter than the side of a square of equal area. Find the side of the square.

12. A freight train leaves a station and travels at the rate of 18 miles per hour. Two hours later a passenger train leaves the same station and travels at the rate of 27 miles per hour. In how many hours will the passenger train have gone 18 miles farther than the freight train?

13. The product of two numbers differing by 4 is 36 more than the square of the less. Find them.

14. A can walk around a rectangle in 10 minutes, and B can walk around it in 12 minutes. If they start at the same time, when will they be together? (Two solutions.)

15. A can walk around a circular track in 7 minutes, and B in 9 minutes. After how many minutes will they be together (*a*) if they go in the same direction? (*b*) if they go in opposite directions?

16. Divide $1 between two persons in the inverse ratio of 13 to 7.

17. The area of the earth's surface is 197,000,000 square miles. The diameter of the earth is 7920 miles, and that of the moon is 2160 miles. Find the area of the surface of the moon. (Similar surfaces are proportional to the squares of their corresponding dimensions.)

18. By investing $4680, partly in 4% stocks at 90 and partly in 3% stocks at 72, I derive an income of $200 a year. How much money is invested in each stock?

19. A and B rent a pasture for \$42. A puts in 10 head of cattle and B 14 head. How much rent should each pay?

20. $\frac{3}{5}$ of A's capital equals $\frac{2}{3}$ of B's capital. What is the ratio of A's capital to B's?

21. A's money is equal to $\frac{2}{3}$ of B's, and B's money is equal to $\frac{3}{4}$ of C's. What part of C's money is A's money?

22. A and B are partners. $\frac{1}{2}$ of A's capital is equal to $\frac{2}{5}$ of B's. Divide a profit of \$1800 between them.

23. In a mile race A can give B 88 yards, and B can give C 80 yards. How many yards can A give C in 1 mile?

24. A crew can row 1 mi. down a stream in 10 min., and can row 1 mi. up the stream in 15 min. How far down stream can they go in order to return in 1 hour?

25. A woman brought a number of turkeys to market. To the first buyer she sold half the turkeys and half a turkey more. To the next she sold half the remainder and half a turkey more. To the next she sold half the number still remaining and half a turkey more. She had then none left. How many did she bring to market?

26. How should goods be marked so that a profit of 25% may be made after giving a discount of 20%?

27. A suit of clothes is sold at a profit of 25%. Had the dealer bought it for \$1 less and sold it for the same price, he would have made $33\frac{1}{3}\%$. Find the cost.

28. Gun metal consists of 90% copper and 10% zinc; English brass of $66\frac{2}{3}\%$ copper and $33\frac{1}{3}\%$ zinc; Dutch brass of $83\frac{1}{3}\%$ copper and $16\frac{2}{3}\%$ zinc. How many pounds of English brass must be melted with 5 lb. of gun metal to make Dutch brass?

Problems in Commercial Arithmetic

EXERCISE 98

Commercial Discount

1. A suit of clothes cost $17 after allowing a discount of 15% on the marked price. Find the marked price.

2. A clothier buys silk at 80 cents per yard. How should he mark it so as to make a profit of 25%? How would he mark it so as to make a profit of 25% after allowing a discount of 20% on the marked price?

HINT. $x - 20\%$ of $x = 80 + 25\%$ of 80.

3. Shoes are bought at $1.68 a pair. How should they be marked so that the dealer may make 25% profit after allowing a discount of $16\frac{2}{3}\%$ on the marked price?

4. How should a dealer mark a rug costing $18, so as to make a profit of 50% after allowing a discount of 10% on the marked price?

Commission

5. An agent sells 80 acres of land, charging 3% commission. If his commission amounts to $144, how much per acre did the land sell for?

6. If a commission of $43.20 is paid for buying 30 acres of land at $36 per acre, what is the rate per cent of commission?

7. An attorney charges 5% for collecting a debt. If

he remits to his principal \$5890, what is the amount of the debt?

8. A tax collector's commission at $2\frac{1}{2}\%$ is \$289. Find the amount of the taxes collected.

9. A dealer sells 500 tons of anthracite coal and remits to his principal after deducting 8% commission, \$3680. Find the selling price of the coal per ton.

Stocks

10. When a share of stock paying $4\frac{1}{2}\%$ dividend is bought for \$90, what rate of interest does money invested in this stock bring?

11. If stock paying a dividend of 7% gives an income at the rate of 5% on the money invested, what is the cost of one share of stock?

12. What should I pay for 5% stock so that I may get 4% interest for my money?

13. A company is able to pay 6% dividend on its entire stock, but \$100,000 of its stock is preferred stock paying 5% dividend, and on this account the company pays on its common stock $6\frac{1}{3}\%$ dividend. Find the amount of the common stock.

14. A person invests equal sums in 8% stock at 80 and 5% stock at 120, and thereby derives an annual income of \$1900. How much money did he invest in each stock?

Taxes

15. In a certain city the tax rate is $1\frac{3}{4}\%$, and on a house and lot the tax is \$63. Find the assessed valuation of the house and lot.

16. If 4% of the taxes of a certain town are spent in collection, find the tax levy so that $166,032 may be available for public purposes.

17. In a certain city the tax for school purposes is 3 mills on $1. If this tax amounts to $24,210, find the assessed valuation of taxable property.

18. If $18,477.90 is raised by taxation at the rate of 40 cents on $100, find the valuation of taxable property.

19. Find the rate of taxation when $29.14 is paid on property assessed at $7285.

United States Customs Duties

20. The cost of 10 opera glasses imported from England after paying a duty of 45% ad valorem is $84.39. (*a*) Find the invoice price in United States currency. (*b*) Find the invoice price in British currency (£1 = $4.85).

21. The duty on treble ingrain carpet is 22¢ per square yard and 40% ad valorem. Suppose the invoice price is $1 per square yard, and the cost of a rug after the payment of duty is $32.40. How many square yards are in the rug?

22. The duty on razors is $1.75 per dozen and 20% ad valorem. The cost to an importer of 6 dozen Sheffield razors is $94.31. (*a*) Find the invoice price in United States currency. (*b*) Find the invoice price in British currency (£1 = $4.85). (*c*) Find the selling price of a razor if the importer makes a profit of 20%. (*d*) If the razors are sold at $2 apiece, what is the rate per cent of profit?

23. A jeweler imports 5 microscopes which cost him after paying a duty of 45% ad valorem $290. (*a*) Find the amount of the invoice. (*b*) How should the microscopes be marked so as to make a profit of 40% after allowing a discount of 10% on the marked price?

24. The duty on lead pencils is 45¢ per gross and 25% ad valorem. (*a*) Find the invoice price per gross of imported pencils which cost the importer $2.45 per gross. (*b*) If these pencils are sold at 3¢ apiece, find the gain per cent.

Insurance

25. The annual premium on a life insurance policy of $2500 is $49.80. Find the rate on $1000 insurance.

26. If the annual premium on a life insurance policy at the rate of $27.10 per $1000 is $130.08, find the amount of the policy.

27. A residence is insured at the rate of $1.10 per $100. If the premium is $37.95, find the amount of the policy.

28. A house is insured at the rate of $1.30 per $100. If the premium is $109.59 and the amount of the policy is $\frac{3}{4}$ of the value of the house, find the value of the house.

29. A man has a life insurance policy at the rate of $29.40 per $1000, and a policy for the same amount at $28.30 per $1000. He pays for both $201.95. Find the amount of each policy.

CHAPTER X

INVOLUTION. SQUARE ROOT

111. The process of raising a quantity to a power is called **involution.**

112. Find the third power of a^2.

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6.$$

Hence, **To find any power of a letter affected with an exponent, write the letter, and for its exponent take the product of the exponent of the power by the exponent of the letter.**

113. Find the fourth power of abc.

$$(abc)^4 = abc \cdot abc \cdot abc \cdot abc = aaaa \cdot bbbb \cdot cccc = a^4b^4c^4.$$

Associative Law.

Hence, **A power of a monomial expression is obtained by raising the factors of the expression to the required power and taking their product.**

114. Raise $\frac{a}{b}$ to the fifth power.

$$\left(\frac{a}{b}\right)^5 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^5}{b^5}.$$

Hence, **A power of a fraction equals the power of the numerator divided by the power of the denominator.**

115. Since
$$(-a)^2 = +a^2,$$
$$(-a)^3 = -a^3,$$
$$(-a)^4 = +a^4,$$
$$(-a)^5 = -a^5,$$
$$(-a)^6 = +a^6.$$

Even powers of negative quantities are positive.
Odd powers of negative quantities are negative.
All powers of positive quantities are positive.

EXERCISE 99

Expand:

1. $(2a)^4$.
2. $(3a^2)^3$.
3. $(4a^3)^4$.
4. $(\frac{1}{2}a^5)^4$.
5. $(-\frac{1}{2}b^3)^5$.
6. $(-\frac{2}{3}x^2)^4$.
7. $(-\frac{3}{4}x^2y)^3$.
8. $(\frac{2}{5}mn^2)^5$.
9. $(-\frac{4}{7}x^2y^3)^3$.
10. $\left(\frac{mn}{a}\right)^6$.
11. $\left(-\frac{by}{c}\right)^7$.
12. $\left(-\frac{xy^2}{c^2}\right)^8$.
13. $\left(\frac{2x^2}{3}\right)^6$.
14. $\left(-\frac{3x^2}{5}\right)^6$.
15. $\left(\frac{x^2y^2z^5}{a^3}\right)^7$.
16. $(-\frac{9}{10}a^3b^5)^5$.

EVOLUTION

116. In involution a quantity is given and a power of the quantity is sought. In evolution a power of a quantity is given and the quantity is sought.

Evolution is the process of finding a root of a quantity. The square root of a quantity is one of its two equal factors. The square root of 16 is 4, since

$$16 = 4 \times 4.$$

Similarly, the cube root of a quantity is one of its three equal factors. Thus, the cube root of 125 is 5, since

$$125 = 5 \times 5 \times 5.$$

What is the square root of 225?

This problem may be solved by resolving 225 into its prime factors.

$$225 = 3 \cdot 3 \cdot 5 \cdot 5 = (3 \cdot 5)(3 \cdot 5).$$

Hence, $\sqrt{225} = 3 \times 5 = 15.$

Since $(+15)(+15) = +225,$

and $(-15)(-15) = +225.$

Therefore, the square root of 225 is ± 15.

The double sign $\pm$ is read plus or minus.

117. Since an even power of a positive or negative quantity is positive, hence, inversely,

An even root of a positive quantity is $\pm$.

Since an odd power of a positive quantity is positive, hence,

An odd root of a positive quantity is positive.

Also, since an odd power of a negative quantity is negative, therefore,

An odd root of a negative quantity is negative.

118. An even root of a negative quantity gives rise to a new kind of number known as an **imaginary**.

For example, $\sqrt{-4}$.

This is neither $+2$ nor -2.

119. Since $(3x^2y^4)^5 = 243x^{10}y^{20},$

therefore, $\sqrt[5]{243x^{10}y^{20}} = 3x^2y^4.$

Any root of a monomial expression is obtained by taking the required root of each of its factors and then multiplying these roots.

EXERCISE 100

Simplify:

1. $\sqrt{16x^2}$.
2. $\sqrt{25y^4}$.
3. $\sqrt[3]{64a^6}$.
4. $\sqrt{\frac{1}{4}x^4}$.
5. $\sqrt{\frac{1}{9}a^8}$.
6. $\sqrt{\frac{4}{25}m^4n^2}$.
7. $\sqrt[4]{16x^8}$.
8. $\sqrt[5]{32x^{10}y^{15}}$.
9. $\sqrt[3]{-a^9b^{12}}$.
10. $\sqrt[5]{-b^{10}y^{20}}$.
11. $\sqrt[3]{\frac{1}{64}x^{12}y^{18}}$.
12. $\sqrt[4]{a^{12}b^{20}c^{28}}$.
13. $\sqrt[5]{-243m^{25}}$.
14. $\sqrt[10]{x^{10}y^{40}}$.
15. $\sqrt[3]{-216x^9y^{15}}$.
16. $\sqrt[4]{81m^{16}n^{24}}$.

120. The identity $(a+b)^2 = a^2 + (2a+b)b$ enables one to extract the square root of arithmetic numbers and polynomial expressions.

Example 1. Extract the square root of $9x^2 + 6xy + y^2$.

$$
\begin{array}{l}
\quad\; 9x^2 + 6xy + y^2 \,(3x + y \\
\quad\; \underline{9x^2} \\
6x + y)\,6xy + y^2 \\
\qquad\quad\;\; \underline{6xy + y^2}
\end{array}
$$

The first term of the root will be $3x$. Squaring $3x$ and subtracting its square there is a remainder $6xy + y^2$. This remainder is twice the product of $3x$ and the second term plus the square of the second term. Hence, in order to get the next term of the root, take as trial divisor twice the part of the root found. Twice $3x$ is $6x$. $6x$ is contained in $6xy$, y times. Write y in the trial divisor and in the root. Multiply $6x + y$ by y.

Example 2. Extract the square root of

$$x^4 - 4x^3 + 6x^2 - 4x + 1.$$

$$\begin{array}{rl}
 & x^4 - 4x^3 + 6x^2 - 4x + 1\,(x^2 - 2x + 1 \\
 & x^4 \\
2x^2 - 2x) & -4x^3 + 6x^2 \\
 & -4x^3 + 4x^2 \\
2x^2 - 4x + 1) & 2x^2 - 4x + 1 \\
 & 2x^2 - 4x + 1
\end{array}$$

The first two terms of the root are obtained as in Example 1. To get the next term double the part of the root found for trial divisor and proceed as before.

EXERCISE 101

Extract the square root of each of the following expressions:

1. $x^2 + 4x + 4.$
2. $x^6 + 6x^3 + 9.$
3. $4x^2 + 20xy + 25y^2.$
4. $x^2 + x + \frac{1}{4}.$
5. $x^2 - \frac{2}{3}x + \frac{1}{9}.$
6. $x^2 - \frac{1}{2}x + \frac{1}{16}.$
7. $x^2 - \frac{2}{5}x + \frac{1}{25}.$
8. $x^2 + \frac{6x}{5} + \frac{9}{25}.$
9. $x^2 + \frac{4x}{3} + \frac{4}{9}.$
10. $x^2 + \frac{4x}{7} + \frac{4}{49}.$
11. $x^4 - 8x^3 + 24x^2 - 32x + 16.$
12. $x^4 - 12x^3 + 54x^2 - 108x + 81.$
13. $x^4 + 2x^3 - x^2 - 2x + 1.$
14. $x^4 + 4x^3 + 2x^2 - 4x + 1.$
15. $x^4 - 6x^3 + 5x^2 + 12x + 4.$
16. $x^4 - 2x^3 - 7x^2 + 8x + 16.$

17. $a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc.$

18. $a^2 + 4\,b^2 + 9\,c^2 - 4\,ab + 6\,ac - 12\,bc.$

121. Beginners should memorize the following:

$1^2 = 1$	$10^2 = 100$	$(.1)^2 = .01$
$2^2 = 4$	$20^2 = 400$	$(.2)^2 = .04$
$3^2 = 9$	$30^2 = 900$	$(.3)^2 = .09$
$4^2 = 16$	$40^2 = 1600$	$(.4)^2 = .16$
$5^2 = 25$	$50^2 = 2500$	$(.5)^2 = .25$
$6^2 = 36$	$60^2 = 3600$	$(.6)^2 = .36$
$7^2 = 49$	$70^2 = 4900$	$(.7)^2 = .49$
$8^2 = 64$	$80^2 = 6400$	$(.8)^2 = .64$
$9^2 = 81$	$90^2 = 8100$	$(.9)^2 = .81$

Example **1.** Extract the square root of 5776.

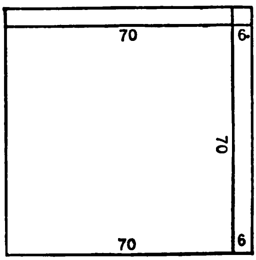
```
         5776(70 + 6
         4900
6(140 + 6))876
          876
```

The square root of 5776 is more than 70 and less than 80. Hence, the tens' digit of the required root is 7. Square 70 and subtract the result from 5776. Double 70 for trial divisor. 140 is contained in 876, 6 times. Write 6 in the root and in the trial divisor. Multiply 140 + 6, *i.e.* 146 by 6. The answer is 76.

In practice the work is contracted as follows:

```
    5776(76
    49
146)876
    876
```

The accompanying cut is a geometric illustration of the square root of 5776.



Example 2. Extract the square root of 136161.

This number equals 1361 hundred +61.

```
   136161(369
     9
 66)461
    396
729)6561
    6561
```

Taking the square root of 1361 hundred, as in Example 1 above, we get 36 tens with a remainder of 65 hundred. Take down 61, and for trial divisor double 36 tens. Since 72 tens is contained in 6561, 9 times, the next figure is 9. Write it in the root and in the divisor, and proceed as before.

Example 3. Extract the square root of 2420.64.

$$\sqrt{2420.64} = \sqrt{\frac{242064}{100}} = \frac{\sqrt{242064}}{\sqrt{100}} = \frac{492}{10} = 49.2.$$

In actual practice we begin at the decimal point and point off the number in periods of two figures each. The work in its contracted form will then stand as follows:

```
      4 9 .2
     2420.64
     16
 89)820
    801
982)1964
    1964
```

122. To extract the square root of a fraction whose denominator is not the square of an integer, reduce the fraction to an equivalent decimal, and then take the root of this decimal to the required degree of approximation.

EXERCISE 102

Extract the square root of:

1. 5476.	4. 9604.	7. 927369.	10. 378.25.
2. 6889.	5. 60516.	8. 3806.89.	11. 1245.3841.
3. 7921.	6. 46656.	9. 734.41.	12. .81378441.

Extract to four places of decimals the square roots of:

13. .1.	15. .02687.	17. $1\frac{1}{2}$.	19. $4\frac{2}{7}$.	21. $1\frac{2}{3}$.
14. .69.	16. 1.0625.	18. $2\frac{1}{3}$.	20. $6\frac{2}{9}$.	22. $82\frac{1}{11}$.

Example 1. The dimensions of a rectangle are in the ratio 25 : 8, and the area of the rectangle is 5 acres. Find the dimensions in yards.

Let $x =$ the number of yards in the width.

$$\frac{25x}{8} = \text{the number of yards in the length.}$$

$$\frac{25x^2}{8} = \text{the number of square yards in the area.}$$

$$5 \times 4840 = \text{the number of square yards in the area.}$$

Hence, $\frac{25x^2}{8} = 24,200.$

$$25x^2 = 24,200 \times 8.$$

$$x^2 = \frac{24,200 \times 8}{25} = \frac{24,200 \times 8 \times 4}{100} = 7844.$$

$$x = \sqrt{7844} = 88.$$

$$\frac{25x}{8} = \frac{25 \times 88}{8} = 275$$

The dimensions are 275 yards, 88 yards.

Check. $\frac{275 \times 88}{4840} = 5.$

EXERCISE 103

1. Find in yards the side of a square whose area is 640 acres.

2. A square garden contains $\frac{1}{10}$ of an acre. Find in yards the length of its side.

3. A dealer sold a suit of clothes at as many per cent profit as the suit cost dollars. If the profit was $4, find the prime cost of the suit of clothes.

4. The width of a rectangle is $\frac{2}{3}$ of its length, and its area is 15 acres. Find its dimensions in yards.

5. Two numbers are in the ratio of 4 : 5, and their product is 1620. Find them.

6. Two numbers are in the ratio of 8 : 15, and the sum of their squares is 7225. Find them.

7. Two numbers are in the ratio of 5 : 13, and the difference of their squares is 5184. Find them.

8. The area of a circle $= \pi r^2$, π being 3.1416, and r the radius of the circle. Calculate the radius of the circle whose area is (*a*) 1809.6, (*b*) 6647.6, (*c*) 24,885.

9. The surface of a sphere $= 4\pi r^2$. Calculate the radius of the sphere whose surface is (*a*) 5026.56 square inches, (*b*) 45,239 square inches, (*c*) 101,788 square inches, (*d*) 123,163 square inches.

10. In the equation $v^2 = 2gs$, v stands for the velocity

in feet per second of a cannon ball, s, the height in feet which the ball will ascend if discharged vertically upward, and $g = 32$ feet. If the ball ascends 5 miles, calculate in feet per second the velocity of discharge.

11. If a body is dropped from a height and falls vertically downward, the number of feet it falls in t seconds is given by the formula $s = \frac{1}{2} gt^2$ ($g = 32$ feet). How far will a body fall in 3 seconds? How far will it fall in 8 seconds?

12. If a stone is dropped from the top of a tower 400 feet high, after how many seconds will it strike the earth?

CHAPTER XI

QUADRATIC EQUATIONS

123. A quadratic equation in one variable is an equation of the form $ax^2 + bx + c = 0$, a, b, and c being known numbers or constants, a not zero.

A quadratic equation is also called an equation of the second degree.

The following are examples of quadratic equations:

$$x^2 = 20.$$

$$x^2 - 3x = 0.$$

$$\frac{1}{2}x^2 + \frac{1}{3}x - 4 = 0.$$

124. A quadratic equation of the form $ax^2 + c = 0$ is called an **incomplete quadratic.** In English text-books this form of quadratic is known as a **pure quadratic.**

A quadratic equation of the form $ax^2 + bx + c = 0$, and b being different from zero, is called a **complete quadratic.** In English text-books a complete quadratic is generally known as an **adfected quadratic.** Some American texts use **affected** where English texts use **adfected.**

Example 1. Solve

$$5(x^2 - 3x + 1) - 3(2x^2 - 5x + 3) + 20 = 0.$$

SOLUTION. Removing parentheses,

$$5x^2 - 15x + 5 - 6x^2 + 15x - 9 + 20 = 0.$$

Combining, $$-x^2 + 16 = 0.$$

Transposing, $-x^2 = -16.$

Dividing by -1, $x^2 = 16.$

$x = \pm 4.$

Check.

$$5(4^2 - 3 \times 4 + 1) - 3(2 \times 4^2 - 5 \times 4 + 3) + 20 = 0.$$

EXERCISE 104

Solve:

1. $x^2 - 9 = 0.$
2. $x^2 - 25 = 0.$
3. $x^2 = 121.$
4. $x^2 = 169.$
5. $4x^2 - 25 = 0.$
6. $9x^2 - 1 = 0.$
7. $16x^2 - 49 = 0.$
8. $\frac{x^2}{2} - \frac{x^2}{3} = 6.$
9. $\frac{x^2}{3} - \frac{x^2}{4} = 3.$
10. $\frac{x^2}{4} + \frac{x^2}{5} = \frac{9}{20}.$
11. $\frac{x^2}{7} - \frac{x^2}{9} = 14.$
12. $(x+3)(x+4) = 7x + 48.$
13. $(x+2)(x-1) = x + 23.$
14. $(3x+4)(4x+3) = 25x + 24.$
15. $\frac{7}{x^2} - \frac{5}{4x^2} = 1\frac{7}{16}.$
16. $(3x-2)(x+3) = 7(x+3).$
17. $(2x-3)^2 - (3x-2)^2 = 0.$
18. $(5x-1)^2 - (x-1)^2 = 8(12-x).$
19. $\frac{x-1}{x+1} + \frac{x+1}{x-1} = 2\frac{1}{2}.$
20. $\frac{x+2}{x-2} + \frac{x-2}{x+2} = 3\frac{1}{3}.$
21. $\frac{1}{x} - \frac{1}{x+5} = \frac{x^2+1}{x(x+5)}.$
22. $\frac{3}{x} - \frac{3}{x+7} = \frac{x^2+5}{x(x+7)}.$

125. Solution of quadratics by factoring.

Example 1. Find the roots of the equation

$$x^2 + 40 = 13x.$$

SOLUTION. Transposing $13x$, $x^2 - 13x + 40 = 0.$

Factoring, $(x-8)(x-5)=0.$

If either of these factors is equal to zero, the equation is satisfied.

If $x-5=0$, then $x=5$.

If $x-8=0$, then $x=8$.

The roots of $x^2-13x+40$ are 5 and 8.

Check. $5^2+40=13\times 5.$

$8^2+40=13\times 8.$

To solve a quadratic equation by factoring

1. *Bring all the terms to the first member of the equation.*
2. *Factor.*
3. *Make each factor equal to zero.*
4. *Solve the resulting simple equations.*

126. Since a quadratic expression can be resolved into two factors of the first degree, a quadratic equation has two and only two roots.

EXERCISE 105

Solve:

1. $x^2-7x+6=0.$
2. $x^2-7x+12=0.$
3. $x^2-10x+9=0.$
4. $x^2+30=11x.$
5. $x^2+55=16x.$
6. $x^2-81=0.$
7. $16x^2-25=0.$
8. $x^2+3x=10.$
9. $x^2+5x=36.$
10. $x^2+6x=91.$
11. $x^2+12x=13.$
12. $x^2+9x=22.$
13. $9x^2-9x+2=0.$
14. $4x^2-17x+4=0.$
15. $10x^2-19x+6=0.$
16. $6x^2-7x=3.$

17. $6x^2 - 7x = 5.$

18. $16x^2 - 8x + 1 = 0.$

19. $4x^2 - 12x + 9 = 0.$

20. $25x^2 - 40x + 16 = 0.$

21. $6x^2 - 17x + 12 = 0.$

22. $4x^2 - 5x - 6 = 0.$

23. $10x^2 + 23x - 5 = 0.$

24. $x^4 - 5x^2 + 4 = 0.$

127. Solution by completing the square.

Since $$(x + a)^2 = x^2 + 2ax + a^2,$$

and, $$(x - a)^2 = x^2 - 2ax + a^2,$$

it follows that the expressions $x^2 + 2ax$ and $x^2 - 2ax$ may each be made a perfect square by the addition of a^2, *i.e.* the square of the half coefficient of x.

Hence, in order to make an expression of the form $(x^2 + bx)$ a perfect square, **add the square of half the coefficient of x.**

Example 1. Solve

$$x^2 - 3x - 40 = 0.$$

Transposing, $$x^2 - 3x = 40.$$

Adding to each member the square of half the coefficient of x, we have

$$x^2 - 3x + \tfrac{9}{4} = 40 + \tfrac{9}{4} = \tfrac{169}{4}.$$

Extracting the square root,

$$x - \tfrac{3}{2} = \pm \tfrac{13}{2}.$$

Transposing, $$x = \tfrac{3}{2} \pm \tfrac{13}{2} = 8 \text{ or } -5.$$

Check. $$8^2 - 3(8) - 40 = 0.$$

$$(-5)^2 - 3(-5) - 40 = 0.$$

Example 2. Solve $6x^2 + x - 2 = 0$.

Transposing, $$6x^2 + x = 2.$$

Making the coefficient of x^2 unity by dividing each member by 6, we have $x^2 + \frac{1}{6}x = \frac{1}{3}$, adding ($\frac{1}{2}$ of $\frac{1}{6})^2$, the square of half the coefficient of x, we have
$$x^2 + \tfrac{1}{6}x + \tfrac{1}{144} = \tfrac{1}{3} + \tfrac{1}{144} = \tfrac{49}{144}.$$

Extracting the square root,
$$x + \tfrac{1}{12} = \pm \tfrac{7}{12}.$$
$$x = -\tfrac{1}{12} \pm \tfrac{7}{12} = \tfrac{1}{2} \text{ or } -\tfrac{2}{3}.$$

The above examples illustrate the method of solving quadratics by completing the square. We should add one more example of a perfectly general character and from it derive a rule for the solution of quadratics.

Example 3. Solve
$$ax^2 + bx + c = 0.$$

Transposing, $ax^2 + bx = -c$.

Dividing by a, $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Adding $\left(\frac{1}{2} \text{ of } \frac{b}{a}\right)^2$,
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root,
$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

128. The expression $b^2 - 4ac$ is called the discriminant of the quadratic. Hence, we have the following rule for solving quadratic equations:

(1) **Reduce the equation to the form $ax^2 + bx + c = 0$.**

(2) **Then x is equal to its own coefficient with an opposite sign $\pm$ the square root of the discriminant of the quadratic.**

(3) **Divide the result of Step 2 by twice the coefficient of x^2.**

Example 1. Solve

$$x^2 - 5x + 6 = 0.$$

$$x = \frac{5 \pm \sqrt{5^2 - 4 \times 1 \times 6}}{2}$$

$$= \frac{5 \pm 1}{2} = 3 \text{ or } 2.$$

Example 2. Solve

$$12x^2 - 23x - 24 = 0.$$

$$x = \frac{23 \pm \sqrt{23^2 - 4(-24)(12)}}{24}$$

$$= \frac{23 \pm 41}{24} = \frac{8}{3} \text{ or } -\frac{3}{4}.$$

EXERCISE 106

Solve:

1. $x^2 - 6x = -8.$
2. $x^2 - 8x + 12 = 0.$
3. $x^2 - 8x + 15 = 0.$
4. $x^2 - 9x + 14 = 0.$
5. $x^2 - 10x - 11 = 0.$
6. $x^2 - 15x + 26 = 0.$
7. $x^2 - 4x - 45 = 0.$
8. $x^2 + 6x - 27 = 0.$
9. $x^2 + 12x - 45 = 0.$
10. $x^2 - 10x + 25 = 0.$
11. $x^2 - x = 12.$
12. $x^2 - x = 20.$

13. $x^2 + x = 30.$

14. $x^2 + 3x = 10.$

15. $x^2 + 5x = 14.$

16. $x^2 + 9x = 22.$

17. $x^2 + 3x = 40.$

18. $x^2 + 7x = 18.$

19. $3x^2 - 10x = -3$

20. $8x^2 - 18x = 5.$

21. $6x^2 - 31x = -35.$

22. $2x^2 - 5x = 3.$

23. $3x^2 - 7x = 20.$

24. $x^2 + x = 1.$

25. $5x^2 - 4x = 11.$

26. $6x^2 - 7x = 17.$

27. $9x^2 - 8x = 19.$

28. $4x^2 - 5x = 14.$

29. $\frac{2x-1}{3x+2} = \frac{3x+3}{7x-1}.$

30. $\frac{2x-5}{3x-4} = \frac{x+2}{3x+4}.$

31. $\frac{4}{x} - \frac{3}{x-1} = -1.$

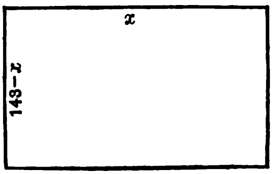
32. $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}.$

The perimeter of a rectangle is 286 yards and its area is 1 acre. Find its dimensions.

Since twice the length of the rectangle + twice its breadth = 286 yards, hence the length + the breadth of the rectangle = 143 yards.

Let $\qquad x =$ the number of yards in the length.

$143 - x =$ the number of yards in the breadth.



Then, $x(143 - x) =$ the number of square yards in the area.

$4840 =$ the number of square yards in the area.

Therefore,

$$x(143 - x) = 4840.$$

$$143x - x^2 = 4840.$$

Transposing and multiplying by -1,

$$x^2 - 143x + 4840 = 0.$$

$$x = \frac{143 \pm \sqrt{143^2 - 4 \times 4840}}{2}$$

$$= \frac{143 \pm 33}{2} = 88 \text{ or } 55.$$

$$143 - x = 143 - (88 \text{ or } 55) = 55 \text{ or } 88.$$

The dimensions are 88 yards, 55 yards.

Check. $88 \times 55 = 4840.$

EXERCISE 107

1. The sum of two numbers is 30 and their product is 209. Find them.

2. The difference of two numbers is 5, and the sum of their squares is 697. Find them.

3. The difference of the cubes of two consecutive numbers is 469. Find them.

4. The length of a rectangle exceeds its breadth by 7 yards, and one diagonal is 17 yards. Find the dimensions. (The square on the diagonal equals the sum of the squares on the length and breadth.)

5. Divide a line 10 inches long into two segments so that the rectangle contained by the whole line and one segment equals the square upon the other segment.

6. A square and a rectangle have the same area. The length of the rectangle exceeds its width by 33 yards. The side of the square is 28 yards. Find the dimensions of the rectangle.

7. The length of a rectangular garden is 50 yards and its width is 40 yards. Around it is a path containing 475 square yards. Find the width of the path.

8. The height of a mirror is two feet greater than its width. The glass cost \$1 per square foot, and the frame cost 50¢ per linear foot inside measure. The cost of the mirror was \$34. Find the dimensions of the mirror.

9. A man buys a farm for \$6400. The number of dollars an acre cost is 25% of the number of acres bought. Find the cost of one acre.

10. "Some bees were sitting on a tree. At one time the square root of half their number flew away, at another time $\frac{8}{9}$ of the whole flew away. There were then 2 bees left. How many bees were there?" (Taken from the Bija Ganita, the second chapter of a Hindu work on astronomy.)

HINT.

Let $2x^2 =$ the number of bees.

Example 1. A farmer bought a number of cattle for \$480. If he had bought 4 less for the same money they would have cost \$6 apiece more. How many did he buy?

Let $x =$ the number of cattle bought.

$\frac{480}{x} =$ the number of dollars each cost.

$x - 4 =$ the number of cattle less 4.

$\frac{480}{x-4} =$ the number of dollars each would have cost had he bought 4 less.

Therefore, $\dfrac{480}{x-4} = \dfrac{480}{x} + 6.$

Dividing by 6, $\dfrac{80}{x-4} = \dfrac{80}{x} + 1.$

Multiplying by $x(x-4)$,

$$80\,x = 80\,x - 320 + x^2 - 4\,x.$$

Transposing,

$$x^2 - 4\,x - 320 = 0.$$

$$x = \frac{4 \pm \sqrt{16 + 1280}}{2}$$

$$= \frac{4 \pm 36}{2} = 20 \text{ or } -16.$$

Here minus 16 is irrelevant. If the problem read "A number of cattle were sold for \$480. If 4 more were sold for the same money, the selling price would have been \$6 apiece less. How many were sold?" the answer would be 16.

EXERCISE 108

1. Two numbers differ by 2 and the sum of their reciprocals is $2\frac{2}{5}$. Find them.

2. A number of men were paid \$112 for doing a piece of work. If two more men had been employed, each man would have received \$1 less. How many men were employed?

3. The circumferences of two wheels differ by 1 foot, and the smaller wheel makes 40 revolutions more than the larger in going a distance of 1 mile. Find the circumference of each.

4. A man sold a mule for \$16 at a loss of as many per cent as the mule cost him dollars. Find the cost.

5. A crew can row 20 miles down a stream and back in $6\frac{1}{2}$ hours. The rate down the stream is 3 miles per hour more than the rate up. Find the rate down, the rate up, and the rate in still water.

6. A and B working together do a piece of work in $7\frac{1}{2}$ days. B alone takes 8 days longer than A alone to do the work. Find A's and B's time.

7. A man bought a number of cattle for \$2000. He sold all but 5 at an advance of \$5 per head for \$2025. How many cattle did he buy?

8. The sum of a number and its reciprocal is $2\frac{1}{6}$. Find the number.

9. A number consists of two digits of which the first exceeds the second by unity, and the number itself exceeds the sum of the squares of its digits by 4. Find the number.

10. A sum of money amounts in 1 year to \$262.50. The rate per cent of interest is $\frac{1}{50}$ of the principal. Find the principal.

CHAPTER XII

RADICALS. THEORY OF EXPONENTS. RADICAL EQUATIONS. GRAPHS OF FUNCTIONS

129. The second of the four fundamental rules, namely subtraction, gave rise to the introduction of zero and the negative number.

To illustrate, take the problem: What number added to b gives a for the sum? The answer to this question is the root of the equation

$$x + b = a,$$

i.e. $$x = a - b.$$

Now if a equals b, $a - b$ is zero.

If b equals $a + c$, c being a positive number,

then $$a - b = a - (a + c) = - c.$$

Division gave rise to the introduction of the fraction. To illustrate, take the problem: What is the fifth part of 22? The answer is the root of the equation

$$5x = 22.$$

130. The four fundamental rules, when applied to integers, give in every instance results known as rational numbers. The general type of a rational number is $\pm \frac{m}{n}$, m and n being positive integers.

131. Evolution gave rise to a new kind of number known as the **irrational** or **surd number.**

What is the square root of 10? The answer is, the number which being multiplied by itself gives 10 for product.

Algebraically the square root of 10 is defined as a root of the equation

$$x^2 = 10.$$

All irrational numbers are not expressible by a finite number of figures. Their values, however, can be expressed to any desired degree of approximation.

Thus, $\sqrt{10} = 3.16228$ correct to five decimal figures. In fact, 3.16228 differs from $\sqrt{10}$ by less than the hundred-thousandth part of one-fourth of a unit.

Indicated roots of positive rational quantities which cannot be expressed by a finite number of terms are called **irrational** or **surd quantities.**

Inasmuch as irrational numbers occur in investigations, it is well to know the rules by which they are combined.

Since $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6,$

and $\sqrt{4 \times 9} = 2 \times 3 = 6,$

therefore, $\sqrt{4} \times \sqrt{9} = \sqrt{4 \times 9}.$

And since $\sqrt[3]{27} \times \sqrt[3]{8} = 3 \times 2 = 6,$

and $\sqrt[3]{27 \times 8} = 3 \times 2 = 6,$

therefore, $\sqrt[3]{27} \times \sqrt[3]{8} = \sqrt[3]{27 \times 8}.$

Illustrations like the above lead one to suspect that

$$\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6},$$

$$\sqrt[3]{2} \times \sqrt[3]{3} = \sqrt[3]{2 \times 3} = \sqrt[3]{6}.$$

And in general that

I. $$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab},$$

a and *b* being real positive numbers, and *n* a positive integer.*

* Real numbers include both rational and irrational.

132. The product of the nth roots of two arithmetic numbers is the nth root of the product of the two numbers.

Proof. Let $x = \sqrt[n]{a} \times \sqrt[n]{b}.$

Then $x^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = ab.$

Extracting the nth root, $x = \sqrt[n]{ab}.$

Therefore, (Ax. 1), $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}.$

Since $$\sqrt{\left(\frac{25}{49}\right)} = \frac{\sqrt{25}}{\sqrt{49}} = \frac{5}{7},$$

and since $$\sqrt[3]{\left(\frac{27}{125}\right)} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5},$$

it appears that

II. $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

133. The nth root of a quotient is the quotient of the nth roots of the dividend and divisor.

Proof. Let $$x = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Hence, $$x^n = \frac{a}{b}.$$

Extracting the nth root of each member,

$$x = \sqrt[n]{\frac{a}{b}}.$$

Therefore, $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Since $(2^3)^2 = 2^6 = 64,$

and since $(2^2)^3 = 2^6 = 64,$

therefore, $\sqrt[6]{64}=\sqrt[3]{(\sqrt{64})}$,

$$\sqrt[6]{64}=\sqrt{(\sqrt[3]{64})}.$$

Expressed in ordinary language, the sixth root of a number may be obtained by extracting the square root of the number and then the cube root of this result, or by extracting the cube root of the number and then the square root of this result. In general,

III. $$\sqrt[mn]{a}=\sqrt[m]{(\sqrt[n]{a})}=\sqrt[n]{(\sqrt[m]{a})}.$$

134. Any root of a number may be obtained by taking in succession the roots indicated by the integral factors of the original root index.

135. A surd is in its simplest form when the number under the radical sign is integral and as small as possible, and when a radical does not occur in the denominator of a fraction. Thus, $\sqrt{32}$ is not in its simplest form, for

$$\sqrt{32}=\sqrt{16\times 2}=\sqrt{16}\times\sqrt{2}=4\sqrt{2}.$$

Example 1. Multiply $\frac{1}{2}\sqrt{3}$ by $\frac{1}{3}\sqrt{2}$.

$$\tfrac{1}{2}\sqrt{3}\cdot\tfrac{1}{3}\sqrt{2}=\tfrac{1}{2}\cdot\tfrac{1}{3}\cdot\sqrt{3}\cdot\sqrt{2}=\tfrac{1}{6}\sqrt{6}.$$

Example 2. Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}+\sqrt{2}$.

$2\sqrt{3}+3\sqrt{2}$ Multiply the multiplicand by $3\sqrt{3}$.
$3\sqrt{3}+\sqrt{2}$ Multiply the multiplicand by $\sqrt{2}$. Add the two results.

$$\begin{array}{r} 2\sqrt{3}+3\sqrt{2} \\ 3\sqrt{3}+\sqrt{2} \\ \hline 18+9\sqrt{6} \\ 2\sqrt{6}+6 \\ \hline 24+11\sqrt{6} \end{array}$$

EXERCISE 109

Simplify:

1. $\sqrt{2} \times \sqrt{3}$.
2. $\sqrt{3} \times \sqrt{3}$.
3. $\sqrt{3} \times \sqrt{5}$.
4. $2\sqrt{3} \times 3\sqrt{5}$.
5. $4\sqrt{5} \times 3\sqrt{7}$.
6. $3\sqrt{2} \times 2\sqrt{3}$.
7. $\sqrt{7} \times \sqrt{14}$.
8. $\sqrt{6} \cdot \sqrt{2}$.
9. $\sqrt{10} \times \sqrt{5}$.
10. $\sqrt{10} \times \sqrt{15}$.
11. $\sqrt{10} \times \sqrt{18}$.
12. $\sqrt{8} \times \sqrt{50}$.
13. $\sqrt{12} \times \sqrt{18}$.
14. $\sqrt{32} \cdot \sqrt{8}$.
15. $(1 + \sqrt{2})^2$.
16. $(\sqrt{2} + \sqrt{3})^2$.
17. $(\sqrt{5} + \sqrt{3})^2$.
18. $(\sqrt{4} - \sqrt{6})^2$.
19. $(\sqrt{2} + 1)(\sqrt{3} + 1)$.
20. $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$.
21. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$.
22. $(2\sqrt{3} - 3\sqrt{2})(2\sqrt{3} + 3\sqrt{2})$.
23. $\sqrt[4]{49}$.
24. $\sqrt[4]{100}$.
25. $\sqrt[6]{8}$.
26. $\sqrt[6]{27}$.
27. $\sqrt[6]{125}$.
28. $\sqrt[8]{16}$.
29. $\sqrt[8]{81}$.

THEORY OF EXPONENTS

136. We have seen in Chapter II that the index of the product of two powers of the same quantity equals the sum of the indices of the multiplier and multiplicand. Thus, $a^2 \times a^5 = a^7$, and generally

$$a^m \times a^n = a^{m+n};\ m \text{ and } n \text{ being positive integers.}$$

Now assume that this law always holds.

Then, $$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a.$$

But, $\sqrt{a} \times \sqrt{a} = a.$ Definition of square root.

Hence, $a^{\frac{1}{2}}$ is $\sqrt{a}.$

Also, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a.$

And, $\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = a.$ Definition of cube root.

Hence, $a^{\frac{1}{3}}$ is $\sqrt[3]{a}.$

Also, $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^2.$

Hence, $a^{\frac{2}{3}}$ is $\sqrt[3]{a^2}.$

Similarly, $a^{\frac{3}{4}}$ is $\sqrt[4]{a^3}.$

$a^{\frac{2}{5}}$ is $\sqrt[5]{a^2}.$

$a^{\frac{m}{n}}$ is $\sqrt[n]{a^m}.$

137. Since, $a^{\frac{3}{4}} = \sqrt[4]{a^3},$

$$a^{\frac{3}{4}} = a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} = (a^{\frac{1}{4}})^3 = (\sqrt[4]{a})^3.$$

It appears that $\sqrt[4]{a^3} = (\sqrt[4]{a})^3.$

Similarly, $\sqrt[5]{a^2} = (\sqrt[5]{a})^2,$

and finally $\sqrt[n]{a^m} = (\sqrt[n]{a})^m.$

The processes of involution and evolution may be applied to the same quantity in any order, and the results are arithmetically the same.

138. The exponent of the quotient of a power of a quantity by a power of the same quantity is equal to the exponent of the dividend minus the exponent of the divisor.

Hence, $\dfrac{a^m}{a^m} = a^{m-m} = a^0.$

But $\dfrac{a^m}{a^m} = 1.$

Hence, $a^0 = 1.$

Since, $a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1.$

Hence, a^{-n} and a^n are reciprocals, or dividing by a^n,

$$a^{-n} = \frac{1}{a^n}.$$

Any quantity affected with zero exponent equals unity.

Any quantity affected with a negative exponent equals the reciprocal of the same quantity affected with a positive exponent of the same absolute value.

The above are not proofs. They merely serve as a justification for the following definitions:

Suppose a is a real positive quantity and m and n positive integers, then,

I. $a^{\frac{m}{n}}$ is defined as $\sqrt[n]{a^m}$.

II. a^0 is defined as 1.

III. a^{-n} is defined as $\dfrac{1}{a^n}$.

The definitions give an additional meaning to the term *power* as hitherto used.

Example 1. Find the value of $8^{-\frac{2}{3}}$.

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{4}.$$

Example 2. Find the value of $(\frac{16}{81})^{-\frac{3}{4}}$.

$$(\tfrac{16}{81})^{-\frac{3}{4}} = (\tfrac{81}{16})^{\frac{3}{4}} = (\sqrt[4]{\tfrac{81}{16}})^3 = (\tfrac{3}{2})^3 = \tfrac{27}{8}.$$

EXERCISE 110

Evaluate:

1. 10^0.	7. $4^{\frac{1}{2}}$.	13. $(\frac{4}{25})^{\frac{1}{2}}$.
2. 10^{-1}.	8. $27^{\frac{1}{3}}$.	14. $(\frac{8}{343})^{\frac{1}{3}}$.
3. 10^{-2}.	9. $125^{\frac{2}{3}}$.	15. $(\frac{8}{125})^{\frac{2}{3}}$.
4. 10^{-3}.	10. $16^{\frac{5}{4}}$.	16. $(\frac{27}{64})^{-\frac{1}{3}}$.
5. 10^{-4}.	11. $8^{\frac{4}{3}}$.	17. $(\frac{4}{49})^{-\frac{3}{2}}$.
6. 10^{-6}.	12. $49^{\frac{3}{2}}$.	18. $10^{\frac{3}{2}}$.

139. All numbers can be expressed as powers of some other number different from zero and ± 1. Thus,

$$10^0 = 1. \qquad 10^{-1} = \frac{1}{10} = .1.$$

$$10^1 = 10. \qquad 10^{-2} = \frac{1}{10^2} = .01.$$

$$10^2 = 100. \qquad 10^{-3} = \frac{1}{10^3} = .001.$$

$$10^3 = 1000. \qquad 10^{-4} = \frac{1}{10^4} = .0001.$$

$$10^4 = 10000.$$

Since $10^0 = 1$, and $10^1 = 10$, it is natural to infer that the power to which 10 must be raised to equal any number greater than 1 and less than 10 is more than 0 and less than 1; *i.e.* it is some proper fraction.

Example 1. Given $10^{.30103} = 2$; $10^{.4771213} = 3$; $10^{.845098} = 7$, find what power of 10 is 28.

SOLUTION. $28 = 2^2 \times 7$.

Hence, $28 = (10^{.30103})^2 \times 10^{.845098} = 10^{.60206} \times 10^{.845098} = 10^{1.447158}$.

Ans. 1.447158.

Example 2. What power of 10 is 5?

SOLUTION. $5 = \frac{10}{2} = \frac{10^1}{10^{.30103}} = 10^{.69897}$. *Ans.* .69897.

EXERCISE III

Given $10^{.30103} = 2$, $10^{.4771213} = 3$, $10^{.845098} = 7$, find what power of 10 is each of the following numbers:

1. 4.	7. 18.	13. 36.	19. 80.	25. 125.
2. 6.	8. 24.	14. 45.	20. 84.	26. 280.
3. 8.	9. 25.	15. 54.	21. 98.	27. 480.
4. 9.	10. 30.	16. 64.	22. 105.	28. 960.
5. 12.	11. 32.	17. 72.	23. 112.	29. 784.
6. 15.	12. 35.	18. 75.	24. 120.	30. 1260.

31. What power of 10 is 2.16?

SOLUTION. $2.16 = \frac{216}{100} = \frac{2^3 \times 3^3}{10^2} = \frac{(10^{.30103})^3 \times (10^{.4771213})^3}{10^2}$

$$= \frac{10^{.90309} \times 10^{1.4313639}}{10^2} = 10^{.334454}.$$ *Ans.* .334454.

32. $1\frac{1}{2}$.	41. 2.5.	50. 24.3.	58. 8.75.
33. $1\frac{3}{4}$.	42. 2.25.	51. 1.6.	59. 1.68.
34. $1\frac{3}{7}$.	43. 1.44.	52. 13.5.	60. 7.35.
35. $2\frac{1}{3}$.	44. 2.94.	53. 40.5.	61. $\sqrt{2}$.
36. $3\frac{1}{2}$.	45. 2.45.	54. 25.2.	62. $\sqrt[3]{5}$.
37. $3\frac{3}{4}$.	46. 1.47.	55. 19.6.	63. $\sqrt{1.68}$.
38. $3\frac{4}{7}$.	47. 4.32.	56. 19.2.	64. $(1.05)^5$.
39. $16\frac{1}{3}$.	48. 31.5.	57. 12.6.	65. $\sqrt[3]{4} \times \sqrt{5}$.
40. $10\frac{5}{7}$.	49. 28.8.		

RADICAL EQUATIONS

140. An equation in which the unknown quantity is affected by the radical sign is called a **radical equation.**

$$E.g.\ \sqrt{x^2 - 5x + 4} = x - 1.$$

Example 1. Solve $\sqrt{x^2 - 7x + 9} - x = -4.$

Making the radical occupy one side of the equation and the other terms the other side, we have

$$\sqrt{x^2 - 7x + 9} = x - 4.$$

Squaring both members, $x^2 - 7x + 9 = x^2 - 8x + 16.$

Transposing, $x = 7.$

Check. $\sqrt{7^2 - 7 \times 7 + 9} - 7 = -4.$

EXERCISE 112

Solve:

1. $\sqrt{x+9} = 4.$
2. $\sqrt{2x+5} = 5.$
3. $\sqrt{3x+9} = 6.$
4. $\sqrt{5x+4} = 7.$
5. $\sqrt{x^2+2x+11} = x+3.$
6. $\sqrt{x^2-5x+11} - x = 2.$
7. $\sqrt[3]{5x+2} = 3.$
8. $\sqrt[3]{9x+8} = 5.$
9. $\sqrt[3]{x^3+3x^2+5x-3} - x = 1.$
10. $\sqrt[3]{x^3+6x^2+44} - x = 2.$
11. $\dfrac{3}{\sqrt{x}} + \sqrt{x} = \sqrt{x+15}.$
12. $\dfrac{\sqrt{x+3}}{\sqrt{2x+4}} = \dfrac{3}{4}.$
13. $\dfrac{\sqrt{3x-5}}{\sqrt{6x+4}} = \dfrac{5}{8}.$
14. $\dfrac{\sqrt{2x+2}}{\sqrt{x+2}} = \dfrac{4}{3}.$
15. $\sqrt[3]{9x} = 3\sqrt[3]{2}.$
16. $\sqrt[3]{10x^2} = 2\sqrt[3]{5}.$
17. $\sqrt{3+\sqrt{5x+1}} = 3.$
18. $\sqrt{12+\sqrt{x+3}} = 4.$

141. *Example* 1. Solve $\sqrt{x+7}+\sqrt{x+2}=5$.

Transposing $\sqrt{x+2}$,

$$\sqrt{x+7}=5-\sqrt{x+2}.$$

Squaring both members,

$$x+7=25+x+2-10\sqrt{x+2}.$$

Making the radical one member of the equation, we have

$$-20=-10\sqrt{x+2}.$$

Dividing by -10, $\quad 2=\sqrt{x+2}$.

Squaring, $\quad 4=x+2$.

$$x=2.$$

Check. $\sqrt{2+7}+\sqrt{2+2}=5$.

When an equation contains two radical expressions in x, let one of the radicals stand on one side of the equation and all other terms on the other side.

EXERCISE 113

Solve:

1. $\sqrt{x+11}+\sqrt{x-1}=6$.
2. $\sqrt{x+8}-\sqrt{x+3}=1$.
3. $\sqrt{x+3}-\sqrt{x-13}=2$.
4. $\sqrt{x+1}+\sqrt{x+10}=9$.
5. $\sqrt{x-4}-\sqrt{x-11}=1$.
6. $\sqrt{x-1}-\sqrt{x-17}=2$.
7. $\sqrt{19-5x}+\sqrt{16-5x}=3$.
8. $\sqrt{9x^2+4\sqrt{1+9x}}=3x-2$.

9. $\sqrt{x^2+\sqrt{1-7x}}+x=1.$

10. $\dfrac{3}{\sqrt{x-2}}+\sqrt{x-2}=\sqrt{x+7}.$

GRAPHS OF FUNCTIONS

142. *Example* 1. Trace the curve $y=x^2-2x-3$.

In this equation

If $x=0$, $y=-3$.	If $x=5$, $y=12$.
If $x=1$, $y=-4$.	If $x=-1$, $y=0$.
If $x=2$, $y=-3$.	If $x=-2$, $y=5$.
If $x=3$, $y=0$.	If $x=-3$, $y=12$.
If $x=4$, $y=5$.	

Plotting these points and drawing a line through the points in succession, we have the curve shown in the figure.

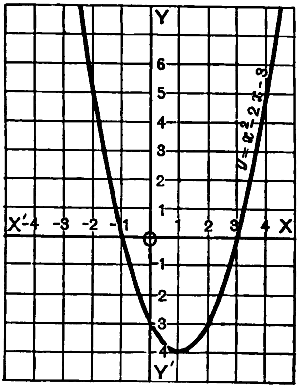


FIG. 1.

It will be noticed that the curve intersects the axis of x in two points $x=3$, $x=-1$.

These values of x are the roots of the equation

$$x^2-2x-3=0.$$

Example 2. Trace the curve

$$xy=5.$$

For $x=\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 5,\ \pm \frac{3}{4},\ \pm 1\frac{1}{2}$,

we find, $y=\pm 5,\ \pm 2.5,\ \pm 1\frac{2}{3},\ \pm 1\frac{1}{4},\ \pm 1,\ \pm 6\frac{2}{3},\ \pm 3\frac{1}{3}$.

Plotting these points, we have two curves, one in the first quadrant and one in the third quadrant. These two are referred to as one curve having two branches.

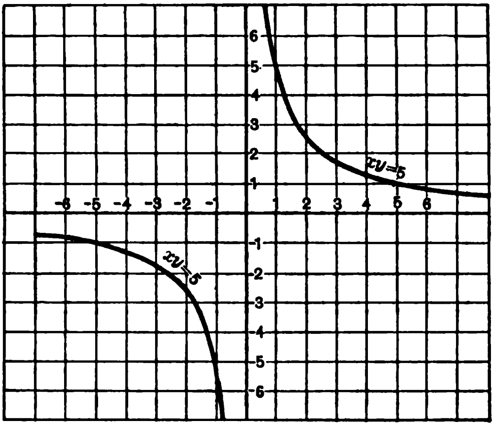


FIG. 2.

Example 3. Trace the curve $x^2 + 2y^2 = 16$.

If $x = 0$, $y = \pm\sqrt{8}$; if $y = 0$, $x = \pm 4$. In the equation $x^2 + 2y^2 = 16$, the value of x does not exceed 4, and the value of y does not exceed $+\sqrt{8}$.

For $x = 0, \pm 1, \pm 2, \pm 3, \pm 3.5, \pm\sqrt{14}, \pm 4,$
we find,

$$y = \pm\sqrt{8}, \pm\sqrt{7.5}, \pm\sqrt{6}, \pm\sqrt{3.5}, \pm\sqrt{1.87}, \pm 1, 0.$$

In plotting these points it will be sufficient to get the values of $\pm\sqrt{8}$, $\pm\sqrt{7.5}$, $\pm\sqrt{6}$, etc., correct to two decimal figures.

Example 4. Trace the curve $x^2 + y^2 = 12$.

For $\quad x = 0, \pm 1, \pm 2, \pm 2.5, \pm 3, \pm\sqrt{12},$

we find, $\quad y = \pm\sqrt{12}, \pm\sqrt{11}, \pm\sqrt{8}, \pm\sqrt{5.75}, \pm\sqrt{3}, 0.$

Plotting the points, we get a circle whose radius is $\pm\sqrt{12}$.

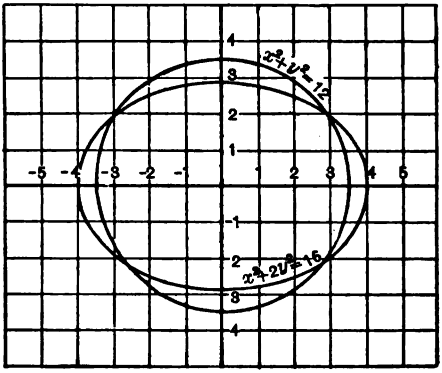


FIG. 3.

143. If we solve the system of equations,

$$x^2 + y^2 = 12, \quad (1)$$

$$x^2 + 2y^2 = 16, \quad (2)$$

we find by subtracting (1) from (2), member by member,

$$y^2 = 4,$$

$$y = \pm 2.$$

Substituting in either equation ± 2 for y, we get

$$x = \pm\sqrt{8}.$$

These values of x and y are the coördinates of the four points of intersection of the two curves. This is as it should be, for the coördinates of every point on a curve satisfy the equation of the curve, and consequently if two curves have points in common, the coördinates of these points must satisfy both equations of the curves.

EXERCISE 114

Trace the curves:

1. $y=x^2$, $y=x^2-1$, $y=x^2+2$.

2. $y=x^2-2x-4$, $y=x^2-2x-2$, $y=x^2-2x$, $y=x^2-2x+4$.

3. $y=\frac{8}{x}$, $\frac{4}{x}$, $\frac{2}{x}$.

4. $y=(x-2)^2$, $y=(x-3)^2$, $y=(x+3)^2$.

5. $x^2+y^2=25$, $x^2+y^2=16$.

6. $x^2-y^2=7$, $x^2-y^2=12$, $x^2-y^2=1$.

7. $y^2=8x$, $y^2=16x$, $y^2=4x$.

8. $2x^2+3y^2=32$, $4x^2+9y^2=36$.

9. $y=x(6-x)$, $y=x(4-x)$, $y=x(1-x)$.

HINT. In the last one take $x=\pm .2$, $\pm .5$, $\pm .7$, ± 1, ± 2, etc.

10. Solve $x^2-x+1=0$. Trace the graph, $y=x^2-x+1$. How is it situated with respect to the x-axis?

11. The number of feet a body falls under the influence of gravity in t seconds is given by the formula $S=16t^2$, where S denotes the number of feet. Trace the graph of $S=16t^2$.

SIMULTANEOUS QUADRATICS

144. *Example* 1. Solve:

$$x^2+2y^2=17, \quad (1)$$

$$x+y=5. \quad (2)$$

From equation (2), we find, $y=5-x$.

Substitute this value of y in equation (1), and we have,

$$x^2+2(5-x)^2=17. \quad (3)$$

Solving equation (3), we find, $x = 3$ or $\frac{11}{3}$.

Substituting this value of x in equation (2), $y = 2$, or $\frac{4}{3}$.

Check. $$3^2 + 2\,(2)^2 = 17.$$
$$(\tfrac{11}{3})^2 + 2\,(\tfrac{4}{3})^2 = 17.$$

Systems of equations in two variables where one equation is quadratic and the other linear may be all solved like equation (1) above.

EXERCISE 115

Solve:

1. $x^2 + y^2 = 13,\quad x + y = 5.$
2. $x^2 + y^2 = 13,\quad x - y = 1.$
3. $x^2 + y^2 = 26,\quad x + y = 6.$
4. $x^2 - y^2 = 21,\quad x + y = 7.$
5. $x^2 - y^2 = 24,\quad x - y = 4.$
6. $xy = 3,\quad x + y = 4.$
7. $xy = 8,\quad x - y = 2.$
8. $xy = 10,\quad x - y = 3.$
9. $x^2 + 2\,y^2 = 24,\quad x - y = 2.$
10. $x^2 + 3\,y^2 = 12,\quad x + y = 4.$
11. $2\,x^2 + y^2 = 59,\quad x + y = 8.$
12. $2\,x^2 - 3\,y^2 = 29,\quad x - y = 3.$
13. $4\,x^2 + 9\,y^2 = 25,\quad x + 2\,y = 4.$
14. $4\,x^2 - 9\,y^2 = 27,\quad 2\,x - 3\,y = 3.$
15. $y = 3\,x - x^2,\quad 2\,x + y = 6.$
16. $x^2 + y^2 = 13,\quad xy = 6.$
17. $2\,x^2 + 3\,y^2 = 21,\quad x^2 + y^2 = 10.$
18. $x^2 - y^2 = 15,\quad xy = 4.$

CHAPTER XIII

SIMULTANEOUS EQUATIONS, MULTIPLICATION, FACTORS, DIVISION, ETC.

EXERCISE 116

Solve and check:

1. $9x - 7y = 31,$
$6x + 5y = 40.$

2. $11x + 4y = 66,$
$3x - 10y = 18.$

3. $2x - 9y = 18,$
$3x - y = 14.5.$

4. $4x - 11y = 36,$
$5x - 3y = 45.$

5. $13x + 4y = 74,$
$14x + 5y = 79.$

6. $\frac{1}{2}x + \frac{1}{3}y = 3,$
$5x - y = 17.$

7. $9x - 5y = 5,$
$7x - 12y = -61.$

8. $3x - 4y = 0,$
$7x - y = 25.$

9. $4x + 2y = 1,$
$9x + 11y = -14.$

10. $13x - 17y = 21,$
$29x - 11y = 43.$

11. $\frac{1}{2}x + \frac{1}{3}y = 30,$
$\frac{1}{3}x - \frac{2}{5}y = 1\frac{1}{3}.$

12. $\frac{2}{11}(3x + y) = 6\frac{8}{11},$
$\frac{4}{15}(4x + 3y) = 14\frac{2}{3}.$

13. $5x - 3y + 2z = 30,$
$4x + 2y - 3z = 10,$
$3x + y - 4z = 0.$

14. $2x - 7y + 4z = 16,$
$3x - 2y + 3z = 32,$
$x + y + 5z = 26.$

15. $3x - 2z = 32,$
$4y - z = 2,$
$2x - 3y = 14.$

16. $4x + 3y = 0,$
$z - y = 1\frac{2}{3},$
$7x - 3z = \frac{1}{2}.$

17. $4y - 5x = 3,$
$4z - 7y = 11\frac{1}{2},$
$3z - 7x = 13.$

EXERCISE 117

Multiply:

1. $6x^2 - x + 5$ by $6x^2 - x - 5$.
2. $4x^2 + 2x - 5$ by $4x^2 - 2x - 5$.
3. $3x^2 - 2x - 3$ by $3x^2 + 2x - 3$.
4. $4x^2 - 9x + 7$ by $4x^2 - 9x - 7$.
5. $9x^2 - 2x + 1$ by $9x^2 - 2x - 1$.
6. $x^2 - 2x + 1$ by $x^2 + 2x + 1$.
7. $a - 4b - 2c$ by $a - 4b + 2c$.
8. $2a - 3b - 3c$ by $2a - 3b + 3c$.
9. $5x - y - 4z$ by $5x - y + 4z$.
10. $x^2 + x - 9$ by $x^2 - x + 1$.
11. $x^2 - 7x + 7$ by $x^2 + x - 7$.
12. $3x^2 + 2x + 2$ by $3x^2 - 2x - 10$.
13. $3x^2 + x - 5$ by $3x^2 - x + 1$.
14. $4x^2 + 2x - 2$ by $4x^2 - 2x - 4$.
15. $3x^2 + x - 7$ by $3x^2 - 3x + 7$.
16. $x^2 + x + 2y$ by $x^2 - x - 4y$.
17. $4a^2 + 6a + 9$ by $2a^2 + 5a - 3$.
18. $x^4 + x^3y + xy^3 + y^4$ by $x^2 - xy + y^2$.
19. Expand $(x^2 - ax + bx - ab)(x^2 + ax - bx - ab)$.
20. Expand $(a + b + c)(b + c - a)(c + a - b)(a + b - c)$.
21. Expand $(a^{2n} - 2a^nb^n + b^{2n})(a^{2n} + 2a^nb^n + b^{2n})$.
22. Expand $(a^{3n} - a^{2n}b^n + a^nb^{2n} - b^{3n})(a^n + b^n)$.

145. The square of a binomial.

$$(a+b)^2 = a^2 + 2\,ab + b^2 = a^2 + b^2 + 2\,ab.$$
$$(a-b)^2 = a^2 - 2\,ab + b^2 = a^2 + b^2 - 2\,ab.$$

The square of a binomial is the sum of the squares of its terms and twice their algebraic product.

146. The square of a polynomial.

$$(a+b+c)^2 = (a+\overline{b+c})^2 = a^2 + 2\,a\cdot\overline{b+c} + \overline{b+c}^2$$
$$= a^2 + 2\,ab + 2\,ac + b^2 + c^2 + 2\,bc$$
$$= a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

The square of a polynomial is the sum of the squares of its terms and twice the algebraic product of each term with every term that follows it.

Example 1. $(4\,x - 3\,y - 5\,z)^2 = 16\,x^2 + 9\,y^2 + 25\,z^2 - 24\,xy - 40\,xz + 30\,yz.$

EXERCISE 118

Expand:

1. $(3\,x^2 - 4\,xy)^2$.
2. $(a^2 - 9\,ab)^2$.
3. $(4\,mn - 9\,n^2)^2$.
4. $(9\,b^2c - 3\,bc^2)^2$.
5. $(p^2 - pq)^2$.
6. $(7\,q^2 - 8\,pq)^2$.
7. $(x^2 - x - 1)^2$.
8. $(2\,a + 3\,b + 4\,c)^2$.
9. $(3\,a^2 - a - 4)^2$.
10. $(3\,y^2 - 2\,y + 5)^2$.
11. $(2\,x^2 - 4\,xy + y^2)^2$.
12. $(a^2 - 2\,ab + 4\,b^2)^2$.
13. $(a - b + c - d)^2$.
14. $(x + y - 2\,z - 3\,u)^2$.
15. $(x^3 - x^2 - x + 1)^2$.
16. $(4\,x^3 + 2\,x^2 - x - 1)^2$.

147. If the terms of a polynomial contain a common quantity, its factors may be obtained by inspection.

Example 1. Find the factors of $3a^2b - 3ab^2 - 6b^3$.

$$3a^2b - 3ab^2 - 6b^3 = 3b(a^2 - ab - 2b^2) = 3b(a - 2b)(a + b).$$

148. Method of grouping terms.

Example 1. Resolve into factors (i) $a^2 - ab + ac - bc$.
(ii) $ax^2 - a^2x - bx + ab$. (iii) $a^2 - b^2 - ac - bc$.

(i) $a^2 - ab + ac - bc = (a^2 - ab) + (ac - bc)$
$= a(a - b) + c(a - b)$ (§ 147)
$= (a - b)(a + c)$. (§ 147)

(ii) $ax^2 - a^2x - bx + ab = (ax^2 - a^2x) - (bx - ab)$
$= ax(x - a) - b(x - a)$ (§ 147)
$= (x - a)(ax - b)$. (§ 147)

(iii) $a^2 - b^2 - ac - bc = (a^2 - b^2) - (ac + bc)$
$= (a + b)(a - b) - c(a + b)$
(§§ 92, 147)
$= (a + b)(a - b - c)$. (§ 147)

Check by multiplying the factors.

EXERCISE 119

Factor:

1. $x^3 + x^2 + 2x + 2$.
2. $x^3 + 2x^2 + 3x + 6$.
3. $x^2 + ax + bx + ab$.
4. $x^3 + x^2 + x + 1$.
5. $x^2 - 2bx + 4cx - 8bc$.
6. $6x^2 - 3xy - 8xz + 4yz$.
7. $x^2 - 3nx + 4mx - 12mn$.
8. $20a^2 + 25ab - 8ac - 10bc$.
9. $21a^2 + 6ab - 28ac - 8bc$.
10. $a^4 - a^2b^2 - a^2c^2 + b^2c^2$.

149. Difference of two squares.

$$x^2 - y^2 = (x + y)(x - y).$$

In this identity substitute $a + b$ for x and $c + d$ for y and get

$$(a + b)^2 - (c + d)^2 = (a + b + c + d)(a + b - c - d).$$

Example 1. $25x^4 - 16y^2z^2 = (5x^2)^2 - (4yz)^2$
$= (5x^2 + 4yz)(5x^2 - 4yz).$

Example 2. $a^4 - 81b^4 = (a^2)^2 - (9b^2)^2$
$= (a^2 + 9b^2)(a^2 - 9b^2)$
$= (a^2 + 9b^2)(a + 3b)(a - 3b).$

Example 3. $x^2 - y^2 + 14yz - 49z^2$
$= x^2 - (y^2 - 14yz + 49z^2)$
$= x^2 - (y - 7z)^2$
$= \{x + (y - 7z)\}\{x - (y - 7z)\}$
$= (x + y - 7z)(x - y + 7z).$

Example 4. $a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2$
$= (a^2 + b^2)^2 - (ab)^2$
$= (a^2 + ab + b^2)(a^2 - ab + b^2).$

EXERCISE 120

1. $x^4 - 16.$
2. $a^4 - 16b^4.$
3. $x^4 - 81y^4.$
4. $x^{2n} - y^{2n}.$
5. $x^6 - x^4.$
6. $5x^7 - 245x.$
7. $(a + b)^2 - c^2.$
8. $b^2 - (x - 1)^2.$
9. $c^2 - (x + 2)^2.$
10. $x^2 - (y + z)^2.$
11. $m^2 - (n + p)^2.$
12. $(a - 2b)^2 - 9c^2.$
13. $a^2 - b^2 + 2bc - c^2.$
14. $x^2 - y^2 - 2yz - z^2.$
15. $a^2 + b^2 - c^2 + 2ab.$
16. $4x^2 + y^2 - z^2 - 4xy.$
17. $a^2 - 4b^2 + 12bc - 9c^2.$
18. $64x^2 - 25(y - z)^2.$

150. The sum or difference of two cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Example 1. $8x^3 + 125y^3 = (2x)^3 + (5y)^3$
$= (2x + 5y)\{(2x)^2 - (2x)(5y) + (5y)^2\}$
$= (2x + 5y)(4x^2 - 10xy + 25y^2).$

Example 2. $x^4 - x = x(x^3 - 1) = x(x^3 - 1^3)$
$= x(x - 1)(x^2 + x + 1).$

Example 3. $x^6 - 64 = (x^3 - 8)(x^3 + 8)$
$= (x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4).$

EXERCISE 121

1. $a^3 - 1.$
2. $a^3 + 1.$
3. $a^3 - 8.$
4. $a^3 + 125.$
5. $a^3 - 64.$
6. $a^3 + 512.$
7. $8x^3 - 1.$
8. $a^3 + 1000b^3.$
9. $8x^3 - 27y^3.$
10. $x^3 - 27y^3.$
11. $27a^3 + b^3.$
12. $125y^3 - 8a^3.$
13. $27a^3 + b^6.$
14. $64a^3 + b^3.$
15. $216a^3 + 125b^3.$
16. $216x^3 - 343y^3.$
17. $125x^3y^3 + z^3.$
18. $729a^3b^3 + c^3.$
19. $16x^4 - 2x.$
20. $512x + x^4.$
21. $x^6 - x^3.$
22. $a^6 + 1.$
23. $a^6 + 64.$
24. $a^9 + 1.$
25. $x^6 + y^6.$
26. $x^{12} + y^{12}.$
27. $a^6 + b^9.$
28. $a^5 + a^2.$
29. $a^3b^3 + a^3c^3.$
30. $a^{3n} - 1.$
31. $x^3 - y^3.$
32. $x^6 - 8.$
33. $b^6 - c^3.$
34. $512 - x^9.$
35. $x^3 + y^3.$
36. $x^3 + 64.$
37. $8x^3 \times 729.$
38. $8x^6 + 125.$

EXERCISE 122

REVIEW OF FACTORING

Resolve the following into as many factors as possible:

1. $8x^2 - 16x$.
2. $ax^2 - bx$.
3. $7x^3 - 28x^2$.
4. $x^3 + x^2 + x$.
5. $mx^2 + mx$.
6. $a^2b - ab^2$.
7. $a(c-1) - c + 1$.
8. $m - n - nm + n^2$.
9. $4 - b - a(4 - b)$.
10. $x^2 - 6bx + 3cx - 18bc$.
11. $x^2 - ax + bx - ab$.
12. $x^3 + x^2y + xy^2 + y^3$.
13. $6a^3 - 4a^2b + 9ab^2 - 6b^3$.
14. $6(a^3 + b^3) - ab(9a + 4b)$.
15. $x^2 - y^2 - (x + y)$.
16. $n(m^2 - 1) - m(n^2 - 1)$.
17. $a(a-1) - b(b-1)$.
18. $x^2 + 8x + 7$.
19. $x^2 + 11x + 10$.
20. $x^2 + 41x + 78$.
21. $x^2 + 44x + 123$.
22. $x^2 - 23x + 132$.
23. $x^2 - 13ax + 40a^2$.
24. $x^2 - 21ax + 108a^2$.
25. $x^2 - 22ax + 96a^2$.
26. $x^2 - x - 72$.
27. $x^2 + x - 30$.
28. $x^2 + 2x - 99$.
29. $x^2 + 3x - 54$.
30. $a^2 - 29ab - 390b^2$.
31. $x^2 - (a - b)x - ab$.
32. $1 - a + b - 72(a - b)^2$.
33. $x^2 - (a+3)x - 3(a+6)$.
34. $4x^2 - 17x + 4$.
35. $3x^2 - 8x - 3$.
36. $6x^2 - 13x + 6$.
37. $6x^2 - 5x - 6$.
38. $20x^2 - 41x + 20$.
39. $20x^2 - 3x - 2$.
40. $9x^2 + 29x - 28$.
41. $3a^2 - 25ay + 42y^2$.
42. $x^2 - 6xy + 8y^2 + 3x - 12y$.
43. $25x^4 - 26x^2 + 1$.

EXERCISE 123

In each of the examples from 1 to 25, the first expression is exactly divisible by the second. Find the quotients by long division.

1. x^3+1; $x+1$.
2. x^3-8; $x-2$.
3. x^3+27; $x+3$.
4. a^6-b^6; a^2-b^2.
5. a^6+b^6; a^2+b^2.
6. x^5-y^5; $x-y$.
7. x^3-3x^2+3x-1; $x-1$.
8. $x^3-6x^2+12x-8$; $x-2$.
9. x^4+x^2+1; x^2-x+1.
10. $a^4+a^2b^2+b^4$; a^2+ab+b^2.
11. $a^4-2a^2b^2+b^4$; $a^2-2ab+b^2$.
12. a^8+a^4+1; a^4-a^2+1.
13. a^6-1; a^4+a^2+1.
14. a^9-1; a^3-1.
15. a^6+a^4-2; a^4+2a^2+2.
16. $2x^5-2x^4+9x^3+3x+4$; x^2-x+4.
17. $x^4-2x^3-6x^2+28x-24$; x^2-4x+6.
18. $36x^4-12x^3+x^2-25$; $6x^2-x-5$.
19. $16x^4-72x^3+81x^2-49$; $4x^2-9x+7$.
20. x^6-4x^3-1; x^2-x-1.
21. x^6+10x^3+27; x^2-2x+3.
22. $8x^6-99x^3-64$; $2x^2-3x-4$.
23. a^6-2a^3+1; a^2+a+1.
24. $27x^6-44x^3y^3-8y^6$; $3x^2-2xy-2y^2$.
25. $a^3+b^3+c^3-3abc$; $a+b+c$.
26. 1 by $1-x$ to 6 terms.

151. Find the remainder when $mx^3 + nx^2 + px + q$ is divided by $x - a$.

$$\begin{array}{r|l} & mx^2 + (ma + n)x + ma^2 + na + p \\ \hline x - a & mx^3 + nx^2 + px + q \\ & \underline{mx^3 - max^2} \\ & (ma + n)x^2 + px \\ & \underline{(ma + n)x^2 - (ma + n)ax} \\ & (ma^2 + na + p)x + q \\ & \underline{(ma^2 + na + p)x - (ma^2 + na + p)a} \\ & ma^3 + na^2 + pa + q \end{array}$$

The remainder is a polynomial in a and may be obtained from the dividend by substituting a for x. This result is perfectly general and is known as the **Remainder Theorem.** It may be stated thus —

When an integral algebraic function of x is divided by $x - a$, the remainder is obtained by substituting a for x in the dividend.

152. From the **Remainder Theorem**, it follows that if the substitution of a for x makes an expression zero, then $x - a$ is a factor of that expression. This is known as the **Factor Theorem.**

Example 1. Show that $x - a$ is a factor of $x^n - a^n$.

Substitute a for x and get $a^n - a^n = 0$.

Hence, $x - a$ is a factor.

Example 2. Show that $x + a$ is a factor of $x^n + a^n$ if n is odd.

Substitute $-a$ for x and get $(-a)^n + a^n = - a^n + a^n = 0$. (§ 115)

Hence, $x + a$ is a factor. (§ 152)

Example 3. Show that $x + a$ is a factor of $x^n - a^n$ if n is even.

Substitute $-a$ for x and get $(-a)^n - a^n = a^n - a^n = 0$. (§ 115)

Hence, $x + a$ is a factor.

Example 4. Find the remainder when $x^3 - 9x^2 - 11x + 19$ is divided by $x - 3$.

Substituting 3 for x, the expression becomes

$$3^3 - 9(3)^2 - 11(3) + 19 = 27 - 81 - 33 + 19 = -68.$$

Example 5. Show that $x - 2$ is a factor of $3x^3 - 4x^2 + 12x - 32$ and find its other factor.

Substituting 2 for x, the expression becomes $24 - 16 + 24 - 32 = 0$.

Hence, $x - 2$ is a factor.

$$3x^3 - 4x^2 + 12x - 32 = (x - 2)(x^2 + 2x + 16).$$

The second factor is obtained by long division.

Example 6. Find the value of a if $x - 3$ is a factor of

$$3x^3 - 2x^2 + ax - 30.$$

Substituting 3 for x, $81 - 18 + 3a - 30 = 0$.

Solving, $a = -11$.

EXERCISE 124

Find the remainder obtained by dividing:

1. $x^3 - 5x^2 + 9x - 11$ by $x - 1$.
2. $2x^3 - 7x^2 + 14$ by $x - 2$.
3. $2x^3 - 25x^2 + 31x - 9$ by $x - 2$.
4. $3x^3 - 2x^2 - 27x - 1$ by $x + 1$.
5. $7x^2 - 19x - 23$ by $x + 4$.
6. $13x^3 - 29x + 40$ by $x + 2$.

Show that:

7. $x - 1$ is a factor of $x^3 - 11x^2 + 19x - 9$.

8. $x - 2$ is a factor of $2x^3 - 17x^2 + 21x + 10$.

9. $x + 3$ is a factor of $4x^3 - 2x^2 + 19x + 183$.

10. $x + 2$ is a factor of $5x^3 - 17x^2 + 108$.

11. $x + 1$ is a factor of $x^4 - 11x^2 + 37x + 47$.

12. If $x - 3$ is a factor of $3x^3 + ax^2 + 8x - 6$, find a.

13. If $x + 2$ is a factor of $ax^3 - 15x^2 + 4x - 12$, find a.

14. Show that $a + b$, $b + c$, $c + a$ are factors of

$$a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc.$$

15. Show that $a - b$, $b - c$, and $c - a$ are factors of

$$a^2(b - c) + b^2(c - a) + c^2(a - b).$$

CHAPTER XIV

H. C. F., L. C. M., AND FRACTIONS

153. The Highest Common Factor of two or more algebraic expressions is the expression of highest degree that is a factor of the given expressions. It is denoted by H. C. F.

154. The Lowest Common Multiple of two or more algebraic expressions is the expression of least degree that is exactly divisible by the given expressions. It is denoted by L. C. M.

Example 1. Find the H. C. F. of $a^3x^2y^4$ and $a^2x^3y^7$.

The H. C. F. is $a^2x^2y^4$, since a^2, x^2, and y^4 are the highest powers of the common factors of the given expressions.

Example 2. Find the L. C. M. of $a^3x^2y^4$ and $a^2x^3y^7$.

The L. C. M. is $a^3x^3y^7$.

155. The H. C. F. of two or more expressions is the continued product of their common factors, each raised to the lowest power in which it occurs in the given expressions.

The L. C. M. is the product of the factors each raised to the highest power in which it occurs in the given expressions.

Example 3. Find the H. C. F. and L. C. M. of $x^3 - x$ and $x^3 - 2x^2 + x$.

$$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1),$$
$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x - 1)^2.$$

The H.C.F. is $x(x-1)$; the L.C.M. is $x(x+1)(x-1)^2$.

Example 4. Find the H.C.F. and L.C.M. of $8a^2b(a^2-b^2)^2$, $12(a^2b+ab^2)^3$.

$$8a^2b(a^2-b^2)^2 = 8a^2b\{(a+b)(a-b)\}^2$$
$$= 8a^2b(a+b)^2(a-b)^2,$$
$$12(a^2b+ab^2)^3 = 12\{ab(a+b)\}^3 = 12a^3b^3(a+b)^3.$$

The H.C.F. is $4a^2b(a+b)^2$; the L.C.M. is $24a^3b^3(a+b)^3(a-b)^2$.

Example 5. Find the H.C.F. and L.C.M. of

$$a^4+a^3-a-1,\ a^3+a^2-a-1.$$

$$a^4+a^3-a-1 = a^3(a+1)-(a+1) = (a+1)(a^3-1)$$
$$= (a+1)(a-1)(a^2+a+1).$$
$$a^3+a^2-a-1 = a^2(a+1)-(a+1) = (a+1)(a^2-1)$$
$$= (a+1)(a-1)(a+1)$$
$$= (a+1)^2(a-1).$$

The H.C.F. is $(a+1)(a-1)$; L.C.M. is $(a+1)^2(a-1)(a^2+a+1)$.

EXERCISE 125

Find the H.C.F. and L.C.M. of:

1. $6a^2b^3$, $9ab^4$, $12a^3b^2$.
2. $8x^3y^5$, $12x^2y^4$, $16x^4y^2$.
3. x^2-x, x^2+x.
4. a^2+a, a^3+2a^2+a.
5. x^3-xy^2, $x^3-2x^2y+xy^2$.
6. x^3-y^3, $x^3+x^2y+xy^2$.
7. a^3+ay^2, a^4-y^4.
8. $16-x^4$, $16+8x^2+x^4$.
9. x^2-4, x^3-8.
10. a^4-3a^3b, ab^3-3b^4.
11. $1+a^3$, $1+a^2+a^4$.
12. $1-a^3$, $1+a^2+a^4$.

13. $a^3 - a^2b$, $a^3 - a^2b + ab^2 - b^3$.

14. $x^4 - y^4$, $x^3 + x^2y + xy^2 + y^3$.

15. $a^4 - b^4$, $a^3 - a^2b + ab^2 - b^3$.

16. $x^6 + y^6$, $x^5 - x^3y^2 + xy^4$.

17. $x^2 - 3x + 2$, $x^2 - x - 2$, $x^2 - 4$.

18. $4x^2 - 1$, $4x^2 + 4x + 1$, $2x^2 - 3x - 2$.

19. $a^2 - 3ab - ac + 3bc$, $a^2 - 2ab - 3b^2$.

20. $x^3 + 1$, $x^2 + 4x + 3$, $x^3 + 2x^2 + 2x + 1$.

21. $a^2 - (b + c)^2$, $b^2 - (a + c)^2$, $c^2 - (a + b)^2$.

22. $a^2 - 2ab + b^2 - c^2$, $a^2 - b^2 - 2bc - c^2$.

156. If the terms of a fraction are both multiplied or both divided by the same number, the value of the fraction remains unchanged. Expressed in symbols the rule is

$$\frac{an}{bn} = \frac{a}{b}, \quad \frac{a \div n}{b \div n} = \frac{a}{b}.$$

Reduce to lowest terms $\dfrac{9a^4 - 54a^3b + 81a^2b^2}{3a^3 - 27ab^2}$.

$$\begin{aligned}\frac{9a^4 - 54a^3b + 81a^2b^2}{3a^3 - 27ab^2} &= \frac{9a^2(a^2 - 6ab + 9b^2)}{3a(a^2 - 9b^2)} \\ &= \frac{9a^2(a - 3b)^2}{3a(a + 3b)(a - 3b)} \\ &= \frac{3a(a - 3b)}{a + 3b}.\end{aligned}$$

The method of reducing consists in factoring the terms of the fraction and then in cancelling the common factors in the numerator and denominator.

EXERCISE 126

Reduce to lowest terms:

1. $\dfrac{a^2-1}{a^2-a}$.

2. $\dfrac{a^2-ab}{a^2+ab}$.

3. $\dfrac{x^2-2x+1}{x^2-3x+2}$.

4. $\dfrac{2a^2-ab}{2ac-bc}$.

5. $\dfrac{3ab-3b^2}{a^2-b^2}$.

6. $\dfrac{2a^2-ab-b^2}{a^2-3ab+2b^2}$.

7. $\dfrac{2xyx-2y^2z}{x-yz-y(1-z)}$.

8. $\dfrac{(a^2-b^2)(a^2+ab+b^2)}{(a-b)^2(a^3-b^3)}$.

9. $\dfrac{x^2+2x-3}{x^2+5x+6}$.

10. $\dfrac{a^3-a^2-a+1}{a^3+a^2+a+1}$.

11. $\dfrac{9a^4+2a^2+1}{9a^4+6a^3+3a}$.

12. $\dfrac{(3x^2+2x-5)^2-(11x+9)^2}{(2x^2+3x+7)^2-(x^2+10x-3)^2}$.

13. $\dfrac{xy+2bx+6ab+3ay}{xy-2bx-6ab+3ay}$.

14. $\dfrac{4x^4-4x^2+9}{(2x^2+3)^2-3x(2x^2+3)-4x^2}$.

157. If a is exactly divisible by b, the Law of Signs is

$$-\frac{a}{b}=-\frac{a}{b};\quad \frac{a}{-b}=-\frac{a}{b};\quad \frac{-a}{-b}=\frac{a}{b}.$$

These relations will be assumed to hold also in the case of all other fractions; that is, when a is not exactly divisible by b.

Thus $$\frac{-1}{a-b}=-\frac{1}{a-b};\quad \frac{-1}{a-b}=\frac{-1}{-(b-a)}=\frac{1}{b-a}.$$

$$\frac{1}{(a-b)(c-b)}=\frac{-1}{(a-b)(b-c)}=-\frac{1}{(a-b)(b-c)}.$$

158. Rules for addition and subtraction of fractions.

1. $$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

2. $$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

Example 1. Simplify

$$\frac{2x+1}{3x+5}-\frac{2x+2}{4x+1}-\frac{2x^2-10x-9}{12x^2+23x+5}.$$

The expression

$$=\frac{2x+1}{3x+5}-\frac{2x+2}{4x+1}-\frac{2x^2-10x-9}{(3x+5)(4x+1)}$$

$$=\frac{(2x+1)(4x+1)-(2x+2)(3x+5)-(2x^2-10x-9)}{(3x+5)(4x+1)}$$

$$=\frac{8x^2+6x+1-6x^2-16x-10-2x^2+10x+9}{(3x+5)(4x+1)}$$

$$=\frac{0}{(3x+5)(4x+1)}=0.$$

Example 2.

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$$

$$=-\frac{1}{(a-b)(c-a)}-\frac{1}{(b-c)(a-b)}-\frac{1}{(c-a)(b-c)} \quad (\S\ 157)$$

$$=\frac{-b+c-c+a-a+b}{(a-b)(b-c)(c-a)}=0.$$

The arrangement, gained by placing a before b, b before c, c before a, is called **cyclic order.**

Example 3. Simplify:

$$\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}$$

$$=\frac{1+x+1-x}{1-x^2}+\frac{2}{1+x^2}+\frac{4}{1+x^4}$$

$$=\frac{2}{1-x^2}+\frac{2}{1+x^2}+\frac{4}{1+x^4}$$

$$=\frac{2(1+x^2)+2(1-x^2)}{1-x^4}+\frac{4}{1+x^4}$$

$$=\frac{4}{1-x^4}+\frac{4}{1+x^4}=\frac{4(1+x^4)+4(1-x^4)}{1-x^8}$$

$$=\frac{8}{1-x^8}.$$

In this example the work is shortened by adding the first two fractions, then by adding this sum and the third fraction, then this latter sum and the fourth fraction.

EXERCISE 127

Simplify:

1. $\frac{1}{x-2}+\frac{1}{x+2}.$

2. $\frac{1}{x+5}-\frac{1}{x+6}.$

3. $\frac{1}{1-x}+\frac{x}{(1-x)^2}.$

4. $\frac{7}{x+1}-\frac{6}{x-1}.$

5. $\frac{5}{x-3}+\frac{4}{x+3}.$

6. $\frac{4}{x+8}-\frac{3}{x-8}.$

7. $\frac{x+1}{x+2}-\frac{x-1}{x}.$

8. $\frac{2x-1}{2x+3}-\frac{2x-5}{2x-1}.$

9. $\frac{x+a}{x-a}-\frac{x-a}{x+a}.$

10. $\frac{7x-9}{14x-15}-\frac{2x-3}{4x-5}.$

11. $\frac{4x-5}{6x-7}-\frac{6x-8}{9x-11}.$

12. $\frac{3x+1}{x-5}+\frac{x+4}{x+6}.$

13. $\frac{3a^2+6b^2}{9a^2+8b^2}-\frac{a^2-3b^2}{3a^2-4b^2}.$

14. $\frac{2a+b}{a-b}-\frac{2a-b}{a+b}.$

15. $\frac{1}{a^2-4}-\frac{1}{a^2+a-2}.$

16. $\frac{1}{(a-b)^2}-\frac{1}{a^2-b^2}.$

17. $\frac{1}{x-2}-\frac{2x}{2x-x^2}$.

18. $\frac{1}{x-y}+\frac{x}{y^2-x^2}$.

19. $\frac{x-y}{2x-y}-\frac{y^2+xy}{xy-2x^2}$.

20. $\frac{2b}{a^2(a+b)}-\frac{1}{a^2-b^2}$.

21. $\frac{x}{4y^2-x^2}-\frac{1}{x+2y}$.

22. $\frac{x+1}{x^2-4}-\frac{x-1}{(x-3)(2-x)}$.

23. $\frac{2a+3}{(a-2)(a-1)}-\frac{5a+2}{(1-a)(3-a)}+\frac{2-3a}{(3-a)(a-2)}$.

24. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

25. $\frac{a}{(a-b)(a-c)}-\frac{b}{(a-b)(b-c)}+\frac{c}{(c-a)(c-b)}$.

26. $\frac{2y^2}{x^2-y^2}+\frac{y}{x+y}+\frac{y}{y-x}$.

27. $\frac{1+a}{(a-b)(a-c)}+\frac{1+b}{(b-c)(b-a)}-\frac{1+c}{(c-a)(b-c)}$.

28. $\frac{1}{x-1}-\frac{1}{x+1}-\frac{1}{x^2-1}-\frac{1}{x^2+1}$.

29. $\frac{a}{a-b}+\frac{a}{a+b}+\frac{2a^2}{a^2+b^2}+\frac{4a^2b^2}{a^4-b^4}$.

30. $\frac{x+y}{x-y}-\frac{x^2+y^2}{x^2-y^2}+\frac{4x^2y^2}{x^4-y^4}$.

31. $\frac{a+b}{a-b}-\frac{2ab}{a^2-b^2}-\frac{2a^2b^2}{a^4-b^4}$.

32. $\frac{2a+b}{a-2b}+\frac{2a-b}{a+2b}+\frac{a^4-40a^2b^2-16b^4}{a^4-16b^4}$.

33. $\frac{1}{x-1}-\frac{x}{x^2-1}-\frac{x^2+1}{x^3-1}$.

159. Rules for multiplication and division of fractions.

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd};\quad \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

EXERCISE 128

Simplify:

1. $\dfrac{x^2-9}{a^2-1}\times\dfrac{a^2-4a+3}{x^2-6x+9}.$

2. $\dfrac{a^2-ab}{ac-bc}\times\dfrac{a^2-b^2}{(a-b)^2}.$

3. $\dfrac{a^2-x^2}{a^2+x^2}\div\dfrac{a-x}{a+x}.$

4. $\dfrac{(x-2)^2}{x+2}\times\dfrac{3x+6}{x-2}.$

5. $\dfrac{c^2-d^2}{m+n}\times\dfrac{m^2+mn}{c+d}.$

6. $\dfrac{a^3+b^3}{a^3}\div\dfrac{a^2-ab+b^2}{2a^2}.$

7. $\dfrac{a^2-4}{a^2+a-6}\times\dfrac{a^2+6a+9}{a^2+2a}.$

8. $\dfrac{x^2-5x+4}{x-1}\times\dfrac{x^2-2x+1}{x^2-16}.$

9. $\dfrac{a^2+ab-2b^2}{a^2-ab-12b^2}\div\dfrac{a^2+3ab+2b^2}{a^2-3ab-4b^2}.$

10. $\dfrac{a^3-b^3}{2b^2}\div\dfrac{a^2+ab+b^2}{4b}.$

11. $\dfrac{4x^2-81}{4x^2-5}\times\dfrac{16x^4-36x^2+20}{8x^3-729}\div\dfrac{4x^2+16x-9}{4x^2+18x+81}.$

12. $\dfrac{x^2+x+1}{x}\times\dfrac{x^2-x+1}{x}\div\dfrac{x^6-1}{x^4-x^2}.$

13. $\dfrac{x^2-4}{x^2+3x+2}\times\dfrac{x^2+2x+1}{x^2-x-12}\div\dfrac{x^2-x-2}{x^2-4x}.$

14. $\dfrac{x^3-y^3}{(x+y)^2}\times\dfrac{x^3+y^3}{x^2+xy-2y^2}\div\dfrac{x^4+x^2y^2+y^4}{x^2+3xy+2y^2}.$

15. $\dfrac{x^3-n^3}{x^2-nx+cx-nc}\times\dfrac{x^2-c^2}{x^2+nx-cx-nc}\div\dfrac{x^2+nx+n^2}{x^2+nx+cx+nc}.$

16. $\dfrac{(a-b)^2-c^2}{ab-b^2-bc}\times\dfrac{c}{a^2+ab-ac}\div\dfrac{ac-bc+c^2}{a^2-(b-c)^2}.$

CHAPTER XV

EQUATIONS

160. *Example* 1. Solve $\frac{2x+3}{x(x-3)}=\frac{4}{x}+\frac{3}{x-3}$.

Multiplying by $x(x-3)$, the L. C. M. of the denominators,

$$2x+3=4x-12+3x.$$

$$\therefore -5x=-15.$$

$$x=3.$$

When $x=3$, the given equation becomes

$$\frac{9}{3(0)}=\frac{4}{3}+\frac{3}{0}.$$

The first and third terms are meaningless; as the value 3 for x causes a denominator to vanish, it cannot be asserted that 3 is a solution of the equation.

Example 2. Solve $\frac{2x-13}{x-3}+\frac{x+3}{x-4}=\frac{x-6}{x+1}+\frac{2x+7}{x}$.

This may be written

$$2-\frac{7}{x-3}+1+\frac{7}{x-4}=1-\frac{7}{x+1}+2+\frac{7}{x}.$$

$$\therefore -\frac{7}{x-3}+\frac{7}{x-4}=-\frac{7}{x+1}+\frac{7}{x}.$$

$$\therefore -\frac{1}{x-3}+\frac{1}{x-4}=-\frac{1}{x+1}+\frac{1}{x}.$$

Combining, $$\frac{1}{(x-3)(x-4)}=\frac{1}{x(x+1)}.$$

$$\therefore x(x+1)=(x-3)(x-4).$$

$$x^2+x=x^2-7x+12.$$

$$8x = 12.$$
$$x = 1\tfrac{1}{2}.$$

This value of x does not make a denominator zero. It will satisfy the equation, and hence it is a solution.

EXERCISE 129

Solve and check:

1. $\dfrac{x}{210 - x} = \dfrac{11}{19}.$

2. $\dfrac{2x + 2}{x + 2} = \dfrac{2x - 1}{x - 1}.$

3. $\dfrac{3x + 3}{x + 7} = \dfrac{3x + 13}{x + 6}.$

4. $\dfrac{x + 4}{x - 1} = \dfrac{x + 8}{x}.$

5. $\dfrac{x}{x - 2} = \dfrac{2x + 6}{2x - 1}.$

6. $\dfrac{17x - 1}{4x - 2} = \dfrac{17x - 16}{4x - 5}.$

7. $\dfrac{7x - 5}{x + 1} = \dfrac{7x - 17}{x - 2}.$

8. $\dfrac{18x - 2}{4x - 1} = \dfrac{18x + 13}{4x + 2}.$

9. $\dfrac{4x - 7}{6x - 17} = \dfrac{6x - 5}{9x - 14}.$

10. $\dfrac{4x - 3}{8x - 5} = \dfrac{6x + 5}{12x + 11}.$

11. $\dfrac{7x - 4}{9x + 13} = \dfrac{7x + 8}{9x + 25}.$

12. $\dfrac{5x - 4}{7x + 9} = \dfrac{15x - 14}{21x + 23}.$

13. $\dfrac{x}{2x - 3} + \dfrac{x}{2x + 3} = 1 - \dfrac{x - 16}{4x^2 - 9}.$

14. $\dfrac{x - 1}{2x - 5} + \dfrac{x + 1}{2x + 5} = 1 - \dfrac{x - 23}{4x^2 - 25}.$

15. $\dfrac{x + 7}{2x + 9} - \dfrac{x + 11}{3x - 8} = \dfrac{x^2 - 27x - 2}{(2x + 9)(3x - 8)}.$

16. $\dfrac{3x}{7x - 2} - \dfrac{4x}{7x + 2} + \dfrac{7x^2 - 12x - 4}{49x^2 - 4} = 0.$

17. $\dfrac{1}{x - 1} + \dfrac{2}{x - 2} - \dfrac{3}{x - 3} = 0.$

161. Simultaneous equations involving the reciprocals of the unknown quantities.

Equations involving $\frac{1}{x}$ and $\frac{1}{y}$ should be solved for $\frac{1}{x}$ and $\frac{1}{y}$, and then from these values the required values should be obtained.

Example 1.

$$\frac{5}{3x}-\frac{2}{5y}=\frac{11}{15}, \quad (1)$$

$$\frac{3}{2x}+\frac{7}{3y}=\frac{4}{3}. \quad (2)$$

Multiplying (1) by 15 and (2) by 6,

$$\frac{25}{x}-\frac{6}{y}=11, \quad (3)$$

$$\frac{9}{x}+\frac{14}{y}=8. \quad (4)$$

Multiplying (3) by 7, and (4) by 3,

$$\frac{175}{x}-\frac{42}{y}=77, \quad (5)$$

$$\frac{27}{x}+\frac{42}{y}=24. \quad (6)$$

$$\therefore \frac{202}{x}=101, \text{ adding (5) and (6).}$$

$$\therefore \frac{1}{x}=\frac{1}{2}.$$

Substituting in (4),

$$9\left(\frac{1}{2}\right)+\frac{14}{y}=8.$$

Solving, $\frac{1}{y}=\frac{1}{4}$. Hence, $x=2$, $y=4$.

Example 2. Solve

$$4x+\frac{3}{y}=17,$$
$$5x-\frac{2}{y}=4.$$

Solving for x and $\frac{1}{y}$, we find $x=2$, $\frac{1}{y}=3$.

$$\therefore\ y=\tfrac{1}{3}.$$

EXERCISE 130

Solve and check:

1. $\frac{3}{x}+\frac{2}{y}=4,$
$\frac{2}{x}+\frac{1}{y}=3.$

2. $\frac{4}{x}+\frac{5}{y}=2.$
$\frac{3}{x}-\frac{2}{y}=\frac{7}{20}.$

3. $\frac{5}{y}-\frac{4}{x}=11,$
$\frac{6}{y}-\frac{5}{x}=13.$

4. $\frac{10}{x}-\frac{11}{y}=\frac{17}{126},$
$\frac{4}{x}-\frac{7}{y}=\frac{25}{126}.$

5. $\frac{9}{2x}-\frac{4}{3y}=\frac{7}{6},$
$\frac{4}{3x}+\frac{5}{4y}=\frac{109}{144}.$

6. $15x+\frac{7}{y}=44,$
$12x-\frac{3}{y}=18.$

7. $3x+5y=\frac{5}{2}xy,$
$\frac{5}{x}+\frac{2}{y}=2.$

8. $4x-\frac{7}{y}=11\tfrac{1}{3},$
$x+\frac{7}{2y}=-\frac{2}{3}.$

9. $\frac{4}{x}+\frac{30}{y}+\frac{5}{z}=-12,$
$\frac{14}{x}-\frac{10}{y}+\frac{5}{z}=18,$
$\frac{4}{x}+\frac{4}{y}+\frac{5}{z}=1.$

10. $\frac{3}{x}-\frac{2}{y}+\frac{3}{z}=\frac{161}{60},$
$\frac{6}{x}-\frac{1}{y}+\frac{2}{z}=\frac{77}{30},$
$\frac{4}{x}-\frac{3}{y}+\frac{4}{z}=\frac{19}{5}.$

162. *Example* 1. Solve $\frac{4x-9}{6x+16}=\frac{7x-9}{5x+16}$.

Multiply by $(6x+16)(5x+16)$.

$$\therefore (4x-9)(5x+16)=(6x+16)(7x-9).$$

Expanding, $20x^2+19x-144=42x^2+58x-144.$

Transposing and dividing by -1,

$$22x^2+39x=0.$$

Factoring, $x(22x+39)=0.$

$$x=0 \text{ or } -\tfrac{39}{22}.$$

163. Higher equations solved like quadratics.

Example 1. Solve $x^4-10x^2+9=0.$

Factoring, $(x^2-1)(x^2-9)=0.$

$$(x-1)(x+1)(x-3)(x+3)=0.$$

$$\therefore x=1,\ -1,\ 3 \text{ or } -3.$$

Example 2. Solve $(x^2+2x+3)(x^2+2x-7)=144.$

Let $y=x^2+2x,$

then $y+3=x^2+2x+3,$

$$y-7=x^2+2x-7.$$

$$\therefore (y+3)(y-7)=144.$$

$$y^2-4y-21=144.$$

$$y^2-4y-165=0.$$

$$(y-15)(y+11)=0.$$

$$y=15 \text{ or } -11.$$

$$\therefore x^2+2x=15 \text{ or } x^2+2x=-11$$

Solving these equations, $x=3,\ -5,\ -1\pm\sqrt{-10}.$

(§ 127)

Example 3. Solve $\frac{x^2+12}{6x}+\frac{6x}{x^2+12}=\frac{25}{12}.$

Let $\frac{x^2+12}{6x}=y$, then $\frac{6x}{x^2+12}=\frac{1}{y}$

and $$y+\frac{1}{y}=\frac{25}{12}.$$

Multiplying by $12y$, $12y^2-12=25y.$

Solving, $$y=\frac{3}{4} \text{ or } \frac{4}{3},$$

i.e. $$\frac{x^2+12}{6x}=\frac{3}{4} \text{ or } \frac{4}{3}.$$

Solving, $x=2,\ 6,\ \frac{9\pm\sqrt{-111}}{4}.$ (§ 127)

Example 4. Find the three cube roots of 1.

If x denotes one of them, then

$$x^3=1,$$

i.e. $$x^3-1=0.$$

Factoring, $(x-1)(x^2+x+1)=0.$

$\therefore\ x-1=0$ and $x^2+x+1=0.$

Solving, $x=1,\ \frac{-1\pm\sqrt{-3}}{2}.$

EXERCISE 131

Solve and check:

1. $6x^2+6=37x.$
2. $9x^2-9=80x.$
3. $4x^2+21=20x.$
4. $3x^2-4x=15.$
5. $5x^2+19x=4.$
6. $2x^2-3x=27.$
7. $6x^2+41x=7.$
8. $2x^2-x=10.$
9. $4x^2-4x=3.$
10. $3x^2-2x=40.$
11. $2x^2-41x=21.$
12. $2x^2-x=6.$

13. $3x^2 - 25x + 42 = 0.$

14. $4x^2 - 32x + 15 = 0.$

15. $9x^2 - 45x + 44 = 0.$

16. $16x^2 - 80x + 19 = 0.$

17. $4x^2 - 24x + 27 = 0.$

18. $9x^2 - 18x - 16 = 0.$

19. $9x^2 - 45x - 34 = 0.$

20. $ax^2 - (a + b)x + b = 0.$

21. $a(4x^2 - 1) = 2(a^2 - 1)x.$

22. $x^2 - ax + ab = b^2.$

23. $acx^2 + adx = bcx + bd.$

24. $mnx^2 - (m^2 + n^2)x + mn = 0.$

25. $x^4 - 24x^2 + 128 = 0.$

26. $x^4 - x^2 - 12 = 0.$

27. $x^4 - 15x^2 - 16 = 0.$

28. $(x^2 - 3)(x^2 - 2) = 42.$

29. $(x^2 - x)(x^2 - x + 3) = 10.$

30. $(2x^2 + 1)^2 + (2x^2 + 1) = 90.$

31. $(x^2 + 3x)^2 + 4(x^2 + 3x) = -4.$

32. $(x^2 + x)^2 - 18(x^2 + x) = -72.$

33. $\left(\frac{x^2+1}{x}\right)^2 + 4\left(\frac{x^2+1}{x}\right) = 12.$

34. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = \frac{29}{10}.$

35. $\frac{x^2-4x}{5x} + \frac{5x}{x^2-4x} = \frac{26}{5}.$

36. $x^3 - 4x^2 + 3x = 0.$

37. $x^3 - 1 = 3(x - 1).$

38. $x^3 + 8 = 12(x + 2).$

39. $x^3 + 8 = 0.$

40. $x^4 + 6x^3 + 9x^2 = 16.$

41. $x^2 - x + \frac{120}{x^2 - x} = 26.$

42. $x^3 = 27.$

43. $(x^2 - 2x + 25)^2 = 24(x^2 - 2x + 25).$

44. $2(x^2 + 2x + 11)^2 = 49(x^2 + 2x + 5).$

45. $(x^2 - 7x + 13)^2 = (x - 3)(x - 4) + 1.$

46. $(2x^2 + 4x)^2 - 9(x + 1)^2 = 0.$

47. $6x^3 - 7x^2 - 7x + 6 = 0.$

164. Constructions of equations.

Example 1. Construct an equation whose roots are a, b, c.

The expression $(x-a)(x-b)(x-c)$ is zero if any one of its factors is zero. Hence, the required equation is

$$(x-a)(x-b)(x-c)=0.$$

Example 2. Construct an equation whose roots are 0, $\frac{3}{4}$, $-\frac{1}{2}$.

An equation having these values for x is

$$x(x-\tfrac{3}{4})(x+\tfrac{1}{2})=0.$$

Multiplying by 8, $x(4x-3)(2x+1)=0.$

Expanding, $8x^3-2x^2-3x=0.$

This is the simplest equation whose roots are 0, $\frac{3}{4}$, $-\frac{1}{2}$.

EXERCISE 132

Construct equations whose roots are:

1. $1, -1$.
2. $3, 4$.
3. $-4, 3$.
4. $5, 1$.
5. $0, 7$.
6. $0, -9$.
7. $5, 5$.
8. $-2, -2$.
9. $a, -a$.
10. $b, 2b$.
11. $n, -m$.
12. $-c, -c$.
13. $\frac{2}{3}, -1$.
14. $\frac{3}{4}, -2$.
15. $\frac{1}{2}, \frac{1}{3}$.
16. $0, -\frac{4}{5}$.
17. $a+2b, a-2b$.
18. $\frac{a}{b}, \frac{b}{a}$.
19. $a, a-b$.
20. $a+b, a-b$.
21. $a, -c$.
22. $-a, -b$.
23. $a, -b, c$.
24. $-a, -b, -c$.
25. $-1, 2, -1$.
26. $\frac{a+b}{a}, \frac{a}{a+b}$.
27. $a-b, b-a$.
28. $-1, \frac{1}{2}, 1, -\frac{1}{2}$.

165. Factors of quadratics.

Any quadratic can be expressed as the difference of two squares and hence factored.

Example 1. Factor x^2+px+q.

Complete the square of x^2+px by adding $\frac{p^2}{4}$.

The addition of $\frac{p^2}{4}-\frac{p^2}{4}$ to x^2+px+q does not alter its value.

$$\therefore\ x^2+px+q = x^2+px+\frac{p^2}{4}-\frac{p^2}{4}+q$$
$$= x^2+px+\frac{p^2}{4}-\left(\frac{p^2}{4}-q\right)$$
$$= \left(x+\frac{p}{2}\right)^2-\frac{p^2-4q}{4}$$
$$= \left(x+\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}\right)\left(x+\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}\right).$$

Example 2. Factor $8x^2-2x-3$.

$$8x^2-2x-3 = 8(x^2-\tfrac{1}{4}x-\tfrac{3}{8})$$
$$= 8(x^2-\tfrac{1}{4}x+\tfrac{1}{64}-\tfrac{1}{64}-\tfrac{3}{8})$$
$$= 8\{(x-\tfrac{1}{8})^2-\tfrac{25}{64}\}$$
$$= 8(x-\tfrac{1}{8}+\tfrac{5}{8})(x-\tfrac{1}{8}-\tfrac{5}{8})$$
$$= 8(x+\tfrac{1}{2})(x-\tfrac{3}{4})$$
$$= 2(x+\tfrac{1}{2})\cdot 4(x-\tfrac{3}{4})$$
$$= (2x+1)(4x-3).$$

This method is perfectly general.

166. *Example* 1. Find the value of x in terms of the other letters in the equation

$$\frac{x-b}{x-a} - \frac{x-a}{x-b} = \frac{2(a-b)}{x-a-b}.$$

Combining the first two fractions,

$$\frac{2(a-b)x-(a^2-b^2)}{(x-a)(x-b)} = \frac{2(a-b)}{x-(a+b)}.$$

Dividing by $a-b$,

$$\frac{2x-(a+b)}{(x-a)(x-b)} = \frac{2}{x-(a+b)}.$$

Clearing of fractions,

$$2x^2 - 3(a+b)x + (a+b)^2 = 2x^2 - 2(a+b)x + 2ab.$$

$$\therefore \quad -(a+b)x = 2ab - (a+b)^2,$$

i.e. $$-(a+b)x = -(a^2+b^2).$$

Dividing by $-(a+b)$, $\therefore x = \dfrac{a^2+b^2}{a+b}.$

EXERCISE 133

Solve:

1. $\dfrac{x-a}{a} + \dfrac{x-b}{b} = \dfrac{x-c}{c} = 0.$

2. $\dfrac{x}{a+b} + \dfrac{x}{a-b} = 1.$

3. $\dfrac{x+a}{b} + \dfrac{x+b}{a} = \dfrac{1}{a} + \dfrac{1}{b} - 2.$

4. $\dfrac{x^2+ax-2a^2}{x-a} + 2x - 5a.$

5. $\dfrac{x}{x-a} + \dfrac{x}{x-b} = 2.$

6. $\dfrac{1}{a-x} - \dfrac{1}{b-x} = \dfrac{b-a}{x^2}.$

7. $\dfrac{2}{x+a} - \dfrac{1}{x+b} = \dfrac{x}{x^2-a^2}.$

8. $\dfrac{a}{ax-1} + \dfrac{b}{bx-1} = \dfrac{2}{x}.$

9. $\dfrac{ax+b}{x+b} + \dfrac{bx+a}{x+a} = a+b.$

10. $(a-b)(x-c) - (b-c)(x-a) = (c-a)(x-b).$

167. One equation of the first degree and the other of the second.

Example 1. Solve

$$2x^2 - 5xy + y^2 + 43 = 0, \quad (1)$$
$$4x - 5y + 9 = 0. \quad (2)$$

Find from (2) the value of x in terms of y.

Substitute this value of x in (1) and get

$$2\left(\frac{5y-9}{4}\right)^2 - 5\left(\frac{5y-9}{4}\right)y + y^2 + 43 = 0.$$

Solve this quadratic in y and get $y = 5$ or -5.

Substitute 5 or -5 for y in (2), $x = 4$, or $-8\frac{1}{2}$.

The solutions may be written $(4, 5)$, $(-8\frac{1}{2}, -5)$.

168. Both equations quadratic, and homogeneous as to *x* and *y* (*i.e.* all terms of the same degree).

Example 1. Solve

$$9x^2 - 9xy + 16y^2 = 91, \quad (1)$$
$$9xy + 4y^2 = 70. \quad (2)$$

Step 1. Eliminate the terms independent of x and y.

Multiply (1) by 10, (2) by 13, subtract, divide by 9 and get

$$10x^2 - 23xy + 12y^2 = 0,$$

i.e.

$$(5x - 4y)(2x - 3y) = 0. \quad (3)$$

Step 2. Solve (1) and (3), or (2) and (3).

Taking the latter, solve

$$9xy + 4y^2 = 70, \quad \text{and} \quad 9xy + 4y^2 = 70,$$
$$5x - 4y = 0, \qquad\qquad 2x - 3y = 0.$$

This is done by the method of § 167. The solutions are

$$x = 2\tfrac{1}{2},\ -2\tfrac{1}{2},\ 3,\ -3,$$
$$y = 2,\ -2,\ 2,\ -2,$$

or $(2\tfrac{1}{2}, 2),\ (-2\tfrac{1}{2}, -2),\ (3, 2),\ (-3, -2).$

EXERCISE 134

Solve and check :

1. $xy = 30,$
   $x + 2y = 19.$

2. $xy + 2x = 20,$
   $2x + y = 12.$

3. $3xy - 2x = 3,$
   $x + 3y = 6.$

4. $xy + x = 25,$
   $2x + y = 14.$

5. $xy = 24,$
   $3x + 2y = 30.$

6. $x^2 + y^2 = 65,$
   $x - 3y = 5.$

7. $x^2 - y^2 = 77,$
   $2x - 3y = 12.$

8. $2x^2 + y^2 = 22,$
   $3x + 2y = 13.$

9. $x^2 - 3y^2 = 13,$
   $x + 2y = 6.$

10. $x^2 - 2y^2 = 14,$
    $x - 2y = 2.$

11. $2x^2 + y^2 = 51,$
    $x + y = 6.$

12. $3x^2 + 2y^2 = 50,$
    $x + y = 5.$

13. $x^2 + y^2 - 2x - 4y = 8,$
    $x + y = 8.$

14. $x^2 + 4x + y^2 - 4y = 5,$
    $x - y + 3 = 0.$

15. $x^2 + y^2 + 6x + 4y = 13,$
    $x + y = 1.$

16. $xy + x - 2y = 16,$
    $x + y = 10.$

17. $xy + 4x - y = 12,$
    $x - y = 7.$

18. $xy + 3x + 5y = 40,$
    $x - y = 4.$

19. $xy - 2x - 3y + 3 = 0,$
    $x + y = 9.$

20. $xy + x - 4y = 28,$
    $x + y = 13.$

21. $x^2 + 2y^2 - 2x - 4y = 21,$
    $x - y = 2.$

22. $2x^2 + y^2 + 4x + 2y = 56,$
    $x + y = 6.$

23. $2x^2 - 3y^2 - 12x - 12y = 23,$
$x - y = 8.$

24. $4x^2 + 9y^2 - 8x - 18y = 12,$
$x + 2y = 7.$

25. $5x^2 + 12y^2 = 32,$
$x + 2y = 4.$

26. $x^2 - 3xy + 2y^2 = 3,$
$4x - 7y = 6.$

27. $3x^2 + y^2 = 49,$
$12x + y = 49.$

28. $2x^2 - y^2 = 46,$
$5x - y = 23.$

29. $x^2 + y^2 = 82,$
$9x + y = 82.$

30. $(x-1)^2 + (y-2)^2 = 10,$
$3x + y = 15.$

31. $4x^2 - 9y^2 = 108,$
$4x - 3y = 18.$

32. $x^2 + 4y^2 = 52,$
$x + 2y = 10.$

33. $7x^2 + 7y^2 + 12y = 115,$
$7x + 3y = -15.$

34. $13x^2 + 13y^2 + 15x + 27y = 188,$
$x - 6y + 10 = 0.$

35. $3x^2 + 16y^2 = 91,$
$x + 2y = 7.$

36. $5x^2 + 4xy = 9,$
$3x + 2y = 3.$

*37. $x^2 + y^2 + 8x - 6y + 16 = 0,$
$x^2 + y^2 + 9x - 7y + 25 = 0.$

38. $2x^2 + 5y^2 = 133,$
$3x^2 + 2y^2 = 62.$

39. $x^2 + y^2 + 9x + 21y + 32 = 0,$
$x^2 + y^2 + 11x + 23y + 34 = 0.$

* Eliminate x^2 and y^2 and get a simple equation.

40. $3x^2 + 7y^2 = 34,$
$2x^2 - 3y^2 = 15.$

41. $x^2 + y^2 - 2x - y = 20,$
$x^2 + y^2 - 4x - 2y = 15.$

42. $x^2 + y^2 = 53,$
$xy = 14.$

43. $3x^2 + 3y^2 + 13x + 5y + 12 = 0,$
$x^2 + y^2 + 6x + 4 = 0.$

44. $x^2 - y^2 = 12,$
$xy = 8.$

45. $31x^2 + 31y^2 + 67x + 25y = 432,$
$2x^2 + 2y^2 + 3x - 5y = 12.$

46. $10x^2 - 2xy - y^2 + 5 = 0,$
$15x^2 - 11xy + 2y^2 = 0.$

47. $10x^2 - 21xy + 170 = 0,$
$20x^2 - 33xy + 10y^2 = 0.$

48. $41x^2 - 46xy + 13y^2 = 4,$
$10x^2 - 11xy + 3y^2 = 0.$

49. $18x^2 - 11xy - 2y^2 = -12,$
$5x^2 - 20xy + 13y^2 = 17.$

50. $3xy - 7x^2 = 2,$
$13xy - 25x^2 - y^2 = 5.$

51. $6x^2 - 9xy + 4y^2 = 34,$
$25x^2 - 39xy + 15y^2 = 85.$

52. $8y^2 - 9xy = 20,$
$4x^2 - 17xy + 12y^2 = 24$

53. $18x^2 - 30xy + 7y^2 = 10,$
$13x^2 - 22xy + 5y^2 = 5.$

54. $11x^2 - 13xy + 4y^2 = 4,$
$21x^2 - 24xy + 7y^2 = 4.$

55. $3x^2 - y^2 = 2,$
$6x^2 + 3xy - 4y^2 + 1 = 0.$

56. $23y^2 - 24xy - 12x^2 = 128,$
$19y^2 - 22xy - 16x^2 = 64.$

57. $23x^2 - 20xy = 567,$
$23y^2 + 30xy - 23x^2 = 81.$

169. Symmetric equations. An expression is *symmetric* with respect to x and y if it is unaltered by interchanging x and y. Examples, $x^2 + xy + y^2$, $x^2y + xy^2$, $x^3 + y^3$.

General Method. If $x + y = s$ and $xy = p$, then $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$, etc., can be expressed in terms of s and p.

$$x^2 + y^2 = (x + y)^2 - 2xy = s^2 - 2p.$$
$$x^3 + y^3 = (x + y)^3 - 3xy(x + y) = s^3 - 3ps.$$
$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = (s^2 - 2p)^2 - 2p^2$$
$$= s^4 - 4ps^2 + 2p^2.$$

Example 1. Solve $x^3 + y^3 = 351$, $x^2y + xy^2 = 126$.

$$x^3 + y^3 = s^3 - 3ps. \quad \therefore s^3 - 3ps = 351, \qquad (1)$$
$$x^2y + xy^2 = xy(x + y) = ps. \quad \therefore ps = 126. \qquad (2)$$

Adding 3 times equation (2) to (1), $s^3 = 729$.

$$\therefore s = 9. \qquad (3)$$

Dividing (2) by (3), $p = 14$.

The solutions of the given equations are those of

$x + y = 9$, $xy = 14$. $\therefore x = 7$, $y = 2$, or $x = 2$, $y = 7$.

Example 2. Solve $x^4 + x^2y^2 + y^4 = 481,$ (1)

$$x^2 + xy + y^2 = 37. \quad (2)$$

Equation (1) may be written

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = 481. \quad (3)$$

$$\therefore 37(x^2 - xy + y^2) = 481.$$

Dividing by 37, $x^2 - xy + y^2 = 13.$ (4)

Solving (4) and (2) by method of § 168 or § 169,

$$x = \pm 3,\ y = \pm 4.$$

Example 3. Solve $x^2y^2 + 2\,xy = 8,$ (1)

$$x + y = 3. \quad (2)$$

Equation (1) may be written

$$(xy + 4)(xy - 2) = 0. \quad (3)$$

Solving, $\begin{cases} x + y = 3, \\ xy + 4 = 0. \end{cases}$; $\begin{cases} x + y = 3, \\ xy - 2 = 0. \end{cases}$.

The required solutions are (1, 2), (2, 1), (4, – 1), (– 1, 4).

EXERCISE 135

Solve and check:

1. $x + y = 7,$
   $x^3 + y^3 = 91.$

2. $(x + y)^2 + x + y = 56,$
   $xy = 10.$

3. $x^2 - xy + y^2 = 21,$
   $x^3 + y^3 = 189.$

4. $x^3 + y^3 = 65,$
   $x^2y + xy^2 = 20.$

5. $x^4 + y^4 = 257,$
   $xy = 4.$

6. $x^4 + y^4 = 257,$
   $x + y = 5.$

7. $x^4 + x^2y^2 + y^4 = 84,$
   $x^2 + xy + y^2 = 14.$

8. $3\,x^3 + 3\,y^3 = 28\,xy,$
   $x + y = 4.$

CHAPTER XVI

SURDS. FRACTIONAL EXPONENTS. RADICAL EQUATIONS, ETC.

170. If n is a positive integer and a is a positive rational number which is not a perfect nth power, then the $\sqrt[n]{a}$ is called a **surd** of the **nth order.**

Thus, $\sqrt{6}$ is a surd of the **second order,**

$\sqrt[3]{2}$ is a surd of the **third order.**

$\sqrt{1.44}$ is not a surd, for $(1.2)^2 = 1.44$.

171. Rules of reckoning.

I. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$. II. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

III. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$. (Chapter XII.)

172. A surd is in its simplest form when the radicand is an integer and is as small as possible.

Simplify (1) $\sqrt{63}$; (2) $\dfrac{5}{\sqrt{8}}$; (3) $\sqrt[6]{16}$.

1. $\sqrt{63} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$.

2. $\dfrac{5}{\sqrt{8}} = \dfrac{5\sqrt{2}}{\sqrt{8} \cdot \sqrt{2}} = \dfrac{5\sqrt{2}}{\sqrt{16}} = \dfrac{5\sqrt{2}}{4}$ or $\dfrac{5}{4}\sqrt{2}$.

3. $\sqrt[6]{16} = \sqrt[6]{4^2} = \sqrt[3]{\sqrt{4^2}} = \sqrt[3]{4}$.

The process in (2) is called **rationalizing the denominator.**

173. Like surds are rational multiples of the same surd factor.

$3\sqrt{2}$, $5\sqrt{2}$, $\frac{3}{2}\sqrt{2}$ are like surds.

174. A surd is said to be expressed as an **entire surd** when its coefficient is unity.

$$3\sqrt{7} = \sqrt{3^2} \cdot \sqrt{7} = \sqrt{9} \cdot \sqrt{7} = \sqrt{9 \cdot 7} = \sqrt{63}.$$

$\sqrt{63}$ is an entire surd.

EXERCISE 136

Simplify:

1. $\sqrt{24}$.
2. $\sqrt{72}$.
3. $\sqrt{50}$.
4. $\sqrt{343}$.
5. $\sqrt{6050}$.
6. $\sqrt{8820}$.
7. $\sqrt[3]{128}$.
8. $\sqrt[3]{375}$.
9. $\sqrt[3]{11664}$.
10. $\sqrt[4]{112}$.
11. $\sqrt[4]{162}$.
12. $\sqrt[4]{1875}$.
13. $\sqrt{a^3b}$.
14. $\sqrt{a^5b^7}$.
15. $\sqrt{18\,x^5y^6}$.
16. $\sqrt{\frac{1}{3}}$.
17. $\sqrt[3]{\frac{1}{3}}$.
18. $\sqrt{\frac{11}{50}}$.
19. $\sqrt[3]{\frac{7}{9}}$.
20. $\sqrt[3]{\frac{27}{5}}$.
21. $\sqrt[3]{\frac{1}{4}}$.
22. $\sqrt[n]{\frac{1}{2^{n-2}}}$.
23. $\sqrt[4]{4}$.
24. $\sqrt[8]{81}$.
25. $\sqrt[6]{100}$.
26. $\sqrt[4]{64}$.
27. $\sqrt[9]{125}$.

Express as entire surds:

28. $4\sqrt{2}$.
29. $3\sqrt{5}$.
30. $2\sqrt{7}$.
31. $2\sqrt[3]{4}$.
32. $6\sqrt[3]{2}$.
33. $3\sqrt[3]{5}$.
34. $\frac{1}{2}\sqrt{8}$.
35. $\frac{1}{3}\sqrt[3]{81}$.
36. $\frac{1}{2}\sqrt[4]{32}$.

Express as surds of the **sixth order**:

37. $\sqrt{2}$. **38.** $\sqrt[3]{4}$. **39.** $\sqrt{11}$.

Express as surds of the same order:

40. $\sqrt{2}, \sqrt[3]{3}$. **42.** $\sqrt{10}, \sqrt[3]{30}$. **44.** $\sqrt[3]{12}, \sqrt{5}$.

41. $\sqrt{7}, \sqrt[3]{4}$. **43.** $\sqrt[6]{80}, \sqrt{3}$. **45.** $\sqrt[4]{a^3}, \sqrt[5]{a^4}$.

175. Multiply (1) $2\sqrt{3}$ by $3\sqrt{5}$; (2) $\sqrt[3]{4}$ by $\sqrt[4]{3}$; (3) $3\sqrt{5}+2\sqrt{7}$ by $2\sqrt{5}-3\sqrt{7}$.

1. $2\sqrt{3}\cdot 3\sqrt{5} = 2\cdot 3\cdot\sqrt{3}\cdot\sqrt{5} = 6\sqrt{15}$.

2. $\sqrt[3]{4} = \sqrt[3]{\sqrt[4]{4^4}} = \sqrt[12]{4^4}$.

$\sqrt[4]{3} = \sqrt[4]{\sqrt[3]{3^3}} = \sqrt[12]{3^3}$.

Hence, $\sqrt[3]{4}\cdot\sqrt[4]{3} = \sqrt[12]{4^4}\cdot\sqrt[12]{3^3} = \sqrt[12]{4^4\cdot 3^3} = \sqrt[12]{6912}$.

3.
$$\begin{array}{rl} 3\sqrt{5}+2\sqrt{7} & \text{Multiply the multiplicand by } 2\sqrt{5}. \\ \underline{2\sqrt{5}-3\sqrt{7}} & \text{Multiply the multiplicand by } -3\sqrt{7}. \\ 30+4\sqrt{35} & \\ \underline{-9\sqrt{35}-42} & \text{Add the two results.} \\ -12-5\sqrt{35} & \end{array}$$

EXERCISE 137

Perform the indicated operations:

1. $\sqrt{2}\cdot\sqrt{8}$. **6.** $\sqrt[3]{16}\cdot\sqrt[3]{32}$. **11.** $\sqrt{3}\cdot\sqrt[3]{2}$.

2. $\sqrt{2}\cdot\sqrt{32}$. **7.** $\sqrt{8}\cdot\sqrt{6}$. **12.** $\sqrt[3]{4}\cdot\sqrt{8}$.

3. $\sqrt{3}\cdot\sqrt{48}$. **8.** $\sqrt{3}\cdot\sqrt{32}$. **13.** $\sqrt[3]{2}\cdot\sqrt[6]{2}$.

4. $\sqrt[3]{4}\cdot\sqrt[3]{2}$. **9.** $\sqrt{12}\cdot\sqrt{24}$. **14.** $\sqrt[3]{16}\cdot\sqrt[6]{128}$.

5. $\sqrt[3]{5}\cdot\sqrt[3]{25}$. **10.** $\sqrt{2}\cdot\sqrt[3]{3}$. **15.** $\sqrt{\frac{5}{7}}\cdot\sqrt{\frac{7}{125}}$.

16. $(1-\sqrt{2})(3+\sqrt{2})$.

17. $(4\sqrt{3}-6\sqrt{2})(4\sqrt{3}+5\sqrt{2})$.

18. $(10\sqrt{5}+6\sqrt{3})(5\sqrt{3}-3\sqrt{5})$.

19. $(2\sqrt{3}+3\sqrt{2})(3\sqrt{3}-\sqrt{2})$.

20. $(3\sqrt{7}-4\sqrt{5})(4\sqrt{7}-2\sqrt{5})$.

21. $(6\sqrt{6}-3\sqrt{10})(2\sqrt{3}-3\sqrt{5})$.

22. $(5\sqrt{5}-2\sqrt{15})(2\sqrt{5}-\sqrt{30})$.

23. $(4\sqrt{8}-3\sqrt{6})(3\sqrt{2}-\sqrt{6})$.

176. Rationalizing factors. If u and v are expressions involving surds, and if uv is rational, then either one is the rationalizing factor of the other.

1. $\sqrt{2}$ is the rationalizing factor of $\sqrt{8}$.
2. $\sqrt[3]{3}$ is the rationalizing factor of $\sqrt[3]{9}$.
3. $\sqrt{a}+\sqrt{b}$ is the rationalizing factor of $\sqrt{a}-\sqrt{b}$.

Example 1. Express $\dfrac{13\sqrt{5}-2}{3\sqrt{5}+4}$ with a rational denominator.

$$\frac{13\sqrt{5}-2}{3\sqrt{5}+4}=\frac{13\sqrt{5}-2}{3\sqrt{5}+4}\cdot\frac{3\sqrt{5}-4}{3\sqrt{5}-4}=\frac{203-58\sqrt{5}}{29}=7-2\sqrt{5}.$$

The process consists in multiplying both the numerator and denominator by the rationalizing factor of the denominator.

EXERCISE 138

Simplify:

1. $\dfrac{10}{\sqrt{2}}$.

2. $\dfrac{5}{\sqrt{10}}$.

3. $\dfrac{3}{\sqrt{18}}$.

4. $\dfrac{1}{5-2\sqrt{6}}$.

5. $\dfrac{1}{3-2\sqrt{2}}$.

6. $\dfrac{8}{\sqrt{11}+\sqrt{3}}$.

7. $\dfrac{40}{4\sqrt{3}+2\sqrt{2}}$.

8. $\dfrac{6}{2\sqrt{6}-3\sqrt{2}}$.

9. $\dfrac{7+5\sqrt{2}}{1+\sqrt{2}}$.

10. $\dfrac{7\sqrt{7}-8}{\sqrt{7}-2}$.

11. $\dfrac{5\sqrt{5}-3\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

12. $\dfrac{3\sqrt{3}-1}{\sqrt{3}-1}$.

13. $\dfrac{27+2\sqrt{2}}{3+\sqrt{2}}$.

14. $\dfrac{8+5\sqrt{5}}{2+\sqrt{5}}$.

15. $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

16. $\dfrac{2\sqrt{3}-2}{2\sqrt{3}+2}$.

17. $\dfrac{2\sqrt{10}+2\sqrt{2}}{\sqrt{10}-\sqrt{2}}$.

18. $\dfrac{29+12\sqrt{5}}{3+2\sqrt{5}}$.

19. What is the rationalizing factor of $\sqrt{3}-\sqrt{2}$?

HINT. Use the identity $x^3-y^3=(x-y)(x^2+xy+y^2)$.

20. What is the rationalizing factor of $\sqrt[3]{3}+\sqrt[3]{2}$?

21. Use the identity, $a^3+b^3=(a+b)^3-3ab(a+b)$ to find the value of

(a) $(\sqrt{2}+1)^3+(\sqrt{2}-1)^3$.

(b) $(\sqrt{1+x}+\sqrt{1-x})^3+(\sqrt{1+x}-\sqrt{1-x})^3$.

22. Given $a^3b^3=(ab)^3$, express in its simplest form

(a) $(\sqrt{2}+1)^3(\sqrt{2}-1)^3$.

(b) $(1+\sqrt{1-x})^3(1-\sqrt{1-x})^3$.

23. Simplify $\dfrac{1}{1+\sqrt{1-x}}+\dfrac{1}{1-\sqrt{1-x}}$.

24. Rationalize and find to four places of decimals the value of:

(a) $\dfrac{1}{\sqrt{2}+1}$.

(b) $\dfrac{1}{2+\sqrt{3}}$.

(c) $\dfrac{1}{8-3\sqrt{7}}$.

(d) $\dfrac{1}{9-4\sqrt{5}}$.

(e) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

(f) $\dfrac{2\sqrt{6}-3\sqrt{2}}{2\sqrt{6}+3\sqrt{2}}$.

25. Find the value of $x^2-6x-10$ when $x=3-\sqrt{19}$.

177. The square root of a binomial quadratic surd.

If a and b are positive rational numbers, then

$$(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}.$$
$$(\sqrt{a}-\sqrt{b})^2 = a+b-2\sqrt{ab}.$$

Hence the square of a binomial quadratic surd is of the form $m \pm 2\sqrt{n}$, where m and n are rational. Conversely,

$$\sqrt{m+2\sqrt{n}} = \sqrt{a}+\sqrt{b},$$
$$\sqrt{m-2\sqrt{n}} = \sqrt{a}-\sqrt{b},$$

and $a+b=m$, and $2\sqrt{ab}=2\sqrt{n}$ or $ab=n$.

Example 1. Extract the square root of $16-5\sqrt{7}$.

$$\sqrt{16-5\sqrt{7}} = \sqrt{a}-\sqrt{b}.$$

$$\therefore\ a+b=16 \text{ and } 2\sqrt{ab}=5\sqrt{7}, \text{ or } 4\,ab=175.$$

Solving, $\quad a+b=16$ and $4\,ab=175$, (§ 167)

$$a=\tfrac{25}{2},\quad b=\tfrac{7}{2}.$$

$$\therefore\ \sqrt{16-5\sqrt{7}} = \frac{5}{\sqrt{2}}-\frac{\sqrt{7}}{\sqrt{2}} = \frac{1}{\sqrt{2}}(5-\sqrt{7}).$$

Alternative solution:

$$\sqrt{16-5\sqrt{7}} = \sqrt{a}-\sqrt{b}, \qquad (1)$$

$$\sqrt{16+5\sqrt{7}} = \sqrt{a}+\sqrt{b}. \qquad (2)$$

$\therefore\ \sqrt{16^2-5^2\cdot 7} = a-b.$ Multiplying (1) by (2),

i.e. $\quad 9 = a-b.$

But $\quad 16 = a+b.$

$$\therefore\ 25 = 2\,a \text{ or } a=\tfrac{25}{2},$$

and $\quad b=\tfrac{7}{2}.$

EXERCISE 139

Extract the square root of:

1. $31 + 12\sqrt{3}$.
2. $24 + 16\sqrt{2}$.
3. $21 - 12\sqrt{3}$.
4. $30 - 12\sqrt{6}$.
5. $37 - 8\sqrt{21}$.
6. $33 - 6\sqrt{10}$.
7. $60 - 40\sqrt{2}$.
8. $\frac{5}{4} + \frac{1}{2}\sqrt{6}$.
9. $2 + \frac{1}{2}\sqrt{15}$.
10. $3 + \frac{4}{3}\sqrt{5}$.
11. $3 + \sqrt{5}$.
12. $\frac{3}{2} + \frac{1}{2}\sqrt{5}$.
13. $21 + 12\sqrt{3}$.
14. $17 + \sqrt{288}$.
15. $2\frac{5}{6} + \sqrt{8}$.
16. Express $\sqrt{(3\sqrt{6} + 4\sqrt{3})}$ in the form $\sqrt[4]{a} \cdot (\sqrt{m} - \sqrt{n})$.

178. Index laws. The following laws hold for all values of m and n:

1. $a^m \cdot a^n = a^{m+n}$. 2. $(a^m)^n = a^{mn}$. 3. $(abc)^n = a^n b^n c^n$.

In agreement with these laws are the following definitions:

1. $a^{\frac{m}{n}}$ is defined as $\sqrt[n]{a^m}$. (§ 138)
2. a^0 is defined as 1, unless a is zero.
3. a^{-n} is defined as $\frac{1}{a^n}$ or $\left(\frac{1}{a}\right)^n$.

Example 1. Find the value of $(0.008)^{-\frac{4}{3}}$.

$$(0.008)^{-\frac{4}{3}} = \{(0.2)^3\}^{-\frac{4}{3}} = (0.2)^{-4} = \left(\frac{1}{0.2}\right)^4 = 5^4 = 625.$$

Example 2. Express in an integral form $\frac{a^{-3}b^2}{c^{-3}a^{-4}}$.

$$\frac{a^{-3}b^2}{c^{-3}a^{-4}} = \frac{ab^2}{c^{-3}} = \frac{ab^2c^0}{c^{-3}} = ab^2c^3. \quad (§\ 39)$$

Example 3. Express $\sqrt{1000}$ in the form 100^x.

$$\sqrt{1000} = \sqrt{10^3} = 10^{\frac{3}{2}}. \qquad 100^x = (10^2)^x = 10^{2x}.$$

$$\therefore\ 10^{2x} = 10^{\frac{3}{2}}. \qquad \therefore\ 2x = \tfrac{3}{2}, \quad i.e.\ x = \tfrac{3}{4}.$$

Hence, $$\sqrt{1000} = 100^{\frac{3}{4}}.$$

Example 4. Simplify $\{\sqrt{(x^{-\frac{4}{3}}y)^6}\}^{-\frac{1}{2}}$.

$$\{\sqrt{(x^{-\frac{4}{3}}y)^6}\}^{-\frac{1}{2}} = \{(x^{-\frac{4}{3}}y)^3\}^{-\frac{1}{2}} = (x^{-\frac{4}{3}}y)^{-\frac{3}{2}} = x^2y^{-\frac{3}{2}}.$$

Example 5. Express with fractional exponents in simplest form:

$$\sqrt[15]{a} \div (\sqrt[5]{a} \cdot \sqrt[3]{a}),$$

$$\sqrt[15]{a} \div (\sqrt[5]{a} \cdot \sqrt[3]{a}) = a^{\frac{1}{15}} \div (a^{\frac{1}{5}} \cdot a^{\frac{1}{3}}) = a^{\frac{1}{15}-\frac{1}{5}-\frac{1}{3}} = a^{-\frac{7}{15}} \text{ or } \frac{1}{a^{\frac{7}{15}}}.$$

EXERCISE 140

Express with radical signs in simplest form:

1. $a^{\frac{1}{2}}b$.
3. $a^{\frac{1}{3}}b^{\frac{1}{4}}$.
5. $7x^{\frac{1}{4}}$.
7. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}}$.
2. $a^{\frac{1}{3}}c$.
4. $a^{\frac{2}{3}}b$.
6. $3a^2b^{\frac{1}{5}}$.
8. $3^{\frac{1}{2}} \cdot 3^{\frac{1}{3}}$.

Express with fractional exponents:

9. $\sqrt{x}$.
10. $\sqrt[3]{ab^2}$.
11. $\sqrt[4]{ab^3}$.
12. $\sqrt[5]{ab^2c^3}$.

Find the numerical value of:

13. $27^{\frac{2}{3}}$.
19. $64^{\frac{5}{6}}$.
25. 10^{-3}.
31. $(.001)^{\frac{2}{3}}$.
14. $16^{\frac{3}{4}}$.
20. $32^{\frac{3}{5}}$.
26. $(0.01)^{\frac{1}{2}}$.
32. $(.001)^{-\frac{2}{3}}$.
15. $8^{\frac{4}{3}}$.
21. $81^{-\frac{1}{4}}$.
27. $(0.09)^{-\frac{1}{2}}$.
33. $(.027)^{-\frac{1}{3}}$.
16. $25^{\frac{3}{2}}$.
22. 10^0.
28. $(0.16)^{-\frac{1}{2}}$.
34. $(.0016)^{\frac{1}{4}}$.
17. $49^{-\frac{1}{2}}$.
23. 10^{-1}.
29. $(abc)^0$.
35. $(.343)^{\frac{2}{3}}$.
18. $36^{-\frac{3}{2}}$.
24. 10^{-2}.
30. $(\frac{1}{2})^{-2}$.
36. $(\frac{1}{8})^{-\frac{2}{3}}$.

Express in an integral form:

37. $\frac{1}{a^{\frac{1}{2}}}$. **39.** $\frac{a^2}{b^{-3}}$. **41.** $\frac{mn}{x^{-2}y^2}$. **43.** $\frac{\sqrt{a}}{\sqrt{b^2}}$.

38. $\frac{x^2}{y-2}$. **40.** $\frac{1}{(a+b)^{-2}}$. **42.** $\sqrt{\frac{a}{b^3}}$.

44. Express as a power of 10: (*a*) $\sqrt{1000}$; (*b*) .01; (*c*) $\frac{1}{100^2}$.

45. Express as a power of 8: (*a*) $\frac{1}{4}$; (*b*) .0625; (*c*) $\sqrt{2}$.

46. Express in the form a^x: (*a*) $2\sqrt{5}$; (*b*) $2^{\frac{1}{3}}+3^{\frac{1}{2}}$; (*c*) $\frac{\sqrt{2}}{\sqrt[3]{4}}$.

47. Express n as a power of n^2; of $\sqrt{n}$; of $n\sqrt[3]{n}$.

Express in simplest exponential form:

48. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$. **51.** $\left(\frac{x^a}{x^{-2a}}\right)^{\frac{1}{3a}}$. **54.** $\left(\frac{x^4}{(a+b)^6}\right)^{-\frac{1}{2}}$.

49. $\left(\frac{8a^3}{b^3}\right)^{-\frac{2}{3}}$. **52.** $(x^m \cdot x^n \cdot x^r)^{\frac{1}{m+n+r}}$. **55.** $(a^2-b^2)^{\frac{1}{2}}\sqrt{a^2+b^2}$.

50. $\left(\frac{x^4}{16b^4}\right)^{-\frac{3}{4}}$. **53.** $\left(\frac{4}{(a+b)^2}\right)^{-\frac{1}{2}}$. **56.** $\sqrt{a\sqrt{a\sqrt{a}}}$.

179. The descending order of magnitude for negative numbers is -1, -2, -3, -4, etc. Thus a^0 is a higher power of a than a^{-1}, and a^{-1} is a higher power than a^{-2}.

The following expression is arranged in descending powers of x:

$$ax^{\frac{1}{2}} + b + cx^{-\frac{1}{2}} + dx^{-1} + ex^{-\frac{3}{2}}.$$

Example 1. Divide

$$4+\frac{2}{\sqrt{x}}+\frac{3}{\sqrt{x}}+\sqrt{x}+2\sqrt[4]{x} \text{ by } 1+\frac{1}{\sqrt{x}}.$$

Writing the expressions in exponential form, and arranging the terms in descending order, we have

$$\begin{array}{l} 1+x^{-\frac{1}{2}})\,x^{\frac{1}{2}}+2x^{\frac{1}{4}}+4+2x^{-\frac{1}{4}}+3x^{-\frac{1}{2}} \\ \qquad\quad \underline{x^{\frac{1}{2}}+1} \\ \qquad\qquad\quad 2x^{\frac{1}{4}}+3+2x^{-\frac{1}{4}} \\ \qquad\qquad\quad \underline{2x^{\frac{1}{4}}\qquad\;\; 2x^{-\frac{1}{4}}} \\ \qquad\qquad\qquad\quad 3+3x^{-\frac{1}{2}} \\ \qquad\qquad\qquad\quad \underline{3+3x^{-\frac{1}{2}}} \end{array}$$

Example 2. Given $(1+x)^n=1+nx$ (approximately) if x is small, find the value of $\sqrt[3]{1.007}$, correct to four places of decimals.

$$\sqrt[3]{1.007}=(1.007)^{\frac{1}{3}}=1+\tfrac{1}{3}(.007)=1.0023.$$

EXERCISE 141

Expand:

1. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$.
2. $(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2$.
3. $(x+x^{-1})^2$.
4. $(e^x+e^{-x})^2$.
5. $(e^x-e^{-x})^2$.
6. $(e^{ix}+e^{-ix})^2$.
7. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.
8. $(x^{\frac{1}{3}}-1)(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)$.
9. $(x^{\frac{1}{3}}+1)(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)$.
10. $(x+1+x^{-1})(x-1+x^{-1})$.
11. $(x^2+x^{-2})^3$.
12. $(x^{2n}+x^ny^n+y^{2n})(x^{2n}-x^ny^n+y^{2n})$.

Divide:

13. $x^n - 1$ by $x^{\frac{1}{2}n} + 1$.

14. $x^{3n} - 1$ by $x^n - 1$.

15. $x - 1$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$.

16. $x + 1$ by $\sqrt[3]{x^2} - \sqrt[3]{x} + 1$.

17. $x - 2x^{\frac{1}{2}} + 1$ by $x^{\frac{1}{3}} + x^{\frac{1}{6}} + 1$.

18. $27x^2 - 98x - 125$ by $3x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 5$.

19. $x^2 + 10x + 27$ by $\sqrt[3]{x^2} - 2\sqrt[3]{x} + 3$.

20. Show that $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}})(a^{\frac{1}{2}} + c^{\frac{1}{2}} - b^{\frac{1}{2}})(b^{\frac{1}{2}} + c^{\frac{1}{2}} - a^{\frac{1}{2}}) = 2ab + 2ac + 2bc - a^2 - b^2 - c^2$, and from this identity determine the rationalizing factor of $\sqrt{a} + \sqrt{b} + \sqrt{c}$.

21. Given $(1 - x)^n = 1 - nx$ (approximately) if x is small, find correct to 4 places of decimals $\sqrt{.9967}$.

22. Extract the square root of $x^{-2} + 4x^{-\frac{3}{2}} + 6x^{-1} + 4x^{-\frac{1}{2}} + 1$.

23. Solve $(9^x - 1)(9^x - 3) = 0$.

24. Find the value of $x^2 - 4x - 6$ if $x = 2 + 10^{\frac{1}{2}}$.

25. Simplify $\{(a + b)^{\frac{1}{2}} + a^{\frac{1}{2}} - b^{\frac{1}{2}}\}\{(a + b)^{\frac{1}{2}} - a^{\frac{1}{2}} + b^{\frac{1}{2}}\}$.

180. Irrelevant or extraneous roots.

Example 1. Solve $\sqrt{7x - 12} + \sqrt{22 - 7x} = \sqrt{2x - 2}$.

Squaring,

$$7x - 12 + 22 - 7x + 2\sqrt{(7x - 12)(22 - 7x)} = 2x - 2.$$

Expanding, transposing, and dividing by 2,

$$\sqrt{-49x^2 + 238x - 264} = x - 6.$$

Squaring, $$-49x^2 + 238x - 264 = x^2 - 12x + 36.$$

$$\therefore -50x^2 + 250x - 300 = 0.$$

Dividing by -50, $$x^2 - 5x + 6 = 0.$$

$$\therefore (x - 2)(x - 3) = 0.$$

$$x = 2 \text{ or } 3.$$

Substituting 2 and 3 for x in the original equation, we get

$$\sqrt{2} + \sqrt{8} = \sqrt{2}.$$

$$3 + 1 = 2.$$

Hence neither 2 nor 3 is a solution. The values 2 and 3 satisfy respectively the equations

$$\sqrt{22 - 7x} - \sqrt{7x - 12} = \sqrt{2x - 2}.$$

$$\sqrt{7x - 12} - \sqrt{22 - 7x} = \sqrt{2x - 2}.$$

***Example* 2.** Clear of radicals,

$$\sqrt{x} + \sqrt{y} - \sqrt{z} = 0.$$

$$\sqrt{x} + \sqrt{y} = \sqrt{z}.$$

Squaring, $$x + y + 2\sqrt{xy} = z,$$

$$i.e.\ 2\sqrt{xy} = z - x - y.$$

Squaring and transposing,

$$2xy + 2yz + 2zx - x^2 - y^2 - z^2 = 0.$$

This equation is identical with

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})(\sqrt{x} + \sqrt{y} - \sqrt{z})(\sqrt{z} + \sqrt{x} - \sqrt{y})$$
$$(\sqrt{y} + \sqrt{z} - \sqrt{x}) = 0,$$

(Example 20, Exercise 141)

which is itself equivalent to the four equations

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 0, \qquad \sqrt{x} + \sqrt{y} - \sqrt{z} = 0,$$

$$\sqrt{x} + \sqrt{z} - \sqrt{y} = 0, \qquad \sqrt{y} + \sqrt{z} - \sqrt{x} = 0.$$

This example shows clearly that the process of squaring both sides of an equation may introduce roots that do not satisfy the original equation. Such roots are known as *irrelevant* or *extraneous* roots.

Example 3. Solve $4x^{\frac{2}{3}} + 9x^{-\frac{2}{3}} = 37$.

Multiplying by $x^{\frac{2}{3}}$, $4x^{\frac{4}{3}} + 9 = 37x^{\frac{2}{3}}$.

$$\therefore 4x^{\frac{4}{3}} - 37x^{\frac{2}{3}} + 9 = 0.$$

$$\therefore (4x^{\frac{2}{3}} - 1)(x^{\frac{2}{3}} - 9) = 0.$$

$$\therefore 4x^{\frac{2}{3}} - 1 = 0, \quad \text{or } x^{\frac{2}{3}} - 9 = 0.$$

$$\therefore x^{\frac{2}{3}} = \tfrac{1}{4}, \quad \text{or} \quad x^{\frac{2}{3}} = 9.$$

Taking square root, $x^{\frac{1}{3}} = \pm\tfrac{1}{2}$, or $x^{\frac{1}{3}} = \pm 3$.

Cubing, $x = \pm\tfrac{1}{8}$, or $x = \pm 27$.

Example 4. Solve $x^2 + 5x + 3\sqrt{x^2 + 5x - 5} = 23$.

Let $y = \sqrt{x^2 + 5x - 5}$, then $y^2 + 5 = x^2 + 5x$.

$$\therefore y^2 + 5 + 3y = 23.$$

$$\therefore y^2 + 3y - 18 = 0.$$

$$\therefore (y + 6)(y - 3) = 0.$$

$$\therefore y = 3, \text{ or } -6,$$

i.e. $\sqrt{x^2 - 5x - 5} = 3$, or 6.

Solving, $x = 2$, or -7. The other roots are extraneous.

EXERCISE 142

Solve and check:

1. $x^{\frac{2}{3}} = 9$.
2. $y^{-\frac{1}{2}} = \frac{1}{3}$.
3. $x^{-\frac{3}{2}} = 8$.
4. $\sqrt[3]{9x} = 3\sqrt[3]{2}$.
5. $\sqrt[3]{10x^2} = 2\sqrt[3]{5}$.
6. $x^{\frac{m}{2}} = 2^m$.
7. $3x + 2x^{\frac{1}{2}} = 1$.
8. $x^{-2} - 2x^{-1} = 8$.
9. $x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 3$.
10. $3x - 4x^{\frac{1}{2}} + 1 = 0$.
11. $x^{2n} + 4x^n - 5 = 0$.
12. $x + 3 + 2\sqrt{x+3} = 15$.
13. $x - 4x^{\frac{3}{4}} + 3x^{\frac{1}{2}} = 0$.
14. $x + 4 + \sqrt{x+4} = 12$.

15. $4x\sqrt{x} - 8x + 3\sqrt{x} = 0.$

16. $4x + 13 + \sqrt{4x + 13} = 30.$

17. $\sqrt{x - 8} + \sqrt{x + 8} = 8.$

18. $\sqrt{2x + 10} + \sqrt{x + 6} = \sqrt{2x + 11}.$

19. $\sqrt{x} + \sqrt{x - 3} = \sqrt{2x + 1}.$

20. $\sqrt{3x + 1} + \sqrt{2x - 7} = \sqrt{5x + 24}.$

21. $\sqrt{5x - 1} - \sqrt{2x + 5} = \sqrt{x - 6}.$

22. $\sqrt{5 - 2x} - \sqrt{5 + 2x} = \sqrt{2 - x}.$

23. $\sqrt{3x + 3} - \sqrt{2x + 3} = \sqrt{x - 10}.$

24. $\sqrt{2x + 1} - \sqrt{2x - 4} = 1.$

25. $\sqrt{x + 1} + \sqrt{x - 8} = \sqrt{4x + 1}.$

26. $\sqrt{3x + 1} - \sqrt{2x - 1} = 1.$

27. $\sqrt{x^2 - 6} + \sqrt{x^2 - 1} = \sqrt{4x^2 - 15}.$

28. $1 - \sqrt{x} = \sqrt{x - \sqrt{1 - x}}.$

29. $\sqrt{x^3 + 1} + \sqrt{2x^3 + 9} = 2\sqrt{x^3 + 8}.$

30. $2\sqrt{1 + \sqrt{x}} = 3.$

31. $x^2 - 5x - \sqrt{x^2 - 5x} = 30.$

32. $x^2 - 6x + \sqrt{x^2 - 6x + 12} + 6 = 0.$

33. $3x^2 - 16x + 8\sqrt{3x^2 - 16x + 21} = 7.$

34. $2x^2 - 6x + 2\sqrt{x^2 - 3x + 5} = 14.$

35. $3x^2 + 3x - \sqrt{x^2 + x + 7} = 119.$

36. $x^2 - 3x + 3\sqrt{2x^2 - 6x + 28} = 42.$

CHAPTER XVII

RATIO, PROPORTION, VARIATION, INEQUALITIES

181. If a, b, c, d are in proportion, *i.e.* $a : b = c : d$, then d is the **fourth proportional** to a, b, c.

If a, b, c are quantities such that $a : b = b : c$, then c is the **third proportional** to a and b, and b is the **mean proportional** to a and c.

182. If a, b, c, d $\cdots$ are such that

$$a : b = b : c = c : d, \cdots$$

then the quantities are said to be in **continued proportion.**

183. Theorems in ratio and proportion.

1. If $a : b = c : d$, then $ad = bc$.

Conversely if $ad = bc$, then $a : b = c : d$.

For if $\frac{a}{b} = \frac{c}{d}$, multiply by bd and it follows $ad = bc$.

Conversely if $ad = bc$, divide by bd and it follows $\frac{a}{b} = \frac{c}{d}$.

2. If $a : b = c : d$, then $b : a = d : c$. (Inversion)

For if $a : b = c : d$, then $bc = ad$. (Th. 1)

Divide these equals by ac and it follows $\frac{b}{a} = \frac{d}{c}$.

3. If $a : b = c : d$, then $a : c = b : d$. (Alternation)

For if $a : b = c : d$, then $ad = bc$. (Th. 1)

Divide these equals by cd and it follows $\frac{a}{c} = \frac{b}{d}$.

4. If $a : b = c : d$, then $a + b : b = c + d : d$.

(Composition)

For if $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$, *i.e.* $\frac{a+b}{b} = \frac{c+d}{d}$.

5. If $a : b = c : d$, then $a - b : b = c - d : d$. (Division)

For if $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$, *i.e.* $\frac{a-b}{b} = \frac{c-d}{d}$.

6. If $a : b = c : d$, then $a + b : a - b = c + d : c - d$. (Composition and Division)

For $\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1}$, and $\frac{c+d}{c-d} = \frac{\frac{c}{d}+1}{\frac{c}{d}-1}$, but $\frac{a}{b} = \frac{c}{d}$.

$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}$. (Ax. 1)

7. If $a : b = c : d = e : f$, then $a + c + e : b + d + f = a : b$.

If a number of ratios are equal, then the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$, then $a = br$, $c = dr$, $e = fr$.

$$\therefore \frac{a+c+e}{b+d+f} = \frac{br+dr+fr}{b+d+f} = r, \text{ but } r = \frac{a}{b}.$$

$$\therefore \frac{a+c+e}{b+d+f} = \frac{a}{b}, \text{ or } a+c+e : b+d+f = a : b.$$

184. *Examples.* (*a*) If $6x^2 - 11xy - 10y^2 = 0$, find $x : y$.

Let $\frac{x}{y} = r$, then $x = yr$. Substitute yr for x,

$$6y^2r^2 - 11y^2r - 10y^2 = 0.$$

$\therefore\ 6r^2 - 11r - 10 = 0$, since y is not zero.

Solving, $r = 2\frac{1}{2}$, or $-\frac{2}{3}$.

(*b*) If $\dfrac{x}{y+z-x} = \dfrac{y}{z+x-y} = \dfrac{z}{x+y-z}$,

prove each ratio $= 1$.

$$\text{Each ratio} = \frac{x+y+z}{(y+z-x)+(z+x-y)+(x+y-z)} = 1.$$

(Th. 7, § 183)

(*c*) If $a : b = c : d$, prove

$a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.

Let $\dfrac{a}{b} = \dfrac{c}{d} = r$, then $a = br$, $c = dr$.

$$\therefore\ \frac{a^2+ab+b^2}{a^2-ab+b^2} = \frac{b^2r^2+b^2r+b^2}{b^2r^2-b^2r+b^2} = \frac{r^2+r+1}{r^2-r+1},$$

and
$$\frac{c^2+cd+d^2}{c^2-cd+d^2} = \frac{d^2r^2+d^2r+d^2}{d^2r^2-d^2r+d^2} = \frac{r^2+r+1}{r^2-r+1}.$$

$\therefore\ a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.

(Ax. 1)

(*d*) If $a : b = b : c = c : d$, then $a^2 : d^2 = a^3 : c^3$.

Let $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = r$, then $c = dr$, $b = dr^2$, $a = dr^3$.

$$\therefore\ \frac{a^2}{d^2} = \frac{d^2r^6}{d^2} = r^6 \text{ and } \frac{a^3}{c^3} = \frac{d^3r^9}{d^3r^3} = r^6.$$

$\therefore\ a^2 : d^2 = a^3 : c^3$. (Ax. 1)

(*e*) Find the mean proportional between a^3b and ab^3.

$a^3b : x = x : ab^3$. $\therefore\ x^2 = a^4b^4$. $\therefore\ x = \pm a^2b^2$.

(*f*) If $a : b = 4 : 7$ and $b : c = 6 : 11$, find $a : c$.

$$\frac{a}{b} = \frac{4}{7},\ \frac{b}{c} = \frac{6}{11}.\quad \therefore\ \frac{a}{b}\cdot\frac{b}{c} = \frac{4}{7}\cdot\frac{6}{11}.\quad \therefore\ \frac{a}{c} = \frac{24}{77}.$$

Notation $a : b : c = x : y : z$ is equivalent to $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$.

EXERCISE 143

1. Find the fourth proportional to 2, 3, 5.

2. Find the third proportional to c, d; to $1-x$, $1-x^2$.

3. If 2, 3 are the first two of four numbers in continued proportion, find the other two.

4. Find the ratio of a speed of 36 miles an hour to one of 44 feet per second.

5. The side of a square is a. Find the ratio of a diagonal to a side of the square.

6. If $4x-2y:2x+y=4:5$, find $x:y$.

7. If $5x^2-11xy+2y^2=0$, find $x:y$.

8. A wheel 7 feet in diameter makes 160 revolutions a minute; find in miles per hour the speed of a point on its rim. Circumference : diameter = 22 : 7.

9. If $a:b=b:c$, show that $a:c=a^2:b^2$.

10. Find the mean proportional to $\sqrt{6}$ and $12+5\sqrt{6}$.

11. If $3x+4y:7x-3y=18:5$, find

$$x^2-3xy+4y^2:x^2+3xy+4y^2.$$

12. Three numbers are related as 3 : 5 : 8, and the sum of the first and third exceeds twice the second by 8. Find them.

13. The sum, difference, and the difference of the squares of two numbers are in the ratios 5 : 1 : 20. Find the numbers.

14. Three numbers are in continued proportion. Their sum is 28 and the sum of their squares is 336. Find them.

15. What angle do the hands of a clock make at 48 minutes past 5 o'clock? at 30 minutes past 2 o'clock?

16. If $a : b = b : c$, prove the relations:

(a) $(a^2 + b^2)(b^2 + c^2) = (ab + bc)^2$.

(b) $a^2 + b^2 : b^2 + c^2 = a : c$.

(c) $(a + b)^2(b + c)^2 = (ab + bc + 2\,ac)^2$.

17. If $a : b = c : d$, prove the relations:

(a) $c^2(a + b)(b - d) = ad(a - c)(c + d)$.

(b) $ma + nb : pa + qb = mc + nd : pc + qd$.

(c) $a^2 + c^2 : ab + cd = ab + cd : b^2 + d^2$.

(d) $a + b + c + d : a - b + c - d = a + b - c - d : a - b - c + d$.

(e) $a + c : b + d = \sqrt{ac} : \sqrt{bd}$.

18. If $\dfrac{by + cz}{b^2 + c^2} = \dfrac{ax + cz}{a^2 + c^2} = \dfrac{ax + by}{a^2 + b^2}$, prove $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

19. If $\dfrac{x}{a - b} = \dfrac{y}{b - c} = \dfrac{z}{c - a}$, prove $x + y + z = 0$.

185. Direct variation.

If the speed of a train is 42 miles an hour, in 3 minutes it will go 2.1 miles, in 4 minutes 2.8 miles, in 5 minutes 3.5 miles.

Consider the two sets of quantities

3	4	5
2.1	2.8	3.5

A number in the second set is obtained from the corresponding one in the first set by multiplying by 0.7. Hence,

$$3 : 4 : 5 = 2.1 : 2.8 : 3.5.$$

One quantity y varies as another quantity x if pairs of corresponding values are proportional to each other; in other words, **y varies as x** if $\frac{y}{x}$ = a constant quantity, *i.e.* $y = cx$ (c, a constant).

186. Inverse variation.

If the area of a rectangle is 40 square feet, then if the base is 4 feet the height is 10 feet, if the base is 5 feet the height is 8 feet, and if the base is 6 feet the height is $6\frac{2}{3}$ feet.

Consider the two sets of quantities

4	5	6
10	8	$6\frac{2}{3}$

The product of corresponding pairs is constant.

One quantity y varies inversely as another quantity x if

$$xy = c, \text{ or } y = \frac{c}{x} \ (c, \text{ a constant}).$$

The symbol to denote variation is $\propto$. Thus $y \propto x$ and $y = cx$ denote precisely the same relation.

187. If $y \propto xz$ or $y = cxz$, y is said to **vary jointly** as x and z.

ILLUSTRATION. The area of a triangle varies jointly as its base and altitude.

If $y \propto \frac{x}{z}$, or $y = C \cdot \frac{x}{z}$, y is said to vary directly as x and inversely as z.

ILLUSTRATION. The time a train takes to run a certain distance varies directly as the distance and inversely as the speed.

188. THEOREM. **If $y \propto x$ when z is constant and if $y \propto z$ when x is constant, then $y \propto xz$ when both x and z vary.**

ILLUSTRATION. Parallelograms of equal altitudes are proportional to their bases; parallelograms of equal bases are proportional to their altitudes, and any two parallelograms having a common angle are proportional to the product of the sides about the common angle.

Example 1. The area of a polygon (A sq. ft.) varies as the square of a side (s ft.). If $A = 108$ when $s = 4$, find A in terms of s; also find A when $s = 12$.

$A \propto s^2$. $\therefore A = cs^2$. $\therefore 108 = c \cdot 4^2$. $\therefore c = 6.75$.

Hence $A = 6.75\, s^2$.

When $s = 12$, $A = 6.75 \times 12^2 = 972$.

Example 2. If y is the sum of two quantities, one of which is proportional to x and the other to x^2, and when $x = 3$, $y = 159$, and when $x = 5$, $y = 425$, find y when $x = 10$.

$$y = ax + bx^2. \quad (a \text{ and } b \text{ constants}) \qquad \text{(I)}$$

Hence, $159 = a \cdot 3 + b \cdot 3^2$, or $3a + 9b = 159$,

$425 = a \cdot 5 + b \cdot 5^2$, or $5a + 25b = 425$.

Solving for a and b, $a = 5$, $b = 16$.

Substitute 10 for x, 5 for a, and 16 for b in (I) and $y = 1650$.

EXERCISE 144

1. The area of a circle whose radius is 10 is 314.16. Find the area of a circle whose radius is 21. $A \propto r^2$.

2. If the side of a regular pentagon is 10, its area is 172.05. Find the area of a regular pentagon, side 15. Area $\propto$ side^2.

3. The volume of a sphere, radius 16 inches is 17157 cubic inches. Find the volume of a sphere, radius 13 inches. $V \propto r^3$.

4. A body falls down an inclined plane 45 feet in 3 seconds. How many feet would it fall in 7 seconds? Distance $\propto$ (time)^2.

5. Two parallelograms have a common angle, and their adjacent sides are respectively 14 and 15, and 16 and 28. The area of the former is 96; find that of the latter.

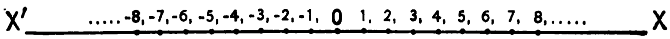
6. A body weighs 180 pounds on the earth's surface. How many pounds would it weigh 500 miles above the earth's surface? Weight is inversely proportional to the square of the distance from the center of the earth. Radius = 4000 miles.

7. Neptune is 30.05 times as far from the sun as the earth. Find its period of revolution. (Period)^2 $\propto$ (distance)^3. Earth's = 365 da.

8. A pendulum vibrates 5 times in 4 seconds; find its length, a second's pendulum being 39.37 inches and $T \propto \sqrt{\text{length}}$.

9. The earth's diameter is 7920 miles, and the moon's, 2160 miles; the earth's mass is 80 times that of the moon, and the value of g on the earth's surface is 32.2 feet. Find g on the moon's surface. $g \propto \frac{\text{mass}}{(\text{radius})^2}$.

189. The points of an unlimited straight line may represent all *real* numbers. To show this, take an unlimited straight line $X'X$ and mark on it a point 0, then take any convenient unit of length and measure off on the line parts equal to this unit, marking them as indicated.



Positive and negative fractions occupy intermediate positions. Thus $2\frac{3}{4}$ lies between 2 and 3. Every point on $X'X$ represents some number, and every number is represented by a point.

A number a is greater than a number b $(a > b)$ if a follows b in this scale. A number a is less than a number b $(a < b)$ if a precedes b in the number scale.

Thus $$-7 < -4 < 0 < 1.$$

From this statement it readily follows that if:

1. $a > b > c$, then $a > c$.
2. $a < b < c$, then $a < c$.
3. $a > b$, then $a \pm n > b \pm n$.
4. $a < b$, then $a \pm n < b \pm n$.
5. $a > b$ and $c > d$, then $a + c > b + d$.
6. $a < b$ and $c < d$, then $a + c < b + d$.
7. $a > b$ and $c > 0$, then $ac > bc$.
8. $a < b$ and $c > 0$, then $ac < bc$.

On the other hand, if c is a negative number, *i.e.* if

9. $a > b$ and $c < 0$, then $ac < ab$.
10. $a < b$ and $c < 0$, then $ac > ab$.

Multiplication by a negative number reverses the sign of inequality, or, as it is sometimes said, it changes the *sense* of the inequality. The statement of these theorems in words, an important matter, is left as an exercise for the student.

190. If a, b, and x are positive numbers, then

$$\frac{a+x}{b+x} > \frac{a}{b} \text{ if } a < b, \text{ and } \frac{a+x}{b+x} < \frac{a}{b} \text{ if } a > b.$$

If $a < b$, then $\frac{a}{b} < \frac{x}{x}$, or $\frac{x}{x} > \frac{a}{b}$.

Let $\frac{a}{b}=r$, then $\frac{x}{x}>r$, and $a=br$, $x>rx$. (7, § 189)

$$\therefore a+x>br+rx. \quad (3, § 189)$$

$$\therefore \frac{a+x}{b+x}>r. \quad (7, § 189)$$

$$\text{But } r=\frac{a}{b}. \quad \therefore \frac{a+x}{b+x}>\frac{a}{b}.$$

The second part is proved in a similar manner.

191. If a, b, c, etc., are positive numbers, and if $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are not all equal to one another, then $\frac{a+c+e}{b+d+f}$ lies between the greatest and the least of the fractions.

Proof. Let $\frac{e}{f}=r$ be the least, then $\frac{c}{d}>r$, $\frac{a}{b}>r$, and

$$e=fr,\ c>dr,\ a>br. \quad (7, § 189)$$

$$\therefore a+c+e>(f+d+b)r.$$

$$\therefore \frac{a+c+e}{b+d+f}>r,\ i.e.\ \frac{a+c+e}{b+d+f}>\frac{e}{f}.$$

By letting $\frac{a}{b}=r$, it is proved in a similar way

$$\frac{a+c+e}{b+d+f}<\frac{a}{b}.$$

192. Solve $5x+7>7x-19$.

Adding 19, $5x+26>7x$. (3, § 189)

Subtracting $5x$, $26>2x$. $\therefore 13>x$, or $x<13$.

The inequality holds for values of x less than 13.

(*b*) Solve $2x^2+3x<27$.

I.e. $2x^2+3x-27<0$, or $(x-3)(2x+9)<0$.

This inequality holds for values of x that make $x-3$ negative and $2x+9$ positive. $\therefore -4\frac{1}{2}<x<3$.

(*c*) If a and b are two numbers, real and unequal, then $a^2 + b^2 > 2ab$.

For $(a - b)^2 > 0$, *i.e.* $a^2 - 2ab + b^2 > 0$.

$\therefore\ a^2 + b^2 > 2ab.$ (3, § 189)

(*d*) If a, b, c are real and unequal, then

$$a^2 + b^2 + c^2 > ab + bc + ca.$$

For $a^2 + b^2 > 2ab$, $b^2 + c^2 > 2bc$, $c^2 + a^2 > 2ca$.

Adding and dividing by 2, $a^2 + b^2 + c^2 > ab + bc + ca$.

EXERCISE 145

1. If $a < b$ and n is a positive number, compare:

$-na$ and $-nb$; na and nb; $\frac{a}{n}$ and $\frac{b}{n}$; $-\frac{a}{n}$ and $-\frac{b}{n}$.

2. If $x < y$ and $x > 0$, $y > 0$, show that $x^2 < y^2$.

3. If $x < y$ and $x < 0$, $y < 0$, show that $x^2 > y_2$.

4. If $a > 0, b > 0$, show that $\frac{a}{b} + \frac{b}{a} > 2$, unless $a = b$.

5. If $a > 0$, $b > 0$, show that $\frac{1}{2}(a + b) > \sqrt{ab} > \frac{2ab}{a+b}$, unless $a = b$.

6. Solve $3x - 9 < x + 2$; $x + 11 < 5x - 53$;
$x^2 + 2x < 8$; $x^2 + x + 1 > 0$.

7. For what values of n is $n > n^2 - 7n + 12$?

8. For what values of n is $n + 11 > n^2 - 9n$?

9. If a, b, c are positive real numbers, prove
$a^3 + b^3 + c^3 > 3abc$. (See Ex. 25, Exercise 123.)

10. If $a^2 + b^2 = 1$, $x^2 + y^2 = 1$, prove that $ax + by < 1$.

11. If $\frac{a}{b} > \frac{c}{d}$, prove $\frac{a}{b} > \frac{ma + nc}{mb + nd} > \frac{c}{d}$.

CHAPTER XVIII

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS

193. A set of real quantities in which the nth quantity is uniquely determined if n is known is called a **series.**

Illustrations :

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ a+4\,d, \text{ etc.},$$

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4, \text{ etc.},$$

$$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6}, \text{ etc.}$$

194. An Arithmetical Progression (A. P.) is a series in which each term differs from the one preceding by a constant quantity. This constant quantity is called the **common difference.**

Illustrations :

$$1,\ 3,\ 5,\ 7,\ 9,\ \cdots \text{ common difference } 2.$$

$$9,\ 5,\ 1,\ -3,\ -7,\ \cdots \text{ common difference } -4.$$

195. Fundamental formulæ. Denote the common difference by d, the first term by a, the number of terms by n, the nth or last term by l, and the sum of the series by s.

The general type of an A. P. is

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ a+4\,d, \cdots.$$

Hence, $$\boldsymbol{l = a + (n-1)d}. \qquad \text{(I)}$$

Also $s = a + (a+d) + (a+2\,d) + \cdots (l-d) + l,$

and $s = l + (l-d) + (l-2\,d) + \cdots (a+d) + a.$

$$\therefore\ 2\,s = (a+l) + (a+l) + (a+l) \cdots \text{ to } n \text{ terms}$$
$$= n(a+l).$$

$$\therefore\ s = \frac{n}{2}(a + l). \qquad \text{(II)}$$

196. If three numbers are in A. P., the second is the **arithmetic mean** of the other two. From § 195, it follows that the arithmetic mean is **half the sum of the other two.**

Example 1. The 10th term of an A. P. is 82 and the 18th is 66; find the first three terms.

$$a + 9\,d = 82,$$
$$a + 17\,d = 66. \qquad \text{(I, § 195)}$$

Solving for a and d, $a = 100$, $d = -2$.

The first three terms are 100, 98, 96.

Example 2. How many terms of the series 90, 86, 82, etc., must be taken that their sum may be 1008?

Here $d = -4$, $a = 90$, and

$$l = 90 + (n - 1)(-4) = 94 - 4\,n.$$

$$\therefore\ s = \frac{n}{2}\{90 + (94 - 4\,n)\}. \qquad \text{(II, § 195)}$$

$$\therefore\ 1008 = \frac{n}{2}(184 - 4\,n).$$

$$\therefore\ n^2 - 46\,n + 504 = 0. \qquad \therefore\ n = 18 \text{ or } 28.$$

Either number is correct. The sum of the terms beginning with the 19th and ending with the 28th is zero.

Example 3. Find the sum of the integers between 22 and 190 that if divided by 8 give in each case 4 for the remainder.

The numbers are 28, 36, 44 ⋯ 188, an A. P.

$$188 = 28 + (n - 1)\,8. \quad \therefore\ n = 21,$$

$$s = \tfrac{21}{2}(28 + 188) = 2268.$$

EXERCISE 146

1. Express in words formulæ I and II, § 195.

2. Find the 12th term of the series 6, 16, 26, ⋯

3. Find the 30th term of the series 75, $73\frac{1}{2}$, 72, ⋯

4. Find the nth term of the series 1, 3, 5, ⋯

5. Find the sum of 10 terms of 6, 10, 14, ⋯

6. Find the sum to 15 terms of $3\frac{1}{2}$, 6, $8\frac{1}{2}$, etc.

7. Find the sum of all the odd numbers less than 50.

8. Find the sum of all the even numbers less than 91.

9. Find the sum of the first n odd numbers.

10. Find the sum of the first n even numbers.

11. Find the sum of the numbers less than 100 and each a multiple of 7.

12. What is the arithmetic mean of 2 and -2? of $x-y$ and $x+y$?

13. Find 4 arithmetic means between 12 and -13, *i.e.* insert 4 numbers in such a way that the entire set will be in A. P.

14. Find 5 arithmetic means between $x-1$ and $x+5$.

15. Find 3 arithmetic means between a and b.

16. How many terms of the series

(*a*) 24, 20, 16, etc., must be taken to make 72?

(*b*) 15, 24, 33, etc., must be taken to make 2010?

17. Given $a=12$, $l=38$, $S=200$, find n.

18. The 3d term of an A. P. is 11, the 7th, $8\frac{1}{8}$ and the sum 55. Find the number of terms.

19. Find the nth term of the A. P. a, b, $2b-a$, etc.

20. Given $S=2535$, $d=5$, and $n=30$, find a.

21. Given $a=\frac{1}{2}$, $d=\frac{1}{2}$, and $s=5$, find n.

197. A Geometrical Progression (G. P.) is a series in which each term is obtained from the one preceding by multiplying it by a constant quantity called the **common ratio.**

Illustrations: 3, 6, 12, 24, 48
45, 30, 20, 40/3, ⋯.

If a is the first term, r the common ratio, and n the number of terms, the standard type of G. P. is

$$a, ar, ar^2, ar^3, \dots ar^{n-1}.$$

A **geometrical progression** and a **continued proportion** are the same. **Geometric mean** and **mean proportional** are synonymous.

198. Two fundamental formulæ.

(I) $l = ar^{n-1}$, (II) $s = a\cdot\dfrac{r^n-1}{r-1}$, or $a\dfrac{1-r^n}{1-r}$.

The first is quite evident, the second follows from the identity $a\cdot\dfrac{1-x^n}{1-x} = a(1+x+x^2+\cdots x^{n-1})$.

Example 1. Find the sum of 7 terms of the series 64, -48, 36 ⋯.

$r = -48 \div 64$ or $36 \div -48$, *i.e.* $-\frac{3}{4}$.

$s = a\cdot\dfrac{1-r^n}{1-r}$. $\therefore s = 64\dfrac{1-(-\frac{3}{4})^7}{1-(-\frac{3}{4})} = \dfrac{2653}{64}$.

Example 2. Insert 4 geometric means between 27 and $3\frac{5}{9}$.

$l = ar^{n-1}$. Here $l = 3\frac{5}{9}$, $a = 27$, and $n = 6$.

$\therefore 3\frac{5}{9} = 27r^5$. $\therefore r^5 = \frac{32}{243}$. $\therefore r = \frac{2}{3}$.

The means are 18, 12, 8, $5\frac{1}{3}$.

EXERCISE 147

1. Find the 6th term and the nth term of 32, 24, 18, ...

2. Find the 7th term and the nth term of 18, -12, 8, ...

3. Find the 9th term and the $(n+1)$th term of a, $-ar$, ar^2 ....

4. Find the 10th term and the nth term of $\sqrt[3]{3}$, $\sqrt[3]{9}$, 3, ...

5. Find the sum of each of the following geometric series:

(a) $1.03 + 1.03^2 + 1.03^3 + \cdots$ to 5 terms.

(b) $1 + 1.04 + 1.04^2 + \cdots$ to 6 terms.

(c) $1 + 1.05 + 1.05^2 + \cdots$ to 8 terms.

(d) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$ to n terms.

(e) $a + a^2 + a^3 + \cdots$ to n terms.

(f) $\sqrt{2} + 2 + 2\sqrt{2}$ to n terms.

6. Find the geometric mean between 2 and 8; a and a^3; $\sqrt{2}$ and $4\sqrt{2}$; a^x and a^{x+2}; 0 and 9; ac and $\frac{a}{c}$.

7. Insert 3 geometric means between 9 and $\frac{1}{9}$.

8. Insert 4 geometric means between 1 and -243.

9. The 5th and 8th terms of a G. P. are $\frac{1}{81}$ and $\frac{1}{2187}$; find the 1st term.

10. The 4th and 8th terms of a G. P. are a and a^3; find the 1st term.

11. If $s = 97\frac{1}{2}$, $r = 1\frac{1}{2}$, and $n = 4$, find a.

12. If $s = 55$, $r = -2$, and $n = 5$, find a.

13. The 1st and 3d terms of a G. P. are 18 and 32 respectively; find the sum of the first 6 terms.

14. The sum of three terms in G. P. is 38 and the sum of their squares is 532. Find the terms.

15. If a, b, c are in G. P. $a^2 + b^2 + c^2 = (a - b + c)(a + b + c)$.

16. Find the sum to n terms of $x+1$, x^2+2, $x^3+3+\cdots$.

17. The continued product of four numbers in G. P. is 1,000,000, and the sum of the means is 70. Find the numbers.

18. The sum of the 1st and 2d terms of a G. P. is 35 and of the second and third is $52\frac{1}{2}$. Find these terms.

199. By an infinite series is meant one that has no last term. The quotient of a by $1 - r$ consists of a boundless number of terms, namely

$$a + ar + ar^2 + ar^3 + \cdots ar^n,\ n = \infty.$$

The symbol ∞ denotes infinity. It is not a definite number.

200. Sum of an infinite geometrical series. To find the sum of an infinite series, in an arithmetic sense, is an impossibility. If the sum of the first n terms of a series tends towards a **constant** quantity as n increases, then this **constant** is called the **limit of the sum of the series. This limit** is what is meant by **sum.**

The sum to infinity of $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} +$ is $\frac{1}{3}$ for

$$\frac{1}{3} = .3333 + \cdots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots \frac{3}{10^n}. \quad n = \infty.$$

If r is numerically less than 1 and $n = \infty$, then

$$a + ar + ar^2 + \cdots ar^{n-1} = \frac{a}{1 - r}.$$

For $s = a \cdot \dfrac{1 - r^n}{1 - r}$ and the limit of r^n or of ar^n is zero.

Hence, $$s = a \cdot \frac{1}{1-r} = \frac{a}{1-r}.$$

Example 1. Find the sum to infinity of $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} +$.

$$s = \frac{a}{1-r},\quad r = \frac{1}{2} \div \frac{3}{4} = \frac{2}{3}. \quad \therefore\ s = \frac{\frac{3}{4}}{1-\frac{2}{3}} = 2\tfrac{1}{4}.$$

Example 2. Evaluate $0.3\dot{8}\dot{1}$, *i.e.* $0.3818181 \cdots$.

$0.3\dot{8}\dot{1} = 0.3818181 + \cdots = .3 + .081 + .00081 + .0000081 +.$

The terms $.081 + .00081 + \cdots$ constitute a geometrical series whose common ratio is .01. Its sum is

$$\frac{.081}{1-.01} = \frac{81}{990} = \frac{9}{110}. \quad \therefore\ .3\dot{8}\dot{1} = \tfrac{3}{10} + \tfrac{9}{110} = \tfrac{21}{55}.$$

EXERCISE 148

Find the sum to infinity:

1. $8, 4, 2, \cdots$.
2. $\frac{4}{3}, 1, \frac{3}{4}, \cdots$.
3. $2, -1, \frac{1}{2}, -\frac{1}{4}, \cdots$.
4. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$.
5. $1, \dfrac{a-1}{a}, \left(\dfrac{a-1}{a}\right)^2, \cdots$.
6. $.8 + .008 + .00008$.
7. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$.
8. $1 - \dfrac{\sqrt{2}}{\sqrt{2}+1} + \dfrac{2}{3+2\sqrt{2}}$.
9. Evaluate: (*a*) $.\dot{3}\dot{6}$, (*b*) $.\dot{6}\dot{3}$, (*c*) $.\dot{9}\dot{0}$, (*d*) $.\dot{8}\dot{0}$.
10. Evaluate: (*a*) $.0784\dot{6}$, (*b*) $.00\dot{6}$, (*c*) $.001\dot{3}$, (*d*) $.\dot{4}2857\dot{1}$, (*e*) $.1\dot{3}\dot{7}$, (*f*) $.0\dot{1}5384\dot{6}$.
11. The sum to infinity of a G. P. is $\frac{2}{3}$ and the ratio is $\frac{2}{3}$; find the first term.

12. Find r if the sum to infinity of a G. P. is $\frac{3}{4}$ and $a = \frac{5}{6}$.

13. The sum of the first 4 terms of a G. P. is $7\frac{2}{9}$ and the sum to infinity is 9. Find the first term.

14. The second term of a G. P. is 6 and its sum to infinity is 32. Find the first term.

15. Show that if $0 < x < \frac{1}{2}$ and if $s = x + x^2 + x^3 + \cdots$ to ∞, then $x = s - s^2 + s^3 - \cdots$ to ∞.

16. Find the sum to infinity of $(1 + a)^2 - (1 - a^2) + (1 - a)^2$. $a < 1$.

17. Prove that the arithmetic mean of two numbers is greater than their geometric mean.

CHAPTER XIX

LOGARITHMS

201. If $a^x = n$, then x is the logarithm of n to the base a. This fact is expressed more shortly by the relation

$$x = \log_a n.$$

The logarithm of a number is the index of the power to which a number called the base must be raised to equal that number.

202. In practical work the base generally used is 10, and in stating logarithms it is as a rule omitted.

Logarithms to base 10 are called **common logarithms.**

Since

$$\begin{aligned} 10^0 &= 1 & \therefore\quad \log 1 &= 0 \\ 10^1 &= 10 & \log 10 &= 1 \\ 10^2 &= 100 & \log 100 &= 2 \\ 10^3 &= 1000 & \log 1000 &= 3 \\ 10^{-1} &= 0.1 & \log 0.1 &= -1 \\ 10^{-2} &= 0.01 & \log 0.01 &= -2 \\ 10^{-3} &= 0.001 & \log 0.001 &= -3 \end{aligned}$$

Consequently the logarithm of a number between

1 and 10, *i.e.* of **1** figure $= \mathbf{0} +$ a fraction
10 and 100, *i.e.* of **2** figures $= \mathbf{1} +$ a fraction
100 and 1000, *i.e.* of **3** figures $= \mathbf{2} +$ a fraction

0.1 and 1 $= -1 +$ a fraction
0.01 and 0.1 $= -2 +$ a fraction
0.001 and 0.01 $= -3 +$ a fraction
0.0001 and 0.001 $= -4 +$ a fraction

Thus $\log 87.43 = 1 + \text{a fraction}$, for $10 < 87.43 < 100$.

$\log 0.0738 = -2 + \text{a fraction}$, for $0.01 < 0.0738 < 0.1$.

The integral and decimal parts of a logarithm are called respectively the **characteristic** and **mantissa** of the logarithm.

From the above illustration it is evident that **the characteristic of the logarithm of a number greater than unity is one less than the number of figures to the left of the decimal point in the number,** and that **the characteristic of a number less than unity is negative, and numerically one more than the number of ciphers between the decimal point and the first significant figure.**

203. Fundamental rules of logarithms.

Let $a^x = n$, *i.e.* $x = \log_a n$,

and $a^y = m$, *i.e.* $y = \log_a m$.

Hence,

$$a^{x+y} = nm, \text{ i.e. } x + y = \log_a nm = \log_a n + \log_a m \qquad \text{I}$$

$$a^{x-y} = \frac{n}{m}, \text{ i.e. } x - y = \log_a \frac{n}{m} = \log_a n - \log_a m \qquad \text{II}$$

$$a^{rx} = n^r, \text{ i.e. } rx = \log_a n^r = r \log_a n. \qquad \text{III}$$

In words: **The logarithm of a product is the sum of the logarithms of the factors.** I

The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor. II

The logarithm of a power is the logarithm of the number multiplied by the exponent of the power. III

Thus $\log 15 = \log 5 + \log 3$,

$\log \frac{5}{7} = \log 5 - \log 7$,

$\log 17^3 = 3 \log 17$,

$\log \sqrt{2} = \log 2^{\frac{1}{2}} = \frac{1}{2} \log 2$,

$$\log 7\sqrt[3]{5} = \log 7 \cdot 5^{\frac{1}{3}} = \log 7 + \tfrac{1}{3}\log 5,$$

$$\log \sqrt{\frac{mnr}{s}} = \tfrac{1}{2}(\log m + \log n + \log r - \log s).$$

The mantissæ of the common logarithms of numbers expressed by the same significant figures are the same.

$$\log 2574 = 3.41061,$$
$$\log 257.4 = \log (2574 \times 10^{-1}) = 3.41061 - 1 = 2.41061,$$
$$\log 25.74 = \log (2574 \times 10^{-2}) = 3.41061 - 2 = 1.41061,$$
$$\log 2.574 = \log (2574 \times 10^{-3}) = 3.41061 - 3 = 0.41061,$$
$$\log 0.2574 = \log (2574 \times 10^{-4}) = 3.41061 - 4 = 0.41061 - 1,$$
$$\log 0.02574 = \log (2574 \times 10^{-5}) = 3.41061 - 5 = 0.41061 - 2.$$

In practice, logarithms are written so that the mantissæ are **positive**.

Thus $\log .02574 = \bar{2}.41061.$

Example 1. What is the common logarithm of $(\sqrt{.001})^5$?

$(\sqrt{.001})^5 = (\sqrt{10^{-3}})^5 = (10^{-\frac{3}{2}})^5 = 10^{-7.5}$. *Ans.* -7.5.

-7.5 is written $\bar{8}.5$, for $-7.5 = 8 - 7.5 - 8 = 0.5 - 8$.

Example 2. Given $\log 2 = .30103$, $\log 3 = .47712$, find $\log 24$.

$$\log 24 = \log (2^3 \times 3) = 3 \log 2 + \log 3$$
$$= 3(.30103) + .47712 = 1.38021,$$

or $$24 = 2^3 \times 3 = (10^{.30103})^3 \times 10^{.47712} = 10^{1.38021}$$
$$\therefore \log 24 = 1.38021.$$

Example 3. Given $\log 17 = 1.23045$, $\log 992 = 2.99651$, find $\log \frac{17}{992}$.

$$\log \tfrac{17}{992} = \log 17 - \log 992 = 1.23045 - 2.99651$$
$$= (3.23045 - 2.99651) - 2 = \bar{2}.23394.$$

Here 2 is added to 1.23045 and afterwards subtracted. This is done to make the mantissa positive.

Example 4. How many ciphers precede the first significant figure in $(\frac{3}{4})^{50}$? Given $\log 2 = .30103$, $\log 3 = .47712$,

$$x = \left(\frac{3}{4}\right)^{50} = \left(\frac{3}{2^2}\right)^{50} = \frac{3^{50}}{2^{100}}.$$

$$\therefore \log x = 50 \log 3 - 100 \log 2 = 50(.47712) - 100(.30103)$$
$$= 23.856 - 30.103 = \overline{7}.753.$$

As the characteristic is one more than the number of ciphers, the required number is 6.

EXERCISE 149

Find the common logarithm of :

1. $\sqrt[3]{10}$.	5. 100^2.	9. $(.1)^3$.	13. $(.01)^{\frac{3}{4}}$.
2. $\sqrt[4]{10}$.	6. 1000^2.	10. $(.1)^4$.	14. $(.001)^{-3}$.
3. $\sqrt[4]{1000}$.	7. 100^3.	11. $(.01)^2$.	15. $(.01)^{-4}$.
4. $\sqrt{10,000}$.	8. 1000^4.	12. $(.01)^{\frac{3}{2}}$.	16. $\sqrt[3]{10,000^{-2}}$.

17. Given $\log 2 = .30103$, $\log 3 = .4771213$, find the logarithms of : 6, 8, 9, 12, 16, 18, 20, 5, 25.

18. Write the following relations in the form $\log_a n = c$: (*a*) $5^2 = 25$, (*b*) $2^4 = 16$, (*c*) $3^5 = 243$, (*d*) $4^6 = 4096$.

19. Write the following relations in the form $a^x = n$:

(*a*) $\log_5 125 = 3$, (*b*) $\log_2 \frac{1}{4} = -2$, (*c*) $\log_2 64 = 6$, (*d*) $\log_4 1024 = 5$.

20. Find the characteristics of the logarithms of 47.28, 91851.1, .0285, .00075, 0.123.

204. The following is an extract from a 5-place table of logarithms. It gives the mantissæ of the logarithms of the natural numbers between 2680 and 2719, both included.

N	0	1	2	3	4	5	6	7	8	9
268	42813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441

The horizontal line beginning with 268 contains the mantissæ of the logarithms of 2680, 2681, 2682, etc., up to 2689. The next line contains the mantissæ of the logarithms of 2690, 2691, 2692, etc., up to 2699, and so on.

Thus $\log 2682 = 3.42846$, $\log 2691 = 3.42991$,
$\log 2705 = 3.43217$, $\log 2717 = 3.43409$.

As the first two figures of the mantissæ are the same for several consecutive numbers, these figures are not set down for each logarithm. In the above table the figures 42 are the initial figures of the mantissæ of the logarithms of the numbers between 2680 and 2691. The figures 43 are the initial figures of the remaining mantissæ. Note that the asterisks serve to remind one to take 43 and not 42. Characteristics of logarithms, being matters of inspection, § 202, are not set down in tables.

205. Interpolation. In the above table the difference between the logarithms of two consecutive integers is 16 units of the fifth decimal order.

For example,

$$\log 2703 = 3.43185,$$
$$\log 2702 = 3.43169.$$

(*a*) Suppose log 27027 is required.

The mantissa of 2702.7 is the same as the mantissa of 27027.

The mantissa of 2702 = .43169.

.7 of 16 units of 5th order = 11.

The mantissa of 2702.7 = .43180.

(*b*) **Number corresponding to a given logarithm (antilogarithm).**

$\log x = 2.43015$, find x.

The mantissa of log 2692 = .43008. (1)

The mantissa of log x = .43015. (2)

The mantissa of log 2693 = .43024. (3)

Hence x is intermediate in value between the sequence 2692 and 2693. As the difference between the first and second is 7 and between the first and third is 16, and as $\frac{7}{16} = .4$ approximately, the required sequence of figures is 26924.

Since the characteristic of log x is 2, x has 3 integral figures.

$\therefore x = 269.24.$

Example 1. Extract the fourth root of .039273.

$$x = \sqrt[4]{.039273},\ \log x = \tfrac{1}{4}\log .039273 = \tfrac{1}{4}(\bar{2}.59409)$$
$$= \tfrac{1}{4}(\bar{4} + 2.59409) = \bar{1}.64852.$$
$$\therefore x = .44516.$$

Note particularly how $\bar{2}.59409$ is divided by 4. The negative characteristic is numerically increased so that 4 is contained an exact number of times in the sum. A positive number of the same numerical value is then added, and the division is performed in the ordinary way.

Example 2. Find $\log_2 6$.

$$x = \log_2 6. \quad \therefore 2^x = 6. \quad \therefore x \log 2 = \log 6. \ (\S\ 203)$$
$$\therefore x(.30103) = .77815. \quad \therefore x = 2.58496.$$

Example 3. If $(.98)^x < .7$, find the least integral value of x.

$$x \log .98 < \log .7. \quad \therefore x(\bar{1}.99123) < \bar{1}.84510.$$
$$\therefore x(-.00877) < -.15490.$$
$$\therefore 877x > 15490, \quad \text{multiplying by } -100{,}000.$$
$$x > 17.6. \qquad \textit{Ans. } 18.$$

EXERCISE 150

1. Multiply 7.639 by 1.298.
2. Multiply 3.275 by 0.9283.
3. Find the area of a circle whose radius is 3.74.
4. Find the volume of a cube whose edge is 4.938.
5. How many meters are there in 1 mile? (1 m. = 39.37 in.)
6. How many kilometers are there in 363 miles?
7. How many acres are there in 1 square kilometer?
8. How many square kilometers in 763 square miles?
9. Evaluate 1.015^{30} given $\log 1.015 = .006466$.
10. Evaluate 1.025^{27} given $\log 1.025 = .0107239$.
11. Evaluate 1.035^{40} given $\log 1.035 = .0149403$.
12. Extract the square root of 1.25; of 0.071.

13. Extract the square root of $\frac{2}{7}$; of $\frac{9}{11}$; of $\frac{17}{28}$.

14. Extract the cube root of $2\frac{1}{8}$; of $11\frac{3}{8}$; of 10.5.

15. Extract the cube root of 0.47; of .0123; of 0.9.

16. Given $x = 36^8 + 39.37^8$, find x.

17. What power of 7 is 2407?

18. Find x: $(1.04)^x = 2$; $(1.05)^x = 2$; $(1.06)^x = 2$; $(1.08)^x = 3$.

19. Show that $(\frac{7}{8})^{100} < 10^{-5}$; $(\frac{41}{40})^{100} > 10$.

20. Find the number of figures in the expansions 2^{82}, 3^{40}, 5^{50}.

21. How many ciphers before the first significant figure in the expansion $(\frac{5}{6})^{100}$? $(\frac{2}{3})^{90}$?

22. Given $\log 2 = .30103$, find $\log .08$, $\log \sqrt[4]{3.2}$, $\log \sqrt[3]{.64}$.

23. Given $\log 2 = .30103$, $\log 21 = 1.32222$, $\log 144 = 2.15836$, find the logarithms of 3, 4, 5, 6, 7, 8, 9, 14, 105.

24. Find without consulting tables $\log_2 .03125$, $\log_9 27$, $\log_3 81$, $\log_2 8$, $\log_8 16$, $\log_{16} 8$, $\log_3 \frac{1}{81}$.

25. Find x, given $\log_x 3 = \frac{1}{2}$, $\log_x 8 = \frac{3}{4}$, $\log_x 4 = -\frac{2}{3}$.

26. Find the least integral value of x for which each inequality holds: $(\frac{9}{10})^x < \frac{1}{2}$, $(\frac{29}{30})^x < \frac{1}{2}$, $(.99)^x < \frac{1}{2}$.

27. Given $(1 + r)^{50} = 10$, find r correct to 3 figures.

28. Given log 14, log 144, and log 21 (see Ex. 23), find the logarithms of the 9 digits.

HINT. Express in terms of log 2, log 3, and log 7 and then solve.

29. Given $1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n}\right) < 10^{-6}$, find n.

30. Solve $3^x - 3^{-x} = 5$.

31. Three numbers are in geometrical progression. Prove their logarithms are in arithmetical progression.

CHAPTER XX

COMPLEX NUMBERS. THEORY OF QUADRATICS. CUBE ROOT. BINOMIAL THEOREM.

206. The roots of the equation

$$x^2 - 2x + 2 = 0$$

are $1 + \sqrt{-1}$ and $1 - \sqrt{-1}$. These numbers are new.

The square of a real number is **not negative.** On the other hand, the square root of a negative number, for example -16, is not real. It is known as a **pure imaginary.** The unit of imaginaries is $\sqrt{-1}$. It is denoted by i. The defining property of i is $i^2 = -1$.

It is subject to all the laws of algebra.

The standard way of writing $\sqrt{-a}$, a being a positive real number, is $i\sqrt{a}$.

For $(i\sqrt{a})^2 = i^2 \cdot a = -a$, and $(\sqrt{-a})^2 = -a$. (Def. of sq. root.)

Illustrations. $\sqrt{-4} = i\sqrt{4} = 2i$, $\sqrt{-17} = i\sqrt{17}$.

207. The sum of a pure imaginary and a real number is called **a complex number.** Its standard type is

$$a + ib. \quad (a \text{ and } b \text{ real.})$$

If $b = 0$, $a + ib$ becomes a. Hence $a + ib$ includes the totality of all real numbers.

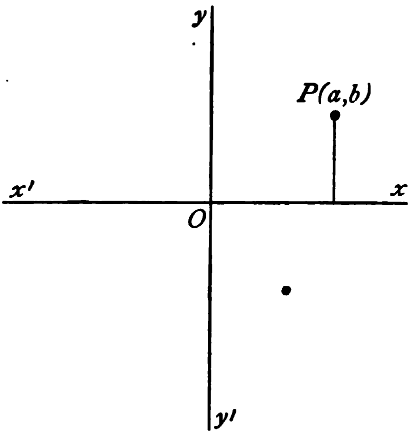
208. As $i^2 = -1$ and i is subject to the laws of algebra, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 \cdot i = -1 \cdot i = -i$, $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$.

The powers of i recur, the period being 4.

$$i^{79} = (i^4)^{19} \cdot i^3 = -i, \quad i^{46} = (i^4)^{11} \cdot i^2 = -1.$$

209. Graphical representation of complex numbers.

If a point P has for its coördinates a and b, then the point P represents the number $a + ib$. Consequently every point in the plane represents some complex number.



The points of the x-axis represent all real numbers. The totality of all real numbers is, therefore, contained in the totality of complex numbers. In all real quantities b, in the expression $a + ib$, is zero. If $a = 0$ and $b = 1$, then $a + ib$ becomes i. The number i is represented by a point on OY, one unit's distance from O. Similarly the number in is a point on OY, n units distant from O. All pure imaginaries are therefore represented by points on the line YOY'.

Example 1. $\sqrt{-2} \cdot \sqrt{-8} = i\sqrt{2} \cdot i\sqrt{8} = i^2\sqrt{16}$
$= -1 \cdot 4 = -4.$

Example 2. $6 \div \sqrt{-4} = 6 \div 2i = 6i \div 2i^2 = 6i \div -2$
$= -3i.$

Example 3. Express in standard form $\frac{1+i}{1-i}$, $\frac{14+5i}{4-i}$.

$$\frac{1+i}{1-i}=\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}=\frac{1+2i+i^2}{1-i^2}=\frac{1+2i-1}{1-(-1)}=i.$$

$$\frac{14+5i}{4-i}=\frac{14+5i}{4-i}\cdot\frac{4+i}{4+i}=\frac{56+34i+5i^2}{16-i^2}=\frac{51+34i}{17}$$
$$=3+2i.$$

Example 4. Construct the equation whose roots are $\frac{5\pm i\sqrt{19}}{2}$.

$$x=\frac{5\pm i\sqrt{19}}{2}. \quad \therefore 2x-5=\pm i\sqrt{19}.$$

$$\therefore (2x-5)^2=-19, \quad i.e. \; x^2-5x+11=0.$$

EXERCISE 151

Represent graphically the following complex numbers:

1. $1+i$.
2. $1-i$.
3. $1+2i$.
4. $1-2i$.
5. $2+3i$.
6. $2-3i$.
7. $3+4i$.
8. $3-4i$.
9. $-3+4i$.
10. $-3-4i$.
11. $5+12i$.
12. $-5+12i$.

Express in standard form:

13. $\sqrt{-4}$.
14. $\sqrt{-9}$.
15. $\sqrt{-25}$.
16. $\sqrt{-144}$.
17. $\sqrt{-100}$.
18. $\sqrt{-10}$.
19. $\sqrt{-24.5}$.
20. $\sqrt{-(a-b)^2}$.

Perform the indicated operations:

21. $\sqrt{-2}\cdot\sqrt{-32}$.
22. $\sqrt{-10}\cdot\sqrt{-10}$.
23. $\sqrt{-3}\cdot\sqrt{-12}$.
24. $\sqrt{-8}\cdot\sqrt{-32}$.
25. $\sqrt{-3}\cdot\sqrt{-5}$.
26. $\sqrt{-a}\cdot\sqrt{-b}$.

27. $(a+ib)^2$.

28. $(\sqrt{x}+\sqrt{-y})^2$.

29. $(2+\sqrt{-3})^2$.

30. $(3-\sqrt{-2})^2$.

31. $(3+\sqrt{-4})(3-\sqrt{-4})$.

32. $(-1-\sqrt{-3})^3$.

33. $(1+\sqrt{-1})^2$.

34. $\sqrt{-4}\div\sqrt{-2}$.

35. $\sqrt{-10}\div\sqrt{-1}$.

36. $\sqrt{-44}\div\sqrt{-11}$.

37. $\sqrt{-63}\div\sqrt{-7}$.

38. $\sqrt{-a^2}\div\sqrt{-b^2}$.

39. $1\div i$.

40. Find the value of x^2-4x+5, when $x=2\pm i$.

41. Find the value of x^2+2x+4, when $x=-1\pm i\sqrt{3}$.

42. Multiply $a+ib$ by $a-ib$, and from the result determine the rationalizing factor of $5-4i$ and of $1+\sqrt{-3}$.

43. Find the value of x^2-x+1, when $x=\frac{1}{2}(1-\sqrt{-3})$.

Simplify and give the results in standard form:

44. $\dfrac{1-i}{1+i}$.

45. $\dfrac{13}{3-2i}$.

46. $\dfrac{i-3}{i+2}$.

47. $1\div i^3$.

48. $\dfrac{(1-i)^2}{2i}$.

49. $\dfrac{1}{(1-i)^2}-\dfrac{1}{(1+i)^2}$.

50. $\dfrac{4}{1-i\sqrt{3}}$.

51. $\dfrac{\sqrt{3}-i\sqrt{2}}{\sqrt{3}+i\sqrt{2}}$.

52. $(1-i)^3$.

53. $\dfrac{1+18i}{3+4i}+\dfrac{7-26i}{3-4i}$.

54. Construct the equation whose roots are $1\pm i$; $2\pm\sqrt{-7}$; $\dfrac{3\pm\sqrt{-59}}{2}$.

210. Consider the typical quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c, \text{ real numbers.}$$

Its roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (page 210)

If $b^2 > 4ac$, the roots are **real.**

If $b^2 = 4ac$, the roots are **equal** and **real.**

If $b^2 < 4ac$, the roots are **imaginary.**

If $b^2 - 4ac$ is a perfect square, the roots are **rational.**

Cor. 1. If $c = 0$, the roots become $\dfrac{-b \pm b}{2a}$, *i.e.* 0 and $\dfrac{-b}{a}$.

If the absolute term is zero, one root of a quadratic is zero.

Cor. 2. If $b = 0$, the roots become $\dfrac{\pm\sqrt{-4ac}}{2a}$.

If the coefficient of x is zero, the roots of a quadratic are equal numerically, but opposite in sign.

211. The equation having m and n for its roots is

$$(x - m)(x - n) = 0,$$

i.e. $$x^2 - (m + n)x + mn = 0. \qquad (1)$$

If m and n are the roots of $ax^2 + bx + c = 0$, then

$$x^2 - (m + n)x + mn = 0, \quad \text{and}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \text{ are identical.}$$

Hence, $$m + n = -\frac{b}{a},$$

$$mn = \frac{c}{a}.$$

$\therefore$ the sum of the roots $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$.

The product of the root $= \dfrac{\text{absolute term}}{\text{coefficient of } x^2}$.

NOTE. These results may be obtained directly from § 210.

Example 1. If m, n are the roots of $x^2 - 2x - 7 = 0$, find the values of (i) $m^2 + n^2$; (ii) $m^4 + n^4$; (iii) $m - n$.

(i) $m^2 + n^2 = (m+n)^2 - 2mn$.

But $$m + n = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2} = -\frac{-2}{1} = 2.$$

$$mn = \frac{\text{absolute term}}{\text{coefficient of } x^2} = \frac{-7}{1} = -7.$$

$\therefore$ (i) $m^2 + n^2 = 2^2 - 2 \cdot -7 = 18$.

(ii) $m^4 + n^4 = (m^2+n^2)^2 - 2m^2n^2 = 18^2 - 2(-7)^2 = 226$.

(iii) $m - n = \sqrt{(m+n)^2 - 4mn} = \sqrt{2^2 - 4(-7)}$

$= \sqrt{32} = 4\sqrt{2}$.

Example 2. Find the equation whose roots are the cubes of the roots of $x^2 - 3x + 2 = 0$.

If the roots of $x^2 - 3x + 2 = 0$ are m and n, the required equation is $$(x - m^3)(x - n^3) = 0,$$

i.e. $$x^2 - (m^3 + n^3)x + m^3n^3 = 0.$$

It now remains to determine $m^3 + n^3$ and m^3n^3.

In this problem $m + n = 3$, $mn = 2$.

$m^3 + n^3 = (m+n)^3 - 3mn(m+n) = 3^3 - 3 \cdot 2 \cdot 3 = 9$.

$m^3n^3 = (mn)^3 = 2^3 = 8$.

$\therefore$ the required equation is $x^2 - 9x + 8 = 0$.

Example 3. Find the least value of $x^2 - 3x + 7$ for real values of x.

$$x^2 - 3x + 7 = x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4} + 7 \qquad (\S\, 156)$$
$$= x^2 - 3x + \tfrac{9}{4} + \tfrac{19}{4} = (x - \tfrac{3}{2})^2 + \tfrac{19}{4}.$$

As the square of a real number is positive, and the least positive number is zero, the required value is $\tfrac{19}{4}$.

EXERCISE 152

Compute the discriminant $(b^2 - 4ac)$ of the expressions:

1. $3x^2 - 2x - 5$.
3. $4x^2 - 4x + 1$.
5. $x^2 - 2x + 4$.
2. $x^2 + x - 1$.
4. $a^2x^2 + 2abx + b^2$.
6. $ax^2 + c$.

Under what condition will a root be zero?

7. $9x^2 - 2x - 4c + 8 = 0$.
9. $x^2 - x - 3c + 4 = 0$.
8. $x^2 + 5x + c^2 - 25 = 0$.
10. $x^2 + 4x + 2c^2 - c - 1 = 0$.

Under what condition is each quadratic a perfect square?

11. $mx^2 + nx + 1$.
12. $x^2 - 2mx + a^2$.
13. $nx^2 - mx - n$.
14. $4x^2 + 4(m-n)x + n^2$.
15. $4kx^2 - (12k - 6)x + 9k = 0$.

16. Unity is a root of the equation

$$(a - b)x^2 + (c - a)x + (b - c) = 0;$$

find by inspection its other root.

17. Construct the equation whose roots are $a + ib$ and $a - ib$, and deduce that the resulting quadratic is the sum of the squares of two real quantities.

18. One root of $x^2 + bx + 8 = 0$ is 4; find the value of b.

19. One root of $x^2 - 15x + c = 0$ is double the other; find c.

20. Find the least possible value of $x^2 - x + 1$.

21. If m and n are the roots of $x^2 - 5x + 9 = 0$, find the values of (a) $m^2 + n^2$, (b) $m^3 + n^3$, (c) $m - n$, (d) $m^{-1} + n^{-1}$, (e) $m^{-2} + n^{-2}$, (f) $mn^{-1} + nm^{-1}$, (g) $(1 + m)(1 + n)$.

22. If m, n are the roots of $x^2 - px + q = 0$, find the equations whose roots are:

(a) $\frac{1}{m}, \frac{1}{n}$, (b) m^2, n^2, (c) $\frac{m}{n}, \frac{n}{m}$, (d) $m + n, mn$.

212. To extract the Cube Root of a Polynomial.

$(a+b)^3 = a^3 + (3a^2 + 3ab + b^2)b.$

$(a+b+c)^3 = a^3 + (3a^2 + 3ab + b^2)b$

$\qquad\qquad + [3(a+b)^2 + 3(a+b)c + c^2]c$, etc.

These identities furnish the rule for extracting the cube root of a polynomial.

Find

$\sqrt[3]{(343x^6 - 294x^5 - 357x^4 + 244x^3 + 153x^2 - 54x - 27)}.$

Cube root $= 7x^2 - 2x - 3.$

$$
\begin{array}{r}
343x^6 - 294x^5 - 357x^4 + 244x^3 + 153x^2 - 54x - 27 \\
343x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
\hline
-294x^5 - 357x^4 + 244x^3 \qquad\qquad\qquad\quad \\
-294x^5 + 84x^4 - 8x^3 \qquad\qquad\qquad\quad \\
\hline
-441x^4 + 252x^3 + 153x^2 - 54x - 27 \\
-441x^4 + 252x^3 + 153x^2 - 54x - 27 \\
\hline
\end{array}
$$

$$
\begin{array}{rr}
3(7x^2)^2 = & 147x^4 \\
3(7x^2)(-2x) = & -42x^3 \\
(-2x)^2 = & 4x^2 \\
\hline
 & 147x^4 - 42x^3 + 4x^2
\end{array}
$$

$$
\begin{array}{rr}
3(7x^2 - 2x)^2 = & 147x^4 - 84x^3 + 12x^2 \\
3(7x^2 - 2x)(-3) = & -63x^2 + 18x \\
(-3)^2 = & +9 \\
\hline
 & 147x^4 - 84x^3 - 51x^2 + 18x + 9
\end{array}
$$

First arrange the terms. Then take the cube root of the first term and set it down for the first term of the root. Next cube this term and subtract. For a trial divisor write down three times the square of the part of the root found, and divide it into the last remainder. This gives the next term. Multiply it by the part of the root found, square it, add the three results and then multiply by the new term, and so on.

EXERCISE 153

Subtract the cube root of:

1. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1.$
2. $8x^6 + 12x^5 + 18x^4 + 13x^3 + 9x^2 + 3x + 1.$
3. $27x^6 + 54x^5 + 63x^4 + 44x^3 + 21x^2 + 6x + 1.$
4. $x^6 - 3x^5 - 3x^4 + 11x^3 + 6x^2 - 12x - 8.$
5. $x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1.$
6. $x^6 + 12x^5 + 45x^4 + 40x^3 - 45x^2 + 12x - 1.$
7. $125x^6 - 150x^5 - 90x^4 + 112x^3 + 36x^2 - 24x - 8.$
8. $64x^6 - 48x^5 + 108x^4 - 49x^3 + 54x^2 - 12x + 8.$
9. $216x^6 - 108x^5 - 306x^4 + 107x^3 + 153x^2 - 27x - 27.$
10. $64x^6 - 240x^5 + 348x^4 - 245x^3 + 87x^2 - 15x + 1.$

POWERS OF A BINOMIAL

213. By actual multiplication it is found that:

$(a + b)^2 = a^2 + 2ab + b^2,$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$

In these expansions note the following facts:

1. The exponents of a decrease by 1, and those of b increase by 1.

2. The terms are homogeneous, the degree being the index of the power.

3. The coefficient of the first term is 1, and that of the second is the index of the power.

4. The coefficient of any term is obtained from the previous term by multiplying its coefficient by the expo-

nent of a in that term and dividing by the number of the previous term. The expansion of $(a+b)^n$ is, accordingly,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2$$

$$+\frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots.$$

1. Write the powers of a, namely a^n, a^{n-1}, a^{n-2}, $\cdots$ a^1.
2. Write the powers of b, namely b, b^2, $b^3 \cdots b^n$.
3. Write the coefficients, namely 1, n, $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{2\cdot 3}$.

Coefficient of 3d term = coefficient of 2d $\times (n-1) \div 2$.
Coefficient of 4th term = coefficient of 3d $\times (n-2) \div 3$.
Coefficient of 5th term = coefficient of 4th $\times (n-3) \div 4$.

This expansion of $(a+b)^n$ is known as the Binomial Theorem. With certain restrictions as to a and b it is true for any value of n.

Example 1. Expand $(3x-2)^5$.

$$(3x-2)^5 = (3x)^5 + 5(3x)^4(-2) + 10(3x)^3(-2)^2$$
$$+ 10(3x)^2(-2)^3 + 5(3x)^1(-2)^4 + (-2)^5$$
$$= 243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32.$$

Example 2. Expand to 4 terms $\frac{1}{1+x}$.

$$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-1-1)}{2}x^2$$
$$+\frac{-1(-1-1)(-1-2)}{2\cdot 3}x^3 = 1 - x + x^2 - x^3 + \cdots.$$

EXERCISE 154

Expand:

1. $(a-x)^5$.
2. $(1-x)^6$.
3. $(1-2x)^5$.
4. $(2x-3)^4$.
5. $(2x-1)^5$.
6. $(x-\frac{1}{2})^4$.
7. $\left(x-\frac{1}{x}\right)^6$.
8. $\left(\frac{x}{3}-3y^2\right)^4$.
9. $(1-.01)^5$.
10. $\frac{1}{(1+x)^3}$.
11. $(1-x)^{-2}$.
12. $\sqrt{1+x}$.

ANSWERS

Exercise 1. — **1.** 14, 49, 77, 133, 175. **2.** 88, 132, 165, 264, 352. **3.** 40, 72, 56, 112. **4.** 63, 144, 180, 270. **5.** 8, 16, 12, 64, 20. **6.** 144, 432, 1728, 576, 1008. **7.** 25, 30, 65, 20. **8.** 7, 1, 11, 9. **9.** 28, 140, 4, $2\frac{3}{7}$, $9\frac{1}{7}$. **10.** 108, 48, 3, $4\frac{1}{2}$. **11.** 37, 35, 30. **12.** 42, 70, 28, 63. **13.** 2, 16, $\frac{1}{2}$, 16. **14.** 5, $33\frac{1}{3}$, 3.6, 6.

Exercise 7. — **1.** 13 rd. **2.** 25 yd. **3.** 19 in. **4.** $37. **5.** $87. **6.** 18. **7.** 91 mi. **8.** 165 mi. **9.** 280 mi. **10.** $$20\frac{2}{3}$. **11.** 60¢. **12.** 60¢. **13.** 63 hr.

Exercise 8. — **1.** 75¢, 25¢. **2.** $2400, $1200. **3.** 39 yr., 13 yr. **4.** 75°, 15°. **5.** 60 A., 20 A. **6.** 180 A., 30 A. **7.** $2300, $1150. **8.** $2640, $880. **9.** 77, 7. **10.** 144°, 36°. **11.** 98 yd., $24\frac{1}{2}$ yd. **12.** 124 rd., 62 rd.

Exercise 9. — **1.** A, $7280, B, $1820. **2.** $2850, $950. **3.** $12, $2.40. **4.** $510, $170. **5.** 99. **6.** $18, $3.60. **7.** 246 A., 41 A. **8.** 480 bu., 20 bu.

Exercise 10. — **1.** 14¢, 11¢. **2.** 27¢, 23¢. **3.** 80, 70. **4.** 1710, 1245. **5.** 146,336, 143,783. **6.** 87,512, 80,217. **7.** 582 yd., 483 yd. **8.** 20.5 in., 15.5 in. **9.** $92.50, $47.50. **10.** 3,829,897 oz., 3,805,500 oz. **11.** 17,170,503 oz., 12,002,087 oz. **12.** $284,479,531.81, $230,810,174.17. **13.** 5612 sq. mi., 2380 sq. mi. **14.** 9377 sq. mi., 8173 sq. mi. **15.** 86,085, 13,371.

Exercise 11. — **1.** $1.25. **2.** $63. **3.** 6 T. **4.** 2175. **5.** 77,580 sq. mi. **6.** 74,031. **7.** 10,727,559. **8.** 483 lb. **9.** 9,815,112 T. **10.** 3366 mi.

Exercise 12. — **1.** 16 da. **2.** 9. **3.** 149 T. **4.** 6. **5.** 15. **6.** 25. **7.** 6. **8.** 215. **9.** 109. **10.** 30.

Exercise 13. — **1.** $93, $57. **2.** $175, $35. **3.** 11. **4.** 916,764, 911,933. **5.** 125,560, 62,559. **6.** 104,863, 33,111. **7.** 5 private banks, 9 state banks, 21 national banks. **8.** 75,924.

Exercise 15. — **1.** $x=3$. **2.** $x=4$. **3.** $x=21$. **4.** $x=8$. **5.** $x=11$. **6.** $x=8$. **7.** $y=9$. **8.** $y=11$. **9.** $x=16$. **10.** $x=11$. **11.** $x=11$. **12.** $x=20$. **13.** $x=16$. **14.** $y=11$. **15.** $y=99$. **16.** $x=80$. **17.** $y=19$. **18.** $y=16$. **19.** $y=11$. **20.** $y=8$. **21.** $y=8$.

22. $y = 10$. **23.** $y = 9$. **24.** $y = 11$. **25.** $y = 6$. **26.** $y = 15$. **27.** $y = 18$. **28.** $x = 16$. **29.** $x = 10$. **30.** $x = 5$. **31.** $x = 5$. **32.** $x = 7$. **33.** $x = 20$. **34.** $y = 16$. **35.** $y = 3$. **36.** $y = 27\frac{1}{2}$. **37.** $y = 17$. **38.** $y = 3$. **39.** $y = 9$. **40.** $y = 9$.

Exercise 22.—**1.** 7. **2.** 3. **3.** 1. **4.** 27. **5.** 55. **6.** 2. **7.** 17. **8.** 36. **9.** 5. **10.** 28. **11.** 21. **12.** 28. **13.** 9. **14.** 1. **15.** 11. **16.** 14. **17.** 6. **18.** 19. **19.** 52. **20.** 2. **21.** 73. **22.** 49. **23.** 500. **24.** 250. **25.** 360. **26.** 600. **27.** 2500. **28.** 3600. **29.** 300. **30.** 3000. **31.** 1500. **32.** 1800. **33.** 15,000. **34.** 18,000. **35.** 9000. **36.** 90,000. **37.** 6000. **38.** 1250. **39.** 2160. **40.** 6000. **41.** 0. **42.** 500. **43.** − 55. **44.** 375. **45.** 391. **46.** 1750.

Exercise 23.—**1.** 0. **2.** $9b$. **3.** $-b$. **4.** $-x$. **5.** $-5c$. **6.** $11c$. **7.** $-3c$. **8.** $20c$. **9.** $-22c$. **10.** $8a$. **11.** $6xy$. **12.** $10xy$. **13.** $22ab$. **14.** $-2ab$. **15.** mn. **16.** $-13ax$. **17.** $8by$. **18.** $16ab$. **19.** $7a^2$. **20.** $-3a^2$.

Exercise 24.—**1.** $2a$. **2.** $4a$. **3.** $9x$. **4.** $9x + 9y$. **5.** $20x - 4y$. **6.** $-4x - y$. **7.** $-5x + 10y$. **8.** $3a + b + c$. **9.** $a + b - c$. **10.** $10a - 3b + 3c$. **11.** $13a - 5b$. **12.** $20a - 10b - 14c$. **13.** $23a - 21b - 5c$. **14.** $-3a - 24b - 5c$. **15.** $a - 5b - 15c$. **16.** $8x^2 - 11xy - 14y^2$. **17.** $18x^2 - 14xy + 18y^2$. **18.** $13a^2 - 10a - 5$. **19.** $6x^2 + 20x - 12$. **20.** $3x^2 - 11x$. **21.** $20x^2 + 10x - 7$. **22.** $18a^2 - 18ab - 13b^2$. **23.** $6a^2 + a + 9$. **24.** $6x^2 - 5xy + 2yz$. **25.** $a + b + c$. **26.** 0. **27.** $17mn + 10my - 7mz$. **28.** $2ab$. **29.** $-4a^2 + 13a$. **30.** $-4x^3 - 10xy + 7y^3$. **31.** $1968. **32.** 25 mi. south. **33.** 5 mi. west. **34.** 6° higher than on Sunday.

Exercise 26.—**1.** $6x + 3$. **2.** $2x + 1$. **3.** $3x + 10$. **4.** $2a - 3b$. **5.** $2b$. **6.** $3a - 4b$. **7.** $-m + 11n$. **8.** $-x + 9y$. **9.** $5xy - 2yz$. **10.** $5xy + xz$. **11.** $-4mn + 11xy$. **12.** $2ab + 2bc$. **13.** $6ad - 11ac$. **14** $3x - 4y - 6z$. **15.** $-x - 2y - 3z$. **16.** $4x^2 - x - 3$. **17.** $-x^2 + 9x + 4$. **18.** $x^2 + 6xy + y^2$. **19.** $4y^2 - 6yz$. **20.** $-3x^3 + 14x^2 + 11x$.

Exercise 27.—**1.** $8x - 2$. **2.** $10x - 11$. **3.** $5x$. **4.** $9x - 22$. **5.** $x - 4$. **6.** $17a - 4b$. **7.** $10a + 2b$. **8.** $2x^2 - x$. **9.** $a + 7b + 7$. **10.** $-a + 11$. **11.** $2a - 2b - 2c$. **12.** $5x - 16$. **13.** $2m + 2n$. **14.** $12y$. **15.** $14a - 10b$. **16.** $5b - 3a$. **17.** $9y - 2x$. **18.** $10 - 6x$. **19.** 0. **20.** y.

Exercise 29.—**1.** $10x + 30$. **2.** $16x + 32$. **3.** $15x - 6$. **4.** $63x - 77$. **5.** $54x - 63$. **6.** $55a^2 - 22$. **7.** $108a^2 - 96$. **8.** $25y^2 - 40y$. **9.** $90n^2 - 80n$. **10.** $28m - 30$. **11.** $-12x + 12$. **12.** $-24x + 28$. **13.** $-30x - 48$. **14.** $-72y^2 + 96$. **15.** $-18 + 4m^2$. **16.** $-6 + 4n^2$. **17.** $-171 + 6a^2$. **18.** $-28 + 14a^3$.

Exercise 30. — 1. 3^6. 2. 2^7. 3. 7^8. 4. 5^{10}. 5. 8^5. 6. 9^{10}. 7. 10^9. 8. 10^{12}. 9. 2^{20}. 10. a^5. 11. a^{11}. 12. b^{18}. 13. b^{18}. 14. a^3b^3. 15. a^5b^6. 16. a^6b^6. 17. a^7b^8. 18. a^6. 19. a^8. 20. a^{14}. 21. $32\,x^2y$. 22. $49\,ab^2$. 23. $-48\,a^3b$. 24. $-72\,a^4b^3$. 25. $-30\,a^6b^5$. 26. $-45\,a^4b^9$. 27. $-120\,a^8b^7$. 28. $-25\,m^4n^4$. 29. $-23\,m^8n^8$. 30. $-12\,m^8n^5$. 31. a^{2n}. 32. x^{n+1}. 33. x^{n+4}.

Exercise 31. — 1. $12\,x-20\,y-16$. 2. $14\,x-21\,a+56$. 3. $9\,x-36\,a-72$. 4. $-10\,x+15\,y+35$. 5. $-12\,a+6\,b+15$. 6. $-24\,a+56\,b+72$. 7. $4\,x^3-8\,x^2-12\,x$. 8. $5\,a^3-15\,a^2-35\,a$. 9. $-a^3+9\,a^2+6\,a$. 10. $-b^3+ab^2+10\,b$. 11. $-16\,x^3+24\,x^2y-8\,xy^2$. 12. $-18\,x^4+45\,x^3y+72\,xy^2$. 13. $4\,x-3\,y-5$. 14. $6\,x-y-3$. 15. $6x-2y-5$. 16. $4x-7y-5$. 17. $-10x-9y+7$. 18. $-16a+6b+15$. 19. $-25\,x+4\,y+15$. 20. $-10\,x+4\,y+40$. 21. $-70\,x+90\,y+79$. 22. $-25\,x+14\,y+41$. 23. $10\,x-37$. 24. $5\,x-23$.

Exercise 32. — 1. 3^2. 2. 4^3. 3. 5^2. 4. 7^4. 5. 3^3. 6. 3^5. 7. 5. 8. 11^3. 9. 8^2. 10. 6^6. 11. 3^4. 12. 9^2. 13. 13^4. 14. 17^2. 15. a. 16. a^3. 17. a^4. 18. a^7. 19. b^3. 20. c^3. 21. c^2. 22. a^7. 23. x^7. 24. x^4. 25. y^4. 26. m^3. 27. n^3. 28. n^9.

Exercise 33. — 1. $4\,x$. 2. $5\,x$. 3. $3\,a$. 4. $-8\,b$. 5. $-4\,c$. 6. $-8\,c^2$. 7. $-3\,n^3$. 8. $2\,x$. 9. $3\,x$. 10. $-4\,x$. 11. $-5\,c^2$. 12. $6\,n^2$. 13. $-10\,n^2$. 14. $2\,a^3$. 15. $-3\,a^2$. 16. $-3\,m^2$. 17. $3\,xy$. 18. $5\,x^2$. 19. $-7\,y^2$. 20. $5\,a^3b$. 21. $-6\,ab^3$. 22. $-10\,c^2n$. 23. $-7\,a^2c^3$. 24. $15\,a^4c^4$. 25. $-2\,a^3c^2$. 26. $-9\,a^6c$. 27. $3\,c^3n^2$. 28. $11\,a^3$.

Exercise 34. — 1. $a+1$. 2. $2\,x+4$. 3. $2\,x+3$. 4. $5\,x-2$. 5. $5\,x-1$. 6. $7\,x-3$. 7. $4\,a-3\,b$. 8. $2\,a-6\,b$. 9. $10\,a-14\,b$. 10. $11\,a^2-13\,b^2$. 11. $3\,a^2-4\,b^2$. 12. $7\,x^2-8\,x-4$. 13. $a+b-c$. 14. a^2+a-2. 15. $a^2+2\,a+1$. 16. $2\,xy-3$. 17. $2\,ab-3\,b^2$. 18. $2\,x-3\,y$. 19. $3\,x^2-4\,x$. 20. $3\,x^3-4\,x^2$. 21. $2\,a^2-3\,ab$. 22. $2\,x^2-3\,x$. 23. $12-13\,a$. 24. $3\,x^2-4$.

Exercise 35. — 1. $5(x+1)$. 2. $7(x-2)$. 3. $6(a-2)$. 4. $8(a+b)$. 5. $9(x-y)$. 6. $11(x-3)$. 7. $12(x-5)$. 8. $a(a+1)$. 9. $m(m+1)$. 10. $y(y-2)$. 11. $x(x-4)$. 12. $x(2\,x-1)$. 13. $x(4\,x-y)$. 14. $5\,a(a-b)$. 15. $7\,a(x-2\,y)$. 16. $9\,a(a-2)$. 17. $4\,a(a^2-2)$. 18. $6\,a^2(a-2)$. 19. $10\,x^2(x-1)$. 20. $xy(x-y)$. 21. $x(1-\frac{1}{100})$. 22. $x(\frac{49}{50})$. 23. $x(\frac{39}{40})$. 24. $\frac{1}{2}(x+1)$. 25. $\frac{1}{2}(x+y+z)$. 26. $\frac{1}{2}\,b(c+d)$.

Exercise 36. — 1. $6\,x$. 2. $12\,x$. 3. $\frac{21\,x}{2}$. 4. $\frac{35\,x}{2}$. 5. $\frac{32\,x}{3}$. 6. $2\,x$. 7. x. 8. $\frac{9\,x^2}{4}$. 9. $\frac{64\,x^2}{17}$. 10. $\frac{13\,a}{2}$. 11. $\frac{21\,a^2}{5}$. 12. $\frac{4}{3\,ax}$. 13. $\frac{x}{2\,a}$. 14. $12\,y$. 15. $\frac{2}{x}$. 16. $\frac{10\,a}{by}$. 17. $\frac{15\,x}{4\,b}$. 18. $\frac{3\,ay}{2\,x}$. 19. $\frac{4\,xy}{b}$.

Exercise 37. — 1. $4x-4$. 2. $14x-56$. 3. $45a-45b$. 4. $21m-21n$. 5. $42x-189$. 6. $99x-231$. 7. $3x-24$. 8. $15x-50$. 9. $-14x+22$. 10. $-27x-6$. 11. $-108+36x$. 12. $9x-21$.

Exercise 38. — 1. $\frac{x}{8}$. 2. $\frac{x}{22}$. 3. $\frac{25x}{144}$. 4. $4x$. 5. $\frac{32x}{225}$. 6. $\frac{xy}{3}$. 7. $\frac{4d}{b^2}$. 8. $\frac{9a}{10}$. 9. $\frac{2a}{x}$. 10. $\frac{6na}{b}$. 11. $\frac{6m}{7a^2bn}$. 12. $\frac{7b}{5a}$.

Exercise 39. — 1. $\frac{3x}{2}$. 2. $\frac{5x}{3}$. 3. $\frac{6a-1}{4}$. 4. $\frac{13a-4}{5}$. 5. $\frac{2a}{3}$. 6. $\frac{2a+1}{3}$. 7. $\frac{11a+4}{7}$. 8. $\frac{22a-5}{6}$. 9. $\frac{13a+3}{5}$. 10. $\frac{a^2+b^2}{a}$. 11. $\frac{a^2-c^2}{a}$. 12. $\frac{a^2+b}{a}$. 13. $\frac{2a^2-b}{a}$. 14. $\frac{x}{x-1}$. 15. $\frac{2x-7}{x-2}$. 16. $\frac{3x-1}{x+1}$. 17. $\frac{b^2}{a}$. 18. $\frac{x^2}{x-2}$.

Exercise 40. — 1. $\frac{13a-7}{12}$. 2. $\frac{17c-11}{12}$. 3. $\frac{8}{15}$. 4. $\frac{4c-1}{18}$. 5. $\frac{16a-73}{12}$. 6. $\frac{16a-27}{18}$. 7. $\frac{23a-28}{36}$. 8. $\frac{11}{36}$. 9. $-\frac{5c+108}{40}$. 10. $\frac{5a+17}{45}$. 11. $\frac{7m-8}{28}$. 12. $\frac{2n+33}{42}$. 13. $\frac{31a-48b}{60}$. 14. $\frac{17x-113y}{36}$. 15. 0. 16. 0. 17. $\frac{10a}{33}$. 18. $\frac{5x+10y}{12}$. 19. ab. 20. 0.

Exercise 41. — 1. $x=5$. 2. $x=8$. 3. $x=4$. 4. $x=5$. 5. $x=1$. 6. $x=6$. 7. $x=10$. 8. $x=4$. 9. $x=10\frac{2}{3}$. 10. $x=12$. 11. $x=4$. 12. $x=4$. 13. $x=-9$. 14. $x=1$. 15. $x=-3$. 16. $x=5$. 17. $x=9$. 18. $x=5$. 19. $x=5$. 20. $x=10$.

Exercise 42. — 1. $x=3$. 2. $x=4$. 3. $x=5$. 4. $x=5$. 5. $x=6$. 6. $x=5$. 7. $x=3$. 8. $x=7$. 9. $x=9$. 10. $x=11$. 11. $x=4$. 12. $x=-34$. 13. $x=15$. 14. $x=19$. 15. $x=0$. 16. $x=3$. 17. $x=6$. 18. $x=0$. 19. $x=7$. 20. $x=0$.

Exercise 44. — 1. 65¢, 25¢. 2. 70 yr., 24 yr. 3. Henry, 17 yr., Thomas, 15 yr., John, 10 yr. 4. 13 boys, 27 girls. 5. 1529, 1992. 6. 70 in., 20 in. 7. 289, 307, 631. 8. 97, 311, 527. 9. A's, $8000, B's, $4500. 10. $8000, $7500, $7000. 11. 20 apples, 30 pears. 12. 15 acres corn, 25 acres wheat. 13. 10 horses, 7 mules. 14. 80 acres, 50 acres. 15. 80 yd., 50 yd. 16. 90 yd., 40 yd. 17. 5 nickels, 3 dimes. 18. 7 dimes, 4 quarters. 19. 11, 6. 20. 49, 51. 21. 32, 33, 34. 22. 34, 36, 38. 23. 16, 17. 24. 11, 16. 25. 14, 18, 22, 26. 26. $4,

\$5, \$20. **27.** 9 hr. **28.** 8 o'clock P.M. **29.** 4 quarters, 12 dimes, 36 nickels. **30.** 70 yd., 40 yd.

Exercise 45. — 1. $x = 18$. **2.** $x = 9$. **3.** $x = 12$. **4.** $x = 30$. **5.** $x = 24$. **6.** $x = 36$. **7.** $x = 12$. **8.** $x = 24$. **9.** $x = 5$. **10.** $x = 20$. **11.** $x = 5$. **12.** $x = 4$. **13.** $x = 4$. **14.** $x = 5$. **15.** $x = 3\frac{3}{4}$. **16.** $x = 5$. **17.** $x = 6$. **18.** $x = 10$. **19.** $x = 7$. **20.** $x = 10$. **21.** $x = 13$. **22.** $x = 6$. **23.** $x = 5$. **24.** $x = 6$. **25.** $x = 15$. **26.** $x = 8$. **27.** $x = 5$. **28.** $x = 4$. **29.** $x = 8$. **30.** $x = 4$. **31.** $x = 5$. **32.** $x = 9$. **33.** $x = 10$. **34.** $x = 5$. **35.** $x = 16$.

Exercise 46. — 1. 60¢. **2.** \$100. **3.** 90. **4.** 450 mi. **5.** 444 mi. **6.** 325 lb. **7.** 42 yr. **8.** \$120 per month. **9.** \$108,000. **10.** 360 bbl. **11.** 1500 bu. **12.** 80 mi. **13.** 400. **14.** \$960. **15.** 500.

Exercise 47. — 1. 48, 72. **2.** 120, 150. **3.** 60, 64. **4.** 110 acres, 70 acres. **5.** 60, 90. **6.** 15¢, 20¢. **7.** 100 yd., 80 yd. **8.** 1250 yd., 500 yd. **9.** A's, \$90, B's, \$20. **10.** 10 yr., 40 yr. **11.** 40 yr., 30 yr. **12.** \$1.45, \$2.60. **13.** 6 quarters, 15 dimes. **14.** 26 days. **15.** \$5100. **16.** 240 yd., 120 yd. **17.** 60 yr., 30 yr. **18.** 45 yr., 9 yr. **19.** A, \$40, B, \$75. **20.** \$75, \$50. **21.** 10 lb. at 60¢, 6 lb. at 75¢. **22.** 30 acres, $22\frac{1}{2}$ acres. **23.** A, \$70, B, \$140, C, \$210.

Exercise 48. — 1. $x = \frac{1}{3}$. **2.** $x = 6$. **3.** $x = \frac{1}{2}$. **4.** $x = 4$. **5.** $x = 8$. **6.** $x = 9$. **7.** $x = 10$. **8.** $x = 7$. **9.** $x = 5$. **10.** $x = 8$. **11.** $x = 9$. **12.** $x = 3$. **13.** $x = 0$. **14.** $x = 10$. **15.** $x = 11$.

Exercise 49. — 1. 5 hr. **2.** 27 hr. **3.** 5 yr. **4.** 5 hr. **5.** 3 o'clock P.M.

Exercise 50. — 1. $x = 8$. **2.** $x = 13$. **3.** $x = 7$. **4.** $x = 5$. **5.** $x = 11$. **6.** $x = 8$. **7.** $x = 13$. **8.** $x = 1$. **9.** $x = 7$. **10.** $x = 0$. **11.** $x = 12$. **12.** $x = -13$. **13.** $x = 15$. **14.** $x = 0$. **15.** $x = 21$. **16.** 360 mi. **17.** 30 mi., 20 mi. **18.** 32 mi. **19.** \$3.50, \$12.25.

Exercise 51. — 1. \$1300. **2.** \$5625. **3.** 112.5 lb. **4.** 31,250. **5.** \$80. **6.** 91 acres. **7.** 450. **8.** 2500 mi. **9.** 42 yr. **10.** \$15,200.

Exercise 52. — 1. $2\frac{1}{2}$%. **2.** $12\frac{1}{2}$%. **3.** 80%. **4.** 5%. **5.** $83\frac{1}{3}$%. **6.** $2\frac{1}{2}$%. **7.** $4\frac{1}{2}$%. **8.** 9.52% nearly. **9.** 45%. **10.** 48%.

Exercise 53. — 1. \$80. **2.** \$2.20. **3.** \$200. **4.** \$30. **5.** \$9600. **6.** 150 ft. **7.** \$19. **8.** \$788.30. **9.** \$450,000,000. **10.** 17,170,000 oz. nearly.

Exercise 54. — 1. 1 yr. **2.** $1\frac{1}{4}$ yr. **3.** $1\frac{1}{2}$ yr. **4.** 1 yr. 3 mo. **5.** 1 yr. 2 mo. 12 da. **6.** 1 yr. 8 mo. **7.** 1 yr. 4 mo. **8.** 1 yr. 1 mo. 10 da. **9.** 1 yr. 4 mo. 24 da. **10.** 3 mo. 6 da.

Exercise 55. — 1. 5%. **2.** 7%. **3.** 9%. **4.** 6%. **5.** 7%. **6.** 8%. **7.** 5%. **8.** 3%. **9.** 4%. **10.** 4%.

Exercise 56. — 1. \$762. 2. \$615. 3. \$773. 4. \$590. 5. \$1370. 6. \$1480. 7. \$1632. 8. \$880. 9. \$638. 10. \$1200.

Exercise 57. — 1. \$720. 2. \$1500. 3. \$1900. 4. \$485. 5. \$456. 6. \$1850. 7. \$1600. 8. 1100. 9. \$381.69. 10. \$575.

Exercise 58. — 1. 4 yr., $3\frac{1}{3}$ yr., $2\frac{1}{2}$ yr. 2. $12\frac{1}{2}$ yr., $8\frac{1}{3}$ yr., 5 yr. 3. 8 yr. 4. $16\frac{2}{3}$ yr., $12\frac{1}{2}$ yr., $14\frac{2}{7}$ yr. 5. 5%. 6. 5%, 4%, 8%. 7. 6%. 8. 10 yr., $8\frac{1}{3}$ yr., $6\frac{1}{4}$ yr.

Exercise 59. — 1. $x = 13$, $y = 5$. 2. $x = 6$, $y = 4$. 3. $x = 11$, $y = 4$. 4. $x = 11$, $y = 8$. 5. $x = 9$, $y = 2$. 6. $x = 6$, $y = 5$. 7. $x = 6$, $y = 7$. 8. $x = 5$, $y = 2$. 9. $x = 1$, $y = 11$. 10. $x = 5$, $y = 4$. 11. $x = 4$, $y = 1$. 12. $x = 10$, $y = 4$. 13. $x = 9$, $y = 1$. 14. $x = 6$, $y = 7$. 15. $x = 8$, $y = 2$. 16. $x = 3$, $y = -1$. 17. $x = 6$, $y = 3$. 18. $x = 4$, $y = -1$. 19. $x = 5$, $y = 2$. 20. $x = 3$, $y = -2$. 21. $x = 7$, $y = -3$. 22. $x = 4$, $y = -4$. 23. $x = 3$, $y = 0$. 24. $x = 5$, $y = 8$. 25. $x = 1$, $y = 1$. 26. $x = 2$, $y = 2$. 27. $x = 2$, $y = -1$. 28. $x = 1$, $y = -2$. 29. $x = 2$, $y = -3$. 30. $x = 3$, $y = -4$.

Exercise 60. — 1. $x = 8$, $y = 6$. 2. $x = 29.4$, $y = 15.2$. 3. $x = 6$, $y = 10$. 4. $x = 9$, $y = 11$. 5. $x = 8$, $y = 5$. 6. $x = 11$, $y = 15$. 7. $x = 7$, $y = 8$. 8. $x = 12$, $y = 4$. 9. $x = 16$, $y = 2$. 10. $x = 8$, $y = 1$. 11. $x = 9$, $y = 2$. 12. $x = 18$, $y = 20$.

Exercise 61. — 1. 35, 24. 2. 56, 24. 3. 15, 5. 4. \$5, \$4. 5. \$5, \$3. 6. \$90, \$50. 7. 55¢, 50¢. 8. 74. 9. 18. 10. 26. 11. 84. 12. 3 \$10 bills, 14 \$5 bills. 13. 7 half-eagles, 5 eagles. 14. 7 measures, 5 measures. 15. A's money, \$80, B's money, \$60. 16. A, \$15, B, \$25. 17. A, \$70, B, \$50. 18. \$5000 at 4%, \$3000 at 5%. 19. 30 mi. per hr., 24 mi. per hr. 20. 15¢ per lb., 18¢ per lb. 21. 35¢, 45¢. 22. 80¢, 65¢. 23. 24, 40. 24. 8 dollars, 4 half-dollars. 25. 54 yr., 18 yr. 26. 19.2 mi. per hr., 28.8 mi. 27. \$3.20, A's wages; \$2.40, B's wages. 28. 40°, 50°. 29. 36°, 54°, 90°. 30. 40 yd., 30 yd.

Exercise 62. — 1. $x = 1$, $y = 2$, $z = 3$. 2. $x = 3$, $y = 2$, $z = 4$. 3. $x = 5$, $y = 1$, $z = 3$. 4. $x = 6$, $y = 2$, $z = 4$. 5. $x = 7$, $y = 3$, $z = 2$. 6. $x = 1$, $y = 2$, $z = 4$. 7. $x = 5$, $y = -4$, $z = 2$. 8. $x = 6$, $y = -1$, $z = -2$. 9. $x = 4$, $y = 6$, $z = 8$. 10. $x = 3$, $y = 2$, $z = 10$. 11. $x = 10$, $y = 8$, $z = -1$. 12. $x = 10$, $y = 9$, $z = -2$.

Exercise 63. — 1. 648. 2. 237. 3. 125. 4. 37, 29, 24. 5. 36, 18, 26. 6. A, \$3, B, \$4, C, \$6. 7. 2 dollars, 3 half-dollars, 6 quarters. 8. 12, 18, 24. 9. Apples, 3¢ apiece, pears, 2¢ apiece, oranges, 4¢ apiece. 10. \$1440 in 3%, \$900 in 5%, \$1080 in 6%.

Exercise 66. — 1. $x^2 + 3x + 2$. 2. $x^2 + 8x + 15$. 3. $x^2 + 10x + 16$. 4. $x^2 + 21x + 108$. 5. $x^2 + 11x + 30$. 6. $x^2 + 16x + 63$. 7. $x^2 - 2x$

-8. **8.** $x^2-5x-24$. **9.** $x^2-11x+18$. **10.** x^2-6x+5. **11.** $x^2-11x+24$. **12.** $x^2-10x+25$. **13.** $4a^2-18a+20$. **14.** $12a^2+a-6$ **15.** $30a^2-61ab+30b^2$. **16.** $9a^2+9ac-28c^2$. **17.** $4a^2-25c^2$. **18.** $64a^2-25c^2$. **19.** $16m^2-8mn-48n^2$. **20.** $27m^2-57mn+20n^2$. **21.** $56c^2-37cd+6d^2$. **22.** $20b^2-57bd+40d^2$. **23.** $72a^2-79ax+14x^2$. **24.** $a^4+3a^2b^2+2b^4$. **25.** $24a^4+42a^2b^2+15b^4$. **26.** $49c^4-16$. **27.** $72x^4-x^3-x^2$. **28.** $42x^4-17x^3-15x^2$. **29.** $55x^4-34x^3-8x^2$. **30.** $36x^4-y^4$. **31.** $16a^4-81b^4$. **32.** $6x^4+7x^2n^2-3n^4$. **33.** x^4+x^3+x+1. **34.** $a^4-a^3-a^2+a$. **35.** x^6-y^6. **36.** $4x^6-9y^6$. **37.** $x^4+2x^2y^2+y^4$. **38.** x^3-1. **39.** x^8-y^8. **40.** a^5-a.

Exercise 67.—**1.** x^3-1. **2.** x^3+1. **3.** x^3-y^3. **4.** x^3+y^3. **5.** x^3-x^2-4x+4. **6.** x^3+x^2-9x-9. **7.** $4x^3+8x^2-x-2$. **8.** $9x^3-27x^2+47x-22$. **9.** $4x^3+2x^2y-3xy^2+18y^3$. **10.** x^6-1. **11.** x^6+1. **12.** a^6-b^6. **13.** a^6+b^6. **14.** $a^6+6a^4+11a^2+6$. **15.** $x^4-3x^3-5x^2-25x$. **16.** $x^3-6x^2y+12xy^2-8y^3$. **17.** $x^3-2x^2y-4xy^2+8y^3$. **18.** $x^3-3x^2y-9xy^2+27y^3$. **19.** $8x^3-36x^2y+54xy^2-27y^3$. **20.** $27x^3-36x^2+9x+2$. **21.** $1-x^4$. **22.** a^4-b^4. **23.** $16x^4-1$. **24.** $81x^4-y^4$. **25.** $x^3+3xy-y^3+1$.

Exercise 68.—**1.** $x-2$. **2.** $x-2$. **3.** $x-2$. **4.** $a-6b$. **5.** $a-5b$. **6.** $a-7b$. **7.** $c+d$. **8.** $x-3y$. **9.** $4a-3b$. **10.** $2a-5b$. **11.** $4a-3b$. **12.** $3a+5b$. **13.** $3a+2b$. **14.** $4x-y$. **15.** $5x+1$. **16.** a^2+a. **17.** $2x^2-x$. **18.** $4x^2-2x$. **19.** $3x^2-5x$. **20.** $3x^2-7x$. **21.** $3a^2-6ab$. **22.** $2mn+3n^2$. **23.** x^2+2x-1. **24.** x^2+3x+5. **25.** a^2-4a+9. **26.** a^2-3a-2. **27.** $x^2-4x+15$. **28.** $2x^2+10x+45$. **29.** $4x^2-3x-3$. **30.** a^2+a+1. **31.** x^2+2x+4. **32.** x^2-3x+9. **33.** a^2-ab+b^2. **34.** $4x^2+10xy+25y^2$. **35.** $9a^2+12ab+16b^2$. **36.** $25x^2-5xy+y^2$. **37.** $36x^2+42xy+49y^2$. **38.** $a-b$. **39.** x^2-1. **40.** $x+3$.

Exercise 69.—**1.** $1+x+x^2$. **2.** $1-x+x^2$. **3.** $1+x+x^2+x^3$. **4.** $4+2a+a^2$. **5.** $9-3b+b^2$. **6.** $8+4a+2a^2+a^3$. **7.** $27+9x+3x^2+x^3$. **8.** $4+4x+2x^2$. **9.** $7+9x+3x^2$. **10.** $4-4x-x^2$. **11.** $1+a+a^2$. **12.** $1-2a+3a^2$.

Exercise 70.—**1.** 5. **2.** 10. **3.** 6. **4.** 8. **5.** 1. **6.** 2. **7.** 48. **8.** $-a$. **9.** $3a$. **10.** $6c$. **11.** $-2b$. **12.** $-3x$. **13.** $8n$. **14.** $3m-2n$. **15.** $5n$. **16.** $9x-12y$.

Exercise 71.—**1.** 24. **2.** -30. **3.** $2a^2$. **4.** 0. **5.** 0. **6.** 0. **7.** $-2a^3$. **8.** $-a^4$. **9.** a^6. **10.** a^6. **11.** a^4. **12.** b^4c^2. **13.** a^4c^6. **14.** $-27a^9b^6$. **15.** $a^2-2ab+b^2$. **16.** $a^2-4ac+4c^2$. **17.** $a^2+b^2+c^2-2ab-2ac+2bc$. **18.** $x^4-2x^3-x^2+2x+1$. **19.** $a^4-4a^3+6a^2-4a+1$. **20.** $x^4-2x^2y^2+y^4$. **21.** 5. **22.** 0. **23.** 1. **24.** 125.

25. 27. **26.** 13. **27.** 5. **28.** $2\frac{2}{3}$. **29.** 32. **30.** 4. **31.** -1. **32.** 26. **33.** 65. **34.** 45. **35.** 6. **36.** 30. **37.** 30. **38.** 60.

Exercise 73. — **1.** $a(a+2)$. **2.** $x(x-1)$. **3.** $x(a+b)$. **4.** $x^2(a+c)$. **5.** $x(x^2-x-1)$. **6.** $x(y+z-u)$. **7.** $x(a^2-b^2-c^2)$. **8.** $x(x-\frac{1}{3})$. **9.** $m(m-\frac{1}{2})$. **10.** $\frac{1}{3}c(m-n)$. **11.** $3x^2(x-2)$. **12.** $6x^2(2x^2-3)$.

Exercise 74. — **1.** $(x+1)^2$. **2.** $(x+2)^2$. **3.** $(x+5)^2$. **4.** $(a+7)^2$. **5.** $(a+10)^2$. **6.** $(2x+y)^2$. **7.** $(3x+y)^2$. **8.** $(4x+3y)^2$. **9.** $(5x+2y)^2$. **10.** $(5x+4y)^2$. **11.** $(7a+b)^2$. **12.** $(8a+3b)^2$. **13.** $(9a+2b)^2$. **14.** $(10a+7b)^2$. **15.** $(a^2+1)^2$. **16.** $(x^2+y^2)^2$. **17.** $(3x^2+y^2)^2$. **18.** $(4x^2+1)^2$. **19.** $(2+7a^2)$. **20.** $(x^3+2)^2$.

Exercise 75. — **1.** $(2x-1)^2$. **2.** $(3x-1)^2$. **3.** $(x-y)^2$. **4.** $(4x-y)^2$. **5.** $(5x-3y)^2$. **6.** $(9a-5b)^2$. **7.** $(10a-3b)^2$. **8.** $(12m-7n)^2$. **9.** $(x^2-2y^2)^2$. **10.** $(x^4-3)^2$. **11.** $(11a-6b)^2$. **12.** $(13a^2-5b^2)^2$.

Exercise 76. — **1.** $(x+1)(x-1)$. **2.** $(a+2)(a-2)$. **3.** $(b+3)(b-3)$. **4.** $(a+6)(a-6)$. **5.** $(a+8)(a-8)$. **6.** $(m+10)(m-10)$. **7.** $(2x+5)(2x-5)$. **8.** $(3x+7)(3x-7)$. **9.** $(4x+9)(4x-9)$. **10.** $(5x+6y)(5x-6y)$. **11.** $(8y+3x)(8y-3x)$. **12.** $(10x+1)(10x-1)$. **13.** $(12y+5z)(12y-5z)$. **14.** $(ab+7c)(ab-7c)$. **15.** $(x^2+5y)(x^2-5y)$. **16.** $(b^2+11c)(b^2-11c)$. **17.** $(5c^2+4d^2)(5c^2-4d^2)$. **18.** $(13a^2+b^2)(13a^2-b^2)$. **19.** $(20+x^2)(20-x^2)$. **20.** $(30+a^2)(30-a^2)$. **21.** $(40+x)(40-x)$. **22.** $(ab+14d)(ab-14d)$. **23.** $(x+\frac{1}{2})(x-\frac{1}{2})$. **24.** $(2x+\frac{1}{3})(2x-\frac{1}{3})$. **25.** $(a+\frac{1}{5})(a-\frac{1}{5})$. **26.** $\left(x+\frac{1}{a}\right)\left(x-\frac{1}{a}\right)$.

Exercise 77. — **1.** $(x+2)(x+1)$. **2.** $(x+3)(x+2)$. **3.** $(x+3)(x+4)$. **4.** $(x+2)(x+6)$. **5.** $(x+1)(x+12)$. **6.** $(x+7)(x+8)$. **7.** $(x+3)^2$. **8.** $(x+4)^2$. **9.** $(x+4)(x+5)$. **10.** $(x+5)(x+6)$. **11.** $(x+2)(x+9)$. **12.** $(x+4)(x+7)$. **13.** $(x-1)(x-5)$. **14.** $(x-2)(x-5)$. **15.** $(x-5)^2$. **16.** $(x-6)^2$. **17.** $(x-6)(x-7)$. **18.** $(x-2)(x-21)$. **19.** $(x-2)(x-25)$. **20.** $(x-2)(x-26)$. **21.** $(x-4)(x-15)$. **22.** $(x-4)(x-18)$. **23.** $(a-1)(a-24)$. **24.** $(a-3)(a-30)$. **25.** $(a+2b)(a+12b)$. **26.** $(c+3d)(c+12d)$. **27.** $(m-4n)(m-12n)$. **28.** $(m-6n)(m-12n)$. **29.** $(a-6b)(a-15b)$. **30.** $(p-7q)(p-14q)$. **31.** $(b+12c)(b+17c)$. **32.** $(b-12c)(b-20c)$. **33.** $(b-13c)(b-14c)$. **34.** $(x-14y)(x-15y)$.

Exercise 78. — **1.** $(x+5)(x-3)$. **2.** $(x+7)(x-4)$. **3.** $(x-7)(x+4)$. **4.** $(x+5)(x-1)$. **5.** $(x+6)(x-2)$. **6.** $(x-6)(x+2)$. **7.** $(x+9)(x-4)$. **8.** $(x-10)(x+5)$. **9.** $(x-7)(x+1)$. **10.** $(x+9)(x-2)$. **11.** $(x-9)(x+2)$. **12.** $(x-6)(x+5)$. **13.** $(x+7)(x-6)$. **14.** $(x+10)(x-9)$. **15.** $(x-12)(x+1)$. **16.** $(x+12)(x-1)$.

17. $(x-12)(x+2)$. 18. $(x+13)(x-9)$. 19. $(x-12)(x+7)$. 20. $(x+12)(x-5)$. 21. $(x-16)(x+6)$. 22. $(x-15)(x+6)$. 23. $(x-15)(x+9)$. 24. $(a+13)(a-8)$.

Exercise 79. — 1. $(3x+5)(x+4)$. 2. $(2x+1)(x+2)$. 3. $(2x+1)(x+5)$. 4. $(2x+3)(x+6)$. 5. $(4x+3)(2x+3)$. 6. $(3x-1)(2x-5)$. 7. $(4x-1)(3x-4)$. 8. $(5x-2)(2x-5)$. 9. $(3x-2)(2x+3)$. 10. $(4x-3)(3x+2)$. 11. $(7x-2)(2x+9)$. 12. $(x+7)(4x-5)$. 13. $(2x-1)(x+8)$. 14. $(x-4)(4x-1)$. 15. $(2x-1)(x+7)$.

Exercise 80. — 1. a^2b^2c. 2. x^3y^2. 3. x^3y^3. 4. $6x^2y^2$. 5. $36x^2y$. 6. $56x^2y^2$. 7. $72x^2y$. 8. $30a^2b^2$. 9. $72x^2y^2$. 10. $42a^4b^3$. 11. $40x^3y^4$. 12. $24a^4b^5$.

Exercise 81. — 1. $x(x+3)$. 2. $3x(x+2)$. 3. $2x(x-1)$. 4. $3x(x-3)$. 5. $x(x^2-1)$. 6. $x(x+1)(x+2)$. 7. $x(x+1)(x+3)$. 8. $x^2(x-2)$. 9. $x^2(x-4)$. 10. $(x-1)(x-2)(x-5)$. 11. $(x-y)(x+y)(x-4y)$. 12. $(x-2y)(x+2y)(x+3y)$. 13. $x(1-x^2)$. 14. $(1+2x)(1-2x)^2$. 15. $(3x+1)(3x-1)^2$. 16. $(x+4y)(x-4y)^2$.

Exercise 82. — 1. $\frac{2}{x+1}$. 2. $\frac{2b}{a+b}$. 3. $\frac{2a^2+b}{a+b}$. 4. $\frac{x^2-2}{x+1}$. 5. $\frac{a^2-2ab}{a-b}$. 6. $\frac{x^3+2}{x+1}$. 7. $\frac{m^2-3n^2}{m+2n}$. 8. $\frac{x^3}{x+y}$. 9. $\frac{2a^2}{a+b}$. 10. $-\frac{1}{x-1}$. 11. $\frac{3b}{a+b}$. 12. $-\frac{8y}{x+y}$. 13. $\frac{m^2-m}{m+3}$. 14. $\frac{2a}{2-a}$. 15. $\frac{x^2-10x}{x-5}$. 16. $\frac{x^2-12x}{6-x}$.

Exercise 83. — 1. $\frac{7x}{12}$. 2. $\frac{18x-11}{20}$. 3. $\frac{7x+12}{10}$. 4. $\frac{8x+4}{15}$. 5. $\frac{3x+55}{24}$. 6. $\frac{x+7}{6}$. 7. $\frac{13x-4}{14}$. 8. $\frac{11x-9}{10}$. 9. $\frac{25x-1}{30}$. 10. $\frac{17a+32b}{15b}$. 11. $\frac{5a-3b}{6b}$. 12. $\frac{3x+22}{6x}$. 13. $\frac{10x+13}{6x}$. 14. $\frac{33x-4}{30x}$. 15. $\frac{5a+72}{24a}$. 16. $\frac{ac-b^2}{ac}$. 17. $\frac{4x-3y}{xy}$. 18. $\frac{7x^2-15y^2}{6xy}$. 19. $\frac{16x^2-3y^2}{12xy}$. 20. 0. 21. $-\frac{2(x+y)}{xy}$. 22. $-\frac{3b+ay}{ab}$. 23. $\frac{1}{x(x+1)}$. 24. $\frac{2}{x^2-1}$. 25. $\frac{4x}{x^2-1}$. 26. $\frac{7}{3x(3x+7)}$. 27. $\frac{20x-35}{2x(2x-7)}$. 28. $\frac{4x+15}{4x(4x+5)}$. 29. $\frac{28}{x(x-4)}$. 30. $\frac{72}{x^2-16}$. 31. $\frac{x^2-xy+y^2}{x(x-y)}$. 32. $\frac{a}{a+b}$. 33. $\frac{2b}{a-b}$. 34. $\frac{3}{x+3}$.

Exercise 84.—1. $\frac{2}{7}$. 2. $\frac{2y}{x}$. 3. $\frac{3}{16}$. 4. $\frac{1}{5x}$. 5. $\frac{1}{3}$. 6. $\frac{1}{x^2}$.
7. $\frac{1}{(x-1)(x-2)}$. 8. $\frac{x}{y}$. 9. $\frac{x+1}{x}$. 10. $x-1$. 11. x^4-x^2.
12. $54x$. 13. $\frac{11x(x-5)}{12}$. 14. $\frac{11a^2-21a-2}{24}$. 15. $1\frac{1}{3}$. 16. $1\frac{1}{2}$.
17. $\frac{x^2}{(x+y)^2}$. 18. b. 19. 1. 20. $\frac{x-3}{3(x+3)}$. 21. $\frac{2}{3}$. 22. 1.
23. $\frac{2(a+1)}{a-1}$. 24. $\frac{x+y}{xy}$.

Exercise 85.—1. $x=60$. 2. $x=12$. 3. $x=15$. 4. $x=6$. 5. $x=11$. 6. $x=13$. 7. $x=10$. 8. $x=4$. 9. $x=12$. 10. $x=0$. 11. $x=3$. 12. $x=6$. 13. $x=5$. 14. $x=4$. 15. $x=2$. 16. $x=7$. 17. $x=10$. 18. $x=8$. 19. $x=1$. 20. $x=6$. 21. $x=25$. 22. $x=1$. 23. $x=4$. 24. $x=3$. 25. $x=1$. 26. $x=1$. 27. $x=8$. 28. $x=4$. 29. $x=12$. 30. $x=3$. 31. $x=2$. 32. $x=1$. 33. $x=6$. 34. $x=4$. 35. $x=-6$. 36. $x=1\frac{2}{19}$. 37. $x=27$. 38. $x=13\frac{1}{2}$. 39. $x=-2\frac{3}{7}$. 40. $x=6$.

Exercise 86.—1. (*a*) 76.95, (*b*) 4.79, (*c*) 23.83, (*d*) 10.74, (*e*) 4.08, (*f*) 12.62, (*g*) 13.23, (*h*) 35.77, (*i*) 5.16. 2. (*a*) 1.013, (*b*) 1.209, (*c*) 1.042, (*d*) 1.093, (*e*) 1.245, (*f*) 1.079, (*g*) 1.075, (*h*) 1.028, (*i*) 1.194. 3. $\frac{9}{32}$ or .28125. 4. 2.54.

Exercise 87.—1. 75, 125. 2. 100, 180. 3. 128, 192. 4. 52, 91. 5. 36, 45. 6. $60, $90. 7. 6 yr. 8. 10 yr. 9. A, $600, B, $800. 10. 35 : 36. 11. 19¼ lb., 2¾ lb. 12. 25.5 lb., 8.5 lb. 13. 22.5 oz., 4.5 oz. 14. 36 A., 32 A. 15. 14, 21. 16. 16 ft., 12 ft.

Exercise 88.—1. 305 mi. 2. $318.50. 3. 4°¾. 4. 63¢. 5. 62½ A. 6. 56 ft. 7. 6.272 in., 5.032 in., 6.736 in. 8. 1 in. to 18 mi. 9. 49 mi. 10. 15 ft. 11. Mercury, 3.9, Venus, 7.2, Mars, 15.2, Jupiter, 52, Saturn, 95.4, Uranus, 191.8, Neptune, 300.5. 12. 59.6 mi., 51.1 mi., 78.7 mi., 42.6 mi. 13. 112 ft. 14. 172.5. 15. 109.3 in. diameter.

Exercise 89.—1. $1133.33. 2. $1410. 3. 1⅓ cu. ft. 4. 12½ da. 5. 15. 6. 5 T. 7. 25 yd. 8. 360. 9. 20 cu. in. 10. 61,500 T.

Exercise 90.—1. 35, 36. 2. 29, 31. 3. 25, 30. 4. 20, 26. 5. 15, 19.

Exercise 91.—1. 25 yd., 20 yd. 2. 40 yd., 30 yd. 3. 16 rd., 12 rd. 4. 36 yd. 5. 24 ft., 12 ft.

Exercise 92.—1. $36,000 at 4%, $14,000 at 5%. 2. $4000 at 6%, $1000 at 5%. 3. $720 in 3%, $1260 in 4%. 4. $12,000. 5. $5400.

Exercise 93.—**1.** 5 lb., 5 lb. **2.** 10 lb., $6\frac{2}{3}$ lb. **3.** $2\frac{1}{2}$ lb. **4.** $2\frac{1}{2}$ gal., $1\frac{1}{3}$ gal. **5.** 4 parts milk, 1 part water. **6.** 12 lb. **7.** $\frac{1}{10}$ ice, $\frac{9}{10}$ butter. **8.** 6 bu. **9.** 13 lb. **10.** 9 oz. gold, 1 oz. copper. **11.** 9 oz. silver, 1 oz. copper.

Exercise 94.—**1.** (*a*) $10\frac{10}{11}$ min. past 2 o'clock, (*b*) $21\frac{9}{11}$ min. past 4 o'clock, (*c*) $38\frac{2}{11}$ min. past 7 o'clock, (*d*) $49\frac{1}{11}$ min. past 9 o'clock. **2.** (*a*) $21\frac{9}{11}$ min. past 1 o'clock, (*b*) $32\frac{8}{11}$ min. past 3 o'clock, (*c*) $10\frac{10}{11}$ min. past 5 o'clock and $43\frac{7}{11}$ min. past 5 o'clock, (*d*) $43\frac{7}{11}$ min. past 11 o'clock. **3.** (*a*) $38\frac{2}{11}$ min. past 1 o'clock, (*b*) $49\frac{1}{11}$ min. past 3 o'clock, (*c*) at no time, (*d*) $16\frac{4}{11}$ min. past 9 o'clock. **4.** 12 min. past 4 o'clock, 48 min. past 4 o'clock. **5.** $23\frac{1}{3}$ min., $4\frac{2}{17}$ min. **6.** 40 min., $13\frac{1}{3}$ min.

Exercise 95.—**1.** 96. **2.** 72. **3.** 63. **4.** 11, 25. **5.** $\frac{7}{29}$. **6.** $\frac{17}{21}$. **7.** 33, 67. **8.** 36, 81.

Exercise 96.—**1.** $4\frac{1}{2}$ mi. per hr., $1\frac{1}{2}$ mi. per hr. **2.** 3 mi., 12 mi. **3.** 6 mi. **4.** 5 mi. per hr. **5.** 16 mi., 8 mi. per hr.

Exercise 97.—**1.** 40 yr., 10 yr. **2.** 45 yr., 15 yr. **3.** $50, $70. **4.** $15. **5.** $9000. **6.** 11, 13. **7.** 156. **8.** 98. **9.** 17 yd., 10 yd. **10.** 30 yd., 20 yd. **11.** 35 yd. **12.** 6 hr. **13.** 9, 13. **14.** 1 hr., $5\frac{5}{11}$ min. **15.** (*a*) $31\frac{1}{2}$ min., (*b*) $3\frac{15}{16}$ min. **16.** 35¢, 65¢. **17.** 14,653,000 sq. mi. nearly. **18.** $1800, $2880. **19.** $17.50, $24.50. **20.** 10 : 9. **21.** $\frac{1}{2}$. **22.** $800, $1000. **23.** 164 yd. **24.** 2.4 mi. **25.** 7 turkeys. **26.** $56\frac{1}{4}$% above cost. **27.** $16. **28.** 2 lb.

Exercise 98.—**1.** $20. **2.** $1; $1.25. **3.** $2.52. **4.** $30. **5.** $60. **6.** 4%. **7.** $6200. **8.** $11,560. **9.** $8. **10.** 5%. **11.** $140. **12.** $125. **13.** $300,000. **14.** $24,000. **15.** $3600. **16.** $172,950. **17.** $8,070,000. **18.** $4,619,475. **19.** 4 mills on $1. **20.** $58.20, £12. **21.** 20 sq. yd. **22.** $69.84, £14 8*s.*, $1.57, 52.7%. **23.** $200, $90.22. **24.** $1.60, 76.33%. **25.** $19.92. **26.** $4800. **27.** $3450. **28.** $11,240. **29.** $3500.

Exercise 101.—**1.** $x + 2$. **2.** $x^3 + 3$. **3.** $2x + 5y$. **4.** $x + \frac{1}{2}$. **5.** $x - \frac{1}{3}$. **6.** $x - \frac{1}{4}$. **7.** $x - \frac{1}{5}$. **8.** $x + \frac{3}{5}$. **9.** $x + \frac{4}{3}$. **10.** $x + \frac{2}{7}$. **11.** $x^2 - 4x + 4$. **12.** $x^2 - 6x + 9$. **13.** $x^2 + x - 1$. **14.** $x^2 + 2x - 1$. **15.** $x^2 - 3x - 2$. **16.** $x^2 - x - 4$. **17.** $a - b - c$. **18.** $a - 2b + 3c$.

Exercise 102.—**1.** 74. **2.** 83. **3.** 89. **4.** 98. **5.** 246. **6.** 216. **7.** 963. **8.** 61.7. **9.** 27.1. **10.** 19.45. **11.** 35.29. **12.** .9021. **13.** .3162. **14.** .8307. **15.** .1639. **16.** 1.0308. **17.** 1.2247. **18.** 1.5275. **19.** 2.0702. **20.** 2.4944. **21.** 1.291. **22.** 9.0604.

Exercise 103.—**1.** 1760 yd. **2.** 22 yd. **3.** $20. **4.** 330 yd., 220 yd. **5.** 36, 45. **6.** 40, 75. **7.** 30, 78. **8.** (*a*) 24, (*b*) 46, (*c*) 89. **9.** (*a*) 20 in., (*b*) 60 in., (*c*) 90 in., (*d*) 99 in. **10.** 1299.8 ft. per sec. **11.** 144 ft., 1024 ft. **12.** 5 sec.

Exercise 104.—**1.** $x = \pm 3$. **2.** $x = \pm 5$. **3.** $x = \pm 11$. **4.** $x = \pm 13$. **5.** $x = \pm \frac{5}{2}$. **6.** $x = \pm \frac{1}{3}$. **7.** $x = \pm \frac{7}{4}$. **8.** $x = \pm 6$. **9.** $x = \pm 6$. **10.** $x = \pm 1$. **11.** $x = \pm 21$. **12.** $x = \pm 6$. **13.** $x = \pm 5$. **14.** $x = \pm 1$. **15.** $x = \pm 2$. **16.** $x = \pm 3$. **17.** $x = \pm 1$. **18.** $x = \pm 2$. **19.** $x = \pm 3$. **20.** $x = \pm 4$. **21.** $x = \pm 2$. **22.** $x = \pm 4$.

Exercise 105.—**1.** $x = 6$ or 1. **2.** $x = 4$ or 3. **3.** $x = 9$ or 1. **4.** $x = 6$ or 5. **5.** $x = 11$ or 5. **6.** $x = \pm 9$. **7.** $x = \pm \frac{5}{4}$. **8.** $x = 2$ or -5. **9.** $x = 4$ or -9. **10.** $x = 7$ or -13. **11.** $x = 1$ or -13. **12.** $x = 2$ or -11. **13.** $x = \frac{2}{3}$ or $\frac{1}{3}$. **14.** $x = 4$ or $\frac{1}{4}$. **15.** $x = \frac{2}{5}$ or $\frac{3}{2}$. **16.** $x = \frac{3}{2}$ or $-\frac{1}{3}$. **17.** $x = \frac{5}{3}$ or $-\frac{1}{2}$. **18.** $x = \frac{1}{4}, \frac{1}{4}$. **19.** $x = \frac{3}{2}, \frac{3}{2}$. **20.** $x = \frac{4}{5}, \frac{4}{5}$. **21.** $x = \frac{4}{3}, \frac{3}{2}$. **22.** $x = 2$ or $-\frac{3}{4}$. **23.** $x = \frac{1}{5}$ or $-\frac{5}{2}$. **24.** $x = \pm 2$ or ± 1.

Exercise 106.—**1.** $x = 2$ or 4. **2.** $x = 6$ or 2. **3.** $x = 5$ or 3. **4.** $x = 7$ or 2. **5.** $x = 11$ or -1. **6.** $x = 13$ or 2. **7.** $x = 9$ or -5. **8.** $x = 3$ or -9. **9.** $x = 3$ or -15. **10.** $x = 5, 5$. **11.** $x = 4$ or -3. **12.** $x = 5$ or -4. **13.** $x = 5$ or -6. **14.** $x = 2$ or -5. **15.** $x = 2$ or -7. **16.** $x = 2$ or -11. **17.** $x = 5$ or -8. **18.** $x = 2$ or -9. **19.** $x = 3$ or $-\frac{1}{3}$. **20.** $x = \frac{5}{2}$ or $-\frac{1}{4}$. **21.** $x = \frac{5}{3}$ or $\frac{7}{2}$. **22.** $x = 3$ or $-\frac{1}{2}$. **23.** $x = 4$ or $-\frac{5}{3}$. **24.** $x = \frac{-1 \pm \sqrt{5}}{2}$. **25.** $x = \frac{2 \pm \sqrt{59}}{5}$. **26.** $x = \frac{7 \pm \sqrt{457}}{12}$. **27.** $x = \frac{4 \pm \sqrt{187}}{9}$. **28.** $x = \frac{5 \pm \sqrt{249}}{8}$. **29.** $x = 5$ or $-\frac{1}{5}$. **30.** $x = 4$ or -1. **31.** $x = \pm 2$. **32.** $x = 2$ or -1.

Exercise 107.—**1.** 11, 19. **2.** 16, 21. **3.** 12, 13. **4.** 15 yd., 8 yd. **5.** 6.18 in., 3.82 in. **6.** 49 yd., 16 yd. **7.** $2\frac{1}{2}$ yd. **8.** 6 ft., 4 ft. **9.** $40. **10.** 72 bees.

Exercise 108.—**1.** $\frac{1}{2}, 2\frac{1}{2}$. **2.** 14 men. **3.** 11 ft., 12 ft. **4.** $80, $20. **5.** 8 mi., 5 mi., $6\frac{1}{2}$ mi. **6.** 12 da., 20 da. **7.** 50 cattle. **8.** $\frac{2}{3}$ or $\frac{3}{2}$. **9.** 65. **10.** $250.

Exercise 109.—**1.** $\sqrt{6}$. **2.** 3. **3.** $\sqrt{15}$. **4.** $6\sqrt{15}$. **5.** $12\sqrt{35}$. **6.** $6\sqrt{6}$. **7.** $7\sqrt{2}$. **8.** $2\sqrt{3}$. **9.** $5\sqrt{2}$. **10.** $5\sqrt{6}$. **11.** $6\sqrt{5}$. **12.** 20. **13.** $6\sqrt{6}$. **14.** 16. **15.** $3 + 2\sqrt{2}$. **16.** $5 + 2\sqrt{6}$. **17.** $8 + 2\sqrt{15}$. **18.** $10 - 4\sqrt{6}$. **19.** $\sqrt{6} + \sqrt{2} + \sqrt{3} + 1$. **20.** 3. **21.** 1. **22.** -6. **23.** $\sqrt{7}$. **24.** $\sqrt{10}$. **25.** $\sqrt{2}$. **26.** $\sqrt{3}$. **27.** $\sqrt{5}$. **28.** $\sqrt{2}$. **29.** $\sqrt{3}$.

Exercise 110.—**7.** 2. **8.** 3. **9.** 25. **10.** 32. **11.** 16. **12.** 343. **13.** $\frac{2}{5}$. **14.** $\frac{2}{7}$. **15.** $\frac{4}{25}$. **16.** $\frac{4}{3}$. **17.** $\frac{343}{8}$. **18.** $\sqrt{1000}$ or 31.6228.

Exercise 111.—1. .60206. 2. .77815. 3. .90309. 4. .95424. 5. 1.07918. 6. 1.17609. 7. 1.25527. 8. 1.38021. 9. 1.39794. 10. 1.47712. 11. 1.50515. 12. 1.54407. 13. 1.55630. 14. 1.65321. 15. 1.73239. 16. 1.80618. 17. 1.85733. 18. 1.87506. 19. 1.90309. 20. 1.92428. 21. 1.99123. 22. 2.02119. 23. 2.04922. 24. 2.07918. 25. 2.09691. 26. 2.44716. 27. 2.68124. 28. 2.98227. 29. 2.89432. 30. 3.10037. 31. .33445. 32. .17609. 33. .24304. 34. .15490. 35. .36798. 36. .54407. 37. .57403. 38. .55284. 39. 1.21307. 40. 1.02996. 41. .39794. 42. .35218. 43. .15836. 44. 46835. 45. .38917. 46. .16732. 47. .63548. 48. 1.49831. 49. 1.45939. 50. 1.38561. 51. .20412. 52. 1.13033. 53. 1.60746. 54. 1.4014. 55. 1.29226. 56. 1.28330. 57. 1.10037. 58. .94201. 59. .22531. 60. .86629. 61. .15051. 62. .23299. 63. .11265. 64. .10595. 65. .55017.

Exercise 112.—1. $x=7$. 2. $x=10$. 3. $x=9$. 4. $x=9$. 5. $x=\frac{1}{2}$. 6. $x=\frac{7}{3}$. 7. $x=5$. 8. $x=13$. 9. $x=2$. 10. $x=3$. 11. $x=1$. 12. $x=6$. 13. $x=10$. 14. $x=7$. 15. $x=6$. 16. $x=\pm 2$. 17. $x=7$. 18. $x=13$.

Exercise 113.—1. $x=5$. 2. $x=1$. 3. $x=22$. 4. $x=15$. 5. $x=20$. 6. $x=26$. 7. $x=3$. 8. $x=\frac{5}{3}$, 0. 9. $x=-\frac{3}{4}$, 0. 10. $x=5$.

Exercise 115.—1. $x=3$ or 2, $y=2$ or 3. 2. $x=3$ or -2, $y=2$ or -3. 3. $x=5$ or 1, $y=1$ or 5. 4. $x=5$, $y=2$. 5. $x=5$, $y=1$. 6. $x=3$ or 1, $y=1$ or 3. 7. $x=4$ or -2, $y=2$ or -4. 8. $x=5$ or -2, $y=2$ or -5. 9. $x=4$ or $-\frac{4}{3}$, $y=2$ or $-\frac{10}{3}$. 10. $x=3$, $y=1$. 11. $x=5$ or $\frac{1}{3}$, $y=3$ or $\frac{23}{3}$. 12. $x=4$ or 14, $y=1$ or 11. 13. $x=2$ or .88, $y=1$ or 1.56. 14. $x=3$, $y=1$. 15. $x=2$ or 3, $y=2$ or 0. 16. $x=\pm 3$ or ± 2, $y=\pm 2$ or ± 3. 17. $x=\pm 3$, $y=\pm 1$. 18. $x=\pm 4$, $y=\pm 1$.

Exercise 116.—1. $x=5$, $y=2$. 2. $x=6$, $y=0$. 3. $x=4\frac{1}{2}$, $y=-1$. 4. $x=9$, $y=0$. 5. $x=6$. $y=-1$. 6. $x=4$, $y=3$. 7. $x=5$, $y=8$. 8. $x=4$, $y=3$. 9. $x=1\frac{1}{2}$, $y=-2\frac{1}{2}$. 10. $x=1\frac{3}{7}$, $y=-\frac{1}{7}$. 11. $x=40$, $y=30$. 12. $x=11.2$, $y=3.4$. 13. $x=5$, $y=1$, $z=4$. 14. $x=9$, $y=2$, $z=3$. 15. $x=4$, $y=-2$, $z=-10$. 16. $x=\frac{1}{2}$, $y=-\frac{2}{3}$, $z=1$. 17. $x=-1$, $y=-\frac{1}{2}$, $z=2$.

Exercise 117.—1. $36x^4-12x^3+x^2-25$. 2. $16x^4-44x^2+25$. 3. $9x^4-22x^2+9$. 4. $16x^4-72x^3+81x^2-49$. 5. $81x^4-36x^3+4x^2-1$. 6. x^4-2x^2+1. 7. $a^2-8ab+16b^2-4c^2$. 8. $4a^2-12ab+9b^2-9c^2$. 9. $25^2-10xy+y^2-16z^2$. 10. $x^4-9x^2+10x-9$. 11. $x^4-6x^3-7x^2+56x-49$. 12. $9x^4-28x^2-24x-20$. 13. $9x^4-13x^2+6x-5$.

14. $16x^4-28x^2-4x+8$. **15.** $9x^4-6x^3-3x^2+28x-49$. **16.** $x^4-2x^2y-x^2-6xy-8y^2$. **17.** $8a^4+32a^3+36a^2+27a-27$. **18.** $x^6+2x^3y^3+y^6$. **19.** $x^4-a^2x^2-b^2x^2+a^2b^2$. **20.** $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$. **21.** $a^{4n}-2a^{2n}b^{2n}+b^{4n}$. **22.** $a^{4n}-b^{4n}$.

Exercise 119. — **1.** $(x+1)(x^2+2)$. **2.** $(x+2)(x^2+3)$. **3.** $(x+a)(x+b)$. **4.** $(x+1)(x^2+1)$. **5.** $(x-2b)(x+4c)$. **6.** $(2x-y)(3x-4z)$. **7.** $(x+4m)(x-3n)$. **8.** $(4a+5b)(5a-2c)$. **9.** $(7a+2b)(3a-4c)$. **10.** $(a+b)(a-b)(a+c)(a-c)$.

Exercise 120. — **1.** $(x^2+4)(x+2)(x-2)$. **2.** $(a^2+4b^2)(a+2b)(a-2b)$. **3.** $(x^2+9y^2)(x+3y)(x-3y)$. **4.** $(x^n+y^n)(x^n-y^n)$. **5.** $x^4(x+1)(x-1)$. **6.** $5x(x^3+7)(x^3-7)$. **7.** $(a+b+c)(a+b-c)$. **8.** $(b+x-1)(b-x+1)$. **9.** $(c+x+2)(c-x-2)$. **10.** $(x+y+z)(x-y-z)$. **11.** $(m+n+p)(m-n-p)$. **12.** $(a-2b+3c)(a-2b-3c)$. **13.** $(a+b-c)(a-b+c)$. **14.** $(x+y+z)(x-y-z)$. **15.** $(a+b+c)(a+b-c)$. **16.** $(2x-y+z)(2x-y-z)$. **17.** $(a+2b-3c)(a-2b+3c)$. **18.** $(8x+5y-5z)(8x-5y+5z)$.

Exercise 121. — **1.** $(a-1)(a^2+a+1)$. **2.** $(a+1)(a^2-a+1)$. **3.** $(a-2)(a^2+2a+4)$. **4.** $(a+5)(a^2-5a+25)$. **5.** $(a-4)(a^2+4a+16)$. **6.** $(a+8)(a^2-8a+64)$. **7.** $(2x-1)(4x^2+2x+1)$. **8.** $(a+10b)(a^2-10ab+100b^2)$. **9.** $(2x-3y)(4x^2+6xy+9y^2)$. **10.** $(x-3y)(x^2+3xy+9y^2)$. **11.** $(3a+b)(9a^2-3ab+b^2)$. **12.** $(5y-2a)(25y^2+10ay+4a^2)$. **13.** $(3a+b^2)(9a^2-3ab^2+b^4)$. **14.** $(4a+b)(16a^2-4ab+b^2)$. **15.** $(6a+5b)(36a^2-30ab+25b^2)$. **16.** $(6x-7y)(36x^2+42xy+49y^2)$. **17.** $(5xy+z)(25x^2y^2-5xyz+z^2)$. **18.** $(9ab+c)(81a^2b^2-9abc+c^2)$. **19.** $2x(2x-1)(4x^2+2x+1)$. **20.** $x(8+x)(64-8x+x^2)$. **21.** $x^3(x-1)(x^2+x+1)$. **22.** $(a^2+1)(a^4-a^2+1)$. **23.** $(a^2+4)(a^4-4a^2+16)$. **24.** $(a+1)(a^2-a+1)(a^6-a^3+1)$. **25.** $(x^2+y^2)(x^4-x^2y^2+y^4)$. **26.** $(x^4+y^4)(x^8-x^4y^4+y^8)$. **27.** $(a^2+b^3)(a^4-a^2b^3+b^6)$. **28.** $a^2(a+1)(a^2-a+1)$. **29.** $a^3(b+c)(b^2-bc+c^2)$. **30.** $(a^n-1)(a^{2n}+a^n+1)$. **31.** $(x-y)(x^2+xy+y^2)$. **32.** $(x^2-2)(x^4+2x^2+4)$. **33.** $(b^2-c)(b^4+b^2c+c^2)$. **34.** $(8-x^3)(64+8x^3+x^2)$. **35.** $(x+y)(x^2-xy+y^2)$. **36.** $(x+4)(x^2-4x+16)$. **37.** $(8-x^3)(64+8x^3+x^6)$. **38.** $(2x^2+5)(4x^4-10x^2+25)$.

Exercise 122. — **1.** $8x(x-2)$. **2.** $x(ax-b)$. **3.** $7x^2(x-4)$. **4.** $x(x^2+x+1)$. **5.** $mx(x+1)$. **6.** $ab(a-b)$. **7.** $(c-1)(a-1)$. **8.** $(m-n)(1-n)$. **9.** $(4-b)(1-a)$. **10.** $(x-6b)(x+3c)$. **11.** $(x-a)(x+b)$. **12.** $(x+y)(x^2+y^2)$. **13.** $(3a-2b)(2a^2+3b^2)$. **14.** $(2a-3b)(3a^2-2b^2)$. **15.** $(x+y)(x-y-1)$. **16.** $(m-n)(mn+1)$. **17.** $(a-b)(a+b-1)$. **18.** $(x+7)(x+1)$. **19.** $(x+10)(x+1)$. **20.** $(x+39)(x+2)$. **21.** $(x+3)(x+41)$. **22.** $(x-12)(x-11)$.

23. $(x-8a)(x-5a)$. **24.** $(x-12a)(x-9a)$. **25.** $(x-16a)(x-6a)$. **26.** $(x-9)(x+8)$. **27.** $(x+6)(x-5)$. **28.** $(x+11)(x-9)$. **29.** $(x+9)(x-6)$. **30.** $(a-39b)(a+10b)$. **31.** $(x-a)(x+b)$. **32.** $(1-9a+9b)(1+8a-8b)$. **33.** $(x-a-6)(x+3)$. **34.** $(4x-1)(x-4)$. **35.** $(3x+1)(x-3)$. **36.** $(3x-2)(2x-3)$. **37.** $(3x+2)(2x-3)$. **38.** $(4x-5)(5x-4)$. **39.** $(4x+1)(5x-2)$. **40.** $(9x-7)(x+4)$. **41.** $(3a-7y)(a-6y)$. **42.** $(x-4y)(x-2y+3)$. **43.** $(5x+1)(5x-1)(x+1)(x-1)$.

Exercise 124. — **1.** -6. **2.** 2. **3.** -31. **4.** 21. **5.** 165. **6.** -6. **12.** -11. **13.** -10.

Exercise 125. — **1.** $3ab^2$, $36a^3b^4$. **2.** $4x^2y^2$, $48x^4y^5$. **3.** x, $x(x-1)(x+1)$. **4.** $a(a+1)$, $a(a+1)^2$. **5.** $x(x-y)$, $x(x+y)(x-y)^2$. **6.** x^2+xy+y^2, $x(x-y)(x^2+xy+y^2)$. **7.** a^2+y^2, $a(a^4-y^4)$. **8.** $4+x^2$, $(2+x)(2-x)(4+x^2)^2$. **9.** $x-2$, $(x+2)(x^3-8)$. **10.** $a-3b$, $a^3b^3(a-3b)$. **11.** $1-a+a^2$, $(1+a)(1+a^2+a^4)$. **12.** $1+a+a^2$, $(1-a)(1+a^2+a^4)$. **13.** $a-b$, $a^2(a-b)(a^2+b^2)$. **14.** $(x+y)(x^2+y^2)$, $(x-y)(x+y)(x^2+y^2)$. **15.** $(a-b)(a^2+b^2)$, $(a-b)(a+b)(a^2+b^2)$. **16.** $x^4-x^2y^2+y^4$, $x(x^6+y^6)$. **17.** $x-2$, $(x^2-1)(x^2-4)$. **18.** $2x+1$, $(2x-1)(2x+1)^2(x-2)$. **19.** $a-3b$, $(a+b)(a-c)(a-3b)$. **20.** $x+1$, $(x+1)(x+3)(x^4+x^2+1)$. **21.** $a+b+c$, $(a-b-c)(b-a-c)(c-a-b)(a+b+c)$. **22.** $a-b-c$, $(a+b+c)(a-b+c)(a-b-c)$.

Exercise 126. — **1.** $\frac{a+1}{a}$. **2.** $\frac{a-b}{a+b}$. **3.** $\frac{x-1}{x-2}$. **4.** $\frac{a}{c}$. **5.** $\frac{3b}{a+b}$. **6.** $\frac{2a+b}{a-2b}$. **7.** $2yz$. **8.** $\frac{a+b}{(a-b)^2}$. **9.** $\frac{x-1}{x+2}$. **10.** $\frac{(a-1)^2}{a^2+1}$. **11.** $\frac{3a^2-2a+1}{3a^2}$. **12.** $\frac{3x^2-9x-14}{x^2-7x+10}$. **13.** $\frac{y+2b}{y-2b}$. **14.** $\frac{2x^2+4x+3}{2x^2+x+3}$.

Exercise 127. — **1.** $\frac{2x}{x^2-4}$. **2.** $\frac{1}{(x+5)(x+6)}$. **3.** $\frac{1}{(1-x)^2}$. **4.** $\frac{x-13}{x^2-1}$. **5.** $\frac{9x+3}{x^2-9}$. **6.** $\frac{x-56}{x^2-64}$. **7.** $\frac{2}{x(x+2)}$. **8.** $\frac{16}{(2x+3)(2x-1)}$. **9.** $\frac{4ax}{x^2-a^2}$. **10.** $\frac{x}{(14x-15)(4x-5)}$. **11.** $\frac{x-1}{(6x-7)(9x-11)}$. **12.** $\frac{4x^2+18x-14}{(x+6)(x-5)}$. **13.** $\frac{25a^2b^2}{(9a^2+8b^2)(3a^2-4b^2)}$. **14.** $\frac{6ab}{a^2-b^2}$. **15.** $\frac{1}{(a-1)(a^2-4)}$. **16.** $\frac{2b}{(a+b)(a-b)^2}$. **17.** $\frac{3}{x-2}$.

18. $\dfrac{y}{x^2-y^2}$. **19.** $\dfrac{x^2+y^2}{x(2x-y)}$. **20.** $\dfrac{2ab-2b^2-a^2}{a^2(a^2-b^2)}$. **21.** $\dfrac{2x-2y}{4y^2-x^2}$.
22. $\dfrac{2x^2-x-5}{(x^2-4)(x-3)}$. **23.** $\dfrac{3}{(1-a)(a-2)(a-3)}$. **24.** 0. **25.** 0.
26. 0. **27.** 0. **28.** $\dfrac{2}{x^4-1}$. **29.** $\dfrac{4a^2}{a^2-b^2}$. **30.** $\dfrac{2xy(x+y)}{(x-y)(x^2+y^2)}$.
31. $\dfrac{a^4+b^4}{a^4-b^4}$. **32.** $\dfrac{5a^2}{a^2+4b^2}$. **33.** $\dfrac{x^3}{(1+x)(1-x^3)}$.

Exercise 128.—1. $\dfrac{(a-3)(x+3)}{(a+1)(x-3)}$. **2.** $\dfrac{a(a+b)}{c(a-b)}$. **3.** $\dfrac{(a+x)^2}{a^2+x^2}$.
4. $3(x-2)$. **5.** $m(c-d)$. **6.** $\dfrac{2(a+b)}{a}$. **7.** $\dfrac{a+3}{a}$. **8.** $\dfrac{(x-1)^2}{x+4}$.
9. $\dfrac{a-b}{a+3b}$. **10.** $\dfrac{2(a-b)}{b}$. **11.** $\dfrac{16x^4-36x^2+20}{(4x^2-5)(2x-1)}$. **12.** 1.
13. $\dfrac{x}{x+3}$. **14.** 1. **15.** $x+c$. **16.** $\dfrac{a-b+c}{ab}$.

Exercise 129.—1. 77. **2.** 0. **3.** $-5\frac{8}{13}$. **4.** $2\frac{2}{3}$. **5.** 4. **6.** 3.
7. 3. **8.** $1\frac{1}{2}$. **9.** -1. **10.** -4. **11.** $-8\frac{1}{2}$. **12.** $5\frac{2}{3}$. **13.** 7.
14. 8. **15.** 17. **16.** 2. **17.** $1\frac{1}{2}$.

Exercise 130.—1. $x=\frac{1}{2}, y=-1$. **2.** $x=4, y=5$. **3.** $x=1$, $y=\frac{1}{3}$. **4.** $x=-21, y=-18$. **5.** $x=3, y=4$. **6.** $x=2, y=\frac{1}{2}$.
7. $x=5, y=2$. **8.** $x=\frac{5}{3}, -\frac{3}{2}$. **9.** $x=1, y=-2, z=-5$.
10. $x=4, y=-\frac{3}{2}, z=5$.

Exercise 131.—1. $6, \frac{1}{6}$. **2.** $9, -\frac{1}{9}$. **3.** $\frac{7}{2}, \frac{3}{2}$. **4.** $3, -\frac{5}{3}$. **5.** $\frac{1}{5}, -4$.
6. $\frac{9}{2}, -3$. **7.** $\frac{1}{6}, -7$. **8.** $\frac{5}{2}, -2$. **9.** $\frac{3}{2}, -\frac{1}{2}$. **10.** $4, -\frac{10}{3}$.
11. $21, -\frac{1}{2}$. **12.** $2, -\frac{3}{2}$. **13.** $6, \frac{7}{5}$. **14.** $7\frac{1}{2}, \frac{1}{2}$. **15.** $\frac{11}{3}, \frac{4}{3}$. **16.** $\frac{19}{4}, \frac{1}{4}$.
17. $\frac{3}{2}, \frac{9}{2}$. **18.** $\frac{5}{3}, -\frac{2}{3}$. **19.** $\frac{17}{3}, -\frac{2}{3}$. **20.** $1, \dfrac{b}{a}$. **21.** $\dfrac{a}{2}, -\dfrac{1}{2a}$.
22. $a-b, b$. **23.** $\dfrac{b}{a}, -\dfrac{d}{c}$. **24.** $\dfrac{m}{n}, \dfrac{n}{m}$. **25.** $\pm 4, \pm 2\sqrt{2}$. **26.** ± 2, $\pm\sqrt{-3}$. **27.** $\pm 4, \pm\sqrt{-1}$. **28.** $\pm 3, \pm\sqrt{-4}$. **29.** $2, -1, \dfrac{1\pm\sqrt{-19}}{2}$.
30. $\pm 2, \dfrac{\pm\sqrt{-22}}{2}$. **31.** $-1, -2, -1, -2$. **32.** $2, -4, \pm 3$.
33. $1, 1, -3\pm 2\sqrt{2}$. **34.** $2, \frac{1}{2}, \dfrac{1\pm 2\sqrt{-6}}{2}$. **35.** 5, 29. **36.** 0, 1, 3.
37. $1, 1, -2$. **38.** $-2, -2, 4$. **39.** $-2, 1\pm\sqrt{-3}$. **40.** $1, -4$, $\dfrac{-3\pm\sqrt{-7}}{2}$. **41.** $3, 5, -2, -4$. **42.** $-3, \dfrac{3\pm\sqrt{-27}}{2}$. **43.** 1, 1,

$1 \pm 2\sqrt{-6}$. **44.** $1, -3, \frac{-2 \pm \sqrt{2}}{2}$. **45.** $4, 3, \frac{7 \pm \sqrt{-3}}{2}$. **46.** $1, -\frac{3}{2}, -3, -\frac{1}{2}$. **47.** $-1, \frac{2}{3}, 1\frac{1}{2}$.

Exercise 132. — **1.** $x^2-1=0$. **2.** $x^2-7x+12=0$. **3.** $x^2+x-12=0$. **4.** $x^2-6x+5=0$. **5.** $x^2-7x=0$. **6.** $x^2+9x=0$. **7.** $x^2-10x+25=0$. **8.** $x^2+4x+4=0$. **9.** $x^2=a^2$. **10.** $x^2-3bx+2b^2$. **11.** $x^2+mx-nx=mn$. **12.** $x^2+2cx+c^2=0$. **13.** $3x^2+x=2$. **14.** $4x^2+5x=6$. **15.** $6x^2-5x+1=0$. **16.** $5x^2+4x=0$. **17.** $x^2-2ax+a^2=4b^2$. **18.** $abx^2-(a^2+b^2)x+ab=0$. **19.** $x^2-(2a-b)x+a^2-ab=0$. **20.** $x^2-2ax+a^2-b^2=0$. **21.** $x^2+(c-a)x=ac$. **22.** $x^2+(a+b)x+ab=0$. **23.** $x^3-(a-b+c)x^2+(ac-ab-bc)x+abc=0$. **24.** $x^3+(a+b+c)x^2+(ab+bc+ac)x+abc=0$. **25.** $x^3-3x-2=0$. **26.** $a(a+b)x^2-(2a^2+2ab+b^2)x+a^2+ab=0$. **27.** $x^2=(a-b)^2$. **28.** $4x^4+1=5x^2$.

Exercise 133. — **1.** $\frac{3abc}{ab+bc+ca}$. **2.** $\frac{a^2-b^2}{2a}$. **3.** $1-a-b$. **4.** $7a$. **5.** $-\frac{2ab}{a+b}$. **6.** $\frac{ab}{a+b}$. **7.** $\frac{a^2-2ab}{2a-b}$. **8.** $\frac{2}{a+b}$. **9.** $\frac{ab(a+b-2)}{a+b-2ab}$. **10.** c.

Exercise 134. — **1.** $(4, 7\frac{1}{2}), (15, 2)$. **2.** $(5, 2), (2, 8)$. **3.** $(3, 1), (1, \frac{5}{3})$. **4.** $(5, 4), (2\frac{1}{2}, 9)$. **5.** $(8, 3), (2, 12)$. **6.** $(8, 1), (-7, -4)$. **7.** $(9, 2)(-18.6, -16.4)$. **8.** $(3, 2), (3, 2)$. **9.** $(4, 1), (-40, 23)$. **10.** $(4, 1), (-8, -5)$. **11.** $(5, 1), (-1, 7)$. **12.** $(4, 1), (0, 5)$. **13.** $(4, 4), (3, 5)$. **14.** $(1, 4), (-4, -1)$. **15.** $(2, -1), (-2, 3)$. **16.** $(9, 1), (4, 6)$. **17.** $(5, -2), (-1, -8)$. **18.** $(6, 2)(-10, -14)$. **19.** $(6, 3), (4, 5)$. **20.** $(10, 3), (8, 5)$. **21.** $(5, 3), (-\frac{1}{3}, -\frac{7}{3})$. **22.** $(4, 2), (-\frac{4}{3}, \frac{20}{3})$. **23.** $(7, -1), (17, 9)$. **24.** $(3, 2), (1.88, 2.56)$. **25.** $(2, 1), (1, \frac{3}{2})$. **26.** $(5, 2), (5, 2)$. **27.** $(4, 1), (4, 1)$. **28.** $(5, 2), (5, 2)$. **29.** $(9, 1), (9, 1)$. **30.** $(4, 3), (4, 3)$. **31.** $(6, 2), (6, 2)$. **32.** $(4, 3), (6, 2)$. **33.** $(-3, 2), (0, -5)$. **34.** $(-4, 1), (2, 2)$. **35.** $(3, 2), (5, 1)$. **36.** $(3, -3), (3, -3)$. **37.** $(-4, 6), (-7, 3)$. **38.** $(\pm 2, \pm 5)$. **39.** $(2, -3)(3, -4)$. **40.** $\pm 3, \pm 1$. **41.** $(0, 5), (4, -3)$. **42.** $\pm 7, \pm 2$. **43.** $(-1, -1), (-2, -2)$. **44.** $(4, 2), (-4, -2), (2\sqrt{-1}, -4\sqrt{-1}), (-2\sqrt{-1}, 4\sqrt{-1})$. **45.** $(2, 2), (-3, 3)$. **46.** $(2, 5), (-2, -5), (1, 3), (-1, -3)$. **47.** $(2, 5), (-2, -5), (5, 4), (-5, -4)$. **48.** $(3, 5), (-3, -5), (2, 4), (-2, -4)$. **49.** $(2, 3), (-2, -3), (1, 2), (-1, -2)$. **50.** $(2, 5)(-2, -5), (1, 3), (-1, -3)$. **51.** Same as 47. **52.** $(3, 4), (-3, -4), (4, 5), (-4, -5)$. **53.** $(1, 4), (-1, -4), (3, 2), (-3, -2)$. **54.** Same as Ex. 48. **55.** $(3, 5), (-3, -5), (1, -1), (-1, 1)$. **56.** $(2, 4), (-2, -4), (3, -2), (-3, 2)$. **57.** $(7, 4)(-7, -4), (3, -6), (-3, 6)$.

16. 4 or 9, (*b*) 20. **17.** 8. **18.** 5 or 33. **19.** $a+(n-1)(b-a)$.
20. 12. **21.** 4.

Exercise 147. — **1.** $\frac{243}{32}$, $\frac{3^{n-1}}{2^{2n-7}}$. **2.** $\frac{128}{81}$, $\frac{(-1)^{n-1}\cdot 2^n}{3^{n-3}}$. **3.** ar^8, $(-1)^n ar^n$. **4.** $27\sqrt[3]{3}$, $3^{\frac{n}{3}}$. **5.** (*a*) 5.4684, (*b*) 6.633, (*c*) 9.5491, (*d*) $\frac{1}{3}\cdot\frac{2^n-(-1)^n}{2^n}$, (*e*) $a\cdot\frac{a^n-1}{a-1}$, (*f*) $(2+\sqrt{2})(2^{\frac{n}{2}}-1)$. **6.** 4, a^2, $2\sqrt{2}$, a^{x+1}, 0, *a*. **7.** 3, 1, $\frac{1}{3}$. **8.** -3, 9, -27, 81. **9.** 1. **10.** $\sqrt[4]{a}$.
11. 12. **12.** 5. **13.** $249\frac{1}{27}$. **14.** 8, 12, 18. **16.** $\frac{x(1-x^n)}{1-x}+\frac{n}{2}(n+1)$.
17. 8, 20, 50, 125. **18.** 14, 21, $31\frac{1}{2}$.

Exercise 148. — **1.** 16. **2.** $5\frac{1}{3}$. **3.** $1\frac{1}{3}$. **4.** 2. **5.** *a*. **6.** $\frac{80}{99}$.
7. 1. **8.** $\frac{3+\sqrt{2}}{7}$. **9.** (*a*) $\frac{4}{11}$, (*b*) $\frac{7}{11}$, (*c*) $\frac{10}{11}$, (*d*) $\frac{80}{99}$. **10.** (*a*) $\frac{1177}{15000}$, (*b*) $\frac{1}{150}$, (*c*) $\frac{1}{750}$, (*d*) $\frac{3}{7}$, (*e*) $\frac{568}{4950}$, (*f*) $\frac{1}{65}$. **11.** $\frac{2}{9}$. **12.** $-\frac{1}{9}$. **13.** 3.
14. 8 or 24. **16.** $\frac{1}{2}(1+a)^8$.

Exercise 150. (Ans. based on 5-place tables.) — **1.** 9.9154. **2.** 3.0402.
3. 43.943. **4.** 120.4. **5.** 1609.3. **6.** 584.19. **7.** 247.1. **8.** 1976.1.
9. 1.563. **10.** 1.9478. **11.** 3.9592. **12.** 1.118, .26646. **13.** .5345, .9045, .8597. **14.** 1.3263, 2.2489, 2.1897. **15.** .7775, .2308, .9655.
16. 764.5. **17.** 4.0013. **18.** 17.68, 14.2, 11.89, 14.28. **20.** 10, 20, 35.
21. 7, 15. **22.** $\bar{2}.90309$, .12629, $\bar{1}.93539$. **24.** -5, $\frac{1}{5}$, 4, 3, $\frac{1}{4}$, $\frac{1}{3}$, -4.
25. 9, 16, $\frac{1}{3}$. **26.** 7, 21,70. **27.** .047. **29.** 20. **30.** 1.4994.

Exercise 151. — **13.** $2i$. **14.** $3i$. **15.** $5i$. **16.** $12i$. **17.** $10i$.
18. $i\sqrt{10}$. **19.** $\frac{7}{3}\sqrt{2}\,i$. **20.** $(a-b)i$. **21.** -8. **22.** -10. **23.** -6.
24. -16. **25.** $-\sqrt{15}$. **26.** $-\sqrt{ab}$. **27.** a^2-b^2+2abi. **28.** $x-y+2i\sqrt{xy}$. **29.** $1+4i\sqrt{3}$. **30.** $7-6i\sqrt{2}$. **31.** 13. **32.** 8. **33.** $2i$.
34. $\sqrt{2}$. **35.** $\sqrt{10}$. **36.** 2. **37.** 3. **38.** $\frac{a}{b}$. **39.** $-i$. **40.** 0. **41.** 0.
42. a^2+b^2. **43.** 0. **44.** $-i$. **45.** $3+2i$. **46.** $i-1$. **47.** i.
48. -1. **49.** i. **50.** $1+i\sqrt{3}$. **51.** $\frac{1}{5}-\frac{2\sqrt{6}}{5}i$. **52.** $-2-2i$.
53. 8. **54.** x^2-2x+2, $x^2-4x+11$, $x^2-3x+17$.

Exercise 152. — **1.** 64. **2.** 5. **3.** 0. **4.** 0. **5.** -12. **6.** $-4ac$.
7. $c=2$. **8.** $c=\pm 5$. **9.** $c=\frac{4}{3}$. **10.** $c=1$ or $-\frac{1}{3}$. **11.** $n^2=4m$.
12. $m=\pm a$. **13.** $m=\pm 2ni$. **14.** $m=0$ or $2n$. **15.** $K=\frac{1}{4}$.

16. $\dfrac{b-c}{a-b}$. **17.** $x^2 - 2ax + a^2 + b^2 = 0$. **18.** $b = -6$. **19.** $c = 50$.
20. $\frac{3}{4}$. **21.** (a) 7, (b) -10, (c) $\sqrt{-11}$, (d) $\frac{5}{9}$, (e) $\frac{7}{81}$, (f) $\frac{7}{9}$, (g) 15.
22. $qx^2 - px + 1 = 0$, (b) $x^2 - (p^2 - 2q)x + q^2 = 0$, (c) $qx^2 - (p^2 - 2q)x + q = 0$, (d) $x^2 - (p+q)x + pq = 0$.

Exercise 153. — **1.** $x^2 + x + 1$. **2.** $2x^2 + x + 1$. **3.** $3x^2 + 2x + 1$. **4.** $x^2 - x - 2$. **5.** $x^2 + 2x - 1$. **6.** $x^2 + 4x - 1$. **7.** $5x^2 - 2x - 2$. **8.** $4x^2 - x + 2$. **9.** $6x^2 - x - 3$. **10.** $4x^2 - 5x + 1$.

Exercise 154. — **1.** $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$. **2.** $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$. **3.** $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$. **4.** $16x^4 - 96x^3 + 216x^2 - 216x + 81$. **5.** The negative of answer to 3. **6.** $x^4 - 2x^3 + \frac{3}{2}x^2 - \dfrac{x}{2} + \dfrac{1}{16}$. **7.** $x^6 - 6x^5 + 15x^2 - 20 + 15x^{-2} - 6x^{-4} + x^{-6}$. **8.** $\dfrac{x^4}{81} - \dfrac{4x^3y^2}{9} + 6x^2y^4 - 36xy^6 + 81y^8$. **9.** 0.95099. **10.** $1 - 3x + 6x^2 - 10x^3 + \cdots$. **11.** $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$. **12.** $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$.