ELEMENTARY ALGEBRA

Pythagoras (about 569–500 B.C.) settled in Croton, a Dorian colony in Southern Italy, where he opened a school in which philosophy and mathematics were taught. He founded a brotherhood, the members of which were afterwards called Pythagoreans. He is said to have taught that the foundation of the theory of the universe was to be found in the science of numbers. The word *mathematics* has been ascribed to him.

ELEMENTARY ALGEBRA

BY

GEORGE H. HALLETT, A.M., PH.D.
PROFESSOR OF MATHEMATICS
THE UNIVERSITY OF PENNSYLVANIA

AND

ROBERT F. ANDERSON, A.M., SC.D.
PROFESSOR OF MATHEMATICS
STATE NORMAL SCHOOL, WEST CHESTER, PA.

SILVER, BURDETT AND COMPANY
BOSTON NEW YORK CHICAGO

PREFACE

This book is designed primarily for the use of those who are beginning the study of algebra; it is, however, sufficiently extensive to serve as a text for a review of algebra during the third or fourth year of the high school course in those schools in which the curricula call for such a review.

In preparing this text the authors have kept in view the fact that no substantial progress in algebra is possible for the student unless due emphasis is placed on the fundamental principles and processes of the subject, and that these essentials must be provided for irrespective of whatever trend the teaching of algebra may have had within the last few years, or may have in the future.

Long experience in teaching has convinced the authors that the technical terms employed and the principles involved in problems of physics and engineering are not sufficiently understood to warrant their inclusion in an elementary course in algebra; that practically the whole of such a course should be devoted to the treatment of the elements of the subject itself, and that the problems referred to may be solved with little difficulty by the student who has mastered the first course in algebra before he begins the study of physics. However, they believe that formulæ drawn from various sources, including physics and engineering, should be used extensively in elementary algebra; for there is, perhaps, no more important practical exercise in the subject than that which comes from determining the actual or approximate numerical value of a literal number which occurs in a formula, when given numerical values are substituted for the remaining letters of the formula.

To make effective provision for attaining the end in view, namely, the furnishing of a textbook from which the student may acquire a thorough grounding in the elements of algebra, the authors have made its prominent features the following:

1. The simplest possible presentation of the topics of elementary algebra.
2. The use of the inductive method in developing fundamental concepts and principles.
3. The use of illustrative problems to make clear the application of the principles.
4. The actual application of the principles by means of numerous, well-graded examples, both sight and written.
5. The extensive use of numerical checks to promote habits of accuracy and to give the student confidence in the results of his work.
6. The copious supply of problems designed to give the student facility in applying the mechanics of algebra.
7. The reviews designed to make the students' knowledge cumulative and coherent and thorough.
8. Such treatment of graphs as is necessary to render the student familiar with the underlying principles of graphical representation so that he may be able to apply them wherever there may be need for their application.
9. The conciseness and exactness of statement of definitions and principles.

The author wishes to gratefully acknowledge the courtesy of the Open Court Publishing Company in permitting the reproduction of the portraits contained in this book.

CONTENTS

CHAPTER		PAGE
I.	Introduction	1
II.	Fundamental Processes	34
III.	Simple Equations	81
IV.	Type Products and Factors	100
V.	Fractions	153
VI.	Fractional and Literal Equations	191
VII.	Systems of Linear Equations	205
VIII.	Ratio, Proportion, and Variation	239
IX.	Graphs	251
X.	Powers, Roots, Radicals, and Exponents	262
XI.	Involution and Evolution	300
XII.	Quadratic Equations	312
XIII.	Systems of Quadratic Equations	350
XIV.	Progressions	365
XV.	General Review	381

ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION

The Notation of Algebra

1. The Notation of Arithmetic. In arithmetic numbers are expressed by the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The numbers represented by these symbols are called integers. The operations of addition, subtraction, multiplication, and division performed on these integers lead either to integers or to fractions. Therefore, primarily, arithmetic treats of integers and fractions and the operations which are indicated by the signs $+$, $-$, $\times$, and $\div$.

2. Algebra. Algebra, like arithmetic, treats of number. The symbols which are used in arithmetic are retained in algebra. In algebra, however, new symbols of number and of relations between numbers are introduced, and, in the written language of algebra, systematic use is made of *letters* to represent numbers. A number which is represented by a letter is called a **literal number.**

3. Use of literal numbers. In arithmetic it is shown that $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}$. This example illustrates a principle which may be expressed in words as follows:

The product of two fractions is equal to a fraction whose numerator is the product of the two given numerators *and whose denominator is the product of the two given denominators.*

This statement may be expressed concisely in the written language of algebra thus:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

Here a denotes the integer which is the numerator of the first fraction and b the integer which is its denominator; also, c denotes the integer which is the numerator of the second fraction and d that which is its denominator. In the algebraic expression $\frac{a}{b}$, the letters a and b represent any integers whatever, whereas in the arithmetical expression $\frac{2}{3}$, the symbols 2 and 3 denote definite integers.

4. Symbols. The language of algebra employs symbols to represent: (1) Numbers themselves; (2) operations on numbers; (3) relations between numbers.

5. Symbols of number. Numbers in algebra are expressed by Arabic numerals, and also by letters; thus:

c may represent the number of cents in the cost of an orange;
a dollars may denote any number of dollars;
m may stand for the number of miles between two towns;
x may stand for any number, which for the sake of brevity is called the *number x*.

6. Symbols of operation. Symbols of operation are as follows:

$+$ is the sign of *addition;* it is read *plus*.

$7 + 3$ denotes the sum of the two numbers 7 and 3; $a + b$ stands for the sum of the two numbers represented by the letters a and b.

$-$ is the sign of *subtraction;* it is read *minus*.

$a - b$ stands for the difference of the numbers represented by the letters a and b.

$\times$ is the sign of *multiplication;* it is read *times* or *multiplied by*.

$a \times b$ stands for the product of the numbers represented by the letters a and b. Multiplication is also indicated by a *dot*. $2 \times a$ may be written $2 \cdot a$. When numbers are represented by letters, the sign of multiplication may be, and usually is, omitted.

Thus, $5ab$ stands for $5 \times a \times b$.

$\div$ or : is the sign of *division;* it is read *divided by.*

$3 \div 5$ or $3 : 5$ denotes the quotient obtained from the division of 3 by 5.

Remark. Neither the sign $\div$ nor the sign : is so frequently used in algebra as formerly. For instance, $a \div b$ is usually written $\frac{a}{b}$.

7. Symbols of relation. Symbols of relation are as follows:

$=$ is the sign of *equality* and is read *equals*, or *is equal to.*

$a = b$ expresses the equality of the numbers represented by the letters a and b.

$>$ and $<$ are signs of *inequality.* $>$ is read *greater than;* $<$ is read *less than.*

$a > b$ means that the number represented by a is greater than the number represented by b. $a < b$ means that the number represented by a is less than the number represented by b.

EXERCISE 1

1. In the following, c stands for cost (meaning, say, the number of cents in the cost), s for selling price, g for gain, and l for loss; read in words:

$c + g = s.$ $\quad$ $s + l = c.$ $\quad$ $s - c = g.$

$c - s = l.$ $\quad$ $s - g = c.$ $\quad$ $c - l = s.$

2. $3 + 2$ expresses the sum of 3 and 2; what then is expressed by $3 + 5$? $5 + a$? $b + 6$? $a + b$?

3. $6 - 2$ expresses the difference of 6 and 2; what then is expressed by $8 - 5$? $7 - a$? $b - 3$? $a - b$?

4. 3×5 expresses the product of 3 and 5; what then is expressed by 4×6? $3 \times a$? $4 \times b$? $a \times b$?

5. What is expressed by $3 \cdot c$? $a \cdot b$? $l \times w$? ab? cd?

6. $\frac{6}{2}$ expresses the quotient of 6 divided by 2; what then is expressed by $\frac{8}{4}$? $\frac{5}{2}$? $\frac{6}{a}$? $\frac{a}{2}$? $\frac{a}{c}$? $\frac{V}{H}$? $\frac{V}{B}$?

7. Using the $+$ sign, express the sum of 6 and 7; 5 and a; c and d.

8. Using the $-$ sign, express the difference of 7 and 5; a and 7; 3 and b; a and b.

9. Using the $\times$ sign, express the product of 2 and 5; 7 and a; 3 and b; a and b.

10. Indicate in three ways the product of 3 and a; 4 and b; c and d.

11. Using the fractional form, indicate the quotient of 9 divided by 4; a divided by 3; b divided by 2; a divided by c.

12. Find the number that is equal to $a + 1$ when a stands in turn for 3; 5; 7; 12; $\frac{1}{2}$; $1\frac{1}{2}$; 0; .3.

13. Find the number that is equal to $b - 1$ when b stands in turn for 2; 3; 7; $1\frac{1}{2}$; 1; $2\frac{1}{4}$; 1.5; 2.25.

14. Find the number that is equal to $2a$ when a stands in turn for 2; 3; 5; $\frac{1}{2}$; $1\frac{1}{2}$; .5; 1.1.

15. Find the number that is equal to $\frac{a}{2}$ when a stands in turn for 2; 1; 3; 6; $\frac{1}{2}$; 5; $1\frac{1}{2}$.

16. When a stands for 2 and b stands for 1, what number is equal to $a + b$? $a - b$? $a \cdot b$? $\frac{a}{b}$?

17. When $x=6$ and $y=3$, what number is equal to $x+y$? $x-y$? $x \cdot y$? $\frac{x}{y}$? $\frac{2 \cdot x}{y}$? $\frac{4}{2 \cdot y}$?

18. When $x=\frac{1}{2}$ and $y=\frac{1}{2}$, what number is equal to $x+y$? $x-y$? $x \cdot y$? $\frac{x}{y}$? $\frac{2 \cdot x}{y}$? $\frac{4}{2 \cdot y}$?

19. If goods cost c dollars and were sold at a gain of g dollars, express the selling price. If $c=7$ and $g=3$, what was the selling price?

20. If goods cost p dollars and were sold at a loss of t dollars, express the selling price. If $p=9$ and $t=2$, what was the selling price?

21. Express the number that is 6 more than a.

22. Express the number that is c more than d.

23. Express the number that is 5 less than c.

24. Express the number that is n less than m.

25. Express the sum of a, b, and c.

26. Express the sum of a and b diminished by c.

27. Find the cost of 3 pounds of sugar at a cents a pound.

28. Find the cost of a yards of cloth at b cents a yard.

29. If p pounds of sugar cost 30 cents, what did one pound cost?

30. If b bushels of grain weigh p pounds, what does one bushel weigh?

31. What did a boy pay for 2 oranges at c cents each and 3 apples at b cents each?

32. How much change should a boy receive from c cents given in payment for a ball that cost b cents?

33. Express the result of adding p times r to l.

34. Express the result of adding t times r to p.

35. Express the result of dividing V by a times b.

36. Write the product of 3, a, b, and c in three different ways.

37. How many quarts are there in a gallons?

38. How many quarts are there in b pecks?

39. How many pecks are there in c bushels?

40. How many quarts are there in m pints?

41. How many units are there in c dozen?

42. How many dozen are there in u units?

43. If l stands for the length of one line in inches and w for the length of another line in feet, what stands for the sum of their lengths in inches? What stands for the difference of their lengths in feet?

44. If l stands for the number of inches in the length of a rectangle and w for the width in inches, what stands for the number of inches in the perimeter?

45. If s stands for the length of one side of an equilateral triangle, what stands for the perimeter?

8. Algebraic expressions. Any symbol or combination of symbols used in algebra to express a number is called an **algebraic expression**, or simply an **expression**.

Thus, $2a$, $a + b$, $\frac{2a}{b}$, $3x - y + z$ are algebraic expressions.

9. Evaluation of an algebraic expression. The process of substituting numbers for letters in an expression and calculating the numerical value of the result is called the **evaluation of the expression** for the given values of the letters.

Thus, to evaluate $2a + b$ when $a = 2$ and $b = 3$, *substitute* the given values of a and b in the expression, obtaining $2 \cdot 2 + 3 = 4 + 3 = 7$.

Remark. It is often convenient in testing the accuracy of algebraic work to evaluate the expression for simple numerical values of the letters. This is termed *checking the result.*

10. Order of operations. Algebraic expressions often contain different signs of operation. In evaluating such expressions it is understood that:

When operations of addition and subtraction are indicated in an expression, they are to be performed in the order of their occurrence from left to right.

Thus, $4 + 5 - 2 + 3 - 6 = 9 - 2 + 3 - 6 = 7 + 3 - 6 = 10 - 6 = 4.$

When operations of multiplication and division are indicated in an expression, they are to be performed in their order from left to right and before the operations of addition and subtraction are performed.

Thus, $6 \times 8 \div 4 + 12 \div 6 \times 2 - 24 \div 4 \div 2 = 48 \div 4 + 2 \times 2 - 6 \div 2$
$= 12 + 4 - 3 = 13.$

EXERCISE 2

1. What is the value of $2a - b$ when $a = 1$ and $b = 1$?

2. What number is $\frac{a+b}{2}$ when $a = 2$ and $b = 4$? When $a = 1$ and $b = 3$? When $a = \frac{1}{2}$ and $b = \frac{1}{2}$?

3. If a stands for the number of units in the altitude, b for the number in the base, and A for the area of a rectangle, read in words the statement, $A = ab$.

4. What is the area of a rectangle when the altitude equals 5 ft. and the base equals 7 ft.?

5. What is the area of a rectangle when $a = 3$ and $b = 4$?

6. A man walked 3 hours at the rate of 4 miles an hour. How far did he walk? If he walked x hours at the same rate, how far did he walk?

7. A man walked a hours at the rate of x miles an hour. How far did he walk?

8. What is the cost of 5 yards of cloth at x cents a yard?

9. How many a's in $a + a$? In $2a + a$? In $3a + a$? In $3a + 2a$? In $4a - a$? In $5a - 2a$? In $7a - 3a$?

In examples 10–16, supply the missing numbers.

10. $4 \times \$10 + 2 \times \$10 + 3 \times \$10 = (\quad) \times \$10$.

11. $4 \times a + 2 \times a + 3 \times a = (\quad) \times a$.

12. 6×10 ft. $+ 5 \times 10$ ft. $- 3 \times 10$ ft. $= (\quad) \times 10$ ft.

13. $6 \times b + 5 \times b - 3 \times b = (\quad) \times b$.

14. $6x + 5x + 4x - 2x = (\quad)x$.

15. $7m + 5m - 3m - 2m = (\quad)m$.

16. $9r + 5r - 3r - r = (\quad)r$.

17. A boy is a years old, his father is $2a$ years old. How many years are there in the sum of their ages?

18. A person is x years old. How old will he be two years hence? How old was he 2 years ago? How old was he a years ago?

19. One number is three times another. If n represents the smaller number, what expression represents their sum? Their difference?

20. Some sugar costing a dollars was sold at a gain of 50%. State the selling price.

21. The circumference of a circle is equal to 2π times the radius; this truth may be briefly expressed thus:

$$c = 2\pi r.$$

Find the length of the circumference if $\pi = \frac{22}{7}$ and $r = 7$; if $\pi = \frac{22}{7}$ and $r = \frac{7}{2}$.

22. The radius of a circle is x ft. What is the circumference?

23. How much greater than a is $a + 2$? How much greater than $a + 1$ is $a + 2$?

24. Read the expression $2a \cdot a$; also, $a \cdot 2a$.

Assume that $a = 5$, $b = 2$, $c = 1$, $x = 3$, $y = 2$, $z = 0$, and evaluate the following:

25. $\dfrac{a+b}{x-y}$.

26. $\dfrac{2a+b}{x-z}$.

27. $2abc - xyz$.

28. $\dfrac{4ab + ac - bc}{2xy}$.

29. $\dfrac{2a}{a+b} + \dfrac{3ax}{2x+1} + \dfrac{2bc}{2y+x}$.

30. $\dfrac{2a \cdot 3b}{a+b+c} - \dfrac{x+1}{4y} - \dfrac{20y}{3x-1}$.

11. Laws of combination in algebra. The laws of combination of algebraic expressions are essentially the same as the laws of combination of numbers in arithmetic; for, in general, by substituting numbers for the letters, the algebraic expressions become arithmetical. Constant reference to this principle will be made in developing the fundamental laws of algebra.

12. Factors. Each of the numbers whose product is a given number is called a **factor** of the given number.

Thus, since $12 = 2 \times 6$, each of the numbers 2 and 6 is a factor of 12. Since $2xy = 2 \cdot x \cdot y$, each of the numbers, 2, x, and y, is a factor of $2xy$.

Remark 1. The expression $2(x + y)$ means 2 times the sum of x and y.

Thus, the factors of $2(x + y)$ are 2 and $(x + y)$.

Remark 2. A number may have different sets of factors.

Thus, sets of factors of 24 are 4 and 6, 2 and 12, 3 and 8; 2, 2, and 6; 2, 3, and 4; and 2, 2, 2, and 3.

13. Coefficient. When a number is the product of two factors, either of these factors is called the coefficient of the other in the product.

Thus, in $5ab$, 5 is the coefficient of ab, and $5a$ is the coefficient of b.

Note 1. A numerical coefficient is usually written first.

Thus, the product of 2 and a is written $2a$, not $a\,2$.

Note 2. The numerical coefficient 1 is usually omitted.

Thus, $1 \times a$ is usually written a.

Remark. When the term coefficient is used, numerical coefficient is usually meant.

Thus, in $3ab$, 3 is understood to be the coefficient, unless otherwise implied.

EXERCISE 3

1. Give factors of 18; 14; 96; 23.

2. Omitting the factors 1 and $3ab$, name six factors of $3ab$.

3. Name the factors of $3(a + b)$.

4. Name three sets of factors of abc.

5. What is the coefficient of ab in $3ab$?

6. What is the coefficient of $2xy$ in $3ab \cdot 2xy$?

7. Each factor in $2xyz$ has a coefficient. Name each factor and give its coefficient.

8. Write the factors of $a(b + c)$; $2a(b + c)$.

9. Write the product $a \times 2 \times b$ in its usual form.

10. Write $1 \cdot a + 1 \cdot b$ in a better form.

11. Is the following a true statement: $1 \cdot a + 2 \cdot a = (1 + 2)a$?

14. Powers. The product of two equal factors is called the **square**, or **second power**, of one of the factors. Similarly, the product of three equal factors is called the **cube**, or **third power**, of one of the factors; and the product of four equal factors is called the **fourth power**, etc.

Thus, 2×2 is the square, or second power, of 2; $a \times a$ is the square of a; aaa is the third power of a; and $aaaaaa$ is the sixth power of a.

15. Base and exponents. For convenience, $a \times a$ is usually written a^2, read *a square;* $a \times a \times a$ is written a^3 read *a cube;* $a \times a \times a \times a$ is written a^4, read *a to the fourth power* or *a with an exponent 4.* In an expression similar to a^4, the number a is called the **base**, and the number 4 is called the **exponent**, or **index**. When the exponent is an integer it shows how many times the base is used as a factor.

Thus, in $2^6 = 64$, the number 64 is the sixth power of 2, and the *exponent*, 6, shows that the *base*, 2, is used six times as a factor in obtaining 64.

Note. The exponent 1 is usually omitted. Thus, a^1 is usually written a.

Remark 1. In a subsequent chapter other numbers than integers will be introduced as exponents. Their use will then be explained.

Remark 2. Care should be taken not to confuse an exponent with a coefficient.

Thus, $3a$ means $a + a + a$, while a^3 means $a \times a \times a$.

Remark 3. It should be emphasized that such expressions as $2ab^2$ mean $2abb$ and not $2ab \times 2ab$. The latter expression is written $(2ab)^2$.

EXERCISE 4

Write examples 1–5 in another form, using exponents and coefficients:

1. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$.

2. $a + a + a + a + a$.

3. $aa + aa$.

4. $xxx + xx + xx + 1$.

5. $aaa + aaa - aa + a + a + a$.

6. Evaluate each of the following when $x = 2$:

$$x^2;\ x^3;\ x^4;\ x^5.$$

7. Evaluate each of the following when $x = 3$:

$$2\,x^2;\ 5\,x^2;\ \tfrac{1}{2}\,x^2;\ 2\,x^3;\ \tfrac{1}{3}\,x^2.$$

8. Evaluate $x^3 + 2\,x^2 - x + 2$ when $x = 3$.

9. Evaluate $a^2 - b^2 + 2\,ab$ when $a = 3$ and $b = 2$.

10. Evaluate $a^3 - b^3$ when $a = 2$ and $b = 1$.

11. Evaluate $3\,a^2 + 2\,b - 5$ when $a = 2$ and $b = 1$.

12. Evaluate $y^2 - 2\,y + 1$ when $y = 2$.

Rewrite these expressions, using exponents when possible:

13. $2\,ab \times 2\,ab \times 2\,ab$.

14. $3\,a^2b \times 3\,a^2b \times 3\,a^2b$.

15. $3\,aa - 2\,bb$; $aaa - 2 \times 3\,bb$; $2 \cdot 2 \cdot 2\,aa - 3 \times 3\,bb$.

16. $2 \times 2\,aa - 3 \times 3\,bbb$; $2 \times 2 \times 2\,xx - 2 \times 3\,yy$; $25\,aa - 49\,bb$.

16. Parentheses. Parentheses, ( ), are used to indicate that the expression inclosed is to be treated as a single number. When different parentheses are used in the same expression, other forms, as **brackets**, [ ], **braces**, { }, and the **vinculum**, ———, are employed.

Thus, $a + (b - c)$ means that the number remaining after subtracting c from b is to be added to a. $(5 + 4)(6 - 2)$ means the product of 9 and 4. In $\dfrac{a + b}{c + d}$ the vinculum groups $a + b$ into one number, and $c + d$ into one number.

Note. The various forms of parentheses are called **signs of aggregation**; their use is illustrated by the following expression:

$$[2 + \{3 - 2\} + \overline{3 - 1} + (2 + 5) + (4 - 2)] = [2 + 1 + 2 + 7 + 2] = 14.$$

EXERCISE 5

Simplify:

1. $(2+3)(4+5)$.

Thus, $(2 + 3)(4 + 5) = 5 \times 9 = 45$.

2. $(5+3)(8-3)$.

3. $5(3+4-5)$.

4. $(18-12)\div 3$.

5. $15-(3\times 2+4)$.

6. $(12+6)\div(8-5)$.

7. $(15-5)\div(3+2)$.

8. $(2+4)(6-4)\div(5+1)$.

9. $(8+4)\div(10-6)(15-5)$.

10. $\dfrac{(12+3)\times(12-7)}{2\times 5}$.

11. $2+[3+(1+15)]$.

12. $3+(2-1)+(3+2)$.

13. $8+[4\div 2+3]$.

14. $(3+1)(2+1)+2$.

15. $5[2+(3-2)+\{3+2+1\}]$.

16. $\overline{15-3}\times 2+\overline{8+6}\div 2$.

17. Monomial. An algebraic expression which does not contain an addition or a subtraction sign is called a **monomial expression**, or simply a **monomial.**

Thus, $3a$, $\dfrac{2a}{b}$, and $\dfrac{5abc}{d}$ are monomials.

Note. An expression within parentheses must be regarded as a monomial, since it is to be taken as a whole.

Thus, $2(a+b)$ and $(a+a)$ are monomials; but $a+b$ is not a monomial.

18. Terms. The monomials of an algebraic expression which are connected by + and − signs are called the **terms** of the expression.

Thus, the terms of $ax + 3b + 2c$ are ax, $3b$, and $2c$.

19. Binomial. An algebraic expression of two terms is called a **binomial.**

Thus, $a+2$, $3a-x$, and $2(a+b)+3(x+y)$ are binomials.

20. Trinomial. An algebraic expression of three terms is called a **trinomial.**

Thus, $x + y + z$, $2a + 3b - c$, and $2(b + c) + 3(x + y) + c(a + b)$ are trinomials.

21. Polynomial. An algebraic expression of two or more terms, is, in general, called a **polynomial.**

Thus, a binomial is a polynomial of two terms, and a trinomial is a polynomial of three terms.

22. Like terms. Terms which do not differ except in their coefficients are called **like terms.**

Thus, $3ab^2$, $5ab^2$, and $\frac{3}{4}ab^2$ are *like terms;* but $3ab^2$ and $5a^2b$ are *unlike terms.*

Remark. Terms may be regarded as like with respect to a certain factor or certain factors, when the remaining factors are regarded as coefficients. Thus, ay and by are like terms with respect to y; also abx and cbx are like with respect to bx.

EXERCISE 6

1. How many terms are there in $2x - 3y + z$?

2. Name the terms in $x - 4y - 3z + 1$.

3. Name the numerical coefficient in each of the monomials $2a^2b$, ab, $3abx$, mn^2, $3xyz$, $4(x + y)$.

4. Of the following monomials name those that are like:

$2xy$, $3x^2y$, $3xy$, $2xy^2$, $4x^2y$, $5xy^2$, xy, xy^2, x^2y, $5xy^2$.

5. State with respect to what letter $4m$, mn, and cm are like. What is the coefficient of each of these monomials?

6. With respect to what factor are axy, bxy, and cxy like, and what is the coefficient of each monomial?

7. With respect to what factor are $2(m + n)$, $c(m + n)$, and $d(m + n)$ like, and what is the coefficient of each monomial?

8. If a stands for one number and b for another, $a + b$ represents their sum; $a - b$, their difference; ab, their product; and $\frac{a}{b}$, the quotient of a divided by b. Which of these expressions are binomials, and which monomials? Evaluate each when $a = 2$ and $b = 1$.

Evaluate the following trinomials when $x = 4$, $y = 2$, and $z = 1$.

9. $x + y + z$. **10.** $x - y + z$. **11.** $x - y - z$.

12. $x - 2y + z$. **13.** $x - y - 2z$. **14.** $x + y - 3z$.

23. The equation. Such statements as,

$$a = b,\ 2x = 3, \text{ and } x + y = 4$$

are called **equations.** The equation $a = b$ means that the number represented by the letter a is equal to the number represented by the letter b. In the equation $a = b$, a is called the **first member** of the equation and b is called the **second member.**

24. Application of the equation.

Problem. $500 is to be divided between two persons, A and B, in such a way that B shall receive $100 more than A. How many dollars should each receive?

Solution. Let $x =$ the number of dollars in A's share.

Then, $x + 100 =$ the number of dollars in B's share.

Hence, $x + x + 100 =$ the number of dollars both are to receive.

That is, $2x + 100 = 500$ {meaning that two times the number of dollars that A receives plus 100 is equal to 500, the number of dollars to be divided.

Hence, $2x = 500 - 100$,

or, $2x = 400$;

$\therefore\ x = 200$, the number of dollars that A is to receive.

$x + 100 = 300$, the number of dollars that B is to receive.

Note. The symbol $\therefore$ is read *therefore.*

In the foregoing solution, the equation $2x = 500 - 100$ was obtained from the preceding equation $2x + 100 = 500$, by means of the principle that *if equal numbers be subtracted from equal numbers the resulting numbers are equal.* Also, the equation $x = 200$ was obtained from the equation $2x = 400$ by means of the principle that *if equal numbers be divided by equal numbers (the number zero excepted) the resulting numbers are equal.*

25. Assumptions. The simplification of all equations depends on such assumptions, or principles, as those stated in section 24. These assumptions are:

1. *If equal numbers be added to equal numbers, the sums will be equal.*
2. *If equal numbers be subtracted from equal numbers, the differences will be equal.*
3. *If equal numbers be multiplied by equal numbers, the products will be equal.*
4. *If equal numbers be divided by equal numbers (zero excepted), the quotients will be equal.*
5. *Numbers which are equal to the same number are equal to each other.*

26. The generalizing spirit of algebra may be illustrated by further consideration of problems similar to that of section 24.

ILLUSTRATION

Instead of dividing $500 between two persons, let the number of dollars divided be represented by the letter n, which may represent any number of dollars, and let the second person receive a dollars more than the first.

Solution. Let $x =$ the number of dollars in A's share.

Then, $x + a =$ the number of dollars in B's share.

Hence, $x + x + a =$ the number of dollars they both receive.

That is, $2x + a = n$,

or, $2x = n - a$. [§ 25, 2]

$$\therefore x = \frac{n-a}{2}.$$ [§ 25, 4]

Hence, $\frac{n-a}{2}$ = the number of dollars in A's share,

and $\frac{n-a}{2} + a$ = the number of dollars in B's share.

27. The resulting value of x in the solution, section 26, namely, $x = \frac{n-a}{2}$, is an illustration of what in algebra is termed a *formula*. This formula gives the solution of a great number of particular problems which are all of the same kind and which differ only in the numerical values assigned to the letters.

Thus, when $n = 500$ and $a = 100$, [§ 24]

$$x = \frac{500 - 100}{2} = 200,$$

and $x + 100 = 300$;

which were the results obtained in § 24.

28. As a general definition we have the following:

A **formula** is a rule of calculation expressed in algebraic symbols.

Remark. Problem 3, Exercise 2, page 7, contains an important geometrical formula, that for the area of a rectangle.

ILLUSTRATIVE EXAMPLES

1. When $x + 3 = 7$, what is the value of x; that is, for what number does x stand?

Solution. $x + 3 = 7$.

Subtracting 3 from each member, $x = 7 - 3$. [§ 25, 2]

$$\therefore x = 4.$$

2. When $3x = 12$, what is the value of x?

Solution. $3x = 12$.

Dividing both members by 3, $x = 4$. [§ 25, 4]

3. When $\frac{1}{4}x = 2$, what is the value of x?

Solution. $\frac{1}{4}x = 2.$

Multiplying both members by 4, $x = 8.$ [§ 25, 3]

4. Solve $3x + 5 = 11$ and check the resulting value of x.

Solution.

$$3x + 5 = 11.$$
$$3x = 11 - 5.$$
$$3x = 6.$$
$$x = 2.$$

Check. $3 \times 2 + 5 = 6 + 5$, or 11.

5. Solve $3x + 2x - x + 7 = 2x + 15$ and check.

Solution.

$$3x + 2x - x + 7 = 2x + 15.$$
$$4x + 7 = 2x + 15.$$

[See Example 9, Exercise 2]

$$4x - 2x + 7 = 2x - 2x + 15.$$ [§ 25, 2]

$$2x + 7 = 15.$$
$$2x = 8.$$
$$x = 4.$$

Check.

$$3 \times 4 + 2 \times 4 - 4 + 7 = 2 \times 4 + 15.$$
$$12 + 8 - 4 + 7 = 8 + 15.$$
$$23 = 23.$$

EXERCISE 7

Solve the following equations and check:

1. $x + 3 = 5.$

2. $x + 7 = 9.$

3. $x + 2 = 3.$

4. $x + 1 = 5.$

5. $x + 1 = 4.$

6. $x + 7 = 8.$

7. $3 + x = 4.$

8. $2 + x = 4.$

9. $4 + x = 6.$

10. $2x + 3 = 7.$

11. $3x + 1 = 10.$

12. $5x + 2 = 12.$

13. $3x + 3 = 9.$

14. $5x + 1 = 11.$

15. $2x + 4 = 6.$

16. $3x + 4 = 5.$

17. $4x + 8 = 12.$

18. $7x + 6 = 20.$

19. $3x + 2x = 10.$

20. $3x + x = 12.$

21. $5x + 2x - x = 12.$

22. $5x = 2x + 3.$

23. $7x - x + 1 = 2x + 9.$

24. $2x + 3x - 4x = 3.$

25. $\frac{1}{2}x = 3.$

26. $\frac{1}{4}x = 3.$

27. $\frac{1}{5}x = 1.$

28. $5x + 1 = 3x + 7.$

29. $5x + 2 = 6x + 1.$

30. $4x - x + 2x = 3x + 4.$

In problems 31–35, denote the unknown number by x. Form the equation and find the value of x.

31. Twice a certain number is 20. What is the number?

32. The sum of 10 and twice a certain number is 50. What is the number?

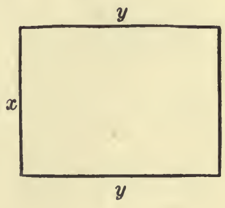
33. If three times a certain number is added to 9, the sum is 15. What is the number?

34. The sum of two consecutive integers is 11. Find the integers.

Suggestion. Consecutive integers are those which differ by 1.

35. If a pound of butter and one of lard together cost 60 cents, what was the price of each if the butter cost three times as much as the lard?

36. The perimeter of the rectangle in the accompanying diagram is given by the formula $P = 2x + 2y$. Find x when $P = 100$ and $y = 30$. Find y when $P = 50$ and $x = 10$.



When A stands for the area of a rectangle, b the number of units in its base, and a the number of units in its altitude,

$$A = a \times b.$$

Using this formula, find the value of the missing letter, given:

37. $b = 6$, $a = 4$. **38.** $b = 10$, $a = 2\frac{1}{2}$.

39. $A = 20$, $a = 4$. **40.** $A = 60$, $b = 10$.

When A stands for the area of a triangle, b the number of units in its base, and a the number of units in its altitude,

$$A = \frac{a \times b}{2}.$$

From this formula find the value of the missing letter, given:

41. $a = 6$, $b = 8$. **42.** $a = 9$, $b = 4$.

43. $A = 12$, $a = 6$. **44.** $A = 20$, $b = 8$.

When r stands for the number of units in the radius of a circle, c the number in the circumference, and A the area, then, (1) $c = 2\pi r$; (2) $A = \pi r^2$.

Using the proper formula,

45. Find c, given $r = 3$.

Thus, $c = 2\pi \times 3$, or 6π.

46. Find A, given $r = 5$.

Thus, $A = \pi \times 5^2$, or 25π.

47. Find r, given $c = 8\pi$.

Thus, $2\pi r = 8\pi$; $2r = 8$; $r = 4$.

Remark. Observe that the equation $2r = 8$ was derived from the equation $2\pi r = 8\pi$ by dividing both members of the latter by π.

48. Find r, given $A = 9\pi$.

Thus, $\pi r^2 = 9\pi$; $r^2 = 9$; $r = 3$.

49. Find c, given $r = 6$. **50.** Find A, given $r = 2$.

51. Find A, given $r = 10$. **52.** Find r, given $c = 16\pi$.

53. Find r, given $A = 64\pi$. **54.** Find r, given $A = \frac{1}{4}\pi$.

55. Using the letters c and n, write a formula for the cost of any number of dozen oranges when one dozen costs 25 cents; when one dozen costs a cents.

56. Write as a formula the rule for finding the simple interest for a given number of years t on a given sum of money s at a given rate of interest per cent per annum r.

57. I am twenty-five years younger than my father, whose age is a years. Write in a formula the rule for finding my age when his age is known.

58. Write a formula for the weight of a bottle of milk, given the weight of the bottle, the weight of a cubic inch of milk, and the quantity in the bottle. (Use the letters W, b, M, and v.)

59. Write a formula for the number of cents in a dollars $+ b$ dimes $+ c$ nickels.

60. Write a formula for the number of inches in a yards b feet c inches.

61. Write an expression for the rate of a train which runs m miles in 5 hours; which runs m miles in h hours.

62. If n articles cost d dollars, find an expression for the number of articles which can be bought for x dollars.

63. Construct a formula for the number N which, when divided by d, gives the quotient q and the remainder r.

64. The sum of three consecutive integers is 18. Find the integers.

65. Write a formula for the volume V of a beam, l feet long, w feet wide, and d feet thick. Apply the formula to find the thickness of a beam which contains 10 cubic feet of timber and is 2 feet wide and 5 feet long.

66. What is the area of a triangle whose base is 5 feet and whose altitude is 10 feet?

67. A room is a feet long and b feet wide. Construct a formula for the cost of carpeting the room with linoleum which costs $2 per square yard.

68. The rule for making tea is: "One teaspoonful of tea for each person and one for the pot." Express this rule by a formula letting t denote the number of teaspoonfuls of tea and p the number of persons.

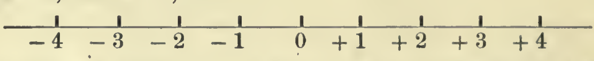
69. A formula for the time in hours required to cook a joint of beef of given weight in pounds is $t = \frac{1}{4}w + \frac{1}{3}$. From this formula state the rule.

Positive and Negative Numbers

29. Algebra makes use of all the numbers of elementary arithmetic and in addition to these it introduces certain other numbers called **positive** and **negative numbers.**

ILLUSTRATION

Let a point on a horizontal line be selected and marked 0; let the point one unit to the right of 0 be marked + 1, the point two units to the right of 0 be marked + 2, and so on. Let the point one unit to the left be marked − 1, the point two units to the left of 0 be marked − 2, and so on; thus:



The points marked + 1, + 2, + 3, etc., represent the *positive* numbers, and those marked − 1, − 2, − 3, etc., represent the *negative* numbers.

30. Two positive or two negative numbers are said to have *like signs.* A positive number and a negative number are said to have *unlike signs.* Positive and negative numbers are called *algebraic numbers.*

31. Whenever quantities exist which are exact opposites, as illustrated in section 29, these quantities may be represented by positive and negative numbers.

Thus, an ordinary thermometer scale may be divided in the manner indicated by the diagram in section 29. When so divided the point marked 0 indicates zero degrees; one degree above zero is marked $+1$, two degrees above zero, $+2$, one degree below zero, -1, two degrees below zero, -2, and so on.

32. The scale of positive and negative numbers, section 29, has various practical applications, such as indicating degrees of latitude north and south from the earth's equator, marked 0°; degrees of longitude west and east from some chosen meridian, as that of Greenwich, which is marked 0°; intervals of time after and before a certain event; gains and losses in business transactions.

Note. Negative numbers are introduced into algebra by a simple convention. In $a-b$ it is convenient to call the expression $-b$ a number and to say that $a-b$ is obtained by adding $-b$ to a. Thus, $a-b$ is written $a+(-b)$. This is a new use of the word *number*. A negative number is, therefore, simply a number which is to be subtracted. In the same way a positive number is a number which is to be added. The numbers of arithmetic are neither positive nor negative.

EXERCISE 8

1. Using the signs + and −, write:

5 positive units; 6 negative units; a positive units; b negative units; $2a$ positive units; $3x$ negative units.

2. State how many and what kind of units there are in each of the following:

$$+3;\ -1;\ +c;\ -b;\ +2x;\ -3b.$$

3. If 10° north latitude is represented by +10°, what number will represent 25° north latitude? 10° south latitude? 30° south latitude?

4. If north latitude is marked + and south latitude −, write, using the signs + and − instead of N. and S.:

5° N.; 6° S.; 9° 30′ N.; 7° 30′ 12″ S.

5. If west longitude is marked + and east longitude −, write, using the signs + and − instead of W. and E.:

20° E.; 5° W.; 4° 15′ W.; 7° 10′ 20″ E.

6. If the year 1916 A.D. is represented by +1916, what will represent the year 2000 A.D.? The year 399 B.C.? The year 646 B.C.?

7. If gains are marked + and losses −, write, using the signs + and −:

$5 gain; $6 loss; $8 loss; $4 gain.

8. If temperature above zero is marked + and temperature below zero −, state what temperature is indicated by each of the following:

$+5°$; $-1°$; $+60°$; $-7°$; $+80°$.

Addition

33. A gain of $10 together with a gain of $6 makes a total gain of $16; also a loss of $10 together with a loss of $6 makes a total loss of $16. Hence, regarding *gain* as *positive* and, therefore, *loss* as *negative*, we may infer that:

1. $(+10)+(+6)=+16.$
2. $(-10)+(-6)=-16.$

Remark. If it is necessary to distinguish a sign of an algebraic number from a sign of operation, the algebraic number is put into parentheses; otherwise, in writing such numbers the positive sign is usually omitted.

34. A gain of $10 combined with a loss of $6 is equivalent to a net gain of $4; also, a loss of $10 combined with a gain of $6 is equivalent to a net loss of $4. Hence, we may infer that:

1. $(+10)+(-6)=+4.$
2. $(-10)+(+6)=-4.$

35. The **absolute value** of an algebraic number is its value without regard to sign.

Thus, the absolute value of $+2$ is 2 and the absolute value of -3 is 3.

Remark. The expressions *arithmetical value* and *numerical value* are sometimes used instead of absolute value.

36. Positive and negative numbers are also called **opposite numbers.** See section 31.

37. The result obtained by combining (adding) two or more algebraic numbers is called the **sum** of the numbers.

From sections 33 and 34 we infer that:

1. *The sum of two numbers with like signs is the sum of their absolute values with their common sign prefixed to the result.*

2. *The sum of two numbers with unlike signs is the difference of their absolute values with the sign of the number which has the greater absolute value prefixed to the result.*

Note 1. To add three or more algebraic numbers with like signs, add the second to the first, to the result add the third, and so on.

Thus, $+3+(+2)+(+4)+(+1)=+5+(+4)+(+1)$
$=(+9)+(+1)=(+10).$

Note 2. To add three or more algebraic numbers with unlike signs, add the positive numbers and the negative numbers separately, and then add the results.

Thus, $+4+(+5)+(+6)+(-2)+(-7)=+15+(-9)=+6.$

Remark. When two numbers have the same absolute value and unlike signs, their sum is zero.

Thus, $(+2)+(-2)=0.$

EXERCISE 9

Name at sight the sum:

1. $+3$	**2.** -2	**3.** $+5$	**4.** -4
$+2$	-1	$+7$	-9

5. $\begin{array}{r} +2 \\ -1 \\ \hline \end{array}$	6. $\begin{array}{r} +3 \\ -5 \\ \hline \end{array}$	7. $\begin{array}{r} -8 \\ +5 \\ \hline \end{array}$	8. $\begin{array}{r} -6 \\ +9 \\ \hline \end{array}$
9. $\begin{array}{r} -4 \\ +4 \\ \hline \end{array}$	10. $\begin{array}{r} -10 \\ -\ \ 1 \\ \hline \end{array}$	11. $\begin{array}{r} +7 \\ -6 \\ \hline \end{array}$	12. $\begin{array}{r} +8 \\ -8 \\ \hline \end{array}$

Add, as indicated, at sight:

13. $+5+(+2)$. **14.** $-7+(-3)$.

15. $-8+(-6)$. **16.** $-8+(-3)$.

17. $+9+(-9)$. **18.** $+10+(-12)$.

19. $+2+(+1)+(+4)$. **20.** $-2+(-3)+(-4)$.

21. $-4+(+1)+(+2)$. **22.** $+6+(-2)+(-1)$.

23. $-1+(-2)+(+1)+(+3)$.

24. $+2+(-3)+(+4)+(-1)$.

25. $-1+(+1)+(+5)+(-3)$.

26. $+3+(-3)+(-4)+(+4)$.

27. $-2+(-3)+(+4)+(-3)$.

28. $6+(-3)+(+2)+(-7)$.

29. $5+(+3)+(-4)+(-8)$.

30. $-7+(-2)+(+8)+(+2)$.

Subtraction

38. In algebra, as in arithmetic, subtraction is the *inverse* operation of addition; that is, subtraction is the *undoing* of an addition.

Thus, $5+3-3=5$, which shows that the addition of 3 has been undone by the subtraction of 3.

Also, $a+3-3=a$, in which a is any algebraic number.

In general, the result of subtracting a number from an equal number, whether it be positive or negative, is zero.

Thus, $a-a=0$, since $0+a-a=0$.

Since $a - a = 0$ and $a + (-a) = 0$ [section 37, remark], the following principle may be inferred:

The result of subtracting a number is the same as the result of adding the opposite number.

ILLUSTRATIVE EXAMPLES

1. Subtract $+3$ from $+5$.

Solution. $(+5)-(+3)=(+5)+(-3)=+2.$

2. Subtract -3 from $+5$.

Solution. $(+5)-(-3)=+5+(+3)=+8.$

3. Subtract $+7$ from $+2$.

Solution. $(+2)-(+7)=+2+(-7)=-5.$

4. Subtract $+3$ from -5.

Solution. $(-5)-(+3)=-5+(-3)=-8.$

5. Subtract -2 from -5.

Solution. $(-5)-(-2)=-5+(+2)=-3.$

Remark. In algebra the terms *minuend*, *subtrahend*, and *difference* have the same meaning as in arithmetic.

Thus, in $(+6)-(+2)=+4$, the minuend is $+6$, the subtrahend is $+2$, and the difference is $+4$.

39. From the solutions of the illustrative examples of section 38 the following rule may be derived:

Rule. *The difference of two algebraic numbers is found by adding to the minuend the subtrahend with its sign changed.*

EXERCISE 10

Name at sight the difference:

1. $\begin{array}{r} +5 \\ \underline{+2} \end{array}$ **2.** $\begin{array}{r} -5 \\ \underline{-3} \end{array}$ **3.** $\begin{array}{r} +3 \\ \underline{+7} \end{array}$ **4.** $\begin{array}{r} -4 \\ \underline{-9} \end{array}$

5. $\begin{array}{r} +9 \\ \underline{-2} \end{array}$ **6.** $\begin{array}{r} -7 \\ \underline{+3} \end{array}$ **7.** $\begin{array}{r} -9 \\ \underline{-1} \end{array}$ **8.** $\begin{array}{r} +1 \\ \underline{+8} \end{array}$

9. $\begin{array}{r} -6 \\ \underline{-6} \end{array}$ **10.** $\begin{array}{r} -6 \\ \underline{+6} \end{array}$ **11.** $\begin{array}{r} 0 \\ \underline{+8} \end{array}$ **12.** $\begin{array}{r} 0 \\ \underline{-5} \end{array}$

Subtract, as indicated, at sight:

13. $+6-(+2)$. **14.** $-4-(-3)$. **15.** $+7-(-1)$.

16. $-9-(+5)$. **17.** $-1-(+1)$. **18.** $-1-(-7)$.

19. $11-(-4)$. **20.** $7-(-10)$. **21.** $0-(+5)$.

22. $0-(-1)$. **23.** $5-(+5)$. **24.** $5-(-5)$.

40. In section 33 it is shown that

$$10+(+6)=16=10+6, \tag{1}$$

also that

$$10+(-6)=4=10-6. \tag{2}$$

Again, from section 39 we have

$$10-(+6)=10+(-6)=10-6, \tag{3}$$

also that

$$10-(-6)=10+(+6)=10+6. \tag{4}$$

Examples (1), (2), (3), and (4) illustrate the following:

Rule of Signs

$+\ +$ *or* $-\ -$ *may be replaced by* $+$.
$+\ -$ *or* $-\ +$ *may be replaced by* $-$.

From examples (1), (2), (3), and (4), it is evident that in addition a number is written down with the sign before it retained, while in subtraction the number is written with its sign changed.

Thus, $(+2)+(+3)+(-4)$ is written $2+3-4=1$;
but, $(+2)-(-6)$ is written $2+6=8$.
Also, $-4+(-3)$ is written $-4-3=-7$;
but, $-4-(-3)$ is written $-4+3=-1$.

Remark. An expression like $4 - 6 = -2$ does not occur in arithmetic. In algebra, this expression means simply [see diagram, § 29] that if we start at the point marked $+4$ and count 6 units to the left we shall end with the point marked -2.

ILLUSTRATIVE EXAMPLES

1. Simplify $(+2)+(-3)+(+4)$.

Solution. $(+2)+(-3)+(+4)=2-3+4=-1+4=3.$

2. Simplify $(+2)+(-3)+(-4)$.

Solution. $(+2)+(-3)+(-4)=2-3-4=-1-4=-5.$

EXERCISE 11

Perform the indicated operations in the following examples:

1. $(+2)-(+1)$.

2. $3-(+2)$.

3. $-5+(-2)$.

4. $(-2)+(+5)$.

5. $2-(-3)$.

6. $-7+(-2)$.

7. $-2-(+1)$.

8. $-3+(-5)$.

9. $-2+(-1)$.

10. $3+(-2)$.

11. $3-(+5)$.

12. $3+5-(+2)$.

13. $(-2)+(-3)-(-4)$.

14. $(-3)+(+2)-(-1)$.

15. $-4+(-4)$.

16. $-2+2+(-2)$.

17. $-5+4+(-1)$.

18. $-1-2-(+3)$.

19. $1+2+(-3)$.

20. $1-1+2+(-2)$.

21. $-3+4-(-5)$.

22. $-1-2+(-3)$.

23. $-(-2)+(-3)-(-1)$.

24. $1+(+9)-(+12)+(-3)+(+17)-(+12)$.

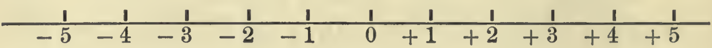
25. $2+(-9)-(-8)+(-1)-(+10)+(+20)$.

26. $8-(-7)+(-14)-(-5)-(+12)+(+7)$.

27. $-6-(-10)+(+20)-(+15)+(-25)$.

Graphic Representation of Addition and Subtraction of Algebraic Numbers

41. It is not possible in arithmetic to subtract a number from a less number. The introduction of negative numbers makes it possible in algebra to subtract in all cases. The diagram of section 29 is here reproduced and used in illustrating the addition and subtraction of algebraic numbers.



1. To add a positive number.

To add + 2 to + 3 begin at the point marked + 3 and count two spaces to the *right*, ending at the point marked + 5. In like manner, to add + 2 to any positive or negative number begin with that number and count two spaces to the right. In general,

In adding a positive number, the counting is to the right.

2. To subtract a positive number.

Since subtraction is the *undoing* of an addition, to subtract + 2 from + 5, begin at the point marked + 5 and count two spaces to the *left*, ending at the point marked + 3. In general,

In subtracting a positive number, the counting is to the left.

3. To add a negative number.

Since the sum of – 2 and – 3 is equal to – 5 [§ 33], to add – 2 to – 3, begin at the point marked – 3 and count two spaces to the *left*, ending at the point marked – 5. In general,

In adding a negative number, the counting is to the left.

4. To subtract a negative number.

Since subtraction is the undoing of an addition, to subtract – 2 from – 5, begin at the point marked – 5 and count two spaces to the *right*, ending at the point marked – 3. In general,

In subtracting a negative number, the counting is to the right.

Remark. Notice that in the two operations of adding a positive and subtracting a negative number, the counting is done to the *right;* while in the two operations of adding a negative number and subtracting a positive number, the counting is done to the *left.* Hence,

The subtraction of a number and the addition of its opposite number lead to the same result.

EXERCISE 12

Using the diagram in section 41, verify the following by counting:

1. $(-2)+(+2)=0.$

2. $(-5)+(+9)=+4.$

3. $(-2)-(+2)=-4.$

4. $(-5)-(-1)=-4.$

5. $0+(+3)=+3.$

6. $0-(+3)=-3.$

7. $0+(-3)=-3.$

8. $0-(-3)=+3.$

9. $0+0=0.$

10. $0-0=0.$

11. $(-1)+(-2)=-3.$

12. $(-4)-(+1)=-5.$

13. $(+3)+(-4)=-1.$

14. $(+5)-(+8)=-3.$

Multiplication

42. In arithmetic, 3×2 means $2+2+2$. In algebra,

1. $(+3)\times(+2)$ means $+2+(+2)+(+2)$; that is, multiplication by a positive integer means that another number is to be repeated positively.

2. $(-3)\times(+2)$ means $-(+2)-(+2)-(+2)$; that is, multiplication by a negative integer means that another number is to be repeated negatively.

Note. $(+3)\times(+2)$ and $(-3)\times(+2)$ are usually written $(+3)(+2)$ and $(-3)(+2)$, respectively.

43. The number repeated is called the *multiplicand;* the number which shows how many times the multiplicand is repeated is called the *multiplier;* the result in multiplication is called the *product.*

44. The possible combinations of signs in algebraic multiplication are given in the following:

1. $(+3)(+2) = +(+2)+(+2)+(+2) = 2+2+2 = +6.$
2. $(+3)(-2) = +(-2)+(-2)+(-2) = -2-2-2 = -6.$
3. $(-3)(+2) = -(+2)-(+2)-(+2) = -2-2-2 = -6.$
4. $(-3)(-2) = -(-2)-(-2)-(-2) = +2+2+2 = +6.$

45. From 1, 2, 3, and 4 of section 44 we may infer the

Rule of Signs in Multiplication

The product of two numbers with like signs is positive; the product of two numbers with unlike signs is negative.

EXERCISE 13

Name, at sight, the product:

1. $(+3)(+2)$. **2.** $(+4)(+3)$. **3.** $(+3)(-2)$.

4. $(-4)(+3)$. **5.** $(-3)(-2)$. **6.** $(-1)(-2)$.

7. $(+1)(-1)$. **8.** $(-1)(-1)$. **9.** $(+2)(-3)$.

10. $(-3)(-3)$. **11.** $(-5)(+2)$. **12.** $(+7)(-3)$.

13. $(+2)(+3)(-4)$.

Suggestion. Multiply $+3$ by $+2$, then find the product of this result and -4.

14. $(+2)(+3)(-2)$. **15.** $(-2)(-3)(-4)$.

16. $(-2)(+3)(-5)$. **17.** $(+5)(-7)(-2)$.

18. $(-1)(+1)(-1)$. **19.** $(+1)(-1)(+1)$.

20. In finding the product of three factors, what is the sign of the product when only one of the factors is negative? When all of the factors are negative? When two of the factors are negative?

Franciscus Vieta (François Viète) (1540–1603) was a French lawyer who gave up most of his leisure to mathematics. He was the author of the earliest work on symbolic algebra. He introduced in this work the use of letters for known and unknown numbers. His solution of the cubic equation continues in use at the present time.

Division

46. In arithmetic, 12 (the dividend) ÷ 3 (the divisor) = 4 (the quotient), because $3 \times 4 = 12$.

That is, *Divisor* × *Quotient* = *Dividend*.

This relation connecting divisor, quotient, and dividend may be employed in establishing the rule of signs in division, thus:

1. $(+12) \div (+4) = +3$, since $(+4)(+3) = +12$.
2. $(+12) \div (-4) = -3$, since $(-4)(-3) = +12$.
3. $(-12) \div (+4) = -3$, since $(+4)(-3) = -12$.
4. $(-12) \div (-4) = +3$, since $(-4)(+3) = -12$.

From 1, 2, 3, and 4, we may infer the

Rule of Signs in Division

The quotient of two numbers with like signs is positive; the quotient of two numbers with unlike signs is negative.

EXERCISE 14

Name, at sight, the quotient:

1. $(+6) \div (+2)$.

2. $(+9) \div (+3)$.

3. $(+6) \div (-2)$.

4. $(-6) \div (+2)$.

5. $(-3) \div (+1)$.

6. $(-3) \div (-1)$.

7. $(-24) \div (-12)$.

8. $(-27) \div (-3)$.

9. $(-14) \div (-2)$.

10. $(+27) \div (+9)$.

11. $(+30) \div (-3)$.

12. $(+56) \div (-8)$.

13. $(+12) \div (+3) \div (+2)$.

Suggestion. Divide $(+12)$ by $(+3)$, then divide the resulting quotient by $(+2)$.

14. $(+24) \div (-2) \div (-3)$.

15. $(-18) \div (-3) \div (-3)$.

CHAPTER II

FUNDAMENTAL PROCESSES

Addition

47. Addition of monomials which have no common factor.

1. The sum of $3a$, $2b$, and $5c$ is $+3a+2b+5c$.

2. The sum of $3a$, $2b$, and $-5c$ is $+3a+2b+(-5c)$, which is equal to $+3a+2b-5c$.

In general,

The sum of two or more monomials which have no common factor is expressed by writing the monomials in turn, each preceded by its own sign.

Remark. For convenience, a plus sign before the first term is usually omitted.

Thus, $+3a+2b+5c$ is written $3a+2b+5c$.

48. Commutative law for addition. In arithmetic, it is not necessary to call attention to the obvious fact that, for instance, $2+3=3+2$. In algebra, it is *assumed* that the sum does not depend on the order in which the terms are taken; this assumption is usually referred to as the **commutative law for addition.** This law is expressed by the formula,

$$\boldsymbol{a+b=b+a.}$$

EXERCISE 15

Name the sum in each of the following:

1. a and c. **2.** x and 1.

3. b and c. **4.** 4 and a.

5. a, b, and c.

6. $2a$, $-b$, and $-c$.

7. $4x$, $2y$, and $-z$.

8. $-4m$, $-5n$, and p.

9. $3(a+b)$ and $-4(c+d)$.

10. $2(x+y)$ and $3(m-n)$.

11. a, b, c, and $-d$.

12. $3(a-b)$ and $4(a+b)$.

13. R^2, r^2, and Rr.

14. $4r$, $-2y$, $2f$, and $-3i$.

49. Addition of monomials which have a common factor.

1. $2a+3a=(2+3)a=5a$. [See problem 9, Exercise 2, page 8.]

2. $2a+3a-4a=(2+3-4)a=a$.

3. $2a+3a-4a-5a=(2+3-4-5)a=-4a$.

In general,

The sum of two or more monomials which have a common factor is equal to the sum of the coefficients of the common factor multiplied by the common factor.

EXERCISE 16

(Solve as many as possible at sight.)

Add:

1.	2.	3.	4.
$4x$	$-4m$	$-m$	$3ab$
$3x$	$-5m$	$-5m$	$4ab$

5.	6.	7.	8.
$-2a$	$3mn$	$9a^2b$	$-4abc$
$5a$	$-8mn$	$-3a^2b$	$9abc$

9.	10.	11.	12.
$-8x^3$	πR^2	$2(a+b)$	$ab(m+n)^2$
$9x^3$	$4\pi R^2$	$3(a+b)$	$-3ab(m+n)^2$

13.	14.	15.	16.
$3a$	$-6m^2$	$2mn^2$	$-xyz$
$4a$	$-8m^2$	$3mn^2$	$-4xyz$
$5a$	$-5m^2$	$5mn^2$	$-2xyz$

17.	18.	19.	20.
$-4m$	$-5a$	$-mn$	$-18xyz$
$5m$	$7a$	$-3mn$	$12xyz$
$3m$	$-3a$	$-5mn$	xyz

21.	22.	23.	24.
$-x^2$	$-10pq$	$-7ts$	ab^2c^2
$3x^2$	pq	$9ts$	$-8ab^2c^2$
x^2	$-11pq$	$-ts$	$-6ab^2c^2$
$-5x^2$	$-19pq$	$-11ts$	$9ab^2c^2$

25.	26.	27.	28.
$2(a+b)$	$-(m+n)$	$(r+1)$	$3(x+y)^2$
$-4(a+b)$	$-3(m+n)$	$-2(r+1)$	$-4(x+y)^2$
$5(a+b)$	$5(m+n)$	$-8(r+1)$	$-8(x+y)^2$
$-6(a+b)$	$4(m+n)$	$6(r+1)$	$11(x+y)^2$

29. $2x$, $3x$, and $5x$.

30. $-mn$, $-2mn$, and $-5mn$.

31. $-4n$, $5n$, and $-n$.

32. b^2c, $-5b^2c$, and $9b^2c$.

33. $-R$, $-6R$, and $8R$.

34. $4bc^2$, $-8bc^2$, and $-3bc^2$.

35. m^2n, $-3m^2n$, $-m^2n$, and $4m^2n$.

36. $2\pi R^2$, $-4\pi R^2$, πR^2, and $3\pi R^2$.

37. $-6xy^2$, $10xy^2$, $-xy^2$, and $-4xy^2$.

38. $-\frac{1}{2}x^2$, $\frac{1}{4}x^2$, $-x^2$, and $\frac{3}{4}x^2$.

39. $(m + n)$, $-3(m + n)$, and $-5(m + n)$.

40. $a(x + y)^3$, $-6a(x + y)^3$, $8a(x + y)^3$.

Just as the sum of $3m$ and $5m$ may be written $(3 + 5)m$, so the sum of am and $2m$ is written $(a + 2)m$. Similarly, $am + bm = (a + b)m$. $am + bm - m = (a + b - 1)m$, and $c(m+n) + d(m+n) = (c+d)(m+n)$.

Add by combining the coefficients:

41. cy and by.

42. ax and $-x$.

43. aby and cby.

44. mxb and $-cyb$.

45. $\frac{1}{2}aB$ and $\frac{1}{2}ab$.

46. $\frac{1}{3}HB$ and $-\frac{1}{3}Hb$.

47. am, $-m$, and cm.

48. πR^2, πr^2, and πRr.

49. $\frac{1}{3}H\pi R^2 + \frac{1}{3}H\pi r^2 + \frac{1}{3}H\pi Rr$.

50. $a(x + y)$ and $-b(x + y)$.

51. $K(y + z)$ and $r(y + z)$.

52. $a(x - y)$ and $(x - y)$.

50. Addition of monomials. The sum of $2a$, $3b$, $-3a$, $5c$, $2b$, $-6a$, and $7b$ may be found as follows:

$$\begin{aligned} &2a + 3b + (-3a) + 5c + 2b + (-6a) + 7b \\ &\quad = 2a + 3b - 3a + 5c + 2b - 6a + 7b && [\S\ 40] \\ &\quad = (2a - 3a - 6a) + (3b + 2b + 7b) + 5c && [\S\ 48] \\ &\quad = (2 - 3 - 6)a + (3 + 2 + 7)b + 5c && [\S\ 49] \\ &\quad = -7a + 12b + 5c. \end{aligned}$$

In general,

The sum of two or more monomials is obtained by finding the sum of those that have a common factor and then adding these sums.

51. Associative law for addition. In arithmetic, it is not necessary to call attention to the obvious fact that, for instance, $2 + 3 + 1 + 6 = (2 + 3) + (1 + 6) = 5 + 7 = 12$. In algebra, it is *assumed* that the sum does not depend on the grouping of terms; this assumption is usually referred to as the **associative law for addition.** This law is expressed by the formula,

$$a + (b + c) = (a + b) + c.$$

EXERCISE 17

Add:

1. a and b. **2.** a and $-b$. **3.** a and $3b$.

4. $2a$ and $3b$. **5.** $3a$ and $-2x$. **6.** a, b, and c.

7. x, y, and $-z$. **8.** $m, -n$, and p. **9.** $2a, 3b, 4a$, and $5b$.

10. $3x, 4x^2, -5x$, and $-8x^2$.

11. $-3a, 4b, 5a, -c, 4c$, and $-a$.

12. $-xy, 2xz, 3yz, 4xy, 5yz$, and $-6xz$.

13. $3m^2, -mn, 2n^2, 3mn, -2n^2, 4m^2$, and $5mn$.

14. $-3y^2, 2xy, 5y^2, 2x^2, -4xy$, and $6x^2$.

15. $a^2, -2ab, b^2, -a^2, -2ab$, and $-b^2$.

16. $4a^2b, 2ab, -5ab^2, -6a^2b, 2a^2b, -ab^2, -4ab$, and $8a^2b$.

17. $2(m+n), 3(m-n), -5(m+n)$, and $-(m-n)$.

18. $4(x+y), -4(x-y), -2(x-y), -(x+y), -3(x+y)$, and $-(x-y)$.

19. $2(x-y), b^2, -4, -(x-y), -7, -4b^2$, and $(x-y)$.

20. $a^2b, ab^2, 3ab, 4a^2b, -6a^2b, -5ab^2$, and $-ab$.

21. $a^2, a, (a+b), -3a, -4a^2, -5(a+b)$, and $4(a+b)$.

22. $x^2y + 3xy - 2xy^2 + 5x^2y - xy^2 - 4xy + 2x^2y$.

52. Addition of polynomials. Since a polynomial is the sum of two or more monomials, no additional rules are necessary for finding the sum of two or more polynomials. However, it is convenient to arrange the terms so that like terms stand in the same column, then to add the columns separately and combine the sums by section 50.

ILLUSTRATIVE EXAMPLES

1. Find the sum of $2x + 3y - 4z$, $3x - y + 2z$, and $4x + 2y - 5z$, and check the work by letting $x = 1$, $y = 1$, and $z = 1$.

Solution	Check
$2x + 3y - 4z$	$2 + 3 - 4 = 1$
$3x - y + 2z$	$3 - 1 + 2 = 4$
$4x + 2y - 5z$	$4 + 2 - 5 = 1$
$9x + 4y - 7z$	$9 + 4 - 7 = 6$

2. Find the sum of $7x - 4(y + z)$, $6x + 2(y + z)$, $2x + (y + z)$, and $x - 3(y + z)$, and check the work by letting $x = 2$, $y = 1$, and $z = 1$.

Solution	Check
$7x - 4(y + z)$	$14 - 8 = 6$
$6x + 2(y + z)$	$12 + 4 = 16$
$2x + (y + z)$	$4 + 2 = 6$
$x - 3(y + z)$	$2 - 6 = -4$
$16x - 4(y + z)$	$32 - 8 = 24$

Remark. Note that the coefficient of $(y + z)$ in the third expression is 1.

3. Find the sum of $ax + by$, $bx - ady$, and $cdx + ay$, and check the work by letting $a = 2$, $b = -1$, $c = 3$, $d = 1$, $x = 1$, $y = 1$, and $z = 1$.

Solution	Check
$ax + by$	$2 - 1 = 1$
$bx - ady$	$-1 - 2 = -3$
$cdx + ay$	$3 + 2 = 5$
$(a + b + cd)x + (b - ad + a)y$	$4 - 1 = 3$

EXERCISE 18

Add the following polynomials and check the results:

1. $a + b$
   $a - b$

2. $a + b - c$
   $a \quad + c$

3.
$$\begin{array}{r} a-b+2c \\ a+b-2c \\ \hline \end{array}$$

4.
$$\begin{array}{r} a+b-c-\ \ d \\ 2a-b+c-2d \\ \hline \end{array}$$

5.
$$\begin{array}{r} x-y+z \\ -x+y-z \\ \hline \end{array}$$

6.
$$\begin{array}{l} x-2y \\ x\qquad -2 \\ \hline \end{array}$$

7.
$$\begin{array}{r} 3m-2n \\ 4m+3n \\ \hline \end{array}$$

8.
$$\begin{array}{r} 4x+7y \\ -5x-8y \\ \hline \end{array}$$

9.
$$\begin{array}{r} 4ab-6c^2d \\ -\ \ ab+8c^2d \\ \hline \end{array}$$

10.
$$\begin{array}{r} 3m^2n-8mn^2 \\ 4m^2n-3mn^2 \\ \hline \end{array}$$

11.
$$\begin{array}{r} a-\ \ b+\ \ c \\ -2a+4b-6c \\ \hline \end{array}$$

12.
$$\begin{array}{r} 4mn-2m^2+\ \ 5n^2 \\ -2mn+3m^2-11n^2 \\ \hline \end{array}$$

13.
$$\begin{array}{r} \frac{1}{2}a+\frac{1}{3}b \\ \frac{1}{2}a+\frac{2}{3}b \\ \hline \end{array}$$

14.
$$\begin{array}{r} \frac{3}{2}a+b \\ -\frac{1}{2}a-b \\ \hline \end{array}$$

15.
$$\begin{array}{r} \frac{2}{3}a+\frac{3}{2}b \\ \frac{3}{2}a+\frac{1}{2}b \\ \hline \end{array}$$

16.
$$\begin{array}{r} -\frac{1}{2}x+\frac{1}{3}y-\frac{1}{3}z \\ -\ \ x+\ \ y-\ \ z \\ \hline \end{array}$$

17.
$$\begin{array}{r} \frac{1}{2}x+\frac{2}{5}y-\frac{1}{2}z \\ -\frac{1}{3}x+\frac{3}{5}y+\ \ z \\ \hline \end{array}$$

18.
$$\begin{array}{r} .5x+.3y \\ .2x+\ \ y \\ \hline \end{array}$$

19.
$$\begin{array}{r} .3a+.2x \\ 5a-.3x \\ \hline \end{array}$$

20.
$$\begin{array}{r} 1.2m+1.5n \\ .3m-\ \ .5n \\ \hline \end{array}$$

21.
$$\begin{array}{r} 2w+4v \\ 6w-7v \\ 8w+2v \\ \hline \end{array}$$

22.
$$\begin{array}{r} 8\pi R^2-2\pi RH \\ -4\pi R^2+\ \ \pi RH \\ -\ \ \pi R^2+\ \ \pi RH \\ \hline \end{array}$$

23.
$$\begin{array}{r} 2A+\ \ B-\ \ C \\ 3A-4B+6C \\ -5A+4B-6C \\ \hline \end{array}$$

24.
$$\begin{array}{r} 2x^2-\ \ xy+\ \ y^2 \\ -4x^2+4xy-2y^2 \\ 3x^2-5xy+3y^2 \\ \hline \end{array}$$

25.
$$\begin{array}{r} 4m-2n+3 \\ 6m+5n-1 \\ -2m-4n-4 \\ \hline \end{array}$$

26.
$$\begin{array}{r} 3a^2b-\ \ ab+4ab^2 \\ -\ \ a^2b+5ab-3ab^2 \\ 2a^2b-6ab\qquad \\ \hline \end{array}$$

27.
$$\begin{array}{rrr} m^3 & & +\,6\,n^3 \\ -\,3\,m^3 & -\,4\,mn & +\,n^3 \\ & -\,2\,mn & -\,5\,n^3 \\ \hline \end{array}$$

28.
$$\begin{array}{rrr} a^3 & +\,3\,ab & \\ -\,3\,a^3 & +\,ab & +\,b^3 \\ 6\,a^3 & & -\,4\,b^3 \\ \hline \end{array}$$

29.
$$\begin{array}{rr} 3(m+n) & -\,3(m-n) \\ 6(m+n) & +\,(m-n) \\ -\,2(m+n) & -\,4(m-n) \\ \hline \end{array}$$

30.
$$\begin{array}{rr} 4(x-y) & +\,(w-v) \\ -\,6(x-y) & -\,8(w-v) \\ -\,4(x-y) & +\,8(w-v) \\ \hline \end{array}$$

31. $2\,x+3\,y-2$, $x-4\,y+4$, and $3\,x-4\,y-8$.

32. $3\,m^2-2\,mn-n^2$, $4\,m^2-2\,mn+4\,n^2$, and $-\,m^2-4\,mn+2\,n^2$.

33. $3\,v^2+vt-2\,t^2$, $t^2-2\,vt+v^2$, and $4\,vt-8\,t^2-v^2$.

34. $r^2-s^2-t^2$, $4\,r^2+2\,t^2$, $6\,s^2-2\,r^2$, and $4\,r^2+2\,t^2-3\,s^2$.

35. x^3+x^2+x-1, $3\,x^2-4\,x+4$, $2\,x^3-7\,x^2-5$, and $2\,x^3-x-1$.

36. $x^3+3\,x^2y+y^2$, $2\,x^2-3\,xy^2-2\,y^2$, $4\,x^2+3\,x^2y+3\,xy^2-y^2$, and $3\,x^2y-2\,xy^2+y^3$.

37. $m^3-3\,m^2n$, $4\,mn^2-2\,n^3$, m^3-n^3, $2\,mn^2-m^2n$, and $n^3+3\,mn^2$.

38. $3(m+n)+2(x-y)$, $-\,2(m+n)-3(x-y)$, $(m+n)+(x-y)$, and $4(m+n)-2(x-y)$.

39. $am+bn$, $bm+cn$, and $dm+en$.

40. ax^2+by^2, cx^2-dy^2, and x^2-y^2.

41. $abx+my$, $cx+npy$, and $dx-ry$.

42. $4\,r+3\,r^2$, $\pi r+\pi r^2$, and $cr+ar^2$.

43. m^2x+b^2y, $-\,n^2x-c^2y$, and p^2x-y.

44. $a(m+n)+b(m-n)$, $c(m+n)-e(m-n)$, and $d(m+n)-h(m-n)$.

45. $ax+bxy-cz$, $2\,x+3\,xy+z$, and $cx-xy+az$.

46. $3\,x^3+2\,x^2y$, $3\,x^2y-4\,xy^2$, $5\,xy^2-4\,y^3$, and $3\,y^3-2\,x^3$.

47. ax^3+bx^2+cx+d and $3\,x^3-2\,x^2+3\,x-5$.

Subtraction

53. Subtraction of monomials. Since monomials are numbers, what is stated in sections 38 and 39 is applicable to monomials. Hence, for subtracting one monomial from another we have, from section 39, the following:

Rule. *Change the sign of the subtrahend and add the result to the minuend.*

ILLUSTRATIVE EXAMPLES

1. From $5m$ take $2m$.

Solution. $5m-(+2m)=5m-2m$ [§ 40]

$=3m.$

2. From $6x$ take $-2x$.

Solution. $6x-(-2x)=6x+2x$ [§ 40]

$=8x.$

3. From $-8y$ take $3y$.

Solution. $-8y-(+3y)=-8y-3y$ [§ 40]

$=-11y.$

4. From $-6xy$ take $-2xy$.

Solution. $-6xy-(-2xy)=-6xy+2xy$ [§ 40]

$=-4xy.$

5. From x take y.

Solution. $x-(+y)=x-y.$ [§ 40]

6. From $-3xy$ take $2yz$.

Solution. $-3xy-(+2yz)=-3xy-2yz.$ [§ 40]

7. From $-2mn$ take $-4rs$.

Solution. $-2mn-(-4rs)=-2mn+4rs$ [§ 40]

$=4rs-2mn.$ [§ 48]

8. From amn take bmn.

Solution. $amn-(+bmn)=amn-bmn$ [§ 40]

$=(a-b)mn.$

EXERCISE 19

(Solve as many as possible at sight.)

In the first three examples subtract the lower monomial from the upper monomial.

1.

$5x$	$-7a$	$5m$	$-5y$	$7z$
$2x$	$-4a$	$7m$	$+2y$	$-4z$

2.

$-5R$	$-\ a^2$	$3r^3$	mn	$-2rs$
$-7R$	$-5a^2$	$7r^3$	$4mn$	$-6rs$

3.

$5x^2y$	$-xyz$	$4(m+n)$	$2(x+y)$	$-2(r+s)$
$-8x^2y$	xyz	$3(m+n)$	$7(x+y)$	$-5(r+s)$

4. From $3x$ take $9x$.

5. From $4y^2$ take $-7y^2$.

6. From $-7mn$ take $2mn$.

7. From $4\pi R^2$ take πR^2.

8. From $-6r^2$ take $-r^2$.

9. From πR^3 take $\frac{1}{3}\pi R^3$.

10. From $5(m+n)$ take $-2(m+n)$.

11. From $(r-1)$ take $-2(r-1)$.

12. From $2a(x+y)$ take $3a(x+y)$.

13. From $b^2(x-y)$ take $-2b^2(x-y)$.

14. From 0 subtract x.

15. From 1 subtract $-a$.

16. From a subtract $-b$.

17. From $-a$ take $-b$.

Subtract as indicated:

18. $m-(-n)$.

19. $2x-(+y)$.

20. $ax-(-5)$.

21. $a^2-(-mn)$.

22. $c^2-(-d^2)$.

23. $-m^2-(-n^2)$.

24. $-4xy-(+3uv)$.

25. $\frac{1}{2}qt^2-(+qt^2)$.

26. $-cm^2-(-dm^2)$.

27. $ax-(-bx)$.

28. $3(x-y)-a(x-y)$.

29. $m(x+y)-n(x+y)$.

54. Subtraction of polynomials. In adding $2a - 3b$ to $3a + 5b - c$, each term of $2a - 3b$ is added to $3a + 5b - c$; the resulting sum is $5a + 2b - c$. Therefore, in the inverse operation of subtracting $2a - 3b$ from $5a + 2b - c$, each term of $2a - 3b$ is subtracted from $5a + 2b - c$; the resulting difference is $3a + 5b - c$.

The work may be arranged thus:

$$\begin{array}{r} 5a + 2b - c \\ \underline{-2a + 3b} \\ 3a + 5b - c \end{array}$$

In accordance with section 39, the sign of each term of the subtrahend, $2a - 3b$, has been changed and the result added to the minuend, $5a + 2b - c$. In general,

To subtract one polynomial from another, change the sign of each term of the subtrahend and proceed as in addition.

ILLUSTRATIVE EXAMPLES

1. Subtract $3x - 2y$ from $5x + 3y$, and check the work by letting $x = 1$ and $y = 1$.

Solution

$$\begin{aligned}(5x + 3y) - (3x - 2y) &= (5x + 3y) + (-3x + 2y) \\ &= 5x - 3x + 3y + 2y \\ &= 2x + 5y.\end{aligned}$$

Another Solution

$$\begin{array}{r} 5x + 3y \\ \underline{-3x + 2y} \\ 2x + 5y. \end{array}$$

Remark. In practice, the change of the signs in the subtrahend and the addition should be performed mentally. Thus, in the above example the work should be arranged as follows:

Solution	**Check**
$5x + 3y$	$5 + 3 = 8$
$3x - 2y$	$3 - 2 = 1$
$2x + 5y$	$2 + 5 = 7$

2. Subtract $3a - 2xy + 5b$ from $5a - 3b + 2c$, and check the work by letting $a = 1$, $b = 1$, $c = 1$, $x = 1$, and $y = 1$.

Solution	Check
$5a - 3b \qquad + 2c$	$5 - 3 \qquad + 2 = 4$
$\underline{3a + 5b - 2xy}$	$\underline{3 + 5 - 2} \quad = \underline{6}$
$2a - 8b + 2xy + 2c$	$2 - 8 + 2 + 2 = -2$

3. Subtract $bxy + 2bz - ct$ from $axy - 2bz + 3t$, and check the work by letting $a = 1$, $b = 1$, $c = 1$, $t = 1$, $x = 1$, and $y = 1$.

Solution	Check
$axy - 2bz + 3t$	$1 - 2 + 3 = 2$
$\underline{bxy + 2bz - ct}$	$\underline{1 + 2 - 1} = \underline{2}$
$(a - b)xy - 4bz + (3 + c)t$	$0 - 4 + 4 = 0$

EXERCISE 20

Subtract and check:

1. $\begin{array}{r} 2x + y \\ \underline{x - y} \end{array}$ **2.** $\begin{array}{r} 3m + 4n \\ \underline{2m + 2n} \end{array}$ **3.** $\begin{array}{r} 5a + 3b \\ \underline{2a - \quad b} \end{array}$

4. $\begin{array}{r} 4m + 3n \\ \underline{2m + 8n} \end{array}$ **5.** $\begin{array}{r} 5x - 3y \\ \underline{3x - 9y} \end{array}$ **6.** $\begin{array}{r} 4a - 3b \\ \underline{5a - 6b} \end{array}$

7. $\begin{array}{r} 5mn - 7 \\ \underline{2mn + 1} \end{array}$ **8.** $\begin{array}{r} 5x^2 - \quad x \\ \underline{7x^2 + 2x} \end{array}$ **9.** $\begin{array}{r} 2m^2n^2 + 4 \\ \underline{-3m^2n^2 + 5} \end{array}$

10. $\begin{array}{r} 2x^2 - 7y^2 \\ \underline{6x^2 - 8y^2} \end{array}$ **11.** $\begin{array}{r} \frac{1}{2}a + \frac{1}{4}b \\ \underline{-\frac{1}{2}a - \frac{3}{4}b} \end{array}$ **12.** $\begin{array}{r} 4m^2 - .4n^2 \\ \underline{- \quad m^2 - .6n^2} \end{array}$

13. $\begin{array}{r} bx + my \\ \underline{cx + my} \end{array}$ **14.** $\begin{array}{r} am^2 - bn^2 \\ \underline{-cm^2 - dn^2} \end{array}$ **15.** $\begin{array}{r} 2\pi RH + \pi R^2 \\ \underline{\pi RH - \pi R^2} \end{array}$

16. $\begin{array}{r} 4x + 3y - 2 \\ \underline{x - 3y + 2} \end{array}$ **17.** $\begin{array}{r} a + b + c \\ \underline{-a - b - c} \end{array}$

18. $\begin{array}{r} 2m + 3n - 4p \\ \underline{3m + 4n - 5p} \end{array}$ **19.** $\begin{array}{r} 3r - 2s - 6t \\ \underline{4r + \quad s + 5t} \end{array}$

20. $\begin{array}{r} -3x^2+2x-4 \\ 4x^2+3x-1 \\ \hline \end{array}$

21. $\begin{array}{l} x+y+z \\ 5x-4y \\ \hline \end{array}$

22. $\begin{array}{l} 4m^2+2m+5 \\ m^2 \qquad -1 \\ \hline \end{array}$

23. $\begin{array}{l} 5x^2 \qquad -3 \\ 2x^2+4x-5 \\ \hline \end{array}$

24. $\begin{array}{r} 2r+s \\ -5t+r-s \\ \hline \end{array}$

25. $\begin{array}{l} am+3bn+4dx \\ cm+bn-3dx \\ \hline \end{array}$

26. $\begin{array}{l} x+y \qquad -3 \\ 4x+2y-5z \\ \hline \end{array}$

27. $\begin{array}{l} -m-5n-p \\ -3m \qquad -2p-3 \\ \hline \end{array}$

28. $\begin{array}{r} 3a+3(m+n) \\ a-2(m+n) \\ \hline \end{array}$

29. $\begin{array}{r} (a+b)+1 \\ -2(a+b)-2 \\ \hline \end{array}$

30. $\begin{array}{l} 2(x+y)+3(m+n) \\ 4(x+y)+2(m+n) \\ \hline \end{array}$

31. $\begin{array}{r} -3(a+b)^2-4(c-d)^2 \\ 2(a+b)^2-4(c-d)^2 \\ \hline \end{array}$

32. $\begin{array}{r} a+b-c-d \\ 3a-b-c+d \\ \hline \end{array}$

33. $\begin{array}{r} (a+b)^2-z+3 \\ 2(a+b)^2+z-2 \\ \hline \end{array}$

34. $\begin{array}{l} 5(10)^2+6(10)+7 \\ 2(10)^2+4(10)+8 \\ \hline \end{array}$

35. $\begin{array}{l} 7\cdot 6^3-3\cdot 6^2+2\cdot 6+1 \\ 6\cdot 6^3-3\cdot 6^2-3\cdot 6+4 \\ \hline \end{array}$

36. From $a+b$ take $3a+4b-c$.

37. From $3m^2+4m-n$ take $4m^2-n^2+n$.

38. From $ax^2+a^2x+2a^2x^2$ take $3a^2x^2-3a^2x+2ax^2$.

39. From $a-1$ take b^2+a^2+a+2.

40. From $x-x^2-x^3$ take $5x^3-3+2x$.

41. From $m+2(a+b)$ take $3(a+b)-2m+4$.

42. Subtract $4xy^2-3x^2y+xy$ from xy.

43. Subtract $3x^2-x+4$ from 0.

44. Subtract $2a^n-3a+4$ from $3a^n-2a+4$.

45. Subtract $\frac{3}{4}a^2 - \frac{1}{2}a + b - 3$ from $4 + \frac{1}{2}b + a^2 + \frac{1}{2}a$.

46. From $ax^2 - by^2 + cz^2$ take $mx^2 - my^2 + 2z^2$.

47. From $m^2 - 3n^2 - p^2$ take $am^2 - bn^2 - cp^2$.

48. Take $3m^2 - n - 4$ from the sum of $m^2 - 3m + 4n$ and $3m^2 - m - 6$.

49. Take $1 + r^2 - s^2 + rs$ from the sum of $3r^2 + rs$ and $4r^2 + rs - 2$.

50. Take the sum of $a^2 - \frac{1}{2}a + 2$ and $-a^3 - \frac{1}{2}a^2 + a$ from 1.

51. Take $x^3 + x^2 - x + 1$ from the sum of $x^3 - x^2y + x^2$ and $-x^3 - xy^2 + 1$.

52. From the sum of πR^2 and $2\pi RH + 2\pi R^2$ take $4\pi R^2 - \pi RH$.

53. From the sum of $a + b - c$ and $2a - 3b + 2c$ take the sum of $-a + 2b$ and $2a - 3b + 2c$.

54. Take the sum of $3x - 4y + 5$ and $3y - 4x - 4$ from 2.

55. From $x + y + z$ subtract the sum of $x - 2y - z$ and $2x - y + 2z$.

If $A = 3x + 2y - 5z^2$, $B = 2x - 3y + 4z^2$, and $C = -x + y + z^2$, find the value of :

56. $A + B - C$. **57.** $A - B + C$. **58.** $B + C - A$.

59. $A - B - C$. **60.** $B - C - A$.

61. From $3a^2bc - 2ab^2 + 5b^2c$ take $2a^2bc + 2ab^2 - 3b^2c + 2abc$.

62. From $4xyz^2 - 2xy^2z + 7x^2yz$ subtract $-5xyz^2 - 13xy^2z + x^2yz$.

63. From $22(a + b)^2 + 5(a + b) - 7$ take $(a + b)^2 - 5(a + b) + 3$.

64. From $5(x + y) + 13(a + b) - 2$ take $7(a + b) - (x + y) + 2$.

65. From $x + (y + z)^2 + t^2$ take $5x + y + 1$.

66. How much greater than $x + 1$ is x^2?

67. How much less than $a + 3b - 4c$ is $2a + b + c - 1$?

68. What must be subtracted from a so that the remainder is b?

69. What must be subtracted from $x^2 - xy + y^2$ so that the remainder is $2x^2 + 3$?

70. From the sum of $11xyz^2 - 13x^2yz + 17xyz$ and $3xyz - 2x^2yz + 5x^2 - 3y$ take the sum of $5xyz - 13xyz^2 + 12y - 19x^2$ and $7x^2yz - 13x^2 + y$.

71. From $ax^2 + bxy + cy^2$ take $\frac{1}{2}ax^2 - \frac{1}{3}bxy + 2cy^2$.

72. Take the sum of $3h^2 - 2hk + 3k^2 + 2h - 3k + 1$ and $4k^2 - 3hk + 5h + 3k + 2$ from $2h^2 + 2hk - 17k^2 + 11h - 16k + 7$.

Parentheses

55. $2a + (3b + c)$ means that the number $(3b + c)$ is to be added to $2a$.

$2a + (3b - c)$ means that the number $(3b - c)$ is to be added to $2a$.

By addition,

$$2a + (3b + c) = 2a + 3b + c;$$

and

$$2a + (3b - c) = 2a + 3b - c.$$

Therefore,

1. *If an expression in parentheses is preceded by the plus sign, the parentheses may be omitted.*

2. *An expression may be inclosed within parentheses preceded by the plus sign.*

Thus,

1. $2a + (3b - c + d)$ may be written $2a + 3b - c + d$.

2. $2a + 3b - c + d$ may be written $2a + (3b - c + d)$, or $2a + 3b + (-c + d)$.

$2a-(3b+c)$ means that the number $(3b+c)$ is to be subtracted from $2a$. $2a-(-b-c)$ means that the number $(-b-c)$ is to be subtracted from $2a$.

Since the result of subtracting a number is the same as the result of adding the opposite number,

$$2a-(3b+c)=2a+(-3b-c)=2a-3b-c;$$

and $2a-(-b-c)=2a+(b+c)=2a+b+c.$

Therefore,

1. *If an expression in parentheses is preceded by the minus sign, the sign of each term within the parentheses must be changed when the parentheses are removed.*

2. *An expression may be inclosed within parentheses preceded by the minus sign, provided that the sign of each term be changed.*

Thus,

1. $2a-(3b+c)$ may be written $2a-3b-c$.

2. $2a-3b+c-d$ may be written $2a-(3b-c+d)$ or $2a-3b-(-c+d)$.

Remark. Observe that when the parentheses preceded by the minus sign are removed, the minus sign before the parentheses is omitted.

ILLUSTRATIVE EXAMPLES

1. Simplify $2+[3-(5-2)]$.

Solution. $2+[3-(5-2)]=2+[3-5+2]$
$=2+3-5+2$, or 2.

2. Simplify $2a-[3a-(2a-b)]$.

Solution. $2a-[3a-(2a-b)]=2a-[3a-2a+b]$
$=2a-3a+2a-b$, or $a-b$.

3. Simplify $1-[2x+\{3y-(4z+5)\}]+[5x-\{4y+(3z-2)\}]$.

Solution. $1-[2x+\{3y-(4z+5)\}]+[5x-\{4y+(3z-2)\}]$
$=1-[2x+\{3y-4z-5\}]+[5x-\{4y+3z-2\}]$
$=1-[2x+3y-4z-5]+[5x-4y-3z+2]$
$=1-2x-3y+4z+5+5x-4y-3z+2$
$=8+3x-7y+z.$

Remark. It will be observed in the solutions of illustrative examples 1, 2, and 3, that the innermost parentheses have been removed first. Although it is not essential to do so, yet the beginner is advised to proceed in this manner.

4. Indicate that from the sum of $2a+3b-c$ and $3a+b+2c$ the sum of $a-b+c$ and $4a+2b-3c$ is to be subtracted; then perform the indicated operations.

Solution.
$[(2a+3b-c)+(3a+b+2c)]-[(a-b+c)+(4a+2b-3c)]$
$=[2a+3b-c+3a+b+2c]-[a-b+c+4a+2b-3c]$
$=[5a+4b+c]-[5a+b-2c]$
$=5a+4b+c-5a-b+2c$
$=3b+3c.$

EXERCISE 21

(Solve as many as possible at sight.)

Remove the parentheses and simplify when possible:

1. $a+(b+c).$ **2.** $x+(y-z).$

3. $x-(y+z).$ **4.** $m-(n-p).$

5. $r-(s-t).$ **6.** $x+(2x-y).$

7. $a-(a+b).$ **8.** $5+(7-2).$

9. $6-(3+2).$ **10.** $8-(5-4).$

11. $3-(-2-1).$ **12.** $3x-(2x-x).$

13. $2m-(-3m-m).$ **14.** $r-(-3r+r).$

15. $7r+\{-2t-s\}.$ **16.** $2k-[m-n].$

17. $8-\overline{5+2}.$ **18.** $9-\overline{7-2}.$

19. $9-(6+2)+3.$ **20.** $7-(8-5)-2.$

21. $8-(6+1)-(8-7).$ **22.** $6-(5-1)-(4-3).$

23. $a-(a-b)+(a-2b)$.

24. $2x-(x-y)-(2x+2y)$.

25. $2y-(3y-6y)-(y-2y+3y)$.

26. $4x+[3x+(3x-2)]$.

27. $7x-[3x-(6y+2z)]$.

28. $2m+3n+\{m-(m-n)\}$.

29. $x-2y-\{2x-(x-3y)\}$.

30. $6r+\{3r-(2r-t)-3t\}$.

31. $4m-[2n-(3m-n)+2n]$.

32. $(3r+s)-\{r-(2r-s)+3r\}$.

33. $(4p-q)-[-p-(3p+q)-4p]-(p+q)$.

34. $x-[x-(y+z)-\{x-(y-z)\}]$.

35. $11-\{10-[9-(8-\overline{7-6x})]\}$.

36. $2-\{-2-[2-(-2-\overline{2+2}-2)-2]\}$.

37. $r-[-r-\{-r-(r+r-\overline{r-r}-r)-r\}-r]$.

In examples 38–44, remove only the inner parentheses and simplify when possible.

Thus, $[a-(b-c)]=[a-b+c]$.

38. $[x-(y+z)]$.

39. $[x+(y+z)]$.

40. $[m-(n-1)]$.

41. $[(m+n)-(p-q)]$.

42. $[(2x+3y)-(2x-3y)]$.

43. $[(x^2-y^2)-(x^2+2y^2)]$.

44. $\{(m+n)-(m-n)\}$.

In examples 45–50, indicate the operations before performing them.

45. To $3a^2-1$ add a^2-3a+1.

46. From $3a$ take $a-2$.

47. Add x^3-2x^2y, $3xy^2-y^3$, and x^2y-2xy^2.

48. From $3y$ take $y - 5$.

49. Add 1 to the sum of $3a - 3$ and $2a + 2$.

50. Take the sum of $3a - 2b + 5$ and $b - 5a - 2$ from 2.

Inclose the last three terms of the following polynomials within parentheses, preceded by a plus sign.

51. $a + b + c + d$.

52. $-a + b - c + d$.

53. $a + b + c - d$.

54. $a + b - c - d$.

55. $m - n + p + 1$.

56. $m - n - p - 1$.

57. $x^2 + y^2 + 2y + 1$.

58. $m^2 - n^2 - 2n - 1$.

59. $r^2 - s^2 + 2s - 1$.

60. $x^2 - y^2 - 2yz - z^2$.

61–70. Inclose the last two terms of the polynomials in examples 51–60 within parentheses, preceded by a plus sign when the sign of the third term is plus, and by a minus sign when it is minus.

71–80. Inclose the last three terms of the polynomials in examples 51–60 within parentheses, preceded by a minus sign.

EXERCISE 22. — GENERAL REVIEW

1. Simplify by collecting like terms:

$$2x - 3y + z - y + 3z + 2x - 3x - y + 4y - 4z.$$

2. Evaluate $x^3 - 2x^2 - x + 1$ when $x = 1$.

3. Evaluate $x^3 - 4x^2 + x + 1$ when $x = 2$.

Given $\pi = 3.1416$, find to three places of decimals:

4. $2\pi r$, when $r = 1$.

5. $2\pi r$, when $r = 1.5$.

6. πr, when $r = 2$.

7. πr, when $r = 2.5$.

8. $4\pi r^2$, when $r = 4$.

9. $\frac{4}{3}\pi r^3$, when $r = 3$.

10. Express the sum of the squares of a and b; the difference of the squares of c and d.

11. If m and n represent two numbers, what does $m+n$ represent? What does $m-n$ represent? m^2+n^2? m^2-n^2? $(m+n)^2$? $(m-n)^2$?

12. Express the sum of the cubes of x and y; the difference.

13. Simplify by collecting like terms:

$$3\,a^2b + 2\,ab - ab^2 - 2\,a^2b - 3\,ab^2 - 2\,a^2b + 2\,ab + 4\,ab^2.$$

14. Add $x+2\,y+3\,z$, $2\,x-y-2\,z$, $y-x-z$, and $z-x-y$.

15. Add $a^4-2\,a^3+3\,a^2$, $a+a^3+a^2$, $2\,a^4+3\,a^3$, $a^2+5\,a-2$, and $-2-3\,a-4\,a^2$.

16. Add $-2(x-y)$, $6(x-y)$, $-4(x-y)$, and $7(x-y)$.

17. Simplify by collecting like terms:

$$3\,x-5(y-z)-2\,x+(y-z)+x-2(y-z)+2\,z.$$

18. From $5\,a$ take $2\,a-3$.

19. From 3 take x^2-x-1.

20. From $a^2+2\,ab+b^2$ take $a^2-2\,ab+b^2$.

21. From 2 take the sum of $2\,a-3\,b-4$ and $2-2\,b+a$.

22. From $3\,abc-2\,a^2bx+7\,ay-2$ take $-3\,abc-3\,a^2bx+7\,ay+2$.

23. What must be added to $m-n+p$ to make $2\,p$?

24. What must be added to $x+y-z$ to make 0?

25. What must be subtracted from $a+b+c$ to make $a-b+c$?

26. If $2\,s=a+b+c$, what expression is equal to $a+b-c$?

27. Given $s=(a+b)\frac{n}{2}$; for what number does s stand when $a=1$, $b=3$, and $n=6$?

28. Simplify by combining the terms having the same powers of m, so as to have the plus sign before each set of parentheses:

$$am^3 + bm^2 + cm - 3m^2 - 4m - 7 - 5m^3.$$

29. Simplify by combining the terms having the same powers of x, so as to have the minus sign before each set of parentheses:

$$-bx^3 - cx - dx^2 + 7 + 4x - 5x^2 + 6x^3.$$

30. If $A = 2x^2 - 3x + 1$, $B = x^2 - 4x - 2$, and $C = x - 4 + x^2$, find the value of $A + B + C$.

31. Evaluate $3x - [4y - (2x - \overline{y - 3y}) - 2x]$ when $x = 1$ and $y = 2$.

32. Subtract $2(x-y)^2 - (x-y) + 1$ from $4(x-y)^2 + (x-y) - 1$.

33. If $A = ax + 2by - cz$ and $B = x - 3by - kz$, find the value of $A - B$.

34. If $A = m(x-y) + r(x+y)$ and $B = n(x-y) - (x+y)$, find the value of $A - B$.

Multiplication

56. Commutative law for multiplication. The product of a and b is expressed by ab. Similarly, the product of a, b, and c is expressed by abc.

In arithmetic it is obvious that

$$3 \times 5 = 5 \times 3.$$

In algebra it is assumed that:

The product does not depend on the order in which the factors are taken.

Thus, it is assumed that

$$a \times b = b \times a.$$

This important principle is referred to as the **commutative law for multiplication.**

57. Associative law for multiplication. In arithmetic it is obvious that

$$2 \times (3 \times 5) = (2 \times 3) \times 5.$$

That is, $\quad 2 \times 15 = 6 \times 5.$

In algebra it is assumed that:

The product does not depend on the way in which the factors are grouped.

Thus, it is assumed that

$$a \times (b \times c) = (a \times b) \times c.$$

This important principle is referred to as the **associative law for multiplication.**

58. Index law. From section 14, we have

$$2^3 = 2 \times 2 \times 2;$$

and $\quad 2^4 = 2 \times 2 \times 2 \times 2.$

Therefore, $2^3 \times 2^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$

$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$= 2^7, \text{ or } 2^{3+4}.$$

Similarly, $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a)$

$$= a \times a \times a \times a \times a \times a \times a$$

$$= a^7, \text{ or } a^{3+4}.$$

In general, when m and n are positive integers,

$$a^m \times a^n = a^{m+n}.$$

The exponent of the product of two powers of the same base is equal to the sum of the exponents of the factors.

Remark. Observe that any power of a positive number is positive and that all even powers of negative numbers are positive, while all odd powers of negative numbers are negative.

For example,

$$(-a)^2 = (-a)(-a) = a^2 \text{ and } (-a)^3 = (-a)(-a)(-a) = -a^3.$$

EXERCISE 23

Name the product in the following examples:

1. x^2x^3. **2.** m^3m. **3.** y^2y^5.

4. r^4r^4. **5.** s^5s. **6.** t^3t^7.

7. $-a^2a^4$. **8.** $a(-a^5)$. **9.** $m^2(-m^5)$.

10. $2^2 \cdot 2^3$. **11.** $3 \cdot 3^2$. **12.** $\pi \times \pi^2$.

13. R^2R. **14.** a^ma. **15.** a^ra^s.

16. x^mx^2. **17.** c^mc^{2m}. **18.** $x^{2a}x^b$.

19. $-x^mx$. **20.** $-x^rx^t$. **21.** $x(-x^m)$.

22. $a^{2m}a^{2n}$. **23.** $-x^mx^{3m}$. **24.** $y^a(-y)$.

25. $(-a)^2(-a^3)$. **26.** $(-a)^2(-a)^3$.

27. $(-a^2)(-a^3)$. **28.** $(x+y)^2(x+y)$.

29. $(x+y)^2(x+y)^3$. **30.** $(x+y-z)(x+y-z)^3$.

31. $a^2 \cdot a^3 \cdot a^4$.

Suggestion. $a^2 \cdot a^3 \cdot a^4 = (a^2a^3)a^4 = a^5a^4 = a^9$.

32. $(ab)^2(ab)^3(ab)^4$. **33.** $(x+y)^2(x+y)^3(x+y)^4$.

34. $(a+b-c)(a+b-c)^3(a+b-c)^5$.

35. $(m-\frac{1}{2})^2(m-\frac{1}{2})^3(m-\frac{1}{2})^7$.

36. $(2x-3y)^3(2x-3y)^3(2x-3y)^2(2x-3y)$.

59. The product of two monomials. When written in full,

$$\begin{aligned} 2ab \times 3ab^2 &= 2 \times a \times b \times 3 \times a \times b \times b \\ &= 2 \times 3 \times a \times a \times b \times b \times b \quad [\S\ 56] \\ &= (2 \times 3) \times (a \times a) \times (b \times b \times b) \quad [\S\ 57] \\ &= 6a^2b^3. \end{aligned}$$

In like manner it may be shown that

$$(-2ab^2c^2)(-4ab^2c) = 8a^2b^4c^3.$$

Also, that

$$(-2ab^2)(-3a^2b^2)(-4ab) = -24a^4b^5.$$

Euclid (330–275 B.C.) was a successful teacher of mathematics in Alexandria. His *Elements* has been the recognized textbook in elementary geometry for 2000 years. In it are to be found geometrical proofs of the *commutative* and *distributive laws*.

From the foregoing illustrations it is evident that:

The product of two or more monomials is equal to the product of their numerical coefficients and all the different literal factors that occur in the monomial factors, each letter having as exponent the sum of the exponents of that letter in the monomial factors.

EXERCISE 24

Multiply:

1. $2a$ by a. **2.** $2R$ by H. **3.** $-3x$ by $4x$.

4. $-6y$ by $-2y$. **5.** $3x^4$ by $2x^3$. **6.** $2R$ by $\frac{1}{2}R$.

7. a^3b^2 by a^2b^3. **8.** $6a$ by $-3b$. **9.** $2R$ by $\frac{1}{2}L$.

10. $-4xyz^2$ by $-2x^2z$. **11.** $4R$ by $\frac{1}{3}R$.

12. $\frac{2}{3}a^2b$ by $-3bc$. **13.** $-2ab^2$ by $-3ax^2$.

14. $-3a^2bc$ by $4bc^2x$. **15.** $(a+b)^2$ by $3(a+b)^3$.

16. $2(x+y)^3$ by $-3(x+y)^2$.

17. $-4x(a+b)^2$ by $3x^2(a+b)^3$.

18. $-5a(x+y)^3$ by $-2a^2(x+y)^4$.

19. $3(x+y)^3$ by $(x+y)$.

20. $-3(x+y)^2$ by $-(x+y)^2$.

Perform the indicated multiplications:

21. $(3a^2b^2)^2$. **22.** $(-2abc^2)^2$. **23.** $(\frac{2}{3}x)^2$.

24. $(-\frac{3}{2}a^3)^2$. **25.** $(-\frac{2}{3}a^2b)^2$. **26.** $(-3a^2b^3c)^2$.

27. $(ab^2)(3a^2b^2)(2a^2b)$.

28. $(3mn^2)(-2m^2n^3)(-4mn)$.

29. $(-xy)(3x^2y^3)(-5x^4y)$.

30. $(\frac{2}{3}a^2b)(-\frac{3}{2}ab)(2a^3b^3)$.

31. $(-y)(-y^3)(-y^4)$. **32.** $(2a^2b)^3$.

33. $(-3x^2)^3$. **34.** $(-\frac{2}{3}ax)^3$.

35. $(-4a^2b^3c^2)^3$. **36.** $(-\frac{1}{2}a^2b^2c^3)^3$.

37. $5(x+y)(x+y)^2(x+y)^3$.

38. $2(a+1)^2(a+1)^3(a+1)^4$.

39. $-3(x+y+z)^2(x+y+z)^2(x+y+z)^5$.

40. $3a(x+y)\cdot 2ab(x+y)\cdot 3bc(x+y)$.

41. $(-3a^2bc)(\frac{2}{3}ab^2)(-4a^3c^2)(-a^2b^3c)$.

42. $(\frac{2}{3}x^3y^2z^4)(-2x^2yz^3)(-\frac{3}{4}xy^3z^2)(-x^4y^2z^5)$.

43. $3x^m\cdot 4x^n$.

44. $5y^r\cdot 4y$.

45. $2a^m\cdot 3a^{2m}$.

46. $5x^{m+1}\cdot 3x^{m-1}$.

47. $3x^{a+b}\cdot 2x^{a-b}$.

48. $-3x^r(-7x^2)$.

49. $-m^{2a+1}(-3m^{2-2a})$.

50. $(-3x^m)^2$.

Find the product of:

51. x, $3x^2$, $2x^5$, and $4x^7$.

52. $2a^2b$, $6b^2c$, and $5c^2a$.

53. $-3h^3k^2$, $5k^3l^2$, and $-7l^3h^2$.

54. 2^3, -3^2m, -4, and 5^2m^3.

55. $3x^2yz$, $-\frac{2}{3}xy^2z$, $\frac{3}{4}xyz^2$, and $-2xyz$.

56. $\frac{5}{2}(x+y)(y+z)^2$, $-\frac{3}{4}(y+z)(z+x)^2$, and $-\frac{2}{3}(z+x)(x+y)^2$.

60. Multiplication of a polynomial by a monomial. It is obvious that

$$3\times(4+5)=3\times 4+3\times 5.$$

Also, that

$$3\times(4+5-2)=3\times 4+3\times 5-3\times 2.$$

In like manner, it is assumed that

$$a(b+c)=ab+ac. \tag{1}$$

Also,

$$a(b+c-d)=ab+ac-ad. \tag{2}$$

Equations (1) and (2) express the fact that *multiplying every term of a polynomial by a monomial multiplies the polynomial by that monomial.* This principle is referred to as the **distributive law for multiplication.**

From equations (1) and (2) we have the following rule for multiplying a polynomial by a monomial :

Rule. *Multiply each term of the polynomial by the monomial and write in succession the resulting products, each with its proper sign.*

ILLUSTRATIVE EXAMPLES

1. Multiply $(b+c-d)$ by $2a$ and verify the result when $a=2$, $b=3$, $c=2$, and $d=1$.

Solution. $2a(b+c-d)=2ab+2ac-2ad.$

Check.
$$2\times 2(3+2-1)=2\times 2\times 3+2\times 2\times 2-2\times 2\times 1$$
$$4\times 4 = 12 + 8 - 4.$$
$$16=16.$$

2. Multiply $(-2x^2+3y-2z^2)$ by $3x^2yz$ and verify when $x=-2$, $y=-1$, and $z=-3$.

Solution	Check
$-2x^2+3y-2z^2$	$-8-3-18 = -29$
$3x^2yz$	$36 = 36$
$-6x^4yz+9x^2y^2z-6x^2yz^3$	$-288-108-648=-1044$

Remark. The form of solution used in example 2 is preferable when the multiplication cannot readily be performed at sight.

3. Simplify $2(a-2b)+3(2a-b)$.

Solution. $2(a-2b)+3(2a-b)=2a-4b+6a-3b.$
$$=8a-7b.$$

EXERCISE 25

(Solve as many as possible at sight.)

1. $a(x+y)$.

2. $x(a-b)$.

3. $2\pi R(H+R)$.

4. $y(a-b+c)$.

5. $x(x-y-z)$.

6. $-a(-a-b-c)$.

7. $ab(a-b-1)$.

8. $x^2(xy-xz+c)$.

9. $\pi(R+r+Rr)$.

10. $m^2(m^2n-m^3p+pn)$.

11. $-2(3a-4b+c)$. **12.** $6(x^2-2y^3-3z^2)$.

13. $2ax(ax-1+3a^2x^3)$. **14.** $-2a^2y(12a^3+3y)$.

15. $3x^2-x+1$ by $3x$. **16.** $2x+y+1$ by $-x$.

17. $3x^2-x+2$ by $3x^2$. **18.** $2ab^2-ab+b^2$ by ab.

19. $a^2-3ab+2b^2$ by $-2ab$.

20. $3x^2y-2xy^2-5y^3$ by xy^2.

21. $ab+bc-ca$ by $-abc$.

22. $v^2-v^3-4v^4$ by vx.

23. $a(x+y)+(m+n)$ by ab.

24. $a(m+n)+b(p+q)+c(x+y)$ by abc.

25. $(x+y)-(x+y)^2+(x+y)^3$ by $(x+y)^2$.

26. $2a(m+n)-3b(m+n)^3+4c(m+n)^4$ by $(m+n)^2$.

27. $4(x+y)^2-3a(x+y)-5c(x+y)^3$ by $2abc(x+y)^4$.

28. $a^2(x+y)-b^3(x+y)^2-abc(x+y)^3$ by $abc(x+y)^2$.

Simplify:

29. $3(a+2b)+2(a-b)$. **30.** $(2-x)y-x(1+y)$.

31. $(2z-3y)xz+(3y-4z)yz-(2z-3x)zy$.

32. $a[1+(b+c)]$.

33. $[a-2(x-a)]x-[x+2(a-x)]a$.

34. $5a^3-2a(2a^2-a+1)-3a(a^2-2a+1)$.

35. $3(1+20+300)$. **36.** $5(2+3x-2y+\frac{1}{5}z)$.

37. $a^n(1+a+a^2)$. **38.** $x^m(x^ny^2+\frac{1}{3}x^2y^m-\frac{1}{2})$.

39. $a^mx^n(-2ax^2+3a^{m+1}x^{n+2}-7x^{n+3})$.

40. $-3m^2n^p(-\frac{2}{3}m^qn^{2p}+\frac{3}{4}m^{q+1}n^{3p+1}-\frac{5}{6})$.

41. $(x-y)^n[(x-y)^2-3(x-y)^n+2(x-y)+\frac{1}{2}]$.

42. $17\left[\frac{a+b}{34}-\frac{2a+b}{51}+\frac{5a-7b}{68}\right]$.

43. $7a^2-2a(a^2+3a-2)-3a(a^2-2a+5)$.

44. To the product of $3a+1$ and $5a$ add $3a-2$, and multiply the sum by a.

45. From the product of $3m^2+3m+2$ and $5m$ subtract the product of $5m^5-2m-1$ and $3m$.

61. Multiplication of a polynomial by a polynomial. The product of two polynomials can be obtained by successive applications of the principle given in section 60; thus:

$$\begin{aligned}(3a-4b)(x-y) &= (3a-4b)x-(3a-4b)y\\ &= (3ax-4bx)-(3ay-4by)\\ &= 3ax-4bx-3ay+4by.\end{aligned}$$

In like manner:

$$\begin{aligned}(3a-4b+2c)(2a-3b) &= (3a-4b+2c)(2a)-(3a-4b+2c)(3b)\\ &= 6a^2-8ab+4ac-9ab+12b^2-6bc\\ &= 6a^2-17ab+4ac+12b^2-6bc.\end{aligned}$$

Remark. From section 60, $z(x-y)=zx-zy$, in which expression the letter z represents any number. Substituting $(3a-4b)$ for z, we have

$$(3a-4b)(x-y)=(3a-4b)x-(3a-4b)y,$$

which is the expression obtained in the first of the preceding examples.

From the foregoing illustrations we have the following rule for multiplying a polynomial by a polynomial:

Rule. *Multiply each term of the multiplicand by each term of the multiplier and add the partial products.*

Note. Before multiplying one polynomial by another, both of them should, if possible, be arranged according to the *ascending* or *descending* powers of a certain letter; that is, in such a manner that the exponents of a certain letter in successive terms *decrease* or *increase* from left to right.

For example, $3x^3-x^2+4x-3$ is arranged according to the descending powers of x; and $a-ay+2by^2-y^4$ is arranged according to the ascending powers of y.

ILLUSTRATIVE EXAMPLES

1. Multiply $3x^2 + 2x - 1$ by $2x - 3$ and verify when $x = 2$.

Solution	Check
$3x^2 + 2x - 1$	$= 15$
$2x - 3$	$= 1$
$6x^3 + 4x^2 - 2x$	
$\qquad - 9x^2 - 6x + 3$	
$6x^3 - 5x^2 - 8x + 3$	$= 15$

2. Multiply $a - b + c$ by $a + b$ and verify when $a = 5$, $b = 3$, and $c = 2$.

Solution	Check
$a - b + c$	$= 4$
$a + b$	$= 8$
$a^2 - ab + ac$	
$\quad + ab \qquad - b^2 + bc$	
$a^2 \qquad + ac - b^2 + bc$	$= 32$

3. Multiply $3x^3 - 2xy^2 + 3x^2y - y^3$ by $-y^2 + xy + x^2$ and verify when $x = 2$ and $y = 1$. (See note, section 61.)

Solution	Check
$3x^3 + 3x^2y - 2xy^2 - y^3$	$= 31$
$x^2 + xy - y^2$	$= 5$
$3x^5 + 3x^4y - 2x^3y^2 - x^2y^3$	
$\quad + 3x^4y + 3x^3y^2 - 2x^2y^3 - xy^4$	
$\qquad - 3x^3y^2 - 3x^2y^3 + 2xy^4 + y^5$	
$3x^5 + 6x^4y - 2x^3y^2 - 6x^2y^3 + xy^4 + y^5$	$= 155$

EXERCISE 26

Multiply, and check results:

1. $x + 3$ by $x + 2$.

2. $2m + 1$ by $m + 3$.

3. $a + 5$ by $2a + 4$.

4. $3x + 5$ by $x - 4$.

5. $5a + 2$ by $2a + 3$.

6. $2m - 4$ by $3m - 3$.

7. $6 - 4x$ by $5 - 2x$.

8. $3a - 5$ by $4 + 6a$.

9. $5 + 2m$ by $3m - 2$.

10. $3a + 2$ by $3a - 2$.

11. $4 - 2y$ by $4 + 2y$.

12. $m + n$ by $m + n$.

13. $a - b$ by $a - b$.

14. $r + s$ by $r - s$.

15. $3m + n$ by $2m + n$.

16. $3r + 2s$ by $r - 3s$.

17. $5x + 4y$ by $3x + 2y$.

18. $8m - 2n$ by $2m + 3n$.

19. $3x^2 + 2y$ by $4x - 3y^2$.

20. $3m^2 + 4n^3$ by $2m^2 - 4n^2$.

21. $2xy + 3z$ by $5x^2y - 4z$.

22. $5x^2y^2 - 2z^2$ by $3x^2y^2 - 3z^2$.

23. $6x^3 - \frac{1}{2}$ by $2x^3 + \frac{1}{3}$.

24. $5x^2y^2 - \frac{1}{2}$ by $6x^2y^2 - \frac{1}{5}$.

25. $m^2 - mn + n^2$ by $m + n$.

26. $2x^2 + 3xy + 4y^2$ by $x + y$.

27. $x^2 + xy + y^2$ by $x + y$.

28. $x^2 + ax + a^2$ by $2x + 3a$.

29. $4r^2 + 6rs + 9s^2$ by $2r + 3s$.

30. $3x^3 + 2x - 1$ by $4x^2 + 2$.

31. $4a^4 - 3a^2 + a$ by $2a^3 - 3a^2$.

32. $m^2n^2 + 2mn + 1$ by $mn + 1$.

33. $x^2y^2 + 4xy + 4z^2$ by $xy + 2z$.

34. $2x^3 + 3x^2 - 4x - 3$ by $3x^2 + 2x$.

35. $2x^3 + 3x^2 - x + 4$ by $2x^2 - 3x + 2$.

36. $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

37. $3m^2 - 2m + 1$ by $m^2 + 3m + 2$.

38. $x^2 + 2x + 2$ by $x^2 - 2x + 2$.

39. $4x^2 + 12xy + 9y^2$ by $2x + 3y$.

40. $x^3 + x^2y + xy^2 + y^3$ by $x - y$.

41. $8x^3 + 12x^2y + 18xy^2 + 27y^3$ by $2x - 3y$.

42. $6m^3n^3 - 2m^2n^2 + 4mn - 1$ by $3m^2n^2 - 2mn - 3$.

Expand:

43. $(4a^2 + 2a + 1)(4a^2 - 2a + 1)$.

44. $(9m^2 - 6mn + 4n^2)(9m^2 + 6mn + 2n^2)$.

45. $(x^3 - x - 1)(x^3 - 2x^2 - x + 1)$.

46. $(x-1)(x-2)(x-3)$.

47. $(x^2 - x + 1)^2$.

48. $(x^3 + 2x^2 + 3x + 1)^2$.

49. $(2m + 4n + \frac{1}{2})^2$.

50. $(2r - s)(r + 2s)(2r + 3s)$.

51. $(m + n + p)^3$.

52. $(a - b - c)^3$.

53. $(x^n + 2)(x^n + 3)$.

54. $(x^n - 4)^2$.

55. $(2a + 3b)^3$.

56. $(3m - 2n)^3$.

57. $(x^{2n} + 4)(x^{2n} - 4)$.

58. $(2a + 3b - c + 4d)^2$.

59. $(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x - y)$.

60. $(3x - 2y + 4z - 1)^2$.

61. What is the area of a rectangle that is $2a + 3$ units long and $3a + 1$ units wide?

62. By how much is the area of a rectangle of base b and altitude a changed by increasing the base by 2 and decreasing the altitude by 1?

Division

62. The quotient of two powers of the same base.

Since $a^2 \times a^3 = a^5$, it follows from the definition of division that

$$a^5 \div a^2 = a^3, \text{ or } a^{5-2}.$$

In general, when m and n are positive integers and n is less than m,

$$a^m \div a^n = a^{m-n}.$$

The exponent of the quotient of two powers of the same base is equal to the exponent of that base in the dividend minus its exponent in the divisor.

EXERCISE 27

Name the quotients in the following examples:

1. $a^2 \div a$.
2. $a^5 \div a^3$.
3. $y^8 \div y^2$.
4. $x^8 \div x^5$.
5. $R^3 \div R$.
6. $m^{12} \div m^6$.
7. $(-x^6) \div (+x^4)$.
8. $(x^8) \div (-x^3)$.
9. $(-x^{12}) \div (-x^2)$.
10. $5^3 \div 5^2$.
11. $4^5 \div 4^3$.
12. $(2^5) \div (-2^3)$.
13. $(a+b)^4 \div (a+b)^3$.
14. $(m+n)^5 \div (m+n)^4$.
15. $(x+y)^7 \div (x+y)^4$.
16. $(r+1)^6 \div (r+1)^3$.
17. $(1+a)^7 \div (1+a)^2$.
18. $(p+q)^{10} \div (p+q)^5$.
19. $(2x+y)^{10} \div (2x+y)^5$.
20. $(a+b-c)^2 \div (a+b-c)$.
21. $(2a+x-y)^7 \div (2a+x-y)^2$.

In examples 22–30, the literal exponents are assumed to be integers and the exponent in the divisor in each case is assumed to be less than the exponent in the dividend.

22. $x^a \div x^b$.
23. $x^m \div x$.
24. $a^{2m} \div a^m$.
25. $y^{4m} \div y^m$.
26. $-a^m \div a^p$.
27. $a^r \div (-a^s)$.
28. $(-m^a) \div (-m^b)$.
29. $r^{3x} \div (-r^x)$.
30. $a^{x+1} \div a^{x-1}$.

63. Meaning of zero exponent. The definition of exponent given in section 15, page 11, does not apply to *zero* used as an exponent. When a is any number other than zero we shall assume that

$$a^0 = 1.$$

Remark. It has been shown in section 62 that $a^m \div a^n = a^{m-n}$, when m and n are positive integers and n is less than m. If n is equal to m, this equation becomes $a^m \div a^m = a^{m-m} = a^0$; but $a^m \div a^m = 1$. Hence, if $a^m \div a^n = a^{m-n}$, in which a is different from zero, is an identity when $n = m$, it is necessary to make the assumption given in section 63; namely, that $a^0 = 1$.

EXERCISE 28

1. Explain why $a^0 = b^0$.
2. Explain why $2^0 = 5^0 = (\frac{1}{2})^0$.

Express in the simplest form:

3. $b \div a^0$; $c \div a^0b^0$; $a^0b^0c^4 \div c^2$.
4. $a^4b^0 \div a^3$; $a^0b^4 \div b^2c^0$; $m^0n^5 \div n^2r^0$.
5. $a^0b^0 \div c^0$; $2^0ab^2 \div a^0b$; $a^5m \div 5^0am^0$.

64. Division of monomials. Since division is the inverse of multiplication, and since

$$2\,ab^2c \times 3\,a^2b^3c^2 = 6\,a^3b^5c^3,$$

it follows that $$6\,a^3b^5c^3 \div 2\,ab^2c = 3\,a^2b^3c^2$$

and $$6\,a^3b^5c^3 \div 3\,a^2b^3c^2 = 2\,ab^2c.$$

Also, since $$-8\,x^2yz \times 2\,xy = -16\,x^3y^2z,$$

it follows that $$-16\,x^3y^2z \div -8\,x^2yz = 2\,xy$$

and $$-16\,x^3y^2z \div 2\,xy = -8\,x^2yz.$$

From these examples we infer the following rule for finding the quotient of two monomials:

Rule. *Divide the numerical coefficient of the dividend by that of the divisor (observing the laws of signs for division) and write after the quotient each letter of the dividend, giving it an exponent equal to its exponent in the dividend minus its exponent in the divisor.*

Note 1. If there occurs in the dividend any letter which is not found in the divisor, it may be understood to occur in the divisor with an exponent 0. If the same power of a letter occurs in both dividend and divisor, the difference of its exponents being 0, the letter is omitted from the quotient; for, $a^0 = 1$.

Note 2. The division of one monomial by another may be expressed in various ways. Thus, $10x^4y^3 \div 5xy^2 = 2x^3y$ may also be expressed:

$$\frac{10x^4y^3}{5xy^2} = 2x^3y \quad \text{or} \quad 5xy^2\overline{)10x^4y^3}\,.\ \ 2x^3y$$

Note 3. When the numerical coefficient in the dividend is not exactly divisible by that in the divisor, the numerical coefficient in the quotient is an arithmetical fraction and should be simplified as in arithmetic.

Thus, $9a^2b^3 \div 6ab^2 = \frac{9}{6}ab$, or $\frac{3}{2}ab$.

EXERCISE 29

(Solve as many as possible at sight.)

Divide:

1. $6x^3$ by $2x$.
2. $-6a^5$ by $3a^2$.
3. $10b^6$ by $-2b^3$.
4. $12a^3b^2$ by $3a$.
5. $-15a^3b^3$ by $-3ab^2$.
6. $16x^2b^5y$ by $4b^3y$.
7. $-21a^5b^2c$ by $7a^3b^2$.
8. $-10xy^2z$ by $-2xyz$.
9. $12x^2y^3z^2$ by $-2xy^2z$.
10. $81a^3bc$ by $9a^3c$.
11. $84x^2y^2z^2$ by $12xyz$.
12. $96x^3y^2z$ by $-8x^2y^2z$.
13. $8(a+b)^3$ by $2(a+b)$.
14. $-12(x+y)^4$ by $-(x+y)$.
15. $-12a^5b^3(m+n)^2$ by $-3a^4b^3(m+n)$.
16. $15x^5y^2z(a+x)^6$ by $-5x^5y^2(a+x)^4$.
17. $\frac{4}{3}\pi R^3$ by πR^2.
18. $\frac{4}{3}\pi R^3$ by $\frac{1}{3}R$.

In examples 19–26 the literal exponents are assumed to be integers, and the exponent in the divisor in each case is assumed to be less than that in the dividend.

19. $6a^x$ by $2a^y$.
20. $9a^x$ by $-3a$.
21. $-12X^m$ by $4X^n$.
22. $-15X^3$ by $15X^m$.
23. $12a^{x+1}$ by $-4a$.
24. $16a^m$ by $4a^{m-2}$.
25. $-18a^{m+1}$ by $9a^{m-1}$.
26. $20a^{x+1}$ by $-5a^x$.

65. Division of a polynomial by a monomial. Since $2a(3b - 4c + 5d) = 6ab - 8ac + 10ad$, and since division is the inverse of multiplication, it follows that

$$(6ab - 8ac + 10ad) \div 2a = 3b - 4c + 5d$$
$$= \frac{6ab}{2a} - \frac{8ac}{2a} + \frac{10ad}{2a}.$$

From the foregoing illustration we infer the following rule:

Rule. *To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and add the quotients.*

Note. $(6ab - 8ac + 10ad) \div 2a$ may be written

$$\frac{6ab - 8ac + 10ad}{2a};$$

or $2a\overline{)6ab - 8ac + 10ad}.$

EXERCISE 30

(Solve as many as possible at sight.)

Divide:

1. $3a^2 - 6a$ by 3.
2. $a^2 + ab$ by a.
3. $6x^3 - 9x$ by $3x$.
4. $2a^2 - 2ab$ by $2a$.
5. $6x^2y - 9xy^2$ by $3xy$.
6. $x^2y^2z^2 - xyz$ by xyz.
7. $4a^2 - 6b^2 - 8c^2$ by 2.
8. $6m^2 - 8mn + 4mn^2$ by $2m$.
9. $3a^2bx - 2ab^2x - 3a^2cx$ by ax.
10. $15x^2y - 20xy^2 - 5xy$ by $5xy$.
11. $a^2 - 2a$ by $-a$.
12. $2a^3b - 3ab^2 + 2ab$ by $-ab$.
13. $3x^2yz - 5xy^2z^3 + 2xyz^2$ by $-xyz$.

Perform the indicated divisions:

14. $\dfrac{\pi RL + \pi R^2}{\pi R}$.

15. $\dfrac{2\,\pi R^2 + \pi R^3}{\pi R}$.

16. $\dfrac{7\,m^2n - 14\,n^2m + 21\,mrn}{7\,mn}$.

17. $\dfrac{ab(m+n)^2 + ab^2(m+n)}{ab}$.

18. $\dfrac{(m+n)^2 + (m+n)}{(m+n)}$.

19. $\dfrac{-18\,m^2nr - 12\,mn^2r + 6\,mnr}{-6\,mnr}$.

20. $\dfrac{r^3 - .5\,r^2 + 1.5\,r}{.5\,r}$.

66. Division of a polynomial by a polynomial. Since division is the inverse operation of multiplication, it follows that the product of two polynomials divided by one of them is equal to the other. Some of the steps involved in dividing one polynomial by another are suggested by a careful inspection of the following multiplication:

$$\begin{array}{rrrrr}
2\,x^2 & -\,3\,x & +\,4 & & \\
x^2 & -\;\;x & -\,1 & & \\
\hline
2\,x^4 & -\,3\,x^3 & +\,4\,x^2 & & \\
 & -\,2\,x^3 & +\,3\,x^2 & -\,4\,x & \\
 & & -\,2\,x^2 & +\,3\,x & -\,4 \\
\hline
2\,x^4 & -\,5\,x^3 & +\,5\,x^2 & -\;\;x & -\,4
\end{array}$$

Let it be required to find $x^2 - x - 1$, given $2\,x^2 - 3\,x + 4$ and $2\,x^4 - 5\,x^3 + 5\,x^2 - x - 4$; that is, to find the quotient of $2\,x^4 - 5\,x^3 + 5\,x^2 - x - 4$ divided by $2\,x^2 - 3\,x + 4$. The quotient, which is the other factor, $x^2 - x - 1$, may be found as follows:

		$x^2 - x - 1$
1. $2x^4 \div 2x^2 = x^2$	$2x^2 - 3x + 4)$	$2x^4 - 5x^3 + 5x^2 - x - 4$
2. Subtract $(2x^2 - 3x + 4) \times x^2 =$		$2x^4 - 3x^3 + 4x^2$
3. $-2x^3 \div 2x^2 = -x$		$-2x^3 + x^2 - x - 4$
4. Subtract $(2x^2 - 3x + 4) \times (-x) =$		$-2x^3 + 3x^2 - 4x$
5. $-2x^2 \div 2x^2 = -1$		$-2x^2 + 3x - 4$
6. Subtract $(2x^2 - 3x + 4)(-1) =$		$-2x^2 + 3x - 4$
		0

Explanation. The explanation of the process may be given thus:

The terms of both dividend and divisor are arranged according to the descending powers of x.

The term of highest power in the dividend being the product of the term of highest power in the quotient and that of the highest power in the divisor, the term of highest power in the quotient is obtained by dividing the term of highest power in the dividend, $2x^4$, by the term of highest power in the divisor, $2x^2$. This gives x^2, the term of highest power in the quotient.

The dividend is formed by multiplying the divisor, $2x^2 - 3x + 4$, by each term of the quotient and adding the resulting products. Hence, reversing the process, $2x^2 - 3x + 4$ is multiplied by x^2, and the result, $2x^4 - 3x^3 + 4x^2$, subtracted from the dividend, $2x^4 - 5x^3 + 5x^2 - x - 4$.

The remainder, $-2x^3 + x^2 - x - 4$, must be the product of the divisor and the part of the quotient to be found. Hence, the term of highest power in the remainder must be the product of the term of highest power in the divisor and the term of the quotient of next higher power to that already found. Therefore, the term of next higher power in the quotient is obtained by dividing the term of highest power in the remainder, $-2x^3$, by $2x^2$. This gives $-x$, the term of next higher power in the quotient.

Multiply the entire divisor by the second term of the quotient, $-x$, and subtract the product, $-2x^3 + 3x^2 - 4x$, from the remainder. This leaves $-2x^2 + 3x - 4$, the second remainder.

The third term of the quotient (-1) is obtained from the second remainder just as the second term is obtained from the first remainder. The entire divisor is then multiplied by (-1) and the product, $-2x^2 + 3x - 4$, subtracted from the second remainder.

The final remainder being zero, the division is said to be *exact*.

The quotient is $x^2 - x - 1$.

From the foregoing explanation we may derive the following rule for dividing one polynomial by another:

Rule. 1. *Arrange the dividend and the divisor according to the descending or ascending powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor; this gives the first term of the quotient.*

3. *Multiply the entire divisor by this first term of the quotient, and subtract the result from the dividend; this gives the first remainder.*

4. *Divide the first term of this remainder by the first term of the divisor; this gives the second term of the quotient.*

5. *Multiply the entire divisor by this second term of the quotient, and subtract the result from this remainder; this gives the second remainder; and so on.*

Remark. Divisor, dividend, and successive remainders should always be arranged in the same order of powers of the same letter.

ILLUSTRATIVE EXAMPLES

1. Divide $2a^3 - 4a - 5a^2 + 3$ by $2a - 1$.

Arrange the terms according to the descending powers of a.

$$\begin{array}{r|l} & a^2 - 2a - 3 \\ 2a - 1 & 2a^3 - 5a^2 - 4a + 3 \\ & \underline{2a^3 - a^2} \\ & - 4a^2 - 4a \\ & \underline{- 4a^2 + 2a} \\ & - 6a + 3 \\ & \underline{- 6a + 3} \\ & 0 \end{array}$$

Check. Let $a = 1$.

Divisor, $2a - 1 = 2 - 1 = 1$.

Dividend, $2a^3 - 5a^2 - 4a + 3 = 2 - 5 - 4 + 3 = -4$.

Dividend ÷ Divisor $= (-4) \div 1 = -4$.

Quotient, $a^2 - 2a - 3 = 1 - 2 - 3 = -4$.

Remark 1. No number should be used in checking which when substituted makes the divisor *zero.*

Remark 2. Observe in the solution of example 1 that only the part of the remainder about to be used is brought down at each stage.

2. Divide $x^4 + y^4$ by $x + y$.

$$
\begin{array}{r|l}
 & x^3 - x^2y + xy^2 - y^3 \\ \hline
x + y & x^4 \qquad\qquad\qquad\quad + y^4 \\
 & \underline{x^4 + x^3y} \\
 & \quad - x^3y \\
 & \quad \underline{- x^3y - x^2y^2} \\
 & \qquad\qquad x^2y^2 \\
 & \qquad\qquad \underline{x^2y^2 + xy^3} \\
 & \qquad\qquad\qquad - xy^3 + y^4 \\
 & \qquad\qquad\qquad \underline{- xy^3 - y^4} \\
 & \qquad\qquad\qquad\qquad\quad 2\,y^4
\end{array}
$$

Remark. The quotient is $x^3 - x^2y + xy^2 - y^3$; the remainder is $2\,y^4$. The complete quotient may be expressed in the same way as in arithmetic. In this instance the complete quotient may be written,

$$x^3 - x^2y + xy^2 - y^3 + \frac{2\,y^4}{x + y}.$$

Check. Let $x = 1$, and $y = 1$.

Dividend, $x^4 + y^4 = 1 + 1 = 2.$

Divisor, $x + y = 1 + 1 = 2.$

Dividend ÷ Divisor $= 2 \div 2 = 1.$

Quotient, $x^3 - x^2y + xy^2 - y^3 + \frac{2\,y^4}{x + y} = 1 - 1 + 1 - 1 + \frac{2}{2}$

$= 1.$

Another Check. As in arithmetic, if the division is correct,

Divisor* × *Quotient* + *Remainder* = *Dividend.

Hence, if we evaluate both members of the equation for the same numerical values of x and y, and the results do not agree, we shall know that there is an error.

3. Divide $abx^2 + (a^2 + b^2)x + ab$ by $ax + b$.

Solution

$$\begin{array}{r|l} & bx + a \\ \hline ax + b & abx^2 + a^2x + b^2x + ab \\ & abx^2 \qquad\quad + b^2x \\ \hline & \qquad\quad a^2x \qquad\quad + ab \\ & \qquad\quad a^2x \qquad\quad + ab \\ \hline & \qquad\qquad\qquad\qquad 0 \end{array}$$

Check

Let $x = 1,\ a = 2,$ and $b = 3.$

Then, $abx^2 + a^2x + b^2x + ab = 6 + 4 + 9 + 6 = 25.$

$ax + b = 5,$ and $bx + a = 5.$

$\frac{25}{5} = 5.$

EXERCISE 31

Divide:

1. $x^2 + 3x + 2$ by $x + 1$.
2. $x^2 + 5x + 6$ by $x + 3$.
3. $x^2 + 7x + 12$ by $x + 4$.
4. $a^2 + 3a + 2$ by $a + 2$.
5. $x^2 - 9x + 20$ by $x - 4$.
6. $y^2 - 7y + 6$ by $y - 1$.
7. $x^2 - x - 20$ by $x - 5$.
8. $m^2 + 2m - 15$ by $m + 5$.
9. $r^2 + 6r - 7$ by $r + 7$.
10. $v^2 - 4v - 21$ by $v - 7$.
11. $x^2 + x - 30$ by $x + 6$.
12. $t^2 - t - 2$ by $t + 1$.
13. $x^4 - 5x^2 + 6$ by $x^2 - 2$.
14. $x^4 + x^2 - 12$ by $x^2 + 4$.
15. $9 + 10m + m^2$ by $1 + m$.
16. $4 - 3k - k^2$ by $4 + k$.
17. $12 - 8r + r^2$ by $2 - r$.
18. $4x^2 - 1$ by $2x + 1$.
19. $9m^2 - 4$ by $3m + 2$.
20. $m^2 - n^2$ by $m + n$.
21. $x^2 - 2xy + y^2$ by $x - y$.
22. $m^2 + 2mn + n^2$ by $m + n$.
23. $y^2 + 2y + 1$ by $y + 1$.
24. $2a^2 + 11ab + 12b^2$ by $2a + 3b$.
25. $6m^2 - mn - 12n^2$ by $3m + 4n$.
26. $10r^2 + 16rs - 8s^2$ by $5r - 2s$.

27. $36y^2 + 24ay - 5a^2$ by $6y + 5a$.

28. $2x^2y^2 + xy - 6$ by $2xy - 3$.

29. $m^2 - 2mn - 15n^2$ by $m - 5n$.

30. $6x^4 - 11x^2 - 35$ by $3x^2 + 5$.

31. $x^3 + 3x^2 + 4x + 4$ by $x^2 + x + 2$.

32. $4x^3 + 10x^2 + 4x - 2$ by $2x^2 + 3x - 1$.

33. $10y^3 - 37y^2 + 13y - 21$ by $5y^2 - y + 3$.

34. $2a^3 + 5a^2 + 4a + 1$ by $2a + 1$.

35. $3m^3 + 7m^2 - 10 + 9m$ by $3m - 2$.

36. $8y^3 - 19y - 10y^2 - 15$ by $2y - 5$.

37. $x^3 - y^3$ by $x - y$.

38. $a^{3m} + b^{3m}$ by $a^m + b^m$.

39. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

40. $a^3 + 1$ by $a + 1$.

41. $8m^3 + 27$ by $2m + 3$.

42. $8y^3 - 64$ by $2y - 4$.

43. $m^3 + 3m^2n + 3mn^2 + n^3$ by $m + n$.

44. $m^4 + m^2 + 1$ by $m^2 - m + 1$.

45. $20r^5 - 36r^4s + 59r^3s^2 - 57r^2s^3 + 29rs^4 - 15s^5$ by $10r^2 - 3rs + 5s^2$.

46. $x^4 - y^4$ by $x - y$.

47. $m^5 + n^5$ by $m + n$.

48. $m^3 + n^3$ by $m - n$.

49. $m^4 + 1$ by $m + 1$.

50. $x^3 + 5x^2 + 10x + 10$ by $x + 2$.

51. $6m^3 - m^2n - 14mn^2 + 3n^3$ by $3m^2 + 4mn - n^2$.

52. $2a^5 + 3a^4 - 12a^3 + 15a^2 - 11a + 3$ by $-3a + 1 + 2a^2$.

53. $3p^6 - 16p^5 + 1$ by $3p^2 - p + 1$.

54. Find the quotient of $\frac{a^6 - 1}{a + 1}$; check the result by letting $a = 2$.

55. Find the quotient of $\frac{a^9 - 1}{a - 1}$; check the result by letting $a = -2$.

56. Find the quotient of $\frac{64a^6 - 1}{2a + 1}$; check the result by letting $a = 1$.

57. Find the quotient of $\frac{a^{13} + 1}{a + 1}$; check the result by letting $a = 1$.

58. What is the remainder when $a^4 - a^3 + 2a + 1$ is divided by $a^2 - 2a + 3$?

59. For what value of m is $a^4 - a^3 - 8a^2 + ma - 3$ exactly divisible by $a^2 + 2a - 3$?

60. For what value of m is $a^5 + a^4 + 2a^3 + a^2 + ma - 2$ exactly divisible by $a^3 - a^2 + 2a - 1$?

61. What are the values of Q (quotient) and R (remainder) in the following expression:

$$\frac{a^2 + 2a - 1}{a + 1} = Q + \frac{R}{a + 1}?$$

62. Show that $\frac{1}{1 - a} = 1 + a + a^2 + a^3 + a^4 + \frac{a^5}{1 - a}$.

63. If $a = \frac{1}{10}$, what error is made in assuming that $\frac{1}{1 - a} = 1 + a + a^2 + a^3 + a^4$?

EXERCISE 32. — GENERAL REVIEW

1. Find the algebraic sum of x, $4y$, and -32.

2. Find the algebraic sum of $3(x - y)$, $-4(x - y)$, $2(x - y)$, and $-5(x - y)$.

3. Simplify $3x - 2a + 4b - 2c - 3a + 2x - b + c - x - 2b + 5a + 3c$.

4. Simplify $ab + 2a^2b - ab^2 + 2ab^2 + 3a^2b - 4a^2b^2 + 5ab - 3ab^2 + a^2b - 6ab + a^2b^2 + 5ab^2 + 3a^2b^2$.

5. Add $-3(a-b)$, $4(a+b)$, $-5(a+b)$, and $6(a-b)$.

6. Add $\frac{1}{4}xy+\frac{1}{3}x^2y+\frac{3}{4}xy^2-.1\,x^2y^2$, $\frac{1}{2}xy^2-\frac{3}{4}xy-\frac{1}{6}x^2y+.7\,x^2y^2$, $\frac{3}{4}xy-.3\,x^2y^2$, and $x^2y-\frac{1}{4}xy^2$.

7. What must be added to $a^2-2ab+b^2$ to give $a^2+2ab+b^2$?

8. What must be added to $x^2+3xy-2y^2+4$ to give $-3x^2+xy-y^2-2$?

9. What must be added to $x^2+2xy+y^2$ to give $x^2-2xy+y^2$?

10. $r^2-2r-7-(-2r^2+r-3)=?$

11. Simplify $1-\{-1-[-1-(-1)-1]-1\}+1$.

12. Inclose the last three terms of $x-y-z+1$ within parentheses preceded by the sign $-$.

13. Name at sight the products:
$ab\cdot ab$; $a^2b\cdot ab^2$; $a^nb^n\cdot ab$; $a^nb^n\cdot a^mb^m$; $2ax^n\cdot 3a^3x$.

14. Expand $(a+1)(a+1)(a-1)(a-1)$.

15. Multiply $2a^2+6ab+3b^2$ by $2a^2-6ab+3b^2$.

16. Simplify $(2x+3y)(2x+3y)-(2x-3y)(2x-3y)$.

17. Multiply $(2m+1)^2$ by $(2m-1)^2$.

18. Evaluate $2\pi R(H+R)$ in terms of π when $R=2$ and $H=3$.

19. Evaluate $\pi R(L+R)$ in terms of π when $R=3$ and $L=5$.

20. Evaluate $\frac{1}{3}\pi H(R^2+Rr+r^2)$ in terms of π when $H=6$, $R=4$, $r=1$.

21. Evaluate $\frac{1}{3}\pi H(R^2+Rr+r^2)$ in terms of π when $H=9$, $R=3$, $r=0$.

22. Evaluate $(mn-rs)(ns-mr)(nr-ms)$ when $m=2$, $n=-1$, $r=0$, and $s=1$.

23. Name the quotients in the following:

$$a^6 \div a^2;\ a^5 \div a^5;\ a^n \div a^n;\ 2^6 \div 2^3.$$

24. Name the quotients in the following:
$a^6b^2 \div a^2b$; $a^3b \div ab$; $-6\,a^6b^4 \div 2\,a^3b^2$; $-(3\,ab)^3 \div -(3\,ab)^2$.

25. Name the quotients in the following:

$$a^nb^n \div ab;\ x^{n+1}y^{n+2} \div x^ny^n;\ x^{2n}y \div x^ny.$$

26. Divide πR^2 by $\frac{1}{2}R$.

27. Divide $2\,\pi RH + 2\,\pi R^2$ by $2\pi R$.

28. Divide $\pi RL + \pi R^2$ by πR.

29. Divide $a^2b - ab^2 + 2\,a^2b^2$ by $-ab$.

30. Divide $(a-b)x + (a-b)^2x^2 - (a-b)(a+b)x^3$ by $(a-b)x$.

31. Divide $m^5 - n^5$ by $m - n$.

32. Divide $1 + 2\,x$ by $1 + x$, carrying the quotient to four terms.

33. Divide $a^3 + b^3 + 3\,a^2b + 3\,ab^2$ by $a + b$.

34. Divide $a^4 + b^4 - 4\,ab^3 - 4\,a^3b + 6\,a^2b^2$ by $a - b$.

35. Express $m - n - p + q - r - a + b - x - y$ in trinomial terms having the last two terms of each trinomial inclosed within parentheses preceded by the sign $-$.

36. Show that $(m-n)(p-q) = (n-m)(q-p)$.

37. Show that $(a-b)^2 = (b-a)^2$.

38. Show that $\dfrac{a^3 - b^3}{a - b} = \dfrac{b^3 - a^3}{b - a}$.

39. Multiply $mx^3 - x^2 + 4\,x - c$ by $mx^3 + x^2 + 4\,x + d$ and arrange the result according to powers of x.

40. In the formula $V = \frac{4}{3}\pi R^3$, find the value of V when $\pi = 3.1416$ and $R = 3$.

41. Divide $3\,a^5 + a^4 - 12\,a^3 + 5\,a^2 - 15\,a - 6$ by $a^3 + 2\,a^2 + 3$.

42. Divide $5\,ax^6 - 13\,a^2x^5 - 7\,a^3x^4 + 5\,a^4x^3 - 6\,a^5x^2$ by $5\,x^3 - 3\,ax^2 + 2\,a^2x$.

43. Multiply $x^2 + (a + b)x + a^2$ by $x - a$, arranging the result according to powers of x.

44. Show that $(ax + by)^2 + (bx - ay)^2 = (a^2 + b^2)(x^2 + y^2)$.

45. If $(x + a)(3\,x^2 - 2\,bx + 3)$ is the same as $3\,x^3 - x^2 - x + 3$, find a and b.

46. The expression ax^3y^4 has the value 32 when $x = 2$ and $y = 1$; find its value when $x = -3$ and $y = -2$.

47. Find the value of $a^{2a} - 3\,a^{a+1} + 2(a + 1)^a$ when $a = 2$.

48. Find the value of $2\,x - 3\,y + z$ when $x = a + b - c$, $y = a - b + 2\,c$, and $z = 3\,a + 2\,b + c$.

49. From the formula $s = at + \frac{1}{2}ft^2$, find:

(1) the value of s when $a = 12$, $t = 2$, $f = 32$.

(2) the value of f when $s = 64$, $a = 64$, $t = 2$.

50. To $\frac{a}{3} + \frac{b}{2}$ add $\frac{a}{2} - \frac{b}{3}$ and take $\frac{3}{2}a - \frac{2}{3}b$ from the sum.

51. Simplify $3(x - y) - 2(y - 2) - 4(t - x) - 7(x + y + t)$.

52. Find the value of n which satisfies the equation, $7(58 - n) = 5(n - 14) - 14(n - 25)$.

53. Subtract $2\,a + 3\,b - 4\,c$ from $5\,a - 3\,b + 2\,c$ and add the remainder to $-3\,a$.

54. Simplify:

$$4\,\{3(\tfrac{3}{2}\,a - \tfrac{2}{3}\,b) + 2(\tfrac{2}{3}\,b - \tfrac{1}{2}\,c) + 6(\tfrac{3}{5}\,c - \tfrac{2}{3}\,a)\}.$$

55. Find the value of $[\,p^2 + (a + 1)p + a] \div (p + a)$.

56. A man has to walk to a place m miles away. How far will he be from his destination in a given number of hours (t) if his rate of walking is r miles an hour?

57. Evaluate a^{n-1}, also $a^n - 1$, when $a = 3$ and $n = 5$.

58. Find the product of $x - 2y$, $x + 2y$, and $x - 6y$.

59. In the year 1900 a man on his birthday found that the number of months he had lived was half of the date of the year of his birth; how old was he?

60. If $(a+1)x + 1 = x + 2a$, express x in terms of a.

61. A man has four sons whose combined ages are equal to his own. In 20 years their combined ages will be double the age of their father; what is his present age?

62. Show that $8a^3 + b^3 + c^3 - 6abc$ is exactly divisible by $2a + b + c$.

63. Show that

$$p^2 = p(p+1) + p(p-1) + (1-p)(1+p) - 1.$$

64. The volume of a circular cone is given by the expression $\frac{1}{3}\pi r^2 h$ where h and r represent, respectively, the measures of the height and the radius of the base in terms of the same unit. Find to two decimal places the volume of a cone 5 ft. high and 12 in. in diameter. (Take $\pi = 3.14$.)

65. Find the value of l, when

$$a^2 - a - l = (a+2)(a-3).$$

66. Find the value of m, when

$$a^2 - ma - 35 = (a+5)(a-7).$$

67. Find the value of n, when

$$na^2 + 3a - 6 = 3(3a+2)(2a-1).$$

68. First indicate and then perform the following series of operations: To the product of $(a+2b)$ and $(2a-b)$ add the product of $(a-2b)$ and $(2a+b)$, and divide the sum by 4.

69. If a gallon of diluted milk contains p pints of water, how much pure milk is there in n gallons of the mixture?

70. Representing the sum of two numbers by s, their difference by d, the larger number by N, and the smaller by n, express by formulæ the following statements: "Half the sum of any two numbers plus half their difference is equal to the greater number, and half their sum less half their difference is equal to the smaller number."

71. If n denotes any positive integer, what kind of integers are completely represented by $2n$? By $2n-1$?

72. If eggs are sold according to quality for 50 cents, 42 cents, and 38 cents per dozen, write a formula for the total cost in dollars of m dozen of the highest grade eggs, n dozen of the medium grade, and p dozen of the lowest grade.

73. What number is represented by $3t^2+2t+1$ when t denotes the number 10? Can any integer of three digits be expressed in the form at^2+bt+c (read from left to right) where a, b, and c denote the digits used to express the number and t denotes the number 10?

74. Using the notation of problem 73, write the six numbers which can be represented by using the digits a, b, and c.

75. a and b are two digits of which a is the larger. Find a formula for the difference between the two numbers which can be represented by them.

76. If a represents the sum of the ages of n persons b years ago, what expression represents the sum of their present ages?

77. Divide $2x^m+3x^{m+1}+14x^{m+2}+32x^{m+3}-98x^{m+4}+39x^{m+5}$ by $2+5x-3x^2$.

CHAPTER III

SIMPLE EQUATIONS

67. Identity. If the two members of an equation are such that the one can be transformed into the other, the equation is an identical equation and is called an **identity**.

Thus, $3a + 4a = 7a$ is an *identity*.

Note. Since the two sides of an identity represent ways of expressing the same number, the statement that they are equal is always true, whatever values may be assigned to the letters involved.

Thus, $3x - 2x = x$ is true whatever value be given to x.

68. Conditional equation. An equation which is true only when the letter or letters involved are restricted to certain values or sets of values is called a **conditional equation.**

Thus, the equation $x - 1 = 0$ is true only when x has the value 1. The equation $x + y = 2$ is true for many sets of values of x and y, but not for all values of x and y.

Remark. When in algebra the word *equation* is used, a conditional equation is usually meant.

69. Notation. In algebra unknown numbers are usually represented by the last letters of the alphabet, as, u, v, x, y, z; and known numbers by the first letters, as, a, b, c. By this convention it is easy to distinguish at a glance between the known and the unknown numbers which occur in the equation.

70. Satisfying an equation. Any set of values of the letters of an equation which reduces the equation to an identity is said to **satisfy the equation.**

Thus, $x = 2$ satisfies the equation $3x = 6$; $x = 2$ and $y = 1$ is a set of values of x and y which satisfies the equation $2x + 3y = 7$.

71. Simple equation in one unknown number. Any equation which can be put in the form $ax + b = 0$ is called a **simple equation in one unknown number.**

Thus, $5x + 2 = 3x + 4$ is a simple equation; it may be written $2x - 2 = 0$ [§ 25].

Remark. The simple equations considered in this chapter contain only one unknown number. Observe that no higher power of the unknown than the first occurs; and that the unknown number does not occur in the denominator of any fraction.

Note. A simple equation is called **an equation of the first degree,** or a **linear equation.**

72. Solution of an equation. **To solve an equation** which contains only one unknown number is to find all values of the unknown which satisfy the equation.

73. Root of an equation. A **root** of an equation which contains only one unknown is any value of the unknown which satisfies the equation.

Thus, 2 is a root, and the only root, of the simple equation $3x - 6 = 0$.

74. Equivalent equations. Two equations in one unknown are said to be **equivalent** when they are satisfied by the same value or values of the unknown; that is, when they have the same roots.

Thus, $3x + 3 = 2x + 5$, and $x + 3 = 5$ are two equivalent equations; each one has the single root 2.

Two equations are therefore equivalent when:

1. *Every solution of the first equation is a solution of the second.*

2. *Every solution of the second equation is a solution of the first.*

Thus, the two equations $x - 1 = 0$ and $x^2 - x = 0$ are not equivalent; for although the one solution of the first equation, namely, $x = 1$, is a solution of the second, yet the second equation has a solution, namely, $x = 0$, which is not a solution of the first.

Note. A simple equation is solved by transforming it into an equivalent equation which shall contain the unknown number alone in one member and its value in the other.

Thus, the equation $3x + 3 = 2x + 5$
is equivalent to $x + 3 = 5,$
which is equivalent to $x = 2.$

75. Transposition. An important principle follows from assumptions 1 and 2, section 25.

Thus:

Let $$ax - b = c. \quad (1)$$

Adding b to each member of equation (1),

$$ax - b + b = c + b. \quad (2)$$

Combining, $$ax = c + b. \quad (3)$$

Observe that equation (3) differs from equation (1) in that the term containing b is in the second member of (3) but in the first member of (1), and that the signs of the terms containing b are different in the two equations.

Again, let $$ax + b = c. \quad (1)$$

Subtracting b from each member of equation (1),

$$ax + b - b = c - b. \quad (2)$$

Combining, $$ax = c - b. \quad (3)$$

Compare equations (1) and (3) and observe as before that the term containing b is in the second member of (3) but in the first member of (1), and that the signs of the terms containing b are different in the two equations.

It follows from the foregoing that:

Any term may be transposed from one member of an equation to the other, provided that its sign is changed.

76. Cancellation of terms in an equation. Cancellation of terms in an equation may be illustrated by solving the equation,

$$x + b = c + b. \quad (1)$$

Transposing, $x = c + b - b.$ (2)

Combining, $x = c.$ (3)

Comparing (1) with (3) we may infer that:

When the same terms preceded by like signs occur in both members of an equation, these terms may be omitted.

77. Change of signs in an equation. It is sometimes convenient to change the signs of all the terms of an equation. This may be done by multiplying both members by -1 [§ 25].

For example, let $2 - x = -5.$ (1)

Multiplying both members of equation (1) by -1,

$$(-1)(2 - x) = (-1)(-5), \quad (2)$$

or, $-2 + x = 5.$ (3)

Observe that equation (3) is equation (1) with the sign of each term of equation (1) changed.

Remark. It is evident that the members of an equation may be interchanged.

ILLUSTRATIVE EXAMPLES

1. Solve the equation $3x - 2 = 2x + 5$.

Solution. $3x - 2 = 2x + 5.$ (1)

Transposing, $3x - 2x = 5 + 2.$ (2)

Combining, $x = 7.$ (3)

Check. $3x - 2 = 2x + 5.$ (4)

Substituting 7 for x, $3 \times 7 - 2 = 2 \times 7 + 5.$ (5)

Simplifying, $19 = 19.$ (6)

2. Solve the equation $x - \frac{3x+2}{4} = \frac{1}{3}(x-2)$.

Solution. $$x - \frac{3x+2}{4} = \frac{1}{3}(x-2). \quad (1)$$

Multiplying each member of the equation by the least common multiple of the denominators,

$$12x - \frac{12(3x+2)}{4} = \frac{12}{3}(x-2). \quad (2)$$

Performing the indicated divisions,

$$12x - 3(3x+2) = 4(x-2). \quad (3)$$

Performing the indicated multiplications,

$$12x - 9x - 6 = 4x - 8. \quad (4)$$

Combining, $$3x - 6 = 4x - 8. \quad (5)$$

Transposing, $$8 - 6 = 4x - 3x. \quad (6)$$

Combining, $$2 = x. \quad (7)$$

Interchanging members, $$x = 2. \quad (8)$$

Check. $$x - \frac{3x+2}{4} = \frac{1}{3}(x-2). \quad (9)$$

Substituting 2 for x,

$$2 - \frac{3\times 2+2}{4} = \frac{1}{3}(2-2). \quad (10)$$

Simplifying, $$2 - 2 = \tfrac{1}{3} \times 0, \quad (11)$$

or, $$0 = 0.$$

3. Solve the equation $2.5x - 3 = .8x + 2.1$.

Solution. $$2.5x - 3 = .8x + 2.1. \quad (1)$$

Expressing the decimals in (1) as common fractions,

$$\tfrac{5}{2}x - 3 = \tfrac{4}{5}x + \tfrac{21}{10}. \quad (2)$$

Transposing, $$\tfrac{5}{2}x - \tfrac{4}{5}x = \tfrac{21}{10} + 3. \quad (3)$$

Combining, $$\tfrac{17}{10}x = \tfrac{51}{10}. \quad (4)$$

Multiplying both members of (4) by $\frac{10}{17}$,

$$x = 3. \quad (5)$$

Check. $$2.5x - 3 = .8x + 2.1. \quad (6)$$

Substituting 3 for x, $$2.5 \times 3 - 3 = .8 \times 3 + 2.1. \quad (7)$$

Simplifying, $$7.5 - 3 = 2.4 + 2.1. \quad (8)$$

Combining, $$4.5 = 4.5. \quad (9)$$

Remark. Various methods of procedure may be resorted to in solving example 3. Thus, each term of equation (1) may be multiplied by 10; also, $.8x$ may be transposed and combined with $2.5x$ and -3 transposed and combined with 2.1. Again, both members of equation (2) may be multiplied by 10, the least common multiple of the denominators; then equation (3) would be replaced by $25x - 30 = 8x + 21$.

EXERCISE 33

Solve the following simple equations, and check each solution :

1. $13x + 7 = 5x - 4$.
2. $5u + 2 = 2u - 4$.
3. $13 - 6a = 13a - 6$.
4. $25c - 13 = -6c + 111$.
5. $3m + 2 = 11m - \frac{10}{3}$.
6. $5p + 12 = 17 - 5p$.
7. $13r - 11 = 2r - 11$.
8. $15 - 6t = 3t - 12$.

Suggestion. Divide each term by 3.

9. $-3y + 17 = 12y - 58$.
10. $13 - 11y = 13y + 253$.
11. $2 + 3(x - 5) = 5 + 4(x - 6)$.
12. $3 - 2(3y - 4) = 5(2y + 3) - 84$.
13. $3(p + 2) - 2(2p - 3) = 11(3 - 7p) + 72p - 1$.
14. $11(1 - x) + 3(2 - x) - 5(3 - x) = 11$.
15. $12t - 5(3t - 2) = 3 - 2t$.
16. $3(2x - 3) = 8 - 5(2x - 3)$.
17. $5(z - 3) + 2(z - 3) - 4(z - 3) = 0$.
18. $2(3y - 5) - 7(2y + 3) = 5(3y - 5) - 8(2y + 3)$.
19. $x(x + 3) = x^2 + 6$.

20. $(x+1)(x+3)=x^2-8x+27.$

21. $(x+3)(2x-5)=2x(x-2).$

22. $(y+1)(y+2)-(y+3)(y+4)+30=0.$

23. $\frac{3}{2}x+\frac{x}{4}=\frac{5}{6}x+11.$

24. $3x-\frac{1}{5}(x+2)=8.$

25. $\frac{x}{2}-\frac{3(x-2)}{5}+1=0.$

26. $\frac{5}{6}(3x-1)=\frac{14}{15}(2x-7).$

27. $\frac{x-3}{10}-\frac{x-2}{15}+\frac{x-5}{20}=0.$

28. $7-\frac{x-7}{7}-(7x+1)=0.$

29. $19+\frac{17}{35}x=3x-\frac{x+162}{28}.$

30. $17-\frac{23x}{55}=2x+\frac{5x+353}{77}.$

31. $.3x+4=.9x-2.$

32. $.8x-1=.1x+2.5.$

33. $1.5x-.5=.7x+.6.$

34. $.9x-2.1=3.9-.1x.$

78. Solution of problems. In solving a particular problem which leads to a simple equation in one unknown number it is necessary to:

1. *Restate the problem in algebraic language in the form of an equation.*

2. *Solve the resulting equation for the unknown number.*

3. *Verify the solution.*

Remark. Many different kinds of problems occur which lead to simple equations each in one unknown number, and only the foregoing very general directions can be given for their solution. However, when any such problem admits of a definite solution, it will be found that there are in it as many distinct statements as unknown numbers. These distinct statements enable us to express all the unknown numbers in terms of one of them. The algebraic form of the final statement is an equation in this one unknown.

ILLUSTRATIVE EXAMPLES

1. What number is as much greater than 10 as it is less than 54?

Solution.

Let $x =$ the required number.

Then, $x - 10 =$ the difference between the required number and 10,

and $54 - x =$ the difference between 54 and the required number.

Since the two differences are equal,

$$x - 10 = 54 - x. \qquad (1)$$

Transposing, $x + x = 54 + 10.$ (2)

Combining, $2x = 64.$ (3)

Therefore, $x = 32.$ (4)

Hence, the required number is 32.

Check. $32 - 10 = 54 - 32.$

2. A dealer bought 500 oranges in two lots; the first lot at the rate of $2\frac{1}{2}$ cents apiece, and the second at the rate of 2 cents apiece. He sold them all at the rate of 30 cents a dozen and gained \$2.25. How many did he buy at each price?

Solution.

Let $x =$ the number of oranges in the first lot.

Then, $500 - x =$ the number of oranges in the second lot.

Then, $\frac{5}{2}x =$ the number of cents in the cost of the first lot.

and $2(500 - x) =$ the number of cents in the cost of the second lot.

$\therefore \frac{5}{2}x + 2(500 - x) =$ the number of cents in the cost of both lots.

$\frac{500}{12} \times 30 =$ the number of cents in the selling price of both lots.

Then, $\frac{5}{2}x + 2(500 - x) + 225 = \frac{500}{12} \times 30.$ (1)

Simplifying, $\frac{5}{2}x + 1000 - 2x + 225 = 1250.$ (2)

Transposing and combining, $\frac{1}{2}x = 25.$ (3)

Whence, $x = 50,$

and $500 - x = 450.$ (4)

Therefore, the dealer bought 50 oranges at $2\frac{1}{2}$ cents apiece and 450 at 2 cents apiece.

Check. $\frac{5}{2} \times 50 + 2(500 - 50) + 225 = \frac{500}{12} \times 30.$

That is, $1250 = 1250.$

3. A man traveled 30 mi. in 6 hr. 40 min., walking part of the distance at the rate of 3 mi. an hour and riding the remaining distance at the rate of 6 mi. an hour. How far did he walk?

Solution. Let $x =$ the number of miles he walked.

Then, $30 - x =$ the number of miles he rode.

Also, $\frac{x}{3} =$ the number of hours he walked,

and $\frac{30 - x}{6} =$ the number of hours he rode.

Therefore, $\frac{x}{3} + \frac{30 - x}{6} = \frac{20}{3}.$ (1)

Multiplying both members of (1) by 6,

$2x + 30 - x = 40.$ (2)

Combining, $x + 30 = 40.$ (3)

Whence, $x = 10.$ (4)

Check. $\frac{10}{3} + \frac{30 - 10}{6} = \frac{20}{3}.$ (5)

That is, $\frac{20}{3} = \frac{20}{3}.$ (6)

4. A number is composed of two digits; the digit in the tens' place is one more than twice that in the units' place, and if 36 is subtracted from the number the resulting number is expressed by the same two digits taken in the reverse order. Find the number.

Solution. Let $x =$ the units' digit.

Then, $(2x + 1) =$ the tens' digit.

Also, $10(2x + 1) + x =$ the number,

and $10x + (2x + 1) =$ the number obtained by writing the digits in the reverse order.

By the conditions of the problem,

$10(2x + 1) + x - 36 = 10x + (2x + 1).$ (1)

Simplifying (1), $9x = 27.$ (2)

Dividing by 9, $x = 3,$ (3)

and $2x + 1 = 7.$ (4)

Therefore, the required number is 73.

EXERCISE 34

1. If x represents a certain number, what represents the number increased by 3?

2. What number increased by 3 is equal to 15?

3. If x represents a certain number, what represents the number diminished by 2?

4. What number diminished by 2 is equal to 10?

5. If x represents a certain number, what represents four times the number?

6. If four times a certain number is 30, what is the number?

7. If x represents a certain number, what represents $\frac{1}{4}$ of the number? What represents $\frac{3}{4}$ of the number?

8. If $\frac{1}{4}$ of a certain number is $2\frac{1}{2}$, what is the number?

9. A certain can filled with lard weighs 42 lb.; if the can weighs 4 lb., what is the weight of the lard?

10. Two boys have 36 cents; if one of them has three times as much as the other, how much has each?

11. Two men bought 100 fruit trees; if one of them bought 10 more than the other, how many did each buy?

12. If the sum of two angles is 90° and one of them is 20°, what is the other?

13. If the sum of two consecutive integers is 21, what are the numbers?

14. The sum of two numbers is 276. One of them is five times the other; what are the numbers?

15. The number of pupils in a certain school is 227 and the number of girls exceeds the number of boys by 21. How many boys are there?

16. I paid $150 for two cows, one costing $30 more than the other. What was the price of each?

17. The difference between two numbers is 7 and their sum is 31. What are the numbers?

18. If from five times a number 21 is subtracted, the remainder is 9. What is the number?

19. Separate 24 into two parts so that one part may be equal to three fifths of the other.

20. A woman bought a certain number of yards of dress goods and one half as many yards of lining. If she bought 24 yards of cloth, how many yards of each did she buy?

21. If one half of a number added to one fourth of the number is $7\frac{1}{2}$, what is the number?

22. Find a number which when 100 is added to it will give a result equal to five times the number.

23. Two dealers together bought 15,000 bushels of wheat, one of them buying three times as many bushels as the other. How many bushels did each buy?

24. A wagon loaded with wheat weighed 6390 lb. If the wagon weighed one half as much as the wheat, what was the weight of the wheat?

25. Three times a certain number is 24 more than $\frac{1}{3}$ of the number. What is the number?

26. The result of subtracting 96 from a certain number is the same as the result of dividing the same number by 13. What is the number?

27. The difference of two numbers is 24 and the smaller is $\frac{5}{8}$ of the larger. What are the numbers?

28. A and B together own 466 acres of woodland. If 22 times A's share is 6 acres less than B's share, how much does each own?

29. A dealer sold an article for \$12, which was at a gain of $\frac{1}{3}$ of the cost. What was the cost?

30. A dealer sold an article for \$12, which was at a loss of $\frac{1}{9}$ of the cost. What was the cost?

31. The wages of a man and his son for one month were \$120. If the son's wages were $\frac{3}{5}$ of the father's, what were the wages of each?

32. Find three consecutive numbers whose sum is 42.

33. The sum of three angles, A, B, C, is 180°. If B is two times C and A three times C, how many degrees are there in each?

34. The sum of the angles of any plane triangle is 180°. If in a triangle ABC, angle A is twice angle B and angle C is $\frac{1}{3}$ of angle B, how many degrees are there in each angle?

35. A storekeeper found that he had \$6.50 in dimes and quarters. How many had he of each if the number of coins of both kinds that he had was 35?

36. A man wishes to divide a straight line 40 ft. long into three parts so that the first part may be 4 ft. less than the second and the second 7 ft. more than the third. Required the length of each part.

37. If $\frac{1}{3}$ of a pole is in mud, $\frac{1}{4}$ of it in water, and the remainder, 15 ft. of it, above water, what is the length of the pole?

38. A baseball team won 63 games, which were $\frac{3}{5}$ of the games that it played. How many games did it play?

39. In sorting melons 27 less than $\frac{2}{3}$ of them were found to be defective. If 45 of the melons were found to be in good condition, how many of them were defective?

40. A man sold 3 acres more than $\frac{2}{5}$ of his lot and had 2 acres less than half of it left. Find the number of acres in the lot.

41. If x represents the number of dollars in the cost of an article, what represents the number of dollars in the gain if the rate of gain is

50%? 25%? 20%? 100%? $12\frac{1}{2}$%? $62\frac{1}{2}$%?

Suggestion. 50% of $x = \frac{50}{100}x = \frac{1}{2}x$, or $\frac{x}{2}$.

42. If x represents the number of dollars in the cost of an article, what represents the number of dollars in the loss if the rate of loss is

5%? 4%? 75%? $37\frac{1}{2}$%? $33\frac{1}{3}$%? $6\frac{1}{4}$%?

43. If x represents the number of dollars in the cost of an article, what represents the number of dollars in the selling price if the rate of gain is

25%? 30%? 80%? $62\frac{1}{2}$%? 200%?

44. If x represents the number of dollars in the cost of an article, what represents the number of dollars in the selling price if the rate of loss is

20%? 10%? $12\frac{1}{2}$%? $16\frac{2}{3}$%? $8\frac{1}{3}$%?

45. A dealer gained 25% by selling a coat at a profit of \$51. Find the cost of the coat.

46. Some lemons were sold at a loss of 6 cents a dozen. If the rate of loss was $16\frac{2}{3}\%$, what was the cost?

47. A farmer sold a horse for $220, which was at a gain of 10%. What was the cost?

48. A used automobile was sold for $600, which was 20% less than cost. What was the cost?

49. The difference between two numbers is 328, and the larger is 42 times the smaller. Find one of the numbers.

50. A tennis court for doubles is 42 ft. longer than its breadth. The distance around the court is 228 ft. Find the length and breadth of the court.

51. An acre of wheat yielded 25,000 lb. more straw than grain. The weight of the grain was $\frac{3}{8}$ of the weight of the straw. What was the weight of the grain?

52. A man bequeathed $45,000 to his wife, daughter, and son. The daughter received $5000 more than the son, and the wife received three times as much as the son. How much did each receive?

53. A man is 27 years older than his son; 12 years hence he will be twice as old as his son will be then. How old is his son?

54. Divide $30 among three persons so that the first person shall receive three times as much as the second, and the third person $5 more than the second.

55. Part of a sum of $3000 was invested at 4% and the remainder at $4\frac{1}{2}\%$; the total income from these investments was $126.25. How much was invested at each rate?

56. Divide 63 into two parts so that one third of one part may equal one fourth of the other.

57. A man is 60 years old and his son is 30; how many years ago was the man just three times as old as his son?

58. What number diminished by $\frac{1}{4}$ of itself equals 1 less than $\frac{5}{6}$ of itself?

59. All school buildings should have the total light space equal to at least 20% of the floor space; what, then, is the greatest amount of floor space that a schoolroom should have whose light space is 180 sq. ft.?

60. A straight line is divided into two parts, one of which is 30 in. longer than the other. Seven times the shorter piece equals two times the longer. How long is the line?

61. A has $3 more than B, and B has $6 more than C; together they have $111. How much has each?

62. On a farm there are twice as many turkeys as there are ducks, and five times as many chickens as there are turkeys. There are 260 of the three kinds in all. How many are there of each?

63. How many pounds of tea at 40 ct. a pound must be mixed with 20 lb. at 75 ct., in order that the mixture may be worth 50 ct. a pound?

64. The perimeter of a rectangle is 1000 yd. and its altitude is four times its base. What is the length of the base?

65. Eight men hired a yacht, but by taking in four more the expense of each was diminished $1; how much did they pay?

66. A man bought 2-cent stamps, 5-cent stamps, and 11-cent stamps, of each the same number; if he paid 72¢ for the lot, how many of each did he buy?

67. A man saved $1350 in three years. He saved twice as much the second year as the first, and three times as much the third as the second. How much did he save the first year?

68. A man spends $\frac{1}{3}$ of his yearly income for board and lodging, $\frac{3}{8}$ of the remainder for clothes and other expenses, and saves \$500 a year. What is his income?

69. What number increased by $\frac{1}{2}$ of itself equals the sum of $\frac{5}{6}$ of the number and 9?

70. A man invests $\frac{3}{5}$ of his capital at 5%, and the remainder of it at $4\frac{1}{2}\%$; his annual income from both investments is \$240. Find his capital.

71. "Eight years ago," said a man to his son, "I was thirteen times as old as you were, and four years hence, I shall, if I live, be four times as old as you will be then." What is the man's age?

72. A merchant mixes 30 lb. of tea which cost 40 ct. a pound, and 20 lb. which cost 60 ct. a pound. What is the mixture worth per pound?

73. If I spend \$70 for rugs and \$36 for chairs, and then have left one fourth of what I had at first, how much have I remaining?

74. A dealer has coffee, some at 20 ct. and some at 35 ct. per pound; he wishes to make a mixture of 100 pounds which shall be worth 30 ct. a pound. How many pounds of each must he use?

75. A train ran from Pittsburgh to Philadelphia in $7\frac{7}{9}$ hours; if it had traveled 10 miles an hour slower, it would have taken 10 hours. Find the distance from Pittsburgh to Philadelphia.

76. It is required to find a number such that if it be multiplied by 3 and the product increased by 7, the result shall be the same as if it were increased by 8, and the sum multiplied by 2.

77. 10 lb. of tea and 12 lb. of coffee together cost $9.60. If a pound of tea cost 30 ct. more than a pound of coffee, find the cost per pound of each.

78. A packer, engaged to pack 500 tumblers, received 3 ct. for every one that arrived at its destination in good condition, and forfeited 15 ct. for every one broken. He received $17.34. How many were broken?

79. A cask contains a mixture of 25 gallons of vinegar and 5 gallons of water; a certain quantity is drawn out and replaced by water and then the mixture consists of 10 gallons of vinegar with 20 gallons of water. How many gallons were drawn out?

Suggestion. If x represents the number of gallons drawn out; then $\frac{5}{6}x$ represents the number of gallons of vinegar drawn out, and $25 - \frac{5}{6}x$ represents the number of gallons of vinegar left in.

80. A bottle contains a mixture of 1 pint of cream and 3 pints of milk; a certain quantity is removed and replaced by milk, and then the mixture contains $\frac{1}{4}$ of a pint of cream. How much was removed?

81. I traveled 22 mi. in 3 hr., walking part of the way at 4 mi. per hour, and riding the rest of it at 10 mi. per hour. How far did I walk?

82. A person wishing to give 50 cents apiece to some boys, finds that he has not money enough by 25 cents; but if he gives them 40 cents apiece he will have 35 cents remaining. Required the number of boys.

83. A workman received $2.50 and his board for each day that he worked, and paid 60 ct. for board for each day that he did not work. For 90 da. he received $132; how many of these days did he work?

84. It is required to find two numbers whose sum is 12, such that if $\frac{1}{2}$ the less be added to $\frac{1}{3}$ the greater, the sum shall be equal to $\frac{1}{2}$ the greater diminished by $\frac{1}{3}$ the less.

85. How many pounds of water must be added to 40 lb. of a 5% solution of salt to obtain a 4% solution?

86. How many pounds of salt must be added to 80 lb. of a 10% solution of salt to obtain an $11\frac{1}{9}$% solution?

87. A certain number consists of two digits, the one in the units' place being twice that in the tens' place. If 18 be added to the number, the resulting number is represented by the same digits reversed. What is the original number?

88. A certain number consists of two digits, the one in the units' place being three times the one in the tens' place. If the order of the digits be reversed and 16 be added to the resulting number, the new number will be three times the original number. What is the original number?

89. Into what two sums can $2700 be divided so that the income from one at 5% shall equal the income from the other at 4%?

90. M's income is $500 a year more then N's and each saves $\frac{1}{5}$ of his income. At the end of 10 years M has saved $1\frac{1}{2}$ times as much as N. What is the yearly income of each?

91. A certain medicine contains 80% alcohol. How much water must be added to 1 quart of it so that the mixture shall contain only 10% alcohol?

92. During one year a traction company carried 3,000,000 fewer passengers than in the preceding year; but, as the average fare had been raised from 4.1 ct. to 5.2 ct., the receipts were $394,000 more. How many were carried in each year?

93. If one machine can grind 10 bu. of grain in $2\frac{1}{2}$ hr. and another can grind 10 bu. in $1\frac{2}{3}$ hr., how long will it take both machines to grind 100 bu. of grain?

94. The digit in the units' place of a number composed of two digits is 4 less than 3 times that in the tens' place; the sum of the digits plus 27 is equal to the number. Find the number.

95. Twice the digit in the tens' place of a number composed of two digits is 7 greater than that in the units' place; if 7 is subtracted from the number, the remainder is 5 times the sum of the two digits. Find the number.

96. If one machine can skim 75 gal. of milk in $1\frac{3}{4}$ hr. and another 60 gal. per hour, how long must both run to skim 300 gal. of milk?

97. A man invested $5500 in two business enterprises. On the first investment he lost 6% and on the second he gained 5%. His net gain was $55. How many dollars did he invest in each enterprise?

98. There is a reservoir which can be supplied with water from three different inlets; from the first it can be filled in 12 hr., from the second in 18 hr., and from the third in 36 hr. In what time will it be filled if it is being supplied from all inlets at the same time?

99. A certain principal will earn $20 more interest in 8 yr. at 6% than it will in 6 yr. at 5%. What is the principal?

100. A certain principal will in 4 yr. at 5% amount to $10 less than the same principal will amount to in 5 yr. at $4\frac{1}{2}$%. What is the principal?

101. The sum of two numbers is 30 and one of them is 6 less than the other. Find the numbers.

CHAPTER IV

TYPE PRODUCTS AND FACTORS

79. Rational operations. Addition, subtraction, multiplication, and division are called the **rational operations** of algebra.

80. Rational expression. An expression which involves only rational operations is called a **rational expression**.

81. Integral expression with respect to any letter. An expression is said to be integral with respect to any letter when it does not involve a division either by that letter or by a polynomial containing that letter.

Thus, $a^2 - 2a + \frac{1}{3}$ is integral, but $\frac{2}{a}$ and $\frac{3a^2 + 2}{2a - 1}$ are not integral, with respect to a.

82. Integral expression. An integral expression is an expression that is integral with respect to each one of the letters which it contains.

Thus, $a^2b - \frac{1}{3}xy + 2$ is an integral expression.

83. Degree of a monomial. The degree of an integral monomial is equal to the number of its literal factors.

Thus, $3x^2y$ is a monomial of the third degree.

84. Degree of an expression. The degree of an integral algebraic expression is the same as the degree of the term or terms of the expression which are of the highest degree.

Thus, $3a^2b + 2abc - 5a$ is an algebraic expression of the third degree, and $x^2 + 5x + 6$ is one of the second degree.

Note. When all the terms of an expression are of the same degree, the expression is called *homogeneous*.

Thus, $3a^2b + 2b^2c - 5c^2a$ is a homogeneous expression.

85. Degree of an expression with respect to a particular letter. The degree of an integral expression with respect to a particular letter is the same as the index of the highest power of that letter in the expression.

Remark. It is convenient to classify expressions according to their degree with respect to a given letter.

Thus, with respect to x:

$ax + b$ is a *linear* expression, or an expression of the *first degree;*

$ax^2 + bx + c$ is a *quadratic* expression, or an expression of the *second degree;*

$ax^3 + bx^2 + cx + d$ is a *cubic* expression, or an expression of the *third degree;*

$ax^4 + bx^3 + cx^2 + dx + e$ is an expression of the *fourth degree.*

EXERCISE 35

1. State the degree of each of the following monomials:

$$3xyz;\ -2x^2y;\ 7x^2;\ -3x^4y^2z.$$

2. State the degrees with respect to x of each of the monomials in example 1.

3. State the degree of the following expressions:

$$2x + 3;\ ax - by;\ ax^2 + bx + c;\ x^3 - y^3;\ x^2y^2z^2 + 2xyz - 3.$$

4. State the degrees with respect to x of each of the expressions in the preceding exercise.

5. Which of the following expressions are integral with respect to each of the letters contained?

$$ax^2 + \frac{2b}{3}x;\ \frac{3t^2 - 5t}{3t + 1};\ \frac{2x^2}{a} + bx - \frac{1}{2};\ \frac{ax^3 + by^3}{ab}.$$

6. State the degree of each of the following products:

$$(x+2)(x+3);\ x(x-1);\ (x^2+1)(x^2-x+1);$$
$$(x^2+1)(x^2+3)(x^2+5).$$

7. State the degree of each of the following quotients:

$$3a^5 \div 2a^2;\ a^2b^3 \div 2ab^2;\ 3x^7 \div 3x;\ x^3y^3z^3 \div xy^2z^3.$$

8. Which of the following expressions are homogeneous?

$$a^2+2a+b^2;\ a+2b+3c;\ x^2+3xy;\ x-y+z^2;$$
$$x+y+1;\ x^2+y^2+z^2-xyz;\ bc+ca+ab.$$

9. Write a homogeneous polynomial of five terms using the letters a, b, and c.

10. Write two homogeneous polynomials of three terms each, find their product, and state whether or not it is homogeneous.

86. The square of a monomial. From section 58, we have,

$$x^m \cdot x^m = x^{m+m}.$$

That is, $(x^m)^2 = x^{2m}$.

Hence,

Rule. *To square a power of a number, multiply its exponent by* 2.

By the commutative law of multiplication, section 56,

$$a^mb^n \cdot a^mb^n = a^ma^m \cdot b^nb^n.$$

That is, $(a^mb^n)^2 = (a^m)^2(b^n)^2 = a^{2m}b^{2n}$.

Hence,

Rule. *To square a monomial, multiply the exponent of each of its factors by* 2.

Remark. When a monomial has a numerical coefficient, it is usually preferable actually to square the coefficient rather than to indicate its square.

Thus, $(3x^2yz)^2 = 3^2(x^2)^2y^2z^2 = 9x^4y^2z^2.$

ILLUSTRATIVE EXAMPLES

1. Square a^3.

Solution. $$(a^3)^2 = a^6.$$

2. Square b^{3m}.

Solution. $$(b^{3m})^2 = b^{6m}.$$

3. Square $-5\,a^3b^2c$.

Solution. $$(-5\,a^3b^2c)^2 = (-5)^2a^6b^4c^2 = 25\,a^6b^4c^2.$$

4. Square $(a+b)^2(c-d)^2$.

Solution. $$[(a+b)^2(c-d)^2]^2 = (a+b)^4(c-d)^4.$$

EXERCISE 36

(Solve as many as possible at sight.)

Find the square of each of the following, as indicated :

1. $(-3)^2$; $(-2\,a)^2$; $(5\,a)^2$; $(-2\times 3\,x^2)^2$.

2. $[(a+b)]^2$; $[2(a+b)]^2$; $[-3(a+b)]^2$.

3. $(-5\,x^3y^4z^5)^2$; $(-2\cdot 3\,ab^3c^2)^2$.

4. $(-2\,a^m)^2$; $(3\,a^{3m})^2$.

5. $[2\,a(b+c)]^2$; $[-3\,a^3(b+c)^4]^2$.

6. $[\frac{2}{3}\,a]^2$; $[-\frac{3}{2}\,a]^2$; $[\frac{2}{5}\,a^2b]^2$.

7. $[-\frac{5}{3}\,a^2(b+c)^2(x+y)^3]^2$.

8. Why is the square of all numbers which we have considered necessarily positive?

9. Why are the exponents of the literal factors which occur in the square of a monomial necessarily even numbers?

10. If the numerical coefficient of a monomial is a perfect square, and the exponents of the literal factors are all even numbers, what can be said of the monomial?

87. The cube of a monomial. From section 58, we have

$$x^m \cdot x^m \cdot x^m = x^{m+m+m}.$$

That is, $$(x^m)^3 = x^{3m}.$$

Hence,

Rule. *To cube a power of a number multiply its exponent by* 3.

By the commutative law of multiplication, section 56,

$$a^mb^n \cdot a^mb^n \cdot a^mb^n = a^ma^ma^mb^nb^nb^n.$$

That is, $$(a^mb^n)^3 = (a^m)^3(b^n)^3 = a^{3m}b^{3n}.$$

Hence,

Rule. *To cube a monomial multiply the exponent of each of its factors by* 3.

Remark. When a monomial has a numerical coefficient, it is usually preferable to cube the coefficient rather than to indicate its cube.

Thus, $$(3\,x^2yz)^3 = 3^3(x^2)^3y^3z^3 = 27\,x^6y^3z^3.$$

ILLUSTRATIVE EXAMPLES

1. Cube a^2.

Solution. $$(a^2)^3 = a^6.$$

2. Cube x^{3m}.

Solution. $$(x^{3m})^3 = x^{9m}.$$

3. Cube $-5\,a^3b^2c$.

Solution. $$(-5\,a^3b^2c)^3 = (-5)^3(a^3)^3(b^2)^3c^3 = -125\,a^9b^6c^3.$$

4. Cube $(a+b)^2(c-d)^3$.

Solution. $$[(a+b)^2(c-d)^3]^3 = (a+b)^6(c-d)^9.$$

EXERCISE 37

(Solve as many as possible at sight.)

Find the cube of each of the following, as indicated:

1. $(2)^3$; $(-3)^3$; $(-2\,a)^3$; $(5\,a)^3$.

2. $(abc^2)^3$; $(2\,ab^2c)^3$; $(-5\,x^3y^2z)^3$.

3. $[(a+b)]^3$; $[2(a+b)]^3$; $[-3(a+b)]^3$.

4. $(-2a^m)^3$; $(3a^{3m})^3$.

5. $[2a(b+c)]^3$; $[-3a(b+c)^2]^3$.

6. $[\frac{2}{3}a]^3$; $[-\frac{3}{2}a]^3$; $[\frac{2}{5}a^2b]^3$.

7. $[-\frac{5}{3}a^2(b+c)^2(x+y)^3]^3$.

8. When is the cube of a number positive? When negative?

9. Why are the exponents of the literal factors which occur in the cube of a monomial necessarily divisible by 3?

10. If the numerical coefficient of a monomial is a perfect cube and the exponents of the literal factors are all multiples of 3, what can be said of the monomial?

88. Square root and cube root of a monomial. When the square of a number a is equal to a given number A, the number a is called a square root of A. Also, when the cube of a number a is equal to a given number A, the number a is called a cube root of A.

Thus, 3 is a square root of 9 since $(3)^2$ is equal to 9. Also, 3 is the cube root of 27, since $(3)^3$ is equal to 27.

There are always two square roots of a number, the one being positive and the other negative.

Thus, since $(+2)^2 = 4$ and $(-2)^2 = 4$, both $+2$ and -2 are square roots of 4.

Since the cube of a positive number is positive, and the cube of a negative number is negative, it follows that the cube root of a positive number is positive and the cube root of a negative number is negative.

Thus, $(+2a)^3 = +8a^3$ and $(-2a)^3 = -8a^3$; therefore, $+2a$ is the cube root of $+8a^3$ and $-2a$ is the cube root of $-8a^3$.

Remark. For the present, a number will be considered as having but one cube root. Later it will be shown that there are three different expressions which when cubed give the same number.

89. Notation. The radical sign $\sqrt{\ }$ is used in algebra to indicate a square root; similarly, $\sqrt[3]{\ }$ is used to indicate a cube root.

Note 1. In such expressions as $\sqrt{}ab$ the sign $\sqrt{\ }$ does not include b. The square root of the whole expression is indicated either by $\sqrt{(ab)}$ or $\sqrt{ab}$, the vinculum over ab serving the purpose of parentheses.

Note 2. The sign $\pm$ is read *plus or minus.*

Thus, $\sqrt{9} = \pm 3$ is read *the square root of 9 is equal to plus or minus 3.* It is agreed, however, that $+\sqrt{9}$ shall mean $+3$ and $-\sqrt{9}$ shall mean -3.

EXERCISE 38

(Solve as many as possible at sight.)

1. Why is it not possible to find a numerical value of the square root of a negative number?

2. In finding a square root of a^2, by what number must the exponent be divided?

3. In finding the square root of x^{2m}, by what number must the exponent be divided?

4. Give a rule for extracting a square root of a number such as x^{2m} [§ 86].

5. Give a rule for extracting a square root of such an expression as $a^{2m}b^{2n}$ [§ 86].

6. How can the result obtained by taking a square root of a number be checked?

7. In finding the cube root of a^3, by what number must the exponent be divided?

8. In finding the cube root of x^{3m}, by what number must the exponent be divided?

9. Give a rule for extracting the cube root of a number such as x^{3m} [§ 87].

10. Give a rule for extracting the cube root of such an expression as $a^{3m}b^{3n}$ [§ 87].

11. How can the cube root of a number be checked?

Find the following roots, as indicated:

12. $\sqrt{a^4}$; $\sqrt[3]{a^3}$; $\sqrt[3]{8a^3}$; $\sqrt{4a^2b^4}$; $\sqrt[3]{8a^3b^6}$; $\sqrt{16a^4b^6}$.

13. $\sqrt{9a^4b^2}$; $\sqrt{36x^2y^4z^6}$.

14. $\sqrt[3]{27a^6b^3}$; $\sqrt[3]{-27a^6b^3}$.

15. $\sqrt{25(a+b)^2}$; $\sqrt[3]{-125(a+b)^3}$.

16. $\sqrt{\frac{9}{4}a^2(b+c)^2}$; $\sqrt[3]{\frac{27}{8}a^3(b+c)^3}$.

17. $\sqrt[3]{-\frac{27}{64}(x^2+y)^6}$.

18. $\sqrt[3]{-1000(a+x)^9(2b-3)^{54}}$.

Type Products

90. Certain algebraic identities which occur in multiplication are specially important owing to their frequent occurrence. They serve as models for other multiplications, and for this reason should be memorized.

91. The distributive law [§ 60].

$$a(b+c)=ab+ac. \quad (1)$$

$$a(b-c)=ab-ac. \quad (2)$$

ILLUSTRATIVE EXAMPLES

1. $x(y-z)=xy-xz$.
2. $12(3x+y)=36x+12y$.
3. $-3x^2y(x-2y)=-3x^3y+6x^2y^2$.
4. $2y(x-\overline{y+z})=2yx-2y(y+z)=2yx-2y^2-2yz$.

EXERCISE 39

Multiply:

1. $-2(a-4)$.
2. $3x(2x-3y)$.
3. $\frac{3}{4}a(\frac{1}{2}b-c)$.
4. $2uv(2u-3v)$.

5. $3ab(3a - 2b)$.
6. $\frac{2}{3}(3x - 6y)$.
7. $-xy(x^2 - y)$.
8. $(5 - a)a^2$.
9. $(-2x + 3)(-2x)$.
10. $a(b + c + d)$.
11. $\frac{p}{3}(b^2 + B^2)$.
12. $(3x^2y - 5z)(-3xyz^2)$.
13. $lm(l + m + 1)$.
14. $2x(y + z - t)$.
15. $(2x - 3y + 5z)(-2x)$.
16. $3a(b - c + d - f)$.
17. $(-2ab)(-2a + 3b)$.
18. $(2abc - 7bcd)(-bd)$.
19. $-3t^2v(11v^2 - 7t^3)$.
20. $(a - b + 2c)(-3bc)$.
21. $-2pq^2(8p^2q - 5pq^2)$.
22. $\frac{1}{3}\pi H(B^2 + b^2 + Bb)$

92. The square of a binomial. By multiplication we find that

$$(a + b)^2 = a^2 + 2ab + b^2. \tag{1}$$

This identity may be expressed in words as follows:

The square of the sum of two numbers is the square of the first plus twice their product, plus the square of the second.

Replacing b by $-b$ in identity (1), we have [§ 67 note]:

$[a + (-b)]^2 = a^2 + 2a(-b) + (-b)^2$. Performing the indicated operations,

$$(a - b)^2 = a^2 - 2ab + b^2. \tag{2}$$

This identity may be expressed in words as follows:

The square of the difference of two numbers is the square of the first minus twice their product, plus the square of the second.

ILLUSTRATIVE EXAMPLES

1. Square 35 by use of identity (1).

Solution.
$$\begin{aligned}\overline{35}^2 &= (30 + 5)^2 \\ &= \overline{30}^2 + 2 \times 5 \times 30 + 5^2 \\ &= 900 + 300 + 25 = 1225.\end{aligned}$$

Leonard Euler (1707–1783) was born at Bâle and died at St. Petersburg. He wrote on almost all the branches of mathematics then known, revising almost all those of pure mathematics. In 1770 he published an algebra at St. Petersburg which was translated into French in 1794 by the celebrated mathematician Lagrange.

2. Square 19 by use of identity (2).

Solution.
$$\overline{19}^2 = (20-1)^2$$
$$= \overline{20}^2 - 2 \times 20 \times 1 + 1^2$$
$$= 400 - 40 + 1 = 361.$$

3. Square $2\,a + 3\,b$.

Solution.
$$(2\,a + 3\,b)^2 = (2\,a)^2 + 2(2\,a)(3\,b) + (3\,b)^2$$
$$= 4\,a^2 + 12\,ab + 9\,b^2.$$

4. Square $xy - 2$.

Solution.
$$(xy - 2)^2 = (xy)^2 - 2(xy)(2) + (2)^2$$
$$= x^2y^2 - 4\,xy + 4.$$

5. Square $\frac{2}{3}\,x - \frac{3}{5}\,y$.

Solution.
$$(\tfrac{2}{3}\,x - \tfrac{3}{5}\,y)^2 = (\tfrac{2}{3}\,x)^2 - 2(\tfrac{2}{3}\,x)(\tfrac{3}{5}\,y) + (\tfrac{3}{5}\,y)^2$$
$$= \tfrac{4}{9}\,x^2 - \tfrac{4}{5}\,xy + \tfrac{9}{25}\,y^2.$$

EXERCISE 40

Write the squares of the following binomials as indicated:

1. $(x+1)^2$.	**2.** $(2\,x-y)^2$.	**3.** $(a-3\,b)^2$.
4. $(2\,c+3\,d)^2$.	**5.** $(7\,a-2)^2$.	**6.** $(3\,b+2)^2$.
7. $(5\,a-3\,b)^2$.	**8.** $(5\,cd-3\,a)^2$.	**9.** $(a+1)^2$.
10. $(2\,x-1)^2$.	**11.** $(3\,ax+2\,by)^2$.	**12.** $(2\,x-9)^2$.
13. $(a+\frac{1}{2})^2$.	**14.** $(3\,m-2\,n)^2$.	**15.** $(ab+3)^2$.
16. $(3\,x+4)^2$.	**17.** $(5\,x-3)^2$.	**18.** $(2\,mn-3\,n)^2$.
19. $(abc+1)^2$.	**20.** $(xyz-1)^2$.	**21.** $\overline{31}^2$.
22. $\overline{11}^2$.	**23.** $\overline{57}^2$.	**24.** $\overline{78}^2$.
25. $\overline{29}^2$.	**26.** $\overline{99}^2$.	**27.** $(15\,x+2\,y)^2$.
28. $(13\,a-3\,b)^2$.	**29.** $(\frac{1}{2}\,x-\frac{1}{3}\,y)^2$.	**30.** $(\frac{3}{2}\,x-\frac{5}{3}\,z)^2$.
31. $(\frac{2}{3}\,x-\frac{3}{4}\,y)^2$.	**32.** $(\frac{1}{2}\,a-1)^2$.	**33.** $(\frac{1}{2}\,m+2)^2$.

34. Is $9\,x^2 + 4\,y^2 + 6\,xy$ a perfect square; that is, the square of a binomial?

35. Give a rule for determining whether or not a trinomial is a perfect square.

Which of the following trinomials are perfect squares?

36. $x^2 + 2x + 4.$

37. $p^2 + 6p + 9.$

38. $m^2 + n^2 - 2mn.$

39. $4x^2 + 1 + 4x.$

40. $9x^2 + 4 + 6x.$

41. $x^2 + 4y^2 - 4xy.$

42. $x^2 - x + \frac{1}{4}.$

43. $4m^2 + 25 - 20m.$

44. $4x^2 + 4x - 1.$

45. $r^2 - 2rs - s^2.$

46. $1 - 2p + p^2.$

47. $4 - 12m + 3m^2.$

93. The square of a trinomial.

Since $a + b + c = a + (b + c),$

$$\begin{aligned}(a + b + c)^2 &= [a + (b + c)]^2 \\ &= a^2 + 2a(b + c) + (b + c)^2 \\ &= a^2 + 2ab + 2ac + b^2 + 2bc + c^2.\end{aligned}$$

Hence,

$$\boldsymbol{(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.}$$

This identity may be expressed in words as follows:

The square of a trinomial is equal to the sum of the squares of its terms plus twice the sum of the products of all pairs of the terms.

Note. In a trinomial there are three pairs of terms.

Thus, in the trinomial $2x - 3y + 5z$ the three pairs of terms are $2x$ and $-3y$, $2x$ and $5z$, $-3y$ and $5z$.

ILLUSTRATIVE EXAMPLES

1. Square $x + y - z.$

Solution. $$\begin{aligned}(x+y-z)^2 &= x^2+y^2+(-z)^2+2xy+2x(-z)+2y(-z) \\ &= x^2 + y^2 + z^2 + 2xy - 2xz - 2yz.\end{aligned}$$

2. Square $2x - 3y - 1$ and check; let $x = 2$ and $y = 1.$

Solution. $(2x-3y-1)^2$
$=(2x)^2+(-3y)^2+(-1)^2+2(2x)(-3y)+2(2x)(-1)+2(-3y)(-1)$
$=4x^2+9y^2+1-12xy-4x+6y.$

Check. $(2x-3y-1)^2=4x^2+9y^2+1-12xy-4x+6y$
$(4-3-1)^2=16+9+1-24-8+6$
$0=0.$

EXERCISE 41

Square the following trinomials, as indicated, and check by substituting $a=1$, $b=2$, $c=3$, $x=1$, $y=-1$, $z=2$.

1. $(x+y+2)^2$.
2. $(x-y-z)^2$.
3. $(a+b+1)^2$.
4. $(a+b+z)^2$.
5. $(x+2y+z)^2$.
6. $(a-x+3c)^2$.
7. $(2a-3b-1)^2$.
8. $(3x-2y-z)^2$.
9. $(a+2b-3)^2$.
10. $(yz+zx+xy)^2$.
11. $(a^2+a+1)^2$.
12. $(x^2-xy+y^2)^2$.

94. The square of a polynomial. When a polynomial contains more than three terms, it may be expressed as a binomial or a trinomial by the use of parentheses.

Thus, the polynomial $2a+3b-c+5d$ may be written,
as a binomial, $(2a+3b)+(-c+5d)$;
as a trinomial, $(2a+3b)-c+5d$.

By repeated application of the rules for finding the square of a binomial or a trinomial, it will be found in every case that the square of a polynomial is expressed by the principle employed to find the square of a trinomial; namely,

The square of a polynomial is equal to the sum of the squares of its terms plus twice the sum of the products of all pairs of the terms.

Note. A systematic way of naming the pairs of terms in a polynomial is as follows: Take the first term with each of the terms that follows it; take the second term with each that follows it; take the third term with each term that follows it; continue this process until next to the last term is taken with the last.

Thus, the sum of the algebraic products of all pairs of terms of the polynomial $(a + b + c - d + e)$ is

$$ab + ac + a(-d) + ae + bc + b(-d) + be + c(-d) + ce + (-d)e.$$

EXERCISE 42

(Solve as many as possible at sight.)

Square the following polynomials as indicated:

1. $(a+b+c+d)^2$. **2.** $(x+y+z-u)^2$.

3. $(m+n-p+q)^2$. **4.** $(r-s+t+u)^2$.

5. $(a-b-c+d)^2$. **6.** $(m-n-p-q)^2$.

7. $(x+y+z+1)^2$. **8.** $(m+3n-p+2)^2$.

9. $(a+b+c+d+e)^2$. **10.** $(m-n-p-q-t)^2$.

95. The product of the sum and difference of two numbers. By multiplication we find that

$$(a+b)(a-b) = a^2 - b^2.$$

This identity may be expressed in words as follows:

The product of the sum and difference of two numbers is equal to the square of the first minus the square of the second.

ILLUSTRATIVE EXAMPLES

1. Find $(a+1)(a-1)$.

Solution. $(a+1)(a-1) = a^2 - 1^2 = a^2 - 1.$

2. Find $(2m+5y)(2m-5y)$.

Solution. $(2m+5y)(2m-5y) = (2m)^2 - (5y)^2 = 4m^2 - 25y^2.$

3. Find $(ax-b)(ax+b)$

Solution. $(ax-b)(ax+b) = (ax)^2 - (b)^2 = a^2x^2 - b^2.$

4. Find 101×99.

Solution. $101 \times 99 = (100+1)(100-1) = (100)^2 - 1$

$= 10000 - 1 = 9999.$

5. Find $(a+b+c)(a+b-c)$.

Solution. $(a+b+c)(a+b-c) = (\overline{a+b}+c)(\overline{a+b}-c)$
$= (a+b)^2 - c^2$
$= a^2 + 2\,ab + b^2 - c^2.$

6. Find $(x+y-z)(x-y+z)$.

Solution. $(x+y-z)(x-y+z) = (x+\overline{y-z})(x-\overline{y-z})$
$= x^2 - (y-z)^2$
$= x^2 - y^2 + 2\,yz - z^2.$

EXERCISE 43

(Solve as many as possible at sight.)

Multiply as indicated:

1. $(x+1)(x-1)$.
2. $(a+2\,b)(a-2\,b)$.
3. $(m-n)(m+n)$.
4. $(2\,a+5\,b)(2\,a-5\,b)$.
5. $(ab-c)(ab+c)$.
6. $(2\,ab+c)(2\,ab-c)$.
7. $(3\,xy-2\,z)(3\,xy+2\,z)$.
8. $(4\,x-7\,y)(4\,x+7\,y)$.
9. $(-3x+5\,y)(3x+5\,y)$.
10. $(-1+x)(x+1)$.
11. $(\frac{1}{2}-2\,y)(\frac{1}{2}+2\,y)$.
12. $(a^2+b^2)(a^2-b^2)$.
13. $(a^2+b^2)(b^2-a^2)$.
14. $(10xyz+3)(10xyz-3)$.
15. 63×57. **Suggestion.** $63 \times 57 = (60+3)(60-3)$.
16. 22×18.
17. 81×79.
18. 37×43.
19. 201×199.
20. 202×198.
21. $(x+y+z)(x+y-z)$.
22. $(a-b+c)(a+b+c)$.
23. $(-a+b+c)(a+b+c)$.
24. $(a+b-c)(a-b+c)$.

Suggestion. $(a+b-c)(a-b+c) = [a+(b-c)][a-(b-c)]$.

25. $(-a+b+c)(a-b+c)$.
26. $(a+1-b)(a-1+b)$.
27. $(2\,a+3\,b-c)(2\,a-3\,b+c)$.
28. $(2\,x-3\,y+2)(2\,x+3\,y-2)$.
29. $[a^2+(b+c)^2][a^2-(b+c)^2]$.

30. $(x^3 + y^3)(x^3 - y^3)$.

31. $(m^4 - n^4)(m^4 + n^4)$.

32. $(r^5 + 1)(r^5 - 1)$.

33. State a rule for telling whether or not a binomial is the product of the sum and difference of the same two numbers.

34. Which of the following may be expressed as the product of a sum and difference?

$x^2 - 1$;	$x^2 + 1$;	$a^2 + b^2 + 1$;
$x^2 + y^2 - z^2$;	$x^2 + 2x + 1 - y^2$;	$a^2 + 2ab + b^2 - 1$;
$x^4 - y^6$;	$a^2 - 2ab + b^2 + 1$;	$x^2 + y^2$.

35. Find in two different ways the value of $(a + b)(a - b)$ when $a = x + 1$ and $b = x - 1$.

36. Find in two different ways the value of $(x + y)(x - y)$, when $x = 2a + 3$ and $y = 2a - 5$.

37. Find in two different ways the value of $(x + y)(x - y)$, when $x = a + b + c$ and $y = a + b - c$.

96. The product of two binomials having a common term. By actual multiplication we have:

$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \underline{\quad\quad + bx + ab} \\ x^2 + (a + b)x + ab \end{array}$$

Hence,

$$\boldsymbol{(x + a)(x + b) = x^2 + (a + b)x + ab.}$$

This identity may be expressed in words as follows:

The product of two binomials having a common term is equal to the square of the common term plus the product of the algebraic sum of the unlike terms and the common term, plus the product of the unlike terms.

ILLUSTRATIVE EXAMPLES

1. Find the product of $x + a$ and $x - b$.

Solution. $(x + a)(x - b) = x^2 + (a - b)x + (a)(-b)$
$= x^2 + (a - b)x - ab.$

2. Find the product of $x - a$ and $x - b$.

Solution. $(x - a)(x - b) = x^2 + (-a - b)x + (-a)(-b)$
$= x^2 - (a + b)x + ab.$

EXERCISE 44

(Solve as many as possible at sight.)

Multiply as indicated:

1. $(a + b)(a + c)$.

2. $(a + 2)(a + 1)$.

3. $(2a + 1)(2a + 3)$.

4. $(3a + 2)(3a - 4)$.

5. $(n + 2)(n + 5)$.

6. $(x + 3)(x + 2)$.

7. $(x + 2)(x - 3)$.

8. $(x + 4)(x - 5)$.

9. $(x - 3)(x - 2)$.

10. $(x - 5)(x - 7)$.

11. $(ab + c)(ab + d)$.

12. $(xy + z)(xy + 1)$.

13. $(2abc + 1)(2abc + 3)$.

14. $(\frac{1}{2}a + 1)(\frac{1}{2}a + 3)$.

15. $(\frac{1}{3}x + 2)(\frac{1}{3}x - 5)$.

16. $(3a - 1)(3a - 2)$.

17. $(x + y + 1)(x + y + 2)$.

18. $(a + b + 2)(a + b + 3)$.

97. The cube of a binomial.

$$(a + b)^3 = (a + b)^2(a + b)$$
$$= (a^2 + 2ab + b^2)(a + b).$$

By actual multiplication we have,

$$\begin{array}{l} a^2 + 2ab + b^2 \\ \underline{a + b \qquad\qquad\qquad\qquad} \\ a^3 + 2a^2b + ab^2 \\ \underline{\qquad + a^2b + 2ab^2 + b^3} \\ a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

Hence,

$$\boldsymbol{(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2.}$$

By the use of this identity the cube of any binomial can at once be written as in the following:

ILLUSTRATIVE EXAMPLES

1. Find the cube of $a - b$.

Solution. $(a-b)^3 = a^3 + (-b)^3 + 3a^2(-b) + 3a(-b)^2$
$= a^3 - b^3 - 3a^2b + 3ab^2.$

2. Find the cube of $2a + 3b$.

Solution. $(2a+3b)^3 = (2a)^3 + (3b)^3 + 3(2a)^2(3b) + 3(2a)(3b)^2$
$= 8a^3 + 27b^3 + 36a^2b + 54ab^2.$

3. Cube 98 by the use of the identity of section 97.

Solution. $(98)^3 = (100-2)^3$
$= (100)^3 + (-2)^3 + 3(100)^2(-2) + 3(100)(-2)^2$
$= 1000000 - 8 - 60000 + 1200$
$= 941192.$

EXERCISE 45

(Solve as many as possible at sight.)

Cube the following, as indicated:

1. $(x+y)^3$. **2.** $(x-y)^3$. **3.** $(a+1)^3$.
4. $(a-1)^3$. **5.** $(x-2)^3$. **6.** $(x+2)^3$.
7. $(2a+b)^3$. **8.** $(2x+3y)^3$. **9.** $(3-2x)^3$.
10. $(4x-3y)^3$. **11.** $(99)^3$. **12.** $(101)^3$.
13. Is $8a^3 + 12a^2b + 6ab^2 + b^3$ a perfect cube?

EXERCISE 46. REVIEW

(Solve as many as possible at sight.)

By use of type identities perform the indicated multiplications in examples 1–33.

1. $a(2-3b)$. **2.** $2b(3x+5y)$. **3.** $(x+1)^2$.
4. $(x-7)^2$. **5.** $(5x+2)^2$. **6.** $(3x-4)^2$.
7. $(5x+3z)^2$. **8.** $(2m-7n)^2$.

9. $(x+5)(x+3)$.

10. $(x-9)(x-1)$.

11. $(x+10)(x-2)$.

12. $(x-11)(x+3)$.

13. $(x-9)(x+9)$.

14. $(2x-11)(2x+11)$.

15. $(4m-n)(4m+n)$.

16. $(x+7)(x+a)$.

17. $(x-b)(x+4)$.

18. $(2r+m)(2r-n)$.

19. $(2x+y+5)^2$.

20. $(2r+1)^3$.

21. $(1+3c)(1+4c)$.

22. $(3+a-b)^2$.

23. $(x-y+1)(x-y+2)$.

24. $(xy-7)(xy+3)$.

25. $(x+3)^3$.

26. $(2-3y)^3$.

27. $(2a-5c)(2a+5c)$.

28. $(1+x-y)(1+x+y)$.

29. $(xy-yz+zx)(xy+yz-zx)$.

30. $(1+a+2a^2)^2$.

31. $(a+\frac{3}{4})(a-\frac{1}{4})$.

32. $(a+\frac{3}{4})(a-2)$.

33. $(\frac{1}{3}a+\frac{1}{2}b)(\frac{1}{3}a-\frac{1}{2}b)$.

34. Verify the identity:

$$(a^2+b^2)(x^2+y^2)=(ax+by)^2+(ay-bx)^2.$$

35. Verify the identity:

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2+(bz-cy)^2 \\ +(cx-az)^2+(ay-bx)^2.$$

Simplify the following by performing the indicated operations and when possible uniting terms:

36. $(1+x)^2-(x-1)(x+1)$.

37. $(a+b)^2(a-b)^2$.

Suggestion. $(a+b)^2(a-b)^2=[(a+b)(a-b)]^2$.

38. $(2x+3y)^2(2x-3y)^2$.

39. $(x^2+x+1)(x^2-x+1)$.

40. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.

41. $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.

Suggestion. $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$
$=[(a+b+c)(a+b-c)][(a-b+c)(-a+b+c)]$.

42. $(a+b+c+d)(a+b+c-d)$.

43. $a(b-c)+b(c-a)+c(a-b)$.

44. $2x(3y-4z)+3y(4z-2x)+4z(2x-3y)$.

45. $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

46. $(a^2+ab+b^2)(a-b)$. **47.** $(a^2-ab+b^2)(a+b)$.

Factors

98. Integral algebraic factors. In this chapter we shall consider only the integral factors of integral expressions of the more common form.

99. Factors of integral expressions. By factors of an integral expression are meant those integral expressions which when multiplied together will produce the given expression.

Note 1. The word integral refers only to the literal part of the expression. Thus, $\frac{3}{4}a$ and $\frac{a+b}{3}$ are integral expressions.

Note 2. A factor of each of two or more expressions is called a **common factor** of the expressions.

100. Prime number and prime expression. In arithmetic a prime number is defined as an integer which is divisible by itself and 1 and by no other integer; as, 3, 5, 7. In elementary algebra an integral expression with integral coefficients is said to be prime when it is divisible by itself and 1, and by no other integral expression or integer.

Thus, (x^2+y^2) is a prime expression.

Note. An integer or an integral expression with integral coefficients may often be expressed in more than one way as a product of factors, but it can be expressed in only one way as the product of its prime factors.

For example, $24 = 12 \times 2 = 8 \times 3 = 2 \times 2 \times 2 \times 3$.

When an expression is to be factored it is understood that, in general, its prime factors are required.

101. Use of type forms. The type forms in multiplication are of great service in giving the factors of an integral expression. In many cases they furnish the clue as to the kind of factors to expect.

102. Factors of monomials. The literal factors of a monomial can always be seen at a glance.

Thus, the factors of $-2a^3b^2c$ are -2, a, a, a, b, b, and c.

Note. Here, as elsewhere, the sign before the monomial is to be regarded as belonging to the numerical coefficient.

Thus, $-a^2b = (-1)a^2b$.

103. The converse of the distributive law. We learned, section 60, that

$$m(a+b+c+d) = ma + mb + mc + md;$$

hence, conversely,

$$\boldsymbol{ma + mb + mc + md = m(a+b+c+d).}$$

From this identity it follows that:

A monomial which is a factor of every term of an expression is a factor of the whole expression.

Note. The first step in factoring an integral expression is to remove all monomial factors.

ILLUSTRATIVE EXAMPLES

1. Factor $3x + 6y$.

Solution. $3x + 6y = 3(x + 2y)$.

2. Factor $mx + my - m$.

Solution. $mx + my - m = m(x + y - 1)$.

3. Factor $a(l-m) + a(m+n)$.

Solution. $a(l-m) + a(m+n) = a[(l-m) + (m+n)]$
$= a[l - m + m + n] = a(l+n)$.

4. Factor $(a+b)(x+y) - (a+b)(y+z)$.

Solution. $(a+b)(x+y) - (a+b)(y+z) = (a+b)[(x+y) - (y+z)]$
$= (a+b)[x + y - y - z]$
$= (a+b)(x-z)$.

5. Factor $(a+b)(b-c)+(a+b)(c-a)+(a+b)(a-b)$.

Solution. $(a+b)(b-c)+(a+b)(c-a)+(a+b)(a-b)$
$$=(a+b)[b-c+c-a+a-b].$$
$$=(a+b)0=0.$$

EXERCISE 47

(Solve as many as possible at sight.)

Factor the following expressions by removing the monomial factors:

1. $4x+12y$.
2. $am-an$.
3. $5x+10y-15z$.
4. $px-py+pz$.
5. $ax+ay+az$.
6. $\frac{3}{4}a+\frac{3}{4}b-\frac{3}{4}c$.
7. $2ax-2ay$.
8. $3am-6an+9ar$.
9. $12x+30xy$.
10. x^2y-xy^2.
11. $\frac{1}{3}ab+\frac{1}{6}ac$.
12. $-3x^2+12x$.
13. $ax+a$.
14. $ax-ay+a$.
15. $4cx+6cy-2c$.
16. $a^3b-2a^2b^2+3ab^3$.
17. $x^2yz+2xy^2z-3xyz^2$.
18. $-abc-bc-b$.
19. $5x^2+10$.
20. $xy-x$.
21. $xy+xz-x$.
22. $a(b+c)+a(x+y)$.
23. $(x+y)(b+c)+(y+z)(b+c)$.
24. $(a+b)x-(a+b)y$.
25. $(a+b)x-(a+b)y+(a+b)z$.
26. $(a+b)c+(a+b)d-(a+b)e$.
27. $(x+y)(2a+b)+(x+y)(b-a)$.
28. $(3a+2b)a+(3a+2b)b-(3a+2b)(a-b)$.
29. $axyz-ayux+axut$.
30. $2(a+b)c+2(a+b)d$.
31. $2(a-b)cd-2(a+b)(a-b)$.
32. $2(x+y)z-4(x+y)t$.

33. $3(a+b)c+6(a+b)d.$

34. $2(3a-2x)(3a+2x)+3(3a-2x)(2x-a).$

35. $a(b+1)+c(b+1).$

36. $x(y-1)+y(y-1)+(y-1).$

37. $x(z-1)+y(z-1)+z-1.$

38. $x(a-1)+y(a-1)-a+1.$

The expression may be written $x(a-1)+y(a-1)-(a-1)$.

39. $a(x-y)-x+y.$

40. $a(y-1)+by-b.$

41. $a(b+c)+2b+2c.$

42. $2(x-y)+4x-4y.$

43. $3a(y+z)-3by-3bz.$

44. $(a+b)(x+y)^2-(a+b)^2(x+y)+(a+b)(x+y).$

45. $(a+b)^2+(a+b)c.$

46. $2(a+b)c^2-2(a+b)^2c.$

47. $(m+n)(p+q)^2+(m+n)(p+q)-(m+n)^2(p+q).$

104. Factors found by grouping terms. Examples 37 to 43 inclusive of exercise 47 furnish simple instances of the grouping of terms. In each of these examples the grouping required merely the insertion of a single set of parentheses. In general, in factoring an integral expression of four or more terms by the aid of rearrangement and grouping of terms, those terms should be grouped together which have a common monomial factor. Two or more ways of grouping may be possible, and some of these ways may lead to a common factor and others may not. It is not possible, however, to give any simple rule for proper grouping when several ways of grouping exist, but a careful study of the following illustrative examples will prove helpful.

ILLUSTRATIVE EXAMPLES

1. Factor $ax + bx + ay + by$.

Solution.
$$\begin{aligned} ax + bx + ay + by &= (ax + bx) + (ay + by) \\ &= x(a + b) + y(a + b) \\ &= (a + b)(x + y). \end{aligned}$$

2. Factor $ax - bx - ay - b + a + by$.

Solution.
$$\begin{aligned} ax - bx - ay - b + a + by &= (ax - ay + a) + (-bx - b + by). \\ &= a(x - y + 1) - b(x - y + 1) \\ &= (x - y + 1)(a - b). \end{aligned}$$

Another Solution.
$$\begin{aligned} ax - bx - ay - b + a + by &= (ax - bx) + (-ay + by) + (-b + a) \\ &= x(a - b) - y(a - b) + (a - b) \\ &= (a - b)(x - y + 1). \end{aligned}$$

3. Factor $a^3 + a^2b + ab^2 + b^3$.

Solution.
$$\begin{aligned} a^3 + a^2b + ab^2 + b^3 &= a^2(a + b) + b^2(a + b) \\ &= (a + b)(a^2 + b^2). \end{aligned}$$

EXERCISE 48

Factor:

1. $2a + 2b + ac + bc$.

2. $ax + x + a + 1$.

3. $am + an - 4m - 4n$.

4. $by + y - 2b - 2$.

5. $lp + qm + mp + ql$.

6. $am - cm + cu - au$.

7. $xy + 2y - 2x - 4$.

8. $6 - 9a + 4b - 6ab$.

9. $6m - 3 - 2am + a$.

10. $6rs - 9r - 2s^2 + 3s$.

11. $9x^2 + 15xy + 6y^2$.

Suggestion.
$$\begin{aligned} 9x^2 + 15xy + 6y^2 &= 3(3x^2 + 5xy + 2y^2) \\ &= 3(3x^2 + 3xy + 2xy + 2y^2) \\ &= 3[(3x^2 + 3xy) + (2xy + 2y^2)] \\ &= 3[3x(x + y) + 2y(x + y)]. \end{aligned}$$

12. $a^2 + ab + a - ca - cb - c + a + b + 1$.

13. $x^2 - xy + x - xz + zy - z + x - y + 1$.

14. $a(x-y)+b(y-x)$.

15. $2(a-b)-x(b-a)$.

16. x^3-x^2+x-1.

17. x^3+x^2+x+1.

18. $8x^3-4x^2+2x-1$.

Suggestion. $8x^3-4x^2+2x-1=(2x)^3-(2x)^2+(2x)-1$.

19. $8x^3+4x^2+2x+1$.

20. $ax^2-bx^2+ax-cx^2-bx-cx$.

21. x^2-3x+2.

Suggestion. $x^2-3x+2=x^2-x-2x+2$.

22. $ax+ay+az+bx+by+bz+cx+cy+cz$.

23. $a^3+ab^2+a^2b+b^3$.

24. $x^3+xy^2-x^2y-y^3$.

25. a^2x-2-a^2+2x.

26. $15-6x-10y+4xy$.

27. $1-abcd+ac-bd$.

28. $m^2n^2+m^2p^2+q^2n^2+q^2p^2$.

29. $(x^2-y^2)z-(y^2-z^2)x$.

Suggestion. First remove the given parentheses, then group terms.

30. $1-abx^2-(a-b)x$.

31. $x^2+(a+b)x+ab$.

32. $(a^2+b^2)c+(b^2+c^2)a$.

33. $a(b^2+c^2)-b(c^2+a^2)$.

105. Trinomial squares. We learned, § 92, that

$(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$; conversely

$$(1)\quad \mathbf{a^2+2ab+b^2=(a+b)^2.}$$
$$(2)\quad \mathbf{a^2-2ab+b^2=(a-b)^2.}$$

From identities (1) and (2) it is obvious that

If two terms of a trinomial are perfect squares and the third term is equal to plus or minus twice the product of the square roots of the other two terms, then the trinomial is the square of a binomial.

When two terms of a trinomial are perfect squares and the third term is equal to plus or minus twice the product of the square roots of the other two terms, the binomial square root — that is, the binomial which, when squared is equal to the given trinomial — may be found by taking the positive square roots of the two terms of the trinomial which are perfect squares, and adding one of these square roots to the other, or subtracting it from the other, according as the third term of the trinomial is positive or negative.

Remark. $(a-b)^2=(b-a)^2$ since $a-b$ and $b-a$ differ only in sign, and $(+a)^2=(-a)^2$ whatever expression may be represented by a. However, it is customary to write $a^2-2ab+b^2$ equal to either $(a-b)^2$ or $(b-a)^2$ at pleasure and not to write it equal to $[\pm(a-b)]^2$. Similarly, $a^2+2ab+b^2$ is written $(a+b)^2$ and not $[\pm(a+b)]^2$.

ILLUSTRATIVE EXAMPLES

1. Factor a^2+2a+1.

Solution. $a^2+2a+1=(a)^2+2(a)(1)+(1)^2$, which satisfies the conditions for a perfect square.

$$\therefore a^2+2a+1=(a+1)^2. \qquad [\S 105, 1]$$

2. Factor x^4-4x^2+4.

Solution. $x^4-4x^2+4=(x^2)^2-2(2)(x^2)+(2)^2$, which satisfies the conditions for a perfect square.

$$\therefore x^4-4x^2+4=(x^2-2)^2. \qquad [\S 105, 2]$$

Remark. $(x^2-2)^2$ may be written $(2-x^2)^2$. [§ 105, Remark]

3. Factor $4x^2-12xy+9y^2$.

Solution.
$$\begin{aligned}4x^2-12xy+9y^2&=(2x)^2-2(2x)(3y)+(3y)^2\\&=(2x-3y)^2.\end{aligned}$$

4. Factor $(a+b)^2-6(a+b)(c-d)+9(c-d)^2$.

Solution.
$$\begin{aligned}&(a+b)^2-6(a+b)(c-d)+9(c-d)^2\\&=(a+b)^2-2(a+b)[3(c-d)]+[3(c-d)]^2\\&=[(a+b)-3(c-d)]^2\\&=(a+b-3c+3d)^2.\end{aligned}$$

5. Factor $x^2 - x + \frac{1}{4}$.

Solution. $x^2 - x + \frac{1}{4} = x^2 - 2(\frac{1}{2})x + (\frac{1}{2})^2$.
$= (x - \frac{1}{2})^2$.

Remark. Although not all of the numerical coefficients of $x^2 - x + \frac{1}{4}$ are integral, yet the terms of the trinomial satisfy the conditions for a perfect square.

EXERCISE 49

(Solve as many as possible at sight.)

Factor the following trinomials:

1. $m^2 + 2mn + n^2$.

2. $r^2 - 2rs + s^2$.

3. $x^2 + 2x + 1$.

4. $a^2 + 4a + 4$.

5. $m^2 - 6m + 9$.

6. $4x^2 + 4x + 1$.

7. $x^2 - 6xy + 9y^2$.

8. $x^2 + x + \frac{1}{4}$.

9. $a^2 - a + \frac{1}{4}$.

10. $9x^2 + 3x + \frac{1}{4}$.

11. $x^4 + 2x^2 + 1$.

12. $4a^4 + 20a^2 + 25$.

13. $25a^4 - 20a^2 + 4$.

14. $x^2y^2 + 2xy + 1$.

15. $4a^2b^2 - 4ab + 1$.

16. $x^2 + 2xy^2 + y^4$.

17. $a^2b^4 - 2ab^2 + 1$.

18. $x^2y^6 - 2xy^3z^2 + z^4$.

19. $x^2 - 6xy^3 + 9y^6$.

20. $4a^2 - 20ab^2 + 25b^4$.

21. $9x^2y^2z^2 + 6axyz + a^2$.

22. $a^2 - 12a + 36$.

23. $25m^2 + 70mn + 49n^2$.

24. $(p+q)^2 + 2(p+q) + 1$.

25. $a^{2m} + 2a^mb^n + b^{2n}$.

Suggestion. $a^{2m} + 2a^mb^n + b^{2n} = (a^m)^2 + 2(a^m)(b^n) + (b^n)^2$.

26. $x^{2p} - 2x^py^q + y^{2q}$.

27. $a^{2m} + 4a^mb^n + 4b^{2n}$.

28. $4ab^3 - 4a^2b^2 + a^3b$.

29. $x^3y - 2x^2y + xy$.

30. $a^2bc^2 + 4abc^2 + 4bc^2$.

31. $x^2y^2z^2 - 14xyz^2 + 49z^2$.

32. $x^3 + 18x^2 + 81x$.

33. $m^2np + 20mnp + 100np$.

34. $(x+y)^2 - 6(x+y)(a+b) + 9(a+b)^2$.

35. $x^2 - 10x(y+z) + 25(y+z)^2$.

36. $a^2 + b^2 + c^2 + 2\,bc + 2\,ca + 2\,ab.$

Suggestion. $a^2 + b^2 + c^2 + 2\,bc + 2\,ca + 2\,ab$
$$= a^2 + 2(b + c)a + (b^2 + 2\,bc + c^2).$$

37. $a^2 + b^2 + c^2 + 2\,bc - 2\,ca - 2\,ab.$

38. $4\,a^2 - 28\,a(b + c) + 49(b + c)^2.$

39. $49\,x^2y^2 + 28\,xy^2(y + z) + 4\,y^2(y + z)^2.$

40. $1 - 6(a - b) + 9(a - b)^2.$

106. The difference of two squares. We learned, section 95, that

$$(a + b)(a - b) = a^2 - b^2; \text{ hence, conversely,}$$
$$\boldsymbol{a^2 - b^2 = (a + b)(a - b).}$$

From this identity we infer the following rule for factoring the difference of two squares:

Rule. *Find the positive square root of each of the two squares and form the sum and the difference of these square roots in the order in which their squares occur in the expression. The sum and the difference of the square roots are the two factors.*

ILLUSTRATIVE EXAMPLES

1. Factor $9\,x^2 - 25$.

Solution. $9\,x^2 - 25 = (3\,x)^2 - (5)^2.$

The positive square roots of the squares are $3\,x$ and 5. The sum of the square roots is $3\,x + 5$ and the difference is $3\,x - 5$.

$$\therefore 9\,x^2 - 25 = (3\,x + 5)(3\,x - 5).$$

2. Factor $a^2 - (b - c)^2$.

Solution. The positive square roots of the squares are a and $b - c$. The sum of the square roots is $a + (b - c)$, and the difference is $a - (b - c)$; that is, $a + b - c$ and $a - b + c$.

$$\therefore a^2 - (b - c)^2 = (a + b - c)(a - b + c).$$

In practice, the work of factoring $a^2-(b-c)^2$ may be arranged thus:

$$a^2-(b-c)^2=[a+(b-c)][a-(b-c)]$$
$$=(a+b-c)(a-b+c).$$

3. Factor $a^2-\frac{4}{9}b^2$.

Solution. $a^2-\frac{4}{9}b^2=a^2-(\frac{2}{3}b)^2$
$$=(a+\tfrac{2}{3}b)(a-\tfrac{2}{3}b).$$

4. Factor a^4-b^4.

Solution. $a^4-b^4=(a^2)^2-(b^2)^2$
$$=(a^2+b^2)(a^2-b^2)$$
$$=(a^2+b^2)(a+b)(a-b).$$

EXERCISE 50

Factor:

1. m^2-n^2. **2.** x^2-y^2. **3.** R^2-r^2.

4. a^2-1. **5.** $1-z^2$. **6.** m^2-4.

7. $9-n^2$. **8.** $4x^2-y^2$. **9.** y^2-16z^2.

10. x^2-100. **11.** $25x^2-64y^2$. **12.** $9m^2-16n^2$.

13. m^4-n^2. **14.** x^6-y^2. **15.** $36x^4-49y^8$.

16. $25a^2b^4-36b^2a^4$. **17.** x^3-x.

18. x^4-4x^2. **19.** x^4-y^4.

20. $x^2-(y+z)^2$. **21.** $(m+n)^2-p^2$.

22. $(a+b)^2-9$. **23.** $x^2-\frac{9}{4}y^2$.

24. $(a+b)^2-(c+d)^2$. **25.** $1-(x-y)^2$.

26. $4(a-b)^2-9(a+b)^2$. **27.** $25(x-y)^2-36(x+y)^2$.

28. $a^4-(b-c)^4$. **29.** $9c^2-(a+b+c)^2$.

30. $a^2+2ac+c^2-b^2$. **31.** $x^2-1-2y-y^2$.

32. $x^2+10x+25-25y^2$. **33.** $1-m^2-2mn-n^2$.

107. Trinomials of the form x^2+cx+d. From section 96 we have,

$$(x+a)(x+b)=x^2+(a+b)x+ab; \text{ conversely,}$$
$$\boldsymbol{x^2+(a+b)x+ab=(x+a)(x+b)}.$$

From this identity we see that:

Any trinomial of the form $x^2 + cx + d$ can be factored when c, the coefficient of x, is the sum of two expressions, and d, the last term, is the product of the same two expressions.

Remark. When c and d are given integers and not too large, it is possible to determine by inspection whether two other integers a and b exist such that $a + b = c$ and $ab = d$. When two such integers are found the factors of $x^2 + cx + d$ are $(x + a)$ and $(x + b)$.

ILLUSTRATIVE EXAMPLES

1. Factor $x^2 + 7x + 12$.

Solution. The two integers whose sum is $+7$ and whose product is $+12$ are evidently $+3$ and $+4$.

$$\therefore x^2 + 7x + 12 = (x + 3)(x + 4). \qquad [\S\ 107]$$

2. Factor $x^2 + x - 12$.

Solution. The two integers whose sum is $+1$ and whose product is -12 are evidently $+4$ and -3.

$$\therefore x^2 + x - 12 = (x - 3)(x + 4). \qquad [\S\ 107]$$

3. Factor $x^2 - 9x + 20$.

Solution. The two integers whose sum is -9 and whose product is $+20$ are evidently -4 and -5.

$$\therefore x^2 - 9x + 20 = (x - 4)(x - 5). \qquad [\S\ 107]$$

4. Factor $x^2 + a(b - c)x - a^2bc$.

Solution. The two expressions whose sum is $ab - ac$ and whose product is $-a^2bc$ are evidently ab and $-ac$.

$$\therefore x^2 + a(b - c)x - a^2bc = (x + ab)(x - ac). \qquad [\S\ 107]$$

5. Factor $x^4 - 2x^2y^2 - 15y^4$.

Solution. The two monomials whose sum is $-2y^2$ and whose product is $-15y^4$ are $-5y^2$ and $3y^2$.

$$\therefore x^4 - 2x^2y^2 - 15y^4 = (x^2 - 5y^2)(x^2 + 3y^2). \qquad [\S\ 107]$$

EXERCISE 51

Factor the following:

1. $a^2 + 3a + 2$.
2. $y^2 + 2y - 3$.
3. $z^2 - z - 6$.
4. $b^2 - 6b + 5$.
5. $p^2 + 6p + 8$.
6. $c^2 - 7c + 12$.
7. $x^2 - 5x - 14$.
8. $a^2 - 4a - 45$.
9. $b^2 - 12b + 32$.
10. $z^2 + 13z + 30$.
11. $m^2 + 4m - 221$.
12. $x^2 + 18x + 72$.
13. $p^2 - 10p - 11$.
14. $x^2 - 10x + 24$.
15. $x^2 - 27x + 50$.
16. $a^2 - 18a + 80$.
17. $x^2 - 18x + 81$.
18. $a^2 + 8a - 209$.
19. $a^2 + 9a - 36$.
20. $x^2 - x - 2$.
21. $b^2 + 27b + 152$.
22. $m^2 - 16m + 55$.
23. $z^2 + 5z - 24$.
24. $x^2 + 10x - 39$.
25. $a^2 - 12a - 133$.
26. $m^2 - 28m + 171$.
27. $x^2 - 78x + 365$.
28. $x^2 + 10x - 119$.
29. $x^2 + 3x - 154$.
30. $z^2 + 6z - 91$.
31. $a^2 + a - 600$.
32. $l^2 + 24l + 23$.
33. $x^2 - 26x - 155$.
34. $a^2 - 18a - 19$.
35. $x^3y^3 + 4x^2y^2 - 5xy$.
36. $x^2 + (a + 3)x + 3a$.
37. $a^2 - (2 - b)a - 2b$.
38. $y^2 - (5 + z)y + 5z$.
39. $a^2 + x(y - z)a - x^2yz$.
40. $x^2 + 3x - ax - 3a$.
41. $x^2 - ax + 3cx - 3ac$.
42. $x^2 + 2bx - cx - 2bc$.
43. $a^2 + 3ab + 2b^2$.
44. $x^2 - 6xy^2 + 5y^4$.
45. $a^4 + 4a^2b - 221b^2$.
46. $x^4 + 8x^2y^3 + 15y^6$.
47. $x^2 - 2ax - 2bx + a^2 + 2ab$.
48. $x^2 + ax - 5x + 6 - 3a$.

108. The general quadratic trinomial $ax^2 + bx + c$. By actual multiplication we have

$$(px + q)(rx + s) = prx^2 + (ps + qr)x + qs;$$

conversely,

$$prx^2 + (ps + qr)x + qs = (px + q)(rx + s).$$

We observe in the first member of this identity that the coefficient of x is the sum of two terms $+ps$ and $+qr$ whose product is $+pqrs$ and that the product of the coefficient of x^2, namely pr, and the last term, namely qs, is also $+pqrs$. These facts furnish a clew which is of assistance in factoring a trinomial of the form $ax^2 + bx + c$ whenever it is possible to express b as the sum of two numbers whose product is equal to ac.

Note. In the foregoing, we have,

$$a = pr, \quad b = ps + qr, \quad c = qs.$$

Therefore, $\quad b^2 = p^2s^2 + 2pqrs + q^2r^2;$

and, $\quad 4ac = 4pqrs;$

hence, $\quad b^2 - 4ac = p^2s^2 - 2pqrs + q^2r^2 = (ps - qr)^2.$

If, therefore, in $ax^2 + bx + c$, the square of the coefficient of x *minus* four times the product of the coefficient of x^2 by the last term is not a perfect square, it is not possible to express the quadratic $ax^2 + bx + c$ as the product of two rational factors.

The converse of this statement (namely, that when $b^2 - 4ac$ is a perfect square, integral values of p, q, r, and s exist such that $a = pr$, $b = ps + qr$, $c = qs$) will be proved in a later chapter. (The letters a, b, c are assumed here to represent positive or negative integers, or integral expressions.)

ILLUSTRATIVE EXAMPLES

1. Factor $6x^2 + 19x + 10$.

Solution. If possible, we must find two integers whose sum is 19 and whose product is 6×10, or 60. These integers are evidently 15 and 4.

Therefore, $6x^2 + 19x + 10 = 6x^2 + 15x + 4x + 10$
$= 3x(2x + 5) + 2(2x + 5)$
$= (2x + 5)(3x + 2).$

That is, $6x^2 + 19x + 10 = (2x + 5)(3x + 2).$

2. Factor $10x^2 - 7x - 12.$

Solution. If possible, we must find two integers whose sum is -7 and whose product is $10(-12)$, or -120. Since their product is negative, one of these integers is positive and the other negative. Moreover, the larger integer is negative, since the sum of the two integers is negative, namely -7. The required integers are evidently -15 and $+8$, since $-15 + 8 = -7$, and $(-15)(8) = -120$.

$$\therefore 10x^2 - 7x - 12 = 10x^2 - 15x + 8x - 12$$
$$= 5x(2x - 3) + 4(2x - 3)$$
$$= (2x - 3)(5x + 4).$$

That is, $10x^2 - 7x - 12 = (2x - 3)(5x + 4).$

(In this example $a = 10$, $b = -7$, and $c = -12$. $b^2 - 4ac = 529 = \overline{23}^2$. Since $b^2 - 4ac$ is a perfect square, the given expression can be expressed as the product of two rational factors.)

3. Factor $acx^2 + (bc - a)x - b.$

Solution. If possible, we must find two expressions whose sum is $bc - a$ and whose product is $-abc$. The required expressions are evidently bc and $-a$. Hence,

$$acx^2 + (bc - a)x - b = acx^2 + bcx - ax - b$$
$$= cx(ax + b) - (ax + b)$$
$$= (ax + b)(cx - 1).$$

Note. See problem 29, exercise 48.

EXERCISE 52

Factor:

1. $6x^2 - x - 1.$

2. $14x^2 - 5x - 1.$

3. $6x^2 + x - 5.$

4. $14x^2 + x - 3.$

5. $10x^2 - 13x - 3.$

6. $9a^2 - 9a - 4.$

7. $5p^2 - 7p - 6.$

8. $6m^2 - 5m - 4.$

9. $6z^2 + 11z + 3.$

10. $8b^2 - 14b - 15.$

11. $2b^2 + 11b - 21.$

12. $14x^2 - 41x + 15.$

13. $6x^2 - x - 2$.

14. $4y^2 + 16y + 15$.

15. $6x^2 + 13x + 6$.

16. $20m^2 + 13m - 15$.

17. $4a^2 + 8a + 3$.

18. $6q^2 + 11q - 10$.

19. $15z^2 + 16z + 4$.

20. $8b^2 - 26b - 45$.

21. $10x^2 + 9x + 2$.

22. $10p^2 + 29p + 10$.

23. $9a^2 + 18a + 8$.

24. $28a^2 + 51a + 20$.

25. $6z^2 + 11z + 4$.

26. $6x^2 + 23x + 20$.

27. $21a^2 + 8a - 4$.

28. $12x^2 + 59x + 55$.

29. $16a^2 + 2a - 3$.

30. $16x^2 + 34x - 15$.

31. $24x^2 + 7x - 6$.

32. $20x^2 + 53x + 35$.

33. $22a^2 + 27a - 9$.

34. $6x^2 + (9 + 2a)x + 3a$.

35. $3x^2 + (a + 6)x + 2a$.

36. $6y^2 + (2a - 9)y - 3a$.

37. $acx^2 + (bc + 2a)x + 2b$.

38. $abx^2 + (a^2 - b^2)x - ab$.

39. $abx^2 + (a^2 + b^2)x + ab$.

40. $3cx^2 + (6 - 2c)x - 4$.

41. $6a^2x^2 - 5ax - 6$.

42. $6mnp^2 + (3m^2 + 2n^2)p + mn$.

109. Sum and difference of two cubes. By actual multiplication we have,

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

and

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3;$$

conversely,

$$(1)\quad \boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2).}$$

$$(2)\quad \boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2).}$$

By use of identities (1) and (2), the factors of any expression which has the form of the sum or the difference of two cubes may be found. It is obvious that *one factor of the sum of the cubes of two numbers is the sum of the numbers* and that *the other factor is a trinomial which is the sum of the squares of the two numbers minus their product;* also, that *one factor of the difference of the cubes of two numbers is the difference of the numbers* and that *the other factor is a trinomial which is the sum of the squares of the two numbers plus their product.*

ILLUSTRATIVE EXAMPLES

1. Factor $a^3 + 8$.

Solution. $$a^3 + 8 = a^3 + (2)^3$$
$$= (a + 2)(a^2 - 2a + 4). \quad [§ 109, 1]$$

2. Factor $a^3 - 216b^3$.

Solution. $$a^3 - 216b^3 = a^3 - (6b)^3$$
$$= (a - 6b)(a^2 + 6ab + 36b^2). \quad [§ 109, 2]$$

3. Factor $x^6 + y^9$.

Solution. $$x^6 + y^9 = (x^2)^3 + (y^3)^3$$
$$= (x^2 + y^3)[(x^2)^2 - (x^2)(y^3) + (y^3)^2]$$
$$= (x^2 + y^3)(x^4 - x^2y^3 + y^6).$$

That is, $$x^6 + y^9 = (x^2 + y^3)(x^4 - x^2y^3 + y^6).$$

4. Factor $x^6 - y^6$.

Solution. $$x^6 - y^6 = (x^3 - y^3)(x^3 + y^3) \quad [§ 106]$$
$$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2).$$

That is, $$x^6 - y^6 = (x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2).$$

EXERCISE 53

Factor:

1. $m^3 + n^3$. **2.** $m^3 - n^3$. **3.** $b^3 + c^3$.

4. $x^3 - 1$. **5.** $r^3 + 1$. **6.** $a^3 - 8$.

7. $1 - y^3$. **8.** $y^3 - x^3$. **9.** $x^3 + 27$.

10. $a^3 + 216b^3$. **11.** $8a^3 + 27b^3$. **12.** $27a^3 - 1$.

13. $8b^3 + 1$. **14.** $x^3 - \frac{1}{8}$. **15.** $27m^3 + 64n^3$.

16. $x^6 + 1$. **17.** $x^6 - 1$. **18.** $m^4 + m$.

19. $x^2 + x^5$. **20.** $ma^4 + mab^3$. **21.** $16r^3s + 2s^4$.

22. $x^6 + y^6$. **23.** $x^4 - 8x$. **24.** $a^3b^3 - c^3$.

25. $m^6 + 8.$ **26.** $m^6 - 8.$ **27.** $a^6 + 27.$

28. $m^3n^3 - a^3b^3.$ **29.** $r^6 - 27.$ **30.** $2m^3n + 128n.$

31. $(x-2)^3 + 1.$ **32.** $1-(x+y)^3.$

33. $(x+y)^3 + (x-y)^3.$

34. Give, at sight, one factor of $125 - (a+4)^3.$

35. Give, at sight, one factor of $(a+b)^3 - (a-b)^3.$

36. Give, at sight, one factor of $(2m-n)^3 + (m+2n)^3.$

37. By use of the factors of $x^6 - y^6$ [See Illustrative Example 4, p. 133], find four factors of 999,999.

Suggestion. $999,999 = 10^6 - 1.$

Factor:

38. $1 + \frac{1}{64}y^3.$ **39.** $(a-b) + (a^3 - b^3).$

40. $a + b + a^3 + b^3.$ **41.** $2(p+q) + p^3 + q^3.$

42. $(1+x) + 3(1+x^3).$ **43.** $(a+b-c)^3 - (a-b+c)^3.$

44. $m^5n^4 + m^2n.$ **45.** $1 + (p+q-1)^3.$

46. $m^3 - mn(m+n) + n^3.$

Suggestion. $m^3 - mn(m+n) + n^3 = (m^3 + n^3) - mn(m+n).$

47. $m^3 - 2m^2n + 2mn^2 - n^3.$ **48.** $2m^3 - 12m^2 + 24m - 16.$

49. $a^3 + \frac{3}{2}a^2b + \frac{3}{4}ab^2 + \frac{1}{8}b^3.$ **50.** $a^3 + a^2 + \frac{1}{3}a + \frac{1}{27}.$

110. Special methods of factoring. There are certain integral expressions whose factors may be found, and which are not classed under any of the preceding general cases of factoring.

ILLUSTRATIVE EXAMPLES

1. Factor $a^4 + a^2 + 1.$

Solution.
$$a^4 + a^2 + 1 = a^4 + 2a^2 + 1 - a^2$$
$$= (a^2+1)^2 - a^2$$
$$= (a^2 + 1 - a)(a^2 + 1 + a).$$

That is, $a^4 + a^2 + 1 = (a^2 + a + 1)(a^2 - a + 1).$

Remark. The solution of example 1 illustrates the method of factoring by the aid of adding and subtracting the same number.

2. Factor $a^2+b^2+4c^2-4bc+4ca-2ab$.

Solution.
$$\begin{aligned} a^2+b^2+4c^2-4bc+4ca-2ab &= a^2+2a(2c-b)+(4c^2-4bc+b^2)\\ &= a^2+2a(2c-b)+(2c-b)^2\\ &=(a+2c-b)^2. \qquad [\S\,105]\end{aligned}$$

3. Factor $bc(b-c)+ca(c-a)+ab(a-b)$.

Solution. In order to arrange this expression according to the powers of a, it is necessary to perform the indicated multiplications in the last two terms. Then we have,

$$\begin{aligned} bc(b-c)+ca(c-a)+ab(a-b) &= bc(b-c)+c^2a-ca^2+a^2b-ab^2\\ &=(b-c)a^2-(b^2-c^2)a+bc(b-c)\\ &=(b-c)[a^2-(b+c)a+bc]\\ &=(b-c)(a-b)(a-c).\end{aligned}$$

Remark. Many expressions when arranged according to the powers of some letter are seen to be factorable. The solutions of examples 2 and 3 illustrate the method of factoring such expressions.

EXERCISE 54

Factor the following:

1. m^4+m^2+1.

2. $m^4+m^2n^2+n^4$.

3. $1+9x^2+81x^4$.

4. $m^8+m^4n^4+n^8$.

5. $p^8+p^4q^2+q^4$.

6. $x^4+x^2y^4+y^8$.

7. x^8+x^4+1.

8. $16y^4+4y^2+1$.

9. $x^4-11x^2y^2+y^4$.

10. $a^4-27a^2b^2+b^4$.

11. m^4-123m^2+1.

12. $x^4+y^4-7x^2y^2$.

13. a^4-34a^2+1.

14. $p^4-14p^2q^2+q^4$.

15. $x^2+4y^2+z^2+4yz-2zx-4xy$.

16. $yz(y-z)+zx(z-x)+xy(x-y)$.

17. $x^2+y^2+1+2y+2x+2xy$.

18. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

19. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Suggestion. Arrange according to powers of a, remove a factor, then arrange the remaining factor according to powers of b, remove a second factor, finally arrange the remaining factor according to powers of c.

20. $9a^4-37a^2b^2+4b^4$.

21. $4x^4-21x^2y^2+9y^4$.

22. $16a^4+36a^2b^2+81b^4$.

23. $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$.

24. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$.

25. $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$.

111. Summary of factoring. The identities, rules, and solutions of illustrative examples as given in this chapter are sufficient to cover the simple cases of factoring which occur in elementary algebra. The following summary will be of assistance in the work of factoring.

I. *As the first step in factoring an integral expression remove all numerical and monomial literal factors.*

II. *In factoring a binomial use one of these identities:*

1. $a^2-b^2=(a-b)(a+b)$.
2. $a^3-b^3=(a-b)(a^2+ab+b^2)$.
3. $a^3+b^3=(a+b)(a^2-ab+b^2)$.

III. *In factoring a trinomial use one of these identities:*

1. $a^2+2ab+b^2=(a+b)^2$.
2. $a^2-2ab+b^2=(a-b)^2$.
3. $x^2+(a+b)x+ab=(x+a)(x+b)$.
4. $ax^2+bx+c=(px+q)(rx+s)$.
5. $a^4+a^2b^2+b^4=(a^2+b^2)^2-(ab)^2$.

IV. *In factoring a polynomial be guided by one or more of the following directions:*

1. *Rearrange and group terms.*

2. *Consider the polynomial the difference of two squares.*

3. *Arrange the polynomial according to the powers of some one letter.*

4. *Consider the polynomial the square of a polynomial.*

EXERCISE 55 — REVIEW

Find the factors of:

1. $2x+2$.
2. x^3+5x^2.
3. $ax+ay-bx-by$.
4. $ab+bc$.
5. $3x(a-b)-2y(b-a)$.
6. p^2-64.
7. $mn^2(x-y)+2m(x-y)n+m(x-y)$.
8. x^3+2x^2-x-2.
9. x^3+3x^2-x-3.
10. $x^2-2x-15$.
11. x^4-a^2.
12. $64a^2+144ab+81b^2$.
13. $9x^2-y^4$.
14. $a^{2n}-b^2$.
15. $1+ac-bd-abcd$.
16. $(x^2+y^2)z+(y^2+z^2)x$.
17. $n(n+1)(n+2)+(n+1)(n+2)(n+3)$.
18. $2x^2+2ax+2ac+2cx$.
19. $2ax-3ay-2bx+3by$
20. $am-2bn+an-2bm$.
21. $(a+1)^2-3(a+1)$.
22. $a^2-2(a-b)-b^2$.
23. $a^3-2(a-b)-b^3$.
24. $a^3+\frac{1}{27}$.
25. $(m+n)^2-p^2$.
26. $49a^2b^2c^2-64a^2d^2$.
27. $49a^2b^2c^2-64a^2b^2d^2$.
28. $25a^2+10a+1$.
29. $3a^2-7ab+4b^2$.
30. $x^2-12x+36$.
31. $p^2x^4+23px^2+130$.
32. x^2+2x-8.
33. $12m^4-7pm^2+p^2$.
34. $8x^3+27$.
35. $4m^2n^2-(m^2+n^2-p^2)^2$.

36. $8x^6 + 63x^3 - 8.$

37. $a^{10} - a^6.$

38. $ab + b^2 + a - 1.$

39. $ab - b^2 - a + 1.$

40. $x^3 + 5x^2 + 5x + 1.$

41. $a^2 - 4a(x+y) + 3(x+y)^2.$

42. $(a^2 - 2a)^2 + 2(a^2 - 2a) + 1.$

43. $a^2(a+1)^2 - (a+1)^2(a+2)^2.$

44. $(a-b)^2 - (b-a)(a+c).$

45. $(a-b)(a+c) - (b-a)(b+c).$

46. $n^8 + n^4 + 1.$

47. $a^3 - 3a^2b - 6ab^2 + 8b^3.$

48. $x^2 + 68x + 1092.$

49. $12a^2 - a - 6.$

50. $4a^4 - 20a^2b^2 + b^4.$

51. $x^4 - 19x^2y^2 + 9y^4.$

52. $1 - 14x - 51x^2.$

53. $x^2 + 7xy - 30y^2.$

54. $7x^2 - 31xy + 12y^2.$

55. $10a^2 - 29ab + 21b^2.$

56. $(2x-y)^3 - (x+y)^3.$

57. $a^4 + 4b^4.$

58. $216 - (x+3)^3.$

59. $a^2 + 2ac - b^2 - 2bc.$

60. $c^2 + 2ab - a^2 - b^2.$

61. $a^6 + 64c^6.$

62. $m^4 - p^2m^2 - n^2m^2 + p^2n^2.$

63. $14x^2 + 11xy - 15y^2.$

64. $x^4y^2 - x^2 - x^2y^2 + 1.$

65. $m^{16} + m^8 + 1.$

66. $2(x-3)^2 + 3(x-3) - 2.$

67. $a(x-y)^2 - c(y-x).$

68. $(2a - 3b)^2 + 11(2a - 3b) + 30.$

69. $ab^n - cb^{n+2} + db^{n+1}.$

70. $ax^2 + axy + xz + bxy + by^2 + yz.$

71. $a^2b^2 - a^2c^2 - b^2c^2 + c^4.$

72. $m(m+p) - n(n+p).$

73. $4a^2b^2 - (a^2 + b^2 - c^2)^2.$

74. $a^2(a+b+c+d) + (bcd + cda + dab + abc).$

75. $a^2 + b^2 + 1 + 2a + 2b + 2ab.$

76. $\frac{1}{4}a^2 + \frac{1}{9}b^2 + \frac{1}{25}c^2 - \frac{1}{3}ab - \frac{2}{15}bc + \frac{1}{5}ca.$

77. $a^2b^2 + 4\,b^2c^2 + 16\,c^2a^2 - 4\,ab^2c - 16\,bc^2a + 8\,ca^2b$.

78. $xy^2z - yz^2x + 2(y - z)$.

79. $2\,ax^2 - (3\,a + 2)x + (a + 2)$.

80. $3\,ax^2 + (2\,a - 5)x - 5(a - 1)$.

112. The remainder theorem. When a polynomial in x is divided by a binomial $x - a$, the remainder may be found by substituting a for x in the polynomial.

Thus, when $2\,x^3 - 3\,x^2 + 2\,x - 5$ is divided by $x - 2$, the remainder is the value of $2\,x^3 - 3\,x^2 + 2\,x - 5$ when 2 is substituted for x; namely,

$$2 \cdot 2^3 - 3 \cdot 2^2 + 2 \cdot 2 - 5 \text{ or } 16 - 12 + 4 - 5, \text{ which is } 3.$$

Similarly, when $3\,x^4 + 2\,x - 5$ is divided by $x - 1$, the remainder is $3 \cdot 1^4 + 2 \cdot 1 - 5$, which is 0. Hence, $3\,x^4 + 2\,x - 5$ is exactly divisible by $x - 1$.

The proof in the case of the first of the foregoing examples is as follows:

We know that $2\,x^3 - 3\,x^2 + 2\,x - 5 = (x - 2)Q + R$, where Q and R represent, respectively, the quotient and remainder when $2\,x^3 - 3\,x^2 + 2\,x - 5$ is divided by $(x - 2)$. Since one member of this equation is the same polynomial as the other, the two members are equal for all values of x. Substituting 2 for x, we have $2 \cdot 2^3 - 3 \cdot 2^2 + 2 \cdot 2 - 5 = (2 - 2)Q + R$ or, since $(2 - 2)Q = 0 \cdot Q = 0$, $2^3 - 3 \cdot 2^2 + 2 \cdot 2 - 5 = R$. A precisely similar proof holds when any polynomial in x is divided by $x - a$.

ILLUSTRATIVE EXAMPLES

1. Find the remainder when $x^2 + 3\,x + 5$ is divided by $x - 3$.

Solution. $R = 3^2 + 3 \cdot 3 + 5 = 23$

2. Find the remainder when $x^3 + 3\,x^2 - 5\,x + 3$ is divided by $x + 2$.

Solution. Since $x + 2 = x - (-2)$, we substitute -2 for x in the polynomial; hence

$$R = (-2)^3 + 3(-2)^2 - 5(-2) + 3 = 17.$$

3. Find the remainder when $2x^3 - 3x + 7$ is divided by x.

Solution. Since the divisor may be written $x - 0$, we have

$$R = 2 \cdot 0 - 3 \cdot 0 + 7 = 7.$$

4. Find the remainder when

$x^3 - (a+2)x^2 + (2a-3)x + 4a$ is divided by $x - a$.

Solution. $R = a^3 - (a+2)a^2 + (2a-3)a + 4a$
$= a^3 - a^3 - 2a^2 + 2a^2 - 3a + 4a = a.$

5. Find the remainder when

$a^2(b-c) + b^2(c-a) + c^2(a-b)$ is divided by $a - b$.

Solution. The given expression may be regarded as a polynomial in a. Hence, the remainder is the value of the polynomial when b is substituted for a. Thus,

$$\begin{aligned} R &= b^2(b-c) + b^2(c-b) + c^2(b-b) \\ &= b^3 - b^2c + b^2c - b^3 + 0 \\ &= 0. \end{aligned}$$

Since the remainder is zero, the polynomial is exactly divisible by $a - b$.

113. The remainder theorem in factoring. The linear factors of a polynomial can often be found by an application of the remainder theorem. In order that a polynomial in x should have $x - a$ as one of its factors, it is sufficient that the polynomial should vanish, that is, should become equal to zero, when a is substituted for x. This is evident since, by the remainder theorem, when the polynomial is divided by $x - a$, the remainder is zero, if the polynomial vanishes when a is substituted for x.

ILLUSTRATIVE EXAMPLES

1. Find the factors of $x^3 + 2x^2 - 5x - 6$.

Solution. If $x - a$, where a is an integer, is a factor of $x^3 + 2x^2 - 5x - 6$, it is necessary that the last term -6 should be divisible by a. We therefore substitute in turn the different factors of -6, namely, $1, -1, 2, -2, 3, -3, 6, -6$, in the polynomial.

When $x = 1$, $x^3 + 2x^2 - 5x - 6$ becomes $1 + 2 - 5 - 6 = -8$.

When $x = -1$, $x^3 + 2x^2 - 5x - 6$ becomes $-1 + 2 + 5 - 6 = 0$.

When $x = 2$, $x^3 + 2x^2 - 5x - 6$ becomes $2^3 + 2 \cdot 2^2 - 5 \cdot 2 - 6 = 0$.

When $x = -2$, $x^3 + 2x^2 - 5x - 6$ becomes $(-2)^3 + 2(-2)^2 - 5(-2) - 6 = 4$.

When $x = 3$, $x^3 + 2x^2 - 5x - 6$ becomes $3^3 + 2 \cdot 3^2 - 5 \cdot 3 - 6 = 24$.

When $x = -3$, $x^3 + 2x^2 - 5x - 6$ becomes $(-3)^3 + 2(-3)^2 - 5(-3) - 6 = 0$.

Hence, when $x^3 + 2x^2 - 5x - 6$ is divided by $x - (-1)$, by $x - 2$ and by $x - (-3)$, the remainder is 0, and therefore the factors of $x^3 + 2x^2 - 5x - 6$ are $x + 1$, $x - 2$, and $x + 3$.

Question. Why is it unnecessary to substitute 6 and -6 for x in the given expression?

2. Show that $x^n - y^n$, where n is a positive integer, is exactly divisible by $x - y$.

Solution. When $x^n - y^n$ is divided by $x - y$, the remainder, R, is the value of $x^n - y^n$ when y is substituted for x.

Hence, $$R = y^n - y^n = 0.$$

3. When is $x^n + y^n$ exactly divisible by $x + y$?

Solution. $R = (-y)^n + y^n$, which is 0 or $2y^n$, according as n is an odd or an even positive integer. Therefore, when n is an odd positive integer, $x^n + y^n$ is exactly divisible by $x + y$ and when n is an even integer it is not exactly divisible by $x + y$.

4. Show that $a^3 + b^3 + c^3 - 3abc$ is exactly divisible by $a + b + c$.

Solution. Since $a + b + c = a - (-b - c)$ the remainder, R, of the division is the value of $a^3 + b^3 + c^3 - 3abc$ when $-b - c$ is substituted for a.

Hence,
$$\begin{aligned} R &= -(b + c)^3 + b^3 + c^3 + 3(b + c)bc \\ &= -b^3 - 3b^2c - 3bc^2 - c^3 + b^3 + c^3 + 3b^2c + 3bc^2 \\ &= 0. \end{aligned}$$

Therefore, the division is exact.

EXERCISE 56

Factor:

1. $x^3 - x^2 - x + 1$.
2. $x^3 - 3x + 2$.
3. $x^3 + x^2 - x - 1$.
4. $x^3 - 3x - 2$.
5. $x^3 - 6x^2 + 11x - 6$.
6. $x^3 + 5x^2 - x - 5$.
7. $2x^3 - x^2 - 5x - 2$.

Suggestion. When two factors have been found, the third factor may be obtained by dividing the given expression by the product of the two known factors.

8. $3x^3 - 2x^2 - 19x - 6$.
9. $2x^3 + x^2 - 13x + 6$.
10. $x^4 + 5x^3 + 5x^2 - 5x - 6$.
11. Show that $x^{2n} - y^{2n}$, where n is a positive integer, is exactly divisible by $x + y$.

Equations Solved by Factoring

114. Rational and integral equations in one unknown number. By transposition, all the terms of an equation can be brought to one member of the equation. The other member then is zero. An equation which contains one and only one unknown number is said to be a rational and integral equation in one unknown, provided that, when all the terms are written in one member, the polynomial which occurs in that member is rational and integral with respect to the unknown.

Thus, $3x^3 - \frac{3}{4}x^2 + \frac{1}{2}x - \frac{1}{4} = 0$ is a rational integral equation.

Remark. When the word *equation* is used in this chapter, it will be understood to refer to a rational integral equation in one unknown.

115. Degree of an equation. Any equation may be put in the form $A = 0$, in which A represents a rational and integral expression with respect to the unknown. Equations are classified according to the degree of the expression, represented by A, with respect to the unknown number. For example,

$ax + b = 0$ is a *linear equation*, or an equation of the *first degree.*

$ax^2 + bx + c = 0$ is a *quadratic equation*, or an equation of the *second degree.*

$ax^3 + bx^2 + cx + d = 0$ is a *cubic equation*, or an equation of the *third degree.*

$ax^4 + bx^3 + cx^2 + dx + e = 0$ is an equation of the *fourth degree.* [§ 85, Remark.]

116. Solution of an equation. An equation in one unknown is said to be solved when all of its roots have been found.

117. Roots of an equation found by factoring. If one member of an equation is zero, the roots of the equation may be found easily, provided that the polynomial in the other member can be expressed as a product of factors, each one of which is of the first degree in the unknown number. This important method of solving an equation is applied and explained in the illustrative examples which follow.

ILLUSTRATIVE EXAMPLES

1. Solve the equation $3x^2 + 5x = 2x^2 - 3x + 33$.

Solution. $3x^2 + 5x = 2x^2 - 3x + 33$. (1)

Transposing, $3x^2 + 5x - 2x^2 + 3x - 33 = 0$. (2)

Combining, $x^2 + 8x - 33 = 0$. (3)

Factoring, $(x - 3)(x + 11) = 0$. (4)

Notice here, that the first member of equation (4) is a product of factors and that a product cannot be equal to zero unless at least one of its factors is zero. Hence, any value of x which satisfies equation (4) must cause at least one of the factors of the first member to vanish. Moreover, any value of x which causes either of the factors to vanish will satisfy the equation. Therefore, the required roots are found by equating each of the factors to zero.

Therefore, $x - 3 = 0.$ (5)
Whence, $x = 3.$ (6)
Also, $x + 11 = 0.$ (7)
Whence, $x = -11.$ (8)
That is, the roots of $3x^2 + 5x = 2x^2 - 3x + 33$ are 3 and -11

2. Solve the equation $3x^3 - 2x^2 - 3x + 2 = 0.$

Solution. $3x^3 - 2x^2 - 3x + 2 = 0.$ (1)
Grouping terms, $(3x^3 - 2x^2) - (3x - 2) = 0.$ (2)
Factoring first term of (2), $x^2(3x - 2) - (3x - 2) = 0.$ (3)
Factoring, $(3x - 2)(x^2 - 1) = 0.$ (4)
Factoring completely, $(3x - 2)(x - 1)(x + 1) = 0.$ (5)
Equating $(3x - 2)$ to 0, $3x - 2 = 0.$ (6)
Equating $(x - 1)$ to 0, $x - 1 = 0.$ (7)
Equating $(x + 1)$ to 0, $x + 1 = 0.$ (8)
Solving (6), $x = \frac{2}{3}.$ (9)
Solving (7), $x = 1.$ (10)
Solving (8), $x = -1.$ (11)
That is, the roots of $3x^3 - 2x^2 - 3x + 2$ are 1, -1, and $\frac{2}{3}$.

3. Solve the equation $x^4 - 8x^2 + 16 = 0.$

Solution. $x^4 - 8x^2 + 16 = 0.$ (1)
That is, $(x^2 - 4)^2 = 0.$ (2)
Factoring, $[(x - 2)(x + 2)]^2 = 0.$ (3)
That is, $(x - 2)(x - 2)(x + 2)(x + 2) = 0.$ (4)
Equating each factor to 0, $x-2=0,\ x-2 = 0,\ x+2 = 0,\ x+2=0.$
Solving simple equations, $x = 2,\quad x = 2,\quad x = -2,\quad x = -2.$
That is, the roots of $x^4 - 8x^2 + 16 = 0$ are 2, 2, -2, and -2.

Note. The equation has four roots, two pairs of equal roots; that is, as many roots as there are linear factors.

Remark. The student should carefully check all roots obtained from the solutions of illustrative examples 1, 2, and 3.

4. Solve $6x^3 - 11x^2 - 35x = 0.$

Solution. $6x^3 - 11x^2 - 35x = 0.$
Factoring, $x(3x + 5)(2x - 7) = 0.$
Equating each factor to 0, $x = 0,\ 3x + 5 = 0,\ 2x - 7 = 0.$
Solving simple equations, $x = 0,\ x = -\frac{5}{3},\ x = \frac{7}{2}.$
That is, the roots of $6x^3 - 11x^2 - 35x = 0$ are 0, $-\frac{5}{3}$, and $\frac{7}{2}$.

From the solutions of the foregoing illustrative examples, the following rule for solving an equation by factoring may be inferred:

Rule. *Transpose all the terms to one member of the equation, factor the resulting expression into its linear factors, equate each factor to zero, and solve the resulting simple equations.*

Note. In solving equations by factoring, care should be exercised to bring all terms to one member of the equation. The following is an example of an error which is the direct result of disregarding this practice.

Solve the equation $(2x+3)(x-1)=(x+2)(x-1)$.

Incorrect Solution. $(2x+3)(x-1)=(x+2)(x-1)$. (1)

Dividing both members of (1) by $(x-1)$,

$$2x+3=x+2. \quad (2)$$

Transposing and combining, $x+1=0$. (3)

Solving, $x=-1$. (4)

Correct Solution. $(2x+3)(x-1)=(x+2)(x-1)$.

Transposing,

$$(2x+3)(x-1)-(x+2)(x-1)=0. \quad (1)$$

Factoring, $(x-1)[(2x+3)-(x+2)]=0$. (2)

Simplifying, $(x-1)(x+1)=0$. (3)

Equating each factor to 0, $x-1=0$, $x+1=0$.

Solving simple equations, $x=1$, $x=-1$.

That is, the roots of $(2x+3)(x-1)=(x+2)(x-1)$ are $+1$ and -1.

The error in the incorrect solution arises from dividing both members of equation (1) by a factor which contains the unknown number, and which vanishes when x has the value 1.

The equations $(2x+3)(x-1)=(x+2)(x-1)$ and $2x+3=x+2$ are not equivalent, the first equation having a root which is not a root of the second. In solving equations, *every transformed equation or set of equations which occurs in the solution must be equivalent to the original equation* [§ 74].

EXERCISE 57

Solve the following equations and check the roots:

1. $x(x-2)=0$.
2. $x^2(3x+2)=0$.
3. $(2x+1)(3x-1)=0$.
4. $x^2-2x+1=0$.
5. $x^2-3x+2=0$.
6. $x^2-9x+20=0$.
7. $x^2+2x-3=0$.
8. $x^2+3x-10=0$.
9. $x^2-13x+42=0$.
10. $x^2-6x-55=0$.
11. $x^2-5x+6=0$.
12. $x^2-4x-21=0$.
13. $x^2+x-30=0$.
14. $x^2-7x+10=0$.
15. $x^2-14x-15=0$.
16. $2x^2-3x-2=0$.
17. $3x^2+2x-8=0$.
18. $4x^2-3x-85=0$.
19. $3x^2-5x-12=0$.
20. $4x^2-3x-45=0$.
21. $5x^2+x-6=0$.
22. $4x^2-7x-147=0$.
23. $7x^2-5x-78=0$.
24. $11x^2-13x+2=0$.
25. $15x^2+2x-56=0$.
26. $13x^2-9x-414=0$.
27. $3x^2+2x+5=5x^2-3x-2$.
28. $x(x+1)=(2x+1)(x+1)$.
29. $x(x+1)(x+2)=x(2x+3)(x+1)$.
30. $(2x^2-3x+1)^2-(x^2-1)^2=0$.

Highest Common Factor—Lowest Common Multiple

118. Highest common factor. The **highest common factor (H. C. F.)** of two or more integral expressions is the integral expression of the highest degree, with greatest numerical coefficient, which exactly divides each of them.

Thus, the H. C. F. of $4a^2b^3$ and $6a^3b^2$ is evidently $2a^2b^2$.

119. Greatest common divisor (G. C. D) in arithmetic. In arithmetic, the greatest common divisor of two or more numbers may be found by expressing each of them as the product of powers of its different prime factors, and then taking the product of the common prime factors of the numbers, giving to each common prime factor the least exponent which it has in any of the numbers.

ILLUSTRATIVE EXAMPLE

Find the greatest common divisor of 180, 252, and 270.

Solution. $180 = 2^2 \times 3^2 \times 5$

$252 = 3^2 \times 7 \times 2^2$

$270 = 2 \times 3^3 \times 5$

G. C. D. of 180, 252, and $270 = 2 \times 3^2$, or 18.

120. Highest common factor of monomials. The highest common factor of two or more literal monomials can, obviously, be found by inspection.

ILLUSTRATIVE EXAMPLE

Find the highest common factor of $12\,ab^2c^3$, $18\,a^2b^3c^3$, and $24\,a^3b^4c$.

The greatest number which will exactly divide 12, 18, and 24 is 6.
The highest power of a which will exactly divide a, a^2, and a^3 is a.
The highest power of b which will exactly divide b^2, b^3, and b^4 is b^2.
The highest power of c which will exactly divide c^3 and c is c.
Evidently, the required H. C. F. is $6\,ab^2c$.

From the above illustration, we have the following:

Rule. *To find the highest common factor of two or more monomials, multiply the product of the lowest powers of their common literal factors by the greatest common divisor of their numerical coefficients.*

Note. The numerical coefficient in the highest common factor is taken as positive.

Thus, the H. C. F. of $-4\,a^2b$ and $6\,ab^2$ is regarded as $2\,ab$ and not $-2\,ab$.

Remark. When the greatest common divisor of the numerical coefficients of two or more monomials cannot be readily seen, it may be found as in section 119.

EXERCISE 58

Find by inspection the highest common factor of:

1. a^3b^5 and ab^4.
2. $3x^2yz$ and $12x^3y$.
3. $6a^4b^2c$ and $4a^3b^2$.
4. a^4c, a^3c^2, and a^2bc^3.
5. $2a^3b^2c$, $4a^2bc^3$, and $6ab^3$.
6. $(x+y)z$ and $(x-y)z$.
7. $-2(a+b)xy$ and $-2(a+b)xz$.
8. $(a+b)^2c^3$ and $(a+b)^3c^2$.
9. $2a^2b(c+d)^2$, $4ab^2(c+d)$, and $10ab(c+d)^3$.
10. $x(x-1)(x-2)$, $x(x+1)(x-1)$, and $3x(x-1)(x-2)$.

121. Highest common factor of polynomials by factoring. Expressions which are completely factored; *i.e.* each of which is expressed as a product of powers of its prime factors, are in the form of monomials, and their highest common factor may be found by inspection, as in exercise 58.

ILLUSTRATIVE EXAMPLE

Find the highest common factor of $18x^2+15x-18$ and $36x^2+78x+36$.

Solution.
$$18x^2+15x-18=3(2x+3)(3x-2)$$
$$36x^2+78x+36=6(2x+3)(3x+2).$$

Therefore, the required highest common factor is the H. C. F. of $3(2x+3)(3x-2)$ and $6(2x+3)(3x+2)$, which is evidently $3(2x+3)$.

Therefore, *to find the highest common factor of two or more polynomials which can be readily factored*, the method of procedure is to *factor each polynomial completely*, thus changing each into the form of a monomial, *and then to find the highest common factor of the monomials by inspection.*

EXERCISE 59

Find the highest common factor of:

1. $(2x+4)(x+2)$ and $(3x+6)(x+2)$.
2. x^2+3x and $2x+6$.

3. $x^2 + 2x + 1$ and $x^2 - 3x - 4$.

4. $x^2 + 3x - 10$ and $x^2 - 5x + 6$.

5. $a^2b + ab^2$ and $a^3 + a^2b$.

6. $x^4 - x^2$ and $x^6 + x^3$.

7. $a^3 + 8b^3$ and $2a^2 + 4ab$.

8. $x^3 - y^3$ and $x^4 + x^2 + 1$.

9. $a^3 + b^3$ and $(2a + 3b)^2 - (a + 2b)^2$.

10. $3(a + b)$, $6a^2 + 6ab$, and $2a^3 + 2a^2b$.

11. $ax - ay + bx - by$ and $ax + bx + by + ay$.

12. $x^2 - 9$, $x^2 - 6x + 9$, and $2x^2 - 5x - 3$.

13. $1155a^2b$, $910ab^2$, and $595ab$.

14. $a^2 - b^2 - c^2 + 2bc$ and $a^2 - b^2 + c^2 + 2ac$.

15. $4x^2 + 12x + 9$, $4x^2 - 9$, and $6x^3 + 13x^2 + 6x$.

16. $a^2 + b^2 + c^2 - 2bc - 2ca + 2ab$ and $a^2 + b^2 - c^2 + 2ab$.

17. $x^3 - 2x^2 + x - 2$ and $x^3 - 3x^2 + x - 3$.

18. $x^2 - 3x + 2$ and $x^3 + 5x^2 - 3x - 3$.

Suggestion. The first expression is the product of two factors. Find whether the second expression is exactly divisible by one or both of these factors.

19. $x^2 + 4x - 5$ and $x^3 + 3x^2 - 9x + 5$.

20. $x^2 - x - 6$ and $x^3 + x^2 - 9x - 9$.

122. Lowest common multiple. The **lowest common multiple (L. C. M.)** of two or more integral expressions, the numerical coefficients of which are integers, is the integral expression of lowest degree with least numerical coefficient which is exactly divisible by each of them.

Thus, the L. C. M. of $2a^3b^4$ and $4a^2b^3$ is evidently $4a^3b^4$.

123. Least common multiple (L. C. M.) in arithmetic. In arithmetic the least common multiple of two or more numbers may be found by expressing each of them as the product of powers of its different prime factors and taking the product of all the different prime factors of the numbers, giving to each different prime factor the greatest exponent which it has in any of the numbers.

ILLUSTRATIVE EXAMPLE

Find the least common multiple of 90, 189, and 300.

Solution.

$$90 = 2 \times 3^2 \times 5$$
$$189 = 3^3 \times 7$$
$$300 = 2^2 \times 3 \times 5^2$$

L. C. M. of 90, 189, and 300 $= 2^2 \times 3^3 \times 5^2 \times 7$, or 18,900.

124. Lowest common multiple of monomials. The lowest common multiple of two or more literal monomials can, obviously, be found by inspection.

ILLUSTRATIVE EXAMPLE

Find the lowest common multiple of $6\,a^2bc^2$, $8\,abc^2$, and $12\,a^3b^3c^3$.

Solution.

By inspection it is readily seen that:
The least number which will contain 6, 8, and 12 is 24.
The lowest power of a which will contain a^2, a, and a^3 is a^3.
The lowest power of b which will contain b and b^3 is b^3.
The lowest power of c which will contain c^2 and c^3 is c^3.
Evidently, the required L. C. M. is $24\,a^3b^3c^3$.

From the above illustration, we have the following:

Rule. *To find the lowest common multiple of two or more monomials, multiply the product of the highest powers of their different literal factors by the least common multiple of their numerical coefficients.*

Remark. When the lowest common multiple of the numerical coefficients of two or more monomials cannot be readily seen, it may be found as in section 123.

EXERCISE 60

Find, at sight, the lowest common multiple of:

1. $2a^2b$ and $3ac^2$.
2. abc, $3a^2c$, and $5ab^2$.
3. $5x$, $3y$, and $2z$.
4. $27x^2y$, $81y^2z$, and $2z^2x$.
5. $6x^2yz$, $15x^3z$, and $18xyz^2$.
6. $2m^2n^3p^5$, $3a^2mp$, and $6abc$.
7. $x(x-1)$ and $y(x-1)$.
8. $x^2(x-1)$ and $xy(x-1)^2$.
9. $x^2(x-1)^2(x+3)$ and $y^2(y-1)^2(x+3)$.
10. $17x^2y^2(x+y)^2$ and $10x^3y(x+y)^3$.

125. Lowest common multiple of polynomials by factoring. Expressions which are completely factored, that is, each of which is expressed as a product of powers of its prime factors, are in the form of monomials, and their lowest common multiple may be found by the rule of section 124.

ILLUSTRATIVE EXAMPLES

1. Find the lowest common multiple of x^2+4x and $3x+12$.

Solution.
$$x^2+4x = x(x+4).$$
$$3x+12 = 3(x+4).$$

Therefore, the required L. C. M. = the L. C. M. of $x(x+4)$ and $3(x+4)$, or $3x(x+4)$.

2. Find the lowest common multiple of $3x^2-27x+60$, x^3-5x^2+x-5, and x^3-4x^2+x-4.

Solution.
$$3x^2-27x+60 = 3(x-4)(x-5).$$
$$x^3-5x^2+x-5 = (x-5)(x^2+1).$$
$$x^3-4x^2+x-4 = (x-4)(x^2+1).$$

Therefore, the required L. C. M. = the L. C. M. of $3(x-4)(x-5)$, $(x-5)(x^2+1)$, and $(x-4)(x^2+1)$, or $3(x-4)(x-5)(x^2+1)$.

3. Find the lowest common multiple of x^3+x^2-6x and $x^3-6x^2+11x-6$.

Solution. $x^3+x^2-6x=x(x+3)(x-2)$.

It is now necessary to find whether or not $x^3-6x^2+11x-6$ is exactly divisible by any of the factors of x^3+x^2-6x. By actual division $x^3-6x^2+11x-6$ is found to have $x-2$ as a factor; hence,

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3).$$

Therefore, the required L. C. M. = the L. C. M. of $x(x+3)(x-2)$ and $(x-1)(x-2)(x-3)$, or $x(x-1)(x-2)(x-3)(x+3)$.

EXERCISE 61

Find by factoring the lowest common multiple of:

1. a^2b+ab^2 and a^3-a^2b.
2. a^4-a^3 and a^2-a.
3. x^2+6x and $4x+24$.
4. $(2x+4)(x+2)$ and $(3x+6)(x+2)$.
5. a^2-b^2 and a^3-b^3.
6. x^2-1 and x^4-1.
7. a^3+b^3 and $a^4+a^2b^2+b^4$.
8. a^2-b^2 and $(a-b)(a^2+b^2)$.
9. x^2+5x-6 and x^2-3x+2.
10. $x^2+5x-14$ and x^2-5x+6.
11. x^2-y^2, $x^2+xy-2y^2$ and $x^2+3xy+2y^2$.
12. $12x^2-18xy+45y^2$ and $18x^2-33xy-30y^2$.
13. x^3-2x^2+x and x^3+x^2-x-1.
14. x^3-x^2y, y^3-xy, and $x^3-3x^2y+xy^2+y^3$.
15. $(x+y)^2-xy$, x^3-y^3, and $x^3+x^2y+xy^2$.
16. $(a+b)^2-c^2$, $(a+c)^2-b^2$, and $(b+c)^2-a^2$.

CHAPTER V

FRACTIONS

126. Definition. A fraction in algebra is the quotient of two numbers or expressions. The expression $\frac{a}{b}$ means $a \div b$ (§ 6, Remark). A fraction, however, is usually regarded as an indicated division. The dividend, a, is called the **numerator**, and the divisor, b, is called the **denominator**. The numerator and denominator are called the **terms** of the fraction. $\frac{a}{b}$ is read *a over b* or *a divided by b*.

127. Laws governing algebraic fractions. Algebraic fractions are subject to the same laws as arithmetical fractions. This is, in part, a direct consequence of the fact that both kinds of fractions remain unchanged in value when both numerator and denominator are multiplied or divided by the same number (excepting zero). For arithmetical fractions this important principle is established in arithmetic. It may be proved in algebra as follows:

Let $\frac{a}{b}$ denote any fraction, and m the number by which its terms are to be multiplied. Representing the quotient of a divided by b by q we have,

$$\frac{a}{b} = q. \qquad (1)$$

Since the dividend is equal to the product of the divisor and the quotient,

$$a = bq. \qquad (2)$$

Multiplying both members of identity (2) by m,

$$am = bmq. \tag{3}$$

Dividing both members of identity (3) by bm,

$$\frac{am}{bm} = q. \tag{4}$$

From identities (1) and (4), we have,

$$\frac{a}{b} = \frac{am}{bm}. \tag{5}$$

Again, $\frac{a}{b}$ may be obtained from $\frac{am}{bm}$ by dividing both terms of $\frac{am}{bm}$ by m, and from identity (5) the value of the fraction remains unchanged; that is, we may write

$$\frac{am}{bm} = \frac{a}{b}. \tag{6}$$

From identities (5) and (6) we have the following principle:

Multiplying or dividing both terms of a fraction by the same number (zero excepted) does not change its value.

Note. The denominator of a fraction cannot be zero. The expression $\frac{a}{0}$ has no meaning, and, therefore, does not represent a number. In other words, division by zero is excluded. Care should be exercised in assigning numerical values to letters to see that the values assigned do not cause the denominator of a fraction to vanish.

128. Signs affecting a fraction. The sign of a fraction is the plus or minus sign before the fraction.

Thus, in the expression $+\frac{-2}{+3}$, the sign which stands first is the sign of the fraction.

There are, therefore, three signs which affect a fraction; namely, the sign of the fraction, the sign of the numerator, and the sign of the denominator. The signs of the numerator and denominator combine according to the rule for signs in division.

Thus,

$$\frac{+2}{+3}=+\frac{2}{3}. \quad (1) \qquad \frac{-2}{-3}=+\frac{2}{3}. \quad (2)$$

$$\frac{+2}{-3}=-\frac{2}{3}. \quad (3) \qquad \frac{-2}{+3}=-\frac{2}{3}. \quad (4)$$

In algebraic symbols,

$$\frac{+a}{+b}=+\frac{a}{b}. \quad (1) \qquad \frac{-a}{-b}=+\frac{a}{b}. \quad (2)$$

$$\frac{+a}{-b}=-\frac{a}{b}. \quad (3) \qquad \frac{-a}{+b}=-\frac{a}{b}. \quad (4)$$

By comparing equation (1) with equation (2) and equation (3) with equation (4), it is evident that

I. *The signs of both terms of a fraction may be changed without altering the value of the fraction.*

Thus, $\frac{+a}{+b}=\frac{-a}{-b}$ and $\frac{+a}{-b}=\frac{-a}{+b}$.

II. *Any two of the three signs affecting a fraction may be changed without altering the value of the fraction.*

The foregoing statement is evident from (I) and from the following:

$$-\frac{-a}{+b}=-\left(-\frac{a}{b}\right)=+\frac{a}{b}. \quad (1)$$

Changing the sign of the fraction and the sign of the numerator in (1), we have

$$+\frac{+a}{+b}=+\left(+\frac{a}{b}\right)=+\frac{a}{b}. \quad (2)$$

Changing the sign of the fraction and the sign of the denominator in (1), we have

$$+\frac{-a}{-b}=+\left(+\frac{a}{b}\right)=+\frac{a}{b}. \quad (3)$$

Comparing identities (1), (2), (3), we have

$-\frac{-a}{+b}=+\frac{+a}{+b}=+\frac{-a}{-b}$, since each is equal to $+\frac{a}{b}$.

From I and II it is evident that

III. *A fraction may be written in at least four ways without changing its value.*

Thus, $$+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{+a}{-b}=-\frac{-a}{+b}.$$

In like manner,

$$+\frac{x-y}{x-y-z}=+\frac{-x+y}{-x+y+z}=-\frac{-x+y}{x-y-z}=-\frac{x-y}{-x+y+z}.$$

Remark. When no sign is written before a fraction, + is understood.

Thus, $\frac{a}{b}$ means $+\frac{a}{b}$.

129. Change of signs of factors in the terms of a fraction. To change the sign of one factor of an expression is equivalent to multiplying that expression by -1. Therefore, when either or both terms of a fraction are expressed as a product of factors, the signs of an even number of these factors may be changed without altering the value of the fraction; but if the signs of an odd number of them are changed the sign of the fraction must be changed in order that its value may not be changed.

Thus, $$\frac{(a-b)(c-d)}{(x-y)(z-w)}=\frac{(b-a)(d-c)}{(x-y)(z-w)} \qquad \text{[Why?]}$$

$$=-\frac{(a-b)(c-d)}{(x-y)(w-z)}. \qquad \text{[Why?]}$$

ILLUSTRATIVE EXAMPLE

Without altering the value of the fraction $-\frac{y-x}{-2a}$, express it in a form in which each of the three signs affecting it is plus.

Solution. Changing the sign of the fraction and the sign of the denominator, we have

$$-\frac{y-x}{-2a}=\frac{y-x}{2a}.$$

EXERCISE 62

Tell, at sight, which of the statements in examples 1–15 are true.

1. $+\frac{x}{y}=+\frac{-x}{-y}$.

2. $+\frac{x}{y}=-\frac{x}{-y}$.

3. $+\frac{x}{y}=-\frac{-x}{-y}$.

4. $-\frac{x}{y}=\frac{-x}{y}$.

5. $-\frac{x}{-y}=-\frac{-x}{y}$.

6. $-\frac{x}{-y}=\frac{-x}{-y}$.

7. $\frac{x}{x-y}=-\frac{x}{y-x}$.

8. $\frac{x}{x-y}=\frac{-x}{y-x}$.

9. $\frac{x}{x-y}=-\frac{-x}{x-y}$.

10. $\frac{a}{(a-b)(c-d)}=\frac{a}{(b-a)(d-c)}$.

11. $\frac{a}{(a-b)(c-d)}=\frac{-a}{(b-a)(c-d)}$.

12. $-\frac{a}{(a-b)(c-d)}=\frac{a}{(b-a)(c-d)}$.

13. $\frac{a+b}{c-d}=\frac{a-b}{d-c}$.

14. $\frac{a+b}{c-d}=\frac{-a-b}{d-c}$.

15. $\frac{a+b}{c-d}=-\frac{-a-b}{c-d}$.

Without altering the value of the fractions in examples 16–30, express each in a form in which the three signs affecting it are plus.

16. $\frac{-a}{-b}$.

17. $\frac{x-y}{-c}$.

18. $-\frac{-x}{y}$.

19. $-\frac{x-y}{-a}$.

20. $\frac{-a-b+c}{ab}$.

21. $\frac{-a(b-c)}{y(a-b)}$.

22. $-\frac{b+c}{c-a}$.

23. $\frac{-a-b}{y-x}$.

24. $\frac{-(a+b)(x-y)}{x+a}$.

25. $-\frac{a+b}{-c-d}$.

26. $\frac{-a}{b(c-a)}$.

27. $-\frac{a}{(c-a)b}$.

28. $-\frac{(a+b)(c+d)(x-y)}{abc}$.

29. $\frac{a+b}{-(m-n)}$.

30. $-\frac{-(x-y)}{-(r-s)}$.

Lowest Terms

130. Numbers or algebraic expressions prime to each other. Two numbers in arithmetic or two algebraic expressions are said to be **prime to each other** when their only common factor is 1.

131. Reduction of fractions to lowest terms. A fraction is said to be in lowest terms when its numerator and denominator are prime to each other.

It was shown in section 127 that both terms of a fraction may be divided by the same number without changing the value of the fraction. Hence, we have the following:

Rule. *To reduce a fraction to lowest terms, cancel all factors common to the numerator and denominator; that is, divide both terms of the fraction by their highest common factor.*

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{36\,ab^2c^3d^4}{28\,a^3bc^5d}$ to lowest terms.

Solution. To reduce the given fraction to lowest terms, we divide its numerator and its denominator by their H. C. F., which is $4\,abc^3d$.

$$\therefore \frac{36\,ab^2c^3d^4}{28\,a^3bc^5d} = \frac{36\,ab^2c^3d^4 \div 4\,abc^3d}{28\,a^3bc^5d \div 4\,abc^3d} = \frac{9\,bd^3}{7\,a^2c^2}.$$

2. Reduce $\frac{x^2-9}{x^2+3\,x-18}$ to lowest terms.

Solution. $\frac{x^2-9}{x^2+3\,x-18} = \frac{(x-3)(x+3)}{(x-3)(x+6)} = \frac{(x+3)}{(x+6)}.$

Note. In practice it is customary to separate the numerator and the denominator into their prime factors and *cancel* the factors common to both.

Thus, $$\frac{3\,m^3-3\,m}{6\,m^5+6\,m^4-12\,m^3} = \frac{\cancel{3\,m(m-1)}(m+1)}{\underset{2\,m^2}{\cancel{6\,m^3(m-1)}}(m+2)} = \frac{m+1}{2\,m^2(m+2)}.$$

Remark. Care should be exercised not to cancel a common *term* of the numerator and denominator of a fraction when they are polynomials.

EXERCISE 63

(Solve as many as possible at sight.)

Reduce the following fractions to lowest terms :

1. $\frac{xy}{xz}$. **2.** $\frac{2\,ab}{4\,bc}$. **3.** $\frac{xyz}{xyt}$.

4. $\frac{a^3}{a^5}$. **5.** $\frac{-x^5}{x^8}$. **6.** $\frac{12\,a^3}{-16\,a^5}$.

7. $\frac{ab^2}{a^2b}$. **8.** $\frac{10\,ab^2c^3}{15\,a^3b^2c}$. **9.** $\frac{6\,pqr^2}{2\,p^2q^2r^2}$.

10. $\frac{-12\,m^2n^3p^5}{-45\,n^5p^2}$. **11.** $\frac{9\,p^2qr}{33\,p^3q^2r^4}$. **12.** $\frac{44\,m^2n^3}{52\,m^3n^2p^4}$.

13. $\frac{\pi R^2 H}{2\pi RH}$.

14. $\frac{4\pi R^3}{12\pi R^2}$.

15. $\frac{x^2(x+z)^2(y+z)}{xy(y+z)^2(x+z)}$.

16. $\frac{x^{2m}}{x^m}$.

17. $\frac{x^m}{x^{2m}}$.

18. $\frac{x^{3m}y^{2n}}{x^{2m}y^{3n}}$.

19. $\frac{2x^{n-1}y^{m+2}}{10x^n y^{m+3}}$.

20. $\frac{x^{n+1}y^{m-1}}{x^{n-1}y^{m-2}}$.

21. $\frac{-35a^{2m-3}b^{2p}}{42a^{3m}b^{3p-1}}$.

22. $\frac{a^2-b^2}{a-b}$.

23. $\frac{(a+b)^3}{(a+b)^5}$.

24. $\frac{x^2-1}{(x-1)(x+2)}$.

25. $\frac{x^3-1}{(x-1)(x^2+1)}$.

26. $\frac{a^2-b^2}{4b+4a}$.

27. $\frac{a^2+2ax+x^2}{a^3+a^2x}$.

28. $\frac{x^2-y^2}{(y-x)^2}$.

29. $\frac{a^2-b^2}{b^3-a^3}$.

30. $\frac{p^6-q^6}{q^3-p^3}$.

31. $\frac{a^2+ax}{2ab+2bx}$.

32. $\frac{x^2-16}{x^2-8x+16}$.

33. $\frac{x^2-2xy+y^2}{x^3-y^3}$.

34. $\frac{a^3+b^3}{a^4+a^2b^2+b^4}$.

35. $\frac{ax+bx+ay+by}{ax+ay-bx-by}$.

36. $\frac{xy}{x^2y+xy^2}$.

37. $\frac{a^3-a}{a^3-1}$.

38. $\frac{3x-3}{4-4x^2}$.

39. $\frac{(a+1)(a+2)(a+3)}{(a+2)(a+3)(a+4)}$.

40. $\frac{2x^2+17x-19}{3x^2-5x+2}$.

41. $\frac{4x^2+3x-22}{5x^2-3x-14}$.

42. $\frac{x^2-(y-z)^2}{(x+y)^2-z^2}$.

43. $\dfrac{(y+z)^2-x^2}{(x+z)^2-y^2}$. **44.** $\dfrac{a+x+y}{x^2-a^2+y(2x+y)}$.

45. $\dfrac{x^3-x^2y+xy^2-x^3}{x^3+x^2y+xy^2+y^3}$. **46.** $\dfrac{1-x+y-xy}{1-x-z+xz}$.

47. $\dfrac{x^3+2x^2+x+2}{x^3+3x^2+x+3}$. **48.** $\dfrac{4b^2c^2-(b^2+c^2-a^2)^2}{4c^2a^2-(c^2+a^2-b^2)^2}$.

Multiplication of Fractions

132. Multiplication of a fraction by an integral expression. Let $\frac{a}{b}$ denote any fraction and c any integral expression. Representing the quotient of $\frac{a}{b}$ by q we have,

$$\frac{a}{b}=q. \tag{1}$$

Since the dividend is equal to the product of the divisor and the quotient,

$$a=bq. \tag{2}$$

Multiplying both members of identity (2) by c,

$$ac=bcq. \tag{3}$$

Dividing both members of identity (3) by b,

$$\frac{ac}{b}=\frac{bcq}{b}=cq=c\times\frac{a}{b} \tag{4}$$

That is,
$$\frac{ac}{b}=c\times\frac{a}{b}. \tag{5}$$

Again,
$$\frac{ac}{b}=\frac{ac\div c}{b\div c}=\frac{a}{b\div c}. \tag{6}$$

Therefore, from identities (5) and (6),

$$\frac{a}{b\div c}=c\times\frac{a}{b}. \tag{7}$$

Identity (5) shows that multiplying the numerator of $\frac{a}{b}$ by c multiplies $\frac{a}{b}$ by c, and identity (7) shows that dividing the denominator of $\frac{a}{b}$ by c multiplies $\frac{a}{b}$ by c; hence,

Rule. *To multiply a fraction by an integral expression, either multiply the numerator or divide the denominator by that expression.*

Note. Divide the denominator when possible.

ILLUSTRATIVE EXAMPLES

1. Find the product of $\frac{m}{n}$ and a.

Solution. $a \times \frac{m}{n} = \frac{am}{n}$. [§ 132]

2. Find the product of $\frac{a}{b}$ and b.

Solution. $b \times \frac{a}{b} = \frac{a}{b \div b} = \frac{a}{1} = a$. [§ 132]

3. Multiply $\frac{5a}{xy}$ by $4ab$.

Solution. $\frac{5a}{xy} \times 4ab = \frac{20a^2b}{xy}$.

4. Multiply $\frac{x^2+y^2}{x^3-y^3}$ by $x-y$.

Solution. $\frac{x^2+y^2}{x^3-y^3} \cdot (x-y) = \frac{x^2+y^2}{(x^3-y^3) \div (x-y)} = \frac{x^2+y^2}{x^2+xy+y^2}$.

5. Multiply $\frac{x^m}{y^n}$ by x^{m+1}.

Solution. $\frac{x^m}{y^n} \cdot x^{m+1} = \frac{x^m x^{m+1}}{y^n} = \frac{x^{2m+1}}{y^n}$.

EXERCISE 64

(Solve as many as possible at sight.)

Multiply as indicated:

1. $2 \times \frac{2}{5}$.

2. $2 \times \frac{3}{8}$.

3. $c \times \frac{m}{n}$.

4. $p \times \frac{q}{p^2}$.

5. $2a \times \frac{a}{b}$.

6. $b \times \frac{a}{2b^2}$.

7. $z \times \frac{-12xy}{z^2}$.

8. $3d^2 \times \frac{2a^2bc}{3d^3}$.

9. $-6ax \times \frac{-5x^2y}{3ab^2}$.

10. $9xyz \times \frac{-2abc^2}{3xyz^2}$.

11. $(a+b) \times \frac{2}{m}$.

12. $(m-n) \times \frac{a}{b(m-n)}$.

13. $(a+b) \times \frac{a-b}{c-d}$.

14. $(x-y) \times \frac{a}{(x-y)^2}$.

15. $(x-y) \times \frac{a}{x^2-y^2}$.

Multiply:

16. $\frac{2a+b}{a^2-b^2}$ by $(a+b)$.

17. $\frac{a^2+b^2}{a^3+b^3}$ by $3(a+b)$.

18. $\frac{x+y}{x^4+x^2+1}$ by x^2-x+1.

19. $\frac{x^m}{y^{n-1}}$ by x^m.

20. $\frac{a^m}{b^n}$ by b.

21. $\frac{a^{2m}}{b^{3n}}$ by a.

22. $\frac{5ab^2c^3d^4}{12xy^2z^3}$ by $-27a^4b^3c^2dx^3y^2z$.

23. $\frac{(a+b)(c+d)}{(a-b)(c-d)}$ by $(a-b)(c-d)$.

24. $\frac{3abc}{(x+y)z}$ by $(x+y)z$.

25. $\frac{a}{b^{m+1}}$ by b.

Addition and Subtraction of Fractions

133. Adding and subtracting fractions with the same denominator. By the distributive law, section 60,

$$m\left(\frac{a}{m}+\frac{b}{m}+\frac{c}{m}\right)=m\left(\frac{a}{m}\right)+m\left(\frac{b}{m}\right)+m\left(\frac{c}{m}\right)$$

$$=a+b+c \text{ [§ 132, Rule].} \qquad (1)$$

Dividing both members of identity (1) by m,

$$\frac{a}{m}+\frac{b}{m}+\frac{c}{m}=\frac{a+b+c}{m}. \qquad (2)$$

Again, $$m\left(\frac{a}{m}-\frac{b}{m}\right)=m\left(\frac{a}{m}\right)-m\left(\frac{b}{m}\right)$$

$$=a-b. \qquad (3)$$

Dividing both members of identity (3) by m,

$$\frac{a}{m}-\frac{b}{m}=\frac{a-b}{m}. \qquad (4)$$

From identities (2) and (4), we have the following

Rule. *To add or subtract fractions which have the same denominator, add or subtract their numerators and place the result over their common denominator.*

134. Lowest common denominator. Two or more fractions whose denominators are not the same may be replaced by other fractions equivalent to them, respectively, each of whose denominators is the lowest common multiple of the denominators of the given fractions.

The lowest common multiple of the denominators of two or more fractions is called their **lowest common denominator** (L. C. D.).

ILLUSTRATIVE EXAMPLES

1. Express as a single fraction $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$.

Solution. The L. C. D. of the fractions is abc.

$$\frac{1}{a}=\frac{bc}{abc},$$
$$\frac{1}{b}=\frac{ac}{abc},$$
$$-\frac{1}{c}=-\frac{ab}{abc}.$$
$$\text{Sum}=\frac{bc+ac-ab}{abc}.$$

2. Express as a single fraction in lowest terms:

$$\frac{1}{x^2+x}+\frac{1}{x^2+3x+2}-\frac{1}{x^2+2x}.$$

Solution. The L. C. D. of the fractions is $x(x+1)(x+2)$.

$$\frac{1}{x^2+x}=\frac{x+2}{x(x+1)(x+2)}$$
$$\frac{1}{x^2+3x+2}=\frac{x}{x(x+1)(x+2)}$$
$$-\frac{1}{x^2+2x}=-\frac{x+1}{x(x+1)(x+2)}$$
$$\text{Sum}=\frac{(x+2)+x-(x+1)}{x(x+1)(x+2)}$$
$$=\frac{x+2+x-x-1}{x(x+1)(x+2)}$$
$$=\frac{1}{x(x+2)}.$$

3. Express as a single fraction $x^2+xy+y^2+\frac{y^3}{x-y}$.

Solution. $x^2+xy+y^2+\frac{y^3}{x-y}=\frac{x^2+xy+y^2}{1}+\frac{y^3}{x-y}$

$$=\frac{x^3-y^3}{x-y}+\frac{y^3}{x-y}$$
$$=\frac{x^3}{x-y}.$$

4. Simplify:

$$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$$

Solution. $$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}$$
$$=-\frac{b+c}{(a-b)(c-a)}-\frac{c+a}{(a-b)(b-c)}-\frac{a+b}{(b-c)(c-a)}.$$

The L. C. D. of the fractions is $(b-c)(c-a)(a-b)$.

$$\frac{b+c}{(a-b)(a-c)}=-\frac{(b+c)(b-c)}{(b-c)(c-a)(a-b)},$$

$$\frac{c+a}{(b-c)(b-a)}=-\frac{(c+a)(c-a)}{(b-c)(c-a)(a-b)},$$

$$\frac{a+b}{(c-a)(c-b)}=-\frac{(a+b)(a-b)}{(b-c)(c-a)(a-b)}.$$

$$\text{Sum}=\frac{-(b^2-c^2)-(c^2-a^2)-(a^2-b^2)}{(b-c)(c-a)(a-b)}$$

$$=\frac{-b^2+c^2-c^2+a^2-a^2+b^2}{(b-c)(c-a)(a-b)}=0.$$

From the solutions of illustrative examples 1, 2, 3, and 4, pages 165 and 166, we have the following:

Rule. *For adding or subtracting fractions:*

1. *Reduce, when necessary, the fractions to their lowest common denominator.*

2. *Find the algebraic sum of the numerators of the resulting fractions.*

3. *Write the algebraic sum of the numerators for the numerator of the result and the lowest common multiple of the denominators for its denominator.*

4. *Simplify the resulting fraction.*

Note. In general, before operations on fractions are performed, each fraction involved should be reduced to its lowest terms.

EXERCISE 65

Express as a single fraction in lowest terms:

1. $\frac{1}{a}+\frac{1}{b}$.
2. $1+\frac{1}{a}$.
3. $\frac{3}{2x}+\frac{2}{3x}$.
4. $2x+\frac{3}{x}$.
5. $x+\frac{1}{2x}-\frac{1}{3x}$.
6. $a+\frac{b}{c}$.
7. $x-\frac{y}{z}$.
8. $\frac{1}{bc}-\frac{1}{ca}$.
9. $\left(\frac{a+b+c}{2}-a\right)+\left(\frac{a+c+b}{2}-b\right)$.
10. $\left(\frac{a+b-c}{2}+a\right)-\left(\frac{a-b+c}{2}+b\right)-\left(\frac{b+c-a}{2}+c\right)$.
11. $\frac{3}{4a}-\frac{1}{6a}$.
12. $\frac{1}{a+1}-\frac{1}{a}$.
13. $\frac{a}{b}+\frac{b}{a}$.
14. $\frac{1}{a+b}-\frac{1}{a-b}$.
15. $\frac{1}{2+a}+\frac{1}{a}$.
16. $\frac{2}{a+2}-\frac{3}{2a}$.
17. $\frac{1}{x+y}+\frac{1}{2x+2y}$.
18. $\frac{3}{x+3}-\frac{2}{x+2}$.
19. $\frac{5}{a+2}-\frac{3}{3a-1}$.
20. $\frac{5}{a-1}+\frac{5}{a+1}$.
21. $\frac{1}{2}+\frac{1}{x}$.
22. $\frac{x}{5y}+\frac{y}{5x}$.
23. $x-\frac{x^2}{x+y}$.
24. $\frac{1}{2a+b}-\frac{1}{3a+b}$.
25. $\frac{x+y}{2x}+\frac{x-y}{2y}$.
26. $\frac{1}{a+b}+\frac{1}{c+d}$.
27. $1+\frac{1}{a-1}$.
28. $\frac{9+2a}{3a}+\frac{9-2b}{3b}$.
29. $\frac{x+y}{a^2b}-\frac{x-y}{ab^2}$.
30. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

31. $\dfrac{a}{(x+y)b}+\dfrac{b}{(x+y)a}$.

32. $\dfrac{m}{a^2-ab}-\dfrac{n}{b^2-ab}$.

33. $\dfrac{7a}{b}-\left(\dfrac{a+5b}{b}-\dfrac{b-2a}{b}\right)$.

34. $\dfrac{m}{ax}-\dfrac{n}{bx}+\dfrac{p}{cx}$.

35. $\dfrac{a^2}{bc}+\dfrac{b^2}{ca}+\dfrac{c^2}{ab}$.

36. $\dfrac{x+y}{xy}-\dfrac{y+z}{yz}+\dfrac{z+x}{zx}$.

37. $\dfrac{x}{a}-\dfrac{2x}{3a}+\dfrac{4x}{9a}$.

38. $\dfrac{1}{2a(x-a)}-\dfrac{1}{2a(x+a)}$.

39. $\dfrac{1}{6(2x-3)}-\dfrac{1}{6(2x+3)}$.

40. $\dfrac{1-a}{a^5}+\dfrac{1}{a^4}$.

41. $a+b-\dfrac{2ab}{a+b}$.

42. $1-\dfrac{1}{a}-\dfrac{a}{a+1}$.

43. $\dfrac{4}{3x^2-3xy}-\dfrac{5}{4y^2-4yx}$.

44. $\dfrac{ax^2}{b}+\dfrac{by}{a}-\dfrac{b^2(x^2+y^2)}{a^2}-\dfrac{3a^2xy}{b^2}$.

45. $\dfrac{1}{b-c}+\dfrac{1}{c-a}+\dfrac{1}{a-b}$.

46. $\dfrac{a}{b-c}+\dfrac{b}{c-a}-\dfrac{c}{a-b}$.

47. $\dfrac{3}{x^2-4}-\dfrac{1}{2-x}$.

48. $\left(\dfrac{1}{a^3}-\dfrac{1}{a^2}+\dfrac{1}{a}\right)\cdot(a^3+a^4)$.

49. $\dfrac{1}{(a-b)(a+c)}+\dfrac{1}{(b-a)(b+c)}$.

50. $\dfrac{1}{(a-b)^3}-\dfrac{3}{(b-a)^3}$.

51. $\dfrac{x^2+xy+y^2}{x^3-y^3}-\dfrac{x^2-xy+y^2}{x^3+y^3}$.

52. $\dfrac{2xy}{x^2-y^2}+\dfrac{2x+y}{x+y}-\dfrac{x+y}{y-x}$.

53. $\dfrac{1}{x}+\dfrac{x+4}{x^2-9}+\dfrac{x+5}{x^2-6x+9}$.

54. $\dfrac{a+c}{(a-b)(a-c)}+\dfrac{b+c}{(b-a)(b-c)}.$

55. $\dfrac{13}{(3x+2)(2x-3)}+\dfrac{2}{(2x-3)(x-2)}+\dfrac{8}{(2-x)(3x+2)}.$

56. $\dfrac{2}{x^2+4x+3}-\dfrac{4}{x^2-2x-3}-\dfrac{5}{1-x^2}.$

57. $\dfrac{7}{2x^2-5x-3}-\dfrac{8}{3x^2-10x+3}+\dfrac{1}{6x^2+x-1}.$

58. $\dfrac{x+1}{6x^2-17x+12}-\dfrac{94x-186}{15x^2-14x-8}+\dfrac{71x-135}{6x^2-17x+12}.$

In examples 59–64, combine not more than two fractions at a time. Thus, in example 59, combine the first two fractions and then the result with the third.

59. $\dfrac{1}{1-a}+\dfrac{1}{1+a}+\dfrac{2}{1+a^2}.$

60. $\dfrac{a}{a+b}+\dfrac{b}{a-b}+\dfrac{a^2-b^2}{a^2+b^2}.$

61. $\dfrac{1}{x-y}+\dfrac{1}{x+y}+\dfrac{2x}{x^2+y^2}+\dfrac{4x^3}{x^4+y^4}.$

62. $\dfrac{a}{a+4}-\dfrac{a+4}{a}+\dfrac{a}{a-4}-\dfrac{a-4}{a}.$

63. $\dfrac{x^2+x+3}{x-1}-\dfrac{x^2-1}{x-2}-\dfrac{2x-8}{x^2-3x+2}.$

64. $\dfrac{1}{x-3}+\dfrac{1}{x+3}-\dfrac{1}{x-2}-\dfrac{1}{x+2}.$

65. $\dfrac{3(2x^2+1)}{x^4+x^2+1}-\dfrac{2x+1}{x^2-x+1}+\dfrac{2x-1}{x^2+x+1}.$

Reduction of Fractions to Integral or Mixed Expressions

135. Simple fraction. When both terms of a fraction are integral, the fraction is called a **simple fraction.**

136. Proper and improper fractions. Simple fractions are classified as proper fractions and improper fractions. A **proper fraction** is a simple fraction in which the degree of the numerator is less than the degree of the denominator. An **improper fraction** is a simple fraction in which the degree of the numerator is either equal to or greater than the degree of the denominator.

Thus, $\frac{a-1}{a^2+b^2}$ is a proper fraction; $\frac{a^2+b^2}{a+b^2}$ is an improper fraction.

Remark. When the terms of a fraction contain more than one literal number, the fraction may be a proper fraction with respect to one of these numbers and an improper fraction with respect to another.

Thus, $\frac{a^2+b^2}{a+b^3}$ is a proper fraction with respect to b, but it is an improper fraction with respect to a.

137. Mixed expression. An expression, some of whose terms are integral and some fractional, is called a **mixed expression.**

Thus, $a+\frac{b}{c}$, $x+y-\frac{x}{x+y}$, and $1+\frac{x}{y}$ are mixed expressions.

Note. An improper fraction can be reduced either to an integral expression or to a mixed expression in which the fractional part is a proper fraction.

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{a^2-b^2-a+2b}{a-b}$ to a mixed expression.

Solution 1.

$$\begin{aligned}\frac{a^2-b^2-a+2b}{a-b} &= \frac{a^2-b^2-a+b+b}{a-b}\\ &= \frac{a^2-b^2}{a-b}+\frac{-a+b}{a-b}+\frac{b}{a-b}\\ &= \frac{a^2-b^2}{a-b}-\frac{a-b}{a-b}+\frac{b}{a-b}\\ &= a+b-1+\frac{b}{a-b}.\end{aligned}$$

Remark. Since $\frac{a+b-c}{m}=\frac{a}{m}+\frac{b}{m}-\frac{c}{m}$, it is evident that a fraction whose numerator is a polynomial can always be written as the algebraic sum of two or more fractions. As a step in reducing a fraction to a mixed expression, it is desirable to express it in this manner, whenever the terms can be grouped at sight in such a way that each numerator with, in general, the exception of the last, is exactly divisible by the denominator.

Solution 2.

$$\begin{array}{r|l} & a-1+b \\ \hline a-b & a^2-a \qquad -b^2+2b \\ & a^2 \qquad -ab \\ \hline & \quad -a+ab-b^2+2b \\ & \quad -a \qquad\qquad +b \\ \hline & \qquad ab-b^2+b \\ & \qquad ab-b^2 \\ \hline & \qquad\qquad\qquad b \end{array}$$

$$\frac{a^2-a-b^2+2b}{a-b}=a-1+b+\frac{b}{a-b}.$$

Remark. Solution 2 is preferable when it is not evident at sight how the terms of the numerator should be grouped.

Check. Let $a=2$, $b=1$.

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}.$$

$$a^2-b^2-a+2b=(a+b-1)(a-b)+b.$$

$$4-1-2+2=(2+1-1)(2-1)+1$$

$$3=2\times 1+1$$

$$3=3.$$

2. Reduce $\frac{2x^3-3x^2y+4xy^2+5y^3}{3x^2+2xy-4y^2}$ to a mixed expression.

Solution.

$$\begin{array}{r|l} & \frac{2}{3}x-\frac{13}{9}y \\ \hline 3x^2+2xy-4y^2 & 2x^3-3x^2y+4xy^2+5y^3 \\ & 2x^3+\frac{4}{3}x^2y-\frac{8}{3}xy^2 \\ \hline & \qquad -\frac{13}{3}x^2y+\frac{20}{3}xy^2+5y^3 \\ & \qquad -\frac{13}{3}x^2y-\frac{26}{9}xy^2+\frac{52}{9}y^3 \\ \hline & \qquad\qquad\qquad \frac{86}{9}xy^2-\frac{7}{9}y^3 \end{array}$$

Hence,

$$\frac{2x^3-3x^2y+4xy^2+5y^3}{3x^2+2xy-4y^2}=\frac{2}{3}x-\frac{13}{9}y+\frac{\frac{86}{9}xy^2-\frac{7}{9}y^3}{3x^2+2xy-4y^2}$$

$$=\frac{2}{3}x-\frac{13}{9}y+\frac{86xy^2-7y^3}{9(3x^2+2xy-4y^2)}.$$

Check: Let $x=2,\ y=1.$

$$2x^3-3x^2y+4xy^2+5y^3=\left(\frac{2}{3}x-\frac{13}{9}y\right)\left(3x^2+2xy-4y^2\right)+\frac{86xy^2-7y^3}{9}.$$

$$16-12+8+5=\left(\frac{4}{3}-\frac{13}{9}\right)(12+4-4)+\frac{172-7}{9}.$$

$$17=-\tfrac{4}{3}+\tfrac{55}{3}=17.$$

EXERCISE 66

(Solve as many as possible at sight.)

Reduce to either an integral or a mixed expression:

1. $\dfrac{2a^2-3a}{a}.$

2. $\dfrac{2m^2n+3mn-5mn^2}{mn}.$

3. $\dfrac{2x^3-3x^2+2x-1}{x}.$

4. $\dfrac{3a^2-2ab+5}{2a}.$

5. $\dfrac{4m+3mn-2m}{n}.$

6. $\dfrac{1+3ab-2a^2b+5b^3}{2ab}.$

7. $\dfrac{x^2-a^2}{x-a}.$

8. $\dfrac{ma+mb}{a+b}.$

9. $\dfrac{x^2-2x+1}{x-1}.$

10. $\dfrac{2\pi RH+2\pi R^2}{H+R}.$

11. $\dfrac{a^2-b^2+a-b}{a-b}.$

12. $\dfrac{a^2+b^2}{a-b}.$

13. $\dfrac{2a^2-b^2+1}{a+b}.$

14. $\dfrac{a^2-b^2+a-2b}{a-b}.$

15. $\dfrac{x^3-y^3}{x+y}.$

16. $\dfrac{2x^2+3xy+y^2}{2x+3y}.$

17. $\dfrac{3x^2 - 2xy + 2}{3x - 2y}$.

18. $\dfrac{3x^2 + 3xy + 2}{x + y}$.

19. $\dfrac{x^3 + 2x^2y + 3xy^2 - 5}{x^2 + 2xy + 3y^2}$.

20. $\dfrac{2x^2 + 3x + 1}{x^2 + 2x - 3}$.

21. $\dfrac{3x^2 - 2x + 2}{2x^2 - 3x + 3}$.

22. $\dfrac{3x^2 - 2xy + 4y^2}{2x^2 + 4xy - 3y^2}$.

Multiplication of Fractions

138. Product of two fractions.

Let $\qquad \dfrac{a}{b} = q \quad (1) \quad$ and $\quad \dfrac{c}{d} = r \quad (2)$

Then, $\qquad a = bq \quad (3) \quad$ and $\quad c = dr \quad (4)$

$$ac = bdqr. \qquad (5) \qquad [\S\ 25,\ 3]$$

Dividing both members of (5) by bd,

$$\frac{ac}{bd} = qr. \qquad (6)$$

But

$$\frac{a}{b} \times \frac{c}{d} = qr. \qquad (7)$$

$$\therefore\ \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}. \qquad (8)$$

From identity (8) we have the following:

Rule. *To find the product of two fractions, multiply the numerators together for the numerator of the product and the denominators for the denominator.*

ILLUSTRATIVE EXAMPLES

1. Find the product of $\dfrac{2a^2b}{3cd^2}$ and $\dfrac{3cd}{4ab^2}$.

Solution. $\dfrac{2a^2b}{3cd^2} \cdot \dfrac{3cd}{4ab^2} = \dfrac{\overset{a}{\cancel{6a^2bcd}}}{\underset{2bd}{\cancel{12ab^2cd^2}}} = \dfrac{a}{2bd}$.

Remark. In practice it is customary to cancel as shown in solution of example 2, which follows.

2. Multiply $\frac{x^2+3x-4}{x^2+5x+6}$ by $\frac{3x+6}{4x-4}$.

Solution. $\frac{x^2+3x-4}{x^2+5x+6}\cdot\frac{3x+6}{4x-4}=\frac{(x-1)(x+4)}{(x+2)(x+3)}\cdot\frac{3(x+2)}{4(x-1)}$

$$=\frac{3(x+4)}{4(x+3)}.$$

Check. Let $x=2$.

$$\frac{x^2+3x-4}{x^2+5x+6}\cdot\frac{3x+6}{4x-4}=\frac{3(x+4)}{4(x+3)}.$$

$$\frac{4+6-4}{4+10+6}\cdot\frac{6+6}{8-4}=\frac{3(2+4)}{4(2+3)}.$$

$$\tfrac{6}{20}\cdot\tfrac{12}{4}=\tfrac{18}{20}.$$

$$\tfrac{9}{10}=\tfrac{9}{10}.$$

Remark. It is evident that $x-1$ and $x+2$ may each be cancelled in the numerator of the one fraction and the denominator of the other [Solution, example 2], for, if expressed in the product, each would be a factor common to the numerator and denominator of the product and hence could be cancelled.

3. Simplify $1-\frac{c^3+y^3}{(c-y)^2}\cdot\frac{c^3-y^3}{c^4+c^2y^2+y^4}\cdot\frac{(c+y)^2}{c^2-y^2}$.

Solution.

$$1-\frac{c^3+y^3}{(c-y)^2}\cdot\frac{c^3-y^3}{c^4+c^2y^2+y^4}\cdot\frac{(c+y)^2}{c^2-y^2}$$

$$=1-\frac{(c+y)(c^2-cy+y^2)}{(c-y)^2}\cdot\frac{(c-y)(c^2+cy+y^2)}{(c^2-cy+y^2)(c^2+cy+y^2)}\cdot\frac{(c+y)^2}{(c-y)(c+y)}$$

$$=1-\frac{(c+y)^2}{(c-y)^2}$$

$$=\frac{(c-y)^2-(c+y)^2}{(c-y)^2}$$

$$=\frac{[(c-y)+(c+y)][(c-y)-(c+y)]}{(c-y)^2}$$

$$=\frac{(2c)(-2y)}{(c-y)^2}=\frac{-4cy}{(c-y)^2}.$$

EXERCISE 67

(Solve as many as possible at sight.)

Simplify:

1. $\frac{2}{3}\cdot\frac{a}{b}$. **2.** $\frac{2a}{3b}\cdot\frac{a^2}{b^2}$. **3.** $\frac{1}{a^2}\cdot\frac{1}{a^3}$. **4.** $\frac{2a}{b}\cdot\frac{3c}{d}$.

5. $\frac{-3}{5}\cdot\frac{20}{21}\cdot\frac{7}{-2}$.

6. $2a\cdot\frac{3b}{c}$.

7. $a^2\cdot\frac{a^3}{b^2}$.

8. $\frac{2a}{3c}\cdot\frac{4b}{9d}$.

9. $\frac{2xy^2}{3ab^2}\cdot\frac{4x}{3y}$.

10. $\frac{-3mn^2p^3}{2ab^2c^3}\cdot\frac{4m}{9a}$.

11. $\frac{5xy^2z^4}{3abc}\cdot\frac{-2xy}{3ac}$.

12. $\frac{2ab^2c^3}{3de^2f^3}\cdot\frac{4a^2bc^3}{9d^2ef^3}$.

13. $\frac{2(x+y)}{3(x-y)}\cdot\frac{(x+y)^2}{(x-y)^3}$.

14. $\frac{-3(a+b)}{2(a-b)}\cdot\frac{5(a+b)^3}{-15(a-b)^2}$.

15. $\frac{x+y}{x-y}\cdot\frac{x-y}{x^2+y^2}$.

16. $\frac{3(x+y)^2}{x^2-y^2}\cdot\frac{x-y}{x+y}$.

17. $\frac{a^2-1}{a^2-b^2}\cdot\frac{(a+b)^2}{a+1}$.

18. $\frac{2a}{b+3}\cdot\frac{2(b+3)}{3a^2}$.

19. $\frac{ab^2}{cd^2}\cdot\frac{cd^2}{e^2f}\cdot\frac{ef}{ab}$.

20. $\frac{4abcd}{7xyz}\cdot\frac{2x^2y^3z^2}{3a^2bc^2d}\cdot\frac{8ac}{3xyz}$.

21. $\frac{ab^2}{3}\left(\frac{1}{a}-\frac{1}{b}\right)$.

22. $-\frac{2a^2b^2}{5}\left(\frac{a^2}{b^2}-\frac{a}{b}\right)$.

23. $\frac{x}{a-1}\cdot\frac{1-a^2}{2x+xy}$.

24. $\frac{a^2+2a+1}{b^3}\cdot\frac{-3b^2}{a+1}$.

25. $x^2y^2\left(\frac{1}{x^2}-\frac{1}{y^2}\right)$.

26. $\frac{1}{2}xy\left(\frac{x}{y}-\frac{y}{x}\right)$.

27. $\frac{3b-6x}{2a+2x}\cdot\frac{2a^4+2a^3b}{9ax+9x}\cdot\frac{6x+3x^2}{2ab-4ax}$.

28. $\dfrac{a^2-2\,ax}{a^2+3\,ab}\cdot\dfrac{ab+3\,b^2}{2\,abx-3\,x^2y}\cdot\dfrac{2\,aby-3\,xy^2}{ax-2\,x^2}.$

29. $\dfrac{2\,m^2-2}{3\,p^2-3\,q}\cdot\dfrac{5\,m^2+5}{ap^2+aq^2}\cdot\dfrac{bp^2+bq^2}{3\,m^2+3}\cdot\dfrac{bp+bq}{cm-c}.$

30. $\dfrac{(x+1)^2}{(y+1)^2}\cdot\dfrac{y^2-1}{x^3+1}.$

31. $\dfrac{b-a}{(x+y)^2}\cdot\dfrac{a^2-b^2}{10}\cdot\dfrac{6(x^2-y^2)}{a+b}\cdot\dfrac{(x+y)^2}{x-y}.$

139. Powers of a fraction. An important case in the multiplication of fractions is that in which the fractions to be multiplied are equal; their product is, therefore, a power of one of the given equal fractions.

Thus, $\left(\dfrac{2}{3}\right)^3=\dfrac{2}{3}\times\dfrac{2}{3}\times\dfrac{2}{3}=\dfrac{(2)^3}{(3)^3}=\dfrac{8}{27}.$

$$\left(\frac{-2}{3}\right)^5=\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}=\frac{(-2)^5}{(3)^5}=\frac{-32}{243}=-\frac{32}{243}.$$

$$\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots\text{ to } n \text{ factors}$$

$$=\frac{a\cdot a\cdot a\cdots\text{ to } n\text{ factors}}{b\cdot b\cdot b\cdots\text{ to } n\text{ factors}}=\frac{a^n}{b^n};\text{ therefore,}$$

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}. \qquad (1)$$

Identity (1) may be expressed in words as follows:

Any power of a fraction is equal to the same power of the numerator divided by the same power of the denominator.

Conversely, we may write,

$$\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n. \qquad (2)$$

ILLUSTRATIVE EXAMPLES

1. Simplify $\left(\frac{a^2-b^2}{x-y}\right)^2 \cdot \left(\frac{x^3-y^3}{a-b}\right)^2$.

Solution.

$$\left(\frac{a^2-b^2}{x-y}\right)^2 \cdot \left(\frac{x^3-y^3}{a-b}\right)^2 = \left[\frac{(a-b)(a+b)}{x-y}\right]^2 \cdot \left[\frac{(x-y)(x^2+xy+y^2)}{a-b}\right]^2$$

$$= \frac{\cancel{(a-b)^2}(a+b)^2}{\cancel{(x-y)^2}} \cdot \frac{\cancel{(x-y)^2}(x^2+xy+y^2)^2}{\cancel{(a-b)^2}}$$

$$= (a+b)^2(x^2+xy+y^2)^2.$$

2. Simplify $\frac{(2x^2+3x+1)^3}{(x+1)^3}$.

Solution. $$\frac{(2x^2+3x+1)^3}{(x+1)^3} = \left(\frac{2x^2+3x+1}{x+1}\right)^3$$

$$= \left[\frac{(2x+1)\cancel{(x+1)}}{\cancel{x+1}}\right]^3 = (2x+1)^3.$$

EXERCISE 68

(Solve as many as possible at sight.)

Raise the following to the indicated powers:

1. $\left(\frac{a^2b}{2c}\right)^3$. **2.** $\left(\frac{-2ab^2}{3c}\right)^2$ **3.** $\left(\frac{5c^2d^3}{-2x}\right)^3$.

4. $\left(\frac{-2a^2b^3c}{3xy^2z}\right)^3$. **5.** $\left(\frac{7abc^2}{5xy^2}\right)^2$. **6.** $\left(\frac{2}{3}\right)^5$.

7. $\left(-\frac{a}{3}\right)^5$. **8.** $\left(\frac{a}{b}\right)^8$. **9.** $\left(\frac{a}{b}\right)^p$.

10. $\left(\frac{a}{b}\right)^{2p}$. **11.** $\left(\frac{-abc^n}{xyz}\right)^3$. **12.** $\left(\frac{1}{x}+\frac{1}{y}\right)^2$.

Suggestion. Add the fractions before squaring.

13. $\left(\frac{a}{b}-\frac{x}{y}\right)^2$. **14.** $\left(\frac{a}{2}+b\right)^2$. **15.** $\left(\frac{x}{y}-a\right)^2$.

16. $\left(1-\frac{1}{x}\right)^3$. **17.** $\frac{(a^3+1)^3}{(a^2-a+1)^3}$. **18.** $\frac{(a^4+ab^3)^4}{(a^3+b^3)^4}$.

19. $\frac{(x^2-xy)^5}{(y-x)^5}$. **20.** $\frac{(x^2-xy)^5}{(x-y)^5}$. **21.** $\frac{(m^4-mn^3)^3}{(m^3-n^3)^3}$.

Write each of the following as the square of a fraction:

22. $\dfrac{a^2+2a+1}{b^2-4b+4}$. **23.** $\dfrac{x^2-2xy+y^2}{x^2+2x+1}$. **24.** $\dfrac{x^4}{y^6}$.

25. $\dfrac{4x^2y^4}{9a^4b^6}$. **26.** $\dfrac{36a^2b^4c^2}{64x^4y^4z^{10}}$. **27.** $\dfrac{4x^2-12x+9}{9x^2+12x+4}$.

Division of Fractions

140. Reciprocal of a number. The reciprocal of a number is 1 divided by that number. Hence, whenever the product of two numbers is equal to 1, either one of these numbers is the reciprocal of the other.

Thus, since $\frac{a}{b} \times \frac{b}{a} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, or $\frac{a}{b} = 1 \div \frac{b}{a}$ and $\frac{b}{a} = 1 \div \frac{a}{b}$. The reciprocal of a fraction is obtained by interchanging the numerator and the denominator of the fraction; that is, by *inverting* the fraction.

141. Quotient of two fractions. The quotient of two fractions may be obtained by use of the identity

$$\textit{Dividend} = \textit{Divisor} \times \textit{Quotient} \tag{1}$$

Thus, let it be required to find the quotient of $\frac{a}{b} \div \frac{c}{d}$.

Multiplying both terms of $\frac{a}{b}$ by cd,

$$\frac{a}{b} = \frac{acd}{bcd}. \tag{2}$$

Expressing the second member of (2) as the product of two fractions,

$$\frac{a}{b} = \frac{c}{d} \times \frac{ad}{bc} \tag{3}$$

Observing that $\frac{a}{b}$ in identity (3) corresponds to dividend in identity (1), that $\frac{c}{d}$ in identity (3) corresponds to divisor in identity (1), and that $\frac{ad}{bc}$ corresponds to quotient in identity (1), and since dividend divided by divisor is equal to quotient, we have,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}. \qquad (4)$$

From identity (4) we have the following:

Rule. *To find the quotient of two fractions multiply the dividend by the reciprocal of the divisor.*

142. Two special cases of division.

1. When the dividend is a fraction and the divisor is an integral expression.

Any integral expression can be written in the fractional form.

Thus, c may be written $\frac{c}{1}$.

Therefore, $$\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}. \qquad (1)$$

That is, $$\frac{a}{b} \div c = \frac{a}{bc}. \qquad (2)$$

Again, $$\frac{a}{bc} = \frac{a \div c}{bc \div c} = \frac{a \div c}{b}. \qquad (3)$$

From identities (2) and (3)

$$\frac{a}{b} \div c = \frac{a \div c}{b}. \qquad (4)$$

From identities (2) and (4) we have the following:

Rule. *To divide a fraction by an integral expression, either divide the numerator or multiply the denominator of the fraction by the integral expression.*

Note. In practice, divide the numerator rather than multiply the denominator whenever the numerator is exactly divisible by the integral expression.

2. When the dividend is an integral expression and the divisor is a fraction.

$$a \div \frac{b}{c} = \frac{a}{1} \div \frac{b}{c} = \frac{a}{1} \times \frac{c}{b} = \frac{ac}{b}. \qquad (1)$$

That is, $$a \div \frac{b}{c} = \frac{ac}{b}. \qquad (2)$$

From identity (2) we have the following:

Rule. *To divide an integral expression by a fraction, find the product of the reciprocal of the fraction and the integral expression.*

ILLUSTRATIVE EXAMPLES

1. Divide $\frac{10\,a^2b^3c}{21\,xy^2z^3}$ by $\frac{-15\,ab^2c^3}{28\,xyz^3}$.

Solution.
$$\frac{10\,a^2b^3c}{21\,xy^2z^3} \div \frac{-15\,ab^2c^3}{28\,xyz^3} = \frac{\overset{2\,ab}{\cancel{10\,a^2b^3c}}}{\underset{3\,y}{\cancel{21\,xy^2z^3}}} \times \frac{\overset{4}{\cancel{28\,xyz^3}}}{\underset{-3\,c^2}{\cancel{-15\,ab^2c^3}}}$$
$$= \frac{2\,ab}{3\,y} \times -\frac{4}{3\,c^2}$$
$$= -\frac{8\,ab}{9\,c^2y}.$$

2. Simplify $\frac{x^2y}{a^3} \times \frac{3\,x^3}{4\,b^2} \div \frac{x^4y^2}{6\,a^2b^4}$.

Solution.
$$\frac{x^2y}{a^3} \times \frac{3\,x^3}{4\,b^2} \div \frac{x^4y^2}{6\,a^2b^4} = \frac{\cancel{x^2y}}{\underset{a}{\cancel{a^3}}} \times \frac{\overset{3\,x}{\cancel{3\,x^3}}}{\underset{2}{\cancel{4\,b^2}}} \times \frac{\overset{3\,b^2}{\cancel{6\,a^2b^4}}}{\underset{y}{\cancel{x^4y^2}}}$$
$$= \frac{1}{a} \times \frac{3\,x}{2} \times \frac{3\,b^2}{y}$$
$$= \frac{9\,b^2x}{2\,ay}.$$

3. Simplify $\frac{(x+y)^2}{x^2-y^2} \times \frac{x^2+xy}{(x+y)^2-xy} \div \frac{(x+y)^3}{x^3-y^3}$.

Solution.
$$\frac{(x+y)^2}{x^2-y^2} \times \frac{x^2+xy}{(x+y)^2-xy} \div \frac{(x+y)^3}{x^3-y^3}$$
$$= \frac{\cancel{(x+y)^2}}{\cancel{(x-y)}\cancel{(x+y)}} \times \frac{x\cancel{(x+y)}}{\cancel{x^2+xy+y^2}} \times \frac{\cancel{(x-y)}\cancel{(x^2+xy+y^2)}}{\underset{x+y}{\cancel{(x+y)^3}}}$$
$$= \frac{1}{1} \times \frac{x}{1} \times \frac{1}{x+y}$$
$$= \frac{x}{x+y}.$$

EXERCISE 69

(Solve as many as possible at sight.)

Simplify:

1. $\frac{m}{n} \div m.$ **2.** $\frac{a^2}{b} \div b.$ **3.** $\frac{a}{b} \div c.$

4. $a \div \frac{1}{c}.$ **5.** $m \div \frac{m}{n}.$ **6.** $r \div \frac{r}{s}.$

7. $\frac{(a+b)^2}{c} \div (a+b).$ **8.** $\frac{a+b}{c} \div (a-b).$

9. $\frac{m}{x-y} \div (x+y).$ **10.** $\frac{m^2-n^2}{x} \div (m-n).$

11. $\frac{m^2+n^2}{m-n} \div (m+n).$ **12.** $\frac{a}{x+1} \div (x-1).$

13. $(x-y) \div \frac{1}{x+y}.$ **14.** $(x^2-y^2) \div \frac{x-y}{a}.$

15. $\frac{(x+y)^3}{a} \div (x+y)^2.$ **16.** $\frac{x+1}{x+2} \div (x+2).$

17. $\frac{x^2+2x+1}{a} \div (x+1).$ **18.** $\frac{a^2+3a+2}{b} \div (a+2).$

19. $\frac{a(b^2-c^2)}{b-1} \div a(b+c).$ **20.** $\frac{mn+n^2}{m} \div (m+n).$

21. $\frac{1}{x} \div \frac{1}{2}.$ **22.** $\frac{3}{a} \div \frac{4}{b}.$ **23.** $\frac{1}{x} \div \frac{1}{y}.$

24. $\frac{a}{2} \div \frac{2}{a}.$ **25.** $\frac{2a^3}{3b^3} \div \frac{4a^2}{9b^3}.$ **26.** $\frac{3}{2a} \div \frac{5}{4a^2}.$

27. $\frac{a^3}{b^3} \div \frac{2a^2}{3b^2}.$ **28.** $\frac{a}{b} \div \frac{-x}{y}.$

29. $\frac{5px^2}{3my^3} \div \frac{15p^2x^3}{2m^2y^3}.$ **30.** $\frac{(a+b)^2}{3(x-y)^3} \div \frac{2(a+b)^3}{5(x-y)^2}.$

31. $\frac{2ab}{c} \div \frac{3b}{c^2}$. **32.** $\frac{a}{bx} \div \frac{2d}{3cx}$. **33.** $\frac{2a}{3b} \div \frac{8a}{3b}$.

34. $\frac{ab}{cd} \div \frac{ac}{bd}$. **35.** $\frac{ax}{by} \div \frac{x}{y}$. **36.** $\frac{2ab}{3cd} \div \frac{2b}{3c}$.

37. $\frac{2ab^2c^3}{9xy^2z^4} \div \frac{3a^3b^2c}{4x^4y^2z}$. **38.** $\frac{2a}{3b} \div 2$.

39. $\frac{3a^2}{x^2} \div 2a$. **40.** $3a \div \frac{b}{c}$.

41. $2xy^2 \div \frac{3x^2y}{2z}$. **42.** $(a+b)^2 \div \frac{2(a+b)^3}{3a}$.

43. $\frac{-2ax}{3y^2} \div \frac{2x}{-a^3y}$. **44.** $\frac{-a^3b^2c^4}{2x^2y^3z} \div \frac{2a^3bc^2}{3x^2y^2z}$.

45. $\frac{a^2b}{x^2y} \div \frac{b^2c}{y^2z} \div \frac{abx}{yz}$. **46.** $(x-1) \div \frac{x^2+x+1}{1-x}$.

47. $(x^2-y^2) \div \frac{x-y}{x+y}$. **48.** $(x+a)^2 \div \frac{x+a}{2x-2a}$.

49. $\frac{a^3+b^3}{a-b} \div (a^2+ab+b^2)$. **50.** $\frac{2(x+y)^2}{3(x-y)^3} \div \frac{3(x+y)^3}{2(x-y)^2}$.

51. $\frac{(2x+1)(3x-2)}{(3x+2)(2x-1)} \div \frac{(3x-2)(2x+1)^2}{(3x+2)(2x-1)^2}$.

52. $\frac{(2x+3)(2x-3)}{(3x+2)(3x-1)} \div \frac{(2x+5)(2x-3)}{3(3x-1)(2x-1)}$.

53. $\frac{6x^2+5x-6}{6x^2-5x-6} \div \frac{2x+3}{3x+2}$.

54. $\frac{9x^2-1}{6x^2-5x+6} \div \frac{3x^2+2x-1}{2x^2-x-3}$.

55. $\frac{x(x^3+y^3)}{y(x-y)} \times \frac{(y^2-x^2)^2}{y^2+yx+x^2} \div \left(\frac{y-x}{y+x}\right)^2$.

56. $\frac{(a+b)^2-c^2}{a^2+ab-ac} \times \frac{(a-b)^2-c^2}{(a-c)^2-b^2} \div \frac{ab-b^2-bc}{a}$.

57. $\dfrac{3x^2y^2+3+6xy}{4x^2y^2+4-8xy} \div \dfrac{2xy+2}{3xy-3}.$

58. $\dfrac{x^3-y^3}{x^2-xy+xz-yz} \times \dfrac{x^2-xy+y^2}{x+y} \div \dfrac{x^2+xy+xz+yz}{x^4+x^2y^2+y^4}.$

59. $\dfrac{(x+y)^2-z^2}{(x-y)^2-z^2} \times \dfrac{z^2-x^2+y(y-2z)}{z^2-x^2+y(2x-y)} \div \dfrac{(y+z)^2-x^2}{(y-z)^2-x^2}.$

60. What is the reciprocal of $-\frac{2}{3}$? Of $-\dfrac{a}{b}$?

61. What is the reciprocal of the reciprocal of a fraction? Illustrate by taking $\frac{2}{3}$.

62. When equal factors are cancelled from the numerator and the denominator of a fraction, what operation are we performing on the terms of the fraction?

63. Why can we not cancel the like terms in the numerator and denominator of $\dfrac{2+x}{1+x}$ and obtain $\dfrac{2+\not{x}}{1+\not{x}}=\dfrac{2+1}{1+1}$ $=\dfrac{3}{2}$?

64. For what value of x has $\dfrac{x+1}{x-2}$ no meaning?

Complex Fractions

143. Definition. A **complex fraction** is a fraction which has one or more fractions in either or both of its terms. A complex fraction is said to be simplified when it is reduced to an equivalent simple fraction or integral expression.

In simplifying a complex fraction it is usually most convenient to express each term of the fraction in its simplest form before attempting to perform the indicated division. Sometimes, however, labor is saved by first multiplying both numerator and denominator by the L. C. D. of all the fractions contained in the terms of the given fraction.

ILLUSTRATIVE EXAMPLES

1. Simplify $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$.

Solution. Multiplying both terms of the given fraction by x,

$$\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}=\frac{x+1}{x-1}.$$

2. Simplify $\dfrac{\dfrac{1}{a}-\dfrac{1}{b+c}}{\dfrac{1}{a}+\dfrac{1}{b+c}}\div\dfrac{\dfrac{1}{b}-\dfrac{1}{a+c}}{\dfrac{1}{b}+\dfrac{1}{a+c}}$.

Solution. Multiplying both terms of the first fraction by $a(b+c)$ and both terms of the second fraction by $b(a+c)$, we have,

$$\frac{\dfrac{1}{a}-\dfrac{1}{b+c}}{\dfrac{1}{a}+\dfrac{1}{b+c}}\div\frac{\dfrac{1}{b}-\dfrac{1}{a+c}}{\dfrac{1}{b}+\dfrac{1}{a+c}}=\frac{b+c-a}{b+c+a}\div\frac{a+c-b}{a+c+b}$$

$$=\frac{b+c-a}{b+c+a}\cdot\frac{a+c+b}{a+c-b}$$

$$=\frac{b+c-a}{a+c-b}.$$

EXERCISE 70

Simplify:

1. $\dfrac{\frac{2}{3}}{-\frac{3}{4}}$.

2. $\dfrac{-\frac{4}{5}}{\frac{3}{5}}$.

3. $\dfrac{-2.5}{-7\frac{1}{2}}$.

4. $\dfrac{\dfrac{a}{b}}{\dfrac{c}{b}}$.

5. $\dfrac{\dfrac{mn}{a^2b^2}}{\dfrac{m^2n}{ab}}$.

6. $\dfrac{m-\dfrac{p}{q}}{m+\dfrac{p}{q}}$.

7. $\dfrac{1+\dfrac{m}{n}}{1-\dfrac{m^2}{n^2}}$.

8. $\dfrac{x^2-y^2}{\dfrac{x-y}{a}}$.

9. $\dfrac{\dfrac{r-1}{r+1}}{\dfrac{r+1}{r-1}}$.

10. $\dfrac{4-\dfrac{a^2}{b^2}}{2-\dfrac{a}{b}}$.

11. $\dfrac{x-\dfrac{a^2}{x}}{\dfrac{x-a}{x}}$.

12. $\dfrac{\dfrac{x}{y}-\dfrac{w}{z}}{\dfrac{x}{y}+\dfrac{w}{z}}$.

13. $\dfrac{\dfrac{x}{x-y}-\dfrac{x}{x-z}}{\dfrac{y}{y-x}-\dfrac{z}{z-x}}$.

14. $\dfrac{\dfrac{1}{a+1}+\dfrac{1}{a-1}}{\dfrac{1}{a-1}-\dfrac{1}{a+1}}$.

15. $\dfrac{\left(\dfrac{x}{y}+1\right)\left(\dfrac{y}{z}+1\right)+\left(\dfrac{x}{y}-1\right)\left(\dfrac{y}{z}-1\right)}{1+\dfrac{x}{z}}$.

16. $\dfrac{\dfrac{a^2}{ab+b^2}+\dfrac{b^2}{a^2-ab}}{\dfrac{ab+b^2}{ab-b^2}}$.

17. $\dfrac{a^2-b^2-c^2+2bc}{\dfrac{a-b+c}{a-b-c}}$.

18. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}} \div \dfrac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2}{\dfrac{1}{x^4}-\dfrac{1}{y^4}}$.

19. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y-z}} \div \dfrac{\dfrac{1}{x}+\dfrac{1}{z-y}}{\dfrac{1}{x}-\dfrac{1}{y+z}}$.

20. $\left(1-\dfrac{b}{a}\right)\left(1+\dfrac{b}{a}\right)\div\left(\dfrac{a}{b}+\dfrac{b}{a}+2\right)$.

21. $x^2\left[1-\dfrac{13x-42}{x^2-4x}\right]\left[1-\dfrac{14x-52}{x^2-3x}\right]\div\dfrac{x^2-27x+182}{x^2}$.

22. $\left(\dfrac{1}{a-b}+\dfrac{1}{a+b}\right)\div\left(\dfrac{a-b}{a+b}+1\right)\div\dfrac{1}{a-b}$.

23. $\left(1-\frac{x}{y+z}\cdot\frac{z}{x+y}\right)\div\left(\frac{x}{y+z}-\frac{z}{x+y}\right).$

24. $x+\cfrac{1}{x+\cfrac{1}{x}}.$

Suggestion. Multiply both terms of the complex fraction by x and then reduce the resulting mixed number to an improper fraction. A complex fraction of the form given in this example is called **a continued fraction.**

25. $1+\cfrac{1}{1+\cfrac{1}{x}}.$ **26.** $\cfrac{3}{2+\cfrac{3}{2+x}}.$

EXERCISE 71. — REVIEW

1. Name at sight the result in each of the following :

$$a\times\frac{b}{c};\ a\times\frac{x}{ac};\ \frac{m^2}{n}\div m;\ \frac{r}{R}\div R.$$

2. Name at sight the result in each of the following :

$(a+b)\frac{c}{(a+b)^2}$; $\frac{m-n}{m+n}\div(m+n)$; $(m+n)\frac{m-n}{m+n}$; $\frac{x^2-y^2}{z}\div(x-y)$.

3. Reduce $\frac{am^2-9\,a}{a^2m+3\,a^2+am+3\,a}$ to lowest terms.

4. Does $\frac{(a-b)^2}{c-d}=-\frac{(b-a)^2}{d-c}$? Why?

5. Reduce $1-\frac{(m-n)^2}{m^2+n^2}$ to an improper fraction.

6. Reduce $\frac{1-x^3}{1+x}$ to a mixed expression.

7. Add $\frac{1}{a}$ to a.

8. Subtract x from $\frac{1}{x}$.

9. Simplify $\left(\frac{a}{b}+\frac{x}{y}\right)+\left(\frac{a}{b}-\frac{x}{y}\right)$.

Simplify:

10. $\frac{1}{m+n}-\frac{1}{m-n}$.

11. $\frac{1}{abc^2}+\frac{1}{ab^3c}$.

12. $\frac{1}{2x-y}-\frac{1}{3x-2y}$.

13. $\frac{max}{nbx+nby}+\frac{nby}{max+may}$.

14. $\frac{1}{3x-4y}+\frac{1}{3x+4y}+\frac{1-6x}{9x^2-16y^2}$.

15. $\frac{1}{a-b}+\frac{2}{a+b}+\frac{b}{a^2-b^2}+\frac{3a^2}{(a-b)^2}$.

16. $\frac{1}{x-1}+\frac{1}{1-x^2}+\frac{1}{x^3-1}$.

17. $\frac{1}{x+a}+\frac{1}{x-a}-\frac{1}{x-b}-\frac{1}{x+b}$.

18. $\frac{x}{x-y}+\frac{y}{y-z}+\frac{z}{z-x}+\frac{x}{x-z}+\frac{y}{y-x}+\frac{z}{z-y}$.

19. $\frac{1}{a+3b}+\frac{1}{a-3b}-\frac{18b^2}{a^3-9ab^2}$.

20. $\left(1-\frac{x-3}{x^2-x-2}\right)\frac{x+1}{x-1}$.

21. $a-\frac{1}{1-a}+\frac{1-3a+a^3}{1-a^2}$.

22. $a+1+\frac{1}{a}-\frac{a^3+3a^2-1}{a^2+2a}.$

23. $\frac{(x-2y)^2}{(x+2y)^2}-\frac{(x+2y)^2}{(x-2y)^2}.$

24. $\frac{1}{a}-\frac{2}{a-1}+\frac{1}{a-2}+\frac{a+1}{a(a-1)(a-2)}.$

25. $\frac{2}{\frac{1}{x}-\frac{1}{y}}+\frac{3}{\frac{1}{x}+\frac{1}{y}}.$

26. $\frac{a^2+2bc}{(a-c)(b-a)}+\frac{b^2+2ca}{(b-a)(c-b)}+\frac{c^2+2ab}{(c-b)(a-c)}.$

Suggestion. Factor the resulting numerator by arranging according to powers of a.

27. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$

28. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$

29. $a+x-\frac{x^2}{a-x}-\frac{a^2+2x^2}{a+x}.$

30. $\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}-\frac{2}{(x+3)(x+1)}.$

31. $\left[1+\frac{x}{x-y}\right]\left[1-\frac{x}{x+y}\right]\left[\frac{x^2-y^2}{2x^2-xy}\right].$

32. $\left[a+1-\frac{a-b}{a+b}\right]\div\left[a+2-\frac{2a}{a+b}\right].$

33. $\left(a^2+a+1+\frac{1}{a-1}\right)\left(a^2-a+1-\frac{1}{a+1}\right)\left(\frac{1}{a^4}-\frac{1}{a^6}\right).$

34. $\frac{1}{x^2-1}+\frac{1}{x^2+1}-\frac{x^2-1}{x^4+x^2+1}-\frac{x^2+1}{x^4-x^2+1}.$

35. $1+\frac{2x+1}{x+1}+\frac{x+1}{(x+3)^2}+\frac{4(x+2)}{(x+3)^2(x+1)}$.

36. $x+3+\frac{2x^2}{(x-1)^2}-\frac{10}{x^2-1}-\frac{4}{(x+1)(x-1)^2}-\frac{x^2+4x+7}{x+1}$.

37. $\left(\frac{2xy+3x-y}{3x-y}+\frac{2xy-3x-y}{3x+y}\right)\div\frac{x^2y}{9x^2-y^2}$.

38. $\dfrac{\frac{x^2+2x+3}{x}-\frac{y^2+2y+3}{y}}{\frac{3x^2+2x+1}{x}-\frac{3y^2+2y+1}{y}}$.

39. $\frac{x^2+x}{x^2+5x+4}\times\frac{x^2+7x+10}{x^2+2x}\times\frac{x^2+7x+12}{x^2+8x+15}$.

40. $\left(a+\frac{b}{c}\right)^2\left(a-\frac{b}{c}\right)^2\left(a^2+\frac{b^2}{c^2}\right)^2$.

41. $\frac{1}{x}-\left[2-\left\{\frac{2x-1}{x}+\frac{1}{3}\left(\frac{x-2}{x+2}-\frac{(x+1)(x-3)}{x(x+2)}\right)\right\}\right]$.

42. $\frac{3}{2h+c}-\frac{8h^2+7c^2}{8h^3+c^3}-\frac{2h-6c}{4h^2-2hc+c^2}$.

43. $\frac{3x-1}{x+3}-\frac{2x+3}{1-x}-\frac{5x^2-2x-1}{x^2+2x-3}$.

44. $\dfrac{a-\frac{a^2}{a+b}}{a-b}+\frac{a^2+ab+b^2}{a^2-b^2}$.

45. $\dfrac{\frac{x^3-y^3}{a^2-b^2}}{\frac{x^2+xy+y^2}{a-b}}\times\dfrac{\frac{x^3+y^3}{a^2-b^2}}{\frac{x^2-xy+y^2}{a+b}}\times\frac{a^2-b^2}{x^2-y^2}$.

46. $2+\frac{1}{c}-\left[2c-3\left(\frac{1}{c+1}-\frac{1}{1-c}\right)-\left(\frac{1}{c-1}-\frac{1}{c}-2\right)\right]$.

47. $\frac{a+b}{a^2+ab+b^2}\times\frac{a^3-b^3}{a^2-b^2}\times\frac{a+b}{a-b}\div\left(1+\frac{a}{a-b}\right)\left(1+\frac{b}{a}\right)$.

48. $\dfrac{\frac{x-y}{x+y}+\frac{x+y}{x-y}}{\frac{x-y}{x+y}-\frac{x+y}{x-y}}\div\left(\frac{1}{x}+\frac{1}{y}\right)$.

49. Is the following a true statement: $\dfrac{\frac{a}{b}}{\frac{c}{b}}=\frac{a}{c}$? What general rule can you derive from it? Apply your rule to find the value of $\dfrac{\frac{4}{21}}{\frac{5}{21}}$.

50. Is the following a true statement: $\dfrac{\frac{a}{b}}{\frac{a}{c}}=\frac{c}{b}$? What general rule can you derive from it? Apply the rule to find the value of $\dfrac{\frac{3}{4}}{\frac{3}{8}}$.

51. Show that

$$(1\times ax)\left(1+\frac{x}{a}\right)\left(1+\frac{a}{x}\right)\left(1+\frac{1}{ax}\right)=\left(a+\frac{1}{a}+x+\frac{1}{x}\right)^2.$$

Reduce each of the following fractions to a mixed number before performing the indicated additions or subtractions:

52. $\frac{2x-1}{x-1}-\frac{2x+1}{x+1}$.

53. $\frac{3x^2+3x+1}{x+1}+\frac{2+3x-3x^2}{x-1}$.

CHAPTER VI

FRACTIONAL AND LITERAL EQUATIONS

Fractional Equations

144. Definition. A rational equation in one unknown number which is not integral with respect to that number must contain the unknown number in the denominator of one or more terms. Such an equation is called a **fractional equation in one unknown number.**

Thus, $\frac{1}{x}+\frac{1}{a}=2$ and $\frac{3x}{x-2}+\frac{2}{x+2}+\frac{5}{2}=0$ are fractional equations.

145. Clearing an equation of fractions. From a fractional equation in which each fraction has been reduced to lowest terms, an integral equation can be derived by multiplying both members of the equation by the lowest common denominator of all its fractions. This process is called **clearing the equation of fractions.**

146. Solution of a fractional equation. The solution of a fractional equation is found by solving the integral equation obtained from it. Hence:

Rule. *To solve a fractional equation, in which each fraction is in its lowest terms:*

1. *Clear the equation of fractions by multiplying both members by the lowest common denominator of all the fractions.*
2. *Solve the resulting integral equation.*
3. *Test the roots by substituting them in the given equation.*

ILLUSTRATIVE EXAMPLES

1. Solve $\frac{2x+1}{x+3}-\frac{5x-2}{3x+1}=0.$

Solution. $$\frac{2x+1}{x+3}-\frac{5x-2}{3x+1}=0. \quad (1)$$

Clearing (1) of fractions by multiplying both members by
$$(x+3)(3x+1),$$
$$(2x+1)(3x+1)-(x+3)(5x-2)=0. \quad (2)$$

Performing the indicated operations in (2),
$$x^2-8x+7=0. \quad (3)$$

Factoring, $(x-7)(x-1)=0.$ (4)

Equating $(x-7)$ to 0, $x-7=0.$ (5)

Equating $(x-1)$ to 0, $x-1=0.$ (6)

Solving (5), $x=7.$ (7)

Solving (6), $x=1.$ (8)

Check. $$\frac{2x+1}{x+3}-\frac{5x-2}{3x+1}=0. \quad (9)$$

Substituting 7 for x in (9),
$$\frac{2\times 7+1}{7+3}-\frac{5\times 7-2}{3\times 7+1}=0. \quad (10)$$

Simplifying, $\frac{3}{2}-\frac{3}{2}=0.$ (11)

Substituting 1 for x in (9),
$$\frac{2\times 1+1}{1+3}-\frac{5\times 1-2}{3\times 1+1}=0. \quad (12)$$

Simplifying, $\frac{3}{4}-\frac{3}{4}=0.$ (13)

2. Solve $\frac{x^2-2x+1}{x^2+3x-4}-\frac{16(x+1)}{5x^2+46x+41}=0.$

Solution. $$\frac{x^2-2x+1}{x^2+3x-4}-\frac{16(x+1)}{5x^2+46x+41}=0. \quad (1)$$

Factoring, $$\frac{(x-1)^2}{(x-1)(x+4)}-\frac{16(x+1)}{(5x+41)(x+1)}=0. \quad (2)$$

Reducing the fractions to lowest terms,
$$\frac{x-1}{x+4}-\frac{16}{5x+41}=0. \quad (3)$$

Clearing (3) of fractions, and dividing by 5,

$$x^2 + 4x - 21 = 0. \tag{4}$$

Factoring, $(x + 7)(x - 3) = 0.$ (5)

Equating $(x + 7)$ to 0, $x + 7 = 0.$ (6)

Equating $(x - 3)$ to 0, $x - 3 = 0.$ (7)

Solving (6), $x = -7.$ (8)

Solving (7), $x = 3.$ (9)

Check. $\dfrac{x^2 - 2x + 1}{x^2 + 3x - 4} - \dfrac{16(x + 1)}{5x^2 + 46x + 41} = 0.$ (10)

Substituting -7 for x in (10),

$$\frac{(-7)^2 - 2(-7) + 1}{(-7)^2 + 3(-7) - 4} - \frac{16(-7 + 1)}{5(-7)^2 + 46(-7) + 41} = 0. \tag{11}$$

Simplifying, $\dfrac{49 + 14 + 1}{49 - 21 - 4} - \dfrac{-96}{245 - 322 + 41} = 0,$ (12)

or, $\frac{8}{3} - \frac{8}{3} = 0.$ (13)

Substituting 3 for x in (10),

$$\frac{(3)^2 - 2 \times 3 + 1}{(3)^2 + 3 \times 3 - 4} - \frac{16(3 + 1)}{5 \times 3^2 + 46 \times 3 + 41} = 0.$$

Simplifying, $\dfrac{9 - 6 + 1}{9 + 9 - 4} - \dfrac{64}{45 + 138 + 41} = 0,$

or $\frac{2}{7} - \frac{2}{7} = 0.$

Note. In certain exceptional cases the integral equation obtained by clearing a given equation (in which each fraction is in its lowest terms) of fractions by multiplying both members by the lowest common multiple of the denominators, is not equivalent to the given equation. It is important, therefore, that each root of the integral equation should be checked by substituting it for the unknown number in the given fractional equation.

The way in which these exceptional cases arise may be seen from a consideration of a particular example. Thus, let it be required to determine the value of x from

$$\frac{x + 1}{x} + \frac{1}{x(x - 1)} = \frac{1}{x - 1}.$$

Clearing of fractions, $(x^2 - 1) + 1 = x.$

Simplifying, $x^2 - x = 0.$

Hence, $x = 0$ or $1.$

On substituting the value 0 for x in the given equation, the first fraction becomes $\frac{1}{0}$, which is without meaning, and, therefore, this value of x does not satisfy the given equation. In like manner, the second root of the integral equation does not satisfy the given equation.

By writing the given equation in the form

$$\frac{x+1}{x}+\frac{1}{x(x-1)}-\frac{1}{x-1}=0$$

and simplifying, we obtain

$$\frac{x^2-x}{x(x-1)}=0.$$

Simplifying,

$$1=0.$$

Now this is an impossible number relation, and the reason for the non-existence of any value of x which will satisfy the given equation is evident. In fact, although the given relation is expressed in the form of an equation, the expression is not actually an equation.

EXERCISE 72

Solve the following equations and check each solution :

1. $\frac{1}{x}=2.$ **2.** $\frac{2}{x}=3.$ **3.** $3=\frac{4}{x}.$

4. $\frac{1}{x}+\frac{2}{x}=\frac{4}{3}.$ **5.** $\frac{3}{2x}=\frac{2x}{3}.$ **6.** $\frac{2}{3x}=\frac{1}{4}.$

7. $\frac{1}{x-2}=\frac{2}{x}.$ **8.** $\frac{2}{x+2}-\frac{3}{x}=0.$

9. $\frac{5}{x}+\frac{6}{3-x}=0.$ **10.** $\frac{4}{x-3}+\frac{3}{x+1}=0.$

11. $\frac{3}{x+4}-\frac{2}{2x+1}=0.$ **12.** $\frac{2}{x+3}-\frac{3}{x-1}=0.$

13. $\frac{4}{x+2}+\frac{3}{3x-1}=0.$ **14.** $\frac{5}{x+1}-\frac{3}{x-1}=0.$

15. $\frac{11}{2x-3}=\frac{5}{3x-2}.$ **16.** $\frac{5}{2x+5}=\frac{7}{2x+7}.$

17. $\frac{5}{3x-2}+\frac{2}{5x+2}=0.$ **18.** $\frac{3}{2x+2}+\frac{4}{3x-3}=0.$

19. $\frac{15}{3x+2}-\frac{1}{x+1}=0.$

20. $\frac{3}{2x}+\frac{1}{3x-9}=\frac{4}{3x}-\frac{1}{x-3}.$

21. $\frac{1}{3x}+\frac{1}{6}=\frac{2}{x(x-2)}.$

22. $\frac{3}{x}+1=\frac{2}{3x}.$ **23.** $\frac{3}{4x}+2=\frac{1}{6x}.$

24. $\frac{3x-4}{2x+1}+\frac{x-1}{x+4}=1.$ **25.** $\frac{2x}{3x-8}=\frac{3x-5}{x+2}.$

26. $\frac{x}{2x-7}+\frac{x-4}{x+3}=\frac{2x^2-7}{2x^2-x-21}.$

27. $\frac{3x-2}{x+1}+\frac{3x+1}{2x-1}-\frac{29x+19}{2x^2+x-1}=0.$

28. $\frac{4(x+2)}{x^2-1}-\frac{7(x-1)}{x^2-x-2}+\frac{3(x+1)}{x^2-3x+2}=0.$

29. $\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{x+3}=\frac{5x^2+30x+11}{(x+1)(x+2)(x+3)}.$

30. $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+5}{x+6}-\frac{x+6}{x+7}.$

Suggestion. Simplify each member before clearing of fractions.

31. A can perform a piece of work in two thirds of the time in which B can; B can perform it in one half of the time in which C can; they all can perform it in 22 days. Find the time in which each alone can do the work.

32. A and B working together can do a certain piece of work in 6 days. A working alone can do the same work in 10 days. In what time can B alone do the work?

Suggestion. Let $x =$ the number of days required.

Then, $\frac{1}{x} =$ the part B can do in 1 day.

33. What number must be added to each term of the fraction $\frac{2}{7}$ so that the resulting fraction shall equal $\frac{2}{3}$?

34. What number must be subtracted from each term of the fraction $\frac{13}{19}$ so that the resulting fraction shall equal $\frac{1}{3}$?

35. Find a proper fraction whose numerator and denominator differ by 1, and such that the result of adding $\frac{13}{10}$ to the fraction is double the result of subtracting the fraction from 2.

36. A can do a piece of work in 10 days and B can do it in 8 days. In what time can they do it working together?

37. A certain pipe can fill a cistern in 10 hours and another can fill it in 12 hours. In what time can they fill it if both pipes run together?

38. A certain fraction is equal to $\frac{3}{4}$. When its numerator is diminished by 5 and its denominator by 8, the resulting fraction is equal to $\frac{5}{6}$. Find the fraction.

Suggestion. Let the fraction be denoted by $\frac{\frac{3}{4}x}{x}$.

39. A can do a certain piece of work in 3 days, B can do it in 4 days, and C can do it in 6 days. In what time can they do it working together?

40. Three pipes are connected with a certain reservoir. The first pipe can fill it in 2 hours, the second in 5 hours, and the third can empty it in 10 hours. In what time will the reservoir be filled if the three pipes are set to flow at the same time?

Literal Equations

147. Numerical equation. A **numerical equation** is an equation in which all the known numbers are expressed by numerals.

Thus, $6x - 1 = 2x + 3$ is a numerical equation.

Remark. The equations which have been considered in the preceding pages are, in general, numerical equations.

148. Literal equation. A **literal equation** is an equation which involves one or more known literal numbers.

Thus, $ax + bx + c = 0$ is a literal equation.

Remark. In section 69 it was stated that in algebra known numbers are usually represented by the first letters of the alphabet.

149. Solution of a literal equation. A literal equation in one unknown number is solved in the same manner as a numerical equation, but the roots of a literal equation are algebraic expressions in the known numbers.

The known numbers in a literal equation are called such to distinguish them from the unknown numbers. In an equation it is sometimes convenient to consider one of the literal numbers as the unknown and at other times another.

Thus, from the formula $A = ab$, which expresses the area of a rectangle in terms of its base and altitude, A, a, or b can be found when the other two are known.

ILLUSTRATIVE EXAMPLES

1. Solve the equation $\frac{x-b}{a-b} - \frac{x-a}{a+b} = \frac{a^2+b^2}{a^2-b^2}$.

Solution.
$$\frac{x-b}{a-b} - \frac{x-a}{a+b} = \frac{a^2+b^2}{a^2-b^2}. \quad (1)$$

Clearing (1) of fractions,

$$(a+b)(x-b)-(a-b)(x-a)=a^2+b^2. \tag{2}$$

Performing the indicated multiplications,

$$ax+bx-ab-b^2-ax+bx+a^2-ab=a^2+b^2. \tag{3}$$

Combining like terms,

$$2\,bx+a^2-2\,ab-b^2=a^2+b^2. \tag{4}$$

Transposing terms, $$2\,bx=2\,ab+2\,b^2. \tag{5}$$

Dividing by $2\,b$, $$x=a+b. \tag{6}$$

2. Solve the equation $\dfrac{a}{x+a}-\dfrac{b}{x+b}=\dfrac{a-b}{x-a+b}$.

Solution. $$\frac{a}{x+a}-\frac{b}{x+b}=\frac{a-b}{x-a+b}. \tag{1}$$

Combining terms in the first member,

$$\frac{(a-b)x}{(x+a)(x+b)}=\frac{a-b}{x-a+b}. \tag{2}$$

Dividing both members of (2) by $(a-b)$,

$$\frac{x}{(x+a)(x+b)}=\frac{1}{x-a+b}. \tag{3}$$

Clearing (3) of fractions,

$$x^2-ax+bx=x^2+ax+bx+ab. \tag{4}$$

Transposing in (4) and uniting like terms,

$$2\,ax=-\,ab. \tag{5}$$

Dividing both members of (5) by $2\,a$,

$$x=-\frac{b}{2}. \tag{6}$$

Check: Substituting $-\frac{b}{2}$ for x in (1),

$$\frac{a}{-\frac{b}{2}+a}-\frac{b}{-\frac{b}{2}+b}=\frac{a-b}{-\frac{b}{2}-a+b}. \tag{7}$$

Simplifying, $$\frac{a-b}{b-2\,a}=\frac{a-b}{b-2\,a}. \tag{8}$$

3. What number must be added to both numerator and denominator of the fraction $\frac{l}{m}$ so that the resulting fraction shall be equal to $\frac{n}{p}$?

Solution. Let x denote the required number.

From the conditions of the problem,

$$\frac{l+x}{m+x}=\frac{n}{p}. \qquad (1)$$

Clearing (1) of fractions,

$$lp+px=nm+nx. \qquad (2)$$

Transposing in (2) and uniting like terms,

$$(p-n)x=nm-lp. \qquad (3)$$

Dividing both members of (3) by $(p-n)$,

$$x=\frac{nm-lp}{p-n}. \qquad (4)$$

Check.

$$\frac{l+x}{m+x}=\frac{l+\dfrac{nm-lp}{p-n}}{m+\dfrac{nm-lp}{p-n}}$$

$$=\frac{pl-ln+nm-lp}{pm-nm+nm-lp}$$

$$=\frac{n(m-l)}{p(m-l)}=\frac{n}{p}.$$

Application of formula. Substitute $l=5$, $m=6$, $n=16$, $p=17$, in formula (4),

$$x=\frac{nm-lp}{p-n},$$

$$x=\frac{6\times16-5\times17}{17-16},$$

$$=96-85, \text{ or } \mathbf{11}.$$

EXERCISE 73

Solve the following and check all roots:

1. $x - b = a.$

2. $\frac{x}{a} = b.$

3. $\frac{1}{x} = \frac{1}{a}.$

4. $ax = b.$

5. $2x + ax = 1.$

6. $ax - x + bx = c.$

7. $ax + bx - x = 0.$

8. $\frac{1}{x} = a.$

9. $\frac{2}{x} = \frac{1}{a}.$

10. $ax = \frac{1}{b}.$

11. $ax + bx = 1.$

12. $ax - 1 = 2 - bx.$

13. $\frac{1}{x-1} = \frac{a}{2}.$

14. $ax - bx + (c + d)x = a - b + c + d.$

15. $ax - b = d - cx.$

16. $\frac{a}{x} - b = \frac{c}{x}.$

17. $\frac{a+b}{x} = \frac{1}{x-1}.$

18. $\frac{ax}{2} - \frac{bx}{3} + \frac{cx}{4} = 1.$

19. $2ax - 3bx = 2bx - 3ax + c.$

20. $(a + b - 1)x - (a - b + 1) = (a + b - 1) - (a - b + 2)x.$

21. $a(x - a) + b(x - b) = 3ax - (a + b)^2.$

22. $a(x + b) + b(x + a) = 2ax + b^2 - (a - b)^2.$

23. $(x + a)^2 = 4a^2 + (x - a)^2.$

24. $(x - a)(x - b) - (x - c)(x - d) = 0.$

25. $a + \frac{b}{x} = c.$

26. $\frac{a}{x} + \frac{b}{x} = c - \frac{d}{x}.$

27. $\frac{a}{x} - \frac{b}{x} + \frac{c}{x} = 1.$

28. $\frac{a+x}{ax} + \frac{b+x}{bx} + \frac{a+b}{ab} = 0.$

29. $\frac{a-x}{a^2x} + \frac{b+x}{b^2x} + \frac{1}{a} + \frac{1}{b} = 0.$

30. $\frac{2}{b} + \frac{b}{x} = \frac{a-2x}{2b-bx}.$

31. $\frac{x-a}{x+a} + \frac{x+b}{x-b} = 2.$

32. $\frac{bx}{x-a} + \frac{a^3x}{ax-b} = a^2 + b.$

33. $\frac{b}{1-ax} + \frac{a}{1-bx} = a + b.$

34. $\frac{a}{ac-x} + \frac{b}{ab-x} = \frac{a}{ab-x}.$

35. $\frac{x+a}{x+b} = \frac{2x+a+c}{2x+b+c}.$

36. $a^2x(bx-1) + b^2x(ax-1) = (a+b)(ax-1)(bx-1).$

EXERCISE 74

1. $A = ab.$
   (*a*) Solve for a.
   (*b*) Solve for b.

2. $C = 2\pi r.$ Solve for r.

3. The volume of a prism is expressed by the formula $V = B \times H.$
   (*a*) Solve for B.
   (*b*) Solve for H.

4. The volume of a rectangular parallelepiped is expressed by the formula $V = abc$.

(*a*) Solve for a.
(*b*) Solve for b.
(*c*) Solve for c.
(*d*) Find a when $V = 100$, $b = 10$, and $c = 5$.

5. The lateral area of a regular pyramid is expressed by the formula $S = \frac{1}{2} L \times P$.

(*a*) Solve for L.
(*b*) Solve for P.
(*c*) Find L when $S = 40$ and $P = 8$.

6. The volume of a pyramid is expressed by the formula $V = \frac{1}{3} B \times H$.

(*a*) Solve for B.
(*b*) Solve for H.
(*c*) Find H when $V = 63$ and $B = 15$.

7. The lateral area of a cylinder of revolution is expressed by the formula $S = 2\pi RH$.

(*a*) Solve for R.
(*b*) Solve for H.
(*c*) Find R when $S = 24$ and $H = 4$.

8. The lateral area of a cone of revolution is expressed by the formula $S = \pi RL$.

(*a*) Solve for R.
(*b*) Solve for L.
(*c*) Find S when $R = 4$ and $L = 5$.
(*d*) Find L when $S = 15$ and $R = 3$.

9. $A = \frac{1}{2}(b + B)a$ is the formula stating the area of a trapezoid.

(*a*) Solve for a.
(*b*) Solve for $(b + B)$.

(*c*) Solve for b.

(*d*) Find a from the following data: $A = 95$, $B = 20$, $b = 18$.

10. $v = gt$ is a formula from physics.

(*a*) Solve for t.

(*b*) Find t, when $g = 32.16$ and $v = 80.4$.

11. $h = \frac{wv^2}{1.25\,r}$ is a formula from engineering.

(*a*) Solve for w.

(*b*) Solve for r.

12. $\frac{1}{f} = \frac{1}{a} + \frac{1}{b}$ is a formula from physics.

(*a*) Solve for f.

(*b*) Solve for a.

13. $A = P + \frac{Prt}{100}$ is a formula from arithmetic.

(*a*) What do A, P, r, and t, respectively, represent in the preceding formula?

(*b*) Solve for P.

(*c*) Solve for t.

(*d*) Solve for $\frac{r}{100}$.

14. $I = \frac{E}{R}$ is a formula used in electrical work.

(*a*) Solve for E.

(*b*) Solve for R.

(*c*) Find I when $E = 200$ and $R = 250$.

(*d*) Find E when $I = 1.5$ and $R = 200$.

(*e*) Find R when $I = .4$ and $E = 120$.

15. The formula for converting a temperature of F degrees Fahrenheit into its equivalent of C degrees

Centigrade is $C = \frac{5}{9}(F - 32)$. Express F in terms of C, and compute F when:

(*a*) $C = 20$.
(*b*) $C = 30$.
(*c*) $C = 27$.

16. What must be added to $x + a$ to make $y - b$?

17. What is the cost of 3 oranges if a oranges cost c cents?

18. Divide the number n into four parts such that, if a is added to the first, a subtracted from the second, the third multiplied by a, and the fourth divided by a, the results will be equal. What are the results if $n = 10$, and $a = 1$?

19. If my age is such that in n years I shall be a times as old as I was m years ago, what is my age?

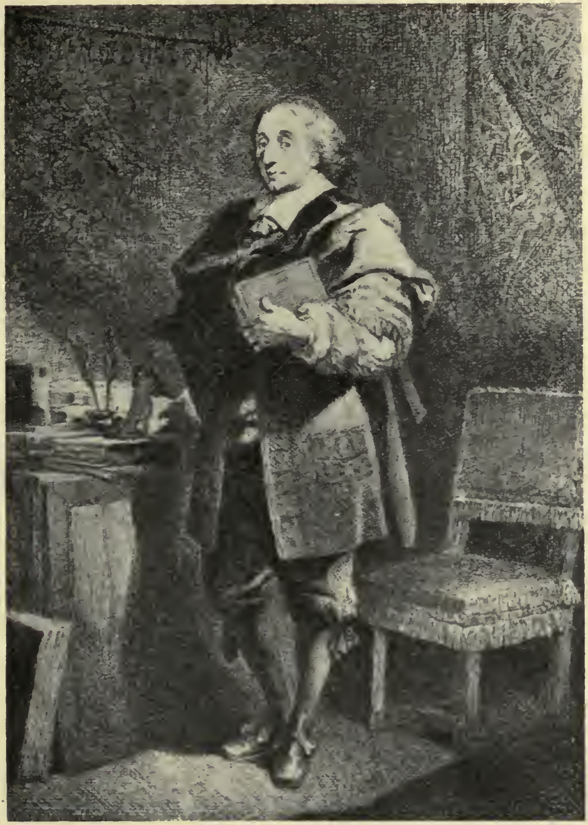
20. If each side of a square were n feet longer, its area would be p^2 square feet greater. Find the length of its side.

21. How many pounds of coffee worth a cents a pound must be mixed with b pounds worth c cents a pound so that the mixture may be worth d cents a pound?

22. Find the number the sum of whose ath and bth parts is c. What is the number when $a = 5$, $b = 7$, and $c = 12$?

23. $t = \dfrac{W}{W + w}\left(1 + \dfrac{V}{v}\right)\dfrac{2a}{v}$.

(*a*) Solve for W.
(*b*) Solve for w.
(*c*) Solve for V.
(*d*) Calculate W when $t = .0019$, $w = 2$, $V = -4$, $v = 100$, $a = 0.1$.



Blaise Pascal (1623–1662) was born at Clermont and died at Paris. Perhaps no man ever displayed greater natural genius than Pascal. His contributions to mathematics were not extensive, but his name will always be mentioned in connection with the arithmetical triangle. Much of his mathematical work was done when he was a boy. His essay on conic sections was written in 1639, when he was but sixteen years old.

CHAPTER VII

SYSTEMS OF LINEAR EQUATIONS

150. An equation may involve more than one unknown number.

Thus, the equation $2x + 3y = 5$ contains two unknown numbers, and the equation $ax + by + cz + d = 0$ contains three.

151. Degree of an equation. By the **degree of a rational integral equation** in two or more unknowns is meant its degree with respect to the unknown numbers.

Thus, $2x + 3y = 5$ is an equation of the first degree in the two unknown numbers x and y; $3x^2 - 2xy = 1$ is an equation of the second degree; $2x^2y + 3z^2 = 5$ is an equation of the third degree.

152. Linear equation. An integral equation of the first degree in any number of unknowns is called a **linear equation**, or a **simple equation.**

Thus, $2x = 6$ and $2x + 3y = 12$ are linear equations.

153. Solution of an equation. A solution of an equation in more than one unknown is *any set of values of the unknown numbers which satisfies the equation.*

Thus, the linear equation $2x + 3y = 5$ has among other solutions the following sets:

$$x = 1,\ y = 1;\ \ x = 0,\ y = \tfrac{5}{3};\ \ x = \tfrac{5}{2},\ y = 0.$$

154. Number of solutions of an equation. The **number of solutions** of an equation in more than one unknown is not determinate.

Thus, in the equation $2x + 3y = 5$, we may assign any value to x and solve the resulting linear equation for y; for example:

Let $\quad x = 7;$

then, $\quad 14 + 3y = 5,$

whence, $\quad y = -3.$

A solution of the equation $2x + 3y = 5$ is, therefore, $x = 7$, $y = -3$.

Remark. It is because the number of solutions of an equation in more than one unknown is not determinate that such an equation is called an **indeterminate equation.**

ILLUSTRATIVE EXAMPLES

Find four solutions of the indeterminate equation $2x + 3y = 10$, and check each solution found.

Solution. $$2x + 3y = 10. \tag{1}$$

Solving (1) for y in terms of x,

$$y = \frac{10 - 2x}{3}. \tag{2}$$

Assigning any four values to x, as 0, 1, -1, 2, we have,

when $x = 0$, $\quad y = \frac{10 - 2 \times 0}{3} = \frac{10}{3};$

when $x = 1$, $\quad y = \frac{10 - 2 \times 1}{3} = \frac{8}{3};$

when $x = -1$, $\quad y = \frac{10 - 2(-1)}{3} = 4;$

when $x = 2$, $\quad y = \frac{10 - 2 \times 2}{3} = 2.$

Four solutions of $2x + 3y = 10$ are, therefore, $(0, \frac{10}{3})$, $(1, \frac{8}{3})$, $(-1, 4)$, $(2, 2)$, in each of which the value of x stands first.

Check. $\quad 2x + 3y = 10.$

Substituting 0 for x and $\frac{10}{3}$ for y in the given equation,

$$2 \times 0 + 3(\tfrac{10}{3}) = 10.$$

Substituting 1 for x and $\frac{8}{3}$ for y,

$$2 \times 1 + 3 \times \tfrac{8}{3} = 10.$$

Substituting -1 for x and 4 for y,

$$2(-1) + 3 \times 4 = 10.$$

Substituting 2 for x and 2 for y,

$$2 \times 2 + 3 \times 2 = 10.$$

EXERCISE 75

Find four solutions of each of the following indeterminate equations and check each solution found.

1. $y = 2x$.
2. $x + y = 4$.
3. $x + 2y = 7$.
4. $2x - 3y = 5$.
5. $3x + 3y = 5$.
6. $y = x + 1$.
7. $y = mx$.
8. $y = mx + b$.

155. Independent equations. Two linear equations in two or more unknowns are called **independent equations** if each has solutions which are not solutions of the other.

Thus, $x + y = 8$ and $x - y = 2$ are two independent linear equations, as $x = 8$ and $y = 0$ is a solution of the first equation which is not a solution of the second; and $x = 2$ and $y = 0$ is a solution of the second equation which is not a solution of the first.

156. Dependent equations. Two linear equations in two or more unknowns are called **dependent equations** when every solution of the one is also a solution of the other.

Thus, $x + y = 1$ and $2x + 2y = 2$ are two dependent linear equations.

157. Inconsistent equations. Two equations which have no common solution are called **inconsistent equations.**

Thus, $x + y = 2$ and $x + y = 1$ are evidently inconsistent, since the sum of two numbers cannot be both 2 and 1 at the same time.

158. Simultaneous equations. Two or more equations in more than one unknown number, when considered together, are said to be **simultaneous** if they have at least one solution in common.

159. System of equations. Equations considered together are called a **system of equations.**

160. Number of solutions of two independent linear equations in two unknown numbers. Two independent and consistent linear equations in two unknown numbers are satisfied by one and only one set of values of the unknown numbers. Such a system of linear equations has, therefore, only one solution.

Note. The student in his later algebraic work will find that the fact stated in section 160 covers a particular case which is included in a more general statement. He can, however, satisfy himself that two independent linear equations in two unknowns have no more than one solution in common. For instance, to show that $x + y = 8$ and $x - y = 4$ have one and only one solution, we may write

$$x + y = 8, \tag{1}$$
$$x - y = 4; \tag{2}$$

then by adding (1) and (2) we have $2x = 12$. Therefore, any value of x which satisfies both equations must be such that when multiplied by 2 the result is 12, and there is but one such number. Moreover, by subtracting (2) from (1), we have $2y = 4$, and there is but one value of y. It will be made evident in the course of this chapter that a similar proof based on the theory of equivalent systems of equations holds for the general equations of the form $ax + by = c$ and $mx + ny = p$.

Remark. Two systems of equations in two or more unknown numbers are said to be equivalent when every solution of the first system is a solution of the second and every solution of the second is a solution of the first.

161. Method of solution. Two independent linear equations in two unknowns are solved by combining these equations in such a way that there results a simple equation in one unknown. One of the unknowns does not appear in the resulting equation; it has been *eliminated*. The process by which the unknown is eliminated is called **elimination.**

162. Elimination by addition or subtraction. The solution of a system of two linear equations by employing the method of elimination by addition or subtraction is explained in the following:

ILLUSTRATIVE EXAMPLES

1. Solve the system $\begin{cases} 2x + 3y = 5, & (1) \\ 3x - 2y = 1. & (2) \end{cases}$

Solution. Multiplying (1) by 2 and (2) by 3 so that the coefficients of y in the two equations shall have the same absolute value,

$$4x + 6y = 10. \qquad (3)$$

$$9x - 6y = 3. \qquad (4)$$

Adding (3) and (4), $$13x = 13. \qquad (5)$$

Dividing by 13, $$x = 1. \qquad (6)$$

Substituting the value of x in (1),

$$2 \times 1 + 3y = 5. \qquad (7)$$

Transposing and uniting terms,

$$3y = 3. \qquad (8)$$

Dividing by 3, $$y = 1. \qquad (9)$$

Therefore, the solution of equations (1) and (2) is $x = 1$, $y = 1$.

Check. Substituting 1 for x and 1 for y in equations (1) and (2), we have,

$$2 \times 1 + 3 \times 1 = 5,$$

$$3 \times 1 - 2 \times 1 = 1.$$

Remark. In the solution of example 1, equations (1) and (2) were replaced by the equivalent system (3) and (4); that is, the solution of (1) and (2) is the same as the solution of (3) and (4). The actual operations of addition, subtraction, multiplication, and division, here as elsewhere in solving equations, depend on the principles stated in section 25.

2. Solve the system

$$\begin{cases} \dfrac{3x + 2y}{3} - \dfrac{3x - 2y}{5} = 1 - \dfrac{2y - 21x}{12}, & (1) \\[2ex] \dfrac{5x - 3y}{2} + \dfrac{5x + 3y}{3} = 6 + \dfrac{x + y}{6}. & (2) \end{cases}$$

Solution. Clearing equations (1) and (2) of fractions, transposing, and uniting terms, we have the equivalent system,

$$81x - 74y = -60. \tag{3}$$

$$6x - y = 9. \tag{4}$$

Multiplying both members of (4) by 74, so that the coefficient of y in the resulting equation shall be equal to the coefficient of y in (3),

$$444x - 74y = 666. \tag{5}$$

Subtracting each member of (3) from the corresponding member of (5),

$$363x = 726. \tag{6}$$

Dividing by 363, $$x = 2. \tag{7}$$

Substituting the value of x in (4),

$$12 - y = 9. \tag{8}$$

Solving (8) for y, $$y = 3. \tag{9}$$

Therefore, the solution of equations (1) and (2) is $x = 2$, $y = 3$.

Check. Substituting 2 for x and 3 for y in equations (1) and (2), we have

$$\frac{3 \times 2 + 2 \times 3}{3} - \frac{3 \times 2 - 2 \times 3}{5} = 1 - \frac{2 \times 3 - 21 \times 2}{12}, \tag{10}$$

$$\frac{5 \times 2 - 3 \times 3}{2} + \frac{5 \times 2 + 3 \times 3}{3} = 6 + \frac{2 + 3}{6}. \tag{11}$$

Simplifying (10), $$4 = 4. \tag{12}$$

Simplifying (11), $$\tfrac{41}{6} = \tfrac{41}{6}. \tag{13}$$

3. Solve the system $\begin{cases} ax + by = m, & (1) \\ bx - ay = n. & (2) \end{cases}$

Solution. Multiplying both members of (1) by a and both members of (2) by b in order that in the two resulting equations the coefficients of y may have the same absolute value,

$$a^2x + aby = am. \tag{3}$$

$$b^2x - aby = bn. \tag{4}$$

Adding (3) and (4), $$(a^2 + b^2)x = am + bn. \tag{5}$$

Dividing by $a^2 + b^2$, $$x = \frac{am + bn}{a^2 + b^2}. \tag{6}$$

Substituting in (1) the value of x as found in (6),

$$\frac{a^2m + abn}{a^2 + b^2} + by = m. \tag{7}$$

Clearing (7) of fractions,

$$a^2m + abn + (a^2 + b^2)by = ma^2 + mb^2. \tag{8}$$

Transposing and collecting,

$$(a^2 + b^2)by = mb^2 - abn. \tag{9}$$

Dividing both members of (9) by b,

$$(a^2 + b^2)y = mb - an. \tag{10}$$

Dividing (10) by $a^2 + b^2$,

$$y = \frac{mb - an}{a^2 + b^2}. \tag{11}$$

Therefore, the solution of equations (1) and (2) is

$$x = \frac{am + bn}{a^2 + b^2},\ y = \frac{mb - an}{a^2 + b^2}.$$

Remark. The value of y might also have been found as the value of x was found; namely, by making the coefficients of x alike, combining the resulting equations so as to eliminate x, and solving the resulting equation for y.

From the solutions of illustrative examples 1, 2, and 3 we may infer the following:

Rule. *To eliminate an unknown number, as y, from two simultaneous linear equations by addition or subtraction, multiply, if necessary, the members of one or both equations by such a number or such numbers as will make the absolute values of the coefficients of y in the two equations the same; then add or subtract the corresponding members of the two resulting equations according as the coefficients of y in the equations have opposite signs or the same sign.*

EXERCISE 76

Solve the following systems of equations, and check the solution of each:

1. $\begin{cases} x + y = 6, \\ 2y = 3. \end{cases}$

2. $\begin{cases} x + y = 10, \\ x - y = 4. \end{cases}$

3. $\begin{cases} 2x + 3y = 7, \\ 5x - 3y = 14. \end{cases}$

4. $\begin{cases} 5x + 7y = 17, \\ 5x + 3y = 13. \end{cases}$

5. $\begin{cases} 2x + 3y = 5, \\ 3x + 2y = 0. \end{cases}$

6. $\begin{cases} 6x + 3y = 4, \\ 8x - 9y = 1. \end{cases}$

7. $\begin{cases} 12x + 4y = 3, \\ 3x + 8y = 1. \end{cases}$

8. $\begin{cases} 3x - 5y = 9; \\ y - 4x = 5. \end{cases}$

9. $\begin{cases} 17y + 2z = 19, \\ 15y - 7z = 8. \end{cases}$

10. $\begin{cases} 11x - 12y = 13, \\ 6y - 13x = 1. \end{cases}$

11. $\begin{cases} 3x + 5y = 9, \\ 2x + 3y = 5. \end{cases}$

12. $\begin{cases} 17x + 15y = 1, \\ 8x + 6y = 1. \end{cases}$

13. $\begin{cases} 4x - 3y = 16, \\ 14x + 5y = 25. \end{cases}$

14. $\begin{cases} 3m + 2n = 7, \\ 4m - 5n = 6. \end{cases}$

15. $\begin{cases} 4m + 3p = 17, \\ 5m - 4p = 2. \end{cases}$

16. $\begin{cases} 5w - 3t = 2, \\ 7t - 15w = 2. \end{cases}$

17. $\begin{cases} 2u - 3v = 4, \\ 5v - 3u = -9. \end{cases}$

18. $\begin{cases} 6u + 5w = -8, \\ 4u + 25w = -1. \end{cases}$

19. $\begin{cases} 11x + 12y = 1, \\ 17x + 19y = 2. \end{cases}$

20. $\begin{cases} 13x + 17y = 4, \\ 31x + 37y = 6. \end{cases}$

21. $\begin{cases} 7x - 3y = -4, \\ -13x + 11y = 2. \end{cases}$

22. $\begin{cases} \dfrac{x}{2} + \dfrac{2y}{3} = \dfrac{5}{2}, \\ \dfrac{3x}{2} + \dfrac{y}{6} = 2. \end{cases}$

23. $\begin{cases} \dfrac{3x}{4} + \dfrac{2y}{3} = 5, \\ y - \dfrac{x}{2} = 1. \end{cases}$

24. $\begin{cases} \dfrac{5x}{3} - \dfrac{3y}{5} = \dfrac{19}{30}, \\ \dfrac{2x}{7} + \dfrac{6y}{5} = \dfrac{19}{35}. \end{cases}$

25. $\begin{cases} \dfrac{3m}{2} + \dfrac{5n}{3} + 2 = 0, \\ 6m + 5n - 1 = 0. \end{cases}$

26. $\begin{cases} 9u - \dfrac{10v}{3} = \dfrac{23}{2}, \\ 6u + 5v + 1 = 0. \end{cases}$

27. $\begin{cases} \dfrac{w}{8} + \dfrac{t}{3} + 3 = 0, \\ \dfrac{w}{2} + \dfrac{2t}{3} = 0. \end{cases}$

28. $\begin{cases} \dfrac{x}{14} - \dfrac{y}{15} = 3, \\ \dfrac{x}{10} - \dfrac{y}{9} = 1. \end{cases}$

29. $\begin{cases} \dfrac{3x}{2} + \dfrac{5y}{17} = 1, \\ \dfrac{9x}{2} - \dfrac{17y}{5} = 3. \end{cases}$

30. $\begin{cases} \dfrac{3x}{17} - \dfrac{2y}{13} = 1, \\ 5x - 6y = 7. \end{cases}$

31. $\begin{cases} x + y = m, \\ x - y = n. \end{cases}$

32. $\begin{cases} y = mx + c, \\ y = lx + d. \end{cases}$

33. $\begin{cases} x + ay = b, \\ ax + y = c. \end{cases}$

34. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1, \\ \dfrac{x-a}{b} + \dfrac{y-b}{a} = 1. \end{cases}$

35. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1, \\ ax - by = c. \end{cases}$

36. $\begin{cases} ax - y = 1, \\ 3x + y = c. \end{cases}$

37. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1, \\ x - y = 1. \end{cases}$

38. $\begin{cases} y - 1 = m(x - 1), \\ m^2y + mx + m^3 + 1 = 0 \end{cases}$

39. $\begin{cases} \dfrac{x+2a}{a+1} - \dfrac{y-2a}{a-1} = 2a, \\ \dfrac{x-y-4a}{2a} = \dfrac{x+y}{a^2+1}. \end{cases}$

40. $\dfrac{x+y}{a} + \dfrac{x-y}{b} = \dfrac{10}{3} = \dfrac{x+y+a}{a} - \dfrac{x-y-b}{b}.$

41. $\begin{cases} x + y = a + b, \\ ax - by = a^2 - b^2. \end{cases}$

42. $\begin{cases} ax + by + c = 0, \\ px + qy + r = 0. \end{cases}$

43. $\begin{cases} 3\,ax - 2\,by = c, \\ a^2x + b^2y = 5\,bc. \end{cases}$

44. $\begin{cases} \frac{a}{b}x + \frac{b}{a}y = \left(\frac{1}{a} + \frac{1}{b}\right)(a^2 + b^2), \\ (x + y)(a^2 + b^2) = ab(x + y) + 2(a^3 + b^3). \end{cases}$

45. $\begin{cases} ax - by = a^2 - 2\,ab - b^2, \\ ax + by = a^2 + b^2. \end{cases}$

46. $\begin{cases} ax + by = a + b, \\ (a + b)x - (a - b)y = 2\,b. \end{cases}$

47. $\begin{cases} (a - b)x + (a + b)y = a^2 + b^2, \\ ax - by = a^2 - b^2. \end{cases}$

48. $\begin{cases} \dfrac{ax}{a + b} + \dfrac{by}{a - b} = \dfrac{a^3 - a^2b + ab + b^2}{a^2 - b^2}, \\ (a + b)x - (a^2 + b^2)y = b(a - b). \end{cases}$

163. Elimination by substitution. The solution of a system of two linear equations by employing the method of elimination by substitution is explained in the following:

ILLUSTRATIVE EXAMPLE

Solve the system $\begin{cases} 3\,x + 5\,y = 21, & (1) \\ 2\,x + 3\,y = 13. & (2) \end{cases}$

Solution. Solving equation (1) for the value of x in terms of y,

$$x = \frac{21 - 5\,y}{3}. \qquad (3)$$

Substituting in (2) the value of x as found in (3),

$$\frac{42 - 10\,y}{3} + 3\,y = 13. \qquad (4)$$

Solving (4), $y = 3.$ (5)

Substituting in (3) the value of y as found in (5),

$$x = \frac{21 - 15}{3} = 2. \qquad (6)$$

Therefore, the solution of (1) and (2) is $x = 2$, $y = 3$.

From the solution of the foregoing illustrative example we may infer the following:

Rule. *To eliminate an unknown number, as x, from two simultaneous linear equations by substitution, solve one of the equations for x and substitute the resulting value of x in the other equation.*

EXERCISE 77

Solve the following systems of equations, using the method of elimination by substitution:

1. $\begin{cases} x - 2y = 3, \\ 7x + 3y = 4. \end{cases}$ **2.** $\begin{cases} 6x - 2y = 5, \\ x + 6y = 4. \end{cases}$

3. $\begin{cases} 5x + 3y = 11, \\ y = 2x. \end{cases}$ **4.** $\begin{cases} 3x - 2y = 4, \\ 6x + 5y = 17. \end{cases}$

5. $\begin{cases} x - y = 1, \\ 5x + 7y = 41. \end{cases}$ **6.** $\begin{cases} 5x - y = 8, \\ 3x + 2y = 10. \end{cases}$

7. $\begin{cases} 2y = 3x, \\ 5x - 3y = 2. \end{cases}$ **8.** $\begin{cases} x + y = 5, \\ 5x - 7y = 1. \end{cases}$

9. $\begin{cases} ax = by, \\ (a+b)x + (a-b)y = a + b. \end{cases}$

10. $\begin{cases} y = (a+b)x, \\ (a - b + 2)x + 3y = 4a + 2b + 2. \end{cases}$

11. $\begin{cases} x - y = 1, \\ ax + by = c. \end{cases}$

12. $\begin{cases} ax + by = d, \\ mx + ny = c. \end{cases}$

(For further practice in solving a system of two linear equations by employing the method of elimination by substitution, one or more of the examples in exercise 76 may be taken.)

164. Elimination by comparison. The solution of a system of two linear equations by employing the method of elimination by comparison is explained in the following:

ILLUSTRATIVE EXAMPLE

Solve the system $\begin{cases} 5x + 3y = 4, & (1) \\ 10x + 9y = 10. & (2) \end{cases}$

Solution. Solving (1) for x,

$$x = \frac{4 - 3y}{5}. \qquad (3)$$

Solving (2) for x, $$x = \frac{10 - 9y}{10}. \qquad (4)$$

Equating the two values of x as found in (3) and (4),

$$\frac{4 - 3y}{5} = \frac{10 - 9y}{10}. \qquad (5)$$

Simplifying (5), $3y = 2.$ (6)

Solving (6), $y = \frac{2}{3}.$ (7)

Substituting in (3) the value of y as found in (6),

$$x = \tfrac{2}{5}. \qquad (8)$$

Therefore, the solution of (1) and (2) is $x = \frac{2}{5}$, $y = \frac{2}{3}$.

From the solution of the foregoing illustrative example we may infer the following:

Rule. *To eliminate by comparison an unknown number, as x, from two simultaneous linear equations, solve each of the two equations for x and equate the two resulting values.*

EXERCISE 78

Solve the following systems of equations, using the method of elimination by comparison:

1. $\begin{cases} 3x + 2y = 5, \\ 2y + 3x = 5. \end{cases}$

2. $\begin{cases} 4x + 3y = 1, \\ 3x + 4y = -1. \end{cases}$

3. $\begin{cases} 5x - 2y = 8, \\ 4x + 3y = 11. \end{cases}$

4. $\begin{cases} \frac{2}{3}x - 3y = 7, \\ \frac{3}{2}x + 4y = 5. \end{cases}$

5. $\begin{cases} 17x - 15y = 2, \\ 13x + 14y = 27. \end{cases}$

6. $\begin{cases} 2x - 3y = 1, \\ 5x + 7y = 46. \end{cases}$

7. $\begin{cases} 5x + 2y = -7, \\ 3x + 5y = 11. \end{cases}$

8. $\begin{cases} 11x - 2y = 1, \\ 13x + 7y = -55. \end{cases}$

9. $\begin{cases} 89x + 3y = 3, \\ 78x - 5y = -5. \end{cases}$ **10.** $\begin{cases} 71x - 23y = 2, \\ 23x + 71y = 236. \end{cases}$

(For further practice in solving a system of two linear equations by employing the method of elimination by comparison, one or more of the examples in exercise 76 may be taken.)

Remark. The three methods of elimination considered in sections 162, 163, and 164 are manifestly applicable to any system of two simultaneous linear equations. Of these methods, that of elimination by addition or subtraction is most frequently employed. However, when one of the equations gives the value of one of the unknowns, as x, in terms of the other, elimination by substitution may be used to advantage.

165. Elimination by use of an undetermined multiplier. The solution of a system of two linear equations by use of an undetermined multiplier is explained in the following :

ILLUSTRATIVE EXAMPLE

Solve the system $\begin{cases} 7x + 3y = 17, & (1) \\ 3x - 5y = 1. & (2) \end{cases}$

Solution. Multiplying (2) by m, $3mx - 5my = m.$ (3)

Adding (1) and (3), $(7 + 3m)x + (3 - 5m)y = 17 + m.$ (4)

Equating the coefficient of y in (4) to 0, $3 - 5m = 0.$ (5)

Solving (5), $m = \frac{3}{5}.$ (6)

Substituting the value of m in (4), $(7 + \frac{9}{5})x + 0 \cdot y = 17 + \frac{3}{5}.$ (7)

Solving (7), $x = 2.$ (8)

Substituting the value of x in (1) and solving, $y = 1.$

Remark. The number m in equation (4) of the foregoing solution is undetermined; that is, it may have any numerical value assigned to it. We assign such a value to m that the coefficient of one of the unknown numbers shall vanish. It is evident that instead of first eliminating y, the coefficient of x might have been placed equal to zero and the value of y determined.

EXERCISE 79

Solve the following systems of equations, eliminating one of the unknowns by use of an undetermined multiplier:

1. $\begin{cases} 4x - 5y = 22, \\ 3x + 2y = 5. \end{cases}$

2. $\begin{cases} 4m + 3p = -1, \\ 6m + 5p = -2. \end{cases}$

3. $\begin{cases} 7u - 11v = 26, \\ 15u + 5v = -30. \end{cases}$

4. $\begin{cases} x + y = 2, \\ 2x - 3y = 54. \end{cases}$

5. $\begin{cases} \frac{1}{5}(m+n) + \frac{1}{3}(m-n) = 2, \\ \frac{5}{2}(m+n) + \frac{3}{2}(m-n) = 17. \end{cases}$

6. $\begin{cases} 15x + 19y = 18, \\ 19x + 15y = 50. \end{cases}$

7. $\begin{cases} 18x + 23y = 13, \\ 23x + 18y = 28. \end{cases}$

8. $\begin{cases} 2.5x + 3.7y = 7.69, \\ 3.6x - 2.9y = 1.20. \end{cases}$

9. $\begin{cases} .05x + .03y = .011, \\ .72x + .93y = .258. \end{cases}$

10. $\begin{cases} ax + by = a^2 + b^2, \\ a^2x - b^2y = a^3 - b^3. \end{cases}$

166. Special systems of simultaneous equations. Certain systems of simultaneous fractional equations, which are of frequent occurrence, should be solved by the methods already employed in this chapter. In such systems, *the equations are not cleared of fractions.*

ILLUSTRATIVE EXAMPLES

1. Solve the system $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = 1, & (1) \\ \dfrac{3}{x} + \dfrac{5}{y} = 2. & (2) \end{cases}$

Solution. Writing (1) and (2) in another form,

$$2\left(\frac{1}{x}\right)+3\left(\frac{1}{y}\right)=1, \tag{3}$$

$$3\left(\frac{1}{x}\right)+5\left(\frac{1}{y}\right)=2. \tag{4}$$

Equations (3) and (4) are in the form of two simultaneous linear equations in the unknown numbers $\frac{1}{x}$ and $\frac{1}{y}$. Multiplying both members of (3) by 5 and those of (4) by 3,

$$10\left(\frac{1}{x}\right)+15\left(\frac{1}{y}\right)=5. \tag{5}$$

$$9\left(\frac{1}{x}\right)+15\left(\frac{1}{y}\right)=6. \tag{6}$$

Subtracting the members of (6) from the corresponding members of (5),

$$\frac{1}{x}=-1. \tag{7}$$

Solving (7) for x, $$x=-1. \tag{8}$$

Substituting in (1) the value of x as found in (8),

$$-2+\frac{3}{y}=1. \tag{9}$$

Solving (9) for y, $$y=1. \tag{10}$$

Therefore, the solution of (1) and (2) is $x=-1$, $y=1$.

Check. Substituting -1 for x and 1 for y in (1) and (2), we have, respectively,

$$-2+3=1, \tag{11}$$

$$-3+5=2. \tag{12}$$

Note. The system (1) and (2) whose solution has just been given is not equivalent to the system obtained by clearing (1) and (2) of fractions; namely the system $\left\{\begin{matrix} 2y+3x=xy \\ 3y+5x=2xy \end{matrix}\right\}$. This new system is not composed of linear equations and it has other solutions than the one obtained from (1) and (2). For example, $x=0$, $y=0$ is evidently a solution of this new system but is not a solution of the system (1) and (2). When, however, two equations of the form $\left\{\begin{matrix} ax+by=cxy \\ mx+ny=pxy \end{matrix}\right\}$ are given, one solution of the system may, in general, be obtained by dividing each member of both equations by xy and proceeding as in the solution of the foregoing illustrative example.

2. Solve the system
$$\begin{cases} \dfrac{3}{x-1}+\dfrac{2}{y+2}=12, & (1)\\ \dfrac{5}{x-1}-\dfrac{3}{y+2}=1. & (2)\end{cases}$$

Solution. Writing (1) and (2) in another form,
$$\begin{cases} 3\left(\dfrac{1}{x-1}\right)+2\left(\dfrac{1}{y+2}\right)=12, & (3)\\ 5\left(\dfrac{1}{x-1}\right)-3\left(\dfrac{1}{y+2}\right)=1. & (4)\end{cases}$$

In equations (3)and (4) we may regard the unknown numbers as $\dfrac{1}{x-1}$ and $\dfrac{1}{y+2}$. Multiplying both members of (3) by 3 and both members of (4) by 2, and adding,
$$19\left(\frac{1}{x-1}\right)=38. \qquad (5)$$

Dividing by 19, $$\frac{1}{x-1}=2. \qquad (6)$$

Clearing (6) of fractions, $$1=2(x-1). \qquad (7)$$

Solving (7), $$x=\tfrac{3}{2}. \qquad (8)$$

Substituting in (1) the value of x as found in (8),
$$6+\frac{2}{y+2}=12. \qquad (9)$$

Solving (9), $$y=-\tfrac{5}{3}. \qquad (10)$$

Therefore, the solution of (1) and (2) is $x=\frac{3}{2}$, $y=-\frac{5}{3}$.

Check. Substituting in (1) and (2),
$$\begin{cases} \dfrac{3}{\frac{3}{2}-1}+\dfrac{2}{-\frac{5}{3}+2}=12,\\ \dfrac{5}{\frac{3}{2}-1}-\dfrac{3}{-\frac{5}{3}+2}=1.\end{cases}$$

Simplifying, $$\begin{cases} 6+6=12,\\ 10-9=1.\end{cases}$$

EXERCISE 80

Solve the following systems of equations, regarding each as a system of simultaneous linear equations in two unknowns:

1. $\begin{cases} \dfrac{2}{x}+\dfrac{1}{y}=3, \\ \dfrac{1}{x}+\dfrac{1}{y}=2. \end{cases}$

2. $\begin{cases} \dfrac{3}{x}+\dfrac{2}{y}=-1, \\ \dfrac{2}{x}-\dfrac{2}{y}=-4. \end{cases}$

3. $\begin{cases} \dfrac{3}{x}+\dfrac{2}{y}=10, \\ \dfrac{3}{x}-\dfrac{2}{y}=2. \end{cases}$

4. $\begin{cases} \dfrac{2}{x+1}+\dfrac{3}{y+1}=5, \\ \dfrac{1}{y+1}=1. \end{cases}$

5. $\begin{cases} \dfrac{7}{x-1}+\dfrac{1}{y+5}=11, \\ \dfrac{5}{x-1}+\dfrac{2}{y+5}=\dfrac{17}{2}. \end{cases}$

6. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=2, \\ \dfrac{a+b}{x}+\dfrac{a-b}{y}=\dfrac{a}{b}+\dfrac{b}{a}. \end{cases}$

7. $\begin{cases} \dfrac{2a}{x}+\dfrac{5a}{y}=\dfrac{3}{2}, \\ \dfrac{5a}{x}-\dfrac{2a}{y}=\dfrac{17}{20}. \end{cases}$

8. $\begin{cases} \dfrac{3a}{bx}+\dfrac{5a}{cy}=\dfrac{7}{4}, \\ \dfrac{7a}{bx}-\dfrac{3a}{cy}=\dfrac{23}{20}. \end{cases}$

9. $\begin{cases} \dfrac{2(a+b)}{x}+\dfrac{3(a-b)}{y}=\dfrac{3}{2}, \\ \dfrac{-3(a+b)}{x}+\dfrac{5(a-b)}{y}=\dfrac{11}{12}. \end{cases}$

10. $\begin{cases} \dfrac{5}{6x}-\dfrac{3a}{5y}=\dfrac{a-18}{6}, \\ \dfrac{2}{3x}+\dfrac{3a}{5y}=\dfrac{2a+45}{15}. \end{cases}$

11. $\begin{cases} \dfrac{a}{x}-\dfrac{b}{y}=m, \\ \dfrac{a}{y}-\dfrac{b}{x}=n. \end{cases}$

12. $\dfrac{b}{4\,ax} + \dfrac{a}{4\,by} = c = \dfrac{2\,b}{3\,ax} - \dfrac{a}{by}.$

13. $\begin{cases} \dfrac{a-c}{x-b} - \dfrac{a-b}{y-c} = 0, \\ \dfrac{a+c}{x-b} - \dfrac{a+b}{y-c} = \dfrac{2\,a(c-b)}{(a-b)(a-c)}. \end{cases}$

14. $\begin{cases} \dfrac{1}{x+b+c} + \dfrac{1}{y+c+a} = \dfrac{2}{a+b+c}, \\ \dfrac{2a+b+c}{x+b+c} - \dfrac{2\,b+c+a}{y+c+a} = \dfrac{a-b}{a+b+c}. \end{cases}$

167. Fractional equations. In the case of systems of simultaneous fractional equations which are not included among those considered in section 166, it is usually best to clear the equations of fractions.

ILLUSTRATIVE EXAMPLE

Solve the system $\begin{cases} \dfrac{2\,x+5\,y}{x+2} = 1, & (1) \\ \dfrac{4\,x-3\,y}{x+y} = 2. & (2) \end{cases}$

Solution. Clearing equations (1) and (2) of fractions and combining like terms,

$$x + 5\,y = 2, \qquad (3)$$

$$2\,x - 5\,y = 0. \qquad (4)$$

Adding (3) and (4), $3\,x = 2.$ (5)

Solving (5), $x = \frac{2}{3}.$ (6)

Substituting in (3) the value of x as found in (5),

$$y = \frac{2 - \frac{2}{3}}{5} = \frac{4}{15}. \qquad (7)$$

Therefore, the solution of (1) and (2) is $x = \frac{2}{3}$, $y = \frac{4}{15}$. These values of x and y are found to satisfy the given equations (1) and (2).

Remark. Before clearing an equation of fractions, each fraction should be expressed in its lowest terms (see also note, page 193).

EXERCISE 81

Solve the following systems of equations and check the results :

1. $\begin{cases} \dfrac{x+y}{x-2} = 3, \\ \dfrac{2x}{x+y} = 3. \end{cases}$

2. $\begin{cases} \dfrac{x+2}{y} = 3, \\ \dfrac{y+2}{x} = 3. \end{cases}$

3. $\begin{cases} \dfrac{x+1}{y-2} = \dfrac{x+3}{y-1}, \\ \dfrac{x+1}{y-1} = 1. \end{cases}$

4. $\begin{cases} \dfrac{x+y}{3y-2} = 4, \\ \dfrac{x}{y} + \dfrac{3}{2} = 0. \end{cases}$

5. $\begin{cases} \dfrac{2y+3}{2x+3} = \dfrac{3y+2}{3x+2}, \\ x+y+2=0. \end{cases}$

6. $\begin{cases} \dfrac{2x-3y}{3x-2y} = \dfrac{2}{3}, \\ \dfrac{3x+2y}{2x-3y} = \dfrac{3}{2}. \end{cases}$

7. $\begin{cases} 1 + \dfrac{x}{y} = \dfrac{2y-3x}{y}, \\ 3x+y=7. \end{cases}$

8. $\begin{cases} \dfrac{x+y}{3x} = \dfrac{53}{39} - \dfrac{13x+5}{26x}, \\ \dfrac{y+1}{x+1} = 1 + \dfrac{15}{x+1}. \end{cases}$

9. $\begin{cases} \dfrac{a}{x+2} = \dfrac{b}{y-3}, \\ \dfrac{x+2}{a} = \dfrac{3-y}{b}. \end{cases}$

10. $\begin{cases} \dfrac{x+2}{a} = \dfrac{y-b}{a-b}, \\ \dfrac{a-y}{x} = 1 - \dfrac{b}{a}. \end{cases}$

11. $\begin{cases} \dfrac{a}{y} = \dfrac{b}{x}, \\ \dfrac{2a}{ax+by} = \dfrac{1}{b}. \end{cases}$

12. $\begin{cases} x-y=a-b, \\ \dfrac{a+c}{x} = \dfrac{b+c}{y}. \end{cases}$

EXERCISE 82

1. The sum of two numbers is 15 and one of them is one greater than the other. What are the numbers?

Suggestion. Let $x =$ the larger number and $y =$ the smaller.

Then, $$\begin{cases} x + y = 15, & (1) \\ x = y + 1. & (2) \end{cases}$$

2. The sum of two numbers is 12, and their difference is 6. What are the numbers?

3. The sum of two numbers is 27, and five times the first number is equal to four times the second. What are the numbers?

4. The difference between two numbers is 5, and the sum of the numbers is twice their difference. Find the numbers.

5. Twice a certain number is 4 greater than 5 times a second number; the sum of the two numbers is 30. Find the numbers.

6. A bushel of corn and a bushel of oats together weigh 88 lb., and the weight of a bushel of corn is 24 lb. greater than the weight of a bushel of oats. What is the weight of a bushel of each?

7. The weight of 3 bu. of bran is equal to the weight of 1 bu. of wheat, and the weight of 1 bu. of wheat exceeds the weight of 1 bu. of bran by 40 lb. What is the weight of a bushel of each?

8. The sum of two numbers is 18, and 4 times the larger is equal to 5 times the smaller. What are the numbers?

9. Three times a certain number is 7 greater than four times the sum of 8 and a second number; the sum of three times the first number and four times the second is 63. Find the numbers.

10. One half of one number is equal to two thirds of a second; the sum of the first number and twice the second is 20. What are the numbers?

11. A classroom has 54 desks, some of which are single and some double; the seating capacity of the room is 72. How many desks of each kind are there?

12. Two opposite numbers which differ by 8 have the same absolute values. What are the numbers?

13. 2 lb. of coffee and 6 lb. of sugar cost $1.18; 5 lb. of coffee and 3 lb. of sugar cost $1.99. Find the cost of a pound of each.

14. If 12 gallons of milk will just fill either 152 bottles and 5 jars, or 32 bottles and 20 jars, what are the separate capacities of a bottle and a jar?

15. A dealer bought 30 bu. of wheat and 10 bu. of rye for $46. He also bought at the same time 50 bu. of wheat and 30 bu. of rye for $87. Find the price of each per bushel.

16. The sum of two numbers is equal to 5.5 diminished by the second number; three times the first number diminished by twice the second number is -1.1. What are the numbers?

17. "Give me five of your marbles," said a boy to his brother, "and I shall have twice as many as you." His brother replied, "Give me five of your marbles and then I shall have as many as you." How many marbles had each?

18. Three years ago a boy was twice as old as his sister, and fifteen years hence $\frac{1}{4}$ of his age will equal $\frac{1}{3}$ of his sister's age. How old is each?

19. A bill amounting to $8.70 was paid with 60 coins, some of which were dimes and the rest quarters; how many of each were there?

20. Divide \$10 between A and B, so that the number of half-dollars in A's share may be ten less than the number of quarter-dollars in B's share.

21. A certain number is equal to seven times the sum of its two digits, and the left-hand digit exceeds the right-hand digit by 2. Find the number.

Suggestion. Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then, $10x + y =$ the number.

Whence,
$$\begin{cases} 10x + y = 7(x + y), & (1) \\ x - y = 2. & (2) \end{cases}$$

22. The length of a room is 25% greater than the width, and the perimeter is 35 ft. Find the dimensions.

23. A merchant has tea worth 50 cents per pound and also tea worth 70 cents per pound; how many pounds of each must he use to make a mixture of 25 pounds worth 62 cents per pound?

24. The cost of sending a day telegram of 17 words from Philadelphia to Richmond, Indiana, is 71 cents, and the cost of sending one of 23 words is 89 cents. What is the rate for the first ten words in such a message and for each additional word?

25. Five first-class fitters and 7 plain sewers earn \$160 a week; 7 first-class fitters and 2 plain sewers earn \$185 a week. Find the weekly wages of a first-class fitter and those of a plain sewer.

26. The sum of two digits of a certain number is 10 and if 18 be added to the number, its digits will change places. Required the number.

Suggestion. Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then, $10x + y =$ the number,
and $10y + x =$ the number with the digits interchanged.

27. One digit is one greater than twice a second digit; the difference between the numbers which can be represented by the two digits is 45. Find the digits.

28. A man rode a certain distance, at a uniform rate, in 7 hr. If the distance had been 4 miles less and his rate per hour 1 mile more, the time required would have been 6 hr. Find the distance and his rate.

29. One man and three boys can do a piece of work in $2\frac{2}{5}$ working days of 10 hours each; two men and one boy could do it in the same time. How many hours would one man alone require to do the work?

Suggestion.

Let $x =$ the number of hours in which a man can do the work,

and $y =$ the number of hours in which a boy can do the work.

Then, $\frac{1}{x} =$ the part of the work the man does in 1 hr.,

and $\frac{1}{y} =$ the part of the work a boy does in 1 hr.

Whence,

$$\frac{1}{x} + \frac{3}{y} = \frac{1}{24}. \quad (1)$$

$$\frac{2}{x} + \frac{1}{y} = \frac{1}{24}. \quad (2)$$

30. One man and two boys can do a piece of work in 9 days; two men and five boys could do it in 4 days. How long would one boy alone take to do the work?

31. If 1 is added to the numerator of a fraction, the value of the fraction becomes $\frac{1}{2}$; if 1 is added to the denominator of the same fraction, the value becomes $\frac{1}{3}$. What is the fraction?

Suggestion. Let $\frac{x}{y} =$ the fraction.

32. If 1 be added to both terms of a fraction the resulting fraction will be $\frac{3}{4}$, but if 1 be subtracted from both terms, the resulting fraction will be $\frac{1}{2}$. What is the fraction?

33. Separate 53 into two parts such that the greater part divided by the less shall give both a quotient and a remainder of 2.

34. A owes \$250 and B owes \$375. A could pay all his debts if in addition to his own money he had $\frac{1}{6}$ of B's; and B could pay all of his debts and have \$25 left if in addition to his own money he had $\frac{1}{2}$ of A's. How much money has each?

35. The base of a rectangle is 10% greater than the altitude, and the perimeter is 126 ft. Find the dimensions.

36. A part of \$3000 is invested at $5\frac{1}{2}$% and the remainder at $4\frac{1}{2}$%. The yearly income from the investments is \$147.25. Find the amount in each investment.

37. In a certain family each son has twice as many sisters as brothers but each daughter has as many brothers as sisters. How many children are in the family?

38. In a certain family each daughter has as many brothers as sisters, but each son has three times as many sisters as brothers. How many children are in the family?

39. A man has \$7000 which he wishes to invest in two enterprises so that his total income will be \$330; if one enterprise pays 5% and the other $4\frac{1}{2}$%, how much must he invest in each?

40. A certain principal will amount to \$260 if loaned at simple interest for 5 yr., and to \$240 if loaned at the same rate for 4 yr. Required the principal and the rate.

41. A certain principal in a given time will amount to \$744 if loaned at simple interest at 6%, and to \$708 if loaned for the same time at $4\frac{1}{2}$%. Required the principal and the time.

42. In an athletic meet the winning team scored 42 points and the second team 35 points. The winning team took first place in 6 events and second place in 4; the second team took 4 first and 5 second places. How many points does a first place count and how many does a second place count?

168. Simultaneous linear equations in three unknown numbers. Three consistent linear equations in three unknown numbers have one and only one solution whenever by elimination two independent and consistent linear equations in two unknowns can be derived from them.

ILLUSTRATIVE EXAMPLES

1. Solve the system
$$\begin{cases} x + 2y + 3z = 4, & (1) \\ 2x + 3y + 4z = 7, & (2) \\ 3x - 4y - 5z = 8. & (3) \end{cases}$$

Solution. Multiplying (1) by 2, $2x + 4y + 6z = 8.$ (4)
Subtracting (2) from (4), $y + 2z = 1.$ (5)
Multiplying (1) by 3, $3x + 6y + 9z = 12.$ (6)
Subtracting (3) from (6), $10y + 14z = 4.$ (7)
Dividing (7) by 2, $5y + 7z = 2.$ (8)

Equations (5) and (8) are two independent equations in two unknowns and are solved by methods previously explained; thus:
Multiplying (5) by 5, $5y + 10z = 5.$ (9)
Subtracting (8) from (9), $3z = 3.$ (10)
Solving (10), $z = 1.$ (11)

Substituting in (5) the value of z as found in (11) and solving resulting equation for y, $y = -1.$ (12)

Substituting in (1) the value of y from (12) and the value of z from (11), and solving for x, $x = 3.$ (13)

Therefore, the solution of the given system is $x = 3$, $y = -1$, $z = 1$.

Check. Substituting 3 for x, -1 for y, and 1 for z in equations (1), (2), and (3), we have respectively,

$$\begin{cases} 3-2+3=4. & (14) \\ 6-3+4=7. & (15) \\ 9+4-5=8. & (16) \end{cases}$$

2. Solve the system

$$\begin{cases} \frac{3}{x}-\frac{2}{y}+\frac{1}{z}=1, & (1) \\ \frac{4}{x}+\frac{2}{y}-\frac{3}{z}=-\frac{2}{3}, & (2) \\ \frac{2}{x}-\frac{5}{y}+\frac{2}{z}=\frac{1}{6}. & (3) \end{cases}$$

Solution. Regard equations (1), (2), and (3) as linear in the three unknowns $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

Multiplying (1) by 3, $\frac{9}{x}-\frac{6}{y}+\frac{3}{z}=3.$ (4)

Adding (2) to (4), $\frac{13}{x}-\frac{4}{y}=\frac{7}{3}.$ (5)

Multiplying (1) by 2, $\frac{6}{x}-\frac{4}{y}+\frac{2}{z}=2.$ (6)

Subtracting (3) from (6), $\frac{4}{x}+\frac{1}{y}=\frac{11}{6}.$ (7)

Multiplying (7) by 4, $\frac{16}{x}+\frac{4}{y}=\frac{22}{3}.$ (8)

Adding (8) to (5), $\frac{29}{x}=\frac{29}{3}.$ (9)

Solving (9), $x=3.$ (10)

Substituting in (7) the value of x as found in (10),

$$\frac{4}{3}+\frac{1}{y}=\frac{11}{6}. \qquad (11)$$

Solving (11), $y=2.$ (12)

Substituting in (1) the value of x from (10) and the value of y from (12), and simplifying, $\frac{1}{z}=1.$ (13)

Solving (13), $z=1.$ (14)

Therefore, the solution of the given system is $x=3$, $y=2$, $z=1$.

Check. Substituting 3 for x, 2 for y, and 1 for z, in equations (1), (2), and (3), we have, respectively,

$$\begin{cases} \frac{3}{3} - \frac{2}{2} + \frac{1}{1} = 1, & (15) \\ \frac{4}{3} + \frac{2}{2} - \frac{3}{1} = -\frac{2}{3}, & (16) \\ \frac{2}{3} - \frac{5}{2} + \frac{2}{1} = \frac{1}{6}. & (17) \end{cases}$$

From the foregoing illustrative examples we may infer the following :

Rule. *To solve three linear equations in three unknown numbers, eliminate any one of the unknowns, as x, from any pair of the equations, and then eliminate the same unknown from another pair ; solve the resulting two linear equations in two unknowns for these unknowns, substitute the values of the two unknowns in one of the given equations, and solve for the third unknown number.*

EXERCISE 83

Solve the following systems of equations, and check the results :

1. $\begin{cases} x + 2y + z = -3, \\ 2x - 3y - z = 12, \\ 3x + y + 2z = 5. \end{cases}$ **2.** $\begin{cases} 2x - 3y + 5z = 15, \\ 3x + 2y - 4z = -7, \\ x + y + z = 2. \end{cases}$

3. $\begin{cases} 3x + 2y - 7z = -14, \\ 3x - 2y + 5z = 38, \\ x + 7y - 2z = -29. \end{cases}$ **4.** $\begin{cases} 5x + 2y + 3z = 4, \\ 3x - 3y + 4z = -19, \\ 2x + 5y - 7z = 47. \end{cases}$

5. $\begin{cases} 11x + 2y + 3z = 24, \\ 5x + 3y - 4z = -18, \\ 2x - 5y + 7z = 42. \end{cases}$ **6.** $\begin{cases} 3x + 2y + 3z = 8, \\ 2x + 3y + 2z = 27, \\ 7x - 5y - 5z = 97. \end{cases}$

7. $\begin{cases} 5x - 3y + 2z = 14, \\ 4x + 4y - 3z = 57, \\ 3x + 2y + 5z = 16. \end{cases}$ **8.** $\begin{cases} 5x - 2y = 3, \\ 3x + 2z = 5, \\ 5y - 3z = 2. \end{cases}$

9. $\begin{cases} x + y + z = 2, \\ 2x - 3y + 11z = 8, \\ 3x + 7y - 2z = 5. \end{cases}$ **10.** $\begin{cases} x + y = 0, \\ y + z = -1, \\ z + x = 1. \end{cases}$

11. $\begin{cases} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}=6, \\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}=-2, \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0. \end{cases}$

12. $\begin{cases} \frac{2}{x}+\frac{3}{y}-\frac{1}{z}=\frac{11}{6}, \\ \frac{3}{x}-\frac{1}{y}+\frac{2}{z}=\frac{7}{6}, \\ \frac{5}{x}+\frac{2}{y}-\frac{1}{z}=\frac{7}{3}. \end{cases}$

13. $\begin{cases} \frac{1}{x}+\frac{1}{y}=5, \\ \frac{1}{y}+\frac{1}{z}=6, \\ \frac{1}{z}+\frac{1}{x}=7. \end{cases}$

14. $\begin{cases} \frac{1}{x}+\frac{1}{y}=2a, \\ \frac{1}{y}+\frac{1}{z}=2b, \\ \frac{1}{z}+\frac{1}{x}=2c. \end{cases}$

15. $\begin{cases} ax+by+cz=a^2+b^2+c^2, \\ (b+c)x+(c+a)y+(a+b)z=2bc+2ca+2ab, \\ bcx+cay+abz=3abc. \end{cases}$

EXERCISE 84

1. I paid 91 ct. for 2 lb. of sugar, 1 lb. of coffee, and 3 lb. of lard. If I had bought 3 lb. of sugar, 1 lb. of coffee, and 2 lb. of lard, my bill would have been 84 ct.; but if I had bought 1 lb. of sugar, 2 lb. of coffee, and 1 lb. of lard, my bill would have been 83 ct. What did I pay for a pound of each?

2. For $6 I can buy 3 lb. of tea, 8 lb. of coffee, and 30 lb. of sugar; or 4 lb. of tea, 7 lb. of coffee, and 25 lb. of sugar; or 8 lb. of tea, 1 lb. of coffee, and 15 lb. of sugar. What are the prices?

3. There are three numbers whose sum is 162; the second exceeds the first as much as the third exceeds the second; 13 times the first equals 5 times the third. What are the numbers?

4. A dealer shipped 100 doz. of eggs on Monday and Tuesday, 110 doz. on Tuesday and Wednesday, and 90 doz. on Monday and Wednesday. How many dozen did he ship each day?

5. A man has a triangular lot which he desires to fence. 100 rods of fencing are required for the sides AC and BC, 111 rods for the sides AB and BC, and 90 rods for the sides AB and AC. Find the number of rods of fencing required for each side.

6. The sides of a certain triangle are denoted by a, b, and c. What is the length of each side of the triangle if the sum of the sides is 42, the sum of a and b is 27, and the sum of b and c is 29?

7. The sum of the three angles of any plane triangle is 180°. If these angles are denoted by A, B, and C, and if C exceeds A by 50° and A exceeds B by 10°, find the size of each angle.

8. The sum of three numbers is 24. The quotient of the first divided by the second is $\frac{3}{4}$ and the quotient of the second divided by the third is $\frac{4}{5}$. Find the numbers.

9. The middle digit of a given three-digit number is equal to the sum of the two remaining digits; a second number which is 594 less than the given number is expressed by the same digits written in the reverse order; if 19 be added to the original number and 13 be added to the second, one of the resulting numbers will be four times the other. Find the original number.

10. I made three shipments of goods from Philadelphia to Chicago. The first consisted of 300 lb. first-class freight, 200 lb. second-class freight, and 200 lb. third-class freight. My freight bill was \$4.65. The second shipment consisted of 1000 lb. first-class freight, 500 lb.

of second-class, and 100 lb. of third-class freight. My freight bill was $11.51. The third shipment consisted of 700 lb. of first-class freight, 800 lb. of second-class freight, and 400 lb. of third. My freight bill was $12.71. Find the rate of shipping 100 lb. of freight of each class from Philadelphia to Chicago.

11. A and B together can do a piece of work in 20 days. After they have worked 12 days on it, they are joined by C, who works twice as fast as B. The three finish the work in 4 days. How long would it take each man alone to do it?

12. A number is composed of three digits, whose sum is 12; the digit in the hundreds' place is one greater than that in the tens' place; if ten times the units' digit is subtracted from the number the remainder is 257. Find the number.

EXERCISE 85. — GENERAL REVIEW

(Solve as many as possible at sight.)

1. Add $x+2y+3z$, $2x-y-2z$, $y-x-z$, and $z-x-y$.

2. Add $4m^2-3mn-2m^2n-n^2+3mn^2$, $2m^2n-4mn^2-m^2+3n^2$, $4m^2n-3m^2+4mn-mn^2$, and $5mn^2-2n^2-2m^2n$.

3. What must be added to $x+y+z$ that the sum may be $c-z$?

4. What must be subtracted from x^2-x+1 that the difference may be x^3-1?

5. Subtract $2(r-s)+1$ from the sum of $7(r-s)$ and $-4(r-s)$.

6. Simplify

$$2-(-2)-[-(-2)]-[-\{-(-2)\}-2].$$

7. Express $m-n-p-q+r-s-t-u-x$ in trinomial terms having the last two terms of each inclosed in parentheses.

If $A=R^2-Rr+r^2$, $B=2R+r+1$, and $C=4R^2-2Rr-2$, find the value of:

8. $A+B+C$. **9.** $A-B+C$. **10.** $A-B-C$.

11. What is the product of m^2n^3p and $2m^4pq$?

12. What is the product of $x^{n-1}y^{n+2}z^n$ and $xy^{2-n}z$?

13. What is the product of $2(x+y)$, $3(x+y)^2$, and $-(x+y)^3$?

14. Divide $6m^6n^2p^3r$ by $2m^3n^2pr$.

15. Divide $6x^{n+1}y^nz^{n+2}$ by $3x^ny^{n-1}z^{n+2}$.

16. Divide $6(m+n)^2-4(m+n)^3+(m+n)$ by $2(m+n)$.

17. Multiply $x^2-2xy+y^2+z^2$ by $x^2+2xy+y^2-z^2$.

18. Multiply $x^m+y^p-2z^n$ by $2x^m-3y$.

19. Multiply $\frac{2}{3}x^2+\frac{2}{3}xy+\frac{1}{3}y^2$ by $\frac{3}{2}x^2-\frac{3}{2}xy+\frac{3}{4}y^2$.

20. Multiply 3^m-4^n by 4^m+3^n.

21. Divide

$$24m^2n^2pq^3-36m^2n^2pr^3+48mn^2r^2x \text{ by } -6mn^2.$$

22. Divide $x^3+y^3+z^3-3xyz$ by $x^2+y^2+z^2-yz-xz-xy$.

23. Divide $1-547x^6+546x^7$ by $1+2x-3x^2$.

24. Divide $3-5x-487x^5+489x^6$ by $1-4x+3x^2$ and find the value of the quotient when $x=-1$.

25. Show by division that $\dfrac{1}{1-x}=1+x+x^2+x^3+\dfrac{x^4}{1-x}$.

26. Show by division that $\dfrac{1}{1+x}=1-x+x^2-x^3+\dfrac{x^4}{1-x}$.

27. When the divisor is m^2+n^2 and the quotient is $m^8-m^6n^2+m^4n^4-m^2n^6+n^8$, what is the dividend?

28. When the divisor is $m - n$, the quotient $m^4 + m^3n + m^2n^2 + mn^3 + n^4$, and the remainder $2n^5$, what is the dividend?

29. Simplify $4m - \{2m - [2n(r+s) - 2n(r-s)]\}$.

30. Divide $x^5 - y^5$ by $x^4 + y^4 + x^3y + xy^3 + x^2y^2$.

31. Divide $(1 - m^4)$ by $[(1-m)(1+m)]$.

32. Divide $1 + a^2 + a^4$ by $1 + a + a^2$.

33. From $(2m + n + p)x$ take $(m+n)x$.

34. From $(r+s)y + (s+t)z$ take $(r-s)y - (s-t)z$.

35. Square as indicated:
$(2a+b)^2$; $(1-x)^2$; $(\frac{1}{2}+m)^2$; $(2m-\frac{1}{2})^2$.

36. Square as indicated: $(m^3n - p)^2$; $(mn - p^2r^2)^2$.

37. $(a+b)(a-b)(a^2+b^2) = ?$

38. $(m - \frac{1}{2})(m + \frac{1}{2}) = ?$

39. $\left(1 - \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) = ?$

40. Express 26×24 in the form of $(a+b)(a-b)$ and state the product.

41. $21 \times 19 = ?$ $51 \times 49 = ?$ $53 \times 47 = ?$ $101 \times 99 = ?$

42. Does $(-n)(+x)(-y)(+z) = n(-x)(+y)(-z)$? Why?

43. Find in the shortest way the value of $748 \times 680 - 748 \times 670$.

44. Find in the shortest way the value of $2\pi RH + 2\pi R^2$ when $\pi = 3.1416$, $R = 1$, and $H = 9$.

45. What must be added to $x^2 + 4x$ that the sum may be $(x+2)^2$?

46. What must be added to $x^2 + x$ that the sum may be $(x + \frac{1}{2})^2$?

47. What must be subtracted from $x^2 + 2xy + y^2$ that the difference may be $(x-y)^2$?

Factor:

48. $m^9 + m^3$.

49. $m(x-y) - n(x-y)$.

50. $r(a-b) - s(b-a)$.

51. $a^2 - b^2 - 2bc - c^2$.

52. $mx + ny - nx - my$.

53. $64 - m^6$.

54. $1 - a - a^2 + a^3$.

55. $1 - x^3 - x + x^2$.

56. $m^2 + \frac{1}{m^2} + 2$.

57. $1 + \frac{2}{x} + \frac{1}{x^2}$.

58. $\frac{m^2}{n^2} - \frac{n^2}{m^2}$.

59. $r^3 - \frac{1}{r^3}$.

60. $R^4 + R^2r^2 + r^4$.

61. $4x^2 - \frac{1}{3}x - \frac{1}{6}$.

62. Find all the factors of $3x^4 + x^3 - 7x^2 - 10x - 8$, being given that $x^2 + x + 1$ is one of them.

63. Find all the factors of $x^4 + x^3 - 7x^2 - x + 6$, being given that two of them are $x - 2$ and $x - 1$.

64. Reduce to lowest terms $\frac{x^2 - y^2}{(x-y)^2}$.

65. Reduce to lowest terms $\frac{1-(r-s)^2}{s - rs + s^2}$.

66. Simplify $2 - \frac{x-y-z}{z+y-x}$.

67. Simplify $\frac{1}{x-y} + \frac{y}{y^2 - x^2} - \frac{1}{x+y}$.

68. Simplify $\frac{(x+1)^2}{x^2-1} \times \frac{x + x^2}{(x+1)^2 - x} \div \frac{(x+1)^3}{x^3-1}$.

69. Simplify $\frac{m^2 + mn + n^2}{m^2 - mn + n^2} \times \frac{m^3 + n^3}{m^3 - n^3}$.

70. Simplify $\frac{x^2-y^2}{(m+n)^2} \div \frac{(x+y)^2(x-y)}{m^3+3n^2m+3nm^2+n^3}$.

71. Find the value of $\frac{m-x}{n-x}$ when $x = \frac{mn}{m+n}$.

72. Given $a = p + prt$; find p in terms of a, r and t.

73. Solve $\frac{1}{m} + \frac{1}{x} = \frac{1}{n} - \frac{1}{x}$.

74. Solve $0.3x - 0.1 + \frac{0.6x - 0.9}{1.2} = 0.4x - 0.05$.

75. Solve $\begin{cases} \frac{1}{x} + \frac{1}{y} = 7, \\ \frac{1}{x} - \frac{1}{y} = -1. \end{cases}$

76. Solve $\begin{cases} \frac{3}{x+y-1} - \frac{2}{x-y+2} = 0, \\ \frac{4}{x-8} - \frac{2}{y-4} = 0. \end{cases}$

77. Solve $\begin{cases} x + y + z = 3, \\ 4x - 3y + 2z = -2, \\ 5x - 2y - 3z = 1. \end{cases}$

78. Solve $\begin{cases} \frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 5, \\ \frac{5}{x} - \frac{2}{y} - \frac{3}{z} = 8, \\ \frac{1}{x} + \frac{4}{y} - \frac{7}{z} = 0. \end{cases}$

CHAPTER VIII

RATIO, PROPORTION, AND VARIATION

Ratio

169. Definition. The **ratio of a number *a* to a number *b*** is the quotient $\frac{a}{b}$ obtained by dividing a by b. The ratio a to b is sometimes written $a : b$.

Note. By the ratio of one quantity to a second quantity is meant the number of times that the first contains the second; as the ratio of 4 quarts to 3 quarts is $\frac{4}{3}$. Obviously, no ratio exists between quantities which are not of the *same kind*, and before the ratio of two quantities which are of the same kind can be found, they must be expressed in terms of the same unit.

Thus, the ratio of one gallon to three quarts is the ratio of four quarts to three quarts, which is $\frac{4}{3}$.

170. Definitions. In the ratio $\frac{a}{b}$ the dividend, or numerator, a, is called the **first term** or **antecedent**, and the divisor, or denominator, b, is called the **second term** or **consequent** of the ratio.

EXERCISE 86

In examples 1–24 express the ratios as fractions and simplify when possible :

1. $2 : 4$. **2.** $6 : 8$. **3.** $9 : 3$.

4. $15 : 10$. **5.** $a^2 : a$. **6.** $x^2 : x^3$.

7. $\frac{1}{2} : \frac{1}{4}$. **8.** $\frac{1}{4} : \frac{3}{4}$. **9.** $ab^2 : b$.

10. $ax : x^2$.

11. $2\frac{1}{2} : 5\frac{3}{4}$.

12. $(x - y) : (x^2 - y^2)$.

13. $(x + y)^2 : (x + y)$.

14. $9\,ax^2 : 15\,a^2x^2b$.

15. 3 pk. : 8 pk.

16. 2 rd. : $16\frac{1}{2}$ rd.

17. 9 in. : 1 ft.

18. 2 yd. : 1 rd.

19. 5 pk. : 2 bu.

20. 5 gal. : 3 qt.

21. 1 min. : 1 hr.

22. $abc : \frac{b}{c}$.

23. \$$x$: y ct.

24 $(a - b)$ yd. : $(a + b)$ ft.

25. Arrange the ratios, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{3}{2}$, in ascending order of magnitude.

26. What number must be added to both terms of the ratio $\frac{3}{5}$ in order to convert it into the ratio $\frac{4}{5}$?

27. Two numbers are in the ratio of 7 to 4; if 3 be subtracted from each number, the differences are in the ratio of 11 to 5. Find the numbers.

Proportion

171. Definitions. Four numbers or quantities are said to be in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Remark. In what follows we shall use the expression, *the ratio of one number to another* instead of, *the ratio of one number or quantity to another.*

If $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d are in proportion. **A proportion** is, therefore, an expressed equality of two ratios. In a proportion, as $\frac{a}{b} = \frac{c}{d}$, the numbers a, b, c, d are called the **terms** of the proportion.

Note. A proportion is often written $a:b=c:d$; read, *the ratio of a to b is equal to the ratio of c to d.* An old form and one less frequently used is $a:b::c:d$.

172. Extremes and means. The first and fourth terms of a proportion are called the **extremes** and the second and third terms the **means.**

Thus, in the proportion $\frac{a}{b}=\frac{c}{d}$, a and d are the extremes and b and c the means.

173. Important identities. If the four numbers, a, b, c, and d, are the four terms of a proportion, they satisfy the identity,

$$\frac{a}{b}=\frac{c}{d}. \tag{1}$$

Multiplying both members of identity (1) by bd, we have,

$$ad=bc. \tag{I}$$

Identity (I) may be expressed in words as follows:

In any proportion the product of the extremes is equal to the product of the means.

Again, if $\frac{a}{b}=\frac{c}{d}$, then the reciprocal of $\frac{a}{b}$ is equal to the reciprocal of $\frac{c}{d}$; that is,

$$\frac{b}{a}=\frac{d}{c}. \tag{II}$$

Since the reciprocal of a fraction is obtained by inverting the fraction, identity (II) is sometimes expressed as follows:

If four numbers are in proportion, they are also in proportion by inversion.

If $\frac{a}{b}=\frac{c}{d}$, we have from identity (I),

$$ad=bc. \tag{2}$$

Dividing both members of identity (2) by ab,

$$\frac{d}{b}=\frac{c}{a}. \qquad \text{(III)}$$

Again, dividing both members of identity (2) by dc,

$$\frac{a}{c}=\frac{b}{d}. \qquad \text{(IV)}$$

Comparing identities (III) and (IV) with the proportion $\frac{a}{b}=\frac{c}{d}$, we observe that if four numbers taken in a certain order are in proportion, they continue to be in proportion when either extremes or means are interchanged. This fact is usually expressed as follows:

If four numbers are in proportion, they are also in proportion by alternation.

If 1 be added to both members of the identity

$$\frac{a}{b}=\frac{c}{d}, \qquad (3)$$

we have,

$$\frac{a}{b}+1=\frac{c}{d}+1. \qquad (4)$$

Combining,

$$\frac{a+b}{b}=\frac{c+d}{d}. \qquad \text{(V)}$$

Subtracting 1 from both members of identity (3),

$$\frac{a}{b}-1=\frac{c}{d}-1. \qquad (5)$$

Combining,

$$\frac{a-b}{b}=\frac{c-d}{d}. \qquad \text{(VI)}$$

Dividing the members of (V) by the corresponding members of (VI),

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad \text{(VII)}$$

Identities (V), (VI), and (VII) are usually expressed in order as follows:

If four numbers are in proportion, they are also in proportion by composition.

If four numbers are in proportion, they are also in proportion by division.

If four numbers are in proportion, they are also in proportion by composition and division.

EXERCISE 87

Test identities (I)–(VII) by the use of the proportions of examples 1–4.

1. $\frac{1}{2} = \frac{3}{6}$.

2. $\frac{1\frac{2}{3}}{3\frac{1}{2}} = \frac{100}{210}$.

3. $\frac{5a}{3b} = \frac{30ab}{18b^2}$.

4. $\frac{ax + ay - bx - by}{ax - ay + bx - by} = \frac{\frac{a-b}{a+b}}{\frac{x-y}{x+y}}$.

Find the value of x in each of the proportions stated in examples 5–12.

5. $\frac{2}{3} = \frac{6}{x}$.

6. $5 : 2 = x : 10$.

7. $3 : x = 6 : 14$.

8. $x : 10 = 5 : 2$.

9. $\frac{2+x}{3+x} = \frac{12+x}{15+x}$.

10. $a + x : b + x = c + x : d + x$.

11. $3 - x : -2 = 3x + 4 : 32$.

12. $a : b :: x : c$.

13. Write by inversion:

(*a*) $\frac{6}{9} = \frac{2}{3}$. (*b*) $\frac{x}{y} = \frac{2}{3}$. (*c*) $\frac{m}{n} = \frac{p}{q}$.

14. Write (*a*), (*b*), and (*c*) of example 13 by alternation.

15. Write (a), (b), and (c) of example 13 by composition.

16. Write (a), (b), and (c) of example 13 by division.

17. Write (a), (b), and (c) of example 13 by composition and division.

18. If four numbers are proportionals (in proportion), prove that either mean is equal to the product of the extremes divided by the other mean.

19. If four numbers are proportionals, prove that either extreme is equal to the product of the means divided by the other extreme.

20. If the product of two numbers is equal to the product of two other numbers, prove that a proportion may be formed by taking one pair of numbers for the extremes and the other pair for the means.

21. Using the statement of example 20, write the eight proportions that may be expressed from the identity $mq = np$.

22. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a+b}{a} = \frac{c+d}{c}$.

Suggestion. Write the given proportion by inversion, and then apply identity (V).

23. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{b-a}{a} = \frac{d-c}{c}$.

174. Continued proportion. Three or more numbers are said to be in **continued proportion** when the ratio of the first to the second is equal to the ratio of the second to the third, and so on.

Thus, a, b, c, d are in continued proportion if,

$$a : b = b : c = c : d.$$

175. Mean proportional. When three numbers are in continued proportion, the second is said to be a **mean proportional** between the other two.

Thus, if $\frac{a}{b}=\frac{b}{c}$, the number b is a mean proportional between the extremes a and c.

176. Third proportional. When three numbers are in continued proportion, the third is said to be a **third proportional** to the other two.

Thus, if $\frac{a}{b}=\frac{b}{c}$, the number c is a third proportional to a and b.

177. Fourth proportional. A fourth proportional to three numbers a, b, c, taken in the order given is the fourth term of the proportion

$$a : b = c : x.$$

178. Composition of equal ratios. Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be equal ratios and each equal to r; that is,

$$\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{e}{f}=r.$$

Then, $a=br,\ c=dr,\ e=fr.$

Adding, $a+c+e=(b+d+f)r.$

Dividing, $\frac{a+c+e}{b+d+f}=r.$

Therefore, $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$

This identity may be expressed in words as follows:

In a number of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

EXERCISE 88

1. If a, b, and c are three numbers in continued proportion, show that $b^2 = ac$.

2. If a, b, and c are three numbers in continued proportion, show that c, the third proportional to a and b, is equal to $\frac{b^2}{a}$.

3. What positive integer is a mean proportional between 256 and 36?

4. Express by proportion the fact that 6 is a mean proportional between 4 and 9.

5. Express by proportion the fact that 18 is the third proportional to 2 and 6.

6. What is the third proportional to 10 and 5?

7. If $\frac{1}{2} = \frac{2}{a} = \frac{a}{b}$, what are the values of a and b?

8. If $\frac{m}{n} = \frac{p}{q} = \frac{r}{s}$, show that $\frac{m+p+r}{n+q+s} = \frac{m}{n} = \frac{p}{q} = \frac{r}{s}$.

9. From the proportion $\frac{2}{3} = \frac{4}{6}$, derive another proportion by inversion; derive two other proportions by alternation; derive another proportion by inversion and composition; derive another proportion by division; derive another proportion by inversion and division.

10. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a}{b} = \frac{c}{d} = \frac{3a+2c}{3b+2d}$.

Suggestion. Since $\frac{a}{b} = \frac{c}{d}$, then $\frac{3a}{3b} = \frac{2c}{2d}$; now see section 178.

11. If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$, and $a+b+c$ is not equal to zero, prove that each fraction is equal to $\frac{1}{2}$. When $a+b+c$ is equal to 0, what is the value of each of the fractions?

Variation

179. Constant. A number which has always the same value is a **constant.**

Thus, 2, $-\frac{1}{2}$, and a, supposing the value of a to be known, or given, are constants; also the root of a simple numerical equation in one unknown number is a constant; as the root of $2x - 4 = 0$.

180. Variable. Many of the numbers of algebra are not constants. A number which is not a constant is called a **variable.**

Thus, if the case of an express train running from New York to Philadelphia be considered, and t denotes the number of minutes which have elapsed since it left New York, s the number of feet that it passed over in t minutes, and v the number of feet that it passed over in each minute, v supposed to be constant, then the formula connecting the time, the rate, and the distance passed over is,

$$s = vt.$$

In this formula v is constant, but t and s are both variables during the whole time that the train is in motion.

Remark. The unknown numbers in an equation in two or more unknowns are called variables.

Thus, in $y = 2x$, any value whatsoever may be assigned to either x or y. The fact that it is not necessary to assign one fixed, or constant, value to x or y, as is the case in a simple equation in one unknown, in order to obtain a solution of the equation, is sufficient reason for calling x and y variables in the equation.

181. Application of terms. In general, a letter is said to be a variable; that is, it represents a variable number if it may have a number of different numerical values in a discussion or problem. It is a constant if it can have only one numerical value.

182. Direct variation. If one variable, y, depends upon another variable, x, in such a way that the ratio of y to x is a constant, *then y is said to vary as x*, or *vary directly as x.*

If y varies as x, then,

$$\frac{y}{x} = c \text{ (a constant).} \tag{1}$$

Hence,
$$\frac{x}{y} = \frac{1}{c} \text{ (a constant).} \tag{2}$$

From (1) and (2) it is evident that if y varies directly as x, then x varies directly as y.

183. Illustrations of direct variation. 1. The number of cents (c) in the cost of cloth bought at the fixed price of 60 cents per yard varies as the number of yards (n) bought, since the ratio $\frac{c}{n}$ is a constant, namely, the number of cents in the cost of one yard.

2. In the formula $s = vt$, in which v is a constant, s varies directly as t; that is, the space passed over varies directly as the time.

Note. In the equation $s = vt$, as elsewhere, s, v, and t represent *numbers*, and the statement, "*space passed over varies directly as the time*," means that the number s varies as the number t.

184. Inverse variation. If the variable y varies as the reciprocal of x; that is, if y is equal to a constant times the reciprocal of x, then y is said to vary inversely as x.

Thus, if $y = \frac{c}{x}$, where c is a constant, y varies inversely as x.

Also, from $y = \frac{c}{x}$ we have $x = \frac{c}{y}$, and hence x also varies inversely as y; moreover, from $y = \frac{c}{x}$ we derive $xy = c$; hence,

If the product of two variables is a constant, the one varies inversely as the other.

185. Illustration of inverse variation. The number of yards of cloth (n) that can be bought for a certain fixed price, as $8, varies inversely as the number of dollars (r) that the cloth costs per yard, since $n = \frac{8}{r}$; that is, n is a constant times the reciprocal of r.

ILLUSTRATIVE EXAMPLES

1. If y varies directly as x, and y is 100 when x is 20, what equation expresses the relation between y and x?

Solution. Since y varies as x, then y is some constant, as c, times x;

Therefore, $$y = cx. \quad (1)$$

Since y is equal to 100, when x is equal to 20,

$$100 = 20c. \quad (2)$$

Solving (2) for c, $$c = 5. \quad (3)$$

Substituting the value of c in (1),

$$y = 5x. \quad (4)$$

2. The number of feet that a body falls from rest under the action of gravity is proportional to (varies directly as) the square of the number of seconds during which it falls. If a body falls 144 feet in three seconds, how far will it fall in five seconds? How many seconds are required for it to fall 192 yards?

Solution. Let s represent the number of feet that the body falls, and t the number of seconds during which it falls; also, let c represent a constant.

Since s varies as t^2, $$s = ct^2. \quad (1)$$

Since $s = 144$, when $t = 3$, $$144 = 9c. \quad (2)$$

Solving for c, $$c = 16. \quad (3)$$

Substituting the value of c in (1),

$$s = 16t^2. \quad (4)$$

To find how far the body will fall in five seconds, substitute 5 for t in (4); then, $$s = 16 \times 25 = 400. \quad (5)$$

Hence, the body falls 400 feet in 5 seconds.

Also, to find the number of seconds required for the body to fall

192 yards, or 576 feet, substitute 576 for s in (4); then,

$$576 = 16\,t^2. \qquad (6)$$

Solving (6) for t, $$t = 6. \qquad (7)$$

Hence, 6 seconds is the time required for the body to fall 192 yards.

EXERCISE 89

Write each of the statements in the first five examples in the form of an equation.

1. The area A of a circle varies as the square of its radius R.

2. The volume V of a sphere varies as the cube of its diameter D.

3. The number N of articles which can be bought for a fixed sum S varies inversely as the cost P of one article.

4. The weight w of a body varies as its mass m.

5. The speed v of a falling body starting from rest varies as the time t during which it falls.

6. If y varies as x, and $y=20$ when $x=5$, find the value of y when $x=10$.

7. If y varies inversely as x, and $y=20$ when $x=5$, find the value of y when $x=100$.

8. If p varies as q, and $p=-2$ when $q=-3$, find the equation connecting p and q.

9. If m varies inversely as n, and $m=30$ when $n=6$, find m when $n=9$.

10. If 7 bu. of corn are worth $4.90, what are 3 bu. of corn of the same quality worth?

Suggestion. Employ direct variation.

11. If 10 men can do a piece of work in 15 days, how long will it take 13 men to do the same work?

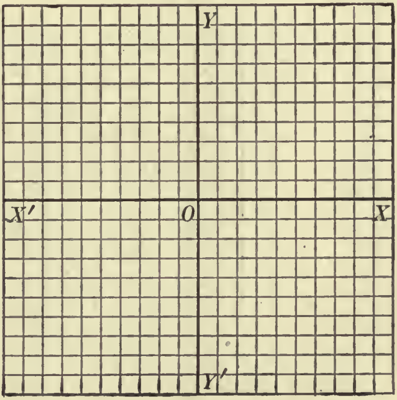
Suggestion. Use inverse variation.

CHAPTER IX

GRAPHS

186. Graphical representation of the relation between two variables. It is convenient to represent corresponding values of two related variables by points in a plane. Such a representation is said to be **graphical.** When a sufficient number of points, which represent pairs of corresponding values of two related variables, have been marked in the plane, a glance at the diagram will reveal to the eye the corresponding changes in the two variable quantities.

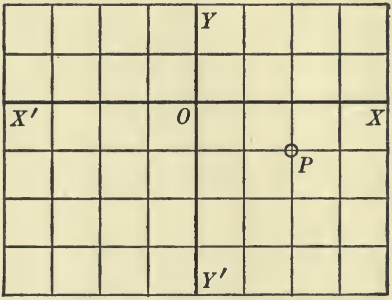
For the purpose of graphical work, paper ruled in small squares, as in the accompanying diagram, is used. Two of the ruled lines of the paper, one horizontal and the other vertical, are selected as **axes of reference.** The point O, at which the axes cross each other is called the **origin.** The axes of reference are called **axes of coördinates.** The horizontal axis $X'X$ is called the **x-axis**, and the vertical axis YY' is called the **y-axis.**



Any point in the plane has two coördinates; namely, its **abscissa,** represented by x, and its **ordinate,** represented by y. The abscissa of a point is its horizontal distance from the y-axis. The ordinate of a point is its vertical distance from the x-axis. *All horizontal distances measured to the right from the y-axis are regarded as positive*, and consequently (see section 31), *all horizontal distances measured in the opposite direction from the y-axis are negative. All vertical distances measured upwards from the x-axis are regarded as positive*, and consequently, *those measured in the opposite direction from the x-axis are negative.*

187. A point fixed by its coördinates. The position of a point in a plane is fixed by its coördinates.

Thus, the position of the point P whose coördinates are $x = 2$ and $y = -1$ is fixed. The sign of its abscissa shows that the point P is to the right of the y-axis and the sign of its ordinate shows that



it is below the x-axis. To actually find the position of P, we count on the x-axis 2 squares to the right of the origin, since the value of the abscissa of P is 2; then we count downward from the x-axis 1 square, since the value of the ordinate of P is -1.

188. Notation. The coördinates of a point are designated by the symbol (x, y).

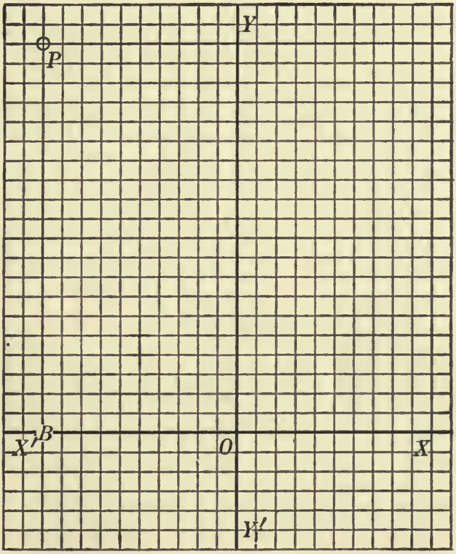
Thus, the point $(2, -3)$ means the point whose x (abscissa) is 2 and whose y (ordinate) is -3. It should be observed that in such a symbol as $(2, -3)$ the number written first is always the abscissa.

189. Plotting a point. The marking of the position of a point on the diagram is called **plotting** the point.

ILLUSTRATIVE EXAMPLES

1. Plot the point $(-2, 4)$.

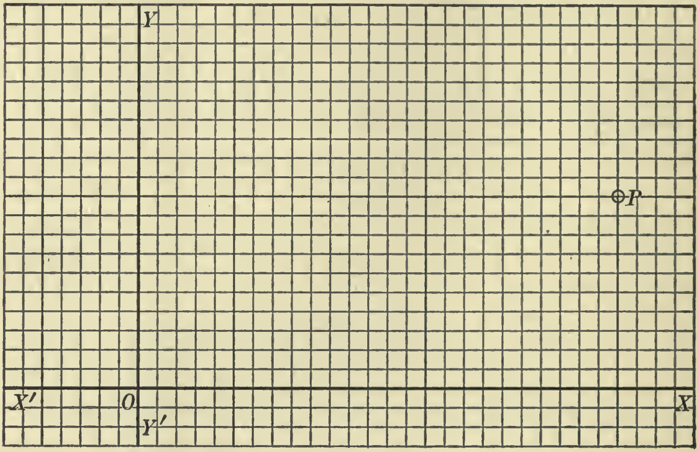
Solution. For convenience, we take five divisions on the coördinate paper as the unit of measure. The abscissa being -2, we



count 10 squares (2 units) to the left of the origin along the x-axis to the point B; then, since the ordinate is $+4$, we count 20 squares (4 units) upwards from the x-axis, locating the position P of the point $(-2, 4)$. We mark the position of P by a small ring.

2. Plot the point (1.25, .5).

Solution. For convenience, we take 20 divisions on the coördinate paper as the unit of measure. Since both coördinates are positive,



we count to the right along the x-axis and upwards from the x-axis. Counting 25 squares (1.25 units) to the right along the x-axis and then 10 squares (.5 units) upwards from the x-axis, we locate the position P of the required point.

EXERCISE 90

(If convenient, use graph paper in plotting the graphs in the examples of this exercise. If such paper is not available, draw two axes. Use an appropriate unit of distance, as $\frac{1}{2}$ inch, $\frac{1}{8}$ inch, $\frac{1}{10}$ inch, or 1 centimeter).

1. Plot the points (4, 3); (−4, 3); (4, −3); (−4, −3).

2. Plot the points (−2, 1); (−2, −1); (4, −4); (4, 1).

3. Plot the points (3, 0); (0, −3); (0, 3); (−3, 0).

4. Plot the points $A(4, 3)$; $B(-4, 4)$; $C(7, 0)$. Connect the points A, B, C with straight lines forming a triangle whose vertices are the given points.

5. Construct the quadrilateral whose vertices taken in order are the points $A(3, 4)$; $B(-2, 6)$; $C(-4, -1)$; $D(4, -2)$.

6. On what straight line are all points located which have for their abscissa 0?

7. On what straight line are all points located which have for their ordinate 0?

8. What are the coördinates of the origin? Plot the point $(0, 0)$.

9. Draw the triangle whose vertices are $(0, -1)$; $(0, +1)$; $(2, 0)$.

10. Plot the points $(-3, -9)$; $(-2, -6)$; $(-1, -3)$; $(1, 3)$; $(2, 6)$; $(3, 9)$. Do these points appear to the eye to be scattered at random over the diagram? How do they appear to lie?

190. A function. In many of the problems of elementary algebra one or more pairs of related variables occur [see examples 1–5, exercise 89, also section 183]. As another illustration of two related variables, let A represent the age of a boy and W his weight. In general, as A changes it is evident that W also changes; this fact may be expressed by stating that the weight of the boy varies with, and depends on, his age.

If one variable varies with another so that when a value of one is given, a corresponding value of the other is determined, the second variable is called a function of the first.

Thus, from the indeterminate equation $y + 2x = 5$, we derive $y = 5 - 2x$. Here y is so related to x that its value is determined for any given value of x; y is, therefore, a function of x. Also, the area of a circle is a function of its radius. If A represents the area and R the radius of a circle and π the well-known constant whose value to four places of decimals is 3.1416, then the relation between the area of a circle and the radius is expressed algebraically by the equation

$$A = \pi R^2$$

191. Graph of a function. When a simple relation between two variables is given, as, for example, the relation expressed by the equation $y = 3x$, numerical values may be assigned to x and the corresponding values of y found. On plotting the points which represent the different pairs of corresponding values of the variables, these points are found to lie on a definite straight line or curve; this straight line or curve is called the **graph of the function.** It is also convenient to speak of the **graph of an equation,** an expression which means that the coördinates of the points plotted and through which the line or curve passes, satisfy the equation.

ILLUSTRATIVE EXAMPLE

Draw the graph of $3x + 5$ for values of x between $x = -4$ and $x = +3$.

Solution. Denoting the function by y, we have

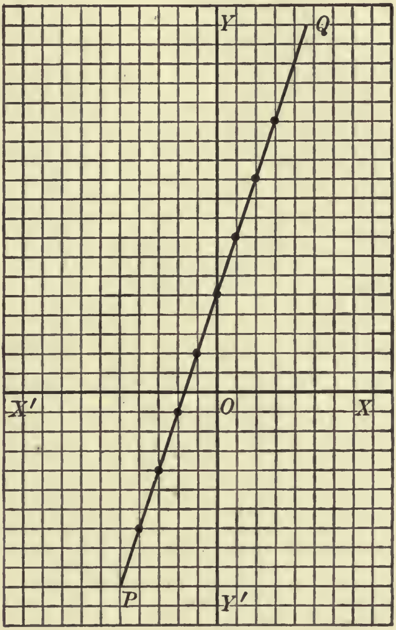
$$y = 3x + 5 \qquad (1)$$

We now make a table showing the corresponding values of x and y for integral values of x between $x = -4$ and $x = +3$, as follows:

x	-4	-3	-2	-1	0	1	2	3
$3x$ 5	-12 5	-9 5	-6 5	-3 5	0 5	3 5	6 5	9 5
y	-7	-4	-1	$+2$	5	8	11	14

Plotting the points which represent the values of x and y (see diagram), these points appear to lie, and in fact do lie, on the straight line PQ. This straight line is the *graph* of the function $3x + 5$, and is also the graph of the equation $y = 3x + 5$.

Any values of x and y which satisfy equation (1) are the coördinates of a point on the graph of equation (1). Conversely, we assume that the coördinates of any point on the graph of the equation will satisfy the equation.



Note. The expressions, *construct the graph of*, *obtain the graph of*, *draw the graph of*, *graph the function*, *plot the curve*, mean the same thing.

EXERCISE 91

Draw the graphs of the following equations for values of x between $x = -4$ and $x = +4$:

1. $y = x$.
2. $y = -x$.
3. $y - 2x = 0$.
4. $y + 4x = 0$.
5. $y = 3x$.
6. $y = x + 1$.

7. $y = x + 2.$ **8.** $y = -x + 2.$ **9.** $y = \frac{3}{2}x + 1.$

10. $2x + 3y = 5.$ **11.** $3x - 2y = 1.$ **12.** $5x - 2y = 4.$

13. $x + y + 1 = 0.$ **14.** $y + \frac{x}{4} - 3 = 0.$ **15.** $3x + 4y + 7 = 0.$

192. Graph of a linear equation in two variables. Any simple equation in two unknowns, as x and y, can be reduced to the form $ax + by + c = 0$. The graph of such an equation is always a straight line. It is for this reason that the simple equation $ax + by + c = 0$ is called a *linear equation.*

Note. It will be assumed here that the graph of any simple equation in two unknowns is a straight line. The proof of this fact is given in analytical geometry.

193. Graphs of simultaneous linear equations in two unknown numbers. When the straight lines which are the graphs of two linear equations in two unknown numbers are plotted on the same diagram, they will, if the given equations are independent and consistent, intersect in one point. The coördinates of the point of intersection satisfy both equations and, therefore, together constitute the solution of the given system.

ILLUSTRATIVE EXAMPLE

Construct the graphs of the equations $2y - x + 4 = 0$ and $2y + 3x - 4 = 0$. Estimate from the diagram the coördinates of their point of intersection. Find whether or not these estimated coördinates satisfy the equation.

Solution.

$$2y - x + 4 = 0. \quad (1)$$

$$2y + 3x - 4 = 0. \quad (2)$$

Solving (1) for y, $$y = \frac{x}{2} - 2. \quad (3)$$

Solving (2) for y, $$y = \frac{-3x}{2} + 2. \quad (4)$$

Assigning to x in (3) any two values, as $x = 0$, and $x = 4$, we have from (3),

$$\text{when } x = 0, \quad y = -2, \tag{5}$$

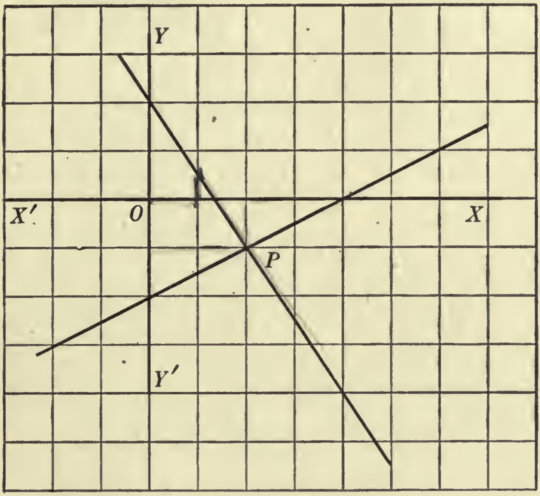
$$\text{when } x = 4, \quad y = 0. \tag{6}$$

Similarly, assigning to x in (4) any two values, as $x = 0$, and $x = 1$, we have from (4),

$$\text{when } x = 0, \quad y = 2, \tag{7}$$

$$\text{when } x = 1, \quad y = \tfrac{1}{2}. \tag{8}$$

Since we know that the graph of (3) is a straight line [§ 192, note], it is necessary to plot two points only and then draw the straight line through these points.



Plotting the points (0, − 2) and (4, 0), the graph of (3) is obtained by drawing the straight line through these points. In like manner, plotting the points (0, 2) and (1, $\frac{1}{2}$) the graph of (4) is the straight line joining these points.

By inspection, the graphs of the two equations appear to intersect in the point P ($x = 2$, $y = -1$). Substituting these values of x and y in (1) and (2), we have

$$2(-1) - 2 + 4 = 0. \tag{9}$$

$$2(-1) + 6 - 4 = 0. \tag{10}$$

Hence the solution of (1) and (2) is $x = 2$, $y = -1$.

EXERCISE 92

Construct the graphs of the equations in each of the following systems. For each system determine by inspection approximate values of the unknown numbers which will satisfy the equations.

1. $\begin{cases} x + y = 4, \\ x - y = 3. \end{cases}$

2. $\begin{cases} x + y = 1, \\ \quad\; y = 2. \end{cases}$

3. $\begin{cases} x + 2y = 3, \\ 3x - 2y = 1. \end{cases}$

4. $\begin{cases} 2x + 3y = 3, \\ 3x + 4y = 4. \end{cases}$

5. $\begin{cases} 3x + y = 2, \\ 2x - y = 3. \end{cases}$

6. $\begin{cases} y + 3x = 6, \\ x + 2y = 3. \end{cases}$

194. Graphs of dependent linear equations in two unknowns. Two linear equations in two unknowns which have more than one solution in common must have all solutions in common, and, therefore, they are dependent equations. This follows directly from the fact that the graph of any such equation being a straight line, it is completely determined when any two points on it are known.

Thus, the equations $x + y = 2$ and $3x + 3y = 6$ have in common the solutions $x = 0$, $y = 2$ and $x = 2$, $y = 0$, and the graph of each is a straight line through these two points. The coördinates of every point on this line satisfy both equations.

195. Inconsistent equations. The graphs of two inconsistent linear equations in two unknowns furnish a simple geometrical reason for the existence of such systems of equations.

Thus, the graphs of the inconsistent equations

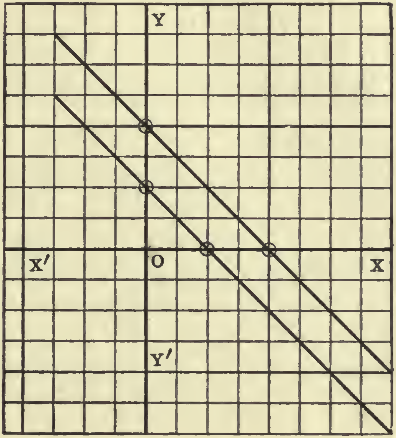
$$x + y = 2 \tag{1}$$
$$x + y = 1 \tag{2}$$

are found to be parallel straight lines.

In the accompanying diagram two divisions on the coördinate paper are taken as the unit of measure. The graph of (1) is the straight

René Descartes (1596–1650) was the first of the modern school of mathematics. He was educated at the Jesuit School of LaFleche, France. Descartes is famous as a mathematician and as a philosopher. The introduction of the fundamental ideas underlying the graphical representation of related variables was his greatest contribution to mathematical science. The introduction of indices as now used was due to Descartes.

line which passes through the points (0, 2) and (2, 0), and the graph of (2) is the straight line which passes through (0, 1) and (1, 0). These lines are evidently parallel.



In order that two linear equations in two unknowns may have a solution in common, it is necessary that their graphs should have a point in common. Two parallel lines have no common point and, therefore, the equations of which the parallel lines are the graphs have no solution in common; they are, therefore, inconsistent equations.

EXERCISE 93

Construct the graphs of the following systems of equations.

1. $\begin{cases} x + y = 1, \\ x + y = 2. \end{cases}$

2. $\begin{cases} x = y, \\ x = 2\,y. \end{cases}$

3. $\begin{cases} x = 1, \\ x = 2. \end{cases}$

4. $\begin{cases} y = -1, \\ y - 1 = 0. \end{cases}$

5. $\begin{cases} x - 2 = 0, \\ y - 3 = 0. \end{cases}$

6. $\begin{cases} x + y = 1, \\ 2\,x + 2\,y = 2. \end{cases}$

CHAPTER X

POWERS, ROOTS, RADICALS, AND EXPONENTS

196. Rational number. The rational operations in algebra are addition, subtraction, multiplication, and division [section 79]. Any number which is either a positive integer or can be obtained from the positive integers by the rational operations is called a **rational number.** Hence, a rational number, when expressed in its simplest form, is either a positive or negative integer or a positive or negative fraction with integral numerator and integral denominator.

Thus, 2, 3.16, -7, and $-\frac{3}{4}$ are rational numbers.

Also, $+\sqrt{4}$ is a rational number, since when expressed in its simplest form, it has a rational value, namely, 2. However, finding the square root of a number is not one of the rational operations of algebra.

197. Irrational number. It is necessary to add to the system consisting of all rational numbers, certain other numbers which are not rational. For instance, in mensuration it is necessary to say that the length of the diagonal of the square is equal to the length of its side multiplied by the $\sqrt{2}$; yet there is no rational number whose square is equal to 2; that is, $\sqrt{2}$ is not a rational number. We shall assume that there is a definite positive number whose square is equal to 2. We call such a number an **irrational number.**

Thus, $\sqrt{3}$, $1-\sqrt{2}$, $\frac{1}{\sqrt{2}}$, $\sqrt{2}+\sqrt{3}$ are irrational numbers.

Note. When we say that we assume that there is a definite positive number whose square is equal to 2, we are extending the meaning of the word *number.* The word *number*, heretofore, has meant *rational number.* Now when we use the word *number* we shall mean irrational number as well as rational.

Remark. We do not call such an expression as $\sqrt{-2}$ an irrational number. This expression is, for the present, without meaning. The square of either a positive or a negative number is necessarily positive; hence, there is no positive or negative number whose square is -2.

Rational numbers and irrational numbers are necessarily positive or negative.

198. Fundamental identities involving powers. The exponents used in the identities of this section are positive integers.

From section 58 we have,

$$a^m \cdot a^n = a^{m+n}. \qquad \text{(I)}$$

By definition, section 14,

$$\begin{aligned}(ab)^m &= ab \cdot ab \cdot ab \cdot \cdots \text{ to } m \text{ factors}\\ &= a \cdot a \cdot a \cdot \cdots \text{ to } m \text{ factors} \times b \cdot b \cdot b \cdot \cdots \text{ to } m \text{ factors}\\ &= a^m \cdot b^m.\end{aligned}$$

That is, $$(ab)^m = a^m b^m. \qquad \text{(II)}$$

Remark. Since $(abc)^m = [(ab)c]^m = (ab)^m c^m = a^m b^m c^m$, it follows that an identity similar to (II) holds for a power of the product of any number of factors.

By definition, section 14, $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots$ to n factors

$$= a^{m+m+m+\cdots \text{ to } n \text{ terms}}$$

$$= a^{mn}.$$

In like manner, $$(a^n)^m = a^{mn}.$$

That is, $$(a^m)^n = (a^n)^m = a^{mn}. \qquad \text{(III)}$$

From identity (II), $$(a^m b^n)^p = (a^m)^p (b^n)^p.$$

From identity (III), $$(a^m)^p (b^n)^p = a^{mp} b^{np}.$$

Therefore, $$(a^m b^n)^p = a^{mp} b^{np}. \qquad \text{(IV)}$$

EXERCISE 94

(Solve as many as possible at sight.)

Find the results of the indicated operations by using the identities of section 198.

1. $2^3 \cdot 2^2$.
2. $2^3 \cdot 2^2 \cdot 2$.
3. $(2 \cdot 3)^2$.
4. $(2 \cdot 3 \cdot 5)^3$.
5. $(2^2)^3$.
6. $(3^3)^2$.
7. $(2^3)^3$.
8. $[(-2)^2]^3$.
9. $[(-2)^3]^2$.
10. $[(-2)^3]^3$.
11. $(2^2 \cdot 3^3)^2$.
12. $(2 \cdot 3^2 \cdot 5^2)^2$.
13. $a^{15} \cdot a^{10}$.
14. $a^5 \cdot a^6 \cdot a^7$.
15. $(ab)^5$.
16. $(2a)^3$.
17. $(-3a)^3$.
18. $(2ab)^3$.
19. $(-3ab)^2$.
20. $(abcd)^5$.
21. $(a^2b^3)^2$.
22. $(2^2b^3)^3$.
23. $[(-2)^3a^2b^3]^2$.
24. $(-3a^2b^3c^2d^4e^5)^4$.
25. $[(a+b)^2]^3$.
26. $(a+b)^2(a+b)$.
27. $(a-b)^2(a-b)^3$.
28. $(a-b)^2(a+b)^2$.

Suggestion. $(a-b)^2(a+b)^2 = [(a-b)(a+b)]^2$.

29. $(a-1)^2(a+1)^2$.
30. $[2(a+b)^2(c-d)^3]^2$.
31. $[2ab(c+d)^2(e-f)^3]^3$.
32. $[(-3)^2]^3 \cdot (3^3)^2$.
33. Show that $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, and, in general, that $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

Suggestion. $\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots$ to m factors. [§ 139]

34. Show that $(\frac{2}{3})^3(\frac{3}{4})^3 = (\frac{2}{3} \times \frac{3}{4})^3$, and, in general, that

$$\left(\frac{a}{b}\right)^m\left(\frac{c}{d}\right)^m = \left(\frac{ac}{bd}\right)^m.$$

Suggestion. $\left(\frac{a}{b}\right)^m\left(\frac{c}{d}\right)^m = \frac{a^m}{b^m} \cdot \frac{c^m}{d^m} = \frac{a^m c^m}{b^m d^m} =$ etc.

35. $(\frac{2}{3})^2(\frac{3}{5})^2(\frac{5}{2})^2$.
36. $(\frac{3}{5})^{99}(\frac{5}{7})^{99}(\frac{7}{3})^{99}$.

199. Roots. When a number is the product of five equal factors, one of these factors is called a fifth root of the number. In general, when a number is the product of n equal factors, one of those factors is called an ***n*th root** of the number.

Thus, if $m^5 = a$, then m is a fifth root of a, which is expressed by $m = \sqrt[5]{a}$.

The two equations,

$$m^5 = a \quad (1) \quad \text{and} \quad m = \sqrt[5]{a} \quad (2)$$

express, therefore, the same relation between a and m. By substituting the value of m from equation (2) in equation (1), we have the identity $(\sqrt[5]{a})^5 = a$. In general, when n is any positive integer we have, by definition,

$$(\sqrt[n]{a})^n = a.$$

Note. The expression $\sqrt[n]{a}$ is read *the nth root of a.*

200. Radical. An indicated root of any number is called a **radical expression**, or simply a **radical.**

Thus, $\sqrt{3}$, $\sqrt[3]{27}$, $\sqrt[3]{\frac{a}{b}}$, $\sqrt[5]{a+b}$ are radicals.

201. Index of a root. The number indicating the root to be taken is called the **index** of the root. The index of a radical is always an integer.

202. Radicand. The number or expression under the radical sign is called the **radicand.**

203. Like roots and unlike roots. Two roots are said to be **like** or **unlike** according as their indices are equal or unequal.

Thus, $\sqrt{a}$ and $\sqrt{b}$ are like roots; $\sqrt{a}$ and $\sqrt[3]{a}$ are unlike roots.

204. Principal root. Any positive number a has one and only one positive nth root. This root is called the **principal *n*th root** of a.

Thus, the principal square root of 4 is 2, the principal cube root of 27 is 3, and the principal square root of 2 is $+\sqrt{2}$.

When the index of a radical is an odd number and the radicand is negative, there is one and only one negative root and no positive root. This negative root is called the **principal *n*th root** of the negative number in the radicand.

Thus, $\sqrt[3]{-8}$ can have no positive value, since the cube of a positive number is positive. It has, however, one negative value, namely -2, since an odd power of a negative number is negative. Also, the principal root of $\sqrt[3]{-27}$ is -3, that of $\sqrt[5]{-32}$ is -2, that of $\sqrt[3]{-2}$ is $-\sqrt[3]{2}$; and if n be any odd integer that of $\sqrt[n]{-a^n}$ is $-a$.

205. A property of positive numbers. *Two positive numbers are equal if any like powers of these numbers are equal.*

For, a positive number has one and only one positive root, namely, its principal root.

Thus, if a is positive and $a^3 = 3^3$, then a is equal to 3, since a is the principal cube root of a^3, or of 27.

206. Notation. In what follows in this chapter the letters a, b, c, and so on, will represent positive numbers or literal expressions which have positive values, except when otherwise stated. By the root of a number we shall mean its principal root; that is, its one positive root when the radicand is positive and its one negative root when the radicand is negative and the index an odd integer. [See section 204.]

Thus, $\sqrt[3]{27} = +3$, and $\sqrt[3]{-27} = -3$.

207. Surds. A **surd** is an irrational number which is a root of a rational number.

Thus, $\sqrt[3]{5}$ and $\sqrt{3}$ are surds; $\sqrt{4}$ and $\sqrt[3]{27}$ are radical expressions, but they are not surds.

208. Order of a surd. The order of a surd is the index of the root involved in the expression.

Thus, $\sqrt{2}$ is a surd of the second order, or a quadratic surd. $\sqrt[3]{5}$ is a surd of the third order, or a cubic surd.

209. Fundamental identities involving roots. Radicals are transformed and combined according to certain rules. These rules may be expressed in the form of algebraic identities.

$$(\sqrt[n]{a})^n = a. \qquad \text{(I)}$$

Identity (I) is simply the definition of a root as expressed in algebraic symbols [section 199].

EXERCISE 95

State at sight the value of each of the following:

1. $(\sqrt{2})^2$.

2. $(\sqrt{3})^2$.

3. $(\sqrt{a})^2$.

4. $(\sqrt[3]{2})^3$.

5. $(\sqrt[3]{-3})^3$.

6. $(\sqrt[4]{5})^4$.

7. $(\sqrt[5]{-10})^5$.

8. $(\sqrt[7]{-a})^7$.

9. $(\sqrt[6]{7})^6$.

10. $(\sqrt[3]{27})^3$.

11. $(\sqrt{a+b})^2$.

12. $(\sqrt[3]{a^2-b^2})^3$.

13. $(\sqrt[4]{(a+b)^4})^4$.

14. $(\sqrt[6]{(a+b)^3})^6$.

15. $\sqrt{7} \cdot \sqrt{7}$.

16. $\sqrt{x} \cdot \sqrt{x}$.

17. $\sqrt[3]{-7} \times \sqrt[3]{-7} \times \sqrt[3]{-7}$.

18. $\sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x}$.

$$\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}. \qquad \text{(II)}$$

From identity (II) we may infer that:

The value of a radical is not changed if its index and the exponent of its radicand are both multiplied by the same positive integer.

Conversely, we may write identity (II) thus:

$$\sqrt[np]{a^{mp}} = \sqrt[n]{a^m}.$$

From the second form of identity (II) we have :

The value of a radical is not changed if its index and the exponent of its radicand are both divided by the same positive integer which is a factor of each.

Note. An important particular case of § 209, I, is expressed by the identity,

$$\sqrt[n]{a^{np}} = a^p.$$

Thus, $\sqrt[n]{a^{np}} = \sqrt[n]{(a^p)^n} = a^p.$

Similarly, $\sqrt[3]{a^6} = a^2$, and $\sqrt{a^8} = a^4.$

ILLUSTRATIVE EXAMPLES

1. $\sqrt{a^3} = \sqrt[6]{a^9}$, and conversely, $\sqrt[6]{a^9} = \sqrt{a^3}$.

2. $\sqrt{a^2} = a$, and conversely, $a = \sqrt{a^2}$.

3. $\sqrt[3]{a^{3m}} = a^m$, and conversely, $a^m = \sqrt[3]{a^{3m}}$.

The proof of identity (II) is as follows:

By identity (I), $(\sqrt[np]{a^{mp}})^{np} = a^{mp}.$

Also, by identity (III) of § 198,

$$(\sqrt[n]{a^m})^{np} = [(\sqrt[n]{a^m})^n]^p$$
$$= (a^m)^p \qquad [§ 209 (I)]$$
$$= a^{mp}. \qquad [§ 198 (III)]$$

We have now shown that the (np)th power of the expression in the first member of identity (II) is equal to the same power of the expression in the second member. It follows by the principle of section 205 that the two expressions are equal.

EXERCISE 96

(Solve as many as possible at sight.)

Simplify the following:

1. $\sqrt[6]{a^{15}}$. **2.** $\sqrt[9]{a^6}$. **3.** $\sqrt[10]{a^{15}}$.

4. $\sqrt[6]{(2)^8}$. **5.** $\sqrt[4]{(a+b)^2}$. **6.** $\sqrt[15]{(c+d)^{10}}$.

7. $\sqrt[9]{-a^{15}}$ **8.** $\sqrt[21]{-a^{35}}$. **9.** $\sqrt[33]{-a^{11}}$.

10. $\sqrt[15]{-(a+b)^5}$. **11.** $\sqrt[4]{a^{12}}$. **12.** $\sqrt[6]{(a+b)^6}$.

13. $\sqrt[5]{(a-b)^{15}}$. **14.** $\sqrt[9]{a^{27}}$. **15.** $\sqrt[4]{a^{4n}}$.

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}. \qquad \text{(III)}$$

From identity (III) we infer that:

Any root of the product of two or more factors is equal to the product of the like roots of the factors.

Note. In identity (III) there are two factors and in the statement of the principle that follows there are *two or more factors.* Here, as in all similar cases, what is true of the product of two factors is true in general. This follows directly from the fact that by grouping factors any product may be expressed in the form (ab).

Thus, in the case of four factors,

$$\begin{aligned}\sqrt[n]{abcd} &= \sqrt[n]{(ab)(cd)}\\ &= \sqrt[n]{ab}\ \sqrt[n]{cd}\\ &= (\sqrt[n]{a}\ \sqrt[n]{b})\ (\sqrt[n]{c}\ \sqrt[n]{d})\\ &= \sqrt[n]{a}\ \sqrt[n]{b}\ \sqrt[n]{c}\ \sqrt[n]{d}.\end{aligned}$$

Writing the converse of identity (III), we have,

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}.$$

This second form of identity (III) may be stated in words as follows:

The product of like roots of two or more numbers is equal to the like root of their product.

ILLUSTRATIVE EXAMPLES

1. $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}\ \sqrt{2} = 2\sqrt{2}.$
2. $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{3^3}\,\sqrt[3]{2} = 3\sqrt[3]{2}.$
3. $\sqrt{3a^2} = \sqrt{a^2}\sqrt{3} = a\sqrt{3}.$
4. $\sqrt[3]{4}\sqrt[3]{2} = \sqrt[3]{8} = \sqrt[3]{2^3} = 2.$
5. $\sqrt[6]{a}\sqrt[6]{a^2}\sqrt[6]{a^3} = \sqrt[6]{a \times a^2 \times a^3} = \sqrt[6]{a^6} = a.$

In order to prove identity (III) we show that the nth power of the positive number expression in the first member is equal to the same power of that in the second.

Thus,

By identity (I) $(\sqrt[n]{ab})^n = ab.$

Also, by identity (II), section 198,

$$(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n, \text{ and by section 209, (I),}$$
$$= ab.$$

Since $(\sqrt[n]{ab})^n$ and $(\sqrt[n]{a}\sqrt[n]{b})^n$ are each equal to ab, it follows that

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$$

EXERCISE 97

(Solve as many as possible at sight.)

Simplify:

1. $\sqrt{12}$. **2.** $\sqrt{27}$. **3.** $\sqrt{32}$.

4. $\sqrt{128}$. **5.** $\sqrt{2a^2}$. **6.** $\sqrt{3a^4}$.

7. $\sqrt{4a^3}$. **Suggestion.** $\sqrt{4a^3} = \sqrt{(4a^2)a}$.

8. $\sqrt{4a^5}$. **9.** $\sqrt{8a^3}$. **10.** $\sqrt{128a^3b^2}$.

11. $\sqrt{x^2z}$. **12.** $\sqrt{4lm^2}$. **13.** $\sqrt{a^2b^3c^4d^5}$.

14. $\sqrt{72a^3}$. **15.** $\sqrt{80a^2b^4}$. **16.** $\sqrt{99a^9}$.

17. $\sqrt{127008}$. **Suggestion.** $127008 = 2^5 \cdot 3^4 \cdot 7^2$.

18. $\sqrt{84672}$. **19.** $\sqrt{27ab^2c^3}$.

20. $\sqrt{81 \times 5 \times a^3b^5}$. **21.** $\sqrt{75x^3y^2}$.

22. $\sqrt{176a^2b^2c^4d^5}$. **23.** $\sqrt{2(a+b)^2}$.

24. $\sqrt{50(a+b)^2(c+d)^4}$. **25.** $\sqrt{108a^2(b+c)^3}$.

26. $\sqrt[3]{16}$. **27.** $\sqrt[3]{54}$.

28. $\sqrt[3]{128}$. **29.** $\sqrt[3]{625}$.

30. $\sqrt[3]{81}$. **31.** $\sqrt[3]{81a}$.

32. $\sqrt[3]{128b^3}$.

33. $\sqrt[3]{-16}$. **Suggestion.** $\sqrt[3]{-16} = -\sqrt[3]{2(2)^3}$.

34. $\sqrt[3]{-54}$. **35.** $\sqrt[3]{-216\,x^3}$. **36.** $\sqrt{9\,x^3y^4z^5}$.

37. $\sqrt[3]{-125\,a^3b^6}$. **38.** $\sqrt[5]{64}$. **39.** $\sqrt{64\,x^5y^{10}}$.

40. $\sqrt[3]{-2}$. **Suggestion.** $\sqrt[3]{-2} = -\sqrt[3]{2}$ [§ 204].

41. $\sqrt[3]{-3}$. **42.** $\sqrt[3]{-5}$. **43.** $\sqrt[5]{-32\,x^3y^6}$.

44. $\sqrt[5]{x^6y^7}$. **45.** $\sqrt[5]{-x^{15}yz^6}$. **46.** $\sqrt[7]{-2^8y^7}$.

47. $\sqrt[5]{-x^{10}}$. **Suggestion.** $\sqrt[5]{-x^{10}} = \sqrt[5]{(-1)^5x^{10}}$.

48. $\sqrt[4]{32\,x^5y^6}$. **49.** $\sqrt[12]{x^{14}}$.

50. $\sqrt[11]{x^{12}}$. **51.** $\sqrt[13]{2^{14}\cdot a^{14}}$.

52. $\sqrt[5]{-243\,a^6b^7}$. **53.** $\sqrt[10]{(a+b)^{12}(c+d)^{10}}$.

54. $\sqrt[7]{2^8\cdot 3^9\cdot a^7x^8y^{10}}$. **55.** $\sqrt[5]{-(x+y)^7(x-y)^2}$.

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}. \qquad \text{(IV)}$$

From identity (IV) we infer that:

Any power of a radical is obtained by multiplying the exponent of the radicand by the exponent of the required power.

Conversely, since $\sqrt[n]{a}$ is one of the m equal factors of $(\sqrt[n]{a})^m$, or of its equal $\sqrt[n]{a^m}$, it follows that $\sqrt[n]{a}$ is the mth root of $\sqrt[n]{a^m}$. Hence:

Rule. *To find the root of the radical, the exponent of whose radicand is exactly divisible by the index of the root, divide the exponent of the radicand by the index of the required root.*

ILLUSTRATIVE EXAMPLES

1. $(\sqrt[4]{8})^3 = (\sqrt[4]{2^3})^3 = \sqrt[4]{2^9} = \sqrt[4]{2^8\times 2} = \sqrt[4]{2^8}\sqrt[4]{2} = 2^2\sqrt[4]{2}$ $= 4\sqrt[4]{2}$.

2. $(\sqrt[3]{4})^6 = \sqrt[3]{4^6} = 4^2 = 16$.

3. $\sqrt{\sqrt[3]{a^6}} = \sqrt{a^2} = a$.

The proof of identity (IV) is as follows:

By identity (I), section 209, $(\sqrt[n]{a^m})^n = a^m$.

Also, by identity (III), section 198, $[(\sqrt[n]{a})^m]^n = [(\sqrt[n]{a})^n]^m = a^m$.

Since $(\sqrt[n]{a^m})^n$ and $[(\sqrt[n]{a})^m]^n$ are each equal to a^m, it follows that $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$ [§ 205].

EXERCISE 98

In simplifying the following, use as many of the preceding identities of this chapter as may be necessary.

1. $(\sqrt[4]{49})^2$.
2. $(\sqrt{2})^4$.
3. $(\sqrt{3})^6$.
4. $(\sqrt{a})^4$.
5. $(\sqrt{3a^3})^3$.
6. $(\sqrt{2a})^4$.
7. $(\sqrt{3a})^6$.
8. $(\sqrt{3a})^5$.
9. $(\sqrt[3]{-a})^2$.
10. $(\sqrt[5]{-2a})^{10}$.
11. $(-\sqrt[3]{9})^4$.
12. $[\sqrt[3]{2(a+b)}]^2$.
13. $[\sqrt[3]{-3(c+d)^2}]^2$.
14. $(\sqrt[5]{2a})^{15}$.
15. $(-\sqrt[5]{-2a})^{15}$.
16. $(-\sqrt[3]{-4ab^2})^6$.
17. $\sqrt[3]{\sqrt{4}}$.
18. $\sqrt[5]{\sqrt{9a^2}}$.
19. $\sqrt[3]{\sqrt[3]{-8}}$.
20. $\sqrt[3]{-\sqrt[5]{-32}}$.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}. \qquad \text{(V)}$$

From identity (V) we infer that:

Any root of a fraction is equal to the like root of the numerator divided by the like root of the denominator.

Conversely, we may write identity (V) thus:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

From this second form of identity (V) we infer that:

Any root of a number divided by the like root of a second number is equal to the like root of the fraction whose numerator is the first number and whose denominator is the second.

ILLUSTRATIVE EXAMPLES

1. $\sqrt{\frac{4}{9}}=\frac{\sqrt{4}}{\sqrt{9}}=\frac{2}{3}.$

2. $\sqrt[3]{\frac{a^6}{b^{12}}}=\frac{\sqrt[3]{a^6}}{\sqrt[3]{b^{12}}}=\frac{a^2}{b^4}.$

3. $\frac{\sqrt{8}}{\sqrt{2}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2.$

4. $\frac{\sqrt[3]{a^5}}{\sqrt[3]{a^2}}=\sqrt[3]{\frac{a^5}{a^2}}=\sqrt[3]{a^3}=a.$

5. $\frac{\sqrt{2}}{\sqrt{3}}=\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{\sqrt{6}}{\sqrt{9}}=\frac{\sqrt{6}}{3}.$

6. $\frac{\sqrt[3]{3}}{\sqrt[3]{2}}=\sqrt[3]{\frac{3}{2}}=\sqrt[3]{\frac{12}{8}}=\frac{\sqrt[3]{12}}{\sqrt[3]{8}}=\frac{\sqrt[3]{12}}{2}.$

The proof of identity (V) is as follows:

By identity (I), § 209, $\left(\sqrt[n]{\frac{a}{b}}\right)^n=\frac{a}{b}.$

By identity in example 33, p. 264,

$$\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n=\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n}=\frac{a}{b}.$$

Since $\left(\sqrt[n]{\frac{a}{b}}\right)^n$ and $\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n$ are each equal to $\frac{a}{b}$, it follows that

$$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}. \qquad [\S\ 205]$$

EXERCISE 99

Reduce the following to equivalent fractions having rational denominators [see illustrative examples 5 and 6, p. 273]:

1. $\dfrac{\sqrt{3}}{\sqrt{2}}$. **2.** $\sqrt{\frac{1}{2}}$. **3.** $\sqrt{\frac{3}{8}}$.

4. $\sqrt{\frac{1}{3}}$. **5.** $\sqrt{\frac{1}{5}}$. **6.** $\dfrac{\sqrt{7}}{\sqrt{3}}$.

7. $\dfrac{\sqrt{3}}{\sqrt{5}}$. **8.** $\dfrac{\sqrt{11}}{\sqrt{5}}$. **9.** $\sqrt{\dfrac{a^2}{b}}$.

10. $\sqrt{\dfrac{b}{a}}$. **11.** $\sqrt{\dfrac{a}{2b}}$. **12.** $\sqrt{\dfrac{1}{3a}}$.

13. $\sqrt{\dfrac{2a}{3bc}}$. **14.** $\dfrac{\sqrt[5]{2}}{\sqrt[5]{81}}$. **15.** $\sqrt[5]{\frac{1}{16}}$.

16. $\sqrt[3]{\frac{2}{3}}$. **17.** $\sqrt[3]{\dfrac{a}{b^2}}$. **18.** $\sqrt[3]{\dfrac{2a}{4c^2}}$.

19. $\sqrt{\dfrac{-3}{4}}$. **20.** $\sqrt[3]{\dfrac{-2a}{3b}}$. **21.** $\sqrt[3]{\dfrac{a^9}{b^6}}$.

22. $\dfrac{\sqrt{a^5}}{\sqrt{a}}$. **23.** $\dfrac{\sqrt{2a^3}}{\sqrt{3a}}$. **24.** $\sqrt[3]{\dfrac{27a^3}{8b^6}}$.

25. $\sqrt[3]{\dfrac{-8a^3b^6}{125c^9}}$. **26.** $\sqrt[4]{\dfrac{16a^4}{81b^8}}$. **27.** $\sqrt[5]{\dfrac{32a^5b^{10}}{243c^5}}$.

28. $\sqrt{\dfrac{(a+b)^3}{(c+d)^6}}$. **29.** $\sqrt[12]{\dfrac{1}{a^{11}}}$. **30.** $\sqrt[n]{\dfrac{a^{2n}}{b^{3n}}}$.

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}. \qquad \text{(VI)}$$

From identity **(VI)** we infer that:

Any root of a root of a number is equal to that root of the number whose index is the product of the given indices.

The converse of identity (VI) shows that when the index of a given root is a composite number, it is possible to express the given root in terms of simpler roots.

Thus, $\sqrt[4]{625}=\sqrt{\sqrt{625}}=\sqrt{25}=5$; also, $\sqrt[6]{a^9}=\sqrt{\sqrt[3]{a^9}}=\sqrt{a^3}=a\sqrt{a}$.

ILLUSTRATIVE EXAMPLES

1. $\sqrt[3]{\sqrt{a}}=\sqrt[6]{a}$

2. $\sqrt[6]{(a+b)^2}=\sqrt[3]{a+b}$; [§ 209 (II)]

or, $\sqrt[6]{(a+b)^2}=\sqrt[3]{\sqrt{(a+b)^2}}=\sqrt[3]{a+b}$. [§ 209 (VI)]

Remark. Since $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$

and $\sqrt[n]{\sqrt[m]{a}}=\sqrt[mn]{a}$,

it follows that $\sqrt[m]{\sqrt[n]{a}}=\sqrt[n]{\sqrt[m]{a}}$.

Thus, $\sqrt{\sqrt[3]{a^2}}=\sqrt[3]{\sqrt{a^2}}=\sqrt[3]{a}$.

Also, $\sqrt{\sqrt{8a^6b^3c^9}}=\sqrt{\sqrt[3]{8a^6b^3c^9}}=\sqrt{2a^2bc^3}$

$=\sqrt{a^2c^2}\sqrt{2bc}=ac\sqrt{2bc}$.

Remark. The proof of identity VI is left as an exercise for the student. See section 205.

210. Simplification of radicals. A surd whose radicand is integral is said to be in its simplest form when the index of the radical is as small as possible and when no factor of the radicand has an exponent which is exactly divisible by the index of the root.

Thus, $\sqrt{2a}$ is in its simplest form; $\sqrt{a^3}$ is not in its simplest form since $\sqrt{a^3}=\sqrt{a^2\times a}=\sqrt{a^2}\sqrt{a}=a\sqrt{a}$; $\sqrt[6]{a^3}$ is not in its simplest form since by section 209 (II), $\sqrt[6]{a^3}=\sqrt{a}$.

From identities (III) and (II), note, section 209, we derive the following rule for simplifying surds:

Rule. *Make the index of the radical as small as possible, then if the exponent of any factor of the radicand is divisible by the index, divide the exponent of that factor by the index and remove the factor from under the radical sign.*

Thus, $\sqrt[3]{54a^8b^7c^3}=\sqrt[3]{2\times 3^3a^6b^6c^3a^2b}=3a^2b^2c\sqrt[3]{2a^2b}$.

Conversely, any factor of the coefficient of the radical may be brought under the radical sign and be made a factor of the radicand provided that its exponent be multiplied by the index of the radical.

Thus, $$2\sqrt{3a} = \sqrt{2^2 \times 3a} = \sqrt{12a}.$$

Any surd whose denominator is irrational is not in the simplest form for the approximate numerical calculation of its value. This may be shown by a particular example, as follows:

Find an approximate value of $\sqrt{\frac{2}{3}}$.

Solution (1) $$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3} = \frac{2.449}{3} = .816^+.$$

Solution (2) $$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{1.114}{1.732} = .816^+.$$

Solution (3) $$\sqrt{\frac{2}{3}} = \sqrt{\frac{4}{6}} = \frac{2}{\sqrt{6}} = \frac{2}{2.449} = .816^+.$$

The actual work required in solutions (2) and (3) is greater than that required in solution (1). The most convenient method of calculation, therefore, is that which proceeds by first making the denominator of the fraction rational.

211. Simplest form of a radical expression. A radical expression is said to be in its simplest form when its denominator is rational and all integral radicands which it contains are reduced to their simplest forms.

212. Simplification of a simple fractional surd. A surd whose radicand is a fraction in its lowest terms is simplified by multiplying the numerator and the denominator of the radicand by the simplest expression which will make its denominator rational.

Thus, $$\frac{\sqrt[3]{3x}}{\sqrt[3]{25y^2z}} = \sqrt[3]{\frac{3x}{25y^2z}} = \sqrt[3]{\frac{3x(5yz^2)}{125y^3z^3}} = \frac{\sqrt[3]{15xyz^2}}{\sqrt[3]{5^3y^3z^3}} = \frac{\sqrt[3]{15xyz^2}}{5yz}.$$

213. Similar, or like, radicals. Two radicals which, when expressed in their simplest forms, differ only in their coefficients, are said to be **similar**, or **like.**

Thus, $\sqrt{8\,a^3b}$ and $-\sqrt{18\,ab^3}$ are similar, since their simplest forms, $2\,a\sqrt{2\,ab}$ and $-3\,b\sqrt{2\,ab}$, differ in their coefficients only.

EXERCISE 100

1. Show that $\sqrt{12}$, $\sqrt{27}$, $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$, are similar.

2. Arrange the following radicals in sets so that those in the same set shall be similar:

$$\sqrt{2\,ab^3};\ \sqrt{4\,a^3b^3};\ \sqrt{\frac{8\,ab}{9}};\ \sqrt{9\,a^4b};\ \sqrt{9\,a^6b};\ \sqrt{\frac{9\,ab}{4}}.$$

3. Write the coefficient as a factor of the radicand in each of the following: $2\sqrt{5}$; $3\sqrt{2}$; $a\sqrt{3}$; $2\,a\sqrt{3}$; $(a+b)\sqrt{3}$.

Suggestion. $2\sqrt{5}=\sqrt{2^2\times 5}=\sqrt{20}$. [§ 210.]

4. Simplify $\sqrt{(a^2-b^2)(a+b)}$, in which expression a is greater than b.

5. Which of the following radical expressions are surds:

$$\sqrt{2}?\quad \sqrt[3]{8}?\quad \sqrt{1+\sqrt{2}}?\quad \sqrt[3]{\sqrt{2}}\,?$$

Reduce each of the following radicals to its simplest form:

6. $\sqrt{72\,ab^2c^3}$. **7.** $\sqrt[3]{-8^2}$. **8.** $\sqrt{a^3b^3}$.

9. $\sqrt[3]{16\,x^6y^3}$. **10.** $\sqrt{\frac{8}{27}}$. **11.** $\sqrt{\frac{a^2}{x}}$.

12. $\sqrt{\frac{x^3}{y}}$. **13.** $\sqrt[4]{32\,x^2y^4z^6}$. **14.** $\sqrt{x^3+x^2y}$.

15. $\sqrt{4\,x^2y^2-8\,x^2z^2}$. **16.** $\sqrt{\frac{x+y}{x-y}}$.

17. $\sqrt{\dfrac{18\,x^2y^3}{27\,a^3}}$.

18. $\sqrt[6]{16\,a^2b^4}$.

19. $\sqrt[n]{a^{2n}}$.

20. $\sqrt{a^{2n+1}}$.

21. $\sqrt{a^{4n+1}}$.

22. $\sqrt[3n]{x^3+x^3y}$.

23. $\sqrt[n]{a^{2n+1}}$.

24. $\sqrt[n]{a^{2n+1}x}$.

25. $\sqrt{\dfrac{a^{11}}{b}}$.

26. $\sqrt[14]{9\,a^{12}}$.

27. $\sqrt[3]{\dfrac{x^2-x+1}{(x+1)^2}}$.

28. $\sqrt[3]{\dfrac{a^2+ab+b^2}{3(a-b)^2}}$.

29. $(\sqrt[6]{a^5})^2$.

30. $\dfrac{3}{\sqrt{3}}$.

31. $\dfrac{2}{\sqrt[3]{5}}$.

32. $\dfrac{1}{\sqrt[4]{8}}$.

Given $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, calculate to two places of decimals the values of the expressions in examples 33, 34, and 35.

33. $\dfrac{2\sqrt{6}}{\sqrt{3}}$.

34. $\sqrt{\frac{1}{5}}$.

35. $\sqrt{\frac{4}{3}}$.

36. Simplify $\sqrt{7}\times\sqrt{7}\times\sqrt{6}\times\sqrt{2}\div\sqrt{3}$.

37. Simplify $\sqrt{63}\times\sqrt{168}\div\sqrt{27}$.

38. Simplify $\sqrt{14}\times\sqrt{15}\div\sqrt{21}$.

39. Simplify $\sqrt{35}\times\sqrt{6}\div\sqrt{30}$.

40. Simplify $\sqrt{12}\times\sqrt{40}\times\sqrt{20}\times\sqrt{48}\times\sqrt{24}$.

41. Simplify $(\sqrt[3]{a^2})^9\times(\sqrt{b^3})^6\times(\sqrt[4]{a^2b^3})^8$.

214. Comparison of surds. For certain purposes it is convenient to express two or more surds in terms of like roots, as, for example, when finding which of two given surds is the greater.

Thus, in finding which of the two surds $\sqrt[3]{5}$ or $\sqrt[4]{7}$ is the greater, we make use of identity (II), section 209, to transform both expressions into surds of the same order as follows:

$$\sqrt[3]{5} = \sqrt[12]{5^4} = \sqrt[12]{625};\ \sqrt[4]{7} = \sqrt[12]{7^3} = \sqrt[12]{343}.$$

It is now evident that $\sqrt[3]{5}$ is greater than $\sqrt[4]{7}$.

EXERCISE 101

1. Express $\sqrt{5}$ and $\sqrt[3]{10}$ as surds of the same order.
2. Which is greater, $\sqrt[6]{5}$ or $\sqrt[4]{3}$?
3. Which is greater, $\sqrt{3}$ or $\sqrt[6]{28}$?
4. Arrange $2\sqrt[6]{27}$, $3\sqrt[4]{4}$, $2\sqrt{6}$ in order of magnitude.

Suggestion. Place the coefficients under the radical signs.

5. When two or more surds are reduced to equivalent surds having a common index, what is their lowest common index?

215. Addition and subtraction of surds. The algebraic sum of two or more surds is expressed in the simplest form when each surd is in its simplest form and all similar surds are combined by adding their coefficients.

ILLUSTRATIVE EXAMPLES

1. Add $5\sqrt{8}$, $-4\sqrt{50}$, $3\sqrt{72}$, $-2\sqrt{98}$, and $\sqrt{128}$.

Solution. The sum of $5\sqrt{8}$, $-4\sqrt{50}$, $3\sqrt{72}$, $-2\sqrt{98}$, and $\sqrt{128}$

$$= 10\sqrt{2} - 20\sqrt{2} + 18\sqrt{2} - 14\sqrt{2} + 8\sqrt{2}$$
$$= (10 - 20 + 18 - 14 + 8)\sqrt{2}$$
$$= 2\sqrt{2}.$$

2. Simplify $2\sqrt{abx^2} - \sqrt{9\,aby^2} - \sqrt{4\,a^2xy} + 3\sqrt{b^2xy}$.

Solution. $2\sqrt{abx^2} - \sqrt{9\,aby^2} - \sqrt{4\,a^2xy} + 3\sqrt{b^2xy}$

$$= 2\,x\sqrt{ab} - 3\,y\sqrt{ab} - 2\,a\sqrt{xy} + 3\,b\sqrt{xy}$$
$$= (2\,x - 3\,y)\sqrt{ab} - (2\,a - 3\,b)\sqrt{xy}.$$

Note. The sum of two unlike surds cannot be expressed as a single surd; such a sum can only be indicated.

Thus, the expression $\sqrt{2} + \sqrt{3}$ is in its simplest form.

Remark. Particular attention should be called to the fact that $\sqrt{3} + \sqrt{2}$ is not equal to $\sqrt{3+2}$, or $\sqrt{5}$.

EXERCISE 102

Simplify each of the following expressions:

1. $3\sqrt{6} - 2\sqrt{6} + 5\sqrt{6}$.

2. $\sqrt{54} - \sqrt{24} + \sqrt{150}$.

3. $\sqrt{50} + \sqrt{32} - \sqrt{18}$.

4. $\sqrt{75} - \sqrt{27} - 2\sqrt{12}$.

5. $3\sqrt{60} - \sqrt{240} - 2\sqrt{15}$.

6. $\sqrt{9\,a^3} - \sqrt{4\,ab^3} + \sqrt{ac^2}$.

7. $\sqrt[3]{250} - 2\sqrt[3]{16} + \sqrt[3]{54}$.

8. $\sqrt[3]{a^5b} - \sqrt[3]{a^5b^4} + \sqrt[3]{a^2b^4}$.

9. $\sqrt{18} - \frac{9}{2}\sqrt{2} + \sqrt{\frac{25}{2}}$.

10. $\dfrac{10}{\sqrt{3}} - \sqrt{27} + \sqrt{\frac{4}{3}}$.

11. $\sqrt{50} + \sqrt{108} + \sqrt{98} + 3\sqrt{12}$.

12. $2\sqrt{90} - \sqrt{176} + \sqrt{704} - \sqrt{160}$.

13. $\sqrt{4\,a^3b} - \sqrt{9\,ab^3} + 2\sqrt{4\,ac^2} - \sqrt{9\,ab^2} + \sqrt{4\,a^3} + \sqrt{16\,abc^2}$.

14. $\dfrac{1}{b}\sqrt{9\,ab} - 2\sqrt{\dfrac{a}{b}} - 3\sqrt{\dfrac{b}{a}} + \dfrac{\sqrt{16\,ab}}{a}$.

15. $2\sqrt[3]{4} + 2\sqrt[3]{32} - 9\sqrt[3]{\frac{1}{2}}$.

16. $2\sqrt[3]{4} - 5\sqrt[3]{32} + 3\sqrt[3]{108}$.

17. $\sqrt{18} - \sqrt[3]{24}$.

18. $\sqrt[3]{6} + \sqrt[3]{\frac{16}{9}} + \sqrt[3]{\frac{2}{9}}$.

19. $\sqrt{6} - \sqrt{\frac{3}{2}} - 4\sqrt{\frac{50}{3}} + \sqrt{\frac{4}{27}}$.

20. $\frac{4}{3}\sqrt[3]{\frac{3}{16}} + \frac{1}{2}\sqrt{\frac{256}{9}}$.

21. $\frac{11}{28}\sqrt[3]{1750} - \frac{3}{2}\sqrt[3]{\frac{7}{4}} - \frac{3}{2}\sqrt[3]{\frac{2}{49}}$.

22. $\sqrt[4]{162\,a} + 2\sqrt[4]{32\,a}$.

23. $x\sqrt{75\,y^2} - \sqrt{48\,x^2y^2} + xy\sqrt{27}$.

24. $\sqrt{3\,x^4} - 2\sqrt{3\,x^2y^2} + \sqrt{3\,y^4}$.

25. $\sqrt{\dfrac{3}{a^2}} - \sqrt{\dfrac{3}{y^2}} + \sqrt{\dfrac{3}{c^2}}$.

26. $\sqrt[3]{8 + 24\,a} + \sqrt[3]{81\,a + 27}$.

27. $\sqrt{12\,a^2x^2 + 8\,a^2} + \sqrt{27\,b^2x^2 + 18\,b^2} - \sqrt{48\,c^2x^2 + 32\,c^2}$.

28. $3\sqrt{3\,a^3} - \sqrt{48\,a} - \sqrt{3\,a^3 + 6\,a^2 + 3\,a}$.

29. $(a+b)^2\sqrt{ab}+(a-b)^2\sqrt{ab}-(a^2+b^2)\sqrt{ab}$.

30. $\sqrt{x^3-2x^2}-\sqrt{4x-8}-\sqrt{(x+2)(x^2-4)}$.

216. Multiplication and division. The rules for the multiplication and division of monomial surd expressions are derived from identities (III) and (V), section 209; namely, from the identities,

$$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}.$$

Remark. Observe that these identities contain surds of the same order only. Hence, before two or more surds are combined by multiplication or division, they must, when necessary, be reduced to surds of the same order. See section 214.

ILLUSTRATIVE EXAMPLES

1. $5\sqrt{20}\times 3\sqrt{45}=10\sqrt{5}\times 9\sqrt{5}=90(\sqrt{5})^2=450.$

2. $2\sqrt{ab}\times 5\sqrt[3]{a^2b^2}=2\sqrt[6]{a^3b^3}\times 5\sqrt[6]{a^4b^4}=10\sqrt[6]{a^7b^7}$
$=10ab\sqrt[6]{ab}.$

3. $\sqrt{6}\div\sqrt{5}=\sqrt{\frac{6}{5}}=\frac{\sqrt{30}}{5}.$

4. $10\sqrt[4]{ab}\div 5\sqrt{ab}=\frac{10\sqrt[4]{ab}}{5\sqrt[4]{a^2b^2}}=2\sqrt[4]{\frac{ab}{a^2b^2}}=2\sqrt[4]{\frac{a^3b^3}{a^4b^4}}$
$=2\frac{\sqrt[4]{a^3b^3}}{\sqrt[4]{a^4b^4}}=\frac{2\sqrt[4]{a^3b^3}}{ab}.$

EXERCISE 103

Simplify each of the following expressions:

1. $\sqrt{5}\times\sqrt{10}$.
2. $\sqrt{10}\div\sqrt{5}$.
3. $ax\sqrt{a^3}\times by\sqrt{b^3}$.
4. $\sqrt{6}\times\sqrt{2}$.
5. $\sqrt{6}\div\sqrt{12}$.
6. $ax\sqrt{a^3}\div by\sqrt{a}$.

7. $\sqrt{5} \times \sqrt{20}$.

8. $\sqrt{90} \div \sqrt{10}$.

9. $\sqrt{a} \times \sqrt{b}$.

10. $2\sqrt{a} \times 3\sqrt{b}$.

11. $2\sqrt[3]{3} \times 3\sqrt[3]{9}$.

12. $\sqrt[3]{320} \div \sqrt[3]{5}$.

13. $\sqrt[4]{24} \div \sqrt[4]{3}$.

14. $2\sqrt[3]{64} \div 4\sqrt[3]{8}$.

15. $\sqrt[3]{18} \times \sqrt[3]{3}$.

16. $\sqrt[3]{\frac{2}{3}} \times \sqrt[5]{\frac{1}{2}}$.

17. $\sqrt{3} \times \sqrt[3]{2}$.

18. $\sqrt{2} \times \sqrt[3]{3}$.

19. $\sqrt[3]{4} \times \sqrt{8}$.

20. $\sqrt[3]{9} \times \sqrt[3]{81^2} \times \sqrt[3]{-9^4}$.

21. $\sqrt[5]{-4} \times \sqrt[5]{8}$.

22. $\sqrt{\frac{1}{9}} \div \sqrt{9}$.

23. $\sqrt{\frac{3}{5}} \div \sqrt{\frac{3}{2}}$.

24. $\frac{3}{7}\sqrt[3]{\frac{2}{3}} \div \frac{5}{2}\sqrt[3]{\frac{9}{4}}$.

25. $20 \div \sqrt{\frac{3}{5}}$.

26. $8\sqrt{x^3} \div 4\sqrt[3]{x}$.

27. $\sqrt[3]{12a^2} \div \sqrt{8a^3}$.

28. $\sqrt[4]{\frac{54x^3}{y}} \div \sqrt[4]{\frac{8y^3}{7}}$.

29. $\sqrt{ab} \div 3\sqrt{bc}$.

30. $m \div \sqrt{n}$.

217. Multiplication of polynomials containing surds. The product of two polynomials involving surds is found by direct application of the distributive law for multiplication; the operation differs in no respect, from that required in the multiplication of rational integral polynomials. Each term of the resulting product must be expressed in its simplest form.

ILLUSTRATIVE EXAMPLE

Multiply $2 + 3\sqrt{5} - 2\sqrt{2}$ by $3\sqrt{5} - 4\sqrt{2}$.

Solution.

$$\begin{array}{l} \phantom{6\sqrt{5}+{}} 2 + 3\sqrt{5} - 2\sqrt{2} \\ \underline{3\sqrt{5} - 4\sqrt{2} \phantom{+ 45 - 6\sqrt{10}}} \\ 6\sqrt{5} + 45 - 6\sqrt{10} \\ \underline{\phantom{6\sqrt{5}} + 16 - 12\sqrt{10} - 8\sqrt{2}} \\ 6\sqrt{5} + 61 - 18\sqrt{10} - 8\sqrt{2}. \end{array}$$

Remark. Observe that it is necessary to simplify the terms of each partial product in order that similar surds may be written in the same column.

EXERCISE 104

Simplify each of the following expressions:

1. $(2+\sqrt{5})(2-\sqrt{5})$. **2.** $(a+2\sqrt{3})(2+\sqrt{3})$.

3. $(3-2\sqrt{2}+\sqrt{3})(\sqrt{6}+\sqrt{2})$.

4. $(a\sqrt{b}+b\sqrt{a})(\sqrt{a}+\sqrt{b})$.

5. $(a+\sqrt{b}+\sqrt{c})(a-\sqrt{b}+\sqrt{c})$.

6. $(\sqrt{a}+\sqrt{b})^2$.

7. $(\sqrt{a}-\sqrt{b})^2-(\sqrt{a}+\sqrt{b})^2$.

8. $(a\sqrt{b}-b\sqrt{a})^3$.

9. Find the product of $3\sqrt{2}-\sqrt{3}$ and $\sqrt{2}+3\sqrt{3}$.

10. Find the product of $9\sqrt{3}+2$ and $2\sqrt{3}+9$.

11. Find the product of $\sqrt{2}-\sqrt{3}+1$ and $\sqrt{2}+\sqrt{3}+1$.

Find the square of:

12. $\sqrt{3}-1$. **13.** $2+\sqrt{2}$.

14. $3\sqrt{3}-\sqrt{2}$. **15.** $\sqrt{5}-2\sqrt{2}$.

Find the cube of:

16. $\sqrt{3}-1$. **17.** $2+\sqrt{2}$.

18. $3\sqrt{3}-\sqrt{2}$. **19.** $\sqrt{5}-2\sqrt{2}$.

20. Simplify $(1+3\sqrt{3})(9-\sqrt{2})(\sqrt{2}+\sqrt{3})(9+\sqrt{2})(3\sqrt{3}-1)(\sqrt{2}-\sqrt{3})$.

218. Conjugate surds. Two binomial quadratic surd expressions which differ in sign of a surd term only are called **conjugate surds.**

Thus, $1+\sqrt{2}$ and $1-\sqrt{2}$, also $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ are conjugate surd expressions.

219. Product of conjugate surds. The product of two conjugate surds is rational.

Thus, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b.$

Note. When the product of two expressions which contain irrational numbers is rational, either expression is called a **rationalizing factor** of the other.

ILLUSTRATIVE EXAMPLE

Rationalize the denominator of the fraction

$$\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{2}}.$$

Solution. Since the product of two conjugate surds is rational [section 219], we multiply the numerator and denominator of the fraction by a conjugate of the denominator. Then we have,

$$\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{(1+\sqrt{2})(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}$$
$$=\frac{\sqrt{5}+\sqrt{10}-\sqrt{2}-2}{3}.$$

EXERCISE 105

Rationalize the denominator of:

1. $\frac{1}{1+\sqrt{2}}$. 2. $\frac{2}{2-\sqrt{2}}$. 3. $\frac{2}{2+\sqrt{2}}$.

4. $\frac{5}{\sqrt{3}-\sqrt{2}}$. 5. $\frac{13}{\sqrt{3}-\sqrt{2}}$. 6. $\frac{x}{2-\sqrt{x}}$.

7. $\frac{m}{n+\sqrt{m}}$. 8. $\frac{x}{\sqrt{x}+\sqrt{y}}$. 9. $\frac{1-\sqrt{5}}{1+\sqrt{5}}$.

10. $\frac{2+\sqrt{3}}{2-\sqrt{3}}$. 11. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. 12. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

13. $\frac{\sqrt{2}}{\sqrt{5}+\sqrt{7}}$. 14. $\frac{4+\sqrt{3}}{4-\sqrt{3}}$. 15. $\frac{\sqrt{6}-2}{\sqrt{12}+\sqrt{8}}$.

Find the sum of:

16. $\dfrac{5}{\sqrt{3}+1}+\dfrac{4}{\sqrt{3}-1}$.

17. $\dfrac{9\sqrt{3}}{2\sqrt{3}+3}-\dfrac{2\sqrt{2}}{3\sqrt{2}+2}$.

Suggestion. Rationalize the denominators.

18. $\dfrac{4}{\sqrt{27}}-\dfrac{5}{\sqrt{48}}-\dfrac{1}{\sqrt{12}}$.

19. $\dfrac{2}{\sqrt{60}}+\dfrac{3}{\sqrt{15}}-\dfrac{7}{\sqrt{135}}$.

Rationalize the denominator of:

20. $\dfrac{4}{2+\sqrt{3}+\sqrt{5}}$.

Suggestion. Multiply both terms by $2+\sqrt{3}-\sqrt{5}$.

21. $\dfrac{12}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

22. $\dfrac{1}{\sqrt{2}+\sqrt{5}-\sqrt{8}}$.

220. Division by polynomials containing surds. To divide an expression by a polynomial containing one or more surds, the dividend should be written as the numerator and the divisor as the denominator of a fraction, which should be transformed into an equivalent fraction with a rational denominator.

EXERCISE 106

Perform the indicated divisions in the following:

1. $\dfrac{1}{\sqrt{2}-1}$.

2. $\dfrac{\sqrt{2}}{\sqrt{2}+1}$.

3. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

4. $\dfrac{1}{\sqrt{5}-\sqrt{3}}$.

5. $\dfrac{2+\sqrt{3}}{1+\sqrt{3}}$.

6. $\dfrac{3-\sqrt{5}}{\sqrt{10}-\sqrt{6}}$.

7. Find the quotient of $(\sqrt{2}+\sqrt{6})\div(1+\sqrt{3})$ without rationalizing the denominator.

8. Mention at sight the numerical value of $\dfrac{3\sqrt{2}+\sqrt{9}}{1+\sqrt{2}}$.

9. State at sight the value of $\dfrac{\sqrt{2}+\sqrt{4}-\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}$.

10. Simplify $\sqrt{3}+\dfrac{\sqrt{2}-3}{1+\sqrt{3}}$.

11. Simplify $\left(\dfrac{1+\sqrt{2}}{\sqrt{3}}\right)^2-\left(\dfrac{1-\sqrt{2}}{\sqrt{3}}\right)^2$.

12. Divide $(3-\sqrt{2})^2+1$ by $3-\sqrt{2}$.

13. Divide $(\sqrt{3}+\sqrt{2})^2+1$ by $\sqrt{3}+\sqrt{2}$.

14. Find the value of $\dfrac{x^3+x^2+x-3}{x}$ when $x=1+\sqrt{2}$.

221. Fractional and negative exponents. The following identities from section 198 had a meaning only when the exponents were positive integers.

$a^m \cdot a^n = a^{m+n}$. (I) $(ab)^m = a^m b^m$. (II)

$(a^m)^n = (a^n)^m = a^{mn}$. (III) $(a^m b^n)^p = a^{mp} b^{np}$. (IV)

In section 62 the identity

$$\frac{a^m}{a^n} = a^{m-n}, \text{ in which } m > n \qquad \text{(V)}$$

was shown to result directly from the definition of division, and in section 63 a meaning was given to the expression a^0, where a denotes any number different from 0. That is, it was shown that,

$$a^0 = 1.$$

It is convenient to extend the meaning of the word *exponent* so that this term shall include, in addition to positive integers, all other rational numbers.

It is evidently desirable that all exponents should combine according to the same laws, and hence to define negative and fractional exponents so that the identities of section 198 may be satisfied for all rational values of m, n, and p. We shall proceed to show that such definitions can be given. In what follows, any base, as a, is supposed to be different from 0.

222. Definition of a^{-m}. If the identity $a^m \cdot a^n = a^{m+n}$, in which m and n are positive integers, is also an identity when n is replaced by $-m$, we shall have

$$a^m \cdot a^{-m} = a^{m+(-m)} = a^0 = 1.$$

We therefore define a^{-m} by the equation $a^m \cdot a^{-m} = 1$; hence, a^{-m} and a^m are reciprocals; that is,

$$a^{-m} = \frac{1}{a^m}; \text{ also, } a^m = \frac{1}{a^{-m}}.$$

223. Definition of $a^{\frac{p}{q}}$. If the identity $(a^m)^n = a^{mn}$, in which m and n are any positive integers, is also an identity when m is replaced by $\frac{p}{q}$, where p and q are positive integers, we shall have, by taking n equal to q,

$$(a^{\frac{p}{q}})^q = a^{\frac{p}{q}\cdot q} = a^p.$$

Therefore $a^{\frac{p}{q}}$ must represent a number whose qth power is equal to a^p. Now the principal qth root of a^p is such a number. We therefore define the symbol $a^{\frac{p}{q}}$ by the equation

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}. \qquad \text{(I)}$$

In this identity, $\sqrt[q]{a^p}$ represents the principal qth root of a^p. In particular, by making p equal to 1,

$$a^{\frac{1}{q}} = \sqrt[q]{a}. \qquad \text{(II)}$$

Since p and q in identity (I) represent any positive integers, it is admissible to change them in this identity to *pm* and *qm*, respectively; we then have from identity (I)

$$a^{\frac{pm}{qm}} = \sqrt[qm]{a^{pm}}.$$

But, $$\sqrt[qm]{a^{pm}} = \sqrt[q]{a^p} \qquad [\S\ 209\ (\text{II})]$$

$$= a^{\frac{p}{q}}.$$

$$\therefore \qquad a^{\frac{pm}{qm}} = a^{\frac{p}{q}}. \qquad (\text{III})$$

From identity (III) it is evident that the value of $a^{\frac{p}{q}}$ is not changed when $\frac{p}{q}$ is replaced by an equivalent fraction.

Note. In the expression $a^{\frac{p}{q}}$, the numerator p indicates a power and the denominator q a principal root. In some cases it may be desirable to take the pth power of the principal qth root of a and in others to take the principal qth root of the pth power of a. That, in both cases, the results are the same is evident from the known identity, section 209, IV,

$$(\sqrt[q]{a})^p = \sqrt[q]{a^p}.$$

Thus, in simplifying $8^{\frac{5}{3}}$, either of two solutions may be given, the first being preferable.

Solution 1. $8^{\frac{5}{3}} = (8^{\frac{1}{3}})^5 = (2)^5 = 32.$

Solution 2. $8^{\frac{5}{3}} = (8^5)^{\frac{1}{3}} = [(2^3)^5]^{\frac{1}{3}} = (2^{15})^{\frac{1}{3}} = 2^5 = 32.$

In simplifying $2^{\frac{2}{3}}$ it is evident that the method followed in solution 2 is preferable; that is,

$$2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}.$$

In simplifying $2^{\frac{5}{3}}$ it is preferable to proceed as follows:

$$2^{\frac{5}{3}} = 2^{1+\frac{2}{3}} = 2 \times 2^{\frac{2}{3}} = 2\sqrt[3]{4}.$$

224. Definition of $a^{-\frac{p}{q}}$. From section 222, $a^{-\frac{p}{q}}$ must be defined by the identity

$$a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}}.$$

225. It will be found that with the foregoing definitions of negative and fractional exponents, identities (I), (II), (III), (IV), and (V), section 221, now hold for all rational values of m, n, and p. We shall agree to enlarge the meaning of the word "power" in accordance with these definitions. For example, by the expressions the "one-third power of a" or "a to the one-third power," we shall mean the number $a^{\frac{1}{3}}$; that is, $\sqrt[3]{a}$.

ILLUSTRATIVE EXAMPLES

1. Simplify $a^{\frac{2}{3}} \times a^{\frac{3}{2}}$.

Solution. $a^{\frac{2}{3}} \times a^{\frac{3}{2}} = a^{\frac{2}{3}+\frac{3}{2}} = a^{\frac{13}{6}}$. [§§ 225, 221 (I)]

2. Simplify $\dfrac{x^{\frac{5}{2}}}{x^{\frac{3}{2}}}$.

Solution. $\dfrac{x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = x^{\frac{5}{2}-\frac{3}{2}} = x$. [§ 221 (V)]

3. Simplify $(a^{\frac{1}{3}}b^{\frac{3}{2}})^6$.

Solution. $(a^{\frac{1}{3}}b^{\frac{3}{2}})^6 = (a^{\frac{1}{3}})^6(b^{\frac{3}{2}})^6$. [§ 221 (II)]

$= a^2b^9$. [§ 221 (III)]

4. Write $\sqrt[6]{\dfrac{1}{x^3y}}$ without using the radical sign.

Solution. $\sqrt[6]{\dfrac{1}{x^3y}} = \left(\dfrac{1}{x^3y}\right)^{\frac{1}{6}}$. [§ 223 (II)]

5. Simplify $m^2n^{-4} \div m^{-3}n^4$.

Solution. $m^2n^{-4} \div m^{-3}n^4 = \dfrac{m^2}{n^4} \div \dfrac{n^4}{m^3} = \dfrac{m^5}{n^8}$. [§ 222]

6. Simplify $\left(\dfrac{1}{x^3y}\right)^{\frac{1}{6}}$.

Solution. $\left(\dfrac{1}{x^3y}\right)^{\frac{1}{6}} = \dfrac{(1)^{\frac{1}{6}}}{(x^3y)^{\frac{1}{6}}} = \dfrac{1}{x^{\frac{3}{6}}y^{\frac{1}{6}}} = \dfrac{1}{x^{\frac{1}{2}}y^{\frac{1}{6}}} = \dfrac{x^{\frac{1}{2}}y^{\frac{5}{6}}}{xy}$.

[Exercise 94, problem 33]

7. Simplify $\left(\dfrac{a^2b^3}{m^3n^2}\right)^{\frac{1}{6}}$.

Solution. $\left(\dfrac{a^2b^3}{m^3n^2}\right)^{\frac{1}{6}} = \dfrac{(a^2b^3)^{\frac{1}{6}}}{(m^3n^2)^{\frac{1}{6}}} = \dfrac{a^{\frac{1}{3}}b^{\frac{1}{2}}}{m^{\frac{1}{2}}n^{\frac{1}{3}}} = \dfrac{a^{\frac{1}{3}}b^{\frac{1}{2}}m^{\frac{1}{2}}n^{\frac{2}{3}}}{mn}$.

[Exercise 94, problem 33]

8. Simplify $\sqrt[3]{\dfrac{a}{\sqrt{a}}}$.

Solution. $\sqrt[3]{\dfrac{a}{\sqrt{a}}} = \left(\dfrac{a}{a^{\frac{1}{2}}}\right)^{\frac{1}{3}} = (a^{1-\frac{1}{2}})^{\frac{1}{3}} = (a^{\frac{1}{2}})^{\frac{1}{3}} = a^{\frac{1}{6}} = \sqrt[6]{a}.$

9. Simplify $\left(\sqrt[6]{\dfrac{1}{x^3y}}\right)(\sqrt[5]{x^2} \times \sqrt[3]{y})$.

Solution

$$\left(\sqrt[6]{\frac{1}{x^3y}}\right)(\sqrt[5]{x^2} \times \sqrt[3]{y}) = \left(\frac{1}{x^3y}\right)^{\frac{1}{6}} x^{\frac{2}{5}}y^{\frac{1}{3}}$$

$$= \frac{x^{\frac{2}{5}}y^{\frac{1}{3}}}{x^{\frac{1}{2}}y^{\frac{1}{6}}}$$

$$= \frac{y^{\frac{1}{6}}}{x^{\frac{1}{10}}}$$

$$= \frac{y^{\frac{1}{6}}x^{\frac{9}{10}}}{x^{\frac{1}{10}}x^{\frac{9}{10}}}$$

$$= \frac{y^{\frac{1}{6}}x^{\frac{9}{10}}}{x}$$

$$= \frac{1}{x}(y^{\frac{5}{30}}x^{\frac{27}{30}}) = \frac{1}{x}\sqrt[30]{x^{27}y^5}.$$

10. Multiply $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$.

Solution.

$$\begin{array}{l} x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\ \underline{x^{\frac{1}{4}} - y^{\frac{1}{4}} \qquad\qquad\qquad} \\ x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} \\ \underline{\qquad - x^{\frac{1}{2}}y^{\frac{1}{4}} - x^{\frac{1}{4}}y^{\frac{1}{2}} - y^{\frac{3}{4}}} \\ x^{\frac{3}{4}} \qquad\qquad\qquad\qquad - y^{\frac{3}{4}} \end{array}$$

11. Divide $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

Solution.

$$\begin{array}{r|l} x^{\frac{1}{3}} - y^{\frac{1}{3}})\, x - y & x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\ \underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}} & \\ x^{\frac{2}{3}}y^{\frac{1}{3}} & \\ \underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}}} & \\ x^{\frac{1}{3}}y^{\frac{2}{3}} - y & \\ \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y} & \end{array}$$

12. Simplify $\dfrac{1}{x^{-1} - y^{-1}} + \dfrac{\sqrt{x^5y^{-1}}(x^{-\frac{1}{2}}y^{\frac{1}{2}})^3}{x - y}$.

Solution. $\dfrac{1}{x^{-1} - y^{-1}} = \dfrac{1}{\frac{1}{x} - \frac{1}{y}} = \dfrac{xy}{y - x}$

$$\sqrt{x^5y^{-1}} = \left(\frac{x^5}{y}\right)^{\frac{1}{2}} = \frac{x^{\frac{5}{2}}}{y^{\frac{1}{2}}}$$

$$(x^{-\frac{1}{2}}y^{\frac{1}{2}})^3 = \left(\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^3 = \frac{y^{\frac{3}{2}}}{x^{\frac{3}{2}}}$$

$$\therefore \frac{1}{x^{-1} - y^{-1}} + \frac{\sqrt{x^5y^{-1}}(x^{-\frac{1}{2}}y^{\frac{1}{2}})^3}{x - y} = \frac{xy}{y - x} + \frac{\frac{x^{\frac{5}{2}}}{y^{\frac{1}{2}}} \cdot \frac{y^{\frac{3}{2}}}{x^{\frac{3}{2}}}}{x - y}$$

$$= \frac{xy}{y - x} + \frac{xy}{x - y} = \frac{xy}{y - x} - \frac{xy}{y - x} = 0.$$

Check. Let $x = 4$ and $y = 1$; then,

$$\frac{1}{x^{-1} - y^{-1}} + \frac{\sqrt{x^5y^{-1}}(x^{-\frac{1}{2}}y^{\frac{1}{2}})^3}{x - y} = \frac{1}{\frac{1}{4} - 1} + \frac{\sqrt{2^{10}}(\frac{1}{2})^3}{3}$$

$$= -\tfrac{4}{3} + \tfrac{4}{3} = 0.$$

EXERCISE 107

(Solve as many as possible at sight.)

Write without negative or fractional exponents, and simplify when possible :

1. a^{-2}. Thus, $a^{-2} = \frac{1}{a^2}$.

2. $4^{\frac{1}{2}}$. Thus, $4^{\frac{1}{2}} = +\sqrt{4} = 2$.

3. $8^{\frac{2}{3}}$. Thus, $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$.

4. $8^{-\frac{2}{3}}$. Thus, $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{4}$.

5. 2^{-2}. **6.** $9^{\frac{1}{2}}$. **7.** $16^{\frac{1}{2}}$. **8.** $4^{\frac{3}{2}}$.

9. x^{-3}. **10.** $9^{\frac{3}{2}}$. **11.** $16^{\frac{3}{4}}$. **12.** $27^{\frac{2}{3}}$.

13. x^{-n}. **14.** $8^{-\frac{2}{3}}$. **15.** $25^{-\frac{3}{2}}$. **16.** $64^{-\frac{2}{3}}$.

17. What verbal statement is expressed in symbols by the formula $a^m a^n = a^{m+n}$?

18. What verbal statement is expressed in symbols by the formula $\frac{a^m}{a^n} = a^{m-n}$?

19. What verbal statement is expressed in symbols by the formula $(a^m)^n = a^{mn}$?

Express without fractional exponents :

20. $a^{\frac{1}{2}}$. **21.** $x^{\frac{1}{3}}$. **22.** $y^{\frac{1}{4}}$.

23. $3\,a^{\frac{1}{2}}$. **24.** $2\,x^{\frac{1}{3}}$. **25.** $4\,y^{\frac{1}{4}}$.

26. $(3\,a)^{\frac{1}{2}}$. **27.** $(2\,x)^{\frac{1}{3}}$. **28.** $(4\,y)^{\frac{1}{4}}$.

29. $a^{\frac{3}{2}}$. **30.** $2\,a^{\frac{3}{2}}$. **31.** $(2\,a)^{\frac{3}{2}}$.

32. $a^{\frac{1}{2}}x$. **33.** $a^{\frac{1}{2}}xy^{\frac{1}{2}}$. **34.** $2\,a^{\frac{1}{2}}x^{\frac{1}{2}}$.

35. $a^{\frac{1}{m}}$.

36. $a^{\frac{n+1}{2}}$.

37. $2^{\frac{1}{2}}x^{\frac{3}{2}}$.

38. $2^{\frac{1}{5}}ax^{\frac{5}{2}}$.

39. $x^{\frac{a-b}{c-d}}$.

40. $\dfrac{ax^{\frac{m}{n}}}{3}$.

41. $\dfrac{ax^{\frac{m}{n}}}{3^{\frac{1}{2}}}$.

42. $(x-y)^{\frac{3}{2}}$.

43. $2\,a^{\frac{2}{5}}b^{\frac{3}{5}}$.

44. $a^{\frac{1}{2}}+b^{\frac{1}{2}}c^{\frac{1}{2}}$.

45. $\left(\dfrac{a}{b}\right)^{\frac{3}{4}}$.

46. $a^{\frac{m-n}{m+n}}b^{\frac{m+n}{m-n}}$.

Express without the radical signs:

47. $\sqrt[3]{x}$.

48. $\sqrt[4]{a}$.

49. $\sqrt[3]{a}$.

50. $4\sqrt[4]{c}$.

51. $3\sqrt{x}$.

52. $5\sqrt[3]{b}$.

53. $\sqrt[3]{3\,a}$.

54. $\sqrt{7\,x}$.

55. $\sqrt[4]{5\,c}$.

56. $\sqrt{4\,x^3}$.

57. $\sqrt{2\,a^3}$.

58. $2\sqrt{b^3}$.

59. $\sqrt{27\,x^3}$.

60. $a\sqrt{x}$.

61. $a\sqrt{bc}$.

62. $7\sqrt{xy}$.

63. $\sqrt[n]{x}$.

64. $\sqrt{2\,a^{m+n}}$.

65. $\sqrt{3\,x^3}$.

66. $c\sqrt[10]{4\,x^{25}}$.

67. $\sqrt[m-n]{a^{m+n}}$.

68. $\dfrac{a}{2}\sqrt[n]{b^m}$.

69. $\dfrac{c\sqrt[n]{x^n}}{\sqrt{z}}$.

70. $\sqrt[3]{(a+b)^2}$.

71. $3\sqrt[7]{a^2b^3}$.

72. $\sqrt{2\,x}+\sqrt{yz}$.

73. $\sqrt[4]{\dfrac{a^3}{b^3}}$.

74. $\sqrt[a+b]{x^{a-b}}\,\sqrt[a-b]{y^{a+b}}$.

75. $\sqrt[n]{a^6}\,\sqrt{b}\,\sqrt{c^5}$.

76. $a\sqrt{g^2}\,\sqrt[m+n]{x^{m^2-n^2}}$.

77. $\sqrt[3]{\sqrt{a^5}}$.

226. Square root of a binomial surd expression. The square of a binomial surd expression of the form $a+\sqrt{b}$ is itself a binomial surd expression of the same form.

Thus, $(2-\sqrt{5})^2 = 9-4\sqrt{5}$ and $(a+\sqrt{b})^2 = a^2+b+2\,a\sqrt{b}$.

When an expression in the form of $a+\sqrt{b}$ is a perfect square, its square root may usually be found by inspection. The method may be seen in the solutions of the following:

ILLUSTRATIVE EXAMPLES

1. Find the square root of $16 + 6\sqrt{7}$.

Solution.

$$\begin{aligned} 16 + 6\sqrt{7} &= 16 + 2\sqrt{63} \\ &= 9 + 2\sqrt{9}\,\sqrt{7} + 7 \\ &= 3^2 + 2 \times 3 \times \sqrt{7} + (\sqrt{7})^2 \\ &= (3 + \sqrt{7})^2. \end{aligned}$$

$$\therefore \sqrt{16 + 6\sqrt{7}} = \sqrt{(3 + \sqrt{7})^2} = 3 + \sqrt{7}.$$

2. Find the square root of $6 - \sqrt{11}$.

Solution.

$$\begin{aligned} 6 - \sqrt{11} &= 6 - 2\sqrt{\tfrac{11}{4}} \\ &= \tfrac{11}{2} - 2\sqrt{\tfrac{11}{2}}\,\sqrt{\tfrac{1}{2}} + \tfrac{1}{2} \\ &= \left(\sqrt{\tfrac{11}{2}}\right)^2 - 2\sqrt{\tfrac{11}{2}}\,\sqrt{\tfrac{1}{2}} + \left(\sqrt{\tfrac{1}{2}}\right)^2 \\ &= \left(\sqrt{\tfrac{11}{2}} - \sqrt{\tfrac{1}{2}}\right)^2. \end{aligned}$$

$$\begin{aligned} \therefore \sqrt{6 - \sqrt{11}} &= \sqrt{\left(\sqrt{\tfrac{11}{2}} - \sqrt{\tfrac{1}{2}}\right)^2} \\ &= \sqrt{\tfrac{11}{2}} - \sqrt{\tfrac{1}{2}} = \frac{\sqrt{22} - \sqrt{2}}{2}. \end{aligned}$$

3. Show that $\sqrt{6 + \sqrt{35}} + \sqrt{6 - \sqrt{35}} = \sqrt{14}$.

Solution.

$$\begin{aligned} \sqrt{6 + \sqrt{35}} &= \sqrt{6 + 2\sqrt{\tfrac{35}{4}}} = \sqrt{\tfrac{7}{2}} + \sqrt{\tfrac{5}{2}} \\ \sqrt{6 - \sqrt{35}} &= \sqrt{6 - 2\sqrt{\tfrac{35}{4}}} = \sqrt{\tfrac{7}{2}} - \sqrt{\tfrac{5}{2}} \\ \text{Sum} & \qquad\qquad\qquad = 2\sqrt{\tfrac{7}{2}} = \sqrt{14}. \end{aligned}$$

227. From the solution of the foregoing illustrative examples we may state the following:

Rule. *A binomial surd expression of the form* $a + 2\sqrt{b}$, *where a and b are rational numbers, is a perfect square when b (the number under the radical sign) is the product of two factors whose sum equals a, the rational term. In this case, the* *square root of the binomial surd expression is equal to the sum or difference of the square roots of the two factors of the number.*

Remark. Any binomial surd expression of the form $a + \sqrt{b}$ can be written $a + 2\sqrt{\frac{b}{4}}$.

228. (1) *A quadratic surd cannot be equal to the algebraic sum of a rational number and a quadratic surd.*

For, if possible, let $\sqrt{a} = b + \sqrt{c}$;

then by squaring, $a = b^2 + c + 2b\sqrt{c}$.

and $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$.

Since $\sqrt{c}$ is a quadratic surd, it is an irrational number and cannot be equal to the rational number $\dfrac{a - b^2 - c}{2b}$. Therefore, the assumption that $\sqrt{a} = b + \sqrt{c}$ is false.

(2) *A binomial surd expression of the form $a + \sqrt{b}$ cannot be equal to another expression $x + \sqrt{y}$ of the same form (where a and x denote rational numbers) unless $a = x$ and $\sqrt{b} = \sqrt{y}$.*

For, if possible, let $a + \sqrt{b} = x + \sqrt{y}$;

then, $\sqrt{b} = (x - a) + \sqrt{y}$.

By (1) this equation can be true only when $x - a = 0$, in which case $\sqrt{b} = \sqrt{y}$.

EXERCISE 108

Find, in surd form, the square root of :

1. $16 - 6\sqrt{7}$.
2. $6 + \sqrt{11}$.
3. $7 + \sqrt{48}$.
4. $32 - \sqrt{700}$.
5. $12 - 6\sqrt{3}$.
6. $8 + 2\sqrt{7}$.
7. Simplify $\dfrac{1}{\sqrt{3 + \sqrt{8}}}$.
8. Simplify $\dfrac{1}{\sqrt{11 + 2\sqrt{18}}} + \sqrt{11 - 3\sqrt{8}}$.
9. Simplify $\sqrt[4]{17 + 12\sqrt{2}}$.

EXERCISE 109. REVIEW

(Solve as many as possible at sight.)

Write without negative exponents:

1. a^{-2}.
2. a^3b^{-3}.
3. $a^{-3}c$.
4. $x^{-3}y^{-2}$.
5. $a^{-5}b^2$.
6. $2a^{-2}b^{-1}$.
7. $3a^{-3}b$.
8. $5^{-1}a$.
9. $(ab)^{-1}$.
10. $5a^{-1}$.
11. $3ab^{-3}$.
12. $x^{\frac{1}{2}}y^{-1}$.
13. $a^4c^{-\frac{3}{5}}$.
14. a^nb^{-n}.
15. $3^{-3}x^n$.
16. $\frac{b^{-1}}{3}$.
17. $a^{-\frac{2}{3}}xy^{-\frac{1}{2}}$.
18. $(5a^{-1})^{-1}$.
19. $\frac{1}{a^{-2}}$.
20. $\frac{1}{x^{-1}}$.
21. $\frac{1}{(2x)^{-1}}$.
22. $\frac{a}{3xy^{-1}}$.
23. $\frac{a}{3(xy)^{-1}}$.
24. $\frac{a}{(3xy)^{-1}}$.
25. $\frac{a^{-1}}{(3xy)^{-1}}$.
26. $x^{n-1}y^{-n-1}$.
27. m^{-n}.
28. $(a+x)^{-2}$.
29. $\frac{1}{a^{-a}}$.
30. $\frac{1}{a^{-\frac{1}{a}}}$.
31. $\frac{2^{-3}}{16^{-\frac{1}{4}}}$.
32. $(-a)^{-3}$.
33. $(-a)^{-2}$.
34. $(-a)^{-4}$.
35. $\left(\frac{1}{x^{-1}}\right)^{-1}$.
36. $\left(\frac{1}{x^{-1}}\right)^{-2}$.
37. $\left(\frac{a}{b}\right)^{-1}$.
38. $a^2\left(\frac{a}{b}\right)^{-2}$.
39. $\frac{xy^{-3}x^2}{a^{-1}b^{-3}c}$.
40. $\frac{4a^{-1}b^3c^4}{5d^{-3}f^2g^{-4}}$.
41. $\frac{(3ab)^{-3}}{(3a^3)^{-3}(2b^4)^{-5}}$.
42. $\frac{1}{a^{-1}+b^{-1}}$.
43. $\frac{a-1}{b^{-1}-c^{-1}}$.
44. $\frac{1}{a^{-1}-1}$.
45. $\frac{x^2-y^2}{x^{-2}-y^{-2}}$.

Perform the indicated operations in examples 46–51:

46. $a^{\frac{2}{3}} \times a^{-\frac{1}{2}}$.

47. $4\,a^{-\frac{6}{8}} \div 2\,x^{-\frac{6}{4}}$.

48. $a^{-\frac{3}{2}}\sqrt[3]{a^2}$.

49. $(a^{\frac{1}{3}})^{\frac{3}{2}} \div (x^{\frac{3}{2}})^2$.

50. $(a^{\frac{1}{3}}b^{-\frac{1}{2}})^{\frac{3}{2}}$.

51. $(a^{-\frac{2}{3}})^{\frac{5}{2}} \times (a^{\frac{3}{4}})^{\frac{2}{3}} \times (a^{-\frac{3}{2}})^{-\frac{4}{9}}$.

52. Find the product of $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{6}}$, $a^{\frac{1}{12}}$, and $a^{\frac{2}{3}}$.

53. Find the product of $m^{\frac{5}{7}}$, $m^{-\frac{3}{4}}$, $m^{\frac{3}{14}}$, $m^{\frac{1}{4}}$, $m^{-\frac{1}{2}}$ and $m^{\frac{1}{14}}$.

54. Express $9\,m^2n^{-3} \div 15\,n^2p^{-3} \times 20\,p^2m^{-3}$ with positive exponents, perform the indicated operations, and reduce the result to simplest form.

55. Express the following with positive exponents, add, and simplify the result:

$$x^{-2}(yz^{-1} - y^{-1}z) + y^{-2}(zx^{-1} - z^{-1}x) + z^{-2}(xy^{-1} - x^{-1}y).$$

Simplify the following:

56. $9^{\frac{3}{2}}$.

57. $4^{\frac{5}{2}}$.

58. $64^{\frac{4}{3}}$.

59. $25^{\frac{5}{2}}$.

60. $-27^{\frac{1}{3}}$.

61. $(-27)^{\frac{1}{3}}$.

62. $(27)^{-\frac{1}{3}}$.

63. $(-27)^{-\frac{1}{3}}$.

64. $32^{\frac{3}{5}}$.

65. $100^{-\frac{3}{2}}$.

66. $\dfrac{2}{4^{-2}}$.

67. $256^{\frac{1}{8}}$.

68. $(-\frac{1}{4})^{-2}$.

69. $(\frac{32}{243})^{-\frac{2}{5}}$.

70. $\dfrac{2^0}{(-1)^{-7}}$.

71. $\dfrac{5^{16}}{5^{12}}$.

72. $4^{-\frac{5}{2}} \times 36^{\frac{3}{2}}$.

73. $\dfrac{1}{4^{-2}} + \dfrac{1}{5^{-3}}$.

74. $64^{-\frac{2}{3}} - 64^{-\frac{1}{3}} + (-64)^{\frac{2}{3}} + 64^{\frac{1}{2}}$.

75. $3^{-\frac{1}{3}} \times \sqrt{81 \times 5^{-\frac{3}{2}}} \times \sqrt{5}$.

76. $(x^{-2})^3$.

77. $(y^2)^{-3}$.

78. $(x^{-1})^{-1}$.

79. $(a^3)^{\frac{1}{3}}$.

80. $\sqrt{\sqrt[3]{a^6}}$.

81. $(64^{\frac{1}{3}})^2$.

82. $\dfrac{1}{4^{-\frac{3}{2}}}+\dfrac{1}{8^{-\frac{2}{3}}}$. **83.** $(64\,a^{-6})^{\frac{5}{6}}$. **84.** $(a^{\frac{4}{5}})^{-2}$.

85. $(-a^3)^5$. **86.** $(-a^{3m})^4$. **87.** $(a^{-\frac{3}{2}}b^{-\frac{5}{3}}c^{\frac{1}{3}})^{\frac{6}{5}}$.

88. $[[a^{-\frac{1}{2}}]^{-\frac{2}{3}}]^{-\frac{1}{2}}$. **89.** $\left(\dfrac{p^n}{q^x}\right)^{\frac{1}{n}}$. **90.** $\left(\dfrac{25\,a^3}{9\,b^4}\right)^{-\frac{3}{2}}$.

91. $8^{-\frac{2}{3}}\times 16^{\frac{3}{4}}\times 2^0$. **92.** $\dfrac{1}{2^{-1}+3^{-1}}$.

93. $(a+b)^0$. **94.** $\left[\left(\dfrac{a^{-\frac{1}{10}}b^{18}}{c^3d^{-\frac{3}{5}}}\right)^{-\frac{1}{2}}\right]^{-\frac{5}{3}}$.

95. $\left(\dfrac{a+b}{a-b}\right)^{-3}\times\left(\dfrac{a-b}{a+b}\right)^{-2}$. **96.** $\left\{\sqrt[3]{(a^{\frac{1}{2}}b^{-1})}\right\}^2$.

97. $2^{n-1}\times 8^{n+1}\div 16^{\frac{n}{2}}$. **98.** $\dfrac{(a^2-b^2)}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}$.

99. $(a^a)^a$. **100.** $\sqrt[4]{a^{-1}\sqrt[3]{a^4}}$.

101. $(a^{x^2-xy}b^{y^2-xy})^{\frac{xy}{x-y}}$. **102.** $[(e^x-e^{-x})^2+4]^{\frac{1}{2}}$.

103. $(x^{\frac{1}{2}}+x^{\frac{1}{4}}+1)(x^{\frac{1}{2}}-x^{\frac{1}{4}}+1)$.

104. $\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}+\dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$. **105.** $[\sqrt{xy^{-2}\sqrt{xy}}]^4$.

106. $[(a^x)^{-z}]^{-\frac{1}{x}}\div[(a^{-2})^y]^{-\frac{1}{z}}$.

107. $(a^{\frac{3}{2}}-ab^{\frac{1}{2}}+a^{\frac{1}{2}}b-b^{\frac{3}{2}})\div(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.

108. $(x^{-\frac{3}{2}}y^{-1}+xy^{\frac{3}{2}})^2$. **109.** $[3\sqrt[a]{x^{-b}y^c}]^{-bc}$.

110. $(-\sqrt[3]{x^{\frac{3}{5}}})^{10}+(2\sqrt{x^{\frac{1}{3}}})^6-x^{-1}\left(\dfrac{3\,x^{\frac{3}{2}}}{\sqrt{x^{-\frac{1}{3}}}}\right)^3$.

111. $(x+2\,x^{\frac{1}{2}}+1)^{\frac{1}{2}}$.

112. $[x-2\,x^{\frac{3}{4}}+3\,x^{\frac{1}{2}}-2\,x^{\frac{1}{4}}+1]^{\frac{1}{2}}$.

Clear of negative exponents and simplify:

113. $\dfrac{m^{-2}+n^{-2}}{m^{-4}-n^{-4}}$.

114. $\dfrac{(m-n)^{-2}}{(m^2-n^2)^{-2}}$.

Arrange according to descending powers of x:

115. $x^{\frac{3}{4}} - 2x^{\frac{5}{6}} + 4x^{\frac{6}{5}} + x^{\frac{2}{3}}$.

116. Since $x^{\frac{1}{2}} - y^{\frac{1}{2}} = (x^{\frac{1}{4}})^2 - (y^{\frac{1}{4}})^2$, express $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ as a product of the sum and difference of the same two numbers.

117. Simplify $\dfrac{1}{a^{\frac{1}{2}}-1} + \dfrac{1}{a^{\frac{1}{2}}+1} - \dfrac{2a^{\frac{1}{2}}}{a-1}$.

118. Evaluate $\sqrt[3]{a^2} \cdot \sqrt[4]{a^{-3}} \cdot \sqrt{a^3} \div \sqrt[12]{a^{-7}}$ when $a = 3$.

119. Evaluate $5^n \cdot 25^{n+1} \cdot 125^{-n+1} \div 5^5$.

120. Simplify $\{1-[1-(1-x^2)^{-1}]^{-1}\}^{\frac{1}{2}}$.

121. Evaluate $(243)^{\frac{4}{5}} + (243)^{\frac{2}{5}}$.

122. Simplify $\left(\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}\right)^8$.

123. Simplify $y^{-\frac{1}{2}}[(x+\sqrt{y})^3 - (x-\sqrt{y})^3]$.

124. Which is the greater $\sqrt[27]{27}$ or $\sqrt[16]{16}$?

125. Simplify $(x^{-\frac{1}{2}})^2 \div (x^{-\frac{2}{3}})^{\frac{3}{4}} \div (x^{-\frac{3}{4}})^{\frac{4}{5}} \div (x^{-\frac{4}{5}})^{\frac{5}{6}}$.

126. Simplify $\left(x^{\frac{1}{(a+b)^2}} \div x^{\frac{1}{(a-b)^2}}\right)^{\frac{a^2-b^2}{4ab}}$.

127. Simplify $\dfrac{x-y}{(x^{\frac{1}{2}}-y^{\frac{1}{2}})(x^{\frac{1}{4}}+y^{\frac{1}{4}})}$.

CHAPTER XI

INVOLUTION AND EVOLUTION

229. Involution. The operation of raising an expression to any positive integral power is called **involution.**

Remark. The involution of monomials has been explained in section 198. We shall now consider the involution of a binomial. Since by grouping the terms of any polynomial it may be expressed as a binomial, the involution of a binomial is an important topic of algebra.

230. Binomial expansion. The process of raising a binomial to a power is called **expanding the binomial,** and the result of the operation is called a **binomial expansion.**

Thus, by expanding $(a + b)^2$ we obtain $a^2 + 2\,ab + b^2$, which is called the expansion of $(a + b)^2$.

231. The binomial formula. By actual multiplication we obtain the following expansions:

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$
$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$
$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$
$$(a + b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

If we examine the above expansions we arrive at the following important conclusions which we here assume are true when a binomial is raised to any positive integral power:

1. *The first term in the expansion is a^n, where a is the first term and n is the exponent of the binomial. The last term is*

Isaac Newton (1642–1727) was perhaps the greatest mathematician of all time. He is best known as the discoverer of the laws of gravitation. In algebra he discovered the binomial theorem and wrote extensively on the theory of equations. His great work "Philosophiæ Naturalis Principia Mathematica" appeared in 1686–87.

b^n, *where* b *is the second term of the binomial. The number of terms is* $n + 1$.

2. *The sum of the exponents of* a *and* b *in any term is* n. *The exponent of* b *in the first term is zero and that of* a *in the last term is zero. The exponent of* a *in the second and following terms is one less than in the preceding term. Hence, the exponent of* b *in any term is one greater than in the preceding term.*

3. *The coefficient of any term is obtained from the coefficient of the preceding term by multiplying that coefficient by the exponent of* a *in that term and dividing the resulting product by one unit more than the exponent of* b *in that term.*

ILLUSTRATIVE EXAMPLES

1. Expand $(2 + x)^3$.

Solution. Since $(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$,

therefore,

$$(2 + x)^3 = 2^3 + 3 \cdot 2^2 \cdot x + 3 \cdot 2 \cdot x^2 + x^3$$
$$= 8 + 12\,x + 6\,x^2 + x^3.$$

2. Expand $(a - 2\,y)^4$.

Solution. Since $(a - b)^4 = a^4 - 4\,a^3b + 6\,a^2b^2 - 4\,ab^3 + b^4$,

$$\therefore (a - 2\,y)^4 = a^4 - 4\,a^3(2\,y) + 6\,a^2(2\,y)^2 - 4\,a(2\,y)^3 + (2\,y)^4$$
$$= a^4 - 8\,a^3y + 24\,a^2y^2 - 32\,ay^3 + 16\,y^4.$$

3. Expand $\left(\frac{1}{a} - \frac{b}{c}\right)^6$.

Solution.

Since $(a - b)^6 = a^6 - 6\,a^5b + 15\,a^4b^2 - 20\,a^3b^3 + 15\,a^2b^4 - 6\,ab^5 + b^6$,

$$\therefore \left(\frac{1}{a} - \frac{b}{c}\right)^6 = \left(\frac{1}{a}\right)^6 - 6\left(\frac{1}{a}\right)^5\left(\frac{b}{c}\right) + 15\left(\frac{1}{a}\right)^4\left(\frac{b}{c}\right)^2 - 20\left(\frac{1}{a}\right)^3\left(\frac{b}{c}\right)^3$$
$$+ 15\left(\frac{1}{a}\right)^2\left(\frac{b}{c}\right)^4 - 6\left(\frac{1}{a}\right)\left(\frac{b}{c}\right)^5 + \left(\frac{b}{c}\right)^6$$
$$= \frac{1}{a^6} - \frac{6\,b}{a^5c} + \frac{15\,b^2}{a^4c^2} - \frac{20\,b^3}{a^3c^3} + \frac{15\,b^4}{a^2c^4} - \frac{6\,b^5}{ac^5} + \frac{b^6}{c^6}.$$

EXERCISE 110

(Solve as many as possible at sight.)

1. $(x+y)^3$. **2.** $(x-y)^3$. **3.** $(a+b)^4$.

4. $(m-n)^4$. **5.** $(x-1)^4$. **6.** $(2-x)^3$.

7. $(a+b)^5$. **8.** $(x-y)^5$. **9.** $(x^2-1)^3$.

10. $(3a-b)^3$. **11.** $(3a+2b)^3$. **12.** $(5x-3y)^3$.

13. $(2m-n)^4$. **14.** $(2a^2-1)^4$. **15.** $(2-3m^2)^4$.

16. $(3x^2+2y^2)^4$. **17.** $(2a+b)^5$. **18.** $(1-a)^5$.

19. $(-m^2-2n^2)^6$. **20.** $(r^2-2)^7$. **21.** $(m^2n^2-1)^8$.

22. $(\frac{1}{2}a^2+1)^5$. **23.** $(\frac{1}{2}x^2-\frac{1}{4}y^2)^5$. **24.** $(\frac{1}{3}x^2+3)^5$.

25. $(\frac{2}{3}m+\frac{3}{4}n)^4$. **26.** $\left(\frac{1}{a}+\frac{b}{c}\right)^5$. **27.** $\left(\frac{a}{b}+\frac{c}{d}\right)^5$.

28. $(a^2-\frac{1}{2})^6$.

29. $\left(1+\frac{1}{x^2}\right)^8$. **30.** $\left(\frac{1}{x^2}+x^2\right)^7$

232. Evolution. The operation of extracting an indicated root is called **evolution**.

233. Square root. In section 88 a square root of a number was defined, and it was shown that any number has two square roots of the same absolute value but with opposite signs.

From section 92 we have $(a+b)^2=a^2+2ab+b^2$; therefore,

$$\sqrt{a^2+2ab+b^2}=\pm(a+b).$$

From section 209 (V), we have

$$\left(\frac{a}{b}\right)^2=\frac{a^2}{b^2};$$

therefore,

$$\sqrt{\left(\frac{a}{b}\right)^2}=\sqrt{\frac{a^2}{b^2}}=\frac{\sqrt{a^2}}{\sqrt{b^2}}=\pm\frac{a}{b}.$$

EXERCISE 111

Find the following square roots by inspection:

1. $\sqrt{4}$.
2. $\sqrt{16\,a^2}$.
3. $\sqrt{9\,a^2b^2}$.
4. $\sqrt{625\,x^2y^2}$.
5. $+\sqrt{16\,x^2z^2}$.
6. $-\sqrt{324\,a^4}$.
7. $+\sqrt{289\,x^{10}}$.
8. $-\sqrt{1024\,a^6x^4}$.
9. $\sqrt{a^2-2\,ab+b^2}$.
10. $-\sqrt{a^2+4\,a+4}$.
11. $\sqrt{4\,x^2-4\,x+1}$.
12. $\sqrt{x^2-6\,x+9}$.
13. $\sqrt{x^2y^2-2\,xy+1}$.
14. $+\sqrt{9\,x^2+12\,xy+4\,y^2}$.
15. $-\sqrt{16\,a^2+24\,ab+9\,b^2}$.
16. $+\sqrt{64\times 81\,a^2b^4c^6}$.
17. $+\sqrt{4\times 25\times 256\,a^4b^8x^{10}y^{16}}$.
18. $-\sqrt{a^{2m}}$.
19. $+\sqrt{a^{2n+2}}$.
20. $+\sqrt{a^2b^{2m}}$.
21. $+\sqrt{\dfrac{9\,a^2}{16}}$.
22. $-\sqrt{\dfrac{16\,x^2y^4}{81\,a^4b^6}}$.
23. $+\sqrt{\dfrac{36\,x^6}{729}}$.
24. $\sqrt{\dfrac{a^2-2\,ab+b^2}{a^2+2\,ab+b^2}}$.
25. $-\sqrt{\dfrac{x^2-6\,x+9}{x^2-10\,x+25}}$.
26. $+\sqrt{\dfrac{4\,x^2+4\,xy+y^2}{9\,x^2-18\,x+9}}$.

234. Square root of a trinomial. The positive square root of any trinomial which is a perfect square may be found by inspection.

Thus, $$+\sqrt{a^2+2\,ab+b^2}=a+b.$$

The actual work of finding the square root of $a^2+2\,ab+b^2$ may be arranged as follows:

		$a^2+2\,ab+b^2$	$a+b$
		a^2	
trial divisor	$2a$	$2\,ab+b^2$	
complete divisor	$2\,a+b$	$2\,ab+b^2$	
		0	

The explanation of the method is as follows:

1. We find the square root of the first term (which is the first term of the result) and subtract its square from the trinomial, obtaining $2\,ab + b^2$.

2. The second term b of the result may be found by dividing $2\,ab$ by $2\,a$. We call $2\,a$ *the trial divisor. The trial divisor is double the part, a, of the root already found.* After b, the second term of the result, has been found, we add it to the trial divisor, obtaining $2\,a + b$, *the complete divisor.*

3. Multiply the complete divisor $2\,a + b$ by b and subtract.

235. Square root of any polynomial. The process employed in finding the square root of $a^2 + 2\,ab + b^2$ is applicable in finding the square root of any polynomial.

ILLUSTRATION

Extract the square root of $x^4 - 2\,x^3y + 3x^2y^2 - 2\,xy^3 + y^4$.

$$\begin{array}{l|l|l}
 & x^4 - 2\,x^3y + 3\,x^2y^2 - 2\,xy^3 + y^4 & x^2 - xy + y^2 \\
 & x^4 & \\
\text{1st trial divisor, } 2(x^2) = 2\,x^2 & -2\,x^3y + 3\,x^2y^2 - 2\,xy^3 + y^4, \text{ 1st remainder} & \\
\text{1st complete divisor, } 2\,x^2 - xy & -2\,x^3y + x^2y^2 & \\
\text{2d trial divisor, } 2(x^2 - xy) = 2\,x^2 - 2\,xy & +2\,x^2y^2 - 2\,xy^3 + y^4, \text{ 2d remainder} & \\
\text{2d complete divisor, } 2\,x^2 - 2\,xy + y^2 & +2\,x^2y^2 - 2\,xy^3 + y^4 & \\
 & 0, \text{ 3d remainder} &
\end{array}$$

Explanation. 1. The square root of x^4 is x^2 (the first term of the result). The first trial divisor is $2\,x^2$ (double the part of the root already found). The first term in the first remainder divided by the trial divisor is $-\,xy$ (the second term of the result), and the first complete divisor is $2\,x^2 - xy$. Here $a = x^2$, $2\,ab = -\,2\,x^3y$, $2\,a + b = 2\,x^2 - xy$. Multiplying $2\,x^2 - xy$ by $-\,xy$ and subtracting, we obtain the second remainder; this corresponds to multiplying $2\,a + b$ by b and subtracting.

2. The second remainder is the result of subtracting the square of the part of the root already found $(x^2 - xy)^2$ from the given polynomial. By taking $a = x^2 - xy$, the second trial divisor is found to be $2(x^2 - xy) = 2\,x^2 - 2\,xy$. The first term of the second remainder divided by the first term of the second trial divisor is y^2 (the third term of the result), and the second complete divisor is $2\,x^2 - 2\,xy + y^2$.

We now multiply $2x^2-2xy+y^2$ by y^2 and subtract. Here $a=x^2-xy$, $2a=2x^2-2xy$, $b=y^2$, and $2a+b=2x^2-2xy+y^2$.

3. The third remainder, 0, is the result of subtracting the square of the part of the root already found $(x^2-xy+y^2)^2$, from the given polynomial. Hence, the given polynomial is the square of x^2-xy+y^2, and x^2-xy+y^2 is the required square root.

Remark. The polynomial should be arranged according to the powers of some one letter before the work of extracting the square root is begun.

EXERCISE 112

Find the square root of the expressions in examples 1–9.

1. $9x^2-6x+1$.
2. $4x^4-12x^3+13x^2-6x+1$.
3. $25x^4-20x^3+34x^2-12x+9$.
4. $x^4+2x^3-3x^2-4x+4$.
5. $x^4-4x^3+5x^2-2x+\frac{1}{4}$.
6. $x^2-2x+3-\dfrac{2}{x}+\dfrac{1}{x^2}$.
7. $x^6-2x^5y+5x^4y^2-14x^3y^3+14x^2y^4-20xy^5+25y^6$.
8. $x^6+2x^4-2x^3+x^2-2x+1$.
9. $x^8+2x^6+1+2x^2+3x^4$.
10. Show that $1+x^4+(2x+4)x^2+(x+1)(x+3)$ is a perfect square.
11. For what value of m is the expression $4x^4-12x^3y+mx^2y^2-6xy^3+y^4$ a perfect square?

Extract the square root of the expressions contained in examples 12 to 26 inclusive.

12. $a^2x^2+b^2y^2+1+2abxy+2ax+2by$.
13. $x^6-6mx^5+15m^2x^4-20m^3x^3+15m^4x^2-6m^5x+m^6$.
14. $(p-q)^4-2(p^2+q^2)(p-q)^2+2(p^4+q^4)$.

15. $x^2 - 4x + \frac{9}{x^2} - \frac{12}{x} + 10.$

16. $\frac{m^2}{n^2} + \frac{n^2}{m^2} + \frac{2m}{n} + \frac{2n}{m} + 3.$

17. $a^2b^{-2} + b^2c^{-2} + c^2a^{-2} + 2ac^{-1} + 2cb^{-1} + 2ba^{-1}.$

18. $a^3 + 2a^2 + a + 1 - 2a^{\frac{3}{2}}(a+1).$

19. $(a^{\frac{5}{2}} + a^2 + 2a^{\frac{3}{2}} + 2a + a^{\frac{1}{2}} + 1)(a^{\frac{1}{2}} - 1)(a-1).$

20. $(p^2 + q^2)(r^2 + s^2) - (ps - qr)^2.$

21. $\frac{x^4}{4} + \frac{x^3}{y} + \frac{x^2}{y^2} - xy - 2 + \frac{y^2}{x^2}.$

22. $\frac{p^2}{q} + \frac{q^2}{4p} + \frac{2p^{\frac{3}{2}} - q^{\frac{3}{2}}}{(pq)^{\frac{1}{4}}}.$

23. $2x^2(y+z)^2 + 2y^2(z+x)^2 + 2z^2(x+y)^2 + 4xyz(x+y+z).$

24. $(a+b)(a+2b)(a+3b)(a+4b) + b^4.$

25. $x + x^{\frac{2}{3}} + x^{\frac{1}{2}} + 2x^{\frac{5}{6}} + 2x^{\frac{3}{4}} + 2x^{\frac{7}{12}}.$

26. $a^{-2} + b^{-4} + c^2 + 2a^{-1}b^{-2} - 2a^{-1}c - 2b^{-2}c.$

27. Show that $a^3 + b^3 + c^3 - 3abc$ is the square root of $(a^2 - bc)^3 + (b^2 - ca)^3 + (c^2 - ab)^3 - 3(a^2 - bc)(b^2 - ca)(c^2 - ab).$

28. Show that $1 + (x-1)x(x+1)(x+2)$ is a perfect square. Hence, write down the square root of $1 + 2 \cdot 3 \cdot 4 \cdot 5.$

29. If $x^4 + 4 - 4x(x^2+2) + 8x^2 = (x^2 + mx + 2)^2$, what is the value of m?

30. Find the square root of $\left(\frac{2a}{a-1} - \frac{2a}{a+1}\right) \div \frac{a-1}{a+1} + 1.$

31. Prove that $x^2+(x+1)^2+(x+2)^2+(x+3)^2-5$ is a perfect square. If the sum of the squares of any four consecutive integers is diminished by 5, what may be said of the form of the result?

236. Square root of arithmetical numbers. We know that $+\sqrt{1}=1$, $+\sqrt{100}=10$, $+\sqrt{10000}=100$, $+\sqrt{1000000}=1000$. Hence, the positive square root of any number between 1 and 100 is between 1 and 10; that of any number between 100 and 10,000 is between 10 and 100, and so on. That is, the integral part of a square root of a number of two figures contains one figure; that of a number of three or four figures contains two figures, and so on. Hence, to find the number of figures in the integral part of the square root of a given number begin at the units' figure and separate the number into *periods*, or *groups*, of two figures each, the last period containing either one or two figures according as the given number contains an odd or an even number of figures in the integral part. There are as many figures in the integral part of the square root of the given number as there are periods.

ILLUSTRATIVE EXAMPLES

1. Find the positive square root of 9604.

Solution. Since there are two periods in the given number, there are two figures in the integral part of the root, which we find in the same manner as we did the square root of $a^2+2ab+b^2$. For a, we take the greatest number of tens whose square is less than 9604; that is, 9 tens, or 90. In practice, we take the greatest number whose square is equal to or less than 96 in the first period at the left. The work is completed as follows:

$$\begin{array}{r|l} & 96'04 \;|\; \underline{90+8} \\ & 81\ 00 \\ 2a=180 & 15\ 04 \\ 2a+b=180+8=188 & \underline{15\ 04} \end{array}$$

Remark. In practice the work is usually arranged thus:

$$\begin{array}{r|l}
\multicolumn{2}{r}{96'04 \,|\underline{98}} \\
\multicolumn{2}{r}{81} \\
\hline
188 & 15\ 04 \\
 & 15\ 04 \\
\hline
\end{array}$$

2. Find the positive square root of 56026.89.

Solution. Observe that if the square root of a number has decimal places, the number itself has double that number of decimal places. For this reason, in extracting the square root, decimals are separated into periods of two figures each beginning at the decimal point.

$$\begin{array}{r|l}
 & 5'60'26.89 \,|\underline{236.7} \\
 & 4 \\
\hline
2a = 2(20) = 40 & 1\ 60 \\
2a + b = 43 & 1\ 29 \\
\hline
2a = 2(230) = 460 & 31\ 26 \\
2a + b = 466 & 27\ 96 \\
\hline
2a = 2(2360) = 4720 & 3\ 30\ 89 \\
2a + b = 4727 & 3\ 30\ 89 \\
\hline
 & 0
\end{array}$$

237. Approximate square root. When a number is not a perfect square, we annex as many periods of zeros as are desired and continue the process of extracting the square root. In this way we obtain a rational number which is an approximate value of the square root of the number.

ILLUSTRATIVE EXAMPLE

Find to two places of decimals the square root of 10.

$$\begin{array}{r|l}
 & 10.00'00'00 \,|\underline{3.162} \\
 & 9 \\
\hline
61 & 1\ 00 \\
 & 61 \\
\hline
626 & 39\ 00 \\
 & 37\ 56 \\
\hline
6322 & 1\ 44\ 00 \\
 & 1\ 26\ 44 \\
\hline
\end{array}$$

Therefore, $\sqrt{10} = 3.16$, correct to two places of decimals.

Remark. Had the third decimal figure been either 5 or greater than 5, we should have taken 3.17 as the approximate square root.

EXERCISE 113

Find the positive square root of:

1. 6724. **2.** 14884. **3.** 5776.

4. 53361. **5.** 110889. **6.** 99856.

7. 591361. **8.** 1723969. **9.** 146.41.

10. 91083.24. **11.** 100.2001. **12.** 493.817284.

Find, to two places of decimals, the square root of:

13. 2. **14.** 3. **15.** 5. **16.** 7.

17. 11. **18.** 13. **19.** 12. **20.** 15.

Find, to three places of decimals, the value of:

21. $(1+\sqrt{2})(1+\sqrt{3})$. **22.** $(7\sqrt{5}-2)(\sqrt{2}+1)$.

23. $(2\sqrt{3}+\sqrt{5})(1+3\sqrt{2})$. **24.** $\dfrac{1}{\sqrt{3}-\sqrt{2}}$.

25. $\dfrac{1}{\sqrt{2}-1}$. **26.** $\dfrac{1}{\sqrt{3}-4}$.

Find, to two places of decimals, approximate values of the roots of the following simple equations:

27. $(x-1)\sqrt{2} = \sqrt{3}+2$.

28. $x(1+\sqrt{2}) = 2x + 3\sqrt{3}$.

29. $\sqrt{2x-1} = \sqrt{5}+2$. **30.** $x = \dfrac{\sqrt{5}-1}{2}$.

Suggestion. $2x - 1 = 9 + 4\sqrt{5}$.

238. Equations solved by finding the square roots of a number. Any equation which can be reduced to the form $ax^2 = b$ in which a and b are positive numbers and $\frac{b}{a}$ is a perfect square, has rational roots which may be found by taking the square root of $\frac{b}{a}$.

ILLUSTRATIVE EXAMPLE

Solve the equation $4x^2 - 9 = 0$.

Solution. $4x^2 - 9 = 0.$ (1)

Solving (1) for x^2, $x^2 = \frac{9}{4}.$ (2)

Now a root of (2) is a number whose square is $\frac{9}{4}$; therefore, it is a square root of $\frac{9}{4}$. Hence, taking the square root of each member of (2),

$$x = \pm \frac{3}{2}. \qquad (3)$$

Therefore, the roots of (2) are $+\frac{3}{2}$ and $-\frac{3}{2}$.

Note. Any root which can be found by the above method can also be found by factoring, as in section 117.

Thus, $4x^2 - 9 = 0.$ (1)

Factoring, $(2x - 3)(2x + 3) = 0.$ (2)

Equating $(2x - 3)$ to 0, $2x - 3 = 0.$ (3)

Equating $(2x + 3)$ to 0, $2x + 3 = 0.$ (4)

Solving (3) and (4) for x, $x = \frac{3}{2}$, and $-\frac{3}{2}$.

EXERCISE 114

Solve the equations in examples 1–14.

1. $x^2 = 4.$

2. $x^2 = 9.$

3. $2x^2 - 32 = 0.$

4. $x^2 = a^2.$

5. $a^2x^2 = b^2.$

6. $18x^2 - 200 = 0.$

7. $4x^2 = 49.$

8. $a^2b^2x^2 = 1.$

9. $(a+b)^2x^2 = a^2 - 2ab + b^2.$

10. $(x + 1)^2 = 16.$

11. $(ax + b)^2 = c^2.$

12. $x^2 + 2x + 1 = a^2 + 2a + 1.$

13. $(2x-3)^2=(a-2)^2.$ **14.** $\left(\frac{x+1}{2}\right)^2=1.$

15. In the illustrative example, page 310, in taking the square root of each member of (2), why is not equation (3) written $\pm x = \pm \frac{3}{2}$?

Calculate, to two places of decimals, the numbers which satisfy the following equations:

16. $x^2=2.$ **17.** $x^2=3.$ **18.** $x^2=5.$

19. $x^2=32.$ **20.** $x^2=27.$ **21.** $x^2=32.16.$

22. Solve the equation $a^2x^2+2\,abx+b^2=c^2.$

23. Solve the equation $4\,p^2x^2-4\,pqx=p^2-q^2.$

24. Solve the equation $x=\frac{4}{x}.$

25. Solve the equation $x-2=\frac{9}{x-2}.$

26. For what values of x will the H. C. F. of $x^3-3\,x^2+x-3$ and $x^3+2\,x^2+x+2$ reduce to 5?

27. Solve the equation $\frac{3\,x-2}{7}=\frac{7}{3\,x-2}.$

28. If $x=2$ satisfies the equation

$$(x-3)(x+1)=3\,x(x+2)-(x+a^2),$$

what are the values of a?

29. Solve the equation $\frac{x-2}{x+4}=\frac{x+4}{x-2}.$

30. When $2\,x^4+10\,x^3+13\,x^2-x-6$ is divided by $x^2+3\,x+2$, the quotient is $2\,x^2+m^2x-3$. Find the values of m.

31. If $m^2-1=x(x+1)(x+2)(x+3)$, find the values of m in terms of x.

CHAPTER XII

QUADRATIC EQUATIONS

239. Definition. An equation whose second member has been reduced to zero by transposition of terms is called an **equation of the second degree**, or simply a **quadratic equation in one unknown number**, as x, when its first member is a polynomial of the second degree in x.

Thus, $5 + 2x^2 - 7x = 0$, and $m - nx^2 + qx + n^2 - p^2 = 0$ are quadratic equations.

240. Notation. It is customary to arrange the polynomial in the first member of a quadratic equation according to the descending powers of x and to render positive, when necessary, the coefficient of the highest power of x by multiplying both members of the equation by -1. The first member of a quadratic equation being a trinomial of the second degree in x, has, in general, three terms, a term in x^2 with positive coefficient, a second term in x, and a third term which does not contain x and which is the constant term in the equation. In any particular equation, however, not all of these terms may occur, as the coefficient of one or more of them may reduce to zero; but we shall assume that in a quadratic equation the coefficient of x^2 is not zero.

241. Standard form of a quadratic equation. Any quadratic equation can be reduced to the *standard form*,

$$ax^2 + bx + c = 0.$$

In this standard form a is positive and different from zero, while b and c may have any values, including zero.

Thus, the equation $mx^2 - 3mx + 5 = nx^2 - 2n + 3px - 4$, when transformed, becomes $(m - n)x^2 - 3(m + p)x + (2n + 9) = 0$. In this last equation the a, b, and c of the standard form have the following values:

$$a = m - n,\quad b = -3(m + p),\quad \text{and}\quad c = 2n + 9.$$

242. Complete quadratic equation. A quadratic equation in which none of the coefficients a, b, or c reduces to zero is called a **complete quadratic equation.** All other quadratic equations are **incomplete quadratic equations.**

243. Solution of incomplete quadratic equations. The roots of an incomplete quadratic equation can, in general, be obtained by methods of solution which have been treated in preceding chapters. These methods are applied in the illustrative examples which follow.

ILLUSTRATIVE EXAMPLES

1. Solve the equation $2x^2 = 0$.

Solution. $2x^2 = 0$ is an example of a quadratic equation in which two of the coefficients, namely b and c, reduce to zero. It is evident that the only value of x which satisfies this equation is 0, since from $2x^2 = 0$, we derive $x^2 = 0$. We say that the equation has two roots each equal to 0, one root corresponding to each factor of x^2.

2. Solve the equation $2x^2 - 3x = 0$.

Solution. $2x^2 - 3x = 0$ is an example of a quadratic equation in which the constant term c reduces to zero. We solve such an equation by the method of section 117. Thus, factoring,

$$2x^2 - 3x = x(2x - 3) = 0. \qquad (1)$$

Equating each factor of (1) to zero, we have (2) and (3),

$$x = 0. \qquad (2)$$

$$2x - 3 = 0. \qquad (3)$$

Solving (3) $$x = \tfrac{3}{2}. \qquad (4)$$

Therefore, the roots of $2x^2 - 3x = 0$ are 0 and $\frac{3}{2}$, which may be verified by substituting each of these results in (1).

Remark. Observe that when the constant term c in a quadratic equation is zero, one root of the equation is zero, and conversely, when one root of a quadratic equation is zero, the constant term is zero.

3. Solve the equation $3x^2 - 2 = 0$.

Solution. $3x^2 - 2 = 0$ is an example of an equation in which b, the coefficient of x, is zero. Such an equation is sometimes called a **pure quadratic** equation.

From $3x^2 - 2 = 0$, we derive $x^2 = \frac{2}{3}$.

The values of x which satisfy this equation are evidently the square roots of $\frac{2}{3}$. Hence, $x = \pm\sqrt{\frac{2}{3}}$, or $\pm\frac{\sqrt{6}}{3}$.

Therefore, the roots of $3x^2 - 2 = 0$ are $\frac{\sqrt{6}}{3}$ and $-\frac{\sqrt{6}}{3}$, which may be verified by substituting these numbers in the given equation.

4. Solve the equation $x^2 + 4 = 0$.

Solution. $x^2 + 4 = 0$ is an example of a pure quadratic equation, both of whose terms have the same sign. No positive or negative number exists which satisfies this equation; for the sum of two positive numbers, one of which is not zero, cannot be zero. This type of quadratic equation will be considered in section 251.

EXERCISE 115

Solve the following incomplete quadratic equations:

1. $5x^2 = 0$.

2. $x^2 - 2x = 0$.

3. $3x^2 - 4x = 0$.

4. $4x^2 - 1 = 0$.

5. $5x^2 - 3x = 0$.

6. $7x^2 - 15 = 0$.

7. $5x^2 - 2 = 0$.

8. $2x^2 - 5x = 0$.

9. $8x^2 - 9 = 0$.

10. $ax^2 - bx = 0$.

11. $9x^2 - 16x = 0$.

12. $3x^2 - 5 = 0$.

13. $6x^2 - 5x = 0$.

14. $20x^2 - 8x = 0$.

15. $2x^2 - 11 = 0$.

16. $a^2x^2 - b^2 = 0$.

17. $(a + b)x^2 - (c + d)x = 0$.

18. $(a + b)x^2 - (a - b)x = 0$.

19. $5ax^2 - 3b = 0.$

20. $\frac{5x}{2} + \frac{3x^2}{5} = 0.$

21. $ax^2 - a = 0.$

22. $x^2 - 5 = 0.$

23. $\frac{1}{x} - x = 0.$

24. $\frac{2}{x} - 8x = 0.$

25. $\frac{4}{5}x^2 + \frac{5}{4}x = 0.$

26. $19x^2 - \frac{7}{2}x = 0.$

244. "Completing the square." From section 92 we know that the trinomials $x^2 + 2ax + a^2$ and $x^2 - 2ax + a^2$ are both perfect squares. We therefore infer that both $x^2 + 2ax$ and $x^2 - 2ax$ are converted into perfect squares by the addition of a^2 to each. Observe in each case that a^2 may be obtained by taking the square of one half the coefficient of x. We therefore have the following rule for completing the square of a binomial of the form $x^2 + 2ax$, in which the coefficient of x^2 is 1:

Rule. *Add the square of one half the coefficient of x.*

EXERCISE 116

Complete the square in each of the following examples, and state the binomial whose square is obtained:

1. $x^2 + 2x.$

2. $x^2 - 2x.$

3. $x^2 + 4x.$

4. $x^2 + 6x.$

5. $x^2 - 6x.$

6. $x^2 + 16x.$

7. $x^2 - 18x.$

8. $x^2 + 20x.$

9. $x^2 + \frac{1}{2}x.$

10. $x^2 - \frac{2}{3}x.$

11. $x^2 - \frac{5}{2}x.$

12. $x^2 + \frac{6}{5}x.$

13. $x^2 + 2cx.$

14. $x^2 - 2(a+b)x.$

15. $x^2 + 2(a-b)x.$

16. $x^2 - ax.$

17. $x^2 + 3x.$

18. $x^2 - (a+b)x.$

19. $x^2 - \frac{3a}{b}x.$

20. $x^2 + \frac{6c}{5d}x.$

21. $x^2 + \frac{a+b}{c+d}x.$

245. Solution of a quadratic equation by "completing the square." When the roots of a quadratic equation are not rational numbers, its solution by factoring [section 117] is not, in general, so convenient as that explained in the following illustrative examples :

ILLUSTRATIVE EXAMPLES

1. Solve the equation $3x^2 - 2x - 2 = 0$.

Solution. $3x^2 - 2x - 2 = 0$. (1)

Dividing both members of (1) by 3 so that the coefficient of x^2 shall be 1, $x^2 - \frac{2}{3}x - \frac{2}{3} = 0$. (2)

Transposing, $x^2 - \frac{2}{3}x = \frac{2}{3}$. (3)

Completing the square by adding the square of one half the coefficient of x to each member of (3),

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}. \quad (4)$$

Since the first member of (4) is a perfect square, we have,

$$(x - \tfrac{1}{3})^2 = \tfrac{7}{9}. \quad (5)$$

Extracting the square root,

$$x - \frac{1}{3} = \pm\frac{\sqrt{7}}{3}. \quad (6)$$

Transposing and combining,

$$x = \frac{1 \pm \sqrt{7}}{3}. \quad (7)$$

Therefore, the roots of $3x^2 - 2x - 2 = 0$ are $\frac{1+\sqrt{7}}{3}$ and $\frac{1-\sqrt{7}}{3}$.

2. Solve the equation $4p^2x^2 - 4mpx + m^2 - m - n = 0$.

Solution. $4p^2x^2 - 4mpx + m^2 - m - n = 0$. (1)

Dividing both members of (1) by $4p^2$,

$$x^2 - \frac{m}{p}x + \frac{m^2 - m - n}{4p^2} = 0. \quad (2)$$

Transposing in (2), $x^2 - \frac{m}{p}x = \frac{m + n - m^2}{4p^2}$. (3)

Completing the square and combining in (3),

$$x^2 - \frac{m}{p}x + \frac{m^2}{4p^2} = \frac{m+n}{4p^2}. \qquad (4)$$

Since the first member of (4) is a perfect square, we have,

$$\left(x - \frac{m}{2p}\right)^2 = \frac{m+n}{4p^2}. \qquad (5)$$

Extracting the square root,

$$x - \frac{m}{2p} = \pm\frac{\sqrt{m+n}}{2p}. \qquad (6)$$

Transposing,
$$x = \frac{m \pm \sqrt{m+n}}{2p}. \qquad (7)$$

Therefore, the two roots of the given equation are $\frac{m+\sqrt{m+n}}{2p}$ and $\frac{m-\sqrt{m+n}}{2p}$.

Remark. The solution, example 2, illustrates the solution of a literal quadratic equation by completing the square. Had the object been merely to solve the equation, the solution would be as follows:

$$4p^2x^2 - 4mpx + m^2 - m - n = 0. \qquad (1)$$

Transposing,
$$4p^2x^2 - 4mpx + m^2 = m + n. \qquad (2)$$

Since the first member of (2) is a perfect square,

$$(2px - m)^2 = m + n. \qquad (3)$$

Whence,
$$2px - m = \pm\sqrt{m+n}$$

and
$$x = \frac{m \pm \sqrt{m+n}}{2p}.$$

EXERCISE 117

Solve the following quadratic equations by the method of completing the square:

1. $x^2 + 4x + 3 = 0.$
2. $x^2 + 4x + 1 = 0.$
3. $x^2 + 2x - 4 = 0.$
4. $x^2 + 3x + 1 = 0.$
5. $x^2 + 5x + 5 = 0.$
6. $x^2 + 10x + 15 = 0.$

7. $x^2 + 11x + 25 = 0.$

8. $x^2 - 3x - 1 = 0.$

9. $x^2 - 5x + 3 = 0.$

10. $x^2 - 7x + 11 = 0.$

11. $x^2 - 11x - 1 = 0.$

12. $x^2 - 13x + 30 = 0.$

13. $x^2 - 15x - 5 = 0.$

14. $x^2 - 10x + 23 = 0.$

15. $x^2 - 6x + 4 = 0.$

16. $4x^2 - 4x - 1 = 0.$

17. $3x^2 + 3x - 2 = 0.$

18. $5x^2 - 5x + 1 = 0.$

19. $3x^2 - 7x + 3 = 0.$

20. $7x^2 - 7x - 5 = 0.$

21. $11x^2 + 7x - 3 = 0.$

22. $5x^2 - 3x - 5 = 0.$

23. $13x^2 - 13x - 3 = 0.$

24. $5x^2 - 5x - 1 = 0.$

25. $2x^2 - 3x - 4 = 0.$

26. $(x - 5)(x - 3) = 1.$

27. $(x - 6)(x - 8) = 4.$

28. $x^2 - (a + 1)x + a = 0.$

29. $bx^2 - a(b + 1)x + a^2 = 0.$

30. $(a + 3)x^2 - 2(a + 4)x + (a + 5) = 0.$

31. $(a + b)x^2 + (b + 2)x - (a - 2) = 0.$

32. $(a - 2)x^2 + x - (a - 3) = 0.$

33. $(2a - b)x^2 - 3ax + (a + b) = 0.$

34. $x^2 + 2(m + 1)x + m(m + 1) = 0.$

35. $4x^2 + 4(m + n)x + m^2 + n^2 = 0.$

36. $x^2 - 2ax + a^2 - a = 0.$

37. $4x^2 - 4bx + b^2 - 4a = 0.$

38. $(a + b)^2x^2 - 2(a + b)^2x + (a + b)^2 - 2 = 0.$

39. $16a^2b^2x^2 - 8ab(a + b)x + (a + b)^2 - 16a^3b^3 = 0.$

40. $(a + b)^2x^2 + 2(a^2 + b^2)x + (a - b)^2 = 0.$

In the following examples, clear the equation of fractions and solve the resulting integral equation, checking the roots found:

41. $\frac{1}{x} + \frac{2}{x - 1} = \frac{4}{3}.$

42. $\frac{2}{x+1}-\frac{3}{x+2}+\frac{1}{12}=0.$

43. $\frac{2}{3x-3}+1+\frac{4}{2x-3}=0.$

44. $\frac{3}{2x+1}+1-\frac{2}{3x+2}=0.$

45. $\frac{1}{6x-5a}-\frac{2}{a}=\frac{5}{a-6x}.$

46. $\frac{2x}{x-2}+\frac{3x+1}{x+2}=8.$

47. $\frac{7x+1}{4x+2}+\frac{4x-8}{3x+5}=7.$

48. $\frac{4x}{6x-5}+\frac{6x+7}{4x+3}=1$

49. $\frac{9x+1}{15x+1}+\frac{6x+5}{3x+5}=\frac{8}{9}.$

50. $\frac{x+1}{2x^2+3x-2}+\frac{x-2}{2x^2+x-1}+\frac{x-2}{x^2+3x+2}=0.$

51. $\frac{x+8}{2x^2+5x+2}+\frac{x+8}{x^2-x-6}+\frac{x-4}{2x^2-5x-3}=0.$

52. $\frac{2x+2}{3x^2+x-2}+\frac{4x+2}{6x^2-13x+6}=\frac{3x+3}{2x^2-x-3}.$

53. $\frac{3}{x^2+3x+2}+\frac{5}{x^2+7x+12}=\frac{6}{x^2+4x+3}.$

246. Approximations. Many of the problems which occur in physics and geometry give rise to quadratic equations. In general, the roots of such quadratics are irrational numbers which appear in the form of radical expressions. For practical purposes we usually require rational results which give approximately the values of the roots. The method used in obtaining such approximations may be seen from the following:

ILLUSTRATIVE EXAMPLE

Approximate to two decimal places the roots of

$$\frac{25x+2}{x+1}+1-\frac{47x+31}{2x-3}=0.$$

Solution. $\frac{25x+2}{x+1}+1-\frac{47x+31}{2x-3}=0.$ (1)

Clearing (1) of fractions and combining,

$$x^2-30x-8=0. \quad (2)$$

Solving (2), $x=15\pm\sqrt{233}$. (3)

That is, $x=15\pm 15.264^+$. (4)

Therefore, the roots of the given equation, correct to two places of decimals, are 30.26 and − .26.

Remark. — In checking the roots of a quadratic equation whose roots are irrational, the expressions in terms of radicals should be substituted for the unknown in the given equation. Thus, in checking the result in the foregoing example, substitute $15\pm\sqrt{233}$ for x in the given equation. The rational numbers obtained as approximations will not satisfy the equation.

EXERCISE 118

Approximate to two places of decimals the roots of the following equations:

1. $x^2-4x+2=0$.

2. $x^2-6x+7=0$.

3. $x^2-22x+118=0$.

4. $x^2-20x+95=0$.

5. $4x^2-12x-3=0$.

6. $2x^2+3x-6=0$.

7. $3x^2-5x-1=0$.

8. $5x^2-7x+1=0$.

9. $7x^2-15x+5=0$.

10. $x^2-32x-1=0$.

11. $x^2-10x+8=0$.

12. $3x^2-8x+1=0$.

13. $2x^2-13x+7=0$.

14. $3x^2+5x-3=0$.

15. $3x^2+7x-2=0$.

16. $8x^2-28x+21=0$.

247. Irrational equations. The unknown number in an equation sometimes occurs in expressions which are found under radical signs. Such equations are called **irrational equations.**

Thus, $6x - \sqrt{3x} + 4 = 20$ is an irrational equation.

We shall consider only equations in which the square root of expressions containing the unknown number is indicated. The solutions of the following examples will illustrate the method of solving such equations.

ILLUSTRATIVE EXAMPLES

1. Solve the equation $5x - \sqrt{3x+7} = 11$.

The equation $5x - \sqrt{3x+7} = 11$ is an example of an equation in which only a single square root occurs.

Solution. $$5x - \sqrt{3x+7} = 11. \quad (1)$$

Transposing terms so that the radical expression stands alone in one member of the equation,

$$-\sqrt{3x+7} = -5x + 11. \quad (2)$$

Squaring both members of (2), $3x + 7 = 25x^2 - 110x + 121$. (3)

Simplifying (3), $25x^2 - 113x + 114 = 0$. (4)

Factoring, $(x-3)(25x-38) = 0$. (5)

Therefore, the roots of equation (4) are 3 and $\frac{38}{25}$.

Substituting 3 in the given equation,

$$15 - \sqrt{9+7} = 11. \quad (6)$$

Simplifying (6), $11 = 11$, which is an identity.

Substituting $\frac{38}{25}$ in the given equation,

$$5 \times \tfrac{38}{25} - \sqrt{3 \times \tfrac{38}{25} + 7} = 11. \quad (7)$$

Simplifying (7), $\frac{21}{5} = 11$,

which is false. Since $\frac{38}{25}$ does not satisfy the given equation, it is not a root of the equation. It may readily be verified that $\frac{38}{25}$ is a solution of the equation $5x + \sqrt{3x+7} = 11$, which equation differs from the given equation only in the sign of the radical expression.

Note. In the process of squaring, as in the above solution, we multiply each member of the equation by a factor containing the unknown number. The resulting rational equation may have solutions which are not solutions of the given equation. It is, therefore, necessary to test each root of the rational equation by substituting it in the given equation.

2. Solve the equation $\sqrt{x+5}-\sqrt{7x+4}+\sqrt{2x+9}=0.$

(The first step in the solution of an irrational equation of this form is to arrange the terms so that one radical shall stand alone in one member. The process of squaring leads to an equation in which a single radical occurs and the solution proceeds as in that of example 1.)

Solution. $\sqrt{x+5}-\sqrt{7x+4}+\sqrt{2x+9}=0.$ (1)

Transposing, $\sqrt{x+5}+\sqrt{2x+9}=\sqrt{7x+4}.$ (2)

Squaring (2) and simplifying, $2x-5=\sqrt{(2x+9)(x+5)}.$ (3)

Squaring (3) and simplifying,

$$2x^2-39x-20=0. \quad (4)$$

Factoring, $(2x+1)(x-20)=0.$ (5)

Therefore, the roots of equation (4) are 20 and $-\frac{1}{2}$.

Substituting 20 in the given equation,

$$\sqrt{25}-\sqrt{144}+\sqrt{49}=0. \quad (6)$$

Simplifying (6), $5-12+7=0$, which is an identity.

Substituting $-\frac{1}{2}$ in the given equation,

$$\sqrt{\tfrac{9}{2}}-\sqrt{\tfrac{1}{2}}+\sqrt{8}=0. \quad (7)$$

Simplifying (7), $3\sqrt{2}=0,$

which is false; hence, $-\frac{1}{2}$ is not a solution of the given equation. It may readily be verified that $-\frac{1}{2}$ is a solution of the equation

$$-\sqrt{x+5}-\sqrt{7x+4}+\sqrt{2x+9}=0.$$

EXERCISE 119

Solve the equations:

1. $\sqrt{x-1}-1=0.$ **2.** $\sqrt{2x-3}-3=0.$

3. $\dfrac{1}{\sqrt{x+1}}=2.$ **Suggestion.** Clear of fractions.

4. $\sqrt{5x+2}=7.$

5. $\dfrac{1}{\sqrt{2x-3}}-5=0.$

6. $x+\sqrt{x}=12.$

7. $x-2\sqrt{x}-8=0.$

8. $x+\sqrt{3x+7}=7.$

9. $x-\sqrt{3x+7}=111.$

10. $5x+\sqrt{5x+4}=52.$

11. $2x-\sqrt{2x+3}=17.$

12. $3x+\sqrt{3x+2}=4.$

13. $5x+\sqrt{x+2}=\frac{11}{4}.$

14. $3\sqrt{x}-\sqrt{x+16}=4.$

15. $\sqrt{2x+9}+7=3\sqrt{2x}.$

16. $\sqrt{x-7}+3=\sqrt{2x}.$

17. $\sqrt{x+24}-\sqrt{x-15}=3.$

18. $\sqrt{9x+1}-\sqrt{4x-3}=3.$

19. $\sqrt{-35-99x}+\sqrt{27+2x}=13.$

20. $\sqrt{7x+4}+2\sqrt{3x}-\sqrt{15x+76}=0.$

21. $\sqrt{2x+11}+2\sqrt{x+2}=\sqrt{20x-19}.$

22. $\sqrt{x-7}-\sqrt{x-11}=\sqrt{3x-29}.$

23. $\sqrt{2x-6}-\sqrt{x-1}+\sqrt{3x-15}=0.$

24. $\sqrt{2x+1}+\sqrt{3x-11}-\sqrt{9x-8}=0.$

25. $\sqrt{7x+1}-\sqrt{3x+10}=1.$

26. $x^{\frac{1}{2}}-12x^{-\frac{1}{2}}+1=0.$ **Suggestion.** Clear of fractions.

27. $\sqrt{\dfrac{x}{2x+1}}+2\sqrt{\dfrac{2x+1}{x}}=3.$

28. $a+b-\sqrt{a^2-x}=\sqrt{b^2+x}.$

29. $\sqrt{ax+b}=\dfrac{2b}{\sqrt{2x}}.$

30. $2\sqrt{x}=\sqrt{a}+\sqrt{4x-c}.$

31. $\sqrt{x}+\sqrt{x+a}=\dfrac{2a}{\sqrt{x+a}}.$

32. $\sqrt{\frac{x+4}{x-4}}+\sqrt{\frac{x-4}{x+4}}=\frac{10}{3}.$

33. $\sqrt{x+2a}-\sqrt{x+2b}=2\sqrt{x}.$

34. $\sqrt{x^2+5ax-2a^2}=x+a.$

35. $\sqrt{x+a^2}=a+\sqrt{x}.$

36. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b.$

248. Solution of a quadratic equation by formulæ. Any quadratic equation can be reduced to the standard form,

$$ax^2+bx+c=0.$$

By this is meant that by assigning particular values to the coefficients a, b, and c, this equation reduces identically to any given quadratic equation. The solution of this general equation, therefore, contains the solution of any given quadratic equation. The solution of a given quadratic equation may, therefore, be found by substituting in the formulæ which give the roots of the general quadratic $ax^2+bx+c=0$, the proper values of the coefficients a, b, and c.

The derivation of the formulæ for the roots of $ax^2+bx+c=0$ is as follows:

$$ax^2+bx+c=0. \quad (1)$$

Since a is not zero, $x^2+\frac{b}{a}x+\frac{c}{a}=0. \quad (2)$

Transposing, $x^2+\frac{b}{a}x=-\frac{c}{a}. \quad (3)$

Completing the square,

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}. \quad (4)$$

That is,
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \quad (5)$$

Extracting the square root,
$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}. \quad (6)$$

Transposing and simplifying,
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (7)$$

Representing the two roots of $ax^2 + bx + c = 0$ by x_1 and x_2, we have, therefore, the following formulae,

$$\boldsymbol{x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.}$$

$$\boldsymbol{x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.}$$

Note. When $(b^2 - 4ac)$ is negative, the given equation, $ax^2 + bx + c = 0$, is satisfied by no positive or negative number. The expression $\sqrt{b^2 - 4ac}$ is, for the present, without meaning when $b^2 - 4ac < 0$. [See section 251.]

ILLUSTRATIVE EXAMPLES

1. Solve the equation $12x^2 + x - 6 = 0$.

Solution. Here $a = 12$, $b = 1$, $c = -6$; hence, by substitution in the foregoing formulæ we find the roots to be

$$\frac{-1 + \sqrt{1 + 288}}{24} \text{ and } \frac{-1 - \sqrt{1 + 288}}{24},$$

which, when simplified, are equal to $\frac{2}{3}$ and $-\frac{3}{4}$, respectively.

2. Solve the equation $2mx^2 - (3m + 1)x + (m + 1) = 0$.

Solution. Here $a = 2m$, $b = -(3m + 1)$, $c = m + 1$; hence, by substitution in the foregoing formulæ we find the roots to be

$$\frac{3m + 1 + \sqrt{(3m + 1)^2 - 8m(m + 1)}}{4m}$$

and
$$\frac{3m + 1 - \sqrt{(3m + 1)^2 - 8m(m + 1)}}{4m},$$

which, when simplified, are equal to 1 and $\frac{m + 1}{2m}$, respectively.

EXERCISE 120

Find the roots of the following equations by substituting, in each case, the proper values of a, b, and c in the foregoing formulæ.

1. $x^2 - 5x - 14 = 0.$

2. $x^2 - 10x + 21 = 0.$

3. $6x^2 - x - 2 = 0.$

4. $x^2 - 3x - 28 = 0.$

5. $20x^2 - 23x + 6 = 0.$

6. $15x^2 - 11x - 12 = 0.$

7. $x^2 + 4x + 2 = 0.$

8. $x^2 - 6x + 6 = 0.$

9. $x^2 - 10x + 18 = 0.$

10. $x^2 - 3x + 1 = 0.$

11. $3x^2 + 4x - 1 = 0.$

12. $25x^2 - 10x - 2 = 0.$

13. $x^2 - (c - d)x - cd = 0.$

14. $x^2 - (2m - 3n)x - 6mn = 0.$

15. $abx^2 - (a^2 + b^2)x + ab = 0.$

16. $4c^2x^2 + a^2 = b^2 + 4acx.$

17. $\frac{1}{x-a} + \frac{1}{x-b} = \frac{1}{a} + \frac{1}{b}.$

18. $bcx^2 + 2cax + ab = 0.$

19. $x - 100 = 10 - \sqrt{x}.$

20. $2(x - 1) = \sqrt{2x}.$

21. $x + \sqrt{x} = 0.$

22. $x^2 - 9 = 0.$

23. $x^2 - 4 = 0.$

24. $x^2 - 3 = 0.$

25. $x^2 - 3x = 0.$

26. $x^2 - x\sqrt{2} = 0.$

27. $x^2 + x\sqrt{3} = 0.$

28. $x^2 - a = 0.$

29. $mx^2 - n = 0.$

30. $a^2x^2 - b^2 = 0.$

249. Relations between roots and coefficients. Taking the standard form of the quadratic equation, we have :

$$ax^2 + bx + c = 0. \qquad (1)$$

Dividing both members of (1) by a (since a is not zero),

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{2}$$

From the formulæ of section 248 we have the roots of (1); namely,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{3}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

From (3) and (4) by addition,

$$x_1 + x_2 = \frac{-2b}{2a} = -\frac{b}{a}. \tag{5}$$

From (3) and (4) by multiplication,

$$x_1x_2 = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{1}{4a^2}[(-b)^2 - (\sqrt{b^2 - 4ac})^2]$$

$$= \frac{1}{4a^2}[b^2 - b^2 + 4ac] = \frac{c}{a}. \tag{6}$$

We see from identity (5) that the sum of the roots of equation (2) differs in sign only from the coefficient of x in that equation. Also, we see from identity (6) that the product of the roots of equation (2) is the constant term in that equation. Hence,

In any quadratic equation of the form $ax^2 + bx + c = 0$:

(I) *The coefficient of* x *with its sign changed divided by the coefficient of* x^2 *is equal to the sum of the roots.*

(II) *The constant term divided by the coefficient of* x^2 *is equal to the product of the roots.*

250. Formation of the equation. The principles of section 249 enable us to form the quadratic equation when its roots are given numbers. For this purpose we suppose the equation written in the form $x^2 + px + q = 0$, in which p and q are written instead of $\frac{b}{a}$ and $\frac{c}{a}$, respectively. From identities (5) and (6), section 249, we have

$$x_1 + x_2 = -\frac{b}{a} = -p.$$

$$x_1 x_2 = \frac{c}{a} = q.$$

ILLUSTRATIVE EXAMPLES

1. Form the equation whose roots are 3 and -2.

Solution. Here $x_1 + x_2 = 3 - 2 = 1 = -p$;

also, $x_1 x_2 = 3 \times (-2) = -6 = q$.

Hence, $p = -1$ and $q = -6$.

Substituting these values of p and q in $x_2 + px + q = 0$, the required equation is $x^2 - x - 6 = 0$.

2. Given that one root of the equation $\frac{x+1}{5} = \frac{x}{6 - 6x}$ is $\frac{2}{3}$; find, without solving the equation, the other root.

Solution. Reducing the given equation to the standard form, we have,

$$6x^2 + 5x - 6 = 0. \quad (1)$$

From (II), section 249, the product of the roots of equation (1) is equal to the constant term divided by the coefficient of x^2; namely, to $\frac{-6}{6}$, or -1.

Since $\frac{2}{3}$ is known to be a root, the other root is equal to $-1 \div \frac{2}{3}$; that is, the second root of the given equation is $-\frac{3}{2}$.

We may check this result by making use of (I), section 249, from which we know that the sum of the roots of this equation is the coefficient of x with the sign changed divided by the coefficient of x^2, which is $-5 \div 6$, or $\frac{-5}{6}$. Since $\frac{2}{3}$ is one root, the other root is equal to $-\frac{5}{6} - \frac{2}{3} = -\frac{9}{6} = -\frac{3}{2}$, which is in agreement with the preceding result.

EXERCISE 121

1. State at sight the sum and the product of the roots of the following six equations. Do not solve the equations.

$x^2 - 2x + 1 = 0.$ $\quad$ $x^2 - 7x + 12 = 0.$

$x^2 + 2x - 3 = 0.$ $\quad$ $12x^2 - 3x - 4 = 0.$

$ax^2 - bx + c = 0.$ $\quad$ $mx^2 + nx + pq = 0.$

2. One root of each of the following six equations is -2. Find the second root in each case. Do not solve the equations.

$x^2 + 3x + 2 = 0.$ $\quad$ $x^2 - x - 6 = 0.$

$3x^2 + 4x - 4 = 0.$ $\quad$ $x^2 + 4x + 4 = 0.$

$ax^2 + (2a + b)x + 2b = 0.$ $\quad$ $px^2 + (2p - q)x - 2q = 0.$

3. In a pure quadratic equation what is the sum of the roots?

Form the quadratic equations which have the following roots:

4. 2 and -3. **5.** 3 and 2. **6.** -1 and 3.

7. 6 and -3. **8.** -2 and -3.

9. $1 + \sqrt{2}$ and $1 - \sqrt{2}$. **10.** $3 - \sqrt{2}$ and $3 + \sqrt{2}$.

11. $m + n$ and $m - n$. **12.** $\frac{a}{b} + \frac{b}{a}$ and 2.

13. $m + \sqrt{n}$ and $m - \sqrt{n}$. **14.** $\frac{5}{2}$ and $-\frac{3}{4}$.

15. $\frac{2}{3}\sqrt{2}$ and $-\frac{2}{3}\sqrt{2}$.

251.* Quadratic equations in which $b^2 - 4ac$ is negative. The equation $x^2 + 4 = 0$ [section 243, example 4], which

* This section may be omitted, if so desired, until the subject is reviewed.

is of the form $ax^2 + bx + c = 0$, where $b^2 - 4ac < 0$ [section 248, note], is an example of a large class of equations which have no rational or irrational solution. There are no rational or irrational numbers which satisfy an equation such as $x^2 + 4 = 0$.

It is evidently desirable that all equations should have solutions, but this is manifestly impossible so long as the number system of algebra includes only rational and irrational numbers.

In agreement with the generalizing spirit of algebra, the number system is so extended that all equations shall have solutions.

252. Pure imaginary number. The square root of a negative number is called a **pure imaginary number.**

Thus, $\sqrt{-4}$, also $\sqrt{-2}$, are pure imaginary numbers.

253. Real numbers. All rational and irrational numbers are called **real numbers.**

Thus, 2, -3, $\frac{3}{4}$, $-\frac{3}{4}$, $\sqrt{4}$, $\sqrt{2}$, are real numbers.

254. The imaginary unit. The pure imaginary number $\sqrt{-1}$ is called **the imaginary unit** and is denoted by the letter i.

Thus, $i = \sqrt{-1}$.

255. A pure imaginary number expressed in terms of i. A pure imaginary number is expressed in terms of the imaginary unit i as follows:

$$\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\,\sqrt{a} = i\sqrt{a}.$$

Whenever a pure imaginary number occurs in any algebraic work, it is to be expressed in terms of the imaginary unit i.

Karl Friedrich Gauss (1777–1855) is called by general agreement the greatest mathematician of modern times. In 1799 he published a proof of the theorem that every algebraic equation has a root of the form $a + bi$. He introduced the symbol i to denote $\sqrt{-1}$ and was the originator of a great part of the modern theory of numbers.

256. Powers of i. We have by definition, $(\sqrt{-1})^2 = -1$, or $i^2 = -1$; therefore,

$$i = \sqrt{-1}.$$
$$i^2 = -1.$$
$$i^3 = i^2 i = (-1)i = -i.$$
$$i^4 = (i^2)^2 = (-1)^2 = +1.$$
$$i^5 = i^4 i = 1 \cdot i = i.$$

From these identities it is inferred that any even power of i is a real number, namely, $+1$ or -1, and that any odd power of i is a pure imaginary number, namely, $+i$ or $-i$.

ILLUSTRATIVE EXAMPLES

1. Find the product of $\sqrt{-4}$ and $\sqrt{-9}$.

Solution. $\sqrt{-4} = \sqrt{(-1)4} = \sqrt{-1}\,\sqrt{4} = 2i.$
$\sqrt{-9} = \sqrt{(-1)9} = \sqrt{-1}\,\sqrt{9} = 3i.$

Hence, $\sqrt{-4}\,\sqrt{-9} = (2i)(3i) = 6i^2 = 6(-1) = -6.$

Note. An error is often made in finding the product of two imaginary numbers by an incorrect use of the identity $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$. In the proof of this identity the numbers a and b were limited to positive numbers; the expressions $\sqrt{-a}$ and $\sqrt{-b}$ were entirely meaningless. This error is avoided by observing that *a pure imaginary number is to be expressed in the form ai, where a is a real number* [§ 255].

2. Simplify $\dfrac{1}{\sqrt{-1}}$.

Solution. $$\frac{1}{\sqrt{-1}} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i.$$

3. Simplify $\sqrt{-2} \times \sqrt{-3} \div \sqrt{-5}$.

Solution. $$\sqrt{-2} \times \sqrt{-3} \div \sqrt{-5} = \frac{i\sqrt{2}\,i\sqrt{3}}{i\sqrt{5}}$$
$$= i\frac{\sqrt{6}}{\sqrt{5}} = \frac{i}{5}\sqrt{30} = \frac{\sqrt{-30}}{5}.$$

257. Simplest form of an expression containing a pure imaginary. An expression which contains pure imaginary numbers is said to be in its simplest form when its denominator is a rational number and its numerator contains no real factor under a radical sign which can be removed.

EXERCISE 122

Simplify :

1. $\dfrac{1}{\sqrt{-2}}$.

2. $\dfrac{1}{\sqrt{-3}}$.

3. $\sqrt{-1}\sqrt{-1}$.

4. $\sqrt{-1}\sqrt{-9}$.

5. $\sqrt{-2}\sqrt{-2}$.

6. $\dfrac{\sqrt{-3}}{\sqrt{3}}$.

7. $\sqrt{-3}\sqrt{-9}$.

8. $\sqrt{-5}\sqrt{-6}$.

9. $\sqrt{-2}\sqrt{-3}\sqrt{-5}$.

10. $\dfrac{1}{\sqrt{-3a}}$.

11. $\dfrac{1}{\sqrt{-4a^2}}$.

12. $\dfrac{\sqrt{-30}}{\sqrt{-5}}$.

13. $\dfrac{\sqrt{-5}}{\sqrt{-30}}$.

14. $\sqrt{-\frac{2}{3}}$.

15. $\sqrt{-\dfrac{a^2}{3}}$.

16. $\sqrt{-\dfrac{2}{b^2}}$.

17. $\dfrac{\sqrt{-2}+\sqrt{-3}}{\sqrt{-2}}$.

18. $\dfrac{\sqrt{-3}+\sqrt{-5}}{\sqrt{-6}}$.

19. $\dfrac{\sqrt{2}+\sqrt{-3}}{\sqrt{2}}$.

20. $\dfrac{\sqrt{-2}+3}{\sqrt{-3}}$.

21. $\dfrac{1}{i^8}$.

22. i^7.

23. $\dfrac{1}{i^{11}}$. **24.** $\sqrt{-125}$.

25. $\sqrt{-2^4 \times 3^6 \times 5^3}$.

258. Complex numbers. A number which can be expressed in the form $a + b\sqrt{-1}$, where a and b are real numbers, neither one of which is zero, is called a **complex number.**

Thus, $4 + \sqrt{-1}$, $2 - \sqrt{-5}$, and $5 + 2\sqrt{-1}$ are complex numbers.

259. Conjugate complex numbers. Two complex numbers which differ in the sign of the imaginary unit only are called **conjugate complex numbers.**

Thus, $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$; $\sqrt{2} + \sqrt{-3}$ and $\sqrt{2} - \sqrt{-3}$; $a + bi$ and $a - bi$, are pairs of conjugate complex numbers.

260. Sum and product of two conjugate complex numbers. Let $a + bi$ and $a - bi$ be any two complex numbers; then:

The sum $= (a + bi) + (a - bi) = 2a.$

The product $= (a + bi)(a - bi) = a^2 - (bi)^2$

$= a^2 - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2.$

Hence both the sum and the product of two conjugate complex numbers are real numbers.

ILLUSTRATIVE EXAMPLES

1. Find the product of $\sqrt{2} + \sqrt{-3}$ and $\sqrt{3} + \sqrt{-5}$.

Solution.

$$\begin{array}{l} \sqrt{2} + i\sqrt{3} \\ \underline{\sqrt{3} + i\sqrt{5}} \\ \sqrt{6} + 3i \\ \underline{\qquad + i\sqrt{10} + i^2\sqrt{15}} \\ \sqrt{6} + i(3 + \sqrt{10}) - \sqrt{15} \end{array}$$

Therefore,

$$(\sqrt{2} + \sqrt{-3})(\sqrt{3} + \sqrt{-5}) = (\sqrt{6} - \sqrt{15}) + (3 + \sqrt{10})\sqrt{-1}.$$

Remark. Observe that the product of the two complex numbers is a complex number.

2. Rationalize the denominator of the fraction $\dfrac{8}{\sqrt{5}+\sqrt{-3}}$.

Solution. Since the product of two conjugate complex numbers is real, we multiply the numerator and denominator of the given fraction by an expression conjugate to the denominator, and have

$$\frac{8}{\sqrt{5}+\sqrt{-3}}\cdot\frac{\sqrt{5}-\sqrt{-3}}{\sqrt{5}-\sqrt{-3}}=\frac{8(\sqrt{5}-\sqrt{-3})}{5+3}=\sqrt{5}-\sqrt{-3}.$$

Note. An imaginary number is sometimes defined as "any even root of a negative number." However, although $\sqrt[4]{-1}$, for example, is evidently not a real number, it is, nevertheless, not a pure imaginary, but a complex number. For, it may be verified that $\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^4$, $\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)^4$, $\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^4$, or $\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)^4$ is equal to -1; hence, each of the expressions within the parentheses is a fourth root of -1.

EXERCISE 123

Simplify:

1. $\sqrt{-16}+\sqrt{-36}$.

2. $\sqrt{-\frac{1}{16}}+3\sqrt{-\frac{1}{144}}$.

3. $\sqrt{-a^4}+\sqrt{-b^2}-\sqrt{-c^4}$.

4. $\sqrt{3}\sqrt{-2}$.

5. $\sqrt{-a^4}\cdot\sqrt{a^8}$.

6. $\sqrt{a^2-b^2}\sqrt{b-a}\,(a>b)$.

7. $(\sqrt{-2}-\sqrt{-4})(\sqrt{-2}+\sqrt{-3})$.

8. $(1+\sqrt{-5})^2$.

9. $(2+\sqrt{-3})(2-\sqrt{-3})$.

10. $(\frac{1}{2}+\frac{1}{2}\sqrt{-3})(\frac{1}{2}-\frac{1}{2}\sqrt{-3})$.

11. $\sqrt{-a}\div\sqrt{-b}$.

12. $(5+i\sqrt{3})\div(5-i\sqrt{3})$.

13. $\dfrac{2}{\sqrt{-3}+1}$.

14. $\dfrac{2}{\sqrt{-2}+\sqrt{-3}}$.

15. $\dfrac{5}{\sqrt{-10}-\sqrt{-5}}$.

261. Quadratic equations with complex roots.

ILLUSTRATIVE EXAMPLE

Solve the equation $2x^2-3x+5=0$.

Solution by completing the square.

$$2x^2-3x+5=0. \quad (1)$$

Transforming (1), $x^2-\frac{3}{2}x+\frac{5}{2}=0.$ (2)

Transposing, $x^2-\frac{3}{2}x=-\frac{5}{2}.$ (3)

Completing the square, $x^2-\frac{3}{2}x+\frac{9}{16}=-\frac{5}{2}+\frac{9}{16}=-\frac{31}{16}.$ (4)

That is, $(x-\frac{3}{4})^2=-\frac{31}{16}.$ (5)

Extracting the square root, $x-\frac{3}{4}=\pm\frac{1}{4}\sqrt{-31}.$ (6)

Solving (6), $x=\dfrac{3\pm\sqrt{-31}}{4}.$ (7)

Solution by formulæ.

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Here $a=2$, $b=-3$, $c=5$; hence, by substitution in the formulæ we find the roots to be

$$\frac{-(-3)+\sqrt{9-40}}{4} \text{ and } \frac{-(-3)-\sqrt{9-40}}{4},$$

which when simplified are equal, respectively, to

$$\frac{3+\sqrt{-31}}{4} \text{ and } \frac{3-\sqrt{-31}}{4}.$$

EXERCISE 124

Solve the following equations, which have either pure imaginary or complex roots:

1. $x^2+1=0.$

2. $x^2+4=0.$

3. $x^2 + 9 = 0.$

4. $x^2 + 2 = 0.$

5. $2x^2 + 3 = 0.$

6. $3x^2 + 5 = 0.$

7. $5x^2 + 2 = 0.$

8. $15x^2 + 17 = 0.$

9. $x^2 + 2x + 2 = 0.$

10. $x^2 + 4x + 5 = 0.$

11. $x^2 - 6x + 11 = 0.$

12. $x^2 + 5x + 7 = 0.$

13. $2x^2 - 3x + 2 = 0.$

14. $3x^2 - 3x + 2 = 0.$

EXERCISE 125

1. A man bought a certain number of oranges at 75 cts. The number of oranges he bought was three times the number of cents he paid for each orange. How many oranges did he buy?

2. The side of one square is three times the side of another and the difference of their areas is 32. What is the side of the smaller square?

3. If the edge of a certain cube be doubled, the area of the entire surface of the cube will be increased by 72 sq. in. What is the edge of the cube?

4. A dealer sold an article at a loss of $6.25 and thereby lost as many per cent as there were dollars in the cost. What was the cost?

5. Find two consecutive integers whose product is 56.

6. Find two consecutive integers whose product is 462.

7. Find a number whose square exceeds 100 times the number by 2684.

8. Two odd integers differ by 2 and the difference of their squares is 56. Find the integers.

9. The quotient of two numbers is $2\frac{1}{3}$ and their product is 756. Find the numbers.

10. The difference of the squares of two consecutive numbers is 197. Find the numbers.

11. The sum of two numbers is 21, and their product is 110. What are the numbers?

12. The difference of two numbers is 4, and their product is 45. What are the numbers?

13. The difference of two numbers is 42, and their quotient is the less number. What are the numbers?

14. A dealer sold an article for \$39 and thereby gained as many per cent as there were dollars in the cost. Find the cost.

15. The plate of a looking-glass is 18 in. by 12 in.; it is to be surrounded by a plain frame of uniform width, whose area shall be equal to that of the glass. Required the width of the frame.

16. Find three consecutive integers the sum of whose products by pairs is 299.

17. If a body be thrown vertically upward from the ground with an initial velocity of 32 ft. per second, when will it be at a height of 7 ft.?

Suggestion. Use the formula $s = at - 16t^2$, in which a represents the initial velocity and s the height at the end of t seconds.

18. The denominator of a given fraction is one greater than its numerator; if $\frac{7}{12}$ be added to the fraction, the sum is equal to the reciprocal of the given fraction. Find the given fraction.

19. The difference between the hypotenuse and base of a right-angled triangle is 6, and the difference between the hypotenuse and altitude is 3. What are the sides?

20. A square garden is surrounded by a path. The area of the path is 12,400 sq. ft. The garden is 290 ft. wider than the path. Find the area of the garden.

21. A field containing one acre is in the form of a rectangle $\frac{2}{5}$ as wide as it is long. The field is enlarged by adding 39,664 sq. ft. in such a way as to increase length and width of the rectangle an equal amount. Find the dimensions of the enlarged field.

22. From the point of intersection of two straight roads which intersect at right angles, two men, A and B, set out simultaneously, A on the one road riding at the rate of 12 mi. per hour, and B on the other walking at the rate of 5 mi. per hour. After how many hours will they be 65 mi. apart?

23. A number of laborers were employed to do a piece of work. If 7 less had been employed, the work would have taken two more days. If 28 men had been employed, the work would have been done in 20 days. How many laborers were employed?

24. A gardener planted a certain number of trees at equal distances apart, and in the form of a square. He found on finishing the planting that he had 5 trees to spare. He then added one of them to each row as far as they would go, and found that he needed 10 trees to complete the square. How many trees had he?

25. Find the price of tea per pound if a rise of 10 cents in the price per pound would reduce by 5 lb. the quantity obtainable for $15.

26. What is the price of eggs per dozen, if a fall of 2 cents in the price would increase by one the number of dozen obtainable for $6.84?

27. Two trains travel without stopping between two stations m miles apart. One train goes a miles an hour faster than the other and takes b hours less time for the journey. Find the speed of each train. What is the speed of each train if $m = 40$, $a = 10$, and $b = \frac{1}{3}$?

28. A man bought a certain number of cows for $1500. He sold 5 less than the whole number of cows for $20 a head more than they cost him and made $100 by the transaction. How many cows did he buy?

29. A merchant sold 7 doz. fresh eggs and 12 doz. storage eggs for $5.81, and found that he had sold 1 doz. more fresh eggs for $2.10 than he had of storage eggs for $1.40. Required the price of each kind per dozen.

30. A merchant draws a certain quantity of vinegar from a full cask containing 63 gallons. Having filled up the cask with water, he draws the same quantity as before. He then finds that the cask contains $\frac{9}{16}$ the original quantity of vinegar. How many gallons did he draw each time?

31. A sum of $30,000 is subject to an inheritance tax of a certain per cent, then to a percentage for fees at a rate one half per cent greater than that of the inheritance tax. When the tax and fees are deducted there remains $27,504. What are the two rates?

262. Utility of the extension of the meaning of the word number. The extension of the meaning of the word *number* so as to include under the term such expressions as $\sqrt{-2}$ and $2 + \sqrt{-3}$ renders possible a greater generality in the statement of algebraic principles and results. As an illustration of this, the statement that *except when* $b^2 - 4ac$ *is negative, the quadratic equation* $ax^2 + bx + c = 0$ *admits of solution* [section 248, note], is replaced by the general statement, *every quadratic equation has two roots.*

263. Nature of the roots of $ax^2 + bx + c = 0$. In this equation the a, b, and c represent rational numbers. The roots of $ax^2 + bx + c = 0$ have been found to be,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

An examination of these formulæ leads to the following important principles:

I. **When $b^2 - 4ac$ is positive and not equal to zero.**
In this case,

1. The roots are real and unequal.

Thus, in the equation $9x^2 - 18x + 7 = 0$ the expression $b^2 - 4ac = 72$, and the roots, $1 \pm \frac{1}{3}\sqrt{2}$, are real and unequal.

2. If $b^2 - 4ac$ is a perfect square, the roots are rational; and, conversely, the roots are rational only when $b^2 - 4ac$ is a perfect square.

Thus, in the equation $2x^2 + 5x - 3 = 0$, the expression $b^2 - 4ac = 49$, and the roots, -3 and $\frac{1}{2}$, are rational.

3. If $b^2 - 4ac$ is not a perfect square, the roots are conjugate quadratic surd expressions.

Thus, in the equation $9x^2 - 18x + 7 = 0$, the expression $b^2 - 4ac = 72$, and the roots $1 + \frac{1}{3}\sqrt{2}$ and $1 - \frac{1}{3}\sqrt{2}$ are conjugate quadratic surd expressions.

4. If c is 0, one root is zero, and the second root is rational.

Thus, in the equation $x^2 - 3x = 0$, the roots are zero and 3.

Note. If a and c have opposite signs, $b^2 - 4ac$ is necessarily positive and the roots are always real.

II. **When $b^2 - 4ac$ is equal to zero.**
In this case,

1. The roots are rational and equal.

Thus, in the equation $9x^2 - 6x + 1 = 0$, the expression $b^2 - 4ac = 0$, and the roots are $\frac{1}{3}$ and $\frac{1}{3}$.

2. The polynomial $ax^2 + bx + c$ is a perfect square.

Thus, $9x^2 - 6x + 1 = 0$ may be written $(3x - 1)^2 = 0$.

3. When c is zero, then b is also zero, and both roots are zero.

Thus, if $b^2 - 4ac = 0$, and $c = 0$, then $b^2 = 0$, or $b = 0$; and the roots of the equation are 0 and 0.

III. **When $b^2 - 4ac$ is negative.**

In this case,

1. The roots are not real.

Thus, in the equation $3x^2 - 2x + 4 = 0$, the expression $b^2 - 4ac = -44$ and the roots are $\frac{1 + \sqrt{-11}}{3}$ and $\frac{1 - \sqrt{-11}}{3}$, which are not real.

2. When b is zero, the roots are pure imaginary numbers.

Thus, in the equation $x^2 + 4 = 0$, the expression $b^2 - 4ac = -4$ and b is zero. The roots are $2i$ and $-2i$, which are pure imaginary numbers.

3. When b is not zero, the roots are conjugate complex numbers.

Thus, in the equation $3x^2 - 2x + 4 = 0$, which is the equation given in 1, the roots are seen to be conjugate complex numbers.

Note. The statement that when $b^2 - 4ac = 0$ the polynomial $ax^2 + bx + c$ is a perfect square, may be proved as follows:

Given $\quad b^2 - 4ac = 0.$

Transposing, $\quad b^2 = 4ac.$

Therefore, $\quad b = \pm 2\sqrt{a}\sqrt{c}.$

Substituting, $\quad ax^2 + bx + c = ax^2 \pm 2\sqrt{a}\sqrt{c}x + c$

$$= (x\sqrt{a} \pm \sqrt{c})^2.$$

EXERCISE 126

Calculate for each of the following equations the value of $b^2 - 4ac$ and determine from principles of section 263 the nature of the roots. The principles should not be memorized, but the reason for each statement made should be clearly understood.

1. $x^2 + 3x - 4 = 0.$
2. $2x^2 - 3 = 0.$
3. $5x^2 = 0.$
4. $x^2 + 1 = 0.$
5. $3x^2 + 2x + 5 = 0.$
6. $3x^2 - 2x - 2 = 0.$
7. $x^2 - x - 1 = 0.$
8. $x^2 + x + 1 = 0.$
9. $9x^2 - 12x + 4 = 0.$
10. $\frac{2}{3}x^2 - \frac{3}{2}x + \frac{4}{5} = 0.$
11. $\frac{2}{5}x^2 - \frac{3}{4}x + \frac{45}{128} = 0.$
12. $4.9x^2 - 7.35x - 22.05 = 0.$
13. $2x^2 - 3x - 20 = 0.$
14. $169x^2 + 442x + 289 = 0.$
15. $x^2 + 2ax + a^2 + b^2 = 0.$
16. $x^2 - 2ax + a^2 + b^2 = 0.$

264. Factors of $ax^2 + bx + c$. By definition, a root of an equation is a number which, when substituted for the unknown, reduces the equation to an identity; that is, satisfies the equation.

Let x_1 represent a root of the equation,

$$ax^2 + bx + c = 0. \qquad (1)$$

Substituting x_1 for x in (1),

$$ax_1^2 + bx_1 + c = 0. \qquad (2)$$

Solving (2) for c,
$$c = -ax_1^2 - bx_1. \qquad (3)$$

Substituting the value of c in (1),

$$ax^2 + bx + c = ax^2 + bx - ax_1^2 - bx_1 \qquad (4)$$
$$= a(x^2 - x_1^2) + b(x - x_1) \qquad (5)$$
$$= (x - x_1)(ax + ax_1 + b) \qquad (6)$$

From the foregoing it may be inferred that:

If x_1 is a root of the equation $ax^2 + bx + c = 0$, the polynomial $ax^2 + bx + c$ is exactly divisible by $x - x_1$.

A second root of $ax^2 + bx + c = 0$ is obtained by equating the second factor of $ax^2 + bx + c$, namely, $ax + ax_1 + b$, to 0 and solving for x [see section 117].

Thus, $$ax + ax_1 + b = 0. \qquad (1)$$

Solving (1) for x, $$x = -x_1 - \frac{b}{a}. \qquad (2)$$

Representing the root $-x_1 - \frac{b}{a}$ by x_2, we have,

$$x_2 = -x_1 - \frac{b}{a}. \qquad (3)$$

Transposing, $$x_1 + x_2 = -\frac{b}{a}. \qquad (4)$$

Identity (4) is in agreement with (I) of section 249. The expression $ax + ax_1 + b$ may be written $a\left(x + x_1 + \frac{b}{a}\right)$; or substituting $-x_2$ for $x_1 + \frac{b}{a}$, we have $a(x - x_2)$. Therefore, the polynomial $ax^2 + bx + c$ is identically equal to $a(x - x_1)(x - x_2)$ where x_1 and x_2 are two roots of the quadratic $ax^2 + bx + c = 0$.

265. Number of roots of a quadratic. Let x_3 be a root of the quadratic equation

$$ax^2 + bx + c = 0;$$

that is, of the equation

$$a(x - x_1)(x - x_2) = 0,$$

in which x_1 and x_2 are two roots of the quadratic. Observe that the existence of at least two roots was shown in section 248.

Substituting x_3 for x in the given equation, $a(x_3 - x_1)(x_3 - x_2) = 0$. From this identity it is evident, since a is not zero, that either $x_3 - x_1 = 0$ or $x_3 - x_2 = 0$; that is, x_3 is equal to either x_1 or x_2; and, therefore, that:

Every quadratic equation has two and only two roots.

266. Quadratic with given roots. The quadratic equation whose roots are x_1 and x_2 is $a(x - x_1)(x - x_2) = 0$ [see section 264]. In this equation a may have any constant value.

Let the two equations $ax^2 + bx + c = 0$ and $mx^2 + nx + p = 0$ have the same roots x_1 and x_2. We may write [see section 249]:

$$\left.\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1x_2 &= \frac{c}{d} \end{aligned}\right\} \text{ and } \left\{\begin{aligned} x_1 + x_2 &= -\frac{n}{m} \\ x_1x_2 &= \frac{p}{m}. \end{aligned}\right.$$

From these identities, $\frac{a}{m} = \frac{b}{n} = \frac{c}{p}$. Therefore:

Two quadratic equations with the same roots have their corresponding coefficients proportional; and conversely, *if two quadratic equations have their corresponding coefficients proportional, they have the same roots.*

ILLUSTRATIVE EXAMPLE

Find the simplest form of a quadratic equation whose roots are $-\frac{2}{3}$ and $\frac{3}{5}$.

Solution. The required equation is $a(x + \frac{2}{3})(x - \frac{3}{5}) = 0$.

That is, $a\left(\frac{3x + 2}{3}\right)\left(\frac{5x - 3}{5}\right) = 0.$

Letting $a = 15$, $(3x + 2)(5x - 3) = 0.$

Expanding, $15x^2 + x - 6 = 0.$

EXERCISE 127

1. By inspection, arrange the following equations in groups so that those in any group shall have the same roots:

$x^2 - 3x + 2 = 0.$ $3x^2 - 9x + 6 = 0.$

$5x^2 - 15x + 10 = 0.$ $10x^2 - 15x - 10 = 0.$

$x^2 - \frac{3x}{2} - 1 = 0.$ $mx^2 - 3mx + 2m = 0.$

Find the quadratic whose roots are:

2. -5 and 6. **3.** $\frac{2}{3}$ and $\frac{3}{4}$. **4.** $-\frac{3}{5}$ and $-\frac{5}{3}$.

5. $-\frac{2}{3}$ and $\frac{7}{6}$. **6.** $\frac{1+\sqrt{2}}{2}$ and $\frac{1-\sqrt{2}}{2}$.

7. $\frac{1+i}{3}$ and $\frac{1-i}{3}$. **8.** $\frac{1+2\sqrt{-2}}{3}$ and $\frac{1-2\sqrt{-2}}{3}$.

9. $\frac{2+3i}{5}$ and $\frac{2-3i}{5}$.

267. Graph of a polynomial of the second degree. Graphs of linear functions of one variable and of linear equations of two unknowns were treated in Chapter IX. We shall now consider the graphs of certain quadratic functions.

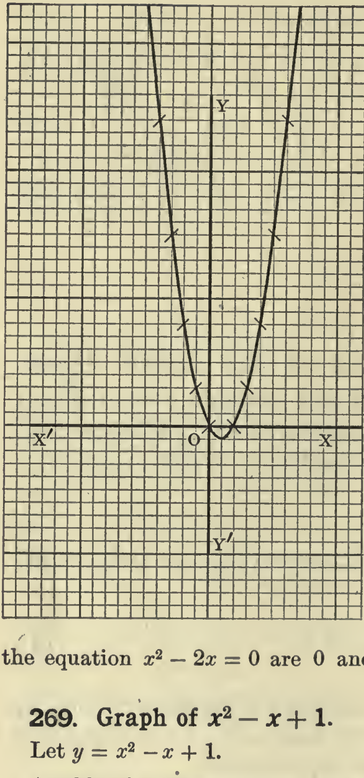
268. Graph of $x^2 - 2x$. Representing the polynomial $x^2 - 2x$ by y; then,

$$y = x^2 - 2x. \tag{1}$$

We construct a table of the corresponding values of x and y by arbitrarily assigning values to x and calculating the corresponding values of y from equation (1). The table follows:

x	-4	-3	-2	-1	0	1	2	3	4	5	6
x^2	16	9	4	1	0	1	4	9	16	25	36
$-2x$	8	6	4	2	0	-2	-4	-6	-8	-10	-12
y	24	15	8	3	0	-1	0	3	8	15	24

Plotting the points (x, y) as given in the table and drawing a smooth curve through these points, we have the required graph of the polynomial $x^2 - 2x$, as indicated in the figure, page 346.



From the graph of the polynomial $x^2 - 2x$; that is, from the graph of the equation $y = x^2 - 2x$, the roots of the quadratic equation $x^2 - 2x = 0$ may be found by inspection. Evidently, those values of x which make y equal to zero are the roots of the equation $x^2 - 2x = 0$, for they satisfy the equation. The points on the graph for which $y = 0$ are those common to the graph and the x-axis; for the ordinate y of a point is zero only when the point is on the x-axis. The roots of the equation $x^2 - 2x = 0$ are, therefore, the x(abscissas) of the points in which the graph of the polynomial $x^2 - 2x$ intersects the x-axis. These points of intersection are $x = 0$ and $x = 2$, and the roots of the equation $x^2 - 2x = 0$ are 0 and 2.

269. Graph of $x^2 - x + 1$.

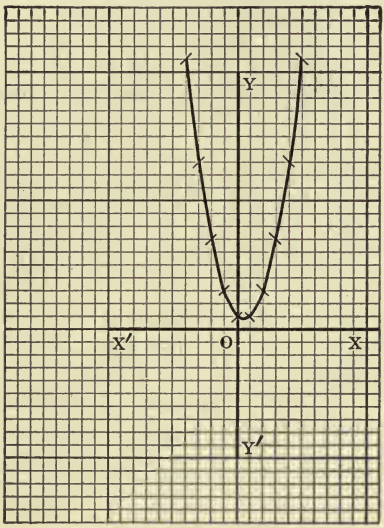
Let $y = x^2 - x + 1$.

A table of corresponding values of x and y is as follows:

x	-4	-3	-2	-1	0	$\frac{1}{2}$	1	2	3	4	5
x^2	16	9	4	1	0	$\frac{1}{4}$	1	4	9	16	25
$-x+1$	5	4	3	2	1	$\frac{1}{2}$	0	-1	-2	-3	-4
y	21	13	7	3	1	$\frac{3}{4}$	1	3	7	13	21

Plotting the points (x, y) as given in the table and drawing a smooth curve through these points, we have the required graph of the polynomial $x^2 - x + 1$, as in the figure, page 347.

We observe that the graph of the equation $y = x^2 - x + 1$ does not intersect the axis of x. This indicates that the quadratic equation $x^2 - x + 1 = 0$ has no real root. The roots of this equation are $\frac{1 \pm \sqrt{-3}}{2}$; that is, conjugate complex numbers. In general, when the graph of a polynomial $ax^2 + bx + c$ does not intersect

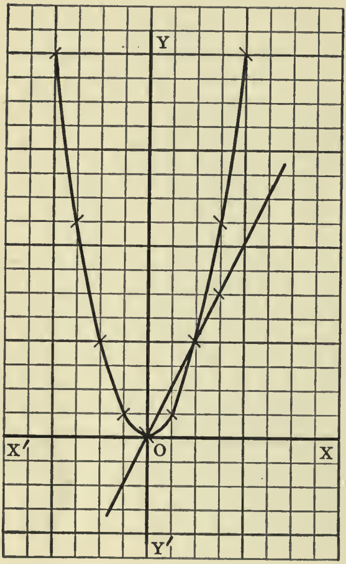


the x-axis, the roots of the equation $ax^2 + bx + c = 0$ are either pure imaginary numbers or conjugate complex numbers.

270. Graph of a system of two simultaneous equations. Given the system of two simultaneous equations,

$$\begin{cases} y = x^2, & (1) \\ y = 2x. & (2) \end{cases}$$

In section 192 we learned that the graph of the linear equation $y = 2x$ is a straight line. It is, therefore, necessary to find the coördinates of two of its points only in order to plot the straight line. We observe from equation (2) that the points (0, 0) and (3, 6) may be taken. The corresponding values of x and y in equation (1); that is, of x and x^2, are given in the preceding table.



The graphs of equations (1) and (2) are seen in the accompanying figure.

The real solutions of equations (1) and (2) are the coördinates of the points of intersection of their graphs; for the coördinates of these points satisfy both equations, and the coördinates of any point not on both graphs do not satisfy the equations. The straight line $y = 2x$ intersects the graph of $y = x^2$ in the points (0, 0) and (2, 4). The two solutions of the simultaneous equations $y = x^2$ and $y = 2x$ are, therefore, (0, 0) and (2, 4).

When the graphs of two equations do not intersect, their solutions involve pure imaginary numbers or complex numbers.

The graphs of $y = x^2$ and $y = 2x$ also show by their intersections the roots of the equation $x^2 - 2x = 0$.

Thus, the ordinate of any point on the straight line whose equation is $y = 2x$ is equal to twice its abscissa, and the ordinate of any point on the graph of $y = x^2$ is equal to the square of its abscissa. At a common point the ordinate is equal to twice its abscissa and also to the square of its abscissa. The abscissa of a common point, therefore, satisfies the equation

$$x^2 = 2x, \text{ or } x^2 - 2x = 0.$$

In like manner the real roots of $ax^2 + bx + c = 0$ are the abscissas of the points common to the graphs of the equations

$$y = ax^2 \text{ and } y = -bx - c.$$

EXERCISE 128

1. Construct the graph of the equation $y = x^2 - 1$. From the resulting graph determine the roots of the equation $x^2 - 1 = 0$.

2. Construct with respect to the same axes of reference the graphs of $y = 2x^2 - 1$ and $y = -3x + 1$. Estimate from the figure the values of x and y which satisfy both equations. Also obtain approximately the roots of $2x^2 + 3x - 2 = 0$.

3. Construct with respect to the same axes of reference the graph of $y + 2x^2 - 3x - 9 = 0$ and $y + x - 3 = 0$. Estimate from the figure the values of x and y which satisfy both equations. Also obtain approximately the roots of $x^2 - 2x - 3 = 0$.

4. Construct the graph of $x^2 - 2$. Estimate from the figure, correct to one decimal place, the value of $\sqrt{2}$.

5. Plot $2y - 3x = 6$ and $2x + 1 = 4y - 4y^2$, using the same axes, and estimate from the graphs the solutions of the equations.

6. Graph $y = 1 + 3x^2$.

7. Construct with respect to the same axes of reference the graphs of $y = 2x^2 + 1$ and $y = x^2 + 3x - 1$. Estimate from the figure the values of x and y which satisfy both equations. Also obtain from the figure the roots of $x^2 - 3x + 2 = 0$.

8. Construct with respect to the same axes of reference the graphs of $x^2 + y - 5 = 0$ and $y^2 + 3y - 2x = 0$. Estimate from the figure the real values of x and y which satisfy both equations.

CHAPTER XIII

SYSTEMS OF QUADRATIC EQUATIONS

271. Systems of two equations in two unknowns. The elimination of one of the unknowns from two equations of the second degree in two unknowns does not, in general, lead to a quadratic equation in one unknown. Certain special systems of two equations, however, neither one of which is of higher degree than the second, are of frequent occurrence and lead to quadratic equations in one unknown. Such systems may be solved by the methods of preceding chapters.

272. A quadratic and a linear equation. A system consisting of a quadratic and a linear equation may always be solved by substitution. The method is indicated in the illustrative examples which follow.

ILLUSTRATIVE EXAMPLES

1. Solve the system $\begin{cases} x^2 + 2xy = 33, & (1) \\ 3x + y = 13. & (2) \end{cases}$

Solution. Solving (2) for y, $y = -3x + 13.$ (3)

Substituting in (1) the value of y from (3),

$$x^2 + 2x(-3x + 13) = 33. \quad (4)$$

Simplifying (4), $5x^2 - 26x + 33 = 0.$ (5)

Factoring (5), $(x - 3)(5x - 11) = 0.$ (6)

From (6), $x = 3.$ (7)

Also from (6), $x = \frac{11}{5}.$ (8)

Substituting 3 for x in (3), $y = 4,$ (9)

One solution of (1) and (2) is, therefore, (3, 4)

Substituting $\frac{11}{5}$ for x in (3), $y = \frac{32}{5}.$ (10)

The second solution of (1) and (2) is, therefore, $(\frac{11}{5}, \frac{32}{5})$.

Check. Substituting 3 and 4 for x and y, respectively, in (1).

$$9 + 24 = 33, \qquad (11)$$

which is an identity.

Substituting $\frac{11}{5}$ and $\frac{32}{5}$ respectively, in (1)

$$\frac{121}{25} + \frac{704}{25} = 33, \qquad (12)$$

which is an identity.

The values of y were obtained from (3), which is another form of (2); it is, therefore, unnecessary to substitute in (2).

2. Solve the system

$$\begin{cases} 2x^2 - 3\,xy + y^2 - 5\,x + 7\,y - 4 = 0, & (1) \\ 4\,x + 3\,y + 1 = 0. & (2) \end{cases}$$

Solution. Solving (2) for y, $\qquad y = -\dfrac{4\,x + 1}{3} \qquad (3)$

Substituting in (1) the value of y from (3),

$$2\,x^2 + 3\,x\left(\frac{4\,x+1}{3}\right) + \left(\frac{4\,x+1}{3}\right)^2 - 5\,x - 7\left(\frac{4\,x+1}{3}\right) - 4 = 0. \qquad (4)$$

Simplifying (4), $\qquad 5\,x^2 - 8\,x - 4 = 0. \qquad (5)$

Factoring (5), $\qquad (x - 2)(5\,x + 2) = 0. \qquad (6)$

From (6), $\qquad x = 2. \qquad (7)$

Also from (6), $\qquad x = -\frac{2}{5}. \qquad (8)$

Substituting 2 for x in (3), $\qquad y = -3. \qquad (9)$

One solution of (1) and (2) is, therefore, $(2, -3)$.

Substituting $-\frac{2}{5}$ for x in (3), $\qquad y = \frac{1}{5}. \qquad (10)$

The solutions of (1) and (2) are, therefore, $(2, -3)$ and $(-\frac{2}{5}, \frac{1}{5})$, which solutions should be verified by substituting in (1) the values of x and y as found.

EXERCISE 129

Solve the following systems, and check each solution:

1. $\begin{cases} x^2 + y^2 = 5, \\ x = 1. \end{cases}$

2. $\begin{cases} 2\,x^2 + 3\,y^2 = 5, \\ y - x = 0. \end{cases}$

3. $\begin{cases} xy = 8, \\ x + 2\,y = 17. \end{cases}$

4. $\begin{cases} 3\,x^2 + 2\,y^2 = \frac{5}{6}, \\ x - y = \frac{5}{6}. \end{cases}$

5. $\begin{cases} 3\,x^2 + 2\,x + y - 11 = 0, \\ 2\,x - y + 7 = 0. \end{cases}$

6. $\begin{cases} xy = 2, \\ x + y = 3. \end{cases}$

7. $\begin{cases} 5xy + 4 = 0, \\ 5x - 5y = 21. \end{cases}$ 8. $\begin{cases} x^2 - y = y^2 + x. \\ x + y = 13. \end{cases}$

9. $\begin{cases} 2(x-1)^2 + 3(y+1)^2 = 5, \\ (y+1) - (x-1) = 0. \end{cases}$

10. $\begin{cases} \dfrac{1}{xy} = 8, \\ \dfrac{1}{x} + \dfrac{2}{y} = 17. \end{cases}$

11. $\begin{cases} x^2 - 4y^2 = 4, \\ 2x + 3y + 4 = 0. \end{cases}$

12. $\begin{cases} (x-1)^2 + (y-3)^2 = 25, \\ x + 2y = 2. \end{cases}$

13. $\begin{cases} x^2 - y^2 + 3x + 2y - 10 = 0, \\ 2x + 3y - 10 = 0. \end{cases}$

14. $\begin{cases} x^2 - 2xy + y^2 + 2x - 2y = 0, \\ x + y - 2 = 0. \end{cases}$

15. $\begin{cases} 2x^2 - 3xy + 4y^2 - 2x + y - 6 = 0, \\ 5x + 4y - 1 = 0. \end{cases}$

16. $\begin{cases} 3x^2 + 2x - 3y - 2 = 0, \\ 3x + y - 4 = 0. \end{cases}$

273. Two quadratic equations, one of which is homogeneous. When one of the given quadratic equations is homogeneous, the given system can be replaced by two systems each of which is of the type considered in section 272. The method of solution is as follows:

ILLUSTRATIVE EXAMPLE

Solve the system

$$\begin{cases} 2x^2 - xy - 15y^2 = 0, & (1) \\ x^2 + x + 6y - 18 = 0. & (2) \end{cases}$$

Expressing the homogeneous polynomial $2x^2 - xy - 15y^2$ as the product of two linear factors, equation (1) may be replaced by

$$(x - 3y)(2x + 5y) = 0. \qquad (3)$$

Since any solution common to equations (2) and (3) must satisfy either $x - 3y = 0$, or $2x + 5y = 0$, we may consider first those solutions of the given systems which satisfy

$$x - 3y = 0, \qquad (4)$$

and $$x^2 + x + 6y - 18 = 0. \qquad (2)$$

Afterwards those that satisfy $$2x + 5y = 0, \qquad (5)$$

and $$x^2 + x + 6y - 18 = 0. \qquad (2)$$

Substituting in (2) the value of x from (4) and simplifying,

$$y^2 + y - 2 = 0. \qquad (6)$$

Solving (6), $y = 1$; also, $y = -2$.

From (4), when $y = 1$, $x = 3$; when $y = -2$, $x = -6$.

Therefore, the solutions of (4) and (2) are (3, 1) and (– 6, – 2).

Substituting in (2) the value of y from (5) and simplifying,

$$5x^2 - 7x - 90 = 0. \qquad (7)$$

Solving (7), $x = 5$; also $x = -\frac{18}{5}$.

From (5), when $x = 5$, $y = -2$; when $x = -\frac{18}{5}$, $y = \frac{36}{25}$.

Therefore, the solutions of (5) and (2) are $(5, -2)$, and $(-\frac{18}{5}, \frac{36}{25})$.

Therefore, the solutions of the given system are (3, 1), (– 6, –2), (5, – 2), and $(-\frac{18}{5}, \frac{36}{25})$.

EXERCISE 130

Solve the following systems:

1. $\begin{cases} x^2 - y^2 = 0, \\ x^2 + 2y^2 - 3x = 0. \end{cases}$

2. $\begin{cases} x^2 - 4y^2 = 0, \\ 3x^2 - 4y = 16. \end{cases}$

3. $\begin{cases} 3x^2 + 7xy = 0, \\ xy + 7x + 7y + 49 = 0. \end{cases}$

4. $\begin{cases} 2x^2 + y^2 + 4y - 23 = 0, \\ 3x^2 + 8xy - 3y^2 = 0. \end{cases}$

5. $\begin{cases} 2x^2 + 3xy - 5y^2 = 0, \\ y^2 + 4xy - 14x - 24 = 0. \end{cases}$

6. $\begin{cases} 2x^2 + 7xy - 3y^2 = 23, \\ 9x^2 - 12xy + 4y^2 = 0. \end{cases}$

274. Two equations without terms of the first degree. A system of two simultaneous quadratic equations, neither one of which contains terms of the first degree, and in which the constant terms do not reduce to zero, may be solved as in the following:

ILLUSTRATIVE EXAMPLE

Solve the system $\begin{cases} 2x^2 + 2xy + y^2 = 1, & (1) \\ 3x^2 + 7xy + 2y^2 = 2. & (2) \end{cases}$

Multiplying both members of (1) by 2 so that the constant term in the resulting equation shall be the same as that in (2),

$$4x^2 + 4xy + 2y^2 = 2. \qquad (3)$$

From (2) and (3) by subtraction, $x^2 - 3xy = 0.$ (4)

The given system can now be replaced by the equivalent system,

$$x(x - 3y) = 0, \qquad (5)$$
$$2x^2 + 2xy + y^2 = 1. \qquad (1)$$

The system (5) and (1) can be solved by the method of section 273.

Solving, we obtain the following solutions:

$$(0, 1), (0, -1), (\tfrac{3}{5}, \tfrac{1}{5}), (-\tfrac{3}{5}, -\tfrac{1}{5}).$$

Remark. If the constant term in one of the given equations is not a multiple of the other, the equations may be multiplied by such numbers as will make the constant terms equal.

EXERCISE 131

Solve the following systems:

1. $\begin{cases} x^2 + 4xy + 3y^2 = 2, \\ x^2 - 4xy + 3y^2 = 12. \end{cases}$

2. $\begin{cases} x^2 - 3xy + 2y^2 = 3, \\ x^2 + 2xy - 3y^2 = 3. \end{cases}$

3. $\begin{cases} 3x^2 + 11xy + 10y^2 = 6, \\ 2xy + 3y^2 = 3. \end{cases}$

4. $\begin{cases} x^2 + 3y^2 = 21, \\ x^2 + 3xy - 2y^2 = 19. \end{cases}$

5. $\begin{cases} x^2 + 3y^2 = 21, \\ x^2 + 2xy - 3y^2 = 15. \end{cases}$

6. $\begin{cases} 2x^2 - 3xy + y^2 = 3, \\ x^2 + xy + y^2 = 7. \end{cases}$

7. $\begin{cases} x^2 - 3xy + 3y^2 = 12, \\ 2x^2 - xy + 4y^2 = 16. \end{cases}$

8. $\begin{cases} 8x^2 + 11xy + 8y^2 = 12, \\ 15x^2 + 18xy + 12y^2 = 20. \end{cases}$

275. Particular systems of equations.

I. *When the sum and product of two unknowns are given*, either of two special methods of solution may be employed; thus:

Solve the system $\begin{cases} x+y=m, & (1) \\ xy=n. & (2) \end{cases}$

Solution 1. From section 249 the roots of the quadratic $x^2 - mx + n = 0$ are two numbers whose sum and product are equal, respectively, to m and n. The roots of the equation, therefore, satisfy the given system.

Solving the quadratic $x^2 - mx + n = 0$, we have,

$$x = \frac{m \pm \sqrt{m^2 - 4n}}{2}.$$

Therefore the two solutions of (1) and (2) are

$$\left(\frac{m+\sqrt{m^2-4n}}{2}, \frac{m-\sqrt{m^2-4n}}{2}\right) \text{ and } \left(\frac{m-\sqrt{m^2-4n}}{2}, \frac{m+\sqrt{m^2-4n}}{2}\right).$$

Solution 2. Squaring (1) $x^2 + 2xy + y^2 = m^2$. (3)

Multiplying (2) by 4, $4xy = 4n$. (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = m^2 - 4n$. (5)

That is, $(x-y)^2 = m^2 - 4n$. (6)

Therefore, $x - y = \pm\sqrt{m^2 - 4n}$. (7)

Adding (1) and (7), $2x = m \pm \sqrt{m^2 - 4n}$. (8)

Subtracting (7) from (1), $2y = m \mp \sqrt{m^2 - 4n}$. (9)

Therefore, when x is $\frac{m+\sqrt{m^2-4n}}{2}$, y is $\frac{m-\sqrt{m^2-4n}}{2}$,

and when x is $\frac{m-\sqrt{m^2-4n}}{2}$, y is $\frac{m+\sqrt{m^2+4n}}{2}$.

EXERCISE 132

Solve the following systems:

1. $\begin{cases} x + y = 2, \\ xy = 1. \end{cases}$

2. $\begin{cases} x + y = -7, \\ xy = 12. \end{cases}$

3. $\begin{cases} x+y=-3, \\ xy=-28. \end{cases}$

4. $\begin{cases} x+y=\frac{1}{2}, \\ xy=\frac{3}{64}. \end{cases}$

5. $\begin{cases} x+y=-\frac{5}{2}, \\ xy=\frac{3}{2}. \end{cases}$

6. $\begin{cases} x+y=3a, \\ xy=2a^2. \end{cases}$

7. $\begin{cases} x+y=\dfrac{4+9a^2}{6a}. \\ xy=1. \end{cases}$

8. $\begin{cases} x+y=0, \\ xy+a^2=0. \end{cases}$

9. $\begin{cases} x+y=\dfrac{3a+2}{3-2a}, \\ xy=\dfrac{1-5a}{2a-3}. \end{cases}$

10. $\begin{cases} x+y=\dfrac{2a+3b}{2b-3a}, \\ xy=\dfrac{2b-16a}{3a-2b}. \end{cases}$

II. *When the difference and product of two numbers are given*, the following will illustrate a method of solution:

Solve the system $\begin{cases} x-y=1, & (1) \\ xy=12. & (2) \end{cases}$

Solution. Introducing an auxiliary number z, defined by the identity $y=-z$, the given equations may be replaced by

$$x+z=1, \qquad (3)$$
$$xz=-12. \qquad (4)$$

The system (3) and (4) now belongs to class I and may be solved by the methods employed therein, the solutions being

$$x=4,\ z=-3, \text{ and } x=-3 \text{ and } z=4.$$

Substituting $-y$ for z, the solutions of the given system are (4, 3), (– 3, – 4).

EXERCISE 133

Solve the following systems:

1. $\begin{cases} x-y=-5, \\ xy+6=0. \end{cases}$

2. $\begin{cases} x-y=2, \\ xy=3. \end{cases}$

3. $\begin{cases} x-y=\frac{4}{3}, \\ 3xy+1=0. \end{cases}$

4. $\begin{cases} x-y=\frac{11}{10}, \\ xy=\frac{3}{5}. \end{cases}$

5. $\begin{cases} x - y = \frac{34}{15}, \\ xy + 1 = 0. \end{cases}$

6. $\begin{cases} x - y = \frac{17}{6}, \\ xy + 2 = 0. \end{cases}$

7. $\begin{cases} x - y = \dfrac{a^2 + b^2}{ab}, \\ xy + 1 = 0. \end{cases}$

8. $\begin{cases} x - y = \frac{23}{6}, \\ xy = 3. \end{cases}$

9. $\begin{cases} x - y = \dfrac{3a - 2b}{a - b}, \\ xy = \dfrac{3b}{a - b}. \end{cases}$

10. $\begin{cases} x - y = \dfrac{a + 2}{3 - 2a}, \\ xy = \dfrac{2a - 2a^2 + 2a^3}{2a - 3}. \end{cases}$

III. *When the sum of the squares and the product, sum, or difference of two numbers are given.* Typical systems are:

$x^2 + y^2 = 5$	(1)	$x^2 + y^2 = 5$	(1)	$x^2 + y^2 = 5$	(1)
$xy = 2$	(2)	$x + y = 3$	(3)	$x - y = 1$	(4)

Solve the system $\begin{cases} x^2 + y^2 = 5, & (1) \\ xy = 2. & (2) \end{cases}$

Solution.

Multiplying (2) by 2, $2xy = 4.$ (5)

Adding (5) and (1), $x^2 + 2xy + y^2 = 9.$ (6)

Subtracting (5) from (1), $x^2 - 2xy + y^2 = 1.$ (7)

From (6), $x + y = \pm 3.$ (8)

From (7), $x - y = \pm 1.$ (9)

Adding (8) and (9), $2x = \pm 3 \pm 1.$ (10)

Dividing (10) by 2, and combining, $x = 2, 1, -1, -2.$

Subtracting (9) from (8), $2y = \pm 3 \mp 1.$ (11)

Dividing (11) by 2 and combining, $y = 1, 2, -2, -1.$

Therefore, the solutions of the system (1) and (2) are

$$(2, 1), (1, 2), (-1, -2), (-2, -1).$$

Note. In solving either the system $\left\{\begin{matrix} x^2 + y^2 = 5 & (1) \\ x + y = 3 & (3) \end{matrix}\right\}$ or $\left\{\begin{matrix} x^2 + y^2 = 5 & (1) \\ x - y = 1 & (4) \end{matrix}\right\}$, it is evident that if either (3) or (4) is squared and combined with (1), the value of $2xy$ will be obtained; it will be found that

$2xy = 4$, as in equation (5) of the foregoing solution; so that the solution of the system (1) and (3) reduces to the case of the solution of a system such as is given under I, and the solution of the system (1) and (4) reduces to the case of the solution of a system such as is given under II. Observe, however, that the system of equations (1) and (3), or (1) and (4) may be solved by the methods of section 272.

EXERCISE 134

Solve the following systems :

1. $\begin{cases} x^2 + y^2 = \frac{17}{2}, \\ x + y = -1. \end{cases}$

2. $\begin{cases} x^2 + y^2 = \frac{53}{18}, \\ x + y = \frac{2}{3}. \end{cases}$

3. $\begin{cases} x^2 + y^2 = \frac{17}{196}, \\ x + y = \frac{3}{14}. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 130, \\ x - y = 8. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 290, \\ x - y = 16. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 13a^2 + 10a + 2, \\ x - y = a. \end{cases}$

7. $\begin{cases} x^2 + y^2 = \frac{82}{9}, \\ xy = 1. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 8, \\ xy = -4. \end{cases}$

9. $\begin{cases} x^2 + y^2 = \dfrac{25a^2 + 16a + 4}{(2a+1)^2}, \\ xy = -\dfrac{6a}{2a+1}. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 14, \\ xy = 1. \end{cases}$

IV. *When the polynomial in x and y (obtained by omitting the constant term) of one equation is a factor of that in the other equation*, a solution of the system can often be obtained by quadratic equations, even though one of the equations is of a degree higher than the second. The solution of the following system will illustrate :

Solve the system

$$\begin{cases} x^3 + y^3 = \frac{7}{27}, & (1) \\ x + y = \frac{1}{3}. & (2) \end{cases}$$

Solution.

Factoring (1), $(x+y)(x^2-xy+y^2)=\frac{7}{27}$. (3)

Substituting $\frac{1}{3}$ for $x+y$ in (3),

$$\frac{1}{3}(x^2-xy+y^2)=\frac{7}{27}. \quad (4)$$

Simplifying (4), $x^2-xy+y^2=\frac{7}{9}$. (5)

Squaring (2), $x^2+2xy+y^2=\frac{1}{9}$. (6)

Subtracting (5) from (6), $3xy=-\frac{2}{3}$. (7)

Dividing, $xy=-\frac{2}{9}$. (8)

Subtracting (8) from (5), $x^2-2xy+y^2=1$. (9)

Therefore, $x-y=\pm 1$. (10)

Adding (10) and (2) and dividing, $x=\frac{2}{3}$ or $-\frac{1}{3}$. (11)

Substituting in (2) the value of x in (11), $y=-\frac{1}{3}$ or $\frac{2}{3}$. (12)

Therefore, solutions of the given systems are $(\frac{2}{3}, -\frac{1}{3})$, and $(-\frac{1}{3}, \frac{2}{3})$.

A solution similar to the foregoing may be given for each one of the following three systems of equations:

$$\begin{Bmatrix} x^3-y^3=7 \\ x-y=1 \end{Bmatrix}, \quad \begin{Bmatrix} x^4+x^2y^2+y^4=21 \\ x^2+xy+y^2=7 \end{Bmatrix}, \quad \begin{Bmatrix} x^2+x^2y^2+y^4=\frac{21}{16} \\ x^2-xy+y^2=\frac{3}{4} \end{Bmatrix}.$$

EXERCISE 135

Solve the systems:

1. $\begin{cases} x^2-y^2=24, \\ x+y=6. \end{cases}$

2. $\begin{cases} x^2-y^2=-39, \\ x-y=-13. \end{cases}$

3. $\begin{cases} x+y=1, \\ x^3+y^3=\frac{13}{4}. \end{cases}$

4. $\begin{cases} x^3-y^3=19, \\ x-y=1. \end{cases}$

5. $\begin{cases} x+y=2, \\ x^3+y^3=98. \end{cases}$

6. $\begin{cases} x-y=1, \\ x^3-y^3=127. \end{cases}$

7. $\begin{cases} x^4+x^2y^2+y^4=21, \\ x^2+xy+y^2=7. \end{cases}$

8. $\begin{cases} x^4+x^2y^2+y^4=91, \\ x^2-xy+y^2=7. \end{cases}$

EXERCISE 136. — REVIEW

Solve the following systems of equations:

1. $\begin{cases}3x^2-2y^2=1,\\ x+y=0.\end{cases}$

2. $\begin{cases}4x^2+xy=15,\\ x+3y=34.\end{cases}$

3. $\begin{cases}x^2+2x-3y-17=0,\\ x+y+1=0.\end{cases}$

4. $\begin{cases}x^2+xy=6,\\ y^2-xy=0.\end{cases}$

5. $\begin{cases}9x^2+4y^2=2,\\ 6xy=1.\end{cases}$

6. $\begin{cases}y^2-4ax=0,\\ y=mx+\dfrac{a}{m}.\end{cases}$

7. $\begin{cases}9x^2+16y^2=1,\\ 16x^2+9y^2=1.\end{cases}$

8. $\begin{cases}y^2=8x,\\ x^2+y^2=20.\end{cases}$

9. $\begin{cases}x^2-16y=0,\\ x^2-y^2=15.\end{cases}$

10. $\begin{cases}x^2-xy=2x+5,\\ x^2-xy=3y+9.\end{cases}$

11. $\begin{cases}4x^2+6xy+2x-6y+1=0,\\ 2x+y-2=0.\end{cases}$

12. $\begin{cases}y^2+3x+11=0,\\ 3x^2+2y+14=0.\end{cases}$

13. $\begin{cases}x^2+y^2=74,\\ xy=35.\end{cases}$

14. $\begin{cases}2x^2-xy=2,\\ 2y^2-xy=12.\end{cases}$

15. $\begin{cases}x^2+3x+y^2-2y=\frac{85}{36},\\ 3x-2y=2.\end{cases}$

16. $\begin{cases}x^2-y^2-3x+y+2=0,\\ 2x-y=102.\end{cases}$

17. $\begin{cases}2x^2-30y-29x+89=0,\\ y-x=2.\end{cases}$

18. $\begin{cases}3xy+2x-3y-10=0,\\ x^2-8xy+7y^2=0.\end{cases}$

19. $\begin{cases}2x^2-3y+20x-100=0,\\ 2x^2+5xy-7y^2=0.\end{cases}$

20. $\begin{cases} 2x^2 + 2xy - y^2 + 3x - 2y - 15 = 0, \\ 3x - 4y - 2 = 0. \end{cases}$

21. $\begin{cases} 2x^2 + 5xy - 3y + 2x + 36 = 0, \\ x + y = -2. \end{cases}$

22. $\begin{cases} 3x^2 - xy + y^2 + 2x - 3y + 5 = 0, \\ 3x^2 + xy - 2y^2 = 0. \end{cases}$

23. $\begin{cases} y^2 + 2xy + y + 6 = 0, \\ 3x^2 + 11xy + 6y^2 = 0. \end{cases}$

24. $\begin{cases} x^2 - y^2 + 3x - 2y + 237 = 0, \\ 5x^2 - 6xy + y^2 = 0. \end{cases}$

25. $\begin{cases} 2xy + 3x - 4y + 19 = 0, \\ 12x - y = 1. \end{cases}$

26. $\begin{cases} 2x^2 - 3xy - 2y^2 = 3, \\ 3x^2 + 8xy - 3y^2 = 8. \end{cases}$

27. $\begin{cases} x^3 + y^3 = -37, \\ x + y = -1. \end{cases}$

28. $\begin{cases} x^2 + y^2 = a^2, \\ x + y = a\sqrt{2}. \end{cases}$

29. $\begin{cases} x^2 + y^2 = a^2, \\ y = 3x + a\sqrt{10}. \end{cases}$

30. $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \\ y - mx = \sqrt{m^2a^2 + b^2} \end{cases}$

31. $\begin{cases} x^2 + xy = a^2, \\ y^2 + xy = b^2. \end{cases}$

32. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 4a^2 + b^2, \\ xy = \dfrac{-1}{2ab}. \end{cases}$

33. $\begin{cases} x^2 - xy + y^2 = 1, \\ x^3 + y^3 = 2. \end{cases}$

34. $\begin{cases} x^4 + x^2y^2 + y^4 = 91, \\ x^2 - xy + y^2 = 7. \end{cases}$

35. $\begin{cases} \dfrac{x}{y} - \dfrac{y}{x} = \dfrac{5}{6}, \\ x - y = 1. \end{cases}$

36. $\begin{cases} x^4 - y^4 = 15, \\ x^2 + y^2 = 5. \end{cases}$

37. $\begin{cases} 7x^2 + 34xy + 39y^2 + x + 3y + 6 = 0, \\ 33x^2 + 140xy + 147y^2 = 0. \end{cases}$

38. $\begin{cases} 9x^2 - 11xy + 2y^2 = 0, \\ 55x^2 - 54xy - 13y^2 + 5x + y + 6 = 0. \end{cases}$

39. $\begin{cases} 5x^2 + 16xy + 13y^2 = 13, \\ 3x^2 + 8xy + 5y^2 = -5. \end{cases}$

EXERCISE 137

1. The sum of two numbers is 25; the sum of their squares is 457. Find the numbers.

2. The sum of the squares of two numbers is 225, and the difference of their squares is 63. Find the numbers.

3. The ratio of two numbers is $\frac{5}{7}$; their product is 315. Find the numbers.

4. The product of two numbers is 637 and their quotient is 13. Find the numbers.

5. The product of the sum and difference of two numbers is 81 and the quotient of their sum divided by their difference is $\frac{9}{4}$. What are the numbers?

6. Find two numbers whose sum, whose product, and the difference of whose squares are equal.

7. The area of a right triangle is 6 sq. ft.; the hypotenuse is 5 ft. Find the sides.

8. The diagonal of a rectangle is 25 in. If the rectangle were 4 in. shorter and 8 in. wider, the diagonal would still be 25 in. Find the area of the rectangle.

9. Two integers are in the ratio 2 : 3. If each is increased by 5, the difference of their squares becomes 40. What are the integers?

10. The combined perimeters of two squares are 68 in. One square contains 51 sq. in. more than the other. Find the area of each.

11. The difference of two numbers is 5; the sum of their reciprocals is $\frac{13}{36}$. Find the numbers.

12. The difference of the cubes of two numbers is 604 and the sum of the numbers is 14. Find the numbers.

13. The difference of the terms of a certain proper fraction is 8 and the product of this fraction by one whose numerator and denominator exceed the numerator and denominator of the given fraction by 1 and 5, respectively, is $\frac{1}{3}$. Find the fraction.

14. The diagonals of two rectangles are 29 ft. and 5 ft., respectively. The ratio of their bases is 7 to 1 and that of their altitudes 5 to 1. What are the dimensions of the larger rectangle?

15. If the length of a rectangle be increased by 4 and the breadth decreased by 2, the area remains unchanged; if the length be decreased by 4 and the breadth by 2, the area is halved. Find the sides of the rectangle.

16. The perimeter of a right triangle is 30 ft.; its area is 30 sq. ft. Find the sides and the hypotenuse.

Suggestion. Let $x =$ the number of feet in the base.

Let $y =$ the number of feet in the altitude.

Then, $$x + y + \sqrt{x^2 + y^2} = 30, \quad (1)$$

and $$xy = 60. \quad (2)$$

Transposing, $$x + y - 30 = -\sqrt{x^2 + y^2}. \quad (3)$$

Squaring, $$x^2 + y^2 + 900 - 60x - 60y + 2xy = x^2 + y^2. \quad (4)$$

Simplifying, and substituting value of xy from (2),

$$x + y = 17. \quad (5)$$

Now solve the system $\begin{cases} x + y = 17, \\ xy = 60. \end{cases}$

17. The perimeter of a right triangle is 70 ft. and its area is 210 sq. ft. Find the three sides of the triangle.

18. The diagonal of a rectangle is 37 ft. If one side were 4 ft. shorter and the other 2 ft. longer, the area of the rectangle would be 14 sq. ft. greater than the area of the original rectangle. Find the sides of the original rectangle.

19. A page is to have a margin at the sides of $\frac{1}{2}$ in. and one of $\frac{3}{4}$ in. at the top and at the bottom; it is to contain 48 sq. in. of printing. How large must the page be if the length is to exceed the width by $2\frac{1}{2}$ inches?

20. The fore wheel of a carriage makes six revolutions more than the rear wheel in going 120 yd.; if the circumference of each wheel be increased one yard, the fore wheel will make four revolutions more than the rear wheel in going the same distance. Find the circumference of each wheel.

21. The circumference of the rear wheel of a carriage is 2 feet greater than the circumference of the fore wheel. The fore wheel makes 64 more revolutions than the rear wheel in traveling 3496 feet. What is the circumference of each wheel?

22. Three men, A, B, and C, can do a piece of work together in $1\frac{1}{3}$ days. To do the work alone A would take twice as long as C and 2 days longer than B. How long would it take each to do the work?

23. A rectangular box is 8 in. long. Its volume is 192 cu. in. and the area of its six faces is 208 sq. in. Find the other two dimensions of this box.

24. The hypotenuse of a certain right triangle is 10 ft., and its area is 24 sq. ft. Find the base and the altitude of the triangle.

CHAPTER XIV

PROGRESSIONS

276. Series. A succession of numbers that proceed according to a fixed law is called a **series.** The numbers which form the series are called the **terms** of the series.

Thus, the sequence of numbers 1, 3, 5, 7, ⋯ is a series in which the first term is 1, and the second term 3. The law of formation in this series is that any term is obtained from the preceding by the addition of the number 2.

277. Arithmetical progression. A series in which each term is obtained from the preceding by the addition of a constant number is called an **arithmetical progression.**

Thus, $-8, -4, 0, 4, 8, 12$ is an arithmetical progression.

278. Common difference. The constant number obtained by subtracting any term of an arithmetical progression from the next succeeding term is called the **common difference**, or simply the **difference.**

Thus, in the arithmetical progression 2, 5, 8, 11, the common difference is 3.

279. General form. The general form of an arithmetical progression is $a, a + d, a + 2d, a + 3d, \cdots$ in which a represents the first term, and d the difference.

280. The general term. By inspection it is seen that any term of an arithmetical progression is equal to the first term plus a multiple of the difference.

Thus, in the general form it is obvious that the coefficient of d in the second term is 1, in the third term 2, in the fourth term 3. In the tenth term the coefficient of d is 9 and in the nth term it is $n-1$. Hence, if the nth term of an arithmetical progression be denoted by a_n, we have the formula,

$$a_n = a + (n-1)d. \qquad (1)$$

ILLUSTRATIVE EXAMPLES

1. Find the fifteenth term of $2, \frac{5}{2}, 3, \cdots$.

Solution. Here $a = 2$, $d = \frac{1}{2}$, $n = 15$.

Substituting in (1), $a_{15} = 2 + (15-1)\frac{1}{2}$, or 9.

2. Write the first three terms of the series whose twelfth term is 6 and whose thirty-fifth term is $\frac{99}{5}$.

Solution. Here $a_{12} = 6 = a + (12-1)d$, (1)

and $a_{35} = \frac{99}{5} = a + (35-1)d$. (2)

From (1), $a + 11d = 6$. (3)

From (2), $a + 34d = \frac{99}{5}$. (4)

Subtracting (3) from (4), $23d = \frac{69}{5}$. (5)

Dividing, $d = \frac{3}{5}$. (6)

From (3), $a = 6 - 11d = -\frac{3}{5}$. (7)

Therefore, the required terms are $-\frac{3}{5}, 0, \frac{3}{5}$.

Remark. Observe that when any two terms of an arithmetical progression are given, the first term and the common difference can be found.

EXERCISE 138

1. Find the fourth term of $1, 2, 3, \cdots$.

2. Find the fifth term of $2, -2, -6, \cdots$.

3. Find the seventh term of $5, 8, 11, \cdots$.

4. Find the fifth and sixth terms of $-6, -4, -2, \cdots$.

5. Find the fourth and fifth terms of a, $a+b$, $a+2b, \cdots$.

6. Find the fourth and fifth terms of x, $2x+1$, $3x+2$, $\cdots$.

7. Find the eighth term of $2, -1, -4, \cdots$.

8. Find the tenth term of $2, 7, 12, \cdots$.

9. Find the ninth term of $\frac{1}{2}, \frac{3}{4}, 1, \cdots$.

10. Find the tenth term of $-2, 0, 2, \cdots$.

11. Find the ninth term of $2x, 4x, 6x, \cdots$.

12. Find the ninth term of $a+b, 2a, 3a-b, \cdots$.

13. Find the nth term of $1, 3, 5, \cdots$.

14. Find the nth term of $-4, -1, 2, \cdots$.

15. Find the nth term of $2, 0, -2, \cdots$.

16. Find the nth term of $2x, 5x, 8x, \cdots$.

17. Find the nth term of $3a-b, 2a, a+b, \cdots$.

18. The third term of an arithmetical progression is 6 and the eighth term is 16. Find the first term and the common difference.

19. The ninth term of an arithmetical progression is 102 and the twenty-second term is 141. Find the first term and the common difference.

20. The seventh term of an arithmetical progression is $\frac{17}{12}$ and the twenty-fifth term is $\frac{11}{5}$. Find the twelfth term.

21. The thirty-first term of an arithmetical progression is 46 and the forty-ninth term is 73. Find the common difference.

22. The sum of the fifth and twenty-fifth terms of an arithmetical progression is 13, and the forty-ninth term is 15. Find the series.

23. The rth term of the series $5, 8, 11, \cdots$ is equal to the rth term of the series $61, 57, 53, \cdots$. Find r.

281. Arithmetical mean. When three numbers are in arithmetical progression, the second number is called the **arithmetical mean** of the other two.

Thus if a, x, c, are in arithmetical progression, x is the arithmetical mean of a and c.

If x is the arithmetical mean of a and c, x may be found in terms of a and c, thus:

Since $x - a$ and $c - x$ are each expressions for the common difference,

$$x - a = c - x$$

therefore, $x = \frac{a + c}{2}$; that is:

The arithmetical mean of two numbers is half their sum.

282. Arithmetical means. In an arithmetical progression, the terms which stand between two given terms are called **arithmetical means** between the given terms.

ILLUSTRATIVE EXAMPLE

Insert four arithmetical means between 2 and 17.

Solution. $a_n = a + (n - 1)d$.

There will be six terms in all, of which 2 is the first term and 17 is the sixth.

Hence, $a_n = a_6$, which is 17.

Therefore, $17 = 2 + (6 - 1)d$.

Whence, $d = 3$.

The arithmetical progression is 2, 5, 8, 11, 14, 17.

EXERCISE 139

Find the arithmetical mean of:

1. 12 and 16.
2. -4 and -12.
3. a and b.
4. $x + y$ and $x - y$.
5. 1 and a.
6. $2a + b$ and $a + 2b$.
7. $\frac{2}{3}$ and $\frac{3}{4}$.
8. $\frac{1}{a}$ and $\frac{1}{b}$.
9. $\frac{1}{x - y}$ and $\frac{1}{x + y}$.

10. Insert three arithmetical means between 1 and $\frac{7}{3}$.

11. Insert three arithmetical means between 230 and 710.

12. Insert three arithmetical means between $x - y$ and $x + y$.

13. Insert six arithmetical means between $-\frac{7}{10}$ and $\frac{14}{15}$.

283. Sum of an arithmetical series. The sum of n terms of an arithmetical progression may be obtained as follows:

Representing the sum of the first n terms of the arithmetical progression by S_n, we have

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n.$$

or, $$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

$$\therefore 2S_n = (a + a_n) + (a + a_n) + (a + a_n) + + (a + a_n) + (a + a_n) + (a + a_n)$$

$$= n(a + a_n).$$

$$\therefore S_n = \frac{n}{2}(a + a_n).$$

That is, the formula for the sum of the first n terms of an arithmetical progression is

$$S_n = \frac{n}{2}(a + a_n). \qquad (2)$$

Formula (2) may be expressed in words as follows:

The sum of the first n terms of an arithmetical progression is equal to the arithmetical mean of the first and last terms multiplied by the number of terms.

By substituting in formula (2) the value of a from formula (1), section 280, we have a second formula for the sum of the first n terms of an arithmetical progression; namely,

$$S_n = \frac{n}{2}[2a + (n - 1)d]. \qquad (3)$$

Note. To find S_n, when n, a, and a_n are given, use formula (2); when n, a, and d are given, use formula (3); when a_n, a, and d are given, find n by formula (1) and then use formula (2).

ILLUSTRATIVE EXAMPLES

1. Find the sum of the first n natural numbers.

Solution. The required sum $= 1 + 2 + 3 + \cdots + n$

$$= \frac{n(n+1)}{2}.$$

2. Find the sum of $1 + 3 + 5 + 7 + +$

Solution. By formula (3) $S_n = \frac{n}{2}[2 + (n-1)2] = n^2.$

Remark. Observe that the result of example 2 expresses the fact that the sum of the first n consecutive odd numbers is a perfect square, namely, n^2.

EXERCISE 140

1. Find the sum of 15 terms of $-1, 4, 9, \cdots$
2. Find the sum of ten terms of $\frac{2}{3}, -\frac{3}{4}, -\frac{13}{6}, \cdots$
3. Find the sum of 12 terms of $3, \frac{17}{4}, \frac{11}{2}, \cdots$
4. Find the sum of 9 terms of $\frac{\sqrt{3}-1}{2}, \sqrt{3}, \frac{3\sqrt{3}+1}{2} \cdots$
5. Find the sum of n terms of $\frac{n+3}{n}, \frac{2n+3}{2n}, 1 \cdots$
6. Find the sum of m terms of $(a-1)^2$, a^2+1, $(a+1)^2, \cdots$
7. Find the sum of all the even numbers between 1 and 101.
8. Find the sum of all the odd numbers between 0 and 100.
9. Find the sum of all the multiples of 5 between 1 and 1001.

10. Find the sum of all the multiples of 7 between 1 and 344.

11. The 12th term of an arithmetical progression is 35, the sum of the first 12 terms is 222. Find the series.

12. The first term of an arithmetical progression is a, the last term is l, and the common difference is 1. Show that $n = l - a + 1$.

13. Find the ratio of the sum of the first n natural numbers to the sum of the first n even numbers beginning with 2.

14. Show that if each term of an arithmetical progression be multiplied or divided by the same number (zero excepted), the result will be in arithmetical progression.

15. A body falls 16.08 feet in the first second, three times as far in the next second, five times as far in the third second, and so on. How far does it fall in 10 seconds?

284. Geometric progression. A series of numbers in which the ratio of each number to the preceding is constant is called a **geometric progression**. The constant ratio is called the **common ratio** of the progression.

Thus, the series of numbers, 3, 6, 12, 24, form a geometric progression of four terms, since $\frac{6}{3} = \frac{12}{6} = \frac{24}{12}$.

285. Increasing progression. The successive terms of a geometric progression increase in absolute value if the ratio is numerically greater than 1, and the progression is said to be an **increasing geometric progression.**

Thus 1, 3, 9, 27, 81 is an increasing geometric progression.

286. Decreasing progression. The successive terms of a geometric progression decrease in absolute value if the ratio is numerically less than 1, and the progression is said to be a **decreasing geometric progression**.

Thus, 128, 64, 32, 16, 8 is a decreasing geometric progression.

287. Form of a geometric progression. From the definition, the general form of a geometric progression is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \cdots$$

in which a represents the first term and r the common ratio of the progression.

288. General term. An inspection of the foregoing general form shows that any term of a geometric progression is obtained by multiplying the first term by a power of the ratio.

Since the exponent of r in the second term is 1, in the third term 2, in the fourth term 3, it is evident that in the tenth term the exponent is 9, and in the general, or nth term, it is $n-1$. Hence, representing the nth term by a_n, we have

$$a_n = ar^{n-1}. \qquad (1)$$

ILLUSTRATIVE EXAMPLE

The third term of a geometric progression is $\frac{4}{27}$ and the sixth term is $\frac{32}{729}$. Find the series.

Solution. Here $a_3 = ar^2 = \frac{4}{27}$,

and $a_6 = ar^5 = \frac{32}{729}$.

Hence, $\frac{ar^5}{ar^2} = \frac{8}{27}$,

or, $r^3 = \frac{8}{27}$,

whence, when r is real, $r = \frac{2}{3}$.

From $ar^2 = \frac{4}{27}$,

we have $\frac{4}{9}a = \frac{4}{27}$, or $a = \frac{1}{3}$.

Therefore, the series is $\frac{1}{3}, \frac{1}{3} \times \frac{2}{3}, \frac{1}{3} \times (\frac{2}{3})^2, \cdots$; that is, $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \cdots$.

Remark. Observe that when any two terms of a geometric progression are given, the first term and the common ratio can be found.

EXERCISE 141

1. Find the fifth term of $1, 2, 4, \cdots$.

2. Find the sixth term of $3, \frac{3}{2}, \frac{3}{4}, \cdots$.

3. Find the sixth term of $1, 5, 25, \cdots$.

4. Find the fifth term of $2, -4, 8, \cdots$.

5. Find the sixth term of $a, ab, ab^2, \cdots$.

6. Find the fifth term of $a, 2a^3, 4a^5, \cdots$.

7. Find the fifth term of $m^2, m(m-1), (m-1)^2, \cdots$.

8. Find the fifth term of $1, \sqrt{2}, 2, \cdots$.

9. Find the fifth term of $\frac{\sqrt{2}-1}{2}, \sqrt{2}, 4(1+\sqrt{2}), \cdots$.

10. Find the nth term of $3, 9, 27, \cdots$.

11. Find the nth term of $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

12. Find the nth term of $a, \sqrt{a}, 1, \cdots$.

13. The fourth term of a geometric progression is 24 and the sixth term is 96. Find the ratio and the first term.

14. The third term of a geometric progression is 16 and the seventh term is $\frac{1}{16}$. Find the tenth term.

15. The sum of the first and fourth terms of a geometric progression is 56 and the sum of the second and third terms is 24. Find the series.

16. The sum of three numbers in a geometric progression is 14 and the sum of their squares is 84. Find the numbers.

17. Each stroke of a certain air pump exhausts one sixteenth of the air in the receiver. How much of the air originally in the receiver is removed in six strokes?

289. Geometric mean. When three numbers are in geometric progression, the second number is called the **geometric mean** of the other two.

Thus, if a, x, b are in geometric progression, x is the geometric mean of a and b.

If x is the geometric mean of a and b, x may be found in terms of a and b thus:

Since $\frac{x}{a} = r$ and $\frac{b}{x} = r$,

we have $$\frac{x}{a} = \frac{b}{x}.$$

Therefore, $$x^2 = ab,$$

and $x = \sqrt{ab}$; that is:

The geometric mean of two numbers is the square root of their product.

Remark. It is evident that the geometric mean of two numbers is the mean proportional between these numbers [§ 175].

290. Geometric means. The terms which stand between any two given terms of a geometric progression are called the geometric means between the given terms.

Thus, in the geometric progression 2, 4, 8, 16, 32, the geometric means between 2 and 32 are 4, 8, and 16.

ILLUSTRATIVE EXAMPLE

Insert three real geometric means between 32 and 2.

Solution. $$a_n = ar^{n-1}.$$

There are five terms in all, of which 32 is the first term and 2 the fifth term.

Here $$a_n = a_5 = 2 = ar^4.$$

$$\therefore 32r^4 = 2.$$

Whence, $$r^4 = \frac{1}{16}, \text{ and } r = \pm\frac{1}{2} \text{ and } \pm\frac{i}{2}.$$

That is, the two real values of r are $\frac{1}{2}$ and $-\frac{1}{2}$.

Corresponding to the real values of r, we have the progressions

$$32, 16, 8, 4, 2, \text{ and } 32, -16, 8, -4, 2.$$

The required means are, therefore, either

$$16, 8, \text{ and } 4 \text{ or } -16, 8, \text{ and } -4.$$

EXERCISE 142

Find the positive geometric mean between:

1. 3 and 48.

2. $\frac{1}{16}$ and $\frac{1}{64}$.

3. $\frac{a}{b}$ and $\frac{b}{a}$.

4. $(x-y)^2$ and $(x^2-y^2)^2$.

5. $3a$ and $27a^3$.

6. a and a^2.

Insert three positive geometric means between:

7. 4 and 64.

8. 48 and 243.

9. $(x-y)$ and $(x^2-y^2)(x+y)^3$.

10. $\frac{a}{b}$ and $\frac{b^3}{a^3}$.

11. The sum of three numbers in geometric progression is 117; the mean is equal to three tenths of the sum of the other numbers. Find the numbers.

291. Sum of a geometric series. Let S_n represent the sum of n terms of a geometric progression; then,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying (1) by r,

$$r S_n = \quad ar + ar^2 + \cdots \quad + ar^{n-2} + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (2) from (1),

$$(1 - r)S_n = a - ar^n.$$

Dividing,

$$S_n = \frac{a - ar^n}{1 - r}.$$

That is, the formula for the sum of n terms of a geometric progression is

$$S_n = \frac{a - ar^n}{1 - r}. \tag{2}$$

In an increasing geometric progression the formula obtained by changing the signs of the terms in formula (2) should be employed; the formula is,

$$S_n = \frac{ar^n - a}{r - 1}. \tag{3}$$

ILLUSTRATIVE EXAMPLES

1. Find the sum of eight terms of 2, 6, 18, ⋯.

Solution. Here $a = 2$, $r = 3$, $n = 8$.

We use the formula $S_n = \frac{ar^n - a}{r - 1}$.

Substituting in the formula,

$$S_8 = \frac{2 \cdot 3^8 - 2}{2} = 3^8 - 1 = 6560.$$

2. The sum of the terms of a geometric progression is 728, the ratio is 3, and the last term is 486. Find the first term and the number of terms.

Solution. Since $a_n = ar^{n-1}$,

$$ra_n = ar^n,$$

and the formula $S_n = \frac{ar^n - a}{r - 1}$

may be written, $S_n = \frac{ra_n - a}{r - 1}$.

Here, $S_n = 728$, $r = 3$, and $a_n = 486$.

Substituting, $728 = \frac{3 \times 486 - a}{3 - 1}$,

whence, $a = 2$.

From $a_n = ar^{n-1}$,

we have $486 = 2 \times 3^{n-1}$,

or, $3^5 = 3^{n-1}$.

Hence, $5 = n - 1$,

or, $n = 6$.

EXERCISE 143

1. Find the sum of 2, 6, 18, $\cdots$ to 6 terms.

2. Find the sum of 4, 2, 1, $\cdots$ to 8 terms.

3. Find the sum of $-3, 9, -27, \cdots$ to 7 terms.

4. Find the sum of $\sqrt{3}$, $3+\sqrt{3}$, $6+4\sqrt{3}$, $\cdots$ to 5 terms.

5. Find the sum of 1, 2, 4, $\cdots$ to n terms.

6. Find the sum of $1, \frac{1}{3}, \frac{1}{9}, \cdots$ to n terms.

7. Find the sum of $a\sqrt{b}, b\sqrt{a}, b\sqrt{b}, \cdots$ to n terms.

8. Find the sum of $3, -6, 12, \cdots$ to $2m+1$ terms.

9. If the sum of the series $1+4+16+\cdots$ is 5461, find the number of terms.

10. The first term of a geometric progression is 1, the last term is 81, and the sum of the series is 121. Find the ratio.

11. Find two numbers whose sum is 52, such that their arithmetical mean exceeds their geometric mean by 2.

12. The first term of a geometric progression is 5, the ratio is 4, and the number of terms is 5. Find the sum of the terms.

13. The first four terms of a geometric progression are the same as the first four terms of a second geometric progression, but in reverse order; the sum of the first eight terms of one is equal to 81 times the sum of the first eight terms of the other. Find the common ratio of each.

292. Infinite geometric series. The terms of a geometric progression in which r is positive and numerically less than 1, become smaller and smaller. By taking n

sufficiently large, we can make the nth term as small as we desire; that is, make it more and more nearly equal to zero.

By this statement we mean that however small a number we please to mention, we can find a term such that it and each one of the succeeding terms of the series is numerically less than the number mentioned.

Thus, in the series 100, 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, ⋯ the successive terms are becoming smaller and smaller. When n is 10, a_n is $\frac{1}{10000000}$, which is a small number; and when n is 103, $a_n = \frac{1}{10^{100}}$, which is an exceedingly small number.

From illustrative example 2, section 291,

$$S_n = \frac{a - ra_n}{1 - r},$$

which may be written, $S_n = \frac{a}{1-r} - a_n\left(\frac{r}{1-r}\right)$.

The term $\frac{a}{1-r}$ is a constant, and the term $a_n\left(\frac{r}{1-r}\right)$ varies with n and becomes smaller and smaller. Hence, as more and more terms of the series are added, S_n differs less and less from $\frac{a}{1-r}$.

The number $\frac{a}{1-r}$ is called the *limit* of the sum of n terms, as n increases without limit.

For convenience we shall call this limit the *sum to infinity* of a decreasing geometric progression, and shall denote it by the symbol S_∞. We therefore have the following identity:

$$\mathbf{S_\infty = \frac{a}{1-r}}.$$

293. Recurring decimal. A decimal in which a figure or set of figures repeats in a certain fixed order is called a **recurring decimal**, or a **repeating**, or **circulating** decimal.

Recurring decimals are illustrations of infinite geometric series: Thus, each of the following is an infinite geometric series:

.6666 ⋯, sometimes written $.\dot{6}$, in which $a = .6$ and $r = .1$.

.3434 ⋯, sometimes written $.\dot{3}\dot{4}$, in which $a = .34$ and $r = .01$.

.304304 ⋯, sometimes written $.\dot{3}0\dot{4}$, in which $a = .304$ and $r = .001$.

ILLUSTRATIVE EXAMPLES

1. Sum to infinity the series $2, 1, \frac{1}{2}, \frac{1}{4}, \cdots$

Solution. Here $a = 2$ and $r = \frac{1}{2}$.

$$S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 4.$$

2. Sum to infinity the series $6, -3, 1\frac{1}{2}, -\frac{3}{4} \cdots$.

Solution. Here $a = 6$ and $r = -\frac{1}{2}$.

$$S_\infty = \frac{a}{1-r} = \frac{6}{1-(-\frac{1}{2})} = \frac{6}{1\frac{1}{2}} = 4.$$

3. Sum to infinity the series $a, \frac{ax^2}{x^2+1}, \frac{ax^4}{(x^2+1)^2}, \cdots$.

Solution. Here $a = a$, and $r = \frac{x^2}{x^2+1}$.

$$S_\infty = \frac{a}{1-r} = \frac{a}{1-\frac{x^2}{x^2+1}} = a(1+x^2).$$

4. Sum to infinity the series .666 ⋯.

Solution. Here $a = .6$ and $r = .1$.

$$S_\infty = \frac{a}{1-r} = \frac{.6}{1-.1} = \frac{.6}{.9} = \frac{2}{3}.$$

5. Sum to infinity the series .24545 ⋯.

Solution. .24545 ⋯ = .2 + .04545 ⋯, in which .2 is not a part of the series.

Here $a = .045$ and $r = .01$.

$$S_\infty = \frac{a}{1-r} = \frac{.045}{1-.01} = \frac{.045}{.99} = \frac{5}{110}.$$

$$\therefore .2\dot{4}\dot{5} = \frac{2}{10} + \frac{5}{110}, \text{ or } \frac{27}{110}.$$

EXERCISE 144

1. Sum to infinity the series $1, \frac{1}{2}, \frac{1}{4}, \cdots$.
2. Sum to infinity the series $100, 10, 1, \cdots$.
3. Sum to infinity the series $5, 1, \frac{1}{5}, \cdots$.
4. Sum to infinity the series $1, \frac{1}{3}, \frac{1}{9}, \cdots$.
5. Sum to infinity the series $6, \frac{3}{4}, \frac{3}{32}, \cdots$.
6. Sum to infinity the series $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \cdots$.
7. Sum to infinity the series $2.5, 1.25, .625, \cdots$.
8. Sum to infinity the series $18, 12, 8, \cdots$.
9. Sum to infinity the series $3.5, .35, .035, \cdots$.

Sum to infinity the following:

10. $.\dot{5}$.   11. $.\dot{5}\dot{4}$.   12. $.\dot{0}\dot{1}$.

13. $.\dot{1}3\dot{5}$.   14. $.2\dot{4}$.   15. $.4\dot{3}\dot{4}$.

16. Show that $4 + \frac{8}{3} + \frac{16}{9} + \cdots = 3 + \frac{9}{4} + \frac{27}{16} + \cdots$.

17. An elastic ball bounces to three fourths the height from which it falls. If it is thrown up from the ground to a height of 20 feet, find the total distance traveled before it comes to rest.

18. A heavy iron ball at the end of a chain is pulled to the right 1 yard out of the vertical and is then released. It swings to a point 0.9 of a yard to the left of the vertical, then to a point 0.9 of a yard to the right of the vertical. The succeeding swings follow the same law. Including the first movement, find the greatest distance the ball could travel before coming to rest.

19. Show that before the ball mentioned in example 18 passes through the vertical for the seventh time after being withdrawn, it has moved more than half its total movement.

CHAPTER XV

GENERAL REVIEW

1. Evaluate $\frac{3\,ab}{c} + \frac{5\,bc}{a} - \frac{3\,ac}{b}$ when $a = 3$, $b = 4$, and $c = 1$.

2. Evaluate $\frac{x^3 - 3\,x^2 + 7\,x + 5}{2\,x^2 - 3\,x + 1}$ when $x = 2$.

3. Evaluate $\frac{(p-q)^2 + (r+s)^2}{p+q+r+s} + \frac{(p+q)^2 + (r-s)^2}{p-q+r-s}$ when $p = 3$, $q = 2$, $r = 1$, and $s = \frac{1}{2}$.

4. Write a formula for the area of a parallelogram. Find the area of a parallelogram whose base is 10 inches and whose altitude is 7 inches.

5. Write a formula for each of two numbers whose sum and difference are known. Find two numbers whose sum is 126 and whose difference is 32.

6. Write a formula to find the weight of a bag containing any given number (n) of bushels of grain, given the weight of the bag and the weight of a single bushel of grain.

7. Write a formula for the area of the wall of a room of length L and width W containing two windows each of length l and width w.

8. Write a formula to find the number of square feet in the four walls of a room, l ft. long, w ft. wide, and h ft. high.

Find the sum of the expressions in examples 9–12.

9. $3x^2 + 2xy - 4y^2 + 18z^2$, $2x^2 - 5xy + 7y^2 - 13z^2$, and $-3x^2 + 2xy - 7y^2 - 5z^2$.

10. $8mn^2 - 7m^2n + 2n^3$, $5m^3 - 2m^2n - 11mn^2$, and $m^3 - 2n^3$.

11. $(p+q)a$, $(q+r)a$, and $(r-q)a$.

12. $\frac{3}{2}x - \frac{3}{4}y + \frac{2}{3}z$, $\frac{1}{2}x + \frac{2}{3}y - \frac{5}{6}z$, and $2x - \frac{3}{4}y - \frac{5}{12}z$.

13. Simplify $3(a - b + c + 2d) - 5(a - 2b + 3c - 3d) + 4(a - 3b + 2c + 4d) - 2(a - 7b - 2c - 7d)$.

14. Evaluate $(3x + 2a)^2 - (2x + 5a)^2$ when $x = 7a$.

15. Evaluate $5(2m - 3n)(3m - 2n) - (2n + 3m)(3n + 2m)$ when $m = 3n$.

16. Simplify $2x - (-3y + z - \{x - y\}) - (3x + 2z - [-2y + 3z])$.

17. Simplify $-\{-[-(-a + b - c)] - c\} - \{-[-(-a + 3b - 4c) + a] - b\}$.

18. From the sum of $3ab - 2xy + 4$ and $2ab - 3xy + 3$ take the difference between $4ab + 3xy - 2$ and $5ab - 2xy$.

19. Add $5p - (3q - 2r)$ and $-(3q - 6p) - 10p$; from the sum subtract $-4p - (3r + q)$.

20. State what value of x will make the expression $3(x + 3) - 2(2x - 3)$ equal to twice the value of x.

21. What number is as much greater than 30 as it is less than 74?

22. Find a number to which if 10 be added the result is equal to 6 times the number.

23. A man, who rode a motorcycle at the rate of m miles an hour, completed a journey from P to Q in h hours, during r of which he rested. Find an expression for the distance from P to Q.

24. Multiply $5x^3 - 2x^2 + 7x - 11$ by $3x^2 + 7x - 3$.

25. Simplify $(ax + by)(cx - dy) - (cx - by)(ax + dy)$.

26. Multiply $5(x + y)^2 - 3(x + y) - 2$ by $3(x + y)$.

27. Multiply $.3x^2 + 1.2x + 1$ by $.5x^2 - 1$.

28. Multiply $x^m + 2x^{m-1} + 3x^{m-2}$ by $2x - 3$.

29. Multiply $a^{a+1} - 2a^a + 3$ by $2a^a - 3$. Verify your result by putting $a = 2$.

30. Simplify $5a - \{3a - [2b(p + q) - 3b(p - q)]\}$.

31. Prove by actual multiplication that
$(a + b + c)^2 + a^2 + b^2 + c^2 = (b + c)^2 + (c + a)^2 + (a + b)^2$.

32. Show that if $x = 1 + a$, $y = 1 + b$, $z = 1 + c$, then
$x^2 + y^2 + z^2 - yz - zx - xy = a^2 + b^2 + c^2 - bc - ca - ab$.

33. Prove by actual multiplication that
$(a^2 - bc)^2 - (b^2 - ca)(c^2 - ab)$ is equal to

$$a(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab).$$

34. Arrange the expression $x^2(x - 1) + (x^2 - 1)^2 - mx(x^2 + 3)$ in descending powers of x.

35. Divide $4a^3 + ab^2 - b^3$ by $a - \frac{b}{2}$.

36. Divide $a^{3p} + b^{3p}$ by $a^p + b^p$.

37. Divide $x^9 + 8x^6 - 192x^3 - 256x + 1024$ by $x^5 - 4x^3 - 16x + 32$.

38. Divide $(x + 5y)^3 - (y + 4z)^3$ by $x + 4(y - z)$.

39. Divide $6x^6 - 9x^5 + 22x^3 - 4x^4 - 4x - 15x^2 + 10$ by $3x^3 - 2x + 5$.

40. Divide $a^{m+n}b^n - 5a^{m+n-1}b^{2n} - 3a^{m+n-2}b^{3n} + 15a^{m+n-3}b^{4n}$ by $a^nb^n - 5a^{n-1}b^{2n}$.

41. Factor $z^4 - 8z^2 - 9$.

42. Factor $(m + 2)^4 - 9(m + 2)^2 + 20$.

43. Factor $a^6 - b^3$.

44. Factor $ax^3 + bx^2 + a - b$.

45. Factor $a^2 + 2ab + 3ac + 6bc$.

46. Factor $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2$.

47. Factor $(x^3 + 3x)^2 - (3x^2 + 1)^2$.

48. Factor $\frac{64a^3}{b^6} - \frac{b^3}{27}$.

49. Factor $px^2 + 2px - 3p + qx^2 + 2qx - 3q$.

50. Factor $3(x - 3) + x(x - 3)(3x + 3)$.

51. Factor $a^4 - 2a^2b^2 + b^4 - a^2 + 2ab - b^2$.

52. Prove that $(x^2 + xy + y^2)^2 - (x^2 + xy - y^2)^2 = 4xy^2(x + y)$.

53. Factor $p^3 - 3p^2 + 4$.

54. Factor $x^3 - x^2y - xy^2 + y^3$.

55. Factor $p^2 - 6pq - 16a^2b^2 + 9q^2$.

56. Factor $(x^2 - 7x)^2 + 18(x^2 - 7x) + 72$.

57. Factor $x^{16} + m^6xy^9$.

58. Factor $3(x + 1)^3 + 4(x + 1)^2 + x + 1$.

59. Express $(5x - 6)(5x + 6) - 4y(10x - 4y)$ as the difference of two squares.

60. Resolve $x^4 - 13x^2 + 36$ into linear factors.

61. Show that

$$(9x - 10)^2 - 2(7x - 10)^2 = (x - 10)^2 - 2(3x - 10)^2.$$

62. Find the H.C.F. and L.C.M. of $x^3 + 1$ and $2x^2 - x - 3$.

63. Find the H.C.F. and L.C.M. of $x^3 + 2x^2 + 2x + 1$ and $x^3 - 2x^2 + 2x - 1$.

64. Reduce to lowest terms: $\frac{a^2 - c^2 + b^2 + 2ab}{a^2 - c^2 - b^2 - 2bc}$.

65. Simplify $$\frac{\frac{a}{b^2} + \frac{b}{a^2}}{\frac{1}{a^2} - \frac{1}{ab} + \frac{1}{b^2}}.$$

66. Simplify

$$\frac{x+5}{(2-x)(3-x)}+\frac{x+2}{(x-3)(x-5)}+\frac{x+3}{(x-2)(x-5)}.$$

67. Simplify

$$\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}-\frac{1}{(a-c)(b-a)}.$$

68. Simplify

$$\left\{\frac{x+y}{x-y}+\frac{x-y}{x+y}-\frac{2x^2}{x^2-y^2}\right\}\div\left\{\frac{x}{x+y}+\frac{y}{x-y}\right\}.$$

69. Simplify $\left\{5-\dfrac{a^2-19b^2}{a^2-4b^2}\right\}\div\left\{3+\dfrac{a-5b}{2b-a}\right\}.$

70. Simplify $\left(\dfrac{a^2}{b}+\dfrac{b^2}{a}\right)\left(\dfrac{1}{b^2-a^2}\right)-\dfrac{b}{a^2+ab}+\dfrac{a}{ab-b^2}.$

71. Simplify $\left(3x-5-\dfrac{2}{x}\right)\left(3x+5-\dfrac{2}{x}\right)\div\left(x-\dfrac{4}{x}\right).$

72. Simplify

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}.$$

73. Simplify $\dfrac{x+4}{x^2+5x+6}-2\left(1-\dfrac{2}{x+2}\right)+\dfrac{x+1}{x^3+27}.$

74. Simplify $\dfrac{2}{3a}-\dfrac{1}{2b}-\dfrac{2a+3}{6a^2}+\dfrac{1}{2a}+\dfrac{3a-2b}{6ab}.$

75. Simplify $\dfrac{abc-a^2b}{a^2bc-ab^2c}\times\dfrac{abc-b^2c}{ab^2c-abc^2}\times\dfrac{abc-c^2a}{abc^2-a^2bc}.$

76. Simplify $\left(m^2+\dfrac{n^4}{m^2-n^2}\right)(m^2+n^2)\div\left(\dfrac{m}{m+n}+\dfrac{n}{m-n}\right).$

77. Evaluate $ax+by+c$ when $x=\dfrac{mc-nb}{ma-pb}$ and $y=\dfrac{pc-na}{pb-ma}.$

78. Simplify $\dfrac{x-3-\dfrac{1}{x-3}}{x-3-\dfrac{4}{x-6}} \times \dfrac{x-5-\dfrac{4}{x-5}}{x-5-\dfrac{1}{x-5}}.$

79. Simplify $\dfrac{a^2-ab}{a^2-ac} \times \dfrac{b^2-bc}{b^2-ba} \times \dfrac{c^2-ca}{c^2-cb}.$

80. Simplify $\dfrac{a-b-\dfrac{2b(a-b)}{a+b}}{\dfrac{a^2+b^2}{ab+b^2}-1}.$

81. Show that $\dfrac{Aa-Bb}{a+b}+\dfrac{Ab+Ba}{a-b}=(A+B)\dfrac{a^2+b^2}{a^2-b^2}.$

82. Simplify $\left(\dfrac{2x}{2x+y}-\dfrac{y}{2x-y}+\dfrac{8x^2}{y^2-4x^2}\right)\div\dfrac{4x-y}{(2x-y)^2}.$

83. Solve the equation $\dfrac{2x+2}{2x-3}+\dfrac{x+1}{x-4}=2.$

84. Solve the equation $\dfrac{3}{3x-4}=\dfrac{5+4x}{6x-1}-\dfrac{2x}{3x-4}.$

85. Solve the equation $\dfrac{1}{m}-\dfrac{1}{x}=\dfrac{1}{x}-\dfrac{1}{n}-\dfrac{1}{p}.$

86. Solve the equation $\dfrac{3x}{x+2}-\dfrac{x}{x+1}=2.$

87. Solve the system $\begin{cases}\dfrac{x+y}{3}-\dfrac{x-y}{4}=11,\\[2ex] \dfrac{x+y}{2}-\dfrac{x-y}{3}=8.\end{cases}$

88. Solve the system $\begin{cases}2xy+3x=6,\\ 3xy+5x=8.\end{cases}$

89. Solve graphically the system $\begin{cases}2x-y=4,\\ 2x+3y=12.\end{cases}$

90. Divide m into two parts, one of which shall exceed the other by n.

91. The difference of the squares of two consecutive odd numbers is 96. Find the numbers.

92. Solve the system $\frac{x-5}{5y}=-2,\ x+7y=8.$

93. Solve the system

$$\frac{2}{x-2}+\frac{3}{y+3}=8,\quad \frac{3}{x-2}+\frac{4}{y+3}=11.$$

94. Solve the system

$$\frac{3}{2x+y}+\frac{5}{x-3y}=11,\quad \frac{2}{2x+y}-\frac{3}{x-3y}=1.$$

95. Solve the system

$$\begin{cases} 4x+8y-3z=6, \\ 3x+\ \ y-\ \ z=7, \\ 4y-5x+4z=8. \end{cases}$$

96. Solve the system

$$\frac{x}{3+y}=\frac{3}{7},\quad \frac{y}{4+z}=\frac{4}{9},\quad \frac{z}{5+x}=\frac{5}{8}.$$

97. Solve the system

$$\begin{cases} \frac{2}{x}-\frac{3}{y}+\frac{4}{z}=8, \\ \frac{1}{x}+\frac{2}{y}+\frac{3}{z}=-4, \\ \frac{3}{x}-\frac{2}{y}+\frac{1}{z}=12. \end{cases}$$

98. Solve the system

$$\begin{cases} \frac{2y-x}{4}-\frac{7+y}{5}=5-3x, \\ \frac{5x-7}{2}+\frac{4y-3}{6}=18-5y. \end{cases}$$

99. Solve for x and y: $\frac{x}{3a}+\frac{y}{2b}=\frac{5}{6}$, $\frac{x}{a}+\frac{y}{b}=2$.

100. A mixture of corn and oats contains $33\frac{1}{3}$ % of oats by weight. How many pounds of corn must be added to 100 lb. of the mixture so that the resulting mixture shall contain only 20 % of oats?

101. A dealer bought 2000 lemons, some of them at the rate of $1\frac{1}{2}$ ct. apiece, and the remainder at the rate of 2 ct. apiece. He sold them all at the rate of 27 ct. per dozen and gained $7.50. How many did he buy at each price?

102. If 7 is added to twice a certain number, the sum is 13. Find the number.

103. If one half of a certain number is added to itself, the sum is 3 less than twice the number. Find the number.

104. Twice a certain number is 9 less than 5 times the number. What is the number?

105. One number is 3 times a second number; the sum of the two numbers is 6 greater than twice the smaller number. What are the numbers?

106. The sum of two numbers is 50; one of the numbers is 5 less than 4 times the other. What are the numbers?

107. The difference between two numbers is 37. The smaller number plus 3 times the larger equals 163. Find the numbers.

108. The sum of two numbers is 60; one number is 17 less than 6 times the other. Find the numbers.

109. One number exceeds another by 30; the smaller is 3 greater than one half of the larger. Find the numbers.

110. One number exceeds another by 101; if 3 times the smaller is added to the greater, the result is 201. Find the numbers.

111. A's share of a business is twice that of his partner B; they sell the business for $12,000. How much should each receive?

112. The sum of two numbers is 280, and their difference is equal to one fourth of the greater. Find the numbers.

113. A house and a garage cost $7000, and twice the cost of the house was equal to five times the cost of the garage. Find the cost of each.

114. A number is composed of two digits; the digit in the tens' place is one less than twice that in the units' place. If 27 is subtracted from the number, the remainder is composed of the same two digits in reversed order. Find the number.

115. The difference between the squares of two consecutive numbers is 13. Find the numbers.

116. If A can perform a piece of work in 3 days, and B in 5 days, in what time should they perform it working together?

117. If a man and 2 boys can do a piece of work in 5 days and the man working alone can do it in 12 days, in what time can one boy working alone do the work, providing the boys do equal amounts?

118. The sum of the two digits of a number is 14; if the order of the digits is reversed, the number is diminished by 18. Find the number.

119. A person has just ten hours at his disposal; how far may he ride at the rate of ten miles an hour, so as to return home on time, walking back at the rate of 4 miles an hour?

120. A train travels from Philadelphia to New York in 2 hours; if it had traveled 15 miles an hour slower, it would have taken one hour longer. Find the distance from Philadelphia to New York.

121. A number of workmen, who receive the same wages, earn together a certain sum. Had there been 6 more workmen, and had each received 10 cents more, their joint earnings would have increased by $19.60. Had there been 3 fewer workmen and had each received 10 cents less, their joint earnings would have decreased by $9.70. How many workmen are there and how much does each receive?

122. The total number of boys and girls attending a certain boarding school is 95. If the number of boys were 30% less and the number of girls 20% more, there would be as many girls in attendance as boys. How many of each are there in attendance?

123. A man has $2.20 in nickels, dimes, and quarters, 15 coins in all. If the number of nickels and quarters were interchanged, he would have $1.80. How many of each has he?

124. A quantity of wheat sufficient to fill three bins of different sizes will fill the smallest bin four times, the second bin three times, or the largest bin twice with 40 bu. to spare. What is the capacity of each bin?

125. A wholesale egg dealer sold on the average 3800 dozen eggs a day for cash. He reduced his price 5%, and found that his average daily cash receipts from sales were increased 10%. How many dozen eggs did he sell daily at the reduced prices?

126. A man has $5000 which he wishes to invest in two enterprises so that his total income will be $180; if one enterprise pays 4% and the other 3%, how much must he invest in each?

127. The circumference of the rear wheel of a carriage is $3\frac{1}{2}$ ft. greater than the circumference of the front wheel. The front wheel makes 98 more revolutions than the rear wheel in traveling 5600 ft. What is the circumference of each wheel?

128. The sum of two fractions is $\frac{22}{15}$ and their difference is $\frac{2}{15}$. What are the fractions?

129. Find a fourth proportional to a^4, ab^2, $6\,a^3b$.

130. Find a mean proportional between $32\,a^2x^3$ and $2\,a^4x$.

131. Two numbers are in the ratio of $m:n$. If c be added to the first and subtracted from the second, the results will be in the ratio of 4 : 5. Find the numbers.

132. What number must be subtracted from each of the numbers 6, 9, 15, and 27, so that the resulting differences shall form a proportion when taken in the given order?

133. If $\frac{a-b}{b-c}=\frac{b}{c}$, prove that b is a mean proportional between a and c.

134. Two numbers have the ratio of 7 : 8; if 21 be added to each, they have the ratio of 10 : 11. Find the numbers.

135. Represent by a graph the distance traveled by an automobile at the rate of 25 miles an hour. What is the equation connecting the distance and the time?

136. Determine the value of x from the proportion $(a-b):(a+b)=x:\left(\frac{1}{a}+\frac{1}{b}\right)$.

137. If b is a mean proportional between a and c, prove that $a-2\,b:b-2\,c=2\,a-3\,b:2\,b-3\,c$.

138. Two numbers, x and y (the first being negative) are in the ratio of $7:-10$. If 15 be subtracted from each one, the resulting numbers are in the ratio of $-10:7$. Find the numbers.

139. If $x+y:a+b=x-y:a-b$, show that x varies as y.

140. Expand $(2-\frac{3}{2}x^2)^4$.

141. Expand $\left(x^{2p}-\dfrac{2}{x^p}\right)^6$ by the binomial theorem.

142. Find the numerical value of $\dfrac{16^{-\frac{1}{2}}\cdot 81^{\frac{1}{4}}}{2^{-2}\cdot 27^{\frac{2}{3}}\cdot 5^0}$.

143. Express in simplest form with positive exponents:

$$\frac{24\,x^{-1}y^5z^{-3}}{40\,x^{-2}y^6z^{-4}}.$$

144. Find one side of a square whose area is represented by the following expression:

$$\frac{x^2}{4}-x+4-\frac{6}{x}+\frac{9}{x^2}.$$ Check the result.

145. Simplify $\dfrac{x^{\frac{1}{2}}\sqrt{x^{-\frac{1}{3}}y^2}}{\sqrt[3]{\dfrac{x^2}{yz^4}}}$.

146. Simplify $\dfrac{5\,a^{-3}b^{\frac{1}{2}}}{a^{\frac{2}{3}}(a-b)^0}\div a^{-1}b^{-\frac{1}{2}}$.

147. If $a=64$, express each of the following as an integer or a fraction: $a^{\frac{1}{2}}$; $\sqrt{a^{-\frac{4}{3}}}$; $(a^0)^{-5}$; $[(\sqrt[3]{a})^{\frac{3}{2}}]^{-2}$.

148. Simplify $\dfrac{4\,x^{-1}y^{-2}}{3\,a^{-2}b^{-4}}\times\dfrac{6\,x^2a^{-1}}{5\,y^{-1}z^2}$.

149. If $a+b+c=2\,s$ and $a=15$, $b=14$, $c=13$, find the value of $\sqrt{s(s-a)(s-b)(s-c)}$.

150. Simplify $ab^{\frac{2}{3}}\left(\dfrac{a^{-\frac{1}{2}}}{b^{-\frac{2}{3}}}\right)^2 \div (a^{\frac{1}{4}}b^{-3})^{-\frac{2}{3}}$.

151. Simplify $(\sqrt[5]{a^{\frac{4}{3}}})^{-\frac{3}{2}}$.

152. Simplify $(25^{-3} \div a^{-4}x^{-2})^{-\frac{1}{2}}$.

153. Evaluate $2^0 \times 9^{\frac{1}{2}} \times 81^{-\frac{3}{4}}$.

154. Simplify, using positive exponents to express the answer,

$$\frac{m^{\frac{1}{2}}\sqrt{n^{-1}m}}{n^0\sqrt[3]{\dfrac{ma^2}{n^4}}}.$$

155. Expand and express in simplest form with positive exponents $(m^{-1}a^{-2} - ma^{\frac{3}{2}})^3$.

156. Simplify $\sqrt{1000}$; $\sqrt[3]{81}$; $\sqrt{\frac{11}{16}}$; $\dfrac{1}{\sqrt{2}}$; $(27^{\frac{1}{2}})^{\frac{5}{3}}$; $2^{\frac{1}{3}} \cdot 4^{\frac{1}{3}}$.

157. Simplify $[a^{\frac{m+n}{p}} \times a^{\frac{m-n}{p}} \div a^{\frac{2m-1}{p}}]^{-p}$.

158. Simplify, using positive exponents to express the answer,

$$\left(\frac{x^{-1}y\sqrt{z^{-3}}}{xy^{-1}}\right)^{\frac{1}{2}} \times \left(\frac{x^{-4}}{zy^{-2}}\right)^{-\frac{3}{4}}.$$

159. Simplify $\sqrt{18}$; $36^{\frac{3}{2}}$; $25^{-\frac{1}{2}}$; $x^{\frac{a}{b}} \cdot x^c$; $\sqrt[3]{27^2}$.

160. Simplify $\dfrac{\sqrt{a^{\frac{1}{6}}\sqrt{b^{-\frac{3}{2}}}}}{b^{\frac{3}{4}}\sqrt[3]{a^{-1}}}$.

161. Simplify $\dfrac{3^{x+1}}{(3^x)^{x-1}} \div \dfrac{9^{x+1}}{(3^{x-1})^{x+1}}$.

162. Simplify $5a^0 - (5a)^0 - 1^5 + \dfrac{2}{9^{\frac{3}{2}}}$.

163. Simplify $5\sqrt{\frac{9}{8}} + \sqrt{\frac{3}{6}} - \sqrt{8}$.

164. Simplify $3\sqrt{12} + \sqrt{75} - \sqrt{108}$.

165. Simplify $\sqrt[4]{\frac{2}{27}} \div \sqrt[3]{\frac{4}{81}}$.

166. Simplify $\sqrt[3]{2} \div \sqrt{2} \div 2^0$.

167. Simplify $\sqrt[3]{\frac{3}{5}} \cdot \sqrt[6]{\frac{1}{45}}$.

168. Simplify $\sqrt[3]{2} + \sqrt[3]{128} - \sqrt[3]{54}$.

169. Solve $\sqrt{x+1} - \sqrt{2x-2} = 0$.

170. Simplify $10\sqrt{\frac{1}{2}} - 3\sqrt{\frac{32}{25}} + 7\sqrt{\frac{1}{8}} - 5\sqrt{2} + 9\sqrt{\frac{1}{98}}$.

171. Simplify $\dfrac{1}{\sqrt{18}+\sqrt{27}}$.

172. Simplify $\sqrt[3]{\frac{7}{5}} - 4\sqrt{\frac{5}{7}} + \sqrt{\frac{9}{35}}$.

173. Simplify $\sqrt{\dfrac{x}{y}} - \sqrt{\dfrac{y}{x}} + \sqrt{\dfrac{x^2+y^2}{xy}+2} - \sqrt{\dfrac{x^2+y^2}{xy}-2}$.

174. Simplify $\dfrac{\sqrt{\frac{2}{3}} - \sqrt{\frac{27}{8}}}{\sqrt{3}\left(\sqrt{2}+\dfrac{1}{\sqrt{2}}\right)}$.

175. Solve $\sqrt{x^2-8} + x - 8 = 0$.

176. Simplify $\sqrt{18} + \sqrt{\dfrac{9}{2}} + \dfrac{\sqrt{2}}{1-\sqrt{2}}$.

177. Simplify $\frac{1}{2}\sqrt{490} + \sqrt{\frac{5}{2}} - \sqrt{160}$.

178. Multiply $\sqrt{3} - 1 - \sqrt{5}$ by $\sqrt{5} - \sqrt{3}$.

179. Simplify $(\sqrt{s+t} - \sqrt{s-t}) \div (\sqrt{s+t} + \sqrt{s-t})$.

180. Simplify $\left(\dfrac{a^{m+n}}{a^n}\right)^n \div \left(\dfrac{a^n}{a^{n-m}}\right)^{m-n}$.

181. Simplify, using positive exponents to express the answer,

$$\left\{\frac{\sqrt[3]{x^2}}{\sqrt[4]{y^{-1}}} \cdot \frac{\sqrt{z^{-3}}}{x^{\frac{1}{3}}} \cdot \frac{y^{-\frac{1}{4}}\sqrt[3]{x}}{z^{-1}}\right\}^{-6}.$$

182. Find the numerical value of the following fraction to two places of decimals: $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

183. Divide $21\,x^{3a} - 27\,x^{a} - 26\,x^{2a} + 20$ by $5 - 3\,x^{a}$.

184. Simplify $\dfrac{1}{\left[x^{\frac{1}{3}} + \dfrac{1}{x^{-\frac{1}{3}}}\right]^{-3}}$.

185. Divide $a^{\frac{3}{5}} - b^{\frac{3}{5}}$ by $\sqrt[5]{a} - \sqrt[5]{b}$.

186. Simplify $2\,x^2\sqrt{9\,x^2 + 81} + 27\sqrt{4\,x^2 + 36}$.

187. Simplify $\dfrac{a^{\frac{2}{3}}(\sqrt{b})^3\sqrt[3]{b - 4}}{a^3\sqrt{b} \div (ab)^{\frac{1}{4}}}$.

188. Simplify $[a + b(1 + \sqrt{-2})][a - b(1 + \sqrt{-2})]$ $[a - b(1 - \sqrt{-2})][a + b(1 - \sqrt{-2})]$.

189. Divide $\sqrt{x^3}$ by $\sqrt[9]{x^8}$.

190. Compute the value of $\dfrac{\sqrt{2} + 2\sqrt{3}}{\sqrt{2} - \sqrt{12}}$ to two decimal places.

191. Simplify $\dfrac{1 + (1 + x^2 + y^2)(x^2 + y^2)^{-\frac{1}{2}} - \sqrt{x^2 + y^2}}{(x^2 + y^2)^{\frac{1}{2}} - y^2(x^2 + y^2)^{-\frac{1}{2}}}$.

192. Find the square root of $a^2 - 2\,b^{\frac{1}{2}}a + 3\,b - 2\,b^{\frac{3}{2}}a^{-1} + b^2a^{-2}$.

193. Simplify $\dfrac{xy(x^{-1}y - xy^{-1})}{y^{\frac{2}{3}} - x^{\frac{2}{3}}}$.

194. Simplify $\left[a^{\frac{2}{3}}\sqrt{\left(\dfrac{a^{\frac{1}{2}}}{\sqrt[3]{a}}\right)^5}\,\right]^{1\frac{3}{5}}$.

195. Simplify $\dfrac{7 - \sqrt{2}}{\sqrt{3} + \sqrt{5}} + \dfrac{7 + \sqrt{2}}{\sqrt{5} - \sqrt{3}}$.

196. Simplify $\dfrac{7 - \sqrt{48}}{2 - \sqrt{3}} - \dfrac{1}{2 + \sqrt{3}}$.

197. Simplify $\dfrac{2}{\sqrt{x + 1} + \sqrt{x - 1}}$.

198. Multiply $a^{\frac{3}{4}}b^{-\frac{3}{4}} + 2 + a^{-\frac{3}{4}}b^{\frac{3}{4}}$ by $a^{-\frac{1}{4}}b^{\frac{1}{4}} - 1 + a^{\frac{1}{4}}b^{-\frac{1}{4}}$.

199. Simplify $\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)^2$.

200. Expand by the binomial theorem and express each term in simplest form $\left(a^{-\frac{2}{3}} - \frac{a^{\frac{1}{2}}}{2}\right)^5$.

201. Simplify $\frac{\sqrt{x+y+z}}{\sqrt{x+y-z}} - \frac{\sqrt{x+y-z}}{\sqrt{x+y+z}}$.

202. Expand $\left(a^2 + \frac{2}{a}\right)^5$.

203. Simplify $\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)^2 - 98$.

204. Write the sum of $\sqrt{28}$, $-\sqrt{63}$, and $\sqrt{700}$ in the simplest form.

205. Simplify $\left(a^{\frac{1}{x^2}} \div a^{\frac{1}{y^2}}\right)^{\frac{xy}{x+1}} \div a^{-\frac{1}{y}}$.

206. Simplify $8^{\frac{2}{3}} - 7\sqrt{98} + \frac{1}{2}\sqrt{\frac{1}{2}} + \frac{4}{\sqrt{2}} + \frac{\sqrt{2}}{3^{-2}} - \frac{1}{3^0}$.

207. Simplify $3^{4m+1} \cdot 9^{2m-1} \div 27^{\frac{8m}{3}}$.

208. Simplify $9^{m-1} \cdot 27^{m+1} \cdot 81^{-m} \cdot 3^{-m-1}$.

209. Evaluate $a^3\sqrt{a^3} + 3\,a\sqrt{a^7} + 2\,a^2\sqrt{a^5}$ when $a = 4$.

210. Simplify $\frac{1}{4}(xy^{-1} - yx^{-1})\left(\frac{y^{-1} - x^{-1}}{y^{-1} + x^{-1}} - \frac{y^{-1} + x^{-1}}{y^{-1} - x^{-1}}\right)$.

211. Simplify $\frac{x^{-2m} - 2 + x^{2m}}{x^{-m} - x^m}$.

212. Divide $5\,x^{\frac{3}{4}} - 6\,x^{\frac{5}{12}} - 4\,x^{-\frac{7}{12}} - 4\,x^{-\frac{1}{4}} - 5\,x^{\frac{1}{12}}$ by $x^{\frac{1}{4}} - 2\,x^{-\frac{1}{12}}$.

213. Show that $(29753)^2 \div 43 - (29581)^2 \div 43 = 237336$.

214. Solve $5x^2 - 3x - 2 = 0$.

215. Solve in two ways $5x^2 + 14x - 55 = 0$.

216. Solve $8x - 15x^2 - 1 = 0$.

217. Solve $ax^2 - bx = c$.

218. Solve $x + 5 + 2\sqrt{x+5} = 15$.

219. Solve by factoring, by completing the square, and by formulæ, $3x^2 - 26x + 35 = 0$.

220. Solve $4x^2 + 8mx = 4mn + n^2$.

221. Solve $m^2x^2 - mx + 1 = x^2$.

222. Solve $\dfrac{2t}{t^2 - 3t + 2} = 1 - \dfrac{5}{t-2}$.

223. What values of x will make the expression $(x+2)(5-3x)$ equal to six times the value of x?

224. Solve and check $\sqrt{x+2} - \sqrt{x-6} = \sqrt{x-3}$.

225. Solve $5x^2 - 3x - 3\sqrt{5x^2 - 3x - 13} = 11$.

226. Solve $6\left(x^2 + \dfrac{1}{x^2}\right) + 5\left(x + \dfrac{1}{x}\right) - 38 = 0$.

227. Solve $72\left(x^2 + \dfrac{1}{x^2}\right) - 6\left(x + \dfrac{1}{x}\right) - 291 = 0$.

228. Solve $4x^2 - 7x + 2 = 0$; give both roots correct to two places of decimals.

229. Solve $(1 - n^2)x^2 - 2mx + m^2 = 0$.

230. Solve $\sqrt{x^2 - 8} + x - 8 = 0$.

231. Solve $3\sqrt{A} - \dfrac{3}{\sqrt{A}} = 8$.

232. Solve $\sqrt{y+5} + y - 1 = 0$.

233. Complete the square in each of the following expressions:

$$x^2 - 12x,\ x^2 + 25,\ x^2 + 9x,\ x^2 + \tfrac{9}{16}.$$

In problems 234–7, write down the quadratic equations whose roots are given.

234. 3 and 5.

235. $1+\sqrt{2}$ and $1-\sqrt{2}$.

236. $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

237. $m+n+\sqrt{m-n}$ and $m+n-\sqrt{m-n} (m > n)$.

238. Solve the system

$$\begin{cases} x+\sqrt{xy}+y=14, \\ x^2+xy+y^2=84. \end{cases}$$

239. Solve the system

$$\begin{cases} x^3-y^3=56, \\ x^2+xy+y^2=28. \end{cases}$$

240. Solve the system

$$\begin{cases} x^2+y^2-6x-2y+6=0, \\ x^2+y^2-12x-8y+42=0. \end{cases}$$

241. Given $p=\frac{0.5d^2}{t}+d$; find value of d. What is the positive root when $t=.21$ and $p=3.2$?

242. The area of a mat of uniform width about a picture 10 inches long by 8 inches wide is one half the area of the picture. What are the outside dimensions of the mat?

243. Solve $\frac{1}{x-a}+\frac{1}{x+2a}-\frac{1}{2a}=0.$

244. Solve $\frac{x+3}{x-6}-\frac{1}{x+1}=\frac{x+2}{x-4}.$

245. A loop of twine 30 inches long is to be stretched over four pegs so as to form a rectangle whose area shall be 50 square inches. What are the sides of the rectangle?

INDEX

[References are to pages.]

Abscissa, 252
Absolute value, 25
Addition, 24 ; associative law for, 37 ; by counting, 30 ; commutative law for, 34 ; graphic representation of, 30 ; of fractions, 164 ; of monomials, 34, 35, 37 ; of polynomials, 38 ; of surds, 279
Algebra, 1 ; laws of combination in, 9 ; symbols of, 2, 3
Algebraic expressions, 6
Algebraic fraction, 153
Alternation, 242
Antecedent, 239, 245
Approximations, 319
Arithmetic, the notation of, 1
Associative laws, 37, 55
Assumptions, 16
Axes of coördinates, 251 ; of reference, 251

Base of a power, 11
Binomial, 13 ; cube of, 115; square of, 108
Binomial expansion, 300
Binomial formula, 300
Binomials, product of two, 114
Braces, 12
Brackets, 12

Cancellation, 84
Checking the result, 7
Clearing of fractions, 191
Coefficient, 10
Common difference, 365
Common ratio, 371
Commutative laws, 34, 54
Completing the square, 315
Consequent, 239
Constant, 247
Coördinates, 252
Cubes, sum and difference of two, 132

Decimal, recurring, 379
Degree of an equation, 82, 142, 143, 205, 312 ; of an expression, 100, 101 ; of a monomial, 100
Descartes, 261
Difference, 27
Distributive law, 58
Dividend, 33, 72
Division, 33, 64 ; by zero, 16, 72, 154 ; of fractions, 178 ; of monomials, 66 ; of polynomials, 68, 69 ; of surds, 281, 285 ; rule of signs in, 33 ; special cases of, 179
Divisor, 33

Equations, 15 ; change of signs in, 84 ; complete quadratic, 313 ; conditional, 81 ; degree of, 82, 142, 143, 205, 312 ; dependent, 207 ; equivalent, 82, 208, 219 ; formation of quadratic, 327 ; fractional and literal, 191, 197, 222 ; graph of dependent, 260 ; graph of inconsistent, 260 ; graph of linear, 258, 260 ; graphs of, 256 ; homogeneous, 352 ; incomplete quadratic, 313, 314 ; inconsistent, 207, 260 ; independent, 207 ; indeterminate, 206 ; irrational, 321 ; linear, 82, 205 ; number of roots of a quadratic, 343 ; number of solutions of, 205, 208 ; numerical, 197 ; particular systems of quadratic, 355 ; quadratic, 312 ; quadratic with complex roots, 335 ; rational and integral, 142 ; relations between roots and coefficients of quadratic, 326 ; roots of, 82, 143, 342 ; satisfying an, 82 ; simple, 82, 205 ; simultaneous, 207, 229 ; solution of, 82, 143, 205, 316,

324 ; standard form of quadratic, 312 ; systems of quadratics, 350 ; systems of linear, 205, 208
Elimination by addition and subtraction, 209 ; by comparison, 215 ; by substitution, 214, 350 ; by undetermined multiplier, 217
Euclid, 57
Euler, 109
Evaluation of an algebraic expression, 6
Evolution, 302
Exponents, 11 ; fractional, 286, 287, 288, 289 ; negative, 286, 287, 288, 289 ; zero, 65
Expressions, algebraic, 6 ; integral, 100 ; mixed, 170 ; prime, 118 ; rational, 100
Extremes, 241

Factoring, 118 ; equations solved by, 142 ; remainder theorem in, 140 ; special methods of, 134 ; summary of, 136
Factors, 9 ; found by grouping terms, 121 ; integral algebraic, 118 ; monomial, 119 ; of algebraic expressions, 118 ; of difference of two squares, 126 ; of general quadratic trinomial, 130 ; of integral expressions, 118 ; of monomials, 119 ; of trinomials of the form $x^2 + cx + d$, 127 ; of trinomial squares, 123 ; sum and difference of two cubes, 132
Formula, 17
Fourth proportional, 245
Fractional equations, 191, 222
Fractions, 153 ; addition and subtraction of, 164 ; change of signs of factors in terms of, 156 ; clearing an equation of, 191 ; complex, 183 ; continued, 186 ; division of, 178 ; laws governing algebraic, 153 ; lowest common denominator of, 164 ; multiplication of, 161, 173 ; powers of, 176 ; proper and improper, 170 ; quotient of two, 178 ; reduction of, 169 ; reduction to lowest terms, 158 ; simple, 169 ; signs affecting, 154, 156.
Function, 255 ; graph of, 256

Gauss, 330
Graphic representation, of addition and subtraction, 30 ; of the relation between two variables, 251
Graphs, 251, 346
Greatest common divisor, 146

Highest common factor, 146 ; of monomials, 147 ; of polynomials by factoring, 148

Identities involving roots, 267
Identity, 81
Imaginary unit, 330
Index, 11 ; law, 55 ; of a radical, 275 ; of a root, 265
Involution, 300

Least common multiple in arithmetic, 149
Linear equations, 82, 205
Literal equations, 197
Lowest common denominator, 164
Lowest common multiple, 149 ; of monomials, 150 ; of polynomials by factoring, 151

Mean, arithmetical, 368 ; geometric, 374 ; proportional, 245, 374
Means, arithmetical, 368 ; geometric, 374 ; of a proportion, 241
Members of an equation, 15
Minuend, 27
Monomial, 13
Multiplicand, 31
Multiplication, 31, 54 ; associative law for, 55 ; combinations of signs in, 32 ; commutative law for, 54 ; distributive law for, 58 ; of a polynomial, 58, 61 ; of fractions, 161, 173 ; of surd expressions, 281, 282 ; rules of signs in, 32
Multiplier, 31

Newton, 301
Notation, 81, 106
Number, complex, 333 ; imaginary, 330 ; irrational, 262 ; literal, 1 ; prime, 118 ; rational, 262 ; reciprocal of a, 178 ; symbols of, 2

Numbers, algebraic, 22; conjugate complex, 333; negative, 22, 23; opposite, 25; positive, 22; prime to each other, 158; property of positive, 266; real, 330; scale of positive and negative, 23; sum of, 25; use of literal, 1

Operations, order of, 7; rational, 100; symbols, 2
Ordinate, 252
Origin, 251

Parentheses, 12, 48
Pascal, 204
Plotting, 253
Polynomial, 14
Portraits, Descartes, 261; Euclid, 57; Euler, 109; Gauss, 330; Newton, 301; Pascal, 204; Pythagoras, ii; Vieta, 33
Powers, 11; ascending and descending, 61; fundamental identities involving, 263; of a fraction, 176; of *i*, 331; quotient of two of the same base, 64
Processes, fundamental, 34
Product, 31; of conjugate surds, 284; of two binomials having a common term, 114; of two complex numbers, 334; of two conjugate complex numbers, 333; of two fractions, 173; of two monomials, 56; of sum and difference of two numbers, 112
Products, type, 107
Progression, arithmetical, 365; decreasing, 372; geometric, 371; increasing, 371; infinite geometric, 377
Proportion, 240; by alternation, 242; by composition, 243; by division, 243; by inversion, 241; continued, 244
Proportional, fourth, 245; mean, 245; third, 245
Pythagoras, ii

Quadratic equations, 312; complete, 313; incomplete, 313; number of roots of, 343; particular systems of, 355; standard form of, 312; systems of, 350; with complex roots, 335
Quadratic surd, 266, 295
Quotient, 33, 72; of two fractions, 178.

Radical, 265
Radicals, 275
Radicand, 265
Ratio, 239
Ratios, composition of, 245
Remainder, 72
Remainder theorem, 139
Review, 52, 75, 137, 186, 234, 296, 381
Root of an equation, 82
Roots, 265; approximate square, 308; cube root of a monomial, 105; found by factoring, 143; like and unlike, 265; nature of the roots of $ax^2 + bx + c = 0$, 340; principal, 265; quadratic equations with complex, 335; square root of arithmetical numbers, 307; square root of a binomial surd expression, 293; square root of a monomial, 105; square root of a polynomial, 304; square root of a trinomial, 303
Rule of signs, in division, 33; in multiplication, 32; in subtraction, 28

Series, 365; sum of arithmetical, 369; sum of geometric, 375
Signs, affecting a fraction, 154; change of, in terms of a fraction, 156; like and unlike, 22; of aggregation, 12; rule of, 28, 32, 33
Square of a binomial, 108; of a monomial, 102; of a polynomial, 111; of a trinomial, 110
Subtraction, 26; of fractions, 164; of monomials, 42; of polynomials, 44; of surds, 279
Subtrahend, 27
Surds, 266; addition and subtraction of, 279; comparison of, 278; conjugate, 283; division by polynomial containing, 285; multiplication and division of, 281; order of a, 266; product of conjugate,

284 ; quadratic, 266, 295 ; simplification of fractional, 276 ; square root of a binomial expression, 293
Symbols, 2 ; of number, 2 ; of operation, 2 ; of relation, 3

Terms, 13 ; like, 14 ; of a fraction, 153 ; of a proportion, 240
Transposition, 83
Trinomial, 14
Type forms, 119

Value, arithmetical, 25 ; numerical, 25
Variable, 247
Variation, 247 ; direct, 247 ; inverse, 248
Vieta, 33
Vinculum, 12

Zero, 23, 25