RUDIMENTARY ALGEBRA

Florian Cajori

ELEMENTARY ALGEBRA

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PREFACE

The object of this book is to provide a complete course in Elementary Algebra that will satisfy the requirements of courses of study in various states and of the College Entrance Board.

Vitality has been given to the subject by imbuing it with the interest that accrues from connection with problems of everyday life and by careful correlation with arithmetic. The utility of algebra is emphasized from the start by showing how much easier it is to solve certain problems by algebra than by arithmetic.

Simplicity is the keynote of the book. This effect is gained by omitting exercises of undue difficulty as well as troublesome phases of the subject that are not essential. A careful development of each new principle anticipates difficulties; and abundant illustrations and examples give further emphasis to the point at issue.

The easy oral exercises assist in developing and fixing in mind the principles and processes. The written exercises are very abundant and range from the simplest type to some of sufficient difficulty to test the pupil's power and to provide adequate drill. Many of the exercises are taken from entrance examination questions set by various colleges and universities, the source being indicated in all such cases.

The problems are practical and varied. They include applications to geometry, physics, engineering, agriculture and commerce, and various interests of everyday life.

The importance of the equation is recognized throughout by abundance of practice.

Thoroughness and accuracy are secured, first, by many reviews, and second, by the emphasis placed on the checking of results.

Exercises designed to encourage an intelligent translation of algebraic language are provided early in the book. The application of algebraic principles to solutions by formulas has also been emphasized.

Graphical representation is introduced in two chapters, but is so arranged that it may be omitted at the option of the teacher without interrupting the sequence of the work.

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ELEMENTARY ALGEBRA

I. INTRODUCTION

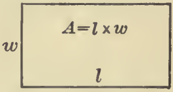
1. **Symbols Representing Numbers.** Algebra, like arithmetic, treats of numbers; but in arithmetic, numbers are usually represented by means of Arabic numerals, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, while in algebra they are represented also by letters, as $a, b, c, \ldots; \ldots x, y, z$.

Numbers represented by letters are called **literal numbers.**

The student has seen how the rules of arithmetic can be abbreviated by the use of letters to represent numbers. When a rule is expressed by means of letters, the result is a **formula.**

The rule, "**the number of square units in the area, *A*, of a rectangle is equal to the product of the number of units in the length, *l*, and the number of units in the width, *w*,**" may be expressed by means of the formula:

$$A = l \times w.$$



Rectangle

1. Find the area of a rectangle whose length is 6 inches and whose width is 4 inches.

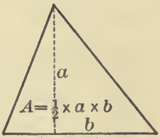
Substitute 6 for l and 4 for w in the formula $A = l \times w$.
Thus, $A = l \times w = 6 \times 4 = 24$, the number of square inches in the area.

2. Find the area of a rectangle whose length is 12 inches and whose width is 9 inches.

3. Explain the use of the following formulas:

Area of a triangle, $A = \frac{1}{2} \times a \times b$,

where b is the number of units in the base and a is the number of units in the altitude.



Triangle

Circumference of a circle, $C = 2 \times \pi \times R$ ($\pi = 3.1416$).

Interest on money invested, $I = p \times r \times t$.

2. Symbols Representing Operations. The following table shows that the symbols of operation used in algebra are the same, with few additions, as those used in arithmetic:

PLUS	MINUS	TIMES	DIVIDED BY
$3+2$	$3-2$	3×2	$4\div 2$, $\frac{4}{2}$, $4:2$
$a+b$	$a-b$	$a\times b$, $a\cdot b$, ab	$a\div b$, $\frac{a}{b}$, $a:b$

The sign for **equality**, $=$, is used as in arithmetic.

Notice that while with arithmetical numbers multiplication is indicated by the sign $\times$, as 3×2, or $3\times 5\times 7$, with literal numbers the sign is usually omitted.

Thus, $3a$, which is read *three a*, means $3\times a$, and $3ab$, which is read *three ab*, means $3\times a\times b$.

$3a+5$ means that 3 times a is to be increased by 5, and is read *three a plus five.*

What does $3a-5$ mean?

If $a=2$, what is the value of $3a$? of $3a+5$? of $3a-5$?

ORAL EXERCISE

3. 1. What is meant by $3x$? by $5b$?

2. What is meant by $3x+4$? by $3x+5y$?

3. If $x=2$, what is the value of $5x$?

4. If $a=2$, and $b=4$, what is the value of $4\div a$? of $b\div a$? of $3ab$?

5. What is meant by $5ab$? $6xyz$? $2mnp+rs$? $2a\cdot 3b$?

6. What is meant by $ab\div c$? $mnp\div r$? $xy\div s$?

7. When $x=1$, and $y=2$, find the value of $3xy$; of $x+y$; of $4x-y$; of $y\div x$.

8. Read $3x+7=12$.

9. Read $m\div n=5xy$. In what other way can the same statement be written?

10. Read $3x = x + x + x$.

11. Read $2abc + b$; $2\pi R$; $I = prt$.

12. Read $3x - 1 = 5$; $2x - 5y = ab$.

13. Read $a + b - ab + \frac{a}{b}$.

14. What operation of arithmetic is suggested by each of the following words: sum? quotient? product? difference?

EXERCISE

4. *Write, using proper algebraic symbols:*

1. The sum of 2 times a increased by b.

2. The sum of $7a$ and $2b$.

3. a times b times c.

4. Two times x diminished by c.

5. The sum of a and two times b.

6. Indicate that two times some number, x, increased by 5 is equal to 25.

7. Indicate the product of the factors 3, a, b, c.

8. Indicate the sum of 3 times b and a times x.

9. Indicate the product of r and s divided by t.

10. Indicate the quotient of d and l increased by f times g.

11. What does $2ab + 3$ equal when $a = 3$ and $b = 4$?

12. Find the value of $3 + 5x$ when $x = 2$.

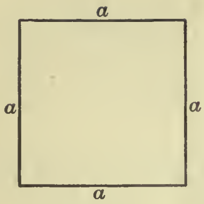
13. Find the value of $4 + 5d$ when $d = 12$.

14. Find the value of $5a + \frac{6}{a} + \frac{a}{2}$ when $a = 2$.

15. Find the value of $\frac{3}{a} + 7b$ when $a = \frac{3}{2}$ and $b = 2$.

16. If t stands for tens and h for hundreds, what number does $6h + 7t + 4$ represent?

17. If y stands for yards, f for feet, and i for inches, how many inches does $14y + 11f + 5i$ represent? How many inches does $19y - 16f + 2i$ represent?



18. The side of a square is a inches. What will represent the distance around it?

19. If the length of a rectangle is l inches and its width w inches, how can you express its area? State this formula for finding the area of a rectangle as a rule. Express the distance around the rectangle. State this result in the form of a rule.

20. The sides of a triangle are x inches, $2x$ inches, and y inches. What is the perimeter, that is, the distance around the triangle?

21. If a, b, and c represent the units', tens', and hundreds' digits of a number, how may the number itself be represented?

SUGGESTION. $543 = 100 \cdot 5 + 10 \cdot 4 + 3$.

22. Find the value of $2n$ when $n = 1; 2; 3; 4; 5$. Does $2n$ always represent an even number when n is any integer?

23. When n is an integer, does $2n - 2$ represent an even or an odd number? $2n + 2$?

24. Do $2n + 1$ and $2n - 1$ represent odd or even numbers when n is any integer?

25. Does $2a + 3a = 5a$ when $a = 2$? when $a = 3$? when $a =$ any number?

26. Does $5a - a = 4a$ when $a = 5$? when $a = 7$? when $a =$ any number?

27. Does $2 \times 3b = 6b$ when $b = 4$? when $b = 10$? when $b =$ any number?

28. Does $10a \div 5 = 2a$ when $a = 1$? when $a = 8$? when $a =$ any number?

29. What number multiplied by 5 equals 25? What number multiplied by 7 equals 35? If $5m = 35$, what is the value of m? If $9h = 72$, what is the value of h?

30. In arithmetic, to find the percentage when the base and the rate are given, we multiply the base by the rate. Express this rule by means of a formula when b, r, and p represent the base, the rate, and the percentage respectively.

5. Factor. If two or more numbers are multiplied together, a **product** is formed and the numbers are **factors** of the product.

Thus, $7xy$ is the product of 7, x, and y; and 7, x, and y are the factors of the product.

6. Exponent. To indicate that the number a has been used as a factor twice in forming the product $a \times a$, we write a^2 instead of $a \times a$; for $a \cdot a \cdot a$ we write a^3. These are read *a square*, and *a cube*, respectively. $4a^3$ means $4 \cdot a \cdot a \cdot a$ and is read *four a cube*. The numbers 2 and 3 in a^2 and a^3 are **exponents.**

7. Square Root and Cube Root. The sign $\sqrt{\ }$ indicates the **square root** of a number, that is, one of the two equal factors of a number.

Thus, $\sqrt{16} = 4$; $\sqrt{a^2} = a$.

The sign $\sqrt[3]{\ }$ indicates the **cube root** of a number, that is, one of its three equal factors.

Thus, $\sqrt[3]{27} = 3$; $\sqrt[3]{x^3} = x$.

8. Symbols of Deduction. In a series of steps one of which is derived from another the symbols of **deduction,** $\therefore$ and $\because$, are used. These symbols are read *therefore* and *since* respectively.

ORAL EXERCISE

9. *Read the following:*

1. $5a^2 + 7ab + 2.$
2. $7 \cdot a + a^2 - 2.$
3. $3x^2 + 3x - 12.$
4. $3x + 7 = 12.$
5. $3x = x + x + x.$
6. $ax^2 + bx + c.$
7. $\sqrt{a} + \sqrt{b} + a^2 + b^2.$
8. $\because 2 \cdot 4 = 8, \therefore 8 \div 2 = 4.$

9. Read $\because 12+3=15$, $\therefore 15-3=12$.

10. What are the factors of ab? of $6x^2y^2$? of $5mnp$? of $9z^2w^3r^4$?

11. Express the square of a; the cube of p.

12. Express the square root of a; of $2m$.

13. Express by using exponents $2\cdot 2\cdot 2$; $a\times a\times a$; $b\cdot b\cdot c\cdot c$; $p\cdot p\cdot p\cdot q\cdot q$.

14. If the side of a square is a, what is its area?

15. If the edge of a cube is a inches, express the formula for finding its volume. State the formula for finding the area of its surface. State these two formulas as rules.

10. Some Simple Operations. From the method of writing algebraic numbers we are justified in performing the following simple operations:

(*a*) $2x+3x=5x$ for all values of x.

For if two times a number is increased by three times that number, the result is five times the number. Compare this with $2\times 4+3\times 4=5\times 4$.

Similarly, $5x-3x=2x$, and $7x-x=6x$. (Note that x is the same as $1x$.)

(*b*) If 2 is subtracted from $5x+2$, the result is $5x$.

For if the sum of two numbers is diminished by one of them, the result is the other.

(*c*) $2\times 3b=2\times 3\times b=6\times b$ or $6b$.

This is similar to $2\times 3\times 4=6\times 4$.

$5\cdot 4y=20y$. $6\cdot 3a=?$

(*d*) $2x\div 2=x$, for if the product of two numbers is divided by one of them, the quotient is the other. Also $16m\div 2=8m$.

ORAL EXERCISE

11. *Perform the indicated operations:*

1. $2\times 3a$; $4\times 5a$; $3\cdot 2b$.
2. $2a\div 2$; $10y\div 5$; $8a\div 4$; $16x\div 8$; $32m\div 4$.

3. $3ab \div a$; $12xy \div 4y$; $18pq \div 3p$; $25abc \div ab$.

4. $4x + 2$ diminished by 2; $7m + 3$ diminished by 3.

5. Add 3 to $5x + 3$. ($5x + 3 + 3 = 5x + 6$.) Add 6 to $8p + 2$.

6. What number subtracted from $3x + 7$ will give $3x$?

7. What must be done to the number $3x + 5$ to get $3x$?

8. What must be done to $3x$ to get x?

9. $2x + 13x = ?$ $3x + x + 5x = ?$

10. $5x - 4x = ?$ $8x - 2x - x = ?$

11. If $5x = 15$, what does x equal?

12. $7x + 2x - 3x = ?$ $9x - 3x + 5x = ?$

13. $10x - 5x + 8 - 3 = ?$

14. $10x - 5x + 8a - 3a = ?$

15. $4x + 7x + 12 - 8 = ?$

16. $4x + 7x + 12b - 8b = ?$

12. Equation. A statement expressing the equality of two numbers is called an **equation.**

Thus, $2x + 3 = 9$ is an equation.

The two equal numbers are the **members** of the equation. The number written at the left of the sign of equality is the **first member,** while the other number is the **second member** of the equation.

13. Unknown Number. A number in the equation whose value is to be found is the **unknown number.**

Thus, in the equation, $2x + 3 = 9$, $2x + 3$ is the first member and 9 is the second member. x is the unknown number.

That value of the unknown number which, if substituted for it in the equation, will make the two members equal **satisfies** the equation.

Thus, if 3 is substituted for x in $2x + 3 = 9$, we have $2 \cdot 3 + 3 = 9$ or $6 + 3 = 9$. Therefore 3 satisfies the equation. Does 2 satisfy $5x + 1 = 11$? Does 3 satisfy $5x + 1 = 11$?

14. Solving Equations. Root. The process of finding the value of the unknown number that satisfies the equation is called **solving the equation.** The value of the unknown number that satisfies the equation is the **root** of the equation.

Thus, 2 is the root of $3x + 1 = 7$ because $3 \cdot 2 + 1 = 7$.

15. Principles used in Solving Equations:

(*a*) If the same number is added to equal numbers, the resulting numbers are equal.

(*b*) If the same number is subtracted from equal numbers, the resulting numbers are equal.

(*c*) If equal numbers are multiplied by the same number, the resulting numbers are equal.

(*d*) If equal numbers are divided by the same number, the resulting numbers are equal.

Note. Division by zero is not included in this last statement.

16. In the solution of $2x + 3 = 9$, the steps are as follows:

1. $2x + 3 = 9$.
2. $\therefore 2x = 6$. Subtract 3 from both members of the equation. See § 15, (*b*).
3. $\therefore x = 3$. Divide both members by 2. See § 15, (*d*).

17. Check. The solution of an equation may be **checked** by putting the root obtained in the place of the unknown number in the equation. When this is done, if the two members are equal, the solution is correct.

Thus, to check the answer 3 in the solution of $2x + 3 = 9$, put 3 for x, then $2 \times 3 + 3 = 9$. Therefore the solution is correct.

EXERCISE

18. *Solve the following equations, explaining each step by the statement of the principle involved. Check each solution.*

1. $3x + 5 = 20$.
2. $2x + 8 = 13$.
3. $5x + 2 = 3$.
4. $3x + 7 = 16$.
5. $4n + 2 = 6$.
6. $3x + 2x + 8 = 23$.

7. $2x+1=4$.
8. $5n+2=52$.
9. $m+1=4$.
10. $3v+6=11$.
11. $9a+1=3$.
12. $5+b=12$.
13. $6y+7=13$.
14. $z+1=5$.
15. $5x+5=x+9$.

SOLUTION. 1. $5x+5=x+9$.
2. $\therefore 4x+5=9$. (Subtracting x from each member.)
3. $\therefore 4x=4$. (Subtracting 5 from each member.)
4. $\therefore x=1$. (Dividing each member by 4.)

CHECK. When $x=1$, $5x+5=10$, and $x+9=10$. Therefore the solution is correct.

16. $3x+3=x+5$.
17. $12y+3=7y+18$.
18. $12y+3=7y+17$.
19. $5x+3x+10=22$.
20. $8x+2x+9=2x+20$.
21. $7z+16=2z+3z+40$.

EXERCISE

19. Writing Algebraic Numbers and Making Equations.

1. If n stands for a number, what will stand for three times this number?

2. If n stands for a number, what will stand for the number increased by 3?

3. If x is an integer, what will stand for the next larger integer?

4. If a room is f feet long, how many inches long is it?

5. How would you express f feet and i inches in inches?

6. Express p pounds and z ounces in ounces.

7. Express the result of multiplying a number x by 3 and adding 2 to the product.

8. Indicate that two times some unknown number x increased by 5 is equal to 17.

9. Find the unknown number in example 8: (a) by arithmetic; (b) by algebra.

10. How can two unknown numbers be expressed if one is double the other?

11. The sum of two numbers is 30, and one of them is twice as large as the other. Find the numbers by arithmetical analysis. Also make and solve an algebraic equation to find them.

SUGGESTION. x and $2x$ may represent the numbers.

12. The sum of two numbers is 45, and one of them exceeds the other by 5. What are the numbers? Solve first by arithmetic, then by algebra.

Notice how much easier it is to solve examples 9, 11, and 12 by algebra than by arithmetic.

13. Five times a certain number, increased by 2, is equal to the result obtained by multiplying the same number by 3 and adding 14 to the product. Find the number.

SOLUTION. Let $x =$ the required number.

Hence $5x + 2 =$ the result of multiplying the number by 5, and adding 2 to the result,

and $3x + 14 =$ . . . (Let the student complete the statement.)

Then $5x + 2 = 3x + 14$. (By the conditions of the problem.)

$\therefore 5x = 3x + 12$. (Why?)

$\therefore 2x = 12$. (Why?)

$\therefore x = 6$.

Therefore the required number is 6.

CHECK. In checking the solution of this problem, it will not do to substitute 6 for x in the equation, for an error might have occurred in forming the equation. The answer should be substituted in the original problem.

EXERCISE

20. *Make and solve equations for the following problems. Check each result by seeing if it satisfies the conditions of the problem.*

1. If a certain number is multiplied by 7 and the product is increased by 5, the result is equal to the original number increased by 83. Find the number.

SOLUTION. Let $x =$ the number.

Hence $7x + 5 = 7$ times the number increased by 5,

and $x + 83 =$ the number increased by 83.

Then $7x + 5 = x + 83$. (By the conditions of the problem.)

$\therefore 7x = x + 78$. (Why?)

$\therefore 6x = 78.$ (Why?)

$\therefore x = 13.$ (Why?)

Therefore 13 is the required number.

CHECK. $7 \times 13 + 5 = 96$, and $13 + 83 = 96$. Therefore 13 is the number required by the conditions of the problem.

2. If two times a certain number is increased by 6, the result is equal to the sum of the original number and 9. Find the number.

3. Find three numbers of which the second is double the first, and the third exceeds the first by 8, their sum being 44.

4. The sum of three numbers is 24. The second is double the first, and the third equals the sum of the other two. Find the numbers.

5. Two men have together \$68. One of them has \$2 more than twice as much as the other. How many dollars has each?

SOLUTION. 1. Let $x =$ the number of dollars one man has.

Hence $2x + 2 =$ the number of dollars the other has,

and $x + 2x + 2 =$ the number of dollars both have.

Then $x + 2x + 2 = 68.$ (By the conditions of the problem.)

or $3x + 2 = 68.$ (Why?)

$\therefore 3x = 66.$ (Why?)

$\therefore x = 22.$ (Why?)

Therefore one man has \$22 and the other man has $2 \times \$22 + \$2$, or \$46.

CHECK. Let the student check the problem.

The student should notice that x was not used to represent one man's money, but the *number* of dollars he had. The dollar sign is not to be placed with any of the numbers in the equation. *The equation is expressed in abstract numbers.*

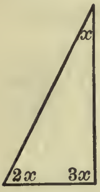
6. The cost of a horse is two times the cost of a cow; the cost of a cow is five times the cost of a sheep. Find the cost of each if a horse, a cow, and a sheep together cost \$208.

SUGGESTION. Let $x =$ the number of dollars one sheep costs.

7. Divide \$55 between A and B so that A shall have \$5 more than four times as much as B.

SUGGESTION. Let $x =$ the number of dollars B has.

8. The sum of the angles of a triangle is 180°. How many degrees are there in each angle if the largest angle is three times as large as the smallest and the other is twice as large as the smallest?



SUGGESTION. Let $x =$ number of degrees in the smallest angle.

9. The sum of the lengths of the three sides of a triangle is 17 inches. The second side is two inches longer than the shortest, and the third is twice as long as the shortest. Find the lengths of the sides.

SUGGESTION. Let $x =$ the number of inches in the shortest side.

10. A piece of rope 106 inches long is to be cut into two parts so that one part shall be 10 inches more than twice as long as the other. How long will each part be?

11. Henry is 5 years older than James, and the sum of their ages is 37. Find the age of each.

12. If $\frac{3}{4}$ of a number is 72, what is the number?

SOLUTION. Let $x =$ the number.

Then $\frac{3}{4}x = 72$.

$\therefore x = 96$. (Dividing both numbers of the equation by $\frac{3}{4}$.)

13. The sum of the ages of three boys is 38 years. The youngest is $\frac{2}{3}$ of the age of the oldest and 3 years younger than the second. How old is each boy?

SUGGESTION. Let $x =$ the number of years in the age of the oldest.

$\therefore x + \frac{2}{3}x + \frac{2}{3}x + 3 = 38$. Explain.

14. If an automobile after being reduced 25 % in price costs \$ 900, what was its original cost?

SUGGESTION. $x - \frac{1}{4}x = 900$. (Why?)

15. A salesman earned \$ 20 at 2 % commission. Find the amount of his sales.

SUGGESTION. $.02x = 20$.

16. A man bought the same number each of 1¢, 2¢, and 4¢ stamps for 70¢. How many of each kind did he buy?

17. If a debt of $144 is paid by using the same number each of $1, $2, $5, and $10 bills, how many of each kind of bills is used?

18. At an election there were two candidates for the office of mayor. They together received 2360 votes. If one candidate was defeated by 328 votes, how many votes did each receive?

19. At an election there were three candidates A, B, and C for a certain office. They together received 3447 votes. If A received twice as many as B, and C 195 more than B, how many votes did each receive?

20. If a field requires 36 pounds of nitrogen for fertilization, how much nitrate of soda containing 18 % of nitrogen will be needed?

21. In an algebra class there are 24 pupils. If there are 6 more girls than boys in the class, how many boys are there?

22. A man buys twice as much hard coal as soft coal and pays $108. If hard coal is $7.50 a ton and soft coal is $3, how many tons of each does he buy?

23. Two trains leave Buffalo at the same time going in opposite directions. One travels 50 miles an hour and the other 40 miles an hour. In how many hours will they be 630 miles apart?

24. Two trains leave Buffalo at the same time going in the same direction. One travels 45 miles an hour and the other 38 miles. In how many hours will they be 35 miles apart?

25. A merchant's profits for the second year increased 25 % over the first year's profits. If the total profits for the two years are $7623, how much are the profits for each year?

SOLUTION. Let $x =$ number of dollars profit the first year.

Hence $x + \frac{1}{4}x =$ number of dollars profit the second year.

Then $x + x + \frac{1}{4}x = 7623$,

or $\frac{9}{4}x = 7623$. (Why?)

$\therefore x = 3388$, the number of dollars profit the first year.

26. A workman's weekly expenses are $\frac{3}{4}$ of his wages. How much does he earn each week if he has \$5 left?

27. Two pupils together solve 28 algebra problems. One of them solves $\frac{3}{4}$ as many as the other. How many problems does each one solve?

28. A rectangular field is $\frac{2}{3}$ as wide as it is long and its perimeter is 40 rods. Find the length and the width.

29. Divide 90 into two such parts that one part equals twice the other.

30. A farmer raised 3000 bushels of corn, wheat, and oats. If he raised 3 times as much corn as wheat and twice as much oats as wheat, how many bushels of each did he raise?

31. A farmer has 4 times as many hogs as cattle and twice as many sheep as hogs and cattle together. If he has 210 animals in all, how many of each kind has he?

32. Three newsboys sold 140 papers. If the first sold $\frac{1}{2}$ as many as the second and the third twice as many as the second, how many did each boy sell?

33. A mason and his helper together earn \$6 a day. If the helper earns $\frac{1}{2}$ as much as the mason, how much does each receive?

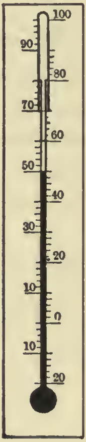
34. A baseball team won 12 games, which was $\frac{3}{4}$ of the number of games played. How many games were played?

35. A boy bought a ball, a bat, and a glove for \$2.50. The ball cost $\frac{3}{5}$ as much as the glove and the bat $\frac{2}{3}$ as much as the ball. How much did each cost?

II. POSITIVE AND NEGATIVE NUMBERS

21. The first numbers with which we became acquainted were the whole numbers used in counting, such as 1, 2, 3. Later it was found necessary to enlarge our idea of numbers and include fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{11}$. Still later it became necessary to express the value of the square roots and cube roots of numbers, as $\sqrt{2}$, $\sqrt[3]{5}$. A still further extension of our number system will now be made, introducing negative numbers.

22. A thermometer scale is marked as in the figure. To indicate that the temperature is 10° below zero we write $-10°$. To indicate that the temperature is 10° above zero we write $+10°$, or simply 10°.



1. At noon on a certain day the temperature was 8° above zero. At night it had fallen 6°. What was the temperature at night? Will the equation $8° - 6° = 2°$, indicate the method of finding the answer?

2. Suppose the temperature is 8° above zero at noon and falls 12° in the next six hours. What is the temperature at 6 o'clock?

The equation, $8° - 12° = -4°$, indicates the method of finding the answer.

3. If the temperature is 10° above zero in the morning and rises 15° during the forenoon, what is the temperature at noon?

$$10° + 15° = 25°.$$

4. If the temperature is 10° below zero in the morning and rises 15° in the forenoon, what is the temperature at noon?

$$-10° + 15° = 5°.$$

ORAL EXERCISE

23. *Explain and give the answers to the following:*

1. $5° + 7° = ?$
2. $-3° + 5° = ?$
3. $-10° + 7° = ?$
4. $-8° + 8° = ?$
5. $7° - 9° = ?$
6. $-3° + 1° = ?$
7. $8° - 5° = ?$
8. $10° - 12° = ?$
9. $18° - 30° = ?$
10. $-5° - 2° = ?$

24. An Extension of Subtraction. In arithmetic the subtrahend must not be larger than the minuend. Such an operation as $8 - 12$ has no arithmetical meaning, for we cannot subtract from a number more units than the number contains. In algebra, however, we do subtract a larger number from a smaller number, and such subtractions give rise to negative numbers.

Thus, $8 - 12 = 8 - 8 - 4 = 0 - 4$, which we write -4.
Also, $5 - 6 = 5 - 5 - 1 = 0 - 1$ or -1.

25. Positive and Negative Numbers. There are many pairs of opposite numbers similar to the numbers of the thermometer scale. The fact that numbers are so related to each other can be conveniently represented by the use of the signs + and −. When thus used to represent the **quality** of a number, these signs are read *positive* and *negative* respectively. Thus, $+5$ is read *positive five* and -7 is read *negative seven.* Numbers preceded by the sign + to indicate the quality of the number are *positive* numbers; numbers preceded by the sign − to indicate the quality of the number are *negative numbers.*

The student will note that each of the signs + and − may have two distinct uses; they may indicate the operations of addition and subtraction, or they may indicate the quality of a number.

26. We usually omit the positive sign before positive numbers, writing and reading them exactly as in arithmetic.

Sometimes, however, for emphasis or for contrast, we write the sign $+$ before a positive number, as $(+5)$ or $+5$. The negative sign before a negative number is *never omitted.* To show that these signs are quality signs, and not operation signs, we often write such numbers within a parenthesis, thus $(-3)+(+5)$, read *negative* 3 *plus positive* 5.

ORAL EXERCISE

27. *Read the following, using "positive" and "negative" as the names of these signs when they indicate quality.*

1. $(-3)+2+(-3)$; $-3+2+(-3)$.
2. $-3+5$; $(-3)+5$; $5+(-3)$.
3. $-7-4$.
4. $(-2)(-3)+4$.
5. $(-a)+b+(-a)$.
6. $x+(-y)+y$.
7. $m-n+(-m)+a$.
8. $23°+(-4°)$.
9. $15¢+(-5¢)$.
10. $-\$40+\$17$.
11. $-2x-3x+(-2x)$.
12. $5+(-7)-a(-b)$.

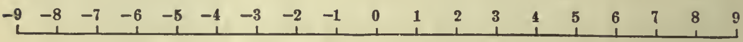
13. If we consider north positive, what should we consider south? If rising temperature is positive, what kind of temperature is negative?

14. What signs would you associate with each of the following: (1) Money earned and money spent? (2) A man's property and his debts? (3) Distance up and distance down? (4) Distance to the right and distance to the left?

28. The Algebraic Number Scale. Draw a straight line and divide it into spaces of equal length. Select some point as zero near the center and name the other points of division as indicated. This arrangement of numbers on a line is the **algebraic number scale.** (See figure, page 18.)

Just as the arithmetical number scale (that part of the algebraic scale that extends from 0 to the right) is conceived as extending indefinitely to the right, so the negative numbers of the algebraic scale extend from 0 indefinitely to the left.

29. Algebraic Numbers. The positive and negative numbers together form the system of **algebraic numbers**, or **signed numbers.**



30. Addition of Signed Numbers on the Number Scale.

1. To add 3 and 5 on the number scale, begin at 3 and count 5 spaces to the right, arriving at the point 8. This gives the result $3 + 5 = 8$.

2. To add (-3) and 5, begin at -3 and count 5 spaces to the right, arriving at the point 2. $\therefore\ -3 + 5 = 2$.

3. Arithmetical numbers can be added in any order; thus, $3 + 2 = 2 + 3$. We shall assume that the order of adding algebraic numbers may be changed in the same way; thus $-3 + 5 = 5 + (-3)$. This suggests that we may add a negative number by counting to the left on the number scale. To verify this begin at 5 and count 3 spaces to the left, arriving at the point 2.

Similarly, $8 + (-3) = 5$ and $5 + (-7) = -2$. Why?

31. These considerations justify the following rules for adding on the number scale:

1. **To add any positive number, *b*, to any number, *n*, begin at *n* and count *b* spaces to the right.**

2. **To add any negative number, $-c$, to any number, *n*, begin at *n* and count *c* spaces to the left.**

EXERCISE

32. *Verify the answers given in examples* 1 *to* 10, *using the above rule, with a number scale:*

1. $3 + 5 = 8$.
2. $-4 + 8 = 4$.
3. $7 + (-5) = 2$.
4. $-7 + 5 = -2$.

5. $5 + (-6) = -1$.

6. $-3 + (-4) = -7$.

7. $-5 + 6 = 1$.

8. $9 + (-3) = 6$.

9. $7 + (-8) = -1$.

10. $-5 + 8 = 3$.

Find the answers to examples 11 *to* 16 *by the use of the number scale:*

11. $2 + (-5) + (-1)$.

12. $(-8) + 7 + (-1)$.

13. $4 + (-3) + 2$.

14. $-7 + 5 + 3 + (-1)$.

15. $-5 + 5 + 6 + (-6)$.

16. $-7 + (-3) + 7 + (-7)$.

33. The essential difference between positive and negative numbers is that they are **opposite quantities.** In adding a positive number we count to the right; in adding a negative number we count to the left. Any number of negative units added to the same number of positive units gives zero. If, in adding a positive and a negative number, the number of positive units exceeds the number of negative units, the sum is a positive number, but if the number of negative units exceeds the number of positive units, the result is a negative number.

EXERCISE

34. 1. \$10 gained and \$12 lost results in an actual loss of \$2, or $\$10 + (-\$12) = -\$2$.

2. Indicate by the addition of signed numbers that a boy has \$4 and owes \$5.

3. Indicate the change in a man's finances, if he spends \$10 in the morning and earns \$12 in the afternoon.

4. Indicate by adding signed numbers that a boy won 12 points in a game and was penalized 3 points. What is his score?

5. In three plays a football team gains 7 yards, is penalized 15 yards, and gains 21 yards. Show, by adding signed numbers, the net result of the three plays.

6. How does the addition of a negative number compare with the subtraction of a positive number containing the same number of units? Illustrate the answer, using $8 - 5$ and $8 + (-5)$. Give another similar illustration.

35. Absolute Value. The **value** of a number without its sign is its **absolute value.** The absolute values of $-2, -3, 3, 5$ are respectively 2, 3, 3, and 5.

ADDITION OF SIGNED NUMBERS

36. The rules given in § 31 for adding positive and negative numbers by means of a number scale would be neither convenient nor practical in adding large numbers, or in adding fractions.

Following are the first six examples of § 32 with their answers:

1. $3 + 5 = 8.$
2. $(-4) + 8 = 4.$
3. $7 + (-5) = 2.$
4. $(-7) + 5 = -2.$
5. $5 + (-6) = -1.$
6. $(-3) + (-4) = -7.$

37. By observing these examples, and others of the same type, we may deduce the following rules:

1. To add two positive numbers, proceed as in arithmetic. (Example 1.)

2. To add a positive and a negative number, subtract the less absolute value from the greater, and prefix the sign of the number having the greater absolute value. (Examples 2, 3, 4, 5.)

3. To add two negative numbers, add their absolute values and prefix the negative sign. (Example 6.)

These rules must be learned.

EXERCISE

38. *Work out the first five examples by the number scale and also by the rules. Solve the remaining examples by the rules.*

1. $3 + (-5).$
2. $-8 + (-2).$
3. $-7 + 5.$
4. $-8 + 8.$

5. $10 + (-8)$.
6. $17 + (-20)$.
7. $-27 + 30$.
8. $-357 + (-258)$.
9. $536.5 + (-233.25)$.
10. $\frac{1}{4} + (-.5)$.
11. $2.3 + (-3.4) + 5.1$.
12. $144 + (-23) + (-7)$.
13. $468 + (-298) + (-200)$.
14. $31.2 + (-2.01) + (-1.11)$.
15. $4.312 + (-25) + 24$.
16. $3 + (-5) + (-2) + 7$.
17. $-8 + 10 + 7 + (-3)$.
18. $22 + (-54) + 7$.
19. $23.1 + (-20.5) + (-1)$.
20. $.4 + (-3) + (-2)$.
21. $-27 + (-5) + 6$.
22. $-5 + (-7) + 11$.
23. $12 + (-2) + (-5)$.
24. $-1 + (-1) + 5$.
25. $-8 + (-7) + 9$.
26. $357 + (-252)$.
27. $-532 + (-5) + 224$.
28. $75 + 2.3 + (-5.2)$.
29. $-78 + 37 + (-24)$.

Add the following:

30.	31.	32.	33.
-5	22	21	-12
3	-52	-15	-7
-12	-31	-17	-5
-7	27	-3	18

34. Augustus Cæsar ruled the Roman Empire 45 years, beginning his reign 31 B.C. Indicate, by the addition of signed numbers, the end of his reign.

35. The Roman historian Livy was born 65 B.C. and lived to be 82 years old. In what year did he die? Indicate by using signed numbers.

39. When several numbers are to be added, they may be added in the order written; or the positive numbers may be added by themselves and the negative numbers by themselves; then the two results may be added.

Thus, $4 + (-3) + 8 + (-5) = 1 + 8 + (-5) = 9 + (-5) = 4$;
or $4 + (-3) + 8 + (-5) = 4 + 8 + (-3) + (-5) = 12 + (-8) = 4$.

EXERCISE

40. *Find the sum of:*

1. $3+(-5)+7+(-2)$.
2. $18+37+(-52)+(-80)$.
3. $25+(-6)+14+(-2)$.
4. $6.4+5.2+(-2.1)+(-.5)$.
5. $-.7+3.2+(-4)+.25$.
6. $\frac{3}{4}+\frac{5}{8}+(-\frac{1}{2})+(-\frac{2}{3})$.
7. $8+(-6)+5+(-11)$.
8. If $x+5+(-1)=14$, find x.
9. What number added to 10 will give 8? If $y+10=8$, what is the value y?
10. What number added to -10 will give 2? If $y+(-10)=2$, what is the value of y?
11. If $a+(-2)+4=6$, what is the value of a?
12. A man has $650 in the bank, $45 in his pocket, and another man owes him $135. He owes one man $240 and another man $325. Indicate by addition of signed numbers his financial standing.

41. Algebraic Sum. The result obtained by adding signed numbers is the **algebraic sum.**

SUBTRACTION OF SIGNED NUMBERS

42. In arithmetic, **subtraction** is defined as the operation of taking one number, the **subtrahend**, from another larger or equal number, the **minuend.** The result of subtraction is the **difference.**

This definition would mean nothing in such algebraic subtractions as, $5-(-8)$, $-2-5$, $6-15$. It is therefore necessary to have a new definition of subtraction that will apply to signed numbers.

The student will recall the relation,

$$\text{difference} + \text{subtrahend} = \text{minuend}.$$

This relation was used to verify answers in subtraction and is the basis of the following definition of subtraction:

Subtraction is the process of finding one of two numbers when their sum, the **minuend**, and the other number, the **subtrahend**, are given.

We shall apply this definition to find answers to a few simple examples in subtraction and from these results shall construct rules for algebraic subtraction.

1. $5-3=?$ By definition this means: What number added to 3 will give 5? We know that $3+2=5$ and therefore $5-3=2$.

2. $4-6=?$ According to the definition, this asks the question: What number added to 6 will give 4? We know that $6+(-2)=4$, and therefore $4-6=-2$.

3. $4-(-6)=?$ The minuend, 4, is the sum of two numbers, and one of the numbers is (-6). Since $(-6)+10=4$, therefore $4-(-6)=10$.

4. $(-4)-6=-10$. Let the student explain by using the definition.

5. $(-4)-(-6)=2$. Why?

The student must make sure that he understands the answers in the preceding examples; that is, he must see that they satisfy the requirements of the definition of subtraction.

43. The method of subtracting by using the definition as a rule would not be practical. We proceed to discover rules that will simplify the process. Collecting, for the sake of comparison, the results of § 42, we have,

1. $5-(+3)=2$. Compare this with $5+(-3)=2$.

2. $4-(+6)=-2$. Compare this with $4+(-6)=-2$.

3. $4-(-6)=10$. Compare this with $4+(+6)=10$.

4. $-4-(+6)=-10$. Compare this with $-4+(-6)=-10$.

5. $-4-(-6)=2$. Compare this with $-4+(+6)=2$.

44. These comparisons indicate that we can change any subtraction to an addition by changing the sign of the subtrahend. Therefore we have the following rule:

To subtract one signed number from another, change the sign of the subtrahend and add the resulting number to the minuend.

EXAMPLES

1. $13-(-4)=13+4=17.$
2. $3-(-4)=3+4=7.$
3. $4-(-10)=4+10=14.$
4. $-5-(+3)=-5+(-3)=-8.$
5. $8-(-3)-(-2)=8+3+2=13.$
6. $5-(-3)+(-2)=5+3+(-2)=6.$
7. $243-(-500)=243+500=743.$
8. $-350-(-250)=-350+250=-100.$

EXERCISE

45. *Find the value of:*

1. $7-(-7)$.
2. $3-10$.
3. $-5-3$.
4. $17-(-3)$.
5. $123-(-21)$.
6. $2.75-(-\frac{3}{4})$.
7. $-37-15$.
8. $.02-(-.1)$.
9. $-22-(-3)$.
10. $3.5-(-2.2)$.
11. $0-(-2)$.
12. $0-(-3)$.
13. $2-(-.2)$.
14. $-4-4-4$.
15. $(-4)-(-4)-(-4)$.
16. $\frac{2}{3}-\frac{5}{6}-(-\frac{3}{4})$.
17. $.5+(-\frac{1}{2})-.5$.
18. $17-(-3)-3$.
19. $0-(-3)+2+16$.
20. $-17+(-3)-16$.
21. $15+(?)=12$.
22. $-15+(?)=12$.
23. $15-(?)=20$.
24. $(?)-10=17$.
25. $(?)-(-10)=17$.
26. $(?)-(-13)=8$.
27. $(?)-(-5)=0$.
28. $0-(-10)$.
29. $-(-4)-(-4)-(-4)$.
30. $-(-5)$.

31. Subtract -7 from 15. Subtract 218.94 from -123.011.

32. Subtract 12 from -26. Subtract -5132 from -2341.

33. What number increased by 15.123 equals 3.102?

34. The minuend is 8.231, the subtrahend is 12.0003; find the difference.

35. The subtrahend is -54.265 and the difference is -2.1981; find the minuend.

MULTIPLICATION OF SIGNED NUMBERS

46. The result of multiplication is the **product**. The numbers multiplied are the **factors** of the product.

47. There are four cases of multiplication of signed numbers. The indicated multiplication 3×4 is to be read "three times four"; that is, the first factor is taken as the multiplier.

1. In arithmetic, 3×4 means that 4 is to be added 3 times.

Thus, $3 \times 4 = 4 + 4 + 4 = 12$, or $(+a) \cdot (+b) = +ab$.

2. Similarly, $3 \times (-4)$ means that (-4) is to be added 3 times.

Thus, $3 \times (-4) = (-4) + (-4) + (-4) = -12$, or $(+a) \cdot (-b) = -ab$.

3. Since to multiply by $+3$ we *add* the multiplicand 3 times, it is reasonable to assume that to multiply by -3 we *subtract* the multiplicand 3 times; that is, $(-3) \times 4$ means that 4 is to be subtracted 3 times.

Thus, $(-3) \times 4 = -4 - 4 - 4 = -12$, or $(-a) \cdot (+b) = -ab$.

4. As in 3,

$$(-3) \times (-4) = -(-4) - (-4) - (-4)$$
$$= 4 + 4 + 4 = 12, \text{ or } (-a) \cdot (-b) = ab.$$

48. Collecting the results in these four cases, we have all possible combinations of signs for two factors.

$$(+3) \times (+4) = 12, \text{ or } (+a) \times (+b) = +ab.$$
$$(+3) \times (-4) = -12, \text{ or } (a) \times (-b) = -ab.$$
$$(-3) \times (+4) = -12, \text{ or } (-a) \times (+b) = -ab.$$
$$(-3) \times (-4) = 12, \text{ or } (-a) \times (-b) = +ab.$$

49. The preceding equations give, in algebraic symbols, the **law of signs** for multiplication, and the method of multiplying two signed numbers.

To find the product of two signed numbers:

1. Find the product of the absolute values of the two numbers.

2. Make the sign of the product positive if the two factors have like signs, and negative if they have unlike signs.

EXAMPLES

1. $3 \times (-7) = -21$. What is the absolute value of the product? Why is the sign of the product negative?

2. $(-8) \times (-7) = 56$. Why is the sign of the product positive?

3. $(-2) \times (-5) \times (-2) = 10 \times (-2) = -20$. Explain.

ORAL EXERCISE

50. *Find the value of:*

1. -3×6; $-3 \times 6a$.
2. $-3 \times (-6)$; $-3a \times (-6)$.
3. $7 \times (-3)$; $7 \times (-3n)$.
4. -10×2.5; $-10g \times 2.5$.
5. $12 \times (-7)$; $12 \times (-7s)$.
6. $(-3) \times (-22)$; $(-3) \times (-22m)$.
7. $8 \times (-6) \times 5$.
8. $12 \times (-2) \times (-.3)$.
9. $(-6) \times 5 \times (-\frac{1}{3}) \times (-4)$.
10. $(-2) \times (-2) \times (-2) \times (-2) \times (-2)$.

11. Given the numbers 2, 5, -3, -2, $\frac{1}{2}$, .5, $-.25$; tell at sight the product of each number multiplied by each one that comes after it.

12. What sign has the product when three negative numbers are multiplied together? four negative numbers? five? Give an answer that will apply to all cases.

DIVISION OF SIGNED NUMBERS

51. Division is the process of finding one of two factors when their product and the other factor are given.

The result of division is the **quotient**.

52. From the definition of division we derive the following:

1. Since $(+7)\cdot(+3) = +21$, therefore $(+\mathbf{21}) \div (+\mathbf{7}) = +\mathbf{3}$.
2. Since $(+7)\cdot(-3) = -21$, therefore $(-\mathbf{21}) \div (+\mathbf{7}) = -\mathbf{3}$.

A negative number divided by a positive number gives a negative quotient.

3. Since $(-7)\cdot(-3) = (+21)$, therefore $(+\mathbf{21}) \div (-\mathbf{7}) = -\mathbf{3}$.

A positive number divided by a negative number gives a negative quotient.

4. Since $(-7)\cdot(+3) = -21$, therefore $(-\mathbf{21}) \div (-\mathbf{7}) = +\mathbf{3}$.

A negative number divided by a negative number gives a positive quotient.

53. Generalizing these results, we have the rule for dividing signed numbers.

To divide one signed number by another:

1. Find the quotient of the absolute value of the dividend divided by the absolute value of the divisor.

2. Make the sign of the quotient positive if the dividend and divisor have like signs and negative if they have unlike signs.

Examples

1. $-12 \div 3 = -4$.
2. $-12a \div 3 = -4a$.
3. $-\frac{2}{3} \div (-\frac{5}{7}) = \frac{14}{15}$.
4. $-10 \div (-5) = 2$.
5. $(-10a) \div (-5a) = 2$.
6. $(-8) \times (-2) \div (-4) = 16 \div (-4) = -4$.
7. $(-2)^2 \div 2^3 = 4 \div 8 = \frac{1}{2}$.
8. $(-2)^3 \div 2^2 = -8 \div 4 = -2$.

ORAL EXERCISE

54. *Perform the operations indicated:*

1. $-12 \div (-3)$.
2. $(-12) \div 4$.
3. $16 \div (-4)$.
4. $-28 \div 7$.
5. $-1 \div \frac{1}{2}$.
6. $3a \div 3$.
7. $-7a \div a$.
8. $-7a \div 7$.
9. $-7a \div (-7)$.
10. $-32 \div (-2)^2$.
11. $-\frac{1}{2} \div (-\frac{1}{3})$.
12. $12x \div (-4)$.
13. $ab \div (-a)$.
14. $7 \times (-6) - (-3)$.
15. $21 \div (-7) - (-3)$.
16. $7 \times (-2r) \div 2$.
17. $12 \div (-4) \times (-1)^3$.
18. $-2 \times (-3) \div (-1)$.
19. $(-\frac{1}{2}) \div 2$.
20. $-5a \div (-1) + (-5a)$.
21. $-32 \div (-8) + (-4)$.
22. $-39 \div 13 \times (-3)$.
23. $4^3 \div (-4)^2$.
24. $(-3)^2 \div 3^3$.
25. $(-3)^3 \div (-3)^2$.
26. $54 \div (-9) \times (-6)$.
27. $(-2)^3 \times (-3) \times (-1)$.
28. $(-5ab) \div (-ab)$.
29. $(-2)^2 + (-3)^2 - (-4)$.
30. $-2 \div \frac{1}{2}$.

55. Order of Operations. In a chain of operations involving the signs, $+$, $-$, $\times$, $\div$, the numbers connected by the signs $\times$ and $\div$ must be operated upon first from left to right in the order in which they occur. The results thus obtained should be added and subtracted as indicated by the signs $+$ and $-$.

Thus, $3 + 4 \times 2 - 6 \div 3 = 3 + 8 - 2 = 9$.

Also $3 + 6 \div 2 \times 5 + 7 = 3 + 3 \times 5 + 7$.

$$= 3 + 15 + 7.$$
$$= 25.$$

EXERCISE

56. *Find the value of:*

1. $2 \times (-3) + (-5) \times 2$.
2. $-7 - (-6) + (-12)$.
3. $-3 \times (-4) + (-5) - 70$.
4. $0 - 2 \times (-3) + 7 - (-1)$.
5. $-1 \times (-2) \times (-3) + 6 \times (-2)$.
6. $-5 + 3 \times 7 - (-5) \times (-4)$.

7. $24 + 8 \times 2 - (-14)$.

8. $60 - 5 \times 3 + 6 \div 3$.

9. $24 \div 3 \times 4 \div 6 \times 5$.

10. $24 \div 3 + 4 \div 6 + 5$.

11. $24 \div 6 + 3 \times 5 \times 4$.

12. $5 + 6 \times 7 - 28 \times 2 + 3 \times (-6)$.

13. $4 - 3 \times 2 + 8 \times (-2) + 4 \div (-1)$.

14. $-8 \times (-2) - 15 \div (-3) + 7 \times 0$.

15. $0 - 4 \times 8 + 7 \times (-2) - (-20)$.

16. $15 + (-3) + (-7)^2 - (-8)^2 + 12$.

17. $120 + (-3)^3 + (-2) \times 8 - 21 - 12$.

18. $-15 \div (-5) + 8 \times (-2) - 7 + (-3)$.

19. $-12 \times (-2) + (-3) \times 8 - 10 \times (-1)$.

20. $0 - 10 \times (-2) - (-4) \times 8 - 7 - (-7)$.

21. $12 - 15 - (-13) - 15 - (-15)$.

22. $12 \times (-1) - (-10) \times (-1) - 8 \times (-1) - (-6)(-1)$.

If $a = 6$, $b = -5$, $c = -3$, $d = -\frac{1}{2}$, $e = -\frac{2}{5}$, *find the values of the expressions in examples* 23 to 43.

23. ab.

24. ac^2.

25. $3bc$.

26. $-abc$.

27. bcd.

28. $a + be$.

29. $b - c^2$.

30. $ab - c^3$.

31. $b - ad$.

32. $-3b + 2c$.

33. $-(ac) - 4d^2$.

34. $de + a + 2b$.

35. $bde + a + b - 1$.

36. $be - bc + 2a$.

37. $bc - be - 2a$.

38. $abcde$.

39. $a^2 - b^2$.

40. $b^2 - (-a)^2$.

41. $6a - 5b - 3c$.

42. $3c - 2d + 5e$.

43. $5a + b - 4d$.

44. Does $3x - 5 = 7x - 9$ when $x = 3$? when $x = 1$?

45. Does $x^2 - 5x + 6 = 0$ when $x = 3$? when $x = 2$?

46. Does $x = -1$ satisfy the equation $x^2 - 2x - 3 = 0$?

REVIEW EXERCISE

57. 1. What quality signs would you associate with each of the following: north latitude, south latitude? rising temperature, falling temperature? debts, credits, money lost, money spent, money earned, money found? A.D., B.C.? points won in a game, points lost, penalties?

2. Compare the addition of a negative number with the subtraction of a positive number having the same absolute value. Illustrate.

3. Indicate the net result of $10 earned, $3 spent, $2 found, and $2 spent.

4. The temperature at 8 o'clock was 28°; at 10 o'clock it had risen 4°; at noon it was 5° warmer than at 10 o'clock; at 2 o'clock it had risen 2° more; at 4 o'clock it was 3° colder than at 2 o'clock; at 6 it was 4° below the temperature at 4 o'clock; and at 8 P.M. it was 7° colder than at 6 o'clock. (1) Indicate by arithmetical additions and subtractions the temperature at 8 P.M. (2) Find the same result by addition of signed numbers.

5. If you walk 3 miles south and 7 miles north, how far and in what direction from the starting point are you? Indicate by adding signed numbers.

6. How far upstream are you if you have rowed 7 miles up and drifted 2 miles down? Indicate the process of finding the answer in two ways.

7. Pikes Peak is 14,108 feet above sea level. A place in Holland is $16\frac{1}{2}$ feet below sea level. How much higher is Pikes Peak than the place in Holland? Indicate two ways of finding the answer.

8. If a gasoline launch can run 14 miles an hour in still water, how fast can it run up a river whose current flows 4 miles an hour? How fast can it run downstream?

9. If a person can swim $2\frac{1}{2}$ miles an hour in still water, represent his rate when swimming against a current of 3 miles an hour. Represent his rate downstream.

10. The Roman Empire fell 476 A.D., 622 years after the fall of Carthage. What was the date of the fall of Carthage?

11. Give the rules for addition, subtraction, multiplication, and division of signed numbers.

12. Define subtraction; define division.

13. What is the basis of the rule for subtraction of signed numbers? of the rule for division?

14. What is the absolute value of a number?

15. What is the sign of $(-1)^{10}$? of $(-1)^{11}$? Can you give an answer that will apply to all such examples?

16. What is the "order of operations"?

When $a = 8$, $b = -3$ *and* $c = -9$, *find the value of:*

17. $a + b + c$.

18. $a - b - c$.

19. $a - bc$.

20. $a - b + c$.

21. $a + b + c + c$.

22. $b - b^2$.

23. $ab - bc$.

24. $abc - c$.

25. $a^2 + b^2 + c$.

26. $ab + bc + ac$.

27. $a^2 - ac$.

28. $-a - b - c$.

III. ADDITION

58. Algebraic Expression. A number represented by algebraic symbols is an **algebraic expression.**

Thus, $2\,ab$, $5-3\,ab$, $4+2\,b$ are algebraic expressions.

59. Monomial, Term. An algebraic expression the parts of which are not separated by either of the signs $+$ or $-$, is a **monomial** or a **term.**

Thus, $2\,ab$, $-xy$, $3\,x \div 7$ are monomials.

60. Polynomial. An algebraic expression consisting of two or more terms is a **polynomial.**

Thus, $3\,ax-4\,c+7$ and $m-n+11\,xy-16$ are polynomials.

The monomials that make up the polynomial are the **terms** of the polynomial.

Thus, $3\,ax$, $-4\,c$, and 7 are the terms of the polynomial $3\,ax-4\,c+7$.

A polynomial of two terms is a **binomial,** and one of three terms is a **trinomial.**

Thus, $2\,a+b$ is a binomial, and $ax-by+c$ is a trinomial.

ORAL EXERCISE

61. *In the following expressions, name (a) the monomials, (b) the binomials, (c) the trinomials, (d) the polynomials:*

1. $3\,a^2x$.	4. $-x^2-2\,ax$.	7. $\frac{1}{2}\,at^2$.
2. $4\,a^2+x$.	5. $4\,a \div b$.	8. $2\,x^2+3\,x-1$.
3. $2\,b-c+3\,d$.	6. $4+a-b$.	9. $a-b-c$.

Name the terms in each of the following polynomials:

10. $3a - b$.
11. $2a^2 - 3ab + c$.
12. $\frac{1}{2}at^2 + 2a + 7$.
13. $ax^2 + bx + c$.
14. $mxn + m \div n - 1$.
15. $-3a - 2b - c$.

62. Coefficient. Any factor of a term, or the product of two or more of the factors of a term, is the **coefficient** (co-factor) of the product of the other factors.

Thus, in $2abc^2$, 2 is the coefficient of abc^2; $2a$ is the coefficient of bc^2, etc.

What is the coefficient of xy^2 in $5xy^2$? of x?

63. Numerical Coefficient. The numerical factor of a term is its **numerical coefficient.**

Thus, the numerical coefficient of $7am$ is 7. $7a$ and $7m$ are **literal coefficients** of m and a respectively.

When we speak of the coefficient of a term we generally mean the numerical coefficient, including the sign preceding the term.

Thus, 2 is the numerical coefficient of $2abc$ and -3 is the numerical coefficient of $-3ax$. Also $\frac{1}{2}$ is the numerical coefficient of $\frac{x}{2}$.

What are the coefficients of x and y in the equation $3x + 4y = 7$? What are the coefficients of x^2 and x in $ax^2 + bx + c = 0$?

If no numerical coefficient is expressed, the coefficient 1 is understood.

Thus, x is the same as $1x$.

What is the numerical coefficient of ab^2? of $-a^2$?

64. Power, Exponent, and Base. The product arising from using a number one or more times as a factor is a **power** of the number.

The number written to the right and above another number to indicate how many times the number is used as a factor is the **exponent** (§ 6) of the power. The repeated factor is the **base.**

Thus, a^4 means *the fourth power of a*, often read *a fourth power*, or simply *a fourth*. 4 is the exponent of the power and a is the base.

a^1 means the same as a. The exponent 1 is never written. a^2 and a^3 are read "a square" and "a cube," since if a represents the length of the side of a square or the edge of a cube, a^2 and a^3 denote the area of the square and the volume of the cube respectively.

65. The student must note carefully the difference between *coefficient* and *exponent.*

Thus, $3a$ means $3 \times a$, while a^3 means $a \cdot a \cdot a$, that is, it means that a is used as a factor three times.

ORAL EXERCISE

66. 1. How would you write $3 \cdot a \cdot a \cdot b \cdot b$ using exponents? $5 \cdot a \cdot a \cdot a \cdot x \cdot x \cdot x$? 1000?

2. How would you write as one term $a + a + a + a$?

Name the numerical coefficients, the exponents, and the base for each exponent:

3. $5a^2$.
4. $2xy^3$.
5. x^2y^n.
6. $4mn$.
7. $-3a^5$.
8. x.
9. $3a - \frac{1}{2}a^3$.
10. $a^3 - 12b^2$.
11. $-a^3$.

Evaluate (that is, find the value of) the following when $a = 2$ and $b = -1$:

12. ab; ab^2; ab^3.

13. $-2a^2b$; $-a^2b^2$; $-a^2b$.

14. $a + b$; $a - b$; $a^2 + b^2$; $a^3 + b^3$.

15. What is the meaning of m^4? of $4m$?

16. Find the value of a^3 when $a = 2$; of $3a$.

17. Find the value of a^2 when $a = -2$; of $-a^2$; of $-a^3$.

67. Similar and Dissimilar Terms. Terms that do not differ at all or that differ only in their numerical coefficients are **like terms** or **similar terms.**

Thus, $2\,ab$, ab, and $6\,ab$ are similar terms.

Terms that differ in other respects than in their numerical coefficients are **unlike terms** or **dissimilar terms.**

Thus, $2\,ab$ and $12\,ab^2$ are dissimilar terms. Why?

ORAL EXERCISE

68. *In the following list, select all terms that are similar to the first; to the second; to the fourth; to the fifth:*

1. $2\,a^2x$.	4. $5\,x$.	7. $5\,a^2x$.	10. $-16\,a^2x$.
2. $4\,abc$.	5. $4\,ax$.	8. $-3\,x$.	11. $5\,ax^2$.
3. $-4\,abc$.	6. $\frac{1}{2}\,abc$.	9. ax.	12. $\frac{2}{3}\,ax$.

ADDITION OF LIKE MONOMIALS

ORAL EXERCISE

69. *Add the following:*

1. 4 and 7; -2 and 4.
2. -4 and 7; -3 and 8; 5 and -4.
3. 4 and -7; 10 and -12; -12 and 10.
4. -4 and -7; -5 and -7; -3 and 3.
5. 13 inches and 5 inches; $13\,i$ and $5\,i$.
6. 4 miles and 7 miles; $4\,m$ and $7\,m$.
7. 5 rods, 6 rods, and 11 rods; $5\,r$, $6\,r$, and $11\,r$.
8. \$5, \$7, \$15; $5\,d$, $7\,d$, $15\,d$.
9. -11, 17, 13; $-11\,a$, $17\,a$, $13\,a$.
10. 8, -9, -5; $8\,x$, $-9\,x$, $-5\,x$.
11. m, $10\,m$, $-7\,m$, $-4\,m$.

Add the following:

12. $7\,ab$, $5\,ab$, $-6\,ab$, $4\,ab$.

13. 3×5, 7×5, -8×5, 2×5.

14. -9×3, -4×3, 13×3, -6×3.

70. These examples suggest the following rule:

To add like monomials, add the numerical coefficients and make their sum the coefficient of the common literal part.

In applying this rule, the numerical coefficients should be added according to the rules for adding positive and negative numbers. The literal part of each term is thought of as the unit of addition.

Terms may be added in any order.

Examples

1. Add $-5\,a$ and $7\,a$.

$-5+7=2$. (Adding numerical coefficients.)

$\therefore -5\,a+7\,a=2\,a$.

2. $4\,a+(-7\,a)=-3\,a$.

3. $ax+(-3\,ax)+(-5\,ax)=-7\,ax$.

Hint. $1+(-3)+(-5)=-7$.

ORAL EXERCISE

71. *Add the following:*

1. $4\,a$, $3\,a$.
2. $4\,a$, $-3\,a$.
3. $-4\,a$, $3\,a$.
4. $-4\,a$, $-3\,a$.
5. $11\,p$, $-7\,p$.
6. $-5\,s$, $-6\,s$.
7. $6\,n$, $5\,n$, $2\,n$.
8. $6\,n$, $-5\,n$, $2\,n$.
9. $3\,m$, $4\,m$, $-5\,m$.
10. $-2\,x$, $5\,x$, $-7\,x$.
11. $4\,r$, $-5\,r$, $6\,r$.
12. $-5\,t$, $2\,t$, $3\,t$.
13. $5\,ab$, $4\,ab$.
14. $-6\,xy$, $-2\,xy$.
15. $7\,mn$, $-2\,mn$.
16. 10×5, 8×5.

17. $11 \times 7, -8 \times 7.$

18. $-6 \times 13, 5 \times 13.$

19.
$$\begin{array}{r} 2b^2 \\ -2b^2 \\ b^2 \\ -3b^2 \\ \underline{4b^2} \end{array}$$

20.
$$\begin{array}{r} 2c \\ -3c \\ -5c \\ -4c \\ \underline{10c} \end{array}$$

21.
$$\begin{array}{r} ax^3 \\ -\ ax^3 \\ -4ax^3 \\ -9ax^3 \\ \underline{-4ax^3} \end{array}$$

22.
$$\begin{array}{r} 7x \\ -3x \\ 2x \\ -5x \\ \underline{12x} \end{array}$$

23. $3a^2+(-5a^2)+(-7a^2)+2a^2+a^2=?$

24. $-5ax+(-3ax)+ax+5ax=?$

25. $\frac{1}{2}d+(-\frac{1}{4}d)+d+(-2d).$

26. Solve $3x+5x-8=16.$

Solution. $3x+5x-8=16.$
$8x-8=16.$ (Why?)
$8x=24.$ (Adding 8 to both members (§ 13, *a*).)
$x=3.$ (§ 13, *d*.)

Solve the following equations:

27. $5x-3=7.$

28. $4x+2x=12+(-3).$

29. $-5y+8y=7+(-3).$

30. $8n+(-3n)+n=12.$

31. $15r+(-r)+2r=20.$

32. $p+(-p)+3p=1.$

33. $-3x+10x=12.$

34. $14x+(-5x)=63+(-9).$

35. $4x+(-3x)=10.$

36. $8x+(-4x)=-4+7.$

37. $3x-5=7.$

38. $7x-2=5.$

39. $\frac{1}{2}x+4=10.$

40. $4x+3=13.$

EXERCISE

72. *Add the following:*

1.
$$\begin{array}{r} .2\ b \\ -3.1\ b \\ 4\ b \\ -\ \ b \\ \underline{.08b} \end{array}$$

2.
$$\begin{array}{r} -3\ x \\ 4\ x \\ -2.5\ x \\ .08x \\ \underline{12.2\ x} \end{array}$$

3.
$$\begin{array}{r} mnx \\ -2mnx \\ -5mnx \\ -4mnx \\ \underline{-7mnx} \end{array}$$

4.
$$\begin{array}{r} 24r \\ -12r \\ -48r \\ 122r \\ \underline{-57r} \end{array}$$

Suggestion. When adding several similar terms, we usually first add the positive numbers, then the negative numbers, then the results.

Add the following:

5. $425\,m,\ -321\,m,\ -m,\ -50\,m.$

6. $4\,a,\ -9\,a,\ a,\ 5\,a,\ -6\,a,\ 2\,a.$

7. $-250\,pq,\ 75\,pq,\ 50\,pq,\ 125\,pq.$

8. $-10.1\,x,\ .2\,x,\ -x,\ -2\,x,\ -5.1\,x.$

9. $210\,p,\ -352\,p,\ 71\,p,\ -83\,p.$

10. $14.3\,q,\ -2.03\,q,\ 17.5\,q,\ -.1\,q,\ q.$

11. $-5\,ab^2,\ -11\,ab^2,\ 14\,ab^2,\ -ab^2.$

12. $13\,r,\ -15\,r,\ -r,\ 73\,r,\ r.$

13. $a,\ -2\,a,\ 3\,a,\ -4\,a,\ 5\,a,\ -6\,a.$

14. $2\,a,\ -4\,a,\ 6\,a,\ -8\,a,\ 10\,a,\ 12\,a.$

15. $6\,x,\ .5\,x,\ -.01\,x,\ -1.72\,x.$

16. $-27\,ab,\ -35\,ab,\ 43\,ab,\ -20\,ab.$

17. $44\,xy,\ -12\,xy,\ -24\,xy.$

18. $75,\ -32,\ -70,\ 23,\ -16.$

19. $784\,x,\ -369\,x,\ -111\,x,\ -53\,x.$

20. $23\,xyz,\ -24\,xyz,\ -36\,xyz.$

21. $1,\ -2,\ 3,\ -4,\ 5,\ -6.$

22. $-3\,x,\ 7\,x,\ 42\,x,\ -13\,x.$

23. What are the units of addition in examples 1 to 8?

24. $(-5\,a)+7\,a+(-12\,a)=-10\,a.$ Is this true when $a=1$? when $a=2$? For what other values of a is it true?

25. In what respect may two like terms differ?

26. What kind of algebraic expression is obtained by adding two like monomials?

27. Solve $2\,x+9\,x+(-3\,x)=17+(-1).$

28. Evaluate $a+b+c+d$, when $a=2,\ b=-3,\ c=-5,\ d=-5.$

29. Simplify $a+b+c+d$ when $a=-3\,x,\ b=-5\,x,\ c=x,\ d=4\,x.$

Solve the following equations:

30. $2.5\,x + 12 = 312.$

31. $18\,x + (-12\,x) = 15 + (-12).$

32. $6\,x + (-5\,x) = 1.7 + 3.3.$

33. $2\,x + (-3\,x) + 4\,x = -5.1.$

34. $18\,x + (-12\,x) + 20\,x = 16 + (-3).$

35. $1.25\,x + (-.75\,x) = .95 + (-.45).$

36. $24\,y + (-12\,y) + 8\,y = 125 + (-32) + (-86) + 48.$

37. $3\,x + 12\,x + 48\,x + (-50\,x) = 48 + 2(-4) + (-1).$

38. $-7\,p + 10\,p + (-p) = 2 \times (-3) + (-3)^2.$

39. $-10\,r + (-8\,r) + (-7) + 7 = 10(-2) + (-5)^2.$

73. In arithmetic we can add several numbers in any order and get the same sum, and we can multiply several factors together in any order and get the same product. Likewise in algebra we can rearrange the terms of a polynomial, or change the order of the factors of a term without changing its value.

Thus, $5 - 2\,a + 3\,b = 3\,b - 2\,a + 5 = 3\,b + 5 - 2\,a.$ Also $6\,m^2np = 6\,nm^2p.$

74. Arranging the Terms of a Polynomial. It is generally convenient, and sometimes necessary, to arrange the terms of a polynomial in some particular order. The most frequent arrangement is in **descending powers** of some letter that occurs in all, or all but one, of the terms.

Thus, $2\,x^3 + 3\,x^2 - 2\,x + 8,$ $4\,x^4 - 3\,x - 7,$ $ax^2 + bx + c$ are all arranged in descending powers of x.

An arrangement in **ascending powers** is sometimes used.

Thus, $-7 - 3\,x + 4\,x^4$ represents one of the above expressions rearranged in ascending powers of x.

If, instead of different powers of the same letter, we have the same power of different letters, we generally arrange the terms alphabetically.

Thus, $a^2 + b^2 + c^2$ is arranged alphabetically.

ORAL EXERCISE

75. *Rearrange the following (a) in descending powers, (b) in ascending powers:*

1. $x^4 - 1 + 2x^2 - 7x$.
2. $3x + 7 - x^3$.
3. $3m^2 - 4m + 6 + m^3$.
4. $b^2 + 16b - 11b^4 + 5b^3 - 1$.
5. $p - 2p^2 + 7$.
6. $bx + ax^2 + c$.

Rearrange the following alphabetically:

7. $2b^2 - a^2 + 3d^2 - 5c^2$.
8. $c - b - a - d$.
9. $n^2 + 2m^2 - l^2$.
10. $h^2 - 14f^2 + 16g^2 + 7k^2$.
11. $y - x + z - v + w$.
12. $p^3 - r^3 + q^3 + n^3$.

ADDITION OF UNLIKE MONOMIALS

76. Just as in arithmetic 2 ft. + 3 ft. = 5 ft., so in algebra $2a + 3a = 5a$; but just as 2 ft. and 3 in. cannot be added without changing them to the same denomination (2 ft. + 3 in. = 24 in. + 3 in. = 27 in.), so the sum of $2a$ and $3b$ must be expressed in the form $2a + 3b$ until we have the values of a and b.

ORAL EXERCISE

77. 1. What is the length of a fence around a field 40 rods long and 30 rods wide? 40 rods long and x rods wide? a rods long and b rods wide?

2. A ship travels 300 miles one day and 320 miles the next day. How far has it gone? If the number of miles had been a and b, how far would it have gone? What is the value of this result if $a = 300$, $b = 320$?

3. There were x boys and y girls in school last term. If a new boys and b new girls enter this term and m boys and n girls leave, how many boys and how many girls are there in school? how many boys and girls together?

78. The following statement may be regarded both as a *definition* of the sum of *unlike terms* and as a *rule* for adding such terms:

The sum of several unlike terms is the algebraic expression obtained by uniting them with their respective signs.

Thus, the sum of $2a$, $(-3b)$, and $11c$ is $2a - 3b + 11c$.

79. It is often necessary to use the rule for adding *like monomials,* along with the above rule for adding unlike monomials.

Thus, $3a + 5b + (-8b) + (-2a) = a - 3b$.

This kind of simplification is usually called **collecting terms.**

ORAL EXERCISE

80. *Add and arrange the terms in proper order:*

1. $-a^2, -4a^2, 2a^2$.
2. $3b, -2a, a, -5b$.
3. $4x^2, -3x, 7, 5x, -2x^2$.
4. $-1, 2x, 3x^3, -x^2, 1$.
5. $\frac{1}{2}a^2, -\frac{2}{3}b^2, \frac{1}{4}a^2, \frac{1}{3}b^2$.
6. $4m^2, -5n^2, .5m^2, 4n^2, 2m^2$.
7. $3x^2, -x, 2, 4x^2, -1, x, 3$.
8. $4b, -2a, -2b, 4c, -a$.

Collect the terms and arrange in order:

9. $4b + (-7) + 3b + 2b + (-3b)$.
10. $8x + y + 2z + 3x + y + 7z$.
11. $7y + x + (-2x) + 2y + (-z)$.
12. $7x + (-y) + (-2x) + (-z)$.
13. $a + 2b + 3c + 2a + (-7b) + (-9c)$.
14. $l + m + n + 2l + n + (-2m) + (-n)$.
15. $5E + 7E = ?$ $5 \cdot 15 + 7 \cdot 15 = ?$
16. $3 \cdot a + 9 \cdot a = ?$ $3 \cdot 7 + 9 \cdot 7 = ?$
17. $\frac{1}{2}a + 3\frac{1}{2}a = ?$ $\frac{1}{2} \cdot 17 + 3\frac{1}{2} \cdot 17 = ?$
18. $17a + 3a = ?$ $17 \cdot 13 + 3 \cdot 13 = ?$
19. $13 \cdot 21 + (-5 \cdot 21) + (-7 \cdot 21) = ?$

EXERCISE

81. *Collect terms and arrange the results in proper order:*

1. $3a + (-5c) + (-a) + 3b + (-c)$.

2. $-5x + (+1.1x) + .7x^2 + (-2.3x^2)$.

3. $22m + 6n + (-14p) + 2m + (-10n) + 4p + (-2m) + 10n + 18p$.

4. $\frac{3}{4}a + \frac{5}{6}b + (-a) + \frac{3}{4}c + (-\frac{5}{6}b) + \frac{1}{4}a + (-\frac{3}{4}c)$.

5. $\frac{4}{5}x + (-\frac{7}{2}y) + \frac{8}{7}x + (-\frac{2}{3}y) + (-x)$.

6. $56a + 58p + 218 + 92p + 36a + 74 + 20p + 360$.

7. $54m + (-62n) + 18x + (-62m) + (-6x) + 42n + 10m + 18n + (-14x)$.

8. $10m + 11 + (-5x) + (-12) + (-4m) + (-3x) + 1 + 9x + (-5m)$.

9. $13x + (-5y) + 8z + (-5x) + 9y + (-11z) + (-3x) + (-6y) + z$.

10. $5.6p + 18.5q + (-7.25p) + 11.5q + 15.5p + (-9.4q)$.

11. $5a + (-3\frac{1}{2}b) + 5\frac{5}{6}c + (-6\frac{7}{8}a) + 9\frac{1}{12}b + 3\frac{2}{3}a + (-2\frac{11}{12}c)$.

ADDITION OF POLYNOMIALS

82. 1. *Add and compare:*

2 ft. + 3 in.	$2f + 3i$
4 ft. + 2 in.	$4f + 2i$
7 ft. + 5 in.	$7f + 5i$
13 ft. + 10 in.	$13f + 10i$

2. Add $2a - 3b + 7c$, $2b + a - 2c$, $c + 2a - 3b$.

$$\begin{array}{r} 2a - 3b + 7c \\ a + 2b - 2c \\ 2a - 3b + c \\ \hline 5a - 4b + 6c \end{array}$$

Rearrange so as to have like terms in the same column; then add.

3. Add $2a^2 - 3a + 7$, $-4a^2 - 6 + 5a$, $8a^2 - 9a - 7$.

$$\begin{array}{r} 2a^2 - 3a + 7 \\ -4a^2 + 5a - 6 \\ 8a^2 - 9a - 7 \\ \hline 6a^2 - 7a - 6 \end{array}$$

ORAL EXERCISE

83. *Add:*

1. $\begin{array}{r} 3a + 2b \\ 4a + 9b \\ \hline \end{array}$

2. $\begin{array}{r} 4x + 3 \\ 7x + 8 \\ \hline \end{array}$

3. $\begin{array}{r} 5m + 3 \\ 8m - 2 \\ \hline \end{array}$

4. $\begin{array}{r} 9x - 9 \\ 8x - 3 \\ \hline \end{array}$

5. $\begin{array}{r} 5r + 3 \\ r - 2 \\ \hline \end{array}$

6. $\begin{array}{r} r + 2s \\ 3r - 5s \\ \hline \end{array}$

7. $\begin{array}{r} 4ab + c \\ -5ab - c \\ \hline \end{array}$

8. $\begin{array}{r} 2a + 3 \\ 5a - 3 \\ \hline \end{array}$

9. $\begin{array}{l} 4p + 7 \\ 5p \\ \hline \end{array}$

10. $\begin{array}{r} 8x + 9 \\ 3 \\ \hline \end{array}$

11. $\begin{array}{r} x \\ -x + 3 \\ \hline \end{array}$

12. $\begin{array}{r} x - 5 \\ -x + 5 \\ \hline \end{array}$

13. $\begin{array}{r} a + b - c \\ 2a - 2b - 3c \\ \hline \end{array}$

14. $\begin{array}{r} 3a - b + 2c \\ a \quad - c \\ \hline \end{array}$

15. $\begin{array}{r} 4a - b - 2c \\ b + 2c \\ \hline \end{array}$

16. $\begin{array}{r} a^2 - 2a + 5 \\ -2a^2 + 3a - 2 \\ \hline \end{array}$

17. $\begin{array}{r} 2x^2 + 3x + 9 \\ x^2 \qquad - 9 \\ \hline \end{array}$

18. $\begin{array}{r} a^2 + a \\ 2a + 1 \\ \hline \end{array}$

19. $\begin{array}{r} a - 3b \qquad \\ 2a \qquad - c \\ 3b + 2c \\ \hline \end{array}$

20. $\begin{array}{r} 4r + 2s - t \\ -3s - 2t \\ 2r + 7s \quad \\ \hline \end{array}$

21. $\begin{array}{r} x^2 + x - 1 \\ -x^2 + x - 1 \\ x^2 - x + 1 \\ \hline \end{array}$

22. $\begin{array}{r} 3x - y + 2z \\ -x + 3y - 3z \\ -2x - 2y + 2z \\ \hline \end{array}$

23. $\begin{array}{r} a + b - c \\ a - b + c \\ -a + b + c \\ \hline \end{array}$

24. $\begin{array}{r} a^2 - 2a + 1 \\ a^2 + 2a + 1 \\ -2a^2 \qquad - 1 \\ \hline \end{array}$

84. Checking Results. $2a + 3a = 5a$ for all values of a. The sum of $3a - 5b$ and $4a + 2b$ equals $7a - 3b$ for all values of a and b. This fact may be used to check the answers.

1. Add $3a - 5b$ and $4a + 2b$ and check the result.

Addition	Check. $a = b = 1$.
$3a - 5b$	$3 - 5 = -2$
$4a + 2b$	$4 + 2 = 6$
$7a - 3b$	$7 - 3 = 4$

The work on the right is the result of putting $a = 1$ and $b = 1$ in the two expressions to be added and in the result. Notice that the final number, 4, is the algebraic sum both of the last line and the right-hand column; that is, $7 - 3 = 4$ and $-2 + 6 = 4$.

2. Add and check the result, $2a^2 - 3a + 4$, $7a + 3$, $-2a^2 + 5$.

Addition	Check. $a = 2$.
$2a^2 - 3a + 4$	$8 - 6 + 4 = 6$
$7a + 3$	$14 + 3 = 17$
$-2a^2 + 5$	$-8 + 5 = -3$
$4a + 12$	$8 + 12 = 20$

85. To add two or more polynomials:

1. Arrange the terms in order of powers of some letter (or alphabetically), writing like terms in the same column.

2. Add the like terms in each column and unite the partial results obtained with their respective signs.

Note. Check, or prove the correctness of the result, by substituting a numerical value for the letters used.

Examples

1. Add and check, $3a - 2$, $a^2 - 1$, $2a + 7a^2 + 3$, $a - 5$.

Addition	Check. $a = 1$.
$3a - 2$	$3 - 2 = 1$
$a^2 - 1$	$1 - 1 = 0$
$7a^2 + 2a + 3$	$7 + 2 + 3 = 12$
$a - 5$	$1 - 5 = -4$
$8a^2 + 6a - 5$	$8 + 6 - 5 = 9$

2. Add and check, $3a^2 - 2ab + b^2$, $-4ab + 2b^2 - a^2$, $a^2 + b^2$.

ADDITION	CHECK. $a = b = 1$.
$3a^2 - 2ab + b^2$	2
$-a^2 - 4ab + 2b^2$	-3
$a^2 + b^2$	2
$3a^2 - 6ab + 4b^2$	1

EXERCISE

86. *Add the following, rearranging when necessary, and check the results in examples* 1 *to* 7:

1.
$$\begin{array}{r} 3a + 2b \\ 4a - 7b \\ -5a + 4b \\ \hline \end{array}$$

2.
$$\begin{array}{r} 4x^2 + 7x - 3 \\ 5x^2 - 7x + 2 \\ 9x^2 - 5x + 1 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 2a^2 - 7ab + b^2 \\ 4a^2 + 7ab \\ -3a^2 - 7b^2 \\ \hline \end{array}$$

4.
$$\begin{array}{r} 7ab - 3ac + 4bc \\ 4ab + 4ac - 5bc \\ -5ab - ac + 2bc \\ \hline \end{array}$$

5.
$$\begin{array}{r} 3a^2 + 4a - 7 \\ -10a^2 - 12a + 11 \\ 7a^2 + 8a - 4 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 5vx - 6v + 4x \\ -11vx + 13v - 7x \\ vx - v + x \\ \hline \end{array}$$

7. $3a^2 + 7ab - 2b^2$, $9b^2 - 3ab + a^2$, $5a^2 + 7b^2$.

8. $14a - 6b + 3c - 5d$, $9a + 7b - 4c - 9d$.

9. $.8a^2 - 3.47ab - 17.25ac + 3.75bc$, $-7.5a^2 + .47ab + 12ac - 7bc$.

10. $1.5x^2 - 3.2x - .07$, $8.04x - 2.1 + 4x^2$, $.3x - .75x^2$.

11. $14a - 6b + 3c - 5d$, $9a + 7b - 4c - 9d$, $5a - b + c + 14d$.

12. $3a^2 - 7b^2 + 10c^2$, $b^2 - 7c^2 + 3a^2$, $21c^2 - 7b^2 + a^2$.

13. $27ab + 5ac + bc$, $-4ac - 21ab$, $ab + 43ac$.

14. $a^2 + x - 10 + 2ax$, $2a^2 - 3x + 20 + ax$, $-5a^2 - 3x + 30 + 5ax$.

15. $5a^2b - 7a^3bc - 13b^2c^4 + 10$, $12a^2b + 8a^3bc - 10b^2c^4 + 2$.

Add the following:

16. $5m^2 - 6n^2$, $-3m^2 + 4mn + 5n^2$, $-m^2 - 3mn + 2n^2$.

17. $7a - 3b + 5c - 10d$, $2b - 3c + d - 4e$, $5c - 6a - 4e + 2d$, $-3b - 8c + 7a - e$, $21e - 16c + a - 5d$.

18. $3ab^2 - 4a^2b + a^3$, $-4ac^2 + 5ab^2 - c^3$, $-7b^3 + 2a^2b - 6ac^2$, $5a^3 - 11ab^2 - 12ac^2$.

19. $a^3 + 3a^2b + 3ab^2 + b^3$, $-5ab^2 + 3a^2b - b^3 + 3a^3$, $3ab^2 - 5a^2b$, $3b^3 - 3a^3$, $-5b^3 + 2a^2b - 4a^3 + 3ab^2$.

20. $5x^2 + 9$, $3x - 5$, $4x^2 - 5x - 1$, $-3x^2 + 2x + 6$.

21. $p^2q - q^3 + p^3 - pq^2$, $q^3 - pq^2 + p^2q - p^3$, $p^3 + pq^2 - p^2q$.

In examples 22 to 31, $X = 2a^2 + 3a - 1$, $Y = -3a^3 + 4a$, $Z = 3a^3 - 5a^2 + 2a$, $P = a^3 - a^2 - a - 1$, $Q = 15a^2 + a - 3$, $R = -a^3 + 7a - 5$.

22. Find $X + Y + Z$.

23. Find $X + Y + Y + R$.

24. Find $X + Y + R$.

25. Find $P + Q + X + Y$.

26. Find $X + Y + R + Q$.

27. Find $R + P + Z + X$.

28. Find $X + Q + (-13a^2 - 4a + 4)$.

29. Find $X + Y + Z + (3a^2 - 5a + 3)$.

30. Find $P + Q + R + (14a^2 - 8a - 10)$.

31. Find $X + Y + Z + P + Q + R$.

REVIEW EXERCISE

87. The four kinds of algebraic addition are:

1. **Addition of positive and negative numbers without literal parts.**
2. **Addition of like monomials.**
3. **Addition of unlike monomials.**
4. **Addition of polynomials.**

1. State the rule for each of the four kinds of addition.

Add the following:

2. $5a$, $(-3a)$, $7a$, $(-32a)$.

3. $-7a$, $(-4b)$, $(-19b)$, $22b$, $8a$.

4. $3x$, $3y$, $-3x$, $-2y$, x.

5. $3a^2 + 2b^2,\ 7a^2 - 8b^2,\ 4b^2,\ 2a^2 + 7b^2$.

6. $-3x^2 + 7x - 2,\ 4x - 7,\ 2x^2 + 9,\ x^2 - x - 1$.

7. $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{4}c,\ 2c - 3d + a,\ \frac{4}{3}b - \frac{3}{2}a + d$.

8. $2a + 7b - 3c,\ -5b - 4a,\ -b - c$.

9. $1.5m^2 - \frac{3}{4}mn - n^2 + (-.5m^2) + \frac{3}{4}mn + 3.5n^2$.

10. $6ab - a^2 + b^2 - 5ab + 2b^2 - 3a^2 + 2ab + 6a^2$.

11. $2.5xy - 3.5xy^2 + 4.5x^2y - 1.25xy + 2xy^2 - 2x^2y$.

12. $pq^2 + p^2q + p^2,\ q^2 + p^2q + 3p^2,\ 4pq^2 - 3q^2 - 2p^2$.

In examples 13 to 17 $A = x^2 - 3x + 7$, $B = 5x^2 + 7x - 7$, $C = x^2 - 6$, $D = 2x^2 - 4x - 3$.

13. Find $A + B + (-2x^2)$.

14. Find $C + B + D$.

15. Find $B + D + C + (-4x^2)$.

16. Find $A + C + B$.

17. Find $A + C + D$.

18. Determine by substituting numerical values whether $a + b - 2c$ is the sum of $3a + 2b - 5c$, $-4a + 7b + c$, and $2a - 8b + 3c$.

19. Add $3x^2 + 1$, $x + 2$, $4x^2 + x + 3$, and check by putting $x = 2$.

20. Collect the following terms: $13a + 5b + (-3a) + (-7b) + 8a + (-3)$.

Add the following:

21. $3a,\ a + 7b - 4c,\ -3a - 5b + 3c,\ -b + c$.

22. $-2x + 5x^2 + 7,\ 3x + 5 - 6x^2,\ 4x^2 + 4x - 3$.

23. $7a - 6b + 8c + 3d,\ 13a - 15b - 7c - 11d,$
$-6d + 5b + 7c - 2a,\ -5c + 10d + 28b - 17a$.

24. $234x + 36y + 18z,\ 24y + 12z - 12x,\ 25y - 44x - 16z,\ 12x - 17z,\ -85y + 6x$.

25. $5n^2 - 7p^2 - 8m^2,\ -2p^2 + m^2 + 3n^2,\ 9m^2 - 8n^2 + 7p^2,\ -2m^2 + 2p^2$.

Add the following:

26. $5.2\,x + .05\,y - 2.1\,z,\ -.6\,x - .5\,y + .1\,z,\ 3.5\,x + .7\,y - .5\,z.$

27. $2\,x^2 - 5 + 8\,x,\ 2 - 4\,x,\ 17 - x - x^2.$

28. $3\,a - 7\,b,\ \ -8\,c + 4\,d - 8\,e,\ \ 7\,a + 6\,e + 9\,c - 5\,d + 8\,b.$

29. $1.34\,m - 7.6\,n - .397\,p,\ -81.7\,p - 9.4\,m - 8.7\,n,$
$9.76\,m + 4.33\,p + 9.3\,n.$

30. $41.6\,q - 43.1\,x + 37.8\,y,\ .09\,y - 5.37\,x - 4.05\,q,$
$1.97\,x - 4.1\,y - .8\,q.$

31. $.3\,x^2 + .1\,y^2 - .3\,yz - .1\,z^2,\ .2\,xy - .3\,y^2 + .3\,yz,$
$-.4\,x^2 - .2\,xy + .1\,y^2 + .1\,z^2.$

32. Solve $3\,x + 2\,x + 5 + 9 = 27 + (-3).$

33. Solve $7\,x + (-3\,x) + (-x) = 5 + (-3) + 12.$

34. Find the sum of five numbers, the first number being $2\,x$ and each succeeding number being $3\,a$ greater than the preceding.

35. If a passenger ticket costs x cents a mile and it costs 4 cents to carry a bicycle each 25 miles, how much is the cost of both for 250 miles?

36. Through how many degrees of longitude does a ship sail in going from $-18°$ to $+37°$?

37. The oldest known mathematical manuscript was written about -1700 (1700 B.C.). How long ago was it written?

38. A merchant's capital was diminished by \$1400 and then amounted to \$4500. What was his capital at first?

SUGGESTION. Let $x =$ number of dollars at first.

39. At a certain election A received 113 more votes than B. The number of votes cast for both was 847. How many votes did each receive?

SUGGESTION. Let $x =$ number of votes B received.
Hence $x + 113 =$ number of votes A received.
Then $x + x + 113 = 847$. (By the conditions.)

40. A rectangular field is twice as long as it is wide and its perimeter is 360 rods. Find the length and the width of the field.

41. A ball team played 20 games and won three times as many as it lost. How many games were won and how many were lost?

SUGGESTION. Let x = number of games lost.

42. A boy paid x cents for a bat, twice as much for a ball, and 20 cents less for a mask than for both ball and bat. How much did each cost him if he spent $2.20 all together?

SUGGESTION. Change $2.20 to 220 cents.

43. The larger of two numbers is three times the smaller and their sum is 84. Find the numbers.

44. The larger of two numbers exceeds the smaller by 10, and the sum of the two numbers is 94. Find the numbers.

45. The girls in a certain high school outnumbered the boys by 122. The entire enrollment was 2742. How many boys were there in the school?

46. A woodworking class spent $32.50 more for jack planes than for try-squares. If both tools together cost $50, find the cost of each kind.

47. One farmer by spraying his potatoes raises 30 bushels more on an acre than his neighbor. If both together raise 400 bushels, how many bushels does each raise?

48. In 1910 Jerry Moore of South Carolina won a prize in a boys' corn raising contest. In 1913 Walker Dunson of Alabama raised 4 bushels more corn on an acre than Jerry Moore's record yield. The total yield on the two acres was 460 bushels. How many bushels did each raise?

49. In 1914 the Allred boys, Luther, Clarence, Elmer, and Arthur, of Georgia, raised on four one-acre plots of land 824 bushels of corn. Clarence raised 10 bushels more than Elmer and 7 bushels less than Luther, while Arthur raised 43 bushels less than Elmer. How many bushels did each raise?

IV. SUBTRACTION

SUBTRACTION OF LIKE MONOMIALS

ORAL EXERCISE

88. 1. Define subtraction. (§ 42.)

2. State the rule for subtracting signed numbers. (§ 44.)

Subtract the following:

3. $3-4$; $7-8$; $10-15$.
4. $3-(-4)$; $9-(-15)$; $4-(-4)$.
5. $-3-5$; $-7-8$; $-15-2$.
6. $-1-(-3)$; $-3-(-4)$; $-5-(-8)$.
7. 7 ft. $-$ 5 ft.; $7f-5f$.
8. $20°-30°$; $20d-30d$.
9. 60 lb. $-$ 35 lb.; $60p-35p$.
10. 85 acres $-$ 42 acres; $85a-42a$.
11. \$73 $-$ \$21; $73d-21d$.
12. $-17°-5°$; $-17d-5d$.
13. $-10x-3x$; $-5a-3a$.
14. $-17x-22x$; $-8x-8x$.
15. $20-(-10)$; $20d-(-10d)$.
16. $8-(-5)$; $8x-(-5x)$.
17. $18-(-1)$; $18r-(-r)$.
18. $49xy-45xy$; $49\cdot 12-45\cdot 12$.

89. From the examples of § 88 we derive the following rule:

To subtract a monomial from a like monomial, change the sign of the subtrahend and add the resulting number to the minuend. (See §§ 42 to 44.)

The student should change the sign mentally.

Examples

1. Subtract $11m$ from $17m$.

$$\begin{array}{r}17m\\ \underline{11m}\\ 6m\end{array}\qquad \text{Add } -11m \text{ to } 17m.$$

CHECK. Add $6m$ to $11m$.

2. Subtract $4x$ from $-6x$.

$$\begin{array}{r}-6x\\ \underline{4x}\\ -10x\end{array}\qquad \text{Add } -4x \text{ to } -6x.$$

CHECK. Add $-10x$ to $4x$.

3. $-9a-4a=-13a$.

The minuend is $-9a$, the subtrahend is $4a$. Therefore add $-4a$ to $-9a$. Let the student check the result.

4. $-5r-(-7r)=2r$.

Add $+7r$ to $-5r$. Let the student check the result.

5. $6a-8a=-2a$. Why?

EXERCISE

90. *Subtract the following:*

1. $\begin{array}{r}15a\\ \underline{2a}\end{array}$	5. $\begin{array}{r}2a\\ \underline{a}\end{array}$	9. $\begin{array}{r}15mn^2\\ \underline{17mn^2}\end{array}$	13. $\begin{array}{r}-mnp^2\\ \underline{mnp^2}\end{array}$
2. $\begin{array}{r}21b\\ \underline{23b}\end{array}$	6. $\begin{array}{r}11m^2\\ \underline{7m^2}\end{array}$	10. $\begin{array}{r}0\\ \underline{-5}\end{array}$	14. $\begin{array}{r}-mnp^2\\ \underline{-mnp^2}\end{array}$
3. $\begin{array}{r}7x\\ \underline{10x}\end{array}$	7. $\begin{array}{r}7xy\\ \underline{-3xy}\end{array}$	11. $\begin{array}{r}0\\ \underline{-5a}\end{array}$	15. $\begin{array}{r}5m^3n^2\\ \underline{-18m^3n^2}\end{array}$
4. $\begin{array}{r}-7x\\ \underline{10x}\end{array}$	8. $\begin{array}{r}-11ab\\ \underline{-2ab}\end{array}$	12. $\begin{array}{r}-21xyz\\ \underline{-4xyz}\end{array}$	16. $\begin{array}{r}-18x^5y^4\\ \underline{-27x^5y^4}\end{array}$

17. $-5bc$ from $-3bc$.

18. $18b^5$ from $-4b^5$.

19. $-13a^2b$ from $24a^2b$.

20. $3ab^2$ from $-7ab^2$.

21. $-5cd^3$ from $14cd^3$.

22. $-2.25m^2n$ from $-3.5m^2n$.

23. From the sum of $7ab^2c$ and $-11ab^2c$ take $-4ab^2c$.

24. Take the sum of m^2n and $-6m^2n$ from the sum of $-4m^2n$ and $3m^2n$.

25. The minuend is 0 and the subtrahend is $-3x$. What is the difference?

26. The minuend is $-27xy$ and the difference is $5xy$. What is the subtrahend?

27. The subtrahend is $-5.2x$ and the difference is $.05x$. What is the minuend?

SUBTRACTION OF UNLIKE MONOMIALS

ORAL EXERCISE

91. 1. What are unlike monomials?

2. What does $a - b$ mean?

3. What length remains if 10 feet are cut from a rope 32 feet long? if x feet are cut from a rope 32 feet long? if b feet are cut from a rope a feet long?

4. How much have you left if you have 16 cents and spend 7 cents? if you have 16 cents and spend x cents? if you have a cents and spend x cents?

5. If you throw a stone vertically upward h feet, how high is it after it has fallen d feet? How high is the stone if $h = 62$ and $d = 21$?

6. If the enrollment in a class is m girls and n boys and there are x girls and y boys absent, what is the attendance?

7. How are unlike monomials added?

92. To subtract a monomial from an unlike monomial, change the sign of the subtrahend and add the resulting number to the minuend.

EXAMPLES

1. Subtract $2a$ from $3x$.

$$\begin{array}{l} 3x \\ \underline{2a} \\ 3x - 2a \end{array}$$

The subtrahend when its sign is changed becomes $-2a$; adding this to $3x$ gives $3x - 2a$.

The result may be checked as usual, $3x - 2a + 2a = 3x$.

2. Subtract -6 from a.

$$\begin{array}{r} a \\ -6 \\ \hline a+6 \end{array}$$

The subtrahend with its sign changed becomes 6. Adding, we have $a + 6$.

3. Subtract -3 from $4x$.

$$\begin{array}{r} 4x \\ -3 \\ \hline 4x+3 \end{array}$$

$4x-(-3)=4x+3$.

EXERCISE

93. *Subtract:*

1. $11b$ from $17a$.
2. $-7x$ from $5y$.
3. $-5b$ from $2a$.
4. b from $-6a$.

5. $\begin{array}{r} 7x \\ \underline{-7} \end{array}$ 6. $\begin{array}{r} 5a^2b \\ \underline{3ab^2} \end{array}$ 7. $\begin{array}{r} q \\ \underline{-p} \end{array}$ 8. $\begin{array}{r} d \\ \underline{-c} \end{array}$

9. $7c^2$ from $5ab$.
10. $-4y$ from $3x^2$.
11. $-5xy^2z$ from $-4x^2yz$.
12. $-x$ from $a+2b+a-2b$.
13. -10 from $3x+2y+y-3x$.
14. 10 from $3x+2y+(-2y)-7x$.

Collect terms:

15. $3x+5y-5z-2x+(-3z)-3z-2x$.
16. $-4+x-(-x)+8$.
17. $4a-5a+(-2a)+7b-(-3b)$.
18. $-5+(-7)-3-8-(-17)-(-b)$.
19 $12a+3x-4z-4x-(-5z)-z+x-6a$.
20. $3x+4y-(-5z)-2x-3y-4z-(-x)+y-(-z)$.
21. $22x-23y-(-24x)$.
22. $44+21z-(-22)$.
23. $17a-(-11a)+(-13b)-16b$.
24. $21c-15c+28d-(-6d)$.
25. $18a-21a-10d+8d$.

SUBTRACTION OF POLYNOMIALS

ORAL EXERCISE

94. *Subtract the following :*

1.	11 lb. 7 oz.	$11p+7z$	4.	12 mi. 20 rd.	$12m+20r$
	8 lb. 5 oz.	$8p+5z$		8 mi.	$8m$
2.	7 ft. 9 in.	$7f+9i$	5.	3 ft. 7 in.	$3f+7i$
	4 ft. 3 in.	$4f+3i$		5 in.	$5i$
3.	13 mi. 40 rd.	$13m+40r$	6.	$5a+2b$	$8x-5y$
	11 mi. 28 rd.	$11m+28r$		$2a+4b$	$7x+4y$
				$3a-2b$	$x-9y$

Let the student check the results in the last example by adding the difference to the subtrahend.

7.	$7x^2-3x$ $4x^2-5x$	11.	$3a+2b+7c$ $a+b+c$
8.	$2ab+7c$ $8ab+9c$	12.	$5a-8b+7c$ $2a+3b-4c$
9.	$17b+2p$ $14b+3p$	13.	$4x+3y-7z$ $5x-2y-3z$
10.	$-5x+7y$ $-3x-8y$	14.	$6x-y-z$ $2x+y-z$

95. From these examples we derive the following rule:

To subtract one polynomial from another, write the subtrahend under the minuend, with like terms in the same vertical column. Change the sign of each term in the subtrahend, and proceed as in addition.

EXAMPLES

1. Subtract $6x^2-3x-12$ from $15x^2+8x+1$.

SUBTRACTION CHECK. $x=1$.

$$\begin{array}{rcr} 15x^2+8x+1 & = & 24 \\ 6x^2-3x-12 & = & -9 \\ \hline 9x^2+11x+13 & = & 33 \end{array}$$

This result might be checked by adding the difference to the subtrahend. The sum should equal the minuend.

2. From $6mn + 3m^2 + n^2$ take $5n^2 + m^2 - 3mn$.

$$\begin{array}{r} 3m^2 + 6mn + n^2 \\ \underline{m^2 - 3mn + 5n^2} \\ 2m^2 + 9mn - 4n^2 \end{array}$$

Arrange the terms in descending powers of m.
CHECK. Add the difference to the subtrahend.

3. Subtract $4a - 3b + c$ from $2b - 3c$.

$$\begin{array}{r} 2b - 3c \\ \underline{4a - 3b + c} \\ -4a + 5b - 4c \end{array}$$

Let the student check the result.

EXERCISE

96. *Subtract, and check results as directed:*

1. $\begin{array}{r} 13m + 40r \\ \underline{11m + 89r} \end{array}$

2. $\begin{array}{r} 8.1r - 1.5s \\ \underline{4r + .2s} \end{array}$

3. $\begin{array}{l} 5a + b \\ \underline{8a} \end{array}$

4. $\begin{array}{l} 3a \\ \underline{2a - b} \end{array}$

5. $\begin{array}{l} 3a - b - c \\ \underline{2a \ + 5c} \end{array}$

6. $\begin{array}{r} a - 2b + c \\ \underline{5a - 5b} \end{array}$

7. $\begin{array}{l} 2.5r - 4s + 3 \\ \underline{1.3r - 2s} \end{array}$

8. $\begin{array}{l} 8a \\ \underline{6a - b + 3c} \end{array}$

9. $\begin{array}{l} 21d + 14m + 27s \\ \underline{16d + 18m + 27s} \end{array}$

10. $\begin{array}{r} 11h + 41m + 56s \\ \underline{7h + 59m + 34s} \end{array}$

11. $\begin{array}{l} 36m + 37r + 42y \\ \underline{25m + 71r + 84y} \end{array}$

12. $\begin{array}{l} 2a - 3b \\ \underline{5a \ - 7} \end{array}$

13. $\begin{array}{r} x^2 + 3x - 7 \\ \underline{5x^2 - x - 1} \end{array}$

14. $\begin{array}{r} 2a + b \\ \underline{a - 3b + 7c} \end{array}$

15. $\begin{array}{r} 5b - 2 \\ \underline{-4a \ - c} \end{array}$

16. Subtract $2a - 3b + c$ from $a + b + c$.

17. $4m^2 - 5mn - 7n^2$ minus $3n^2 + m^2 - 5mn$.

18. The subtrahend is -2, the minuend is $a + b - 2$. Find the difference.

19. The difference is $a + b - c$ and the subtrahend is $2a - 3b + c$. Find the minuend.

20. From the sum of $4a^2 + 3ab + 7b^2$ and $b^2 - 7a^2$ take $a^2 + b^2 - ab$.

Collect terms in examples 21 *to* 25.

21. $(-7) + 3 - (-5) + 12 + (-12) - 8$.

22. $5a + 2b - 3b + (-2b) - (-7a)$.

23. $-8m + (-3n) + 4m + 6n - 7m + n$.

24. $10 + (-14) - 6 + 11 + (-4) - (-13)$.

25. $7p + 8q - (-3p) + (-7q) + 4p - 11q$.

26. From $4x^2 + 2xy - 3y^2$ take $x^2 - xy + 2y^2$.

27. From $a^3 + 3a^2b + 3ab^2 + b^3$ take $a^3 - 3a^2b + 3ab^2 - b^3$.

28. From the sum of $m^2 + 3mn - n^2$ and $2m^2 - 5mn + n^2$ subtract $2m^2 - 2mn + n^2$.

29. Subtract the sum of $ax^2 + bx + c$ and $2ax^2 - 3bx - 2c$ from $4ax^2 - 2bx + c$.

30. What must be subtracted from $a+b+c$ to give $a+b-c$?

31. What number added to $3ax + 4by - 7cz$ will give $ax + by + cz$?

32. The sum of two algebraic expressions is $3x^3 - 4$ and one of them is $x^3 + x^2 + 1$. What is the other?

33. If dates B.C. are considered negative and A.D. positive, how many years are there from -509 to -27? from -34 to $+48$? from -480 to $+60$?

34. From the sum of $2a^2 - 3ab + 4b^2$, $a^2 + 2ab - 2b^2$, and $2ab$ subtract $4a^2 - ab - 2b^2$ plus $a^2 + ab + b^2$.

35. From $a^2x^3 + abx^3 + bcx^2$ subtract $a^2x^3 - abx^3 - bcx^2$.

36. Write five consecutive numbers of which x is (1) the largest, (2) the smallest, (3) the middle number. Add the 5 numbers in each of the parts (1), (2), (3). Which gives the simplest result?

37. Write five consecutive numbers of which $2n$ is (1) the largest, (2) the smallest, (3) the middle number. Add the five numbers in each part. Which gives the simplest result?

38. Write five consecutive odd numbers of which $2n+1$ is the largest. Write five consecutive odd numbers of which $2n-1$ is the smallest.

39. Add the five numbers in example 38, first part.

40. A man's salary is x dollars. How much was it 5 years ago if it has been increased b dollars each year?

41. A machine cuts pieces 3 inches long from a rod 10 feet long. How much is left after x cuts?

42. If an automobile is worth m dollars and depreciates $2n$ dollars in value the first year and n dollars each succeeding year, what is its value at the end of 5 years?

Evaluate the answer if $m = 1850$, $n = 231.50$.

43. From $x^4 + 3ax^3 - 2bx^2 + 3cx - 4d$ subtract

$$3x^4 + ax^3 - 4bx^2 + 6cx + d.$$

If $m=a^2+b^2+c^2$, $n=a^2+b^2-c^2$, $p=a^2-b^2+c^2$, $q=b^2+c^2-a^2$, evaluate 44 to 55:

44. $m+p$.

45. $0-p$.

46. $m-n-p-q$.

47. $m-n$.

48. $m+n+p+q$.

49. a^2-p+q.

50. $m-p$.

51. $m-n-p+q$.

52. b^2+p-q.

53. b^2-n.

54. $m-n+p-q$.

55. $p+q+p+q$.

PARENTHESES

97. In algebra, as in arithmetic, it is frequently necessary to group several numbers that are to be regarded as a single number, or to indicate that the result of several operations is to be taken as a whole. The **parenthesis**, ( ), is generally used to inclose such a group of numbers.

Thus, $a^2 + b^2 - (2a^2 + ab - 5b^2)$ means that the expression $2a^2 + ab - 5b^2$ is to be subtracted from $a^2 + b^2$.

It is often necessary to inclose within a parenthesis parts of an expression already inclosed within a parenthesis. For this purpose the **brackets**, [ ], and the **braces**, { }, are used. The use of the **vinculum**, $\overline{\quad}$, is avoided as far as possible on account of the difficulty in printing it. All these symbols have the same use as the parenthesis and are generally referred to as parentheses.

98. It has already been explained in § 55, that in a series of indicated operations, the multiplications and divisions are to be performed in the order given before the additions and subtractions.

Thus, $12 + 3 \times 5 = 12 + 15 = 27$; $6 \div 2 \times 3 + 7 \times 2 = 9 + 14 = 23$.

If the operations are not to be performed in this accepted order, certain numbers of a series may be inclosed within a parenthesis.

The operations within a parenthesis take precedence over all others. When these operations are performed the resulting number takes the place of the parenthesis in the series of operations.

Thus, $12 + 3 \times 5$ must not be confused with $(12 + 3) \times 5$, for $12 + 3 \times 5 = 27$, while $(12 + 3) \times 5 = 15 \times 5 = 75$.

Also, $27 \div 3 + 6$ must not be confused with $27 \div (3 + 6)$, for $27 \div 3 + 6 = 9 + 6 = 15$, while $27 \div (3 + 6) = 27 \div 9 = 3$.

EXERCISE

99. *Write and evaluate examples* 1 *to* 3.

1. 76 diminished by the sum of 27 and 13.
2. 25 increased by the difference between 23 and 6.
3. 86 diminished by the difference between 118 and 97.
4. What is the difference in meaning between the expressions, $a - b + c$ and $a - (b + c)$?
5. What is the difference in meaning between $a - (b - c)$ and $a - b - c$?

Evaluate examples 6 *to* 10.

6. $12-7-(2+1)$ and $12-7-2+1$.
7. $12-(7-2+1)$ and $12-(7-2)+1$.
8. $12-7-(2+1)$ and $12-(7-2+1)$.
9. $63-(24-15-8)$ and $63-24-(15-8)$.
10. $79-(38-17-14-2+9)$.

100. Removal of Parenthesis. The expression $5+(7-4)$ means that $7-4$ is to be added to 5;

$$\text{or } 5+(7-4)=5+3=8.$$
$$\text{But } 5+7-4=8.$$
$$\therefore 5+(7-4)=5+7-4.$$

Also $a+(2a-3b+c)$ means that $2a-3b+c$ is to be added to a,

$$\text{or } a+(2a-3b+c)=a+2a-3b+c$$
$$=3a-3b+c.$$

A parenthesis inclosing any number of terms and preceded by a plus sign may be removed without changing the signs of the terms inclosed in the parenthesis.

ORAL EXERCISE

101. *Remove the parentheses and collect the terms as much as possible:*

1. $7+(4+5)$.
2. $7d+(4d+5d)$.
3. $19f+(12f+7i)$.
4. $17x+(10x-y)$.
5. $7t+(14t-5r)$.
6. $8x+(10a-x)$.
7. $18b+(7b-9b)$.
8. $2r+(10-3r)$.
9. $(2x-3y)+(-x-y)$.
10. $(5p-7q)+(-3p+5q)$.

102. The expression $11-(8+2)$ means that $8+2$ is to be subtracted from 11,

$$\text{or } 11-(8+2)=11-10=1.$$
$$\text{But } 11-8-2=1.$$
$$\therefore 11-(8+2)=11-8-2.$$

The expression $3a + 2b - (a - 3b)$ means that $a - 3b$ is to be subtracted from $3a + 2b$. The rule for subtraction is: "Change the signs of the subtrahend and add the result to the minuend." To apply this rule in the present case we may remove the parenthesis, changing the signs of the terms within the parenthesis, and collect terms.

$$3a + 2b - (a - 3b) = 3a + 2b - a + 3b$$
$$= 2a + 5b.$$

Also, $3x^2 - x - (2x^2 - 2x + 7) = 3x^2 - x - 2x^2 + 2x - 7$
$$= x^2 + x - 7.$$

The student may verify the answer by ordinary subtraction.

$$\begin{array}{r} 3x^2 - x \phantom{{}+7} \\ 2x^2 - 2x + 7 \\ \hline x^2 + x - 7 \end{array}$$

A parenthesis inclosing any number of terms and preceded by a minus sign may be removed provided the sign of each term inclosed by the parenthesis is changed.

ORAL EXERCISE

103. *Remove the parentheses and simplify as much as possible:*

1. $32 - (17 + 6)$.
2. $32 - (17 - 6)$.
3. $15 - (13 + 11)$.
4. $15 - (13 - 11)$.
5. $a - (b + c)$.
6. $a - (b - c)$.
7. $12p - (3p + q)$.
8. $12p - (-3p - q)$.
9. $15m - (6m + 2m)$.
10. $15m - (-6m - 2m)$.
11. $17r - (21r - 14r)$.
12. $17r - (-21r + 14r)$.
13. $-(7s - 13s) - 15s$.
14. $-(x + y) - p + q$.
15. $(a - b) + (-a + b)$.
16. $(a - b) - (-a + b)$.
17. $(2 + 3m) - (3 - 2m)$.
18. $(2 - 3m) + (3 + 2m)$.
19. $23a - (16 + 5a)$.
20. $23a + (-16 - 5a)$.

104. Sometimes one or more parentheses are inclosed within a parenthesis. In this case either the outer or the inner parenthesis may be removed first. The beginner will find it advisable to remove the inner parenthesis first.

$$18x - \{4y - [9x - (2y + 3x) + y]\}$$
$$= 18x - \{4y - [9x - 2y - 3x + y]\} \quad \text{(Removing ( ).)}$$
$$= 18x - \{4y - 9x + 2y + 3x - y\} \quad \text{(Removing [ ].)}$$
$$= 18x - 4y + 9x - 2y - 3x + y \quad \text{(Removing \{ \}.)}$$
$$= 24x - 5y \quad \text{(Collecting terms.)}$$

EXERCISE

105. *Simplify by removing parentheses and combining like terms.*

1. $(x - y - z) - (2x + y - 3z)$.
2. $25 - (3 + 4 \times 2) + 6$.
3. $1 + m - n - (21 - m + 2n)$.
4. $(x^2 - x) - (x^2 - 2x + 3) - (x^2 + 2x - 6)$.
5. $(x^2 + x) - (x^2 - 1)$.
6. $x^2 + 2ax + a^2 - (x^2 - 2ax + a^2)$.
7. $8m - (4m + 2n) + (5m - 6n)$.
8. $a^2 - (a^2 - 2ab) + (-2ab + a^2)$.
9. $(4p - q) - [2p - (q - p) + 2p]$.
10. $y + [(m - n) + (m + p)]$.
11. $y + [(m + n) - (n + p)]$.
12. $y - [(m - n) - (p - n)]$.
13. $y - [(m + n) - (n - p)]$.
14. $7a - 2b - [(3a - c) - (2b - 3c)]$.
15. $2a - (3b + 2c) - \{5b - 3a - (a + b) + 5c - [2a - (c - 2b)]\}$.
16. $16 - x - \{-7x - [8 - 9x - (3 - 6x)]\}$.
17. $x^4 - \{4x^3 - [6x^2 - (4x - 1)]\} - (x^4 + 4x^3 + 6x^2 + 4x + 1)$.
18. $4.04a - [.275y - (.5b - 3.875a) + 3.6y] - (.165a - .375y)$.
19. $ab - [(3bce - 2ab) - (5bce - bef) + (3ab - 3bef)]$.

Simplify:

20. $1-[-(2-x)]+[4x-(3-6x)]+4-(6x-5)$.

21. $3m-38n-(57p+15q)-(12p-38q+48n-50m)$.

106. *Inserting parenthesis:*

$3-2+3=3+(-2+3)$ and $3-2+3=3-(2-3)$.

Also $a+b-c+d=a+b+(-c+d)$

and $a+b-c+d=a+b-(c-d)$.

Let the student verify these results by removing the parenthesis according to the rules of §§ 100 and 102.

From these results we draw two conclusions:

1. The value of a polynomial is not changed if any of its terms are inclosed within a parenthesis preceded by a plus sign.

2. The value of a polynomial is not changed if any of its terms are inclosed in a parenthesis preceded by a minus sign, provided the sign of each term inclosed is changed.

EXERCISE

107. *Inclose in a parenthesis preceded by the plus sign the* 3*d and* 4*th terms in examples* 1 *to* 5.

1. $a+b-c+m$.
2. $4m-3x+y-2a+c$.
3. $am+bx-ac-mx+2b$.
4. $a^2+b^2+m^2-2mxc+2ax$.
5. $y^3+3ax^2+3a^2x+a^3-m^2+b^2$.

Inclose in a parenthesis preceded by the minus sign, the 2d, 3d, and 4th terms of examples 6 to 10.

6. $p^2+q-pq^2+m^2-n^2-3mny$.
7. $m-n-p+pq-7y$.
8. $m^2+2mn+n^2+p^2-2pq$.
9. $p^3-3p^2q+q^3-n^3$.
10. $y^2+2pq^2+p^2-n^2+3pq$.

11. Inclose in a parenthesis preceded by the minus sign, the last three terms of examples 6, 7, 8.

12. Inclose the last four terms of examples 8 and 10 in a parenthesis preceded by a minus sign.

EQUATIONS INVOLVING ADDITION, SUBTRACTION, AND PARENTHESES

ORAL EXERCISE

108. 1. If $4x + 3x = 7$, what does x equal?

2. If $4x - 3x = 7$, what does x equal?

3. If $2x = x + 2$, what does x equal?

4. If $3x + 2 = 2x + 3$, what does x equal?

Solve the following equations and check the results:

5. $x + 2x = 3$.
6. $2x - x = 5$.
7. $7p - 5p = 6$.
8. $7p + 5p = 12$.
9. $11x - 7x = 8 - 4$.
10. $2 + 3x = 8$.
11. $2r + 5 = 6$.
12. $4m - 4 = 4$.
13. $8q = 15 + (-7)$.
14. $16x + 9x = 50$.
15. $x + 3 = 12$.
16. $2x + 6 = 12$.
17. $2v + 5v = 21$.
18. $17n - 7n = 30$.
19. $15x + 6x = 42$.
20. $15x - 6x = 18$.

109. Solve $5x - (12 - x) = 3 + (x + 3)$.

SOLUTION. $5x - (12 - x) = 3 + (x + 3)$.

$5x - 12 + x = 3 + x + 3$. (Removing parentheses.)

$6x - 12 = 6 + x$. (Collecting terms.)

$5x - 12 = 6$. (Subtracting x from both members.)

$5x = 18$. (Adding 12 to both members.)

$x = 3\frac{3}{5}$. (Dividing both members by 5.)

The result may be checked by substituting $3\frac{3}{5}$ for x and simplifying.

$$5 \times 3\tfrac{3}{5} - (12 - 3\tfrac{3}{5}) = 3 + (3\tfrac{3}{5} + 3).$$
$$18 - 8\tfrac{2}{5} = 3 + 6\tfrac{3}{5}.$$
$$9\tfrac{3}{5} = 9\tfrac{3}{5}.$$

EXERCISE

110. *Solve the following equations and check the results:*

1. $5x - 1 = 14$.
2. $8x + \frac{1}{2} = 16\frac{1}{2}$.
3. $7p - \frac{2}{3} = 13\frac{1}{3}$.
4. $9p - 5p = 36$.

Solve and check:

5. $m-(-2m+3)=6.$
6. $2s-5=4-(5-s).$
7. $5y-(3y+3)=3.$
8. $15a-10a=45.$
9. $27b=31-(-20b+4).$
10. $11x+14=x-16.$
11. $16c-(5+11c)=5.$
12. $5x+5=2x+6.$
13. $11n-23=7n-4.$
14. $l-(-l+3)=3.$
15. $4y-11=3-y.$
16. $2x+(3+4)=x-(5+6).$
17. $m+(3+4m)=3m+3.$
18. $x-14=14-x.$
19. $4x+(2x+2)=2x+(x+1).$

PROBLEMS SOLVED BY MEANS OF ALGEBRAIC EQUATIONS

111. If there are two unknown numbers to be found in a problem, two distinct relations of the numbers must either be given or implied. Generally the method of making the equation is as follows:

1. Introduce some letter as *x*, to represent one of the unknown numbers, preferably the smaller one.

2. Express the other unknown in terms of *x* by using one of the two given relations.

3. Make an equation by using the other relation.

EXERCISE

112. 1. The sum of two numbers is 24, and one of them is twice as large as the other. Find the two numbers.

Solution. Let $x=$ the smaller number.

Hence $2x=$ the larger number.

Then $x+2x=24$, (By the first condition stated.)

or $3x=24.$

$\therefore x=8$, the smaller number,

and $2x=16$, the larger number.

2. The sum of two numbers is 18 and one is five times as large as the other. Find the numbers.

3. The sum of two numbers is 40, and the larger exceeds the smaller by 10. Find the two numbers.

4. The difference of two numbers is 10, and the larger number is 3 times the smaller. Find the numbers.

5. Find two parts of 53, one of which exceeds the other by 11.

6. Find two parts of 28, one of which exceeds twice the other by 4.

SUGGESTION. If $x =$ the smaller part, $2x + 4 =$ the larger part.

7. Find two numbers whose sum is 23 and whose difference is 8.

8. Find two consecutive numbers whose sum is 73.

SUGGESTION. Since the numbers are consecutive the larger exceeds the smaller by 1.

9. Find two consecutive numbers whose sum is 33.

10. Find three consecutive numbers whose sum is 33.

11. Separate 153 into two parts of which the larger exceeds two times the smaller by 30.

12. If the sum of two consecutive numbers is 45, find the numbers.

13. A rectangle is 20 feet longer than it is wide, and its perimeter is 160 feet. Find its length and width.

SUGGESTION. Let $x =$ the number of feet in the width.
Hence $x + 20 =$ the number of feet in the length.
Then $x + x + (x + 20) + (x + 20) = 160$.
Let the student solve the equation.

14. A rectangle is twice as long as it is wide and its perimeter is 150 feet. Find its length and width.

15. The length of a rectangular lot exceeds twice the width by 50 feet and the perimeter is 364 feet. Find its dimensions.

16. If n is the middle one of five consecutive numbers, how would you represent the other four numbers? Find five consecutive numbers whose sum is 45.

17. Three men divide $300 so that the second has $25 less than the first, and the third $50 more than the second. How many dollars does each man get?

SUGGESTION. Let $x =$ the number of dollars the first receives.
Hence $x - 25 =$ the number of dollars the second receives,
and $x + 25 =$ the number of dollars the third receives.
Then $x + (x - 25) + (x + 25) = 300$.
Let the student solve the equation.

18. The combined weight of the largest steam locomotive and the largest electric locomotive in the United States is 381 tons. The steam locomotive weighs 57 tons more than 3 times the weight of the electric locomotive. What is the weight of each?

19. The combined cost of the Panama and Suez Canals was approximately 394 million dollars. The Panama Canal cost 5 million dollars less than 20 times as much as the Suez Canal. What was the approximate cost of each?

20. It is 1274 miles further from London to New Orleans than it is from London to New York, and the sum of the two distances is 7740 miles. Find the distance from London to each place.

21. Two day-rate telegrams were sent from New York, one to Detroit and one to Winnipeg, Manitoba. The two messages cost $ 1.10. The message to Detroit cost 35 cents less than the one to Winnipeg. Find the cost of each.

HINT. Change $ 1.10 to 110 cents.

22. Two six-word Marconigrams (wireless telegrams) were sent from London, one to New York and one to St. Louis. The message to St. Louis cost (in United States money) 36 cents more than the one to New York, and the total cost was $ 2.10. Find the cost of each.

V. MULTIPLICATION

ORAL EXERCISE

113. 1. What is the law of signs in multiplication? (§ 49, 2.)

Find the products:

2. $(-8)(-2)\cdot 2$; $(-2)^2$; $(-2)^3$.
3. $(-1)^2$; $(-1)^3$; $(-1)^4$; $(-1)^5$.
4. $(-2)(-3)^2$; $(-2)^2(-3)$; $(-2)^2\cdot 3$.
5. 2×3 yd.; $2\times 3y$; 3×5 mi.; $3\times 5m$.
6. $4\times 10x$; $5\times 8y$; $7\times 2a$; $4\times 3ab$.
7. $a\cdot b$; $a(-b)$; $(-a)\ b$; $(-a)(-b)$.
8. $7\cdot(-3)$; $7\cdot(-3a)$; $5\cdot(-4xy)$; $8\cdot(-3ab)$.
9. $-2\cdot 3$; $-2\cdot 3a$; $-4\cdot 7b$; $-5\cdot 2ab$.
10. $-4\cdot(-5)$; $-4\cdot(-5a)$; $-4\cdot(-7ab)$; $-7\cdot(-5xy)$.
11. $2\cdot(-3b)$; $-4\cdot 3ab$; $-5\cdot(-9x)$; $4\cdot(-2y)$.
12. $3\times 4\times(-2)$; $3\times 4\times(-2a)$; $-4\times 3b\times 2$.
13. $2\times(-3)\times 4$; $2\times(-3)\times(-4)$; $-2\times(-3)\times(-4)$.

114. The Law of Exponents in Multiplication. Define exponent and base. (§ 64.)

Since $2^2 = 2\cdot 2$ and $2^3 = 2\cdot 2\cdot 2$,

therefore $2^2\times 2^3 = (2\cdot 2)\times(2\cdot 2\cdot 2) = 2^5$ or 2^{2+3}.

Similarly $a^2\cdot a^3 = (a\cdot a)\times(a\cdot a\cdot a) = a^5$ or a^{2+3}.

Similarly $a^2\cdot a^2\cdot a^3 = (a\cdot a)\times(a\cdot a)\times(a\cdot a\cdot a) = a^7$ or a^{2+2+3}.

Also since $a^m = a\cdot a\cdots$ to m factors and $a^n = a\cdot a\cdots$ to n factors therefore $a^m\cdot a^n = a^{m+n}$.

The equation, $a^m \cdot a^n = a^{m+n}$, is the **law of exponents for multiplication** stated in algebraic symbols. In words we have:

In multiplying powers of the same base the exponent of any base in the product is equal to the sum of its exponents in the factors.

Also since $(a^3)^2 = a^3 \cdot a^3 = a^6$, we have $(a^3)^2 = a^{3\times 2}$, and in general
$$(a^m)^n = a^{mn}.$$

115. The student must note that the law $a^m \cdot a^n = a^{m+n}$ applies only when the bases are the same. It must also be remembered that when no exponent is written, the exponent 1 is understood.

Thus, $x \cdot x^2 = x^{1+2} = x^3$.

The base for any given exponent is the number symbol immediately preceding it.

Thus, $3\,ab^2$ means $3\,a \cdot b \cdot b$.

If it is desired that the exponent shall affect other preceding numbers, a parenthesis is used.

Thus, $(3\,ab)^2$ means $3\,ab \times 3\,ab$, and is read "the square of $3\,ab$." $(a+b)^2$ is read "the square of the binomial, $a+b$."

1. $x^2 \cdot x^3 = x^5$.
2. $(xy)^3 \cdot (xy)^4 = (xy)^7$.
3. $(-3)^5 \cdot (-3)^3 = (-3)^8$.
4. $2 \cdot 2^n = 2^{n+1}$.
5. $(x+2)^2(x+2)^5 = (x+2)^7$.
6. $(a^4)^5 = a^{20}$.

ORAL EXERCISE

116. *Find the indicated products:*

1. $x^2 \cdot x^3$.
2. $a^3 \cdot a^4$.
3. $m^2 \cdot m^5$.
4. $y^7 \cdot y^3$.
5. $p^8 \cdot p^9$.
6. $q^{11} \cdot q^4$.
7. $r^2 \cdot r^{13}$.
8. $b \cdot b^6$.
9. $x^p \cdot x$.
10. $n^r \cdot n^2$.
11. $c^{n+1} \cdot c^3$.
12. $x^y \cdot x^{2y}$.
13. $a^x \cdot a^y$.
14. $d^m \cdot d^n$.
15. $a^{2m} \cdot a^2 \cdot a$.
16. $a^{2m} \cdot a^m$.
17. $m^{2x} \cdot m^{5x}$.
18. $x^2 \cdot x \cdot x^3$.
19. $m^2 \cdot m^3 \cdot m^3$.
20. $(a+b)^2 \cdot (a+b)^3$.
21. $r^2 \cdot r^3 \cdot r^4$.
22. $(2\,a)^2 \cdot (2\,a)^3$.
23. $(2+x)^2 \cdot (2+x)^6$.
24. $4^3 \cdot 4^2 \cdot 4$.

MULTIPLICATION OF MONOMIALS

117. Remember that in multiplying two or more monomials together:

The factors of a product may be arranged in any order without changing the value of the product.

Thus, $2 \times 3 = 3 \times 2$; $2 \times 4 \times 3 = 2 \times 3 \times 4$; etc.

The product of an even number of negative factors is positive, and the product of an odd number of negative factors is negative.

Thus, $(-a) \times (-b) = ab$; $(-m)(-m)(-m)(-m)$ or $(-m)^4 = m^4$; $(-a)(-b)(-c) = -abc$; $(-a)^3 = -a^3$.

1. $2a^2b \times (-3a^3b^4c) = 2 \cdot (-3) \cdot a^2 \cdot a^3 \cdot b \cdot b^4 \cdot c = -6a^5b^5c.$
2. $-5ax^2y \times (-3x^3y^n) = (-5)(-3)a \cdot x^2 \cdot x^3 \cdot y \cdot y^n$
   $= 15ax^5y^{n+1}.$

118. The preceding laws together with the law of exponents, § 114, give the rule.

To multiply monomials:

1. Find the product of the numerical coefficients, keeping in mind the law of signs for multiplication.

2. Write after this product the product of the literal factors, giving to each letter an exponent equal to the sum of its exponents in the factors.

Examples

1. $(-3a^2b) \cdot 5ab^2c = -15a^3b^3c.$
2. $4a^nb \cdot a^nb^2 = 4a^{2n}b^3.$
3. $a^nb \cdot ab = a^{n+1}b^2.$

ORAL EXERCISE

119. *Find the products of the following:*

1. $a^2a^2.$
2. $b^2b^4.$
3. $x^3x^5.$
4. $c^5c^5.$
5. $4y^2 \cdot 3y^3.$
6. $5a^4 \cdot 3a^2.$
7. $a^2b \cdot ab.$
8. $x^3y \cdot x^3y^5.$
9. $b^2c^2 \cdot b^3c.$
10. $-b^2c^2 \cdot (-b^3).$
11. $3a^b \cdot a^c.$
12. $x^{2a} \cdot x^a.$

Find the products:

13. $a^2 \cdot a^3 \cdot a^4$.
14. $x \cdot x^2 \cdot x^3$.
15. $2y \cdot y(-y^2)$.
16. $ab \cdot ab^2$.
17. $3rs \cdot 3rs \cdot rs$.
18. $5pq \times (-p^2q^2)$.
19. $c^2d(-cd^2)(-c)$.
20. $x^2y^3 \cdot 2x^3y^2$.
21. $(-p)(-q)(-r)$.
22. $(-p)(-q)2r$.
23. $(-x)^3 \cdot x$.
24. $(-hk)^4$.
25. $(-5)^3$.
26. $(-4)^4$.
27. $-3ab(-c)$.
28. $-7p(-p)(-r)$.
29. $(2a)(-2b)(-c)$.
30. $a^2(-a)^3$.

EXERCISE

120. *Find the products of the following:*

1. $x^2y^3 \cdot 2x^3y^2$.
2. $\frac{1}{2}x^4(-\frac{2}{3}x^7)$.
3. $3a \cdot 2a^2 \cdot 6a^3$.
4. $a^2b(-3a^3c)(-b^3c^2)$.
5. $3(a+b)^2 \cdot (a+b)^3 \cdot (-4)$.
6. $5x^2z^3 \cdot 2nx^3$.
7. $(-3\frac{1}{2}a^2y) \cdot \frac{6}{7}ay^5$.
8. $(-3a^2) \times (-7ax^5)$.
9. $(-2mx)(1.2b^2x^5)$.
10. $.32xy(-11x^3y^4)$.
11. $(-80mx^3)(.05m^2x)$.
12. $(-.2x^4y^2) \cdot 5xy^2$.
13. $2bx(-5.5b^2y^3)$.
14. $12a^2x \cdot \frac{2}{3}ax^4$.
15. $.33\frac{1}{3}m^2n \cdot 15mn^3 \cdot mn$.
16. $.6px^3 \cdot 5b^4p^3x^2$.
17. $(-15a^2n) \cdot 3b \cdot (-22ab)2a^2b$.
18. $(-3ac)(.33\frac{1}{3}a^2y^5)(acy)$.
19. $.4b \cdot .2b^2c \cdot 2bc^2$.
20. $4\frac{1}{2}a^2c^3 \cdot 2\frac{2}{3}ac^4 \cdot \frac{1}{12}a^4c$.
21. $a^m \cdot a^n \cdot a^p$.
22. $a^{x+1}a^{x+2}$.
23. y^3y^x.
24. $3a^n \cdot \frac{2}{3}a$.
25. $(-40a^m)(-.05a^p)$.
26. $m^{p-2} \cdot m^2$.
27. $.8y^{m-4}(-.4y)$.
28. $c^x \cdot d^{y-1} \cdot c \cdot d^{2-y}$.
29. $c^xd^{y+1} \cdot cd^{2+y}$.
30. $x^{n-2}y^{m-3}(-x^{n-1}y^{m-1})$.

31. $(-\frac{1}{2}a^{m+1}) \cdot 7a \cdot 2a^{1-m}$.

32. $6d^{m-n} \cdot 2d^{2n-m}$.

33. $(-b^pc^q) \cdot 9b^qc^p$.

34. $a^{m+n} \cdot a^{m-n}$.

35. $5a^{1-2x} \cdot 3a^{4x}$.

36. $y^2z^p \cdot 3y^{2n-2} \cdot z^{n-p}$.

37. $(-2a^{m+2n}) \cdot \frac{3}{2}a^{4-m}$.

38. $a^{4n}x^{2m} \cdot a^3x^2 \cdot a^nx^m$.

39. $c^{x-1}d^{y+4} \cdot 3c^{2-x}d^{4-y}$.

40. $-m^{x-2}n^3 \cdot 10m^{3-x}n^{y-3}$.

41. $3(x^2)^3(-2xy^2)$.

42. $3(2x^2)^2 \cdot (-3x)$.

43. $4(2x^3)^3(-4x^2)$.

44. $6(3x^2)^2(-4x^2)^2$.

45. $6(2ab)^2(-a^2b)^2$.

46. $5(3xy)^2(-xy^2)^2$.

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

121. $$4(3+5) = 4 \cdot 8 = 32.$$

This result might have been found by multiplying the numbers within the parenthesis separately by 4.

Thus, $4(3+5) = 4 \cdot 3 + 4 \cdot 5 = 12 + 20 = 32$.

$$\begin{array}{r} 53 \\ 7 \\ \hline 21 \\ 350 \\ \hline 371 \end{array}$$

Ordinary arithmetical multiplication, if done without the abbreviating process of "carrying," shows the same principle. The multiplication at the left, if written in a line, is

$$7(50+3) = 350 + 21 = 371.$$

The algebraic law that covers this case may be expressed in algebraic symbols thus,

$$a(b+c) = ab + ac.$$

122. In words this law gives us the following rule:

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and unite the results with their respective signs.

Examples

1. Multiply $2a^2b - b^2c + c^2$ by $-3a^2b^2$.

Multiplication	Check. $a = b = c = 2$.
$2a^2b - b^2c + c^2$	$= 12$
$-3a^2b^2$	$= -48$
$-6a^4b^3 + 3a^2b^4c - 3a^2b^2c^2$	$= -576$

2. Multiply $a^{m-1} + b^{n-1} - c^{p-1}$ by $-abc$.

$$\begin{array}{l} a^{m-1} + b^{n-1} - c^{p-1} \\ \underline{-abc \qquad\qquad\qquad} \\ -a^m bc - ab^n c + abc^p. \end{array}$$

ORAL EXERCISE

123. *Multiply the following:*

1. $a(b + c)$.
2. $-x(a + b)$.
3. $-a(x - y)$.
4. $2b(3a + c)$.
5. $4a^2(3x^2 - y)$.
6. $-3a^2(2a + b)$.
7. $x(a + b + c)$.
8. $-x(a - b - c)$.
9. $-3x(a + b - c)$.
10. $-4c(-2c + 3d - 6e)$.
11. $-5x(x + y + z)$.
12. $-5x(-x + y - z)$.
13. $-p(pq - r)$.
14. $r(st - rs)$.
15. $pq(p^2q - pq^2)$.
16. $a^3(bc - ac + a)$.
17. $-3h(k - hk + h^2k^2)$.
18. $2x(xy - xz - yz)$.
19. $xy(x^{n-1} - y^{n-1})$.
20. $-3x^2y(x^{n-2}y - xy^{n-1})$.

EXERCISE

124. *Multiply the following:*

1. $rs(r + s)$.
2. $a^2(ab - ac)$.
3. $-2x(x^2 - 2x - 1)$.
4. $-3ab(a^2 + b^2)$.
5. $-4x(2x - .5y - 3z)$.
6. $-3a^2b^2(a + b + c)$.
7. $a(a^2 + b^2 + ab)$.
8. $.2m(.2m^2 + .02mn + .002n^2)n$.
9. $-3x(-4x^2 + 2n - \frac{1}{3})$.
10. $-3x^2y^2(2x^4y - 3x^2y^2 + 4x^2y^3)$.
11. $3m^2(2m^3 - 7m^2 - m)$.
12. $-xy(21x^2y^2 - 14xy + 7)(-1)$.
13. $3pqr(-pq - 5pr - 7qr)$.
14. $5^2(5 + 5^2 + 5^3)$.

15. $-9\,abc\left(-\frac{1}{3}a-\frac{1}{6}b-\frac{1}{9}c\right).$

16. $5\cdot 137=5(100+30+7)=?$

17. $\left(\frac{2}{3}a-\frac{5}{2}b-7\right)\left(-\frac{6}{5}ab\right).$

Multiply the following:

18. $$\begin{array}{l} 3\,a^2-b^2+7\,c^2 \\ \underline{-2\,a^2b^2c^2} \end{array}$$

19. $$\begin{array}{l} 6\,x^3-3\,x^2-9\,x+18 \\ \underline{.3\,x} \end{array}$$

20. $$\begin{array}{l} .6\,x^3-.8\,x^2y+2\,xy^2-2\,y^3 \\ \underline{.5\,x^2y^3} \end{array}$$

21. $$\begin{array}{l} 2\,a^2b-3\,cd^3+\frac{1}{2}\,ac^3 \\ \underline{-6\,ac^2d^4} \end{array}$$

Simplify the following:

22. $4(2\,x-7\,y)+2(x+14\,y).$

23. $4\,a(ab+bc+ca)-2\,b(a^2+2\,ac).$

24. $x(x^2-xy+y^2)+y(x^2-xy+y^2).$

25. $x(x^2+xy+y^2)-y(x^2+xy+y^2).$

26. $(x^3-3\,x^2y)y^3-(y^3-3\,xy^2)x^3.$

27. $12(\frac{1}{2}\,x-\frac{1}{3}\,y+\frac{1}{4}\,z)-16(\frac{1}{4}\,x+\frac{1}{2}\,y-\frac{1}{8}\,z).$

28. $a-2[3\,a-b-2(b-a)+3(a-2\,b)].$

29. $2\,x-8\,z-3[2\,y-(2\,x-z)]-3(x-y-z).$

30. $4\,a(b-3)-5\,b(a-2)+ab+7(a-b).$

31. $a(b-a+d)-b(a+c-d)+c(a+b-d).$

32. $3\cdot 6\,a(b-c)-2\,b(9\,a-c)-2\,c(b-9\,c).$

33. $p[q(s+t)-st]-s(pq-1)+t(4-pq)+pst.$

34. $(8\,c^2+24\,cd^3-12\,c^2x-3)\frac{2}{3}\,c^nd^n.$

35. $(3\,b^2-6\,c^2+9\,bc)(-\frac{2}{3}\,b^pc^q).$

Simplify:

36. $(7\,a^n - 3\,a^{n-1} - 2\,a^{n-2})(-\,.4\,a^{n-2})$.

37. $(9\,x^p y^q - 4\,x^{p-1}y^{q-1} + 3\,x^{p-2}y^{q-2})x^2y^2$.

38. $(8\,a^{1-2m} + b^{3-n})(-\,5\,a^{3m}b^n)$.

39. $(x^m + y^p + z^q)xyz$.

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

125. 1. To multiply 32 by 4, we may first multiply 2 by 4, then multiply 30 by 4, and add the partial products.

$$\begin{array}{rl} \text{Thus, } 32 = & 30 + 2 \\ 4 = & \,4 \\ \hline 128 = & 120 + 8. \end{array}$$

2. To multiply 32 by 24, we may first multiply $30 + 2$ by 4 and then by 20, and add the partial products.

$$\begin{array}{rl} \text{Thus, } 32 = & 30 + 2 \\ 24 = & 20 + 4 \\ \hline 128 = & 120 + 8 \\ 64 = & 600 + 40 \\ \hline 768 = & 600 + 160 + 8. \end{array}$$

3. To multiply $2\,a + 3\,b$ by $3\,a + b$, we first multiply $2\,a + 3\,b$ by $3\,a$ and then by b and add the partial products.

$$\begin{array}{l} \text{Thus, } 2\,a + 3\,b \\ \phantom{\text{Thus, }} 3\,a + b \\ \hline 6\,a^2 + 9\,ab \\ 2\,ab + 3\,b^2 \\ \hline 6\,a^2 + 11\,ab + 3\,b^2. \end{array}$$

4. Multiply $x^3 - 2\,x^2 - 3\,x + 2$ by $2\,x - 3$.

$$\begin{array}{l} x^3 - 2\,x^2 - 3\,x + 2 \\ 2\,x - 3 \\ \hline 2\,x^4 - 4\,x^3 - 6\,x^2 + 4\,x \\ - 3\,x^3 + 6\,x^2 + 9\,x - 6 \\ \hline 2\,x^4 - 7\,x^3 + 13\,x - 6. \end{array}$$

An orderly arrangement of the terms of the polynomial and of the partial products is desirable. It is usual to arrange in descending powers of some letter (x in the present case), and to arrange the terms of the partial products with like terms in the same vertical column.

126. These examples lead to the rule.

To multiply one polynomial by another:

1. Arrange the terms in descending powers of some letter (or alphabetically).

2. Multiply the multiplicand by each term of the multiplier, writing like terms in the same column.

3. Add the columns of like terms and join the results obtained with their respective signs.

127. To check the answers we may proceed as in addition (§ 84), by using arbitrary values of the letters.

EXAMPLES

1. Multiply $x^2 - 3x - 2$ by $x + 5$.

MULTIPLICATION — CHECK. $x = 2$.

$$
\begin{array}{llll}
x^2 - 3x - 2 & & & = -4 \\
x + 5 & & & = 7 \\
\hline
x^3 - 3x^2 - 2x & & & -28. \\
\quad 5x^2 - 15x - 10 & & & \\
\hline
x^3 + 2x^2 - 17x - 10 = -28. & & &
\end{array}
$$

Substituting 1 for x in the above example will not check the exponents. (Why ?) Hence it is better to use some other small number, as 2.

2. Multiply $3a^3b - 2a^2b^2 + ab^3$ by $2a^2 - 5b^2 - ab$.

MULTIPLICATION — CHECK. $a = b = 2$.

$$
\begin{array}{llll}
3a^3b - 2a^2b^2 + ab^3 & & & = 32 \\
2a^2 - ab - 5b^2 & & & = -16 \\
\hline
6a^5b - 4a^4b^2 + 2a^3b^3 & & & -512. \\
\quad - 3a^4b^2 + 2a^3b^3 - a^2b^4 & & & \\
\quad\quad - 15a^3b^3 + 10a^2b^4 - 5ab^5 & & & \\
\hline
6a^5b - 7a^4b^2 - 11a^3b^3 + 9a^2b^4 - 5ab^5 = -512. & & &
\end{array}
$$

EXERCISE

128 *Arrange conveniently for multiplication:*

1. $(3x^3 - 4y^2x - x^2y + 4y^3)(2xy - 3y^2 + x^2)$.

2. $(4ab^2 + 6b^3 - 3a^2b + 7a^3)(b^2 + ab + a^2)$.

3. $(3 - x^3 + 2x^2 - x)(x - 3x^2 + 2)$.

Multiply the following:

4. $\begin{array}{r} a+3 \\ \underline{a-5} \end{array}$

5. $\begin{array}{r} x+3 \\ \underline{x+2} \end{array}$

6. $\begin{array}{r} c-2 \\ \underline{c-6} \end{array}$

7. $\begin{array}{r} x+y \\ \underline{x-y} \end{array}$

8. $\begin{array}{r} r+2 \\ \underline{r+s} \end{array}$

9. $\begin{array}{r} m+n \\ \underline{m+p} \end{array}$

10. $\begin{array}{r} 2x+1 \\ \underline{2x-1} \end{array}$

11. $\begin{array}{r} x+1 \\ \underline{x+1} \end{array}$

12. $\begin{array}{l} a^2-a+2 \\ \underline{a\ \ +2\quad} \end{array}$

13. $\begin{array}{l} x^2+x+1 \\ \underline{x\ \ -1\quad} \end{array}$

14. $\begin{array}{l} 2a^2-3a-5 \\ \underline{5a\ \ -7\qquad} \end{array}$

15. $\begin{array}{l} 2b^2-2.4b+1.6 \\ \underline{10b\ \ +20\qquad} \end{array}$

16. $\begin{array}{l} a+b-2 \\ \underline{a-b+2} \end{array}$

17. $\begin{array}{l} m^2-mn+n^2 \\ \underline{m\ \ +n\qquad} \end{array}$

18. $(2x^2+1)(2x^2-1)$.

19. $(x+1)^2$.

20. $(x-4)^2$.

21. $(x+1)(x+3)$.

22. $(2x+3)(4x^2-6x+9)$.

23. $(3x-7)(3x+7)$.

24. $(2R+3)(R-1)$.

25. $(2m+p)(-2m+p)$.

26. $(1-3x^2+x)(x^2+1-x)$.

27. $(5a^2-3ab-2b^2)(a^2+2ab)$.

28. $(x^2+xy+y^2)(x^2-xy+y^2)$.

29. $(3a^2-5ab-2b^2)(a^2-7ab)$.

30. $(x^2+7x-5)(x^2+5-7x)$.

31. $(x^2-5ax-2a^2)(x^2+3ax+2a^2)$.

32. $(c^4-c^2)(c^3+c)$.

33. $(7p^2-3q^2)(4p^2+q^2)$.

34. $(x^4-x^3y+x^2y^2-xy^3+y^4)(x+y)$.

35. $(m^2+n^2+p^2-mn-mp-np)(m+n+p)$.

36. $(3x^2-2xy+y^2)(3x^2+2xy-y^2)$.

37. $(a^3-a^2x^3+ax^6-x^9)(a+x^3)$.

38. $(8a^3+4a^2b+2ab^2+b^3)(2a-b)$.

39. $(a+b+x+y)(a+b-x-y)$.

40. $(a+b+c+d)(a-b+c-d)$.

41. $(x^2+1)(x^2+2)(x^2+3)$.

42. $(2x-3)(3x+7)(6x-5)$.

43. $(3x+5)(7x+5)(2x-1)$.

44. $(3a+2b)(a-b)+(4a+5b)(2a+3b)$.

45. $(u+v)(2v-u)+(u-v)(v+2u)$.

46. $(x+4)(x-2)-(x+2)(x-1)$.

47. $(3x+5)(2x-3)(x-1)-(x-1)(x+2)(x-3)$.

48. $(a+b)(c-d)-(a-b)(c+d)$.

49. $(2ax^2+3ay^2)(2ax^2-3ay^2)$.

50. $(4b^2x^2+5c^2y^2)^2$.

51. $(a+b)^3$.

52. $(3x+4y)^3$.

53. $(3ax-4by)^3$.

54. $(a^x-b^y)(a^x+b^y)$.

55. $(x^a+y^b)^2$.

56. $(x^a+x)^2$.

57. $(x^{2a}+x^a+1)(x^a-1)$.

58. $(a^{n+1}+a^n+a^{n-1})(a^2+a)$.

59. $(a^{2n}+2a^n+4)(a^n-2)$.

In examples 60 to 71,

$A=x^2-2x+4,\ B=x^2+2x+4,\ C=x-2,\ D=x+2.$

Perform the indicated operations:

60. $A\cdot B$.

61. $A\cdot D$.

62. A^2.

63. $A\cdot D^2$.

64. $B\cdot C^2$.

65. B^2.

66. $C^2\cdot D^2$.

67. $A\cdot D-BC$.

68. D^2-A.

69. C^2-B.

70. $Bx-C^3$.

71. $Dx+Cx-(A+B)$.

72. From the product of $x-3$ times x^2+x-2, subtract the sum of x^3+5, and $3x^2-7x$.

73. Multiply $2x-x^3+7$ by the sum of $7-x$ and $2x-10$.

74. Solve $7(x+2)-3(x-1)=2(x-1)+25$.

Solution.

$7(x+2)-3(x-1)=2(x-1)+25.$

$7x+14-3x+3=2x-2+25.$ (Why?)

$4x+17=2x+23.$ (Collecting terms.)

$2x+17=23.$ (Subtracting $2x$ from both members of equation.)

$2x=6.$ (Subtracting 17 from both members of equation.)

$x=3.$ (Why?)

Solve the following equations:

75. $2(x+5)=20.$
76. $2(x+2)+3(3x-3)=6.$
77. $3(x-1)+7=11.$
78. $2(r-1)+2(r-2)=3(r+3).$
79. $3(x-5)+8=18.$
80. $15(x-3)-17=103.$
81. $8(5x-37)-4(3x-17)=20.$
82. $6(x-5)+2x=6x-2(x+10).$
83. $17(m-17)-17=-51.$
84. $18a-2(3+5a)=10.$
85. $7(3p-2)+5(p-3)-4(p-17)=110.$
86. $3(x-5)-4(x-2)+6x=15.$

TYPE FORMS IN MULTIPLICATION

129. Certain multiplications occur so frequently that it is helpful to be able to write the products at sight. Seven such special products are given. They form a sort of algebraic multiplication table, and should be thoroughly learned and understood.

130. Types I and II. The Square of the Sum or of the Difference of Two Numbers.

$$\text{I.}\quad \begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad\;\; +ab+b^2} \\ a^2+2ab+b^2 \end{array} \qquad\qquad \text{II.}\quad \begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad\;\; -ab+b^2} \\ a^2-2ab+b^2 \end{array}$$

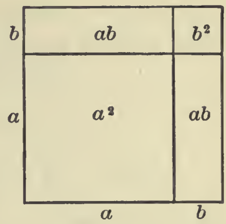
$$(a+b)^2=a^2+2ab+b^2. \qquad (a-b)^2=a^2-2ab+b^2.$$

Type I may be stated in words:

The square of the sum of two numbers equals the square of the first plus twice the product of the first by the second plus the square of the second.

Let the student state Type II in words.

The product $(a + b)^2 = a^2 + 2\,ab + b^2$ may be represented by a figure.



EXAMPLES

1. $(2\,x^2 + 3)^2 = (2\,x^2)^2 + 2(2\,x^2)\cdot 3 + 3^2$
$= 4\,x^4 + 12\,x^2 + 9.$

In applying the type form to find the square of the binomial $2\,x^2 + 3$ we note that the a of the type is to be replaced by $2\,x^2$, and b by 3. Therefore in the second member of the equation, $(a + b)^2 = a^2 + 2\,ab + b^2$, we shall put $4\,x^4$ for a^2, $12\,x^2$ for $2\,ab$, and 9 for b^2.

2. $(2\,x - 3\,y)^2 = (2\,x)^2 - 2(2\,x)(3\,y) + (3\,y)^2 = 4\,x^2 - 12\,xy + 9\,y^2$. Explain.

3. $(x^2 - 4\,y)^2 = x^4 - 8\,x^2y + 16\,y^2$. Explain.

4. $13^2 = (10 + 3)^2 = 100 + 60 + 9 = 169$. Explain

5. $(-a + b)^2 = [(-a) + b]^2 = a^2 - 2\,ab + b^2$. Explain.

6. $(-a - b)^2 = [(-a) + (-b)]^2 = a^2 + 2\,ab + b^2$. Explain.

EXERCISE

131. *Square the following binomials by inspection, using Types I and II:*

1. $(x + y)^2$; $(x - y)^2$; $(y - x)^2$.
2. $(c - a)^2$; $(c + a)^2$; $(a - c)^2$.
3. $(r + s)^2$; $(r - s)^2$; $(-r + s)^2$.
4. $(m + n)^2$; $(m - n)^2$; $(-m - n)^2$.
5. $(p + q)^2$; $(p - q)^2$; $(q - p)^2$.
6. $52^2 = (50 + 2)^2$; $48^2 = (50 - 2)^2$.
7. $25^2 = (20 + 5)^2$; $25^2 = (30 - 5)^2$.
8. $(a + 2)^2$; $(a - 2)^2$; $(2 - a)^2$.
9. $(4 + b)^2$; $(4 - b)^2$; $(b - 4)^2$.

10. $(x+5)^2$; $(x-5)^2$; $(-x-5)^2$.

11. $(6+n)^2$; $(6-n)^2$; $(-6-n)^2$.

12. $(x+7)^2$; $(x-7)^2$; $(7-x)^2$.

13. $(x^2+3)^2$; $(x^2-3)^2$; $(3-x^2)^2$.

14. $(2a+3b)^2$; $(3a-2b)^2$; $(2b-3a)^2$.

15. $(ax+y)^2$; $(ax+y^2)^2$; $(ax+y^3)^2$.

16. $(3a+2)^2$.

17. $(4+y)^2$.

18. $(8-m)^2$.

19. $(xy+z)^2$.

20. $(-2+x^2)^2$.

21. $97^2=(100-3)^2$.

22. $49^2=(50-1)^2$.

23. $(-a-b)^2$.

24. $(4a-5b)^2$.

25. $(a-10)^2$.

26. $(7c-4d^2)^2$.

27. $(2u+v)^2$.

28. $(5a^3-2b^4)^2$.

29. $(-9p-3q)^2$.

30. $(2x^3+3)^2$.

31. $(m^2-r^3)^2$.

32. $(7-4x^2)^2$.

33. $(7a^2y-3ax)^2$.

34. $(20\frac{1}{2})^2=(20+\frac{1}{2})^2$.

35. $(3bc-2cd)^2$.

36. $(\frac{2}{3}a-\frac{4}{7})^2$.

37. $(2x+\frac{1}{2}y)^2$.

38. $32^2=(30+2)^2$.

39. $65^2=(60+5)^2$.

Of what binomial is each of the following trinomials the square?

40. $x^2+2xy+y^2$.

41. $m^2+12mp+36p^2$.

42. x^2+4x+4.

43. $r^2-14r+49$.

44. $x^2y^2-6xy+9$.

45. $9m^2-24mp+16p^2$.

46. $9-6a+a^2$.

47. $4a^2-4a+1$.

48. x^4+2x^2+1.

49. $x^2y^2-16xy+64$.

132. Type III. The Product of the Sum and the Difference of Two Numbers.

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad\ -ab-b^2} \\ a^2 \qquad\quad -b^2 \end{array}$$

$$(a+b)(a-b)=a^2-b^2$$

Type III may be stated in words :

The product of the sum and the difference of two numbers equals the square of the first minus the square of the second.

EXAMPLES

1. $(p+q)(p-q)=(p^2-q^2)$.
2. $(3x-5y)(3x+5y)=(3x)^2-(5y)^2=9x^2-25y^2$.

In this example a of the type form is replaced by $3x$, and b, by $5y$. Therefore, in the product we shall have to replace a^2 by $(3x)^2$, and b^2 by $(5y)^2$. This will give the result obtained above.

3. $(x+2y)(2y-x)=(2y+x)(2y-x)=4y^2-x^2$.
4. $(-2+x)(-2-x)=[(-2)+x][(-2)-x]=4-x^2$.

EXERCISE

133. *Multiply by Type III:*

1. $(x+y)(x-y)$.
2. $(c+d)(c-d)$.
3. $(r+s)(r-s)$.
4. $(a+5)(a-5)$.
5. $(4+a)(4-a)$.
6. $(2a+b)(2a-b)$.
7. $(x^2+2)(x^2-2)$.
8. $(3a^2+2)(3a^2-2)$.
9. $(2x+y)(2x-y)$.
10. $(3a+c)(3a-c)$.
11. $(p+2q)(p-2q)$.
12. $(h+5k)(5k-h)$.
13. $(1+4m)(4m-1)$.
14. $(2a+3b)(2a-3b)$.
15. $(3x+2y)(2y-3x)$.
16. $22\times 18=(20+2)(20-2)$.
17. $27\times 33=(30-3)(30+3)$.
18. 49×51.
19. 68×72.
20. 103×97.

In which of the examples 21 to 28, *may we apply Type III? Give reason in each case.*

21. $(2a+3b)(2a-3b)$.
22. $(2a+b)(2a^2-b)$.
23. $(x+y)(x^2-y^2)$.
24. $(\frac{1}{2}+x)(\frac{1}{2}-x)$.
25. $(-a+b)(b+a)$.
26. 29×31.
27. 25×35.
28. $(m-n)(a-b)$.

Multiply:

29. $(a^2 - b)(a^2 + b)$.

30. $(2x - z^2)(2x + z^2)$.

31. $(\frac{1}{2}m + \frac{1}{3}r)(\frac{1}{2}m - \frac{1}{3}r)$.

32. $(2a^3 - b^2)(2a^3 + b^2)$.

33. $(ab + cd)(cd - ab)$.

34. $(3a^2 - b)(3a^2 + b)$.

35. $(-2 + 3a)(2 + 3a)$.

36. $87 \cdot 93$.

37. $(a + b + c)(a + b - c)$.

SOLUTION. $(a + b + c)(a + b - c)$
$= [(a + b) + c][(a + b) - c]$
$= (a + b)^2 - c^2$
$= a^2 + 2ab + b^2 - c^2$.

38. $(a - b + c)(a + b + c)$.

HINT. $(a - b + c)(a + b + c) = [(a + c) - b][(a + c) + b]$, etc.

39. $(a - b + c)(a + b - c) = [a - (b - c)][a + (b - c)]$, etc.

40. $(a - b + c)(a - b - c)$.

41. $(m - n + p)(m + n - p)$.

42. $(p - 2q + 3r)(p + 2q + 3r)$.

43. $(2x - y - 3z)(2x - y + 3z)$.

44. $(2x - y - 3z)(2x + y + 3z)$.

45. $(x^2 + y^2 + xy)(x^2 + y^2 - xy)$.

46. $(a - x)(a + x)(a^2 + x^2)$.

47. $(x - y)(x + y)(x^2 + y^2)(x^4 + y^4)$.

48. $(x^n + y^n)(x^n - y^n)$.

49. $(m^{2x} - m^{2y})(m^{2x} + m^{2y})$.

50. $(x^2 + p^2)(x + p)(x - p)$.

51. $[(a - 1)(a + 1)]^2$.

52. $(2b - c)(2b + c) + (c - 2b)(c + 2b)$.

53. $[(x + y)(x - y)]^2 + [(y - x)(y + x)]^2$.

54. $(2r - 3q^2)(3q^2 + 2r)$.

55. $(-r - 3s)(r - 3s)$.

56. $(2r^2 - 4s)(-4s - 2r^2)$.

Write each of the following expressions as the product of two binomials:

57. $x^2 - 4$.
58. $9a^2 - 16b^2$.
59. $36 - 81x^4$.
60. $4a^2 - 16b^2$.
61. $x^2y^2 - a^2b^2$.
62. $x^2y^2z^2 - 25$.
63. $25 - 16$.
64. $x^{2n} - 1$.

134. Type IV. The Product of Two Binomials having a Common Term.

The two binomials are of the form $x + a$ and $x + b$. By multiplying, and arranging in order of powers of x, we have

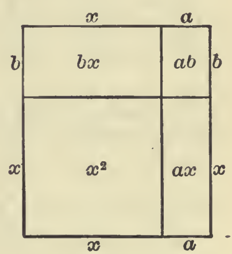
$$\begin{array}{ll} x + 2 & \qquad x + a \\ \underline{x + 3} & \qquad \underline{x + b} \\ x^2 + 2x & \qquad x^2 + ax \\ \underline{\quad\;\; 3x + 6} & \qquad \underline{\qquad\; bx + ab} \\ x^2 + 5x + 6. & \qquad x^2 + (a + b)x + ab. \end{array}$$

This gives $\boldsymbol{(x + a)(x + b) = x^2 + (a + b)x + ab}$.

In words:

The product of two binomials having a common term equals the square of the common term, plus the product of the common term by the sum of the other terms, plus the product of the other terms.

The product $(x + a)(x + b) = x^2 + (a + b)x + ab$, which is the same as $x^2 + ax + bx + ab$, may be represented by the figure.



EXAMPLES

1. $(x + 3)(x + 7) = x^2 + 10x + 21$.

Here $a = 3$ and $b = 7$, $a + b = 10$ and $ab = 21$.

2. $(x - 3)(x + 2) = x^2 + [(-3) + 2]x + (-3) \times 2 = x^2 - x - 6$.
3. $(x + b)(x - 2) = x^2 + (b - 2)x - 2b$.
4. $(x - p)(x - r) = x^2 - (p + r)x + pr$.
5. $(m^2 - 3p)(m^2 + 5p) = m^4 + 2m^2p - 15p^2$.

EXERCISE

135. *Which of the following can be multiplied by Type IV? Give reason for each answer.*

1. $(a+7)(a+3)$.
2. $(a^2-7)(a+3)$.
3. $(m+2p)(m+2p)$.
4. $(3a+7)(4a+10)$.
5. $(3a-7)(3a+7)$.
6. $(5+a)(5+b)$.
7. $(5+a)(6+b)$.
8. $(5+a)(6+a)$.
9. $(5+a)(5-a)$.
10. $(3a^2+5)(3a^2-7)$.

Multiply by Type IV:

11. $(x+7)(x+3)$.
12. $(x-2)(x-3)$.
13. $(x-1)(x-6)$.
14. $(x+1)(x-6)$.
15. $(x-1)(x+6)$.
16. $(x+3)(x-2)$.
17. $(x-3)(x+2)$.
18. $(x+3)(x+2)$.
19. $(x+4)(x-5)$.
20. $(a+8)(a-5)$.
21. $(a+3)(a+2)$.
22. $(a+3)(a-2)$.
23. $(a+6)(a+1)$.
24. $(a+6)(a-1)$.
25. $(r-3)(r-5)$.
26. $(r-3)(r+5)$.
27. $(r+3)(r-5)$.
28. $(a^2-9)(a^2+5)$.
29. $(a^2+9)(a^2-5)$.
30. $(b^2-3)(b^2-2)$.
31. $(a+b)(a+2b)$.
32. $(a-b)(a+2b)$.
33. $(a+b)(a-2b)$.
34. $(a-2x)(a-3x)$.
35. $(a+2x)(a-3x)$.
36. $(r+2s)(r+5s)$.
37. $(r-2s)(r+5s)$.
38. $(p-2rs)(p-3rs)$.
39. $(m+5r^2)(m+2r^2)$.
40. $(m-5r^2)(m+2r^2)$
41. $(b^2-7)(b^2+12)$.
42. $(a^2-3)(a^2-b)$.
43. $(a+c)(a-d)$.
44. $(2a+3)(2a+5)$.
45. $(3x^2-5)(3x^2-12)$.
46. $(3x^2-5y^2)(3x^2+5y^2)$.
47. $(3x^2-5y^2)(3x^2+12y^2)$.
48. $(xy-5)(xy+7)$.
49. $(xy-55)(xy+1)$.
50. $(xy-5z)(xy-3z)$.

51. $(2x + 7y)(2x + 3y)$.

52. $(2x + 7y)(2x + 5)$.

53. $(ar + 5x)(ar - 7x)$.

54. $(ar + 5x)(ar - 7y)$.

55. $(a^2x + 5x)(a^2x + 3x)$.

56. $(10 + 2)(10 + 8)$.

57. $(60 + 2)(60 + 1)$.

58. $(30 + 5)(30 + 6)$.

59. $(10 + 6)(10 + 2)$; 14×16.

60. $(20 + 3)(20 + 4)$; 26×28.

61. Explain how Types I, II, III may be regarded as special cases of Type IV.

136. Type V. The Square of a Polynomial.

$$\begin{array}{llllll} a + b + c \\ \underline{a + b + c} \\ a^2 + & ab + & ac \\ & ab & & + b^2 + & bc \\ & & ac & + & bc + c^2 \\ \hline a^2 + & 2ab + & 2ac + & b^2 + & 2bc + c^2 \end{array}$$

Rearranging for convenience,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Similarly, $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$

In words:

The square of a polynomial equals the sum of the squares of its terms plus twice all products formed by multiplying each term by each succeeding term.

It is to be observed that the squares of the terms will all be positive numbers (why?), but that the double products may be positive or negative according to the requirements of the law of signs in multiplication.

EXAMPLES

1. $(a + b + 2)^2 = a^2 + b^2 + 4 + 2ab + 4a + 4b$.

2. $(2a - b + 1)^2 = 4a^2 + b^2 + 1 - 4ab + 4a - 2b$.

3. $(x^2 - x + 2)^2 = x^4 + x^2 + 4 - 2x^3 + 4x^2 - 4x$
$= x^4 - 2x^3 + 5x^2 - 4x + 4$.

EXERCISE

137. *Multiply by Type V:*

1. $(x+y+z)^2$.
2. $(a+b-1)^2$.
3. $(x^2+y^2-z^2)^2$.
4. $(ab+bc+ac)^2$.
5. $(x^2+x+1)^2$.
6. $(3a+2b-1)^2$.
7. $(x^2+y^2-1)^2$.
8. $(x-xy+2)^2$.
9. $(m^2+mn-n^2)^2$.
10. $(2p+3w+4r)^2$.
11. $(a^2-2ab+b^2)^2$.
12. $(a+b+c+d)^2$.
13. $(m+n-p-q)^2$.
14. $(x-y-z-w)^2$.
15. $(10m+5n+6)^2$.
16. $(2m+3n-4q+p)^2$.
17. $(a-b+2c+d)^2$.
18. $(3a-2b+5c)^2$.
19. $(xy-yz-zx)^2$.
20. $(ab-bc+cd-da)^2$.

138. Types VI and VII. The Cube of a Binomial.

By actual multiplication we can find the value of $(a+b)^3$ to be $a^3+3a^2b+3ab^2+b^3$. Let the student perform the multiplication.

This gives us Type VI: $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

Similarly, Type VII: $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

In words:

The cube of the sum of two numbers equals the cube of the first plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second plus the cube of the second.

Let the student make the corresponding statement for Type VII.

EXAMPLES

1. $(3x+2y)^3 = (3x)^3+3(3x)^2(2y)+3(3x)(2y)^2+(2y)^3$
   $= 27x^3+54x^2y+36xy^2+8y^3$.
2. $(2a^2-b)^3 = (2a^2)^3-3(2a^2)^2b+3(2a^2)b^2-b^3$
   $= 8a^6-12a^4b+6a^2b^2-b^3$.
3. $(-x+2)^3 = (2-x)^3 =$ etc.

4. $(-a-b)^3 = [(-a)+(-b)]^3$
$= (-a)^3 + 3(-a)^2(-b) + 3(-a)(-b)^2 + (-b)^3$
$= -a^3 - 3a^2b - 3ab^2 - b^3.$

EXERCISE

139. *Multiply by Types VI and VII:*

1. $(x+y)^3.$
2. $(x-y)^3.$
3. $(c+d)^3.$
4. $(c-d)^3.$
5. $(m+2)^3.$
6. $(m-3)^3.$
7. $(2a+1)^3.$
8. $(a^2+1)^3.$
9. $(a^2-2)^3.$
10. $(4+a)^3.$
11. $(a+1)^3.$
12. $(a+2)^3.$
13. $(2a-1)^3.$
14. $(a^2+a)^3.$
15. $(-a+b)^3.$
16. $(-a-2b)^3.$
17. $(2x+3)^3.$
18. $(ab+bc)^3.$
19. $(abc-1)^3.$
20. $(b^2c-ab^2)^3.$
21. $(bc-ab^2)^3.$
22. $(6+5a^3)^3.$
23. $(9m^3-5p^3)^3.$
24. $(10-x^2)^3.$
25. $(2p^2-3)^3.$
26. $(6-5p^4)^3.$
27. $(m^2-p^2q)^3.$
28. $(7a^2-2)^3.$
29. $(ab-b^3)^3.$
30. $(xyz-3x^2z)^3.$

SUMMARY OF TYPE FORMS

140. The student should carefully memorize the following type forms that have been developed in this chapter:

I. $(a+b)^2 = a^2 + 2ab + b^2.$

II. $(a-b)^2 = a^2 - 2ab + b^2.$

III. $(a+b)(a-b) = a^2 - b^2.$

IV. $(x+a)(x+b) = x^2 + (a+b)x + ab.$

V. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

$(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$

VI. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

VII. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

Let the student state each type form in words.

REVIEW EXERCISE

141. *Give the answers to examples 1 to 40 by referring each to the proper type form:*

1. $(ax - by)(ax + by)$.
2. $(ax - 3)(ax - 4)$.
3. $(ax - 3)(ax - 3)$.
4. $[(a + b)(a - b)]^2$.
5. $(5\,dy - 3)^2$.
6. $(1 - 2\,m)^2$.
7. $(ab - 3\,c)^2$.
8. $(a^2 - 3\,x)[-(3\,x - a^2)]$.
9. $(2\,p + q)(-2\,p + q)$.
10. $(2\,a - 7)(2\,a - 9)$.
11. $(2\,a + b - 3\,c)^2$.
12. $(3\,xy - 2\,z)^2$.
13. $(3\,xy - 2\,z)^3$.
14. $(3\,xy + 2\,z)(3\,xy - 2\,z)$.
15. $(4 + 3\,a^2)^2$.
16. $(m - n)(m + n)(m^2 + n^2)$.
17. $(m - n)(m + n)(m^2 - n^2)$.
18. $(2\,a - b^2)^3$.
19. $(a + b + 3)^2$.
20. $(3\,xy - 4\,z)(-3\,xy - 4\,z)$.
21. $(x + y - z)(x + y + z)$.
22. $(x - y - z)(x + y + z)$.
23. $(x - pr + q)^2$.
24. $(v + 5\,wv)(v - 5\,wv)$.
25. $(v + 5\,wv)^2$.
26. $(2\,mn - 7\,pq)^2$.
27. $(a - b + x)(a + b - x)$.
28. $(x^2 + 7)(x^2 - 8)$.
29. $(7\,m - 6\,y)^2$.
30. $(y + 2)(y - m)$.
31. $(a + m)(a + p)$.
32. $(2\,c - 1 + d)(2\,c - 1 - d)$.
33. $(2\,c - 1 + d)^2$.
34. $(2\,a^2 - 3\,b)^3$.
35. $(r + 2\,t - 3)^2$.
36. $(d^n - b)^2$.
37. $(a^2x^2 + 5)(a^2x^2 - 3)$.
38. $(x + 2\,y - c - 2\,d)^2$.
39. $(x^a + x)^2$.
40. $(x^a + y)^2$.

Perform the operations indicated in the following, using type forms wherever possible:

41. $(a + b)^2 + (a - b)^2 + (a - b)(a + b)$.
42. $(x^2 + x + 1)(x - 1) - (x^2 - x + 1)(x + 1)$.
43. $(a + 2\,b - c)(a - c) - (a^2 + 2\,ab + c^2)$.
44. $(x^4 - x^3 + x^2 - 1)(x + 1)$.

45. $(v+w)(vw^2-v^2w+v^3-w^3)$.

46. $16(3a-2b)-5(9a-7b)-3(a-4b)-11b$.

47. $(a^6+3a^3+9)(a^3-3)$.

48. $(m^2+mn+n^2)(m-n)$.

49. $(p^2-pq+q^2)(p+q)$.

50. $(a^2+b^2+1-ab-a-b)(a+b+1)$.

51. $(x^2+y^2+z^2-xy-xz-yz)(x+y+z)$.

52. $(x+y+z)^2-(x+y-z)^2+(x-y+z)^2-(-x+y+z)^2$.

53. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

54. $5[3(a+2b-c)+4(a-b-c)]-19(a-b-c)$.

55. $(x^2-y^2)(2x^3-4x^2y-5xy^2)$.

56. $(a^2-b^2)(2a-3b+5c)-b(3a^2-2ab+3b^2)$.

57. $15x^2+24y^2-(3x+2y)(5x+6y)$.

58. $(a^2-b^2)(2b-3a)+(a+b)(8b-7a)a$.

59. $(x+2)^2-(x-1)^2-33(x-3)$.

60. $(1+x-2x^2)(1+x+2x^2)$.

HINT: $(1+x-2x^2)(1+x+2x^2)=[(1+x)-2x^2][(1+x)+2x^2]$.

61. $(m+n+p+q)(m-n+p-q)$.

62. $(m+2p+q)^2+(m-n+p-q)^2$.

63. $(a^2+b^2-ab)(a^2+b^2+ab)$.

64. $(16m^4+4m^2y^4+y^8)(4m^2-y^4)$.

65. $(49x^6+56x^3y+64y^2)(7x^3-8y)$.

66. $(2a+4b-5c+2d)^2$.

67. $(\frac{1}{2}x^2+\frac{2}{3}y^2+6z^2+2)^2$.

68. $(a^6-3a^3+9)(a^3-3)$.

69. State the two binomials whose product is (1) p^2-q^2; (2) $m^2+4mn+4n^2$; (3) $a^2-b^2c^2$; (4) $x^2+10x+25$; (5) $m^2+2mn+n^2$; (6) $9x^2-24xy+16y^2$; (7) $1-\frac{1}{4}a^2$.

Solve the following equations:

70. $(a+2)(a+3)-(a+1)^2=9.$

SOLUTION.
$$(a+2)(a+3)-(a+1)^2=9.$$
$$(a^2+5a+6)-(a^2+2a+1)=9.$$
$$a^2+5a+6-a^2-2a-1=9.$$
$$3a+5=9.$$
$$3a=4.$$
$$a=\tfrac{4}{3}.$$

71. $(x+1)^2=x^2+12.$

72. $12m^2+2m+1-12(m^2+1)=0.$

73. $3+y^2-(9+y^2)+2y=0.$

74. $4(x-1)+1=7-2(2x-3).$

75. $17(1+x)-8(x+2)=26.$

76. $-3(x+2)+7(x+1)=3.$

77. $3(m+1)=2(m+3).$

78. $7(x-1)-2(x+2)=x-3.$

79. $5(x-2)+2(x+3)=17+2(1-x).$

80. A certain fertilizer contains $1\frac{1}{2}$ times as much potash as nitrogen and 4 times as much phosphoric acid as nitrogen. Find the amount of each element in 130 pounds of fertilizer.

81. If 10 is added to a certain number, the sum is three times the original number. Find the number.

82. One number is 32 greater than another. When 3 is added to each number the greater is 5 times the smaller. Find the original numbers.

SOLUTION. Let $x =$ the smaller number.
Hence $x + 32 =$ the larger number.
Also $x + 3 =$ the smaller number increased by 3,
and $x + 35 =$ the larger number increased by 3.
Then $x + 35 = 5(x + 3)$, (By the conditions of the problem.)
or $x + 35 = 5x + 15$. (Why?)
$\therefore 35 = 4x + 15$. (Why?)
$\therefore 20 = 4x$. (Why?)
$\therefore x = 5$, the smaller number,
and $5 + 32 = 37$, the larger number.

83. If a certain number is multiplied by 8 and the product is increased by 14, the result exceeds 5 times the original number by 28. What is the number?

84. A boy had twice as much money as his sister; but after each had spent 12 cents he found that he had 3 times as much as his sister. How much had each at first?

85. One number is 5 times another. If 15 is added to each number, the greater will be 3 times the less. Find the original numbers.

86. A rectangle is 3 times as long as it is wide. If both dimensions are increased by 4 inches, it will be twice as long as it is wide. Find its dimensions.

87. A rectangle is 3 inches longer than it is wide. If both dimensions are increased by 3 inches the area will be increased by 54 square inches. Find the dimensions.

88. A box of candy contained a certain quantity at 35 cents a pound, twice as much at 50 cents a pound, and 3 times as much at 55 cents a pound. If the mixture cost $3, how many pounds of each quality did it contain?

SOLUTION. Let $x =$ the number of pounds @ 35 ¢.
Hence $2x =$ the number of pounds @ 50 ¢,
and $3x =$ the number of pounds @ 55 ¢.
Then $35x + 50 \cdot 2x + 55 \cdot 3x = 300$.

Let the student complete the solution.

89. A grocer blended a certain quantity of coffee at 35 cents a pound with twice as much at 32 cents a pound and 4 times as much at 25 cents a pound. If the total value was $15.92, find the number of pounds of each in the mixture.

90. A certain number of 4 ¢ stamps, 3 times as many 2 ¢ stamps, and 10 times as many 1 ¢ stamps cost $2.00. How many of each were bought?

VI. DIVISION

142. **Division** has been defined as the process of finding one of two factors when their product and the other factor are given. The product is the **dividend**, the given factor is the **divisor**, and the factor sought is the **quotient.**

143. What is the rule for dividing signed numbers? (See § 53).

ORAL EXERCISE

144. *Divide the following:*

1. $8 \div (-2)$; $-8 \div 2$; $-8 \div (-2)$.
2. $-18 \div 6$; $-18 \div (-6)$; $18 \div (-6)$.
3. $36 \div (-9)$; $-36 \div (-6)$; $36 \div (-4)$.
4. $8 \div (-12)$; $-9 \div 12$; $-5 \div (-15)$.
5. $-10 \div 5$; $5 \div (-10)$; $-5 \div 10$.
6. 12 ft. $\div 3$; $12f \div 3$; $12a \div 3$.
7. \$10 $\div 2$; $10d \div 2$; $10x \div 2$.
8. 12 yd. $\div 3$; $12y \div 3$; $12b \div 3$.
9. 20 mi. $\div 4$; $20m \div 4$; $20x \div 4$.
10. $3 \times 6a$; $18a \div 3$; $12b \div 3$.
11. $5 \times 7r$; $35r \div 5$; $27k \div 3$.
12. $21a \div 7$; $28p \div 2$; $50s \div 25$.
13. \$18 $\div$ \$9; $18d \div 9d$; $15d \div 5d$.
14. 26 ct. $\div$ 2 ct.; $26c \div 2c$; $18r \div 6r$.
15. 24 T $\div$ 4 T; $24t \div 4t$; 28 L $\div$ 7 L.
16. $5 \times 8a$; $40a \div 8a$; $40a \div 5$.

17. $8 \times 7k$; $56k \div 7k$; $56k \div 8$.

18. $4 \times (-2a)$; $-8a \div (-2a)$; $-8a \div 4$.

19. $7 \times (-3a)$; $-21a \div (-3a)$; $-21a \div 7$.

20. $-8 \times 7t$; $-56t \div (-8)$; $-56t \div 7t$.

21. $-2 \times (-5r)$; $10r \div (-2)$; $10r \div (-5r)$.

22. $3a \times (-2a)$; $-6a^2 \div 3a$; $-6a^2 \div (-2a)$.

23. $21k \div (-3k)$; $-8a^2 \div 2a$; $-18x \div (-6x)$.

145. Integral Algebraic Expression. An algebraic expression is an **integral algebraic expression** if there are no literal numbers in a denominator.

Thus, $a^2 + 2ab$, $\frac{1}{2}x - \frac{y^2}{2}$, $\frac{a+b}{3}$ are integral algebraic expressions,

and $\frac{a}{3b}$, $\frac{1}{a}$, $\frac{a+b}{c}$ are fractional algebraic expressions.

146. The Law of Signs in Division.

The student should remember that in dividing one number by another:

1. **The quotient of two numbers having like signs is positive.**
2. **The quotient of two numbers having unlike signs is negative.**

147. The Law of Exponents in Division.

Since $a^3 \cdot a^2 = a^5$, therefore $a^5 \div a^2 = a^3$ or a^{5-2},
and $a^5 \div a^3 = a^2$ or a^{5-3}.

Similarly $\because a^8 \cdot a^3 = a^{11}$, $\therefore a^{11} \div a^8 = a^3$ or a^{11-8}.
and $a^{11} \div a^3 = a^8$ or a^{11-3}.

In general, $a^m \div a^n = a^{m-n}$.

The equation $a^m \div a^n = a^{m-n}$, gives in algebraic symbols, the *law of exponents in division*.

In words, this law may be stated:

The exponent of any base in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

EXAMPLES

1. $a^7 \div a^5 = a^{7-5}$ or a^2.
2. $a^2 \div -a = -a$. (Why?)
3. $-k^6 \div -k^2 = k^4$. (Why?)
4. $-2^7 \div 2^4 = -2^3$. (Why?)
5. $a^{3k} \div a^k = a^{2k}$. (Why?)
6. $a^{r+2} \div a^r = a^{r+2-r} = a^2$.

ORAL EXERCISE

148. *Find the quotients:*

1. $a^4 \div a^2$.
2. $m^6 \div m$.
3. $x^4 \div (-x^3)$.
4. $y^7 \div y^4$.
5. $-p^6 \div p^4$.
6. $-r^{10} \div (-r^6)$.
7. $-h^5 \div (-h)$.
8. $-k^9 \div -k^5$.
9. $b^8 \div (-b^5)$.
10. $-t^9 \div t^7$.
11. $c^7 \div c^6$.
12. $-d^8 \div (-d^3)$.
13. $(ab)^3 \div ab$.
14. $5^7 \div 5^4$.
15. $-2^8 \div 2^7$.
16. $(2a)^8 \div (2a)^6$.
17. $(a+b)^5 \div (a+b)^3$.
18. $x^{2n} \div x^n$.
19. $x^{5r} \div x^{3r}$.
20. $m^{k+3} \div m^3$.
21. $m^{k+3} \div m^k$.
22. $3^{n+1} \div 3$.
23. $a^k \div a^l$.
24. $a^{m+n} \div a^{m-n}$.

DIVISION OF MONOMIALS

149. State the definition of division. Define dividend, divisor, and quotient. (§§ 51, 142.)

$8a^2b^3c \div (-2a^2b) = -4b^2c$ is an immediate consequence of the definition of division since $(-2a^2b) \times (-4b^2c) = 8a^2b^3c$.

ORAL EXERCISE

150. *Using only the definition of division give answers to the following and explain:*

1. $21 \div 7$.
2. $a^5 \div a$.
3. $3ab \div a$.
4. $5x^2y^2z \div xyz$.
5. $-7m^3n^2p \div mn^2p$.
6. $a^5b^4c^3 \div ab^2c^3$.
7. $6p^7q^5 \div 2p^6q^4$.
8. $-18x^2y^2z^5 \div 9xyz$.
9. $42ab^2 \div (-7ab)$.
10. $(-33ab^2c^3) \div (-11abc)$.

151. When the examples are simple, the definition of division along with our previous practice in multiplication will enable us to find the quotients. A rule can be stated, however, that will help us to perform divisions.

To divide a monomial by a monomial:

1. Divide the numerical coefficient of the dividend by that of the divisor, keeping in mind the law of signs.

2. Subtract the exponent of any letter in the divisor from the exponent of that letter in the dividend to find its exponent in the quotient.

3. Omit from the quotient any letter whose exponent in the dividend is the same as its exponent in the divisor and write unchanged in the quotient any letter that occurs only in the dividend.

EXAMPLES

1. $-28\,a^5bc^3 \div 7\,abc^2 = -4\,a^4c.$

Why is the sign of the quotient negative? How is the literal part of the answer obtained?

2. $-15\,c^8d^3f \div (-5\,cd^3f) = 3\,c^7.$ Explain.

3. $2^{n+4} \div 2^{n-2} = 2^{n+4-(n-2)} = 2^6 = 64.$

4. $-3\,x^5(a-b)^4 \div x^2(a-b) = -3\,x^3(a-b)^3.$

EXERCISE

152. *Find the quotients:*

1. $15\,x^3y^2 \div 3\,x^2y.$
2. $3\,axy \div 2\,ay.$
3. $-7\,m^3n \div 3\,mn.$
4. $27\,a^2b^5c^3 \div (-9\,abc).$
5. $\frac{11}{2}\,a^4bc^3 \div (-\frac{7}{9}\,a^3bc).$
6. $\frac{15}{16}\,m^5y^4 \div 4\,m^4y^3.$
7. $-y^5z^8 \div (-y^4z^8).$
8. $-ax^9 \div 3\,ax^6.$
9. $7\,a^2b^2c \div (-8\,abc).$
10. $5040\,a^qb^p \div 720\,ab^2.$
11. $x^my^n \div (-xy).$
12. $-5\,a^xb^{m-1} \div (-15\,a^xb^{1-m}).$
13. $\dfrac{3\,a^5bc}{15\,abc}.$
14. $\dfrac{-2\,a^2bc}{4\,a^2bc}.$
15. $\dfrac{4\,xy^2z}{-2\,xyz}.$
16. $\dfrac{33\,x^3y}{(-3\,x)(-y)}.$

Find the quotients:

17. $\dfrac{-4\,ab\cdot(-7\,a^2b^3)}{(-2)^2\cdot a^3b^3}$.

18. $\dfrac{a^5b^6(-ab)^2}{-a^4b^4}$.

19. $\dfrac{2^{12}a^{7n+2}}{2^{10}a^2}$.

20. $\dfrac{(-a^3)(-b^4)(-c)^4}{(-a)^2(-b)^3(-c)^4}$.

21. $\dfrac{(-a)^3(-a^3)}{(-a^2)(-a)^2}$.

22. $\dfrac{-3(-ab)^2}{ab}$.

23. $\dfrac{3^{n+2}a^{n+8}}{3^2a^3}$.

24. $\dfrac{-3^4\cdot 5^8}{3^3(-5)^2}$.

25. $\dfrac{2^{n+2}\cdot 3^{n+3}}{2^n\cdot 3^{n+2}}$.

26. $3\,ax^2y\cdot 6\,ay^3 \div (-9\,a^2x^2y^2)$.

Write the work for this example as follows:

$$3\,ax^2y\cdot 6\,ay^3 = 18\,a^2x^2y^4$$
$$18\,a^2x^2y^4 \div (-9\,a^2x^2y^2) = ?$$

27. $4\,x^3y \div 2\,xy + 17\,x^4y^2 \div 2\,x^2y^2$.

28. $x^3(a-b)^7 \div x^2(a-b)^4$.

29. $4\,x^5(8-m)^3 \div 4\,x^3(8-m)$.

30. $12\,b(y-z)^5 \div [-4\,b(y-z)^4]$.

31. $15\,m^2(x^2-1)^4 \div 3\,m(x^2-1)^3$.

32. $15\,cd^4 \div 5\,d^3 + 22\,c^5d^8 \times cd \div 11c^4d^8$.

33. $10\,a^2b \times ab^2 \div 5\,ab - 5\,a^4b^3 \div (-5\,a^2b)$.

34. $a^m \div a^n$.

35. $2\,a^m \div a^n$.

36. $a^{m+n} \div a^n$.

37. $a^{2n} \div a^n$.

38. $33\,a^{s+2}b^3 \div 3\,a^sb$.

39. $28\,a^{s+2}b^3 \div (-4\,a^2b)$.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

153. 1. Since $2\times(a+b)=2\,a+2\,b$, therefore

$$(2\,a+2\,b)\div 2 = a+b,$$

by the definition of division.

2. Since $a(x+y)=ax+ay$, therefore $(ax+ay)\div a = x+y$.

3. $(2x + 4y) \div 2 = x + 2y.$

CHECK. $2(x + 2y) = 2x + 4y.$

4. $(4a^2 + 10a) \div a = 4a + 10.$

5. $(5a - 10b + 15c) \div (-5) = -a + 2b - 3c.$

ORAL EXERCISE

154. *Find the quotients:*

1. $(cy + dy) \div y.$
2. $(pq + rq) \div q.$
3. $(cd - bd) \div d.$
4. $(4a - 8b) \div 4.$
5. $(a^2 + 2a) \div a.$
6. $(a^2 + a) \div a.$
7. $(15p + 20q) \div 5.$
8. $(18h - 27k) \div (-9).$
9. $(21x^3 - 14x) \div 7x.$
10. $(-18m^2 - 24m) \div 6m.$
11. $(ax + bx + cx) \div x.$
12. $(dx - dy - dz) \div (-d).$
13. $(2x^3 - 6x^2 + 4x) \div 2x.$
14. $(-9p^2 + 12p) \div (-3p).$
15. $(a^2b + ab^2) \div ab.$
16. $(2a^2 - 8a + 10) \div (-2).$
17. $(abc - 2b^2c) \div (-bc).$
18. $(14a - 16b + 18c) \div (-2).$
19. $(a^n + a^{2n}) \div a^n.$
20. $(a^{n+1} - a^n) \div a^n.$

155. From these examples we derive the following rule:

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and unite the results with their respective signs.

EXAMPLES

1. $(3a^2 - 6a) \div 3a.$

$$3a\overline{)3a^2 - 6a}$$
$$a - 2$$

2. $(15a^2 - 5a + 5) \div (-5).$

$$-5\overline{)15a^2 - 5a + 5}$$
$$-3a^2 + a - 1$$

CHECK. $3a(a - 2) = 3a^2 - 6a.$

3. $(3a^3b^3 - 12a^2b^2 - 3ab) \div (-3ab).$

$$-3ab\overline{)3a^3b^3 - 12a^2b^2 - 3ab}$$
$$-a^2b^2 + 4ab + 1$$

The simplest verification of such exercises is by using the relation, divisor × quotient = dividend. $(d \times q = D.)$

EXERCISE

156. *Find the quotients :*

1. $4a^3 - 6a^2b - 12ab^2) \div (-2a)$.
2. $(2x^3 - 8x^2y + 10xy^2) \div 2x$.
3. $(21abc - 35bcd - 42acd) \div (-7c)$.
4. $(15a^2bc - 27ab^2c - 33abc^2) \div (-3abc)$.
5. $(17a^2b - 13ab^2) \div 2ab$.
6. $(21ax^2 + 15a^2x) \div 7ax$.
7. $(3m^2 + 4mn - 9n^2) \div (-3)$.

8. $\dfrac{14a - 7b + 7}{7}$.

9. $\dfrac{apq - 3bpq + pq}{-2pq}$.

10. $\dfrac{5p - 5q}{\cdot 5}$.

11. $\dfrac{m(a+b) - 2m(2a - b)}{m}$.

12. $D = 3x^2y + 5xy^2$, $d = 2xy$, find q.
13. $q = a + b - 3$, $d = -2ab$, find D.
14. $D = 2a^2x^2 - 6ax$, $d = 2ax$, find q.
15. $D = 15a^2 - 9a^5 + 18a^9$, $d = 3a^2$, find q.
16. $(8a^5b - 24a^4b^3 + 16a^7b^8) \div (-8a^4b)$.
17. $(25a^3x^2 + 60a^2x^3 - 25xy^2) \div (-5x)$.
18. $(21a^3 - 14a^2 - a) \div (-a)$.
19. $(36x^3y^2 - 24x^2y^2z - 18xy^2z) \div (-6xy^2)$.
20. $(36a^{10} - 24a^6 + 21a^5) \div (-6a^5)$.
21. $(100a^2bc - 75ab^2c + 50abc^2) \div (-25abc)$.
22. $(35c^2xy + 42cx^2 - 56cxy) \div 7cx$.
23. $(12a^{n+3} - 15a^{n+2} - 27a^{n+1}) \div 3a$.
24. $(12a^{n+3} - 15a^{n+2} - 27a^{n+1}) \div (-3a^n)$.
25. $(12a^{n+3} - 15a^{n+2} - 27a^{n+1}) \div (-3a^{n+1})$.
26. Show that $(5^{n+3} + 5^{n+2} + 5^{n+1}) \div 5^n = 155$.
27. $(a^{2x+3}b^{x+3} - 2a^{2x+1}b^{x+1}) \div (-a^{2x}b^x)$.
28. $(2^{2n+3} - 2^{2n+2}) \div 2^{2n+1}$.

29. $(a^{3x+y}b^{3m+n} - a^{2x+2y}b^{2m+2n}) \div a^{2x+y}b^{2m+n}$.

30. $[(x+y)a+(x+y)b] \div (x+y)$.

SOLUTION.
$$x+y\overline{)(x+y)a+(x+y)b}$$
$$a + \qquad b$$

31. $[(a+b)x+(a+b)y] \div (a+b)$.

32. $[r^2(m+n) - 2\,r(m+n) + (m+n)] \div (m+n)$.

33. $[12\,x^2(a+b)^3 - 32\,xy(a+b)^2] \div [-4\,x(a+b)^2]$.

34. $[2\,m^2(x-y^2)^3 - 3\,m(x-y^2)^2 - (x-y^2)] \div (x-y^2)$.

35. $[-8\,a^2b(x-y)^2 + 9\,ab^2(x-y)] \div ab(x-y)$.

36. $[x^5(a^2+b^2) - 2\,x^2(a^2+b^2)] \div x^2(a^2+b^2)$.

37. $[12\,b(x^2-y^2) - 15\,b^2(x^2-y^2)] \div 3\,b(x^2-y^2)$.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

157. This kind of division will be understood best by studying an example.

Divide $2\,x^3 - 7\,x^2 + 10\,x - 8$ by $x - 2$.

1.		$2\,x^3 - 7\,x^2 + 10\,x - 8$	$x - 2$
2.	$2\,x^2(x-2) =$	$2\,x^3 - 4\,x^2$	$2\,x^2 - 3\,x + 4$
3.		$-3\,x^2 + 10\,x - 8$	
4.	$-3\,x(x-2) =$	$-3\,x^2 + 6\,x$	
5.		$4\,x - 8$	
6.	$4(x-2) =$	$4\,x - 8$	

1. Both dividend and divisor are arranged in descending powers of x.

2. The first term of the dividend is divided by the first term of the divisor to obtain the first term of the quotient, $2\,x^2$. The entire divisor is then multiplied by the first term of the quotient.

3. The product obtained is subtracted from the dividend.

4. The first term of the remainder is divided by the first term of the divisor, to obtain the next term of the quotient, $-3\,x$. The entire divisor is then multiplied by this second term of the quotient.

5. The product is subtracted from the last remainder.

6. The process described in the last two steps is repeated until, in exact division, a remainder zero is obtained.

158. The explanation just given may be regarded as a **rule** for the division of a polynomial by a polynomial. It is of the greatest importance that a proper arrangement of the terms of the polynomials be made at the beginning and that the same arrangement be observed in all the remainders obtained in the course of the work.

Let the student explain how the next term of the quotient is obtained. Also explain all the operations involved in steps 5 and 6.

To check examples in long division the relation $d \cdot q = D$ may be used, or arbitrary values of the letters may be substituted. If the latter method is employed, values of the letters should be chosen which will not make the divisor 0.

Examples

1. Divide $(x^2 + 3x - 4)$ by $(x - 1)$.

Division

$$\begin{array}{r|l} x^2 + 3x - 4 & x - 1 \\ \underline{x^2 - x} & x + 4 \\ 4x - 4 & \\ \underline{4x - 4} & \end{array}$$

Check: When $x = 2$, $D = 6$, $d = 1$, $q = 6$, and $6 \div 1 = 6$.

2. Divide $8a^3 + 27b^3$ by $4a^2 - 6ab + 9b^2$.

$$\begin{array}{l|l} 8a^3 + 27b^3 & 4a^2 - 6ab + 9b^2 \\ \underline{8a^3 - 12a^2b + 18ab^2} & 2a + 3b \\ \qquad 12a^2b - 18ab^2 + 27b^3 & \\ \qquad \underline{12a^2b - 18ab^2 + 27b^3} & \end{array}$$

Check. Multiply the divisor by the quotient.

EXERCISE

159. *Find the quotients :*

1. $(x^2 - 5x + 6) \div (x - 2)$.
2. $(a^2 - 8a + 15) \div (a - 3)$.
3. $(4b^2 - 4b + 1) \div (2b - 1)$.
4. $(a^2 - b^2) \div (a - b)$.
5. $(a^2 - ab) \div (b - a)$.
6. $(7a - 14) \div (2 - a)$.

7. $(3x^2 - 4x^3 + 20) \div (x - 2)$.
8. $(6a^3 - 23a^2b + 25ab^2 - 6b^3) \div (2a - 3b)$.
9. $(30ap - 6bp + 12cp) \div (5a - b + 2c)$.
10. $(20ac - 15ad - 12bc + 9bd) \div (5a - 3b)$.
11. $(3abd - 3cd + abc - c^2) \div (ab - c)$.
12. $(6a^3b + 9ab^2 + 3abc + 2a^2c + 3bc + c^2) \div (3ab + c)$.
13. $(x^4 + x^3 - 4x^2 + 5x - 3) \div (1 - x + x^2)$.
14. $(27x^3 - 8y^3) \div (3x - 2y)$.
15. $(8a^3b^3 - c^3d^3) \div (4a^2b^2 + 2abcd + c^2d^2)$.
16. $(a^2 + b^2 + c^2 + 2ab - 2ac - 2bc) \div (a + b - c)$.
17. $(5a^6 + 15a^5 + 5a + 15) \div (a + 3)$.
18. $(2a^4 - 6a^3 + 3a^2 - 3a + 1) \div (a^2 - 3a + 1)$.
19. $(42a^4 + 41a^3 - 9a^2 - 9a - 1) \div (7a^2 + 8a + 1)$.
20. $(2m^4 - 6m^3 + 3m^2 - 3m + 1) \div (m^2 - 3m + 1)$.
21. $(6a^3x - 17a^2x^2 + 14ax^3 - 3x^4) \div (2a - 3x)$.
22. $(2x^4 + x^3y - 13x^2y^2 - 3xy^3 + y^4) \div (x^2 - 2xy - y^2)$.
23. $(15a^5 + 10a^4b + 4a^3b^2 + 6a^2b^3 - 3ab^4) \div (5a^3 + 3ab^2)$.
24. $(21a^4 - 16a^3b + 16a^2b^2 - 5ab^3 + 2b^4) \div (3a^2 - ab + b^2)$.
25. $(20a^6 - 53a^7 + 45a^9 - a^8) \div (4a^2 - 5a^3)$.
26. $(x^5 - 5x^4y - 10x^2y^3 + 10x^3y^2 + 5xy^4 - y^5) \div (x^2 - 2xy + y^2)$.
27. $(a^4 + 2a^2x^2 + x^4 - b^4) \div (a^2 + x^2 + b^2)$.
28. $(6a^2 + ab + 7ac - 12b^2 + 19bc - 5c^2) \div (2a + 3b - c)$.
29. $(15x^2 - 29xy + 12y^2 - 22yz - 60z^2) \div (5x - 3y + 10z)$.
30. $(48x^2y^4 - 80x^3y^3 - 8xy^5 + 200x^4y^2) \div (20x^2y^2 - 4xy^3)$.
31. $(343a^3x^3 - 64b^3x^6) \div (49a^2x^2 + 28abx^3 + 16b^2x^4)$.
32. $(20x^4 + 32x - 51x^3 - 12x^2) \div (4x^2 - 7x - 8)$.
33. $(32a^2 + 45b^2 + 60c^2 + 76ab + 88ac + 104bc)$
$\div (8a + 9b + 10c)$.
34. $(1.2a^2 + 1.17ab - 11.34b^2) \div (1.5a + 5.4b)$.
35. $[x^2 + (a + c)x + ac] \div (a + x)$.
36. $[y^2 - (a - b)y - ab] \div (a - y)$.

160. Division with a Remainder. If the dividend is not the product of the divisor multiplied by some integral algebraic expression, we shall have a remainder.

1. Divide $6x^2 - 13x - 3$ by $2x + 1$.

DIVISION

$$\begin{array}{r|l} 6x^2 - 13x - 3 & 2x + 1 \\ \underline{6x^2 + 3x} & 3x - 8, \text{ quotient} \\ -16x - 3 & \\ \underline{-16x - 8} & \\ 5, \text{ remainder.} & \end{array}$$

CHECK

$$\begin{array}{r} 3x - 8 \\ \underline{2x + 1} \\ 6x^2 - 16x \\ \underline{3x - 8} \\ 6x^2 - 13x - 8 \\ \underline{5} \\ 6x^2 - 13x - 3 \end{array}$$

2. Divide $x^3 + 3x^2 + 7$ by $x^2 - 2x + 2$.

$$\begin{array}{r|l} x^3 + 3x^2 + 7 & x^2 - 2x + 2 \\ \underline{x^3 - 2x^2 + 2x} & x + 5, \text{ quotient.} \\ 5x^2 - 2x + 7 & \\ \underline{5x^2 - 10x + 10} & \\ 8x - 3, \text{ remainder.} & \end{array}$$

Unless otherwise directed, perform all divisions in descending powers of some letter, and continue the division until the exponent of the highest power of the letter of arrangement in a remainder is less than that of the highest power of that letter in the divisor.

EXERCISE

161. *Divide, and check by the relation* $d \cdot q + r = D$.

1. $(x^2 - 3x + 5) \div (x + 1)$.
2. $(4 - 3x^2 + 2x) \div (2 + x)$.
3. $(x^3 - 1) \div (x^2 - x + 1)$.
4. $(3a - a^3 + 2) \div (1 - a^2)$.
5. $(7a^2 + 6a^3 + 5a - 7) \div (3a - 1)$.
6. $(7a^2 + 6a^3 + 5a - 7) \div (2a^2 + 3a + 2)$.
7. $(x^3 - 8a^3 - 2a^2x) \div (2a - x)$.

8. $(-73\,x^2 - 25 + 56\,x^4 + 95\,x - 59\,x^3)$
$\div(-11\,x + 7\,x^3 - 3\,x^2 + 1).$

9. $(49\,x^3 - 72\,xy^2 + 28\,y^3) \div (7\,x - 3\,y).$

10. $(4\,m^4 - m^2n^2 + 6\,mn^3 - 9\,n^3) \div (2\,m^2 - mn + 3\,n^2).$

11. $x^3 \div (x - 1).$

12. For what value of k is $x^2 - 3\,x + k$ exactly divisible by $x + 1$?

SOLUTION.

$$\begin{array}{l|l} x^2 - 3\,x + k & x + 1 \\ \underline{x^2 + x} & x - 4 \\ \quad -4\,x + k & \\ \quad \underline{-4\,x - 4} & \\ \qquad\qquad k + 4 = \text{the remainder.} & \end{array}$$

The division is exact if the remainder is 0, or if $k + 4 = 0$, that is, if $k = -4$.

13. Determine k so that $x^3 + 3\,x^2 + 2\,x + k$ shall be exactly divisible by $x - 2$.

14. Determine k so that $x + 1$ shall be an exact divisor of $x^3 + k$.

15. For what value of k is $x - 1$ an exact divisor of $x^3 + k$?

VII. SIMPLE EQUATIONS

162. An **equation** is a statement expressing the equality of two numbers. (Review §§ 12–16.)

There are two essentially different kinds of algebraic equations as illustrated by the following:

$$1.\quad (x+2)(x-2)=x^2-4.$$
$$2.\quad x+2=5.$$

Equation 1 is true for all values of x; equation 2 is satisfied when x equals 3, and not otherwise.

163. Identity. An equation that is true for all values of the letters involved is an **identical equation**, or simply an **identity.**

The symbol $\equiv$ is sometimes used to indicate an identity.

The most frequent use of the identical equation is to indicate the result of some operation performed upon algebraic expressions.

The following are examples of identical equations:

$$2x+3x+5x\equiv 10x.$$
$$(x+3)^2\equiv x^2+6x+9.$$
$$(a-b)(a+b)\equiv a^2-b^2.$$
$$6m(m-n)\equiv 6m^2-6mn.$$

In the identical equation, if the indicated operations are performed and the like terms are collected in each member, the two members will be exactly alike.

164. Conditional Equation. An equation that is true for only certain values of the letters involved is a **conditional equation** or simply an equation.

A conditional equation expresses a relation between an unknown number and certain known numbers. The problem suggested by a conditional equation is that of finding for what value of the unknown number the relation expressed in the equation is true.

The following are examples of conditional equations:

$2x - 7 = x + 3.$	True when $x = 10$, and not otherwise.
$3a + 7 = 4a + 7.$	True when $a = 0$ and not otherwise.
$2ax = 4a^2.$	True when $x = 2a$ and not otherwise.

ORAL EXERCISE

165. **1.** Is $x + 1 = 2$ a conditional equation or an identity? $2x + 3 = 7$?

2. Is $2x - (x + 1) = x - 1$ a conditional equation or an identity? $(x - 1)(x + 1) = x^2 - 1$? $2x - 1 = x$?

3. State the four principles used in solving equations. (See § 13.)

4. What is the root of an equation? (See § 16.)

5. What is the root of $x + 2 = 7$? of $x - 2 = 7$? of $2x = 3$? of $\frac{1}{2}x = 5$?

6. What value of x satisfies the equation $x - 2 = 3$?

Show that the following are identities by reducing the two members to the same expression:

7. $a(x - y) = ax - ay.$

8. $(x + a)(x + b) = x^2 + (a + b)x + ab.$

9. $5y + 3 - 4y = y + 3.$

10. $11z - (5 + 10z) = z - 5.$

Solve the following conditional equations:

11. $x - 3 = 2.$

12. $y + 7 = 9 - y.$

13. $2x - 1 = 5.$

14. $\frac{1}{3}x + 1 = 4.$

15. $3x - 4 = 5.$

16. $2z - 8 = 3.$

17. $w + 4 = -10.$

18. $2n = -6.$

19. $4x - 2x + 3 = -3.$

20. $5n - 4 = -14.$

EXERCISE

166. *Show that equations* 1 *to* 4 *are identical equations by reducing the two members to the same expression.*

1. $a(b-c)+b(a-c)=2\,ab-c(a+b)$.
2. $(a+b-c)(a+b+c)=a^2+2\,ab+b^2-c^2$.
3. $(x^2-x-2)(x^2+x-2)=(x^2-3\,x+2)(x^2+3\,x+2)$.
4. $(x+y)(y+z)(z+x)+xyz=(x+y+z)(xy+yz+zx)$.

By substituting 1, 2, *and* 3 *for x in equations* 5 *to* 8, *show that each is a conditional equation.*

5. $2\,x-5=x-3$.
6. $(x-4)^2+2=(x-5)^2-3$.
7. $(x-1)(x+2)=x^2$.
8. $8\,x+7-x=14$.

Solve the following equations:

9. $x-4\,x+3=0$.
10. $5\,p+18=3(p+10)-2$.
11. $31-7\,x=41-8\,x$.
12. $5\,x+13-2\,x=100-20\,x-18+12\,x-15$.
13. $16\,v+10-21\,v=45-10\,v-15$.
14. $7\,y-9-3\,y+5=11\,y-2(3+2\,y)$.
15. $-40=5-30\,x+35-40\,x$.
16. $b=b-9+3\,b+2$.

167. It is a common practice in algebra to use x, y, and z to represent the unknown numbers in an equation and to use a, b, c etc. to represent numbers that are regarded as known.

Thus, in the equation $ax=3\,a^2b$, x is the unknown number and the value of x is to be found in terms of the other letters involved. The value of x is found by dividing both members of the equation by a, giving $x=3\,ab$. The equation $x=3\,ab$ can be solved for a or for b. Thus, dividing both members by $3\,b$ gives $a=\frac{x}{3\,b}\cdot$ Solve the equation for b.

168. Integral Equation. An equation in which the unknown number does not occur in any denominator is an **integral equation.**

Thus, $3x - 4 = \frac{2}{3}x - 5$ and $ax + b - \frac{x}{a} = 3b$ are integral equations.

In the present chapter all equations are integral.

169. Solving Equations.

1. Solve $3x - (5 - x) = 11$.

SOLUTION. $3x - (5 - x) = 11$.

$3x - 5 + x = 11$. (Removing parentheses.)

$-5 + 4x = 11$. (Collecting like terms.)

$4x = 16$. (Adding 5 to both members.)

$x = 4$. (Dividing both members by 4.)

Let the student check the answer by putting 4 for x in the original equation.

$4x = 16$ is the simplified form of the equation and the work done to reduce the equation to this form is called **simplifying** the equation.

2. Solve $(x - 3)(x - 2) = (x - 4)^2$.

SOLUTION. $(x - 3)(x - 2) = (x - 4)^2$.

$x^2 - 5x + 6 = x^2 - 8x + 16$. (Multiplying.)

$-5x + 6 = -8x + 16$. (Why?)

$8x - 5x + 6 = 16$. (Why?)

$3x = 10$. (Why?)

$x = \frac{10}{3}$. (Why?)

Let the student check the answer as in example 1.

What is the simplified form of this equation?

170. Simple Equation. An equation that can be reduced to an integral form containing the first power of the unknown number and no higher power is a **simple equation.**

Thus, $5x - 2(3x - 1) = 1$ is a simple equation. Also $x(x - 5) = (x - 3)(x - 7)$ is a simple equation, for it reduces to $5x = 21$.

Simple equations are frequently called **first degree equations,** also **linear equations.**

171. The type form of the simplified equation of the first degree is $\boldsymbol{ax = b}$. By this we mean that x, with any coefficient it may have (represented in the type form by a), constitutes the first member of the equation, and the known term or terms, represented by b, constitute the second member.

Thus, $3x = 7$ is in the form $ax = b$; here $a = 3$, and $b = 7$.

172. The steps required to reduce an equation to the form $ax = b$ are illustrated in the solution of examples 1 and 2 of § 169. The principles involved are those stated in § 15.

EXERCISE

173. *Solve the first 20 equations orally.*

1. $2x = 15$.
2. $x - 2 = 0$.
3. $-x = 2$.
4. $3x = 0$.
5. $-x + 3 = 0$.
6. $x + 1 = 5$.
7. $2x + 1 = 5$.
8. $x + 3 = -7$.
9. $3x + 6 = 9$.
10. $-x - 7 = 0$.
11. $-2x + 8 = 0$.
12. $7x = .56$.
13. $.4x = 2$.
14. $5x - 7 = 13$.
15. $-x - 7 = -8$.
16. $7 = 2 + x$.
17. $-5x = .5$.
18. $ax = 2a^2$.
19. $bx - 2ab = 0$.
20. $-ax + 7a = 0$.
21. $\frac{1}{2}x - 3 = 2$.
22. $x - 3 = 2 - x$.
23. $x - 5 = 5 - x$.
24. $-2x - \frac{1}{2} = 0$.
25. $17 - (16 - x) = 1$.
26. $-(18 + x) = 19$.
27. $.5x - .05 = .2$.
28. $-100 = 20 + (2x - 25)$.
29. $.82x + .1x = .3x$.
30. $2x - 7 = 4 - 2x$.
31. $4x - (7 - 2x) = 4x + 5$.
32. $6x + 4 = 3(x + 3) + 2(3 - x)$.
33. $5(2x - 3) - 2(3 - 2x) + 2 = 0$.
34. $2x - (4x - 7) - 3(5 - 7x) = 100$.
35. $7x - [3 - 2(x - 5) + 3] = 2$.
36. $5 - 2[2x - (3 - 10x) - 4] = -225$.

174. It is sometimes desired to solve equations when some of the numbers that are regarded as known numbers are represented by letters. The method is the same as that used in the equations already solved.

EXAMPLE. Solve $3ax - a(2c + x) = 2ab - 4ac$.

SOLUTION.
$$3ax - a(2c + x) = 2ab - 4ac.$$
$$3ax - 2ac - ax = 2ab - 4ac.$$
$$2ax - 2ac = 2ab - 4ac.$$
$$2ax = 2ab - 2ac.$$
$$x = b - c.$$

EXERCISE

175. *Solve the following equations, regarding the last letters of the alphabet as the unknown numbers:*

1. $3nx - nx = 2n^2 - 4n$.
2. $4cx - 5c = cx + c$.
3. $-ax + 7a = a^2$.
4. $bx - 2ba = b^2$.
5. $cx - cd + c^2 = c^3$.
6. $3x - (4a + 7) = 2a + x$.
7. $5x - 3(2c - 4d) = 2d + 4c$.
8. $3ax - 2(dx - a^2) = 3a^2 - 2dx$.
9. $(a - 4)x = a^2 - 8a + 16$.
10. $5ax - [2ax - (a^2 - ax)] = 5a^2 - 10a$.
11. $4x - (2a - 3b - x) = 3x + 6a - 9b$.
12. $2ax = 3a^2 - a(a + x)$.
13. $4x + 3 - 5x = a - 2$.
14. $ax - 3ab = 2ax + 7ab$.
15. $(a - 1)x = a^2 - 1$.
16. $3(5x - a) - 2(4x - 5a) = 0$.
17. $12(y + n) = 45 - 3(y + n)$.
18. $5 - (x + 15) + c = 8 - b + 2c$.
19. $2abz - a^2b = a(bz + ab)$.
20. $a(b - x) + b(c - x) = b(a - x) + bc$.

21. If the same term occurs in both members of an equation, it may be dropped from both members. Why? Would the statement be true if the terms were preceded by opposite signs?

176. The known and the unknown numbers are generally found distributed through the two members of an equation. In order to solve an equation, it is necessary to reduce it to the form $ax = b$.

Consider the equation $5x + 3 = 2x - 5$.

SOLUTION.	1.	$5x + 3 = 2x - 5$.	
	2.	$5x = 2x - 5 - 3$.	(Subtracting 3 from both members.)
	3.	$5x - 2x = -5 - 3$.	(Subtracting $2x$ from both members.)
	4.	$3x = -8$.	(Collecting terms gives type form $ax = b$.)
		$x = -\frac{8}{3}$.	

If we compare equation 3 with equation 1, we shall see that the term containing x in the second member of equation 1 appears in the first member of equation 3 with its sign changed; also that the term not containing x, that was in the first member, is now in the second member with its sign changed.

177. Transposing Terms. Any term of an equation may be changed from one member of the equation to the other by changing its sign. This process is known as **transposing terms.**

In writing the solution of equations the student may omit step 2 as given in § 176, and write equation 3 immediately from equation 1, describing the process, as "*transposing all terms containing the unknown number to the first member and all terms not containing the unknown number to the second member.*"

The mechanical process of transposing is a simple one, but *great care must be taken not to lose sight of the principles which underlie the process.*

178. The rule for solving linear equations will now be stated more fully.

To solve linear equations :

1. Perform all indicated operations, removing all parentheses in both members.

2. Transpose so that all terms containing the unknown shall be in the first member and all terms not containing the unknown shall be in the second member of the equation.

3. Collect the terms.

4. Divide both members by the coefficient of the unknown.

Examples

1. Solve $7x - 8 = 4 - (2 - 10x)$.

Solution.
$$7x - 8 = 4 - (2 - 10x).$$
$$7x - 8 = 4 - 2 + 10x. \quad \text{(Removing parenthesis.)}$$
$$7x - 10x = 4 - 2 + 8. \quad \text{(Transposing terms.)}$$
$$-3x = 10. \quad \text{(Collecting terms.)}$$
$$x = -3\tfrac{1}{3}. \quad \text{(Dividing by } -3.\text{)}$$

Check. Substitute $x = -3\frac{1}{3}$ in both members of the original equation, $7x - 8 = 4 - (2 - 10x)$.

Thus,
$$7(-3\tfrac{1}{3}) - 8 = 4 - [2 - 10(-3\tfrac{1}{3})].$$
$$-\tfrac{94}{3} = -\tfrac{94}{3}.$$
$x = -3\frac{1}{3}$ satisfies the equation.

2. Solve $(x - 5)(x - 7) = (x - 4)^2 - 1$.

Solution.
$$(x - 5)(x - 7) = (x - 4)^2 - 1.$$
$$x^2 - 12x + 35 = x^2 - 8x + 16 - 1. \quad \text{(Why?)}$$
$$-12x + 8x = 16 - 35 - 1. \quad \text{(Why?)}$$
$$-4x = -20. \quad \text{(Why?)}$$
$$x = 5. \quad \text{(Why?)}$$

Let the student check the result.

3. Solve $x(x - 6) = (x - 3)(x - 2) - 6$.

Solution.
$$x(x - 6) = (x - 3)(x - 2) - 6.$$
$$x^2 - 6x = x^2 - 5x + 6 - 6. \quad \text{(Why?)}$$
$$-6x + 5x = 0. \quad \text{(Why?)}$$
$$-x = 0. \quad \text{(Why?)}$$
$$x = 0. \quad \text{(Why?)}$$

Let the student check the result.

EXERCISE

179. *Solve the following equations:*

1. $3x - 5 = x + 2$.
2. $4y - 7 = 12y + 2$.
3. $(5a - 3) - (6a + 8) = 0$.
4. $-[x - (2 - 3x)] = 14$.

Solve:

5. $7x - 25 = 15(21 - 3x) + 24.$ 6. $4 - (2x - 7) = 3 - 4(4 - 5x).$

7. $m(m - 5) = (m + 2)^2 + 5.$

8. $4(x + 2) - 2(x + 1) - 3(7 - x) = 0.$

9. $(p - 4)(p + 4) - (p + 2)(p + 3) = -23.$

SOLUTION. $p^2 - 16 - (p^2 + 5p + 6) = -23.$ (Why?)
$p^2 - 16 - p^2 - 5p - 6 = -23.$ (Why?)
$-5p = -23 + 16 + 6.$ (Why?)
$-5p = -1.$ (Why?)
$p = \frac{1}{5}.$ (Why?)

NOTE. In simplifying the product $-(p + 2)(p + 3)$, it is better to perform the multiplication first $-(p^2 + 5p + 6)$ and remove the parenthesis afterward, as the sign $-$ affects the whole result of the multiplication.

10. $(v + 2)^2 - (v + 1)^2 = (v - 2)(v - 1) - v^2.$

11. $(2x - 1)(3x + 1) - (6x - 12)(x + 3) = 0.$

12. $(4x - 7)(9x - 48) = 12(3x + 1)(x - 6).$

13. $(2b - 5)(2b + 5) - (4b - 11)(b + 1) = 0.$

14. $(8x + 5)(2x + 7) - (4x - 3)(4x + 3) = 0.$

15. $(2x - 1)(144x + 5) - 26x = (8x + 1)36x + 11.$

16. $(t + 4)^2 - t(t + 6) = 22.$

17. $(2x)^2 + 5x(x + 7) = (3x)^2 + 70.$

18. $(x + 1)^2 + (x - 2)^2 - (x - 1)(x + 5) - x^2 = 0.$

19. $(q - 1)^2 + (q - 3)^2 - 2(q - 7)(q + 15) = 0.$

20. $x(x - 1)(x + 7) - (x + 1)(x + 2)(x + 3) = 0.$

21. $(z + 1)^3 = z^3 + 10 + 3z(z + 2).$

22. $\frac{7}{6}v - \frac{1}{4} - \frac{17}{18} = \frac{11}{36}(3v + 1).$

SOLUTION. $\frac{7}{6}v - \frac{1}{4} - \frac{17}{18} = \frac{11}{36}(3v + 1).$
$\frac{7}{6}v - \frac{1}{4} - \frac{17}{18} = \frac{11}{12}v + \frac{11}{36}.$ (Why?)
$\frac{7}{6}v - \frac{11}{12}v = \frac{11}{36} + \frac{1}{4} + \frac{17}{18}.$ (Why?)
$\frac{1}{4}v = \frac{3}{2}.$ (Why?)
$v = \frac{3}{2} \div \frac{1}{4} = 6.$

Let the student check the answer.

23. $\frac{1}{4}x - \frac{1}{5}x + \frac{1}{3}x - \frac{1}{6}x = 13.$

24. $\frac{8}{9}(2+5x)=\frac{1}{2}(9x+2)$.

25. $30(m-2)+\frac{1}{3}m=\frac{1}{16}(5m+1)+30$.

26. $\frac{1}{2}(5x+1)-\frac{1}{3}(4x+5)=\frac{1}{4}(3x-1)-\frac{1}{20}(6x+4)$.

27. $\frac{1}{7}(5x-1)-8=\frac{1}{3}(4x-2)$.

28. $\frac{1}{4}(1-x)-\frac{1}{5}(2-x)=\frac{1}{6}(3+x)$.

29. $.2(x-.3)-(x-.1)^2=x(.25-x)+.005$.

SOLUTION. $.2(x-.3)-(x-.1)^2=x(.25-x)+.005.$

$(.2x-.06)-(x^2-.2x+.01)=(.25x-x^2)+.005.$ (Why?)

$.2x-.06-x^2+.2x-.01=.25x-x^2+.005.$ (Why?)

$.2x+.2x-.25x=.06+.01+.005.$ (Why?)

$.15x=.075.$ (Why?)

$x=.5.$

Let the student check the answer.

30. $.25(4c-6)-.4(5c-7)=0$.

31. $5x-1.7=5.4x+0.8$.

32. $0.123d+0.138=0.876-0.123d$.

33. $7.5x-2.5-1.5x=4.5$.

34. $.45y-.75=-.125-.45y$.

35. $.7(.8x-3)-.39=1.11x-3(.2x-.5)$.

36. $.02g=2(.6-.04g)-.2(.5g-2)$.

37. $2\frac{2}{3}x+\frac{1}{3}x=2\frac{1}{2}+5\frac{1}{4}$.

38. $0=.75k-2k-.6k+5k-9$.

39. $\frac{2}{3}(7x-10)-\frac{1}{2}(50-x)=20$.

40. $5=3v+\frac{2}{5}(v+3)-\frac{1}{2}(11v-37)$.

41. $\frac{2}{3}(3x-5)-1=\frac{2}{3}(11-2x)+x$.

42. $1-3(7\frac{1}{2}+x)+7(\frac{2}{3}x-\frac{5}{2})+\frac{8}{3}x=0$.

43. $3-(.3-.07d)+.5(.1d+1)=4-.2(7-.3d)$.

44. $(1+6x)^2+(2+8x)^2-(1+10x)^2=0$.

45. $9(2x-7)^2+(4x-27)^2=13(4x+15)(x+6)$.

46. $9[7(5+\{3x-2\}-4)-6]-8=1$.

Solve:

47. $3[3(3+\{3h-2\}-2)-2]-2=1.$

48. $2+2(2x+3)+3(x+2)=12-(x+1).$

49. $\frac{1}{4}(2k+3)-23=\frac{1}{8}(6k-5).$

50. $.25(2x+1)+.2(3x-1)=\frac{1}{3}(7x-1)-x.$

51. $(s-9)(s+9)=(s+6)^2+s.$

52. $0=4(10-2x)-3(x-5).$

53. $0=3(9-2g)-5(2g-9).$

54. $7(4x-3)+3(7-8x)=1.$

55. $8(3i-2)-7i-5(12-3i)=131.$

56. $7(3x-6)+5(x-3)-4(x-17)=11.$

57. $6x-7(11-x)+11=4x-3(20-x).$

58. $44x-32=84+31.5x+4.2x+16.8.$

EXERCISE

180. The equation $x-15=24$ states in algebraic language that some number diminished by 15 is equal to 24, or that some number is 15 greater than 24.

In the same way translate into verbal language each of the following algebraic equations:

1. $x+10=25.$
2. $x+296=5x.$
3. $3y-14=40.$
4. $2x-15=3x.$
5. $25-5x=11.$
6. $72-2x=9x.$

Write in algebraic language, that is, make algebraic equations for the following:

7. 36 is greater than some unknown number by 10.

8. Three times a number is 14 less than 5 times the number.

9. 25 is divided into two parts the larger of which is n. What is the smaller part? Make an equation that indicates that 3 times one of the parts is equal to 2 times the other part.

10. 786 diminished by two times an unknown number is 270.

11. If 296 is added to an unknown number, the sum is 5 times the number.

12. A boy is three times as old as his sister and the sum of their ages is 16 years.

Let $x =$ the number of years in the sister's age.

13. If 7 is taken from 5 times a number, the remainder is 53.

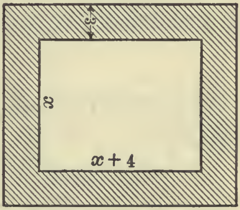
14. If 42 is added to 7 times an unknown number, the sum is 54.

15. A rectangle whose length is a feet, and whose width is 3 feet less than its length, has an area of 54 square feet.

16. 60 is divided into two parts, the smaller of which is $\frac{2}{3}$ of the larger.

17. 50 is divided into two parts such that 40 % of one part is equal to 20 % of the other part.

18. A house and lot are together worth \$ 8500. The value of the house exceeds 3 times the value of the lot by \$ 1000. Find the value of each.



19. The frame of a picture is 3 inches wide. The picture is 4 inches longer than it is wide and the area of the frame is 252 square inches. Find the dimensions of the picture.

HINT. The area of the picture is $x(x+4)$. The area of the picture and frame is $(x+6)(x+10)$. (Why?) The area of the frame is the difference of these areas.

20. A square lot has a walk around it that is 6 feet wide. The surface of the walk contains 2256 square feet. Find the length of a side of the square inside the walk.

THE SOLUTION OF PROBLEMS

181. A **problem** is a question proposed for solution. It involves the finding of one or more unknown numbers from relations stated in the problem.

182. In solving problems the following suggestions will be found useful:

1. The problem should be *carefully read* and the conditions stated in the problem should be *carefully analyzed.* Before the solution is attempted, the student should see clearly the relations existing between the unknown number and the known numbers.

2. Represent one of the unknown numbers by some letter. If more than one unknown number is involved, represent them in terms of the same letter.

3. Translate the verbal language of the problem into algebraic language in the form of an equation.

4. Solve the equation.

5. Check the result by testing whether the number or numbers found by solving the equation satisfy the conditions stated in the problem. It is not sufficient to determine whether these numbers satisfy the equation obtained, as an error might occur in forming the equation.

PROBLEMS

183. The following problems will illustrate the above suggestions:

1. If to 7 times a given number 12 is added, the sum is 54. What is the number?

SOLUTION. There is but one unknown number involved.

Let $n =$ this number.

Then by the conditions of the problem, $7n + 12$, or 7 times the number with 12 added $= 54$, or the sum.

Therefore the verbal language of the problem translated into algebraic language gives the following equation:

$$7n + 12 = 54.$$
$$\therefore 7n = 54 - 12, \quad \text{(Why ?)}$$
$$\text{or } 7n = 42. \quad \text{(Why ?)}$$
$$\therefore n = 6. \quad \text{(Why ?)}$$

CHECK. $7 \times 6 + 12 = 54.$

2. A certain number is 3 times another number. The sum of the two numbers is 28 less than twice the larger number. What are the numbers?

SOLUTION. Two unknown numbers are involved in this problem.

Let $x =$ the smaller number.

Hence $3x =$ the larger number.

Then $x + 3x = 6x - 28$. (By the conditions of the problem.)

$\therefore -2x = -28$. (Why ?)

$\therefore x = 14$, the smaller number,

and $3x = 42$, the larger number.

CHECK. $14 + 42 = 84 - 28$, or $56 = 56$.

3. I bought for my library 3 volumes at a certain price, 5 volumes at double the price, and 4 volumes at $\frac{3}{4}$ the price. For all I paid \$24. How much did each volume cost?

SOLUTION. Three unknown numbers are involved in this problem.

Let $x =$ the number of dollars paid for one of the 3 volumes.

Hence $2x =$ the number of dollars paid for one of the 5 volumes,

and $\frac{3}{4}x =$ the number of dollars paid for one of the 4 volumes.

Then $3x + 10x + 3x = 24$. (Why ?)

Let the student finish the solution and check the answer.

4. The sum of the digits of a number of two figures is 9. By interchanging the digits the resulting number will be 27 greater than the original number. What is the number?

SOLUTION. Two unknown numbers are involved in this problem.

Let $x =$ the units' digit.

Hence $9 - x =$ the tens' digit.

Therefore the original number is $10(9 - x) + x$, (Why ?)

and the number with the digits interchanged is $10x + (9 - x)$.

The second number is 27 greater than the original number.

Then $10(9 - x) + x = 10x + (9 - x) - 27.$

Let the student solve and check.

5. If a certain number is doubled and 7 is added, the result is -1. Find the number.

6. The number of boys in a certain school after being doubled and further increased by 10 is 60. What was the number at first?

7. A man walked for a certain number of hours at 4 miles an hour and then for twice as long a time at 3 miles an hour, covering 20 miles in all. How long did he walk at each rate?

HINT. Let $x =$ the number of hours at 4 miles an hour.
Hence $2x =$ the number of hours at 3 miles an hour,
and $4x =$ the number of miles at 4 miles an hour.
Let the student complete the solution and check.

8. Find a number such that 10 times the number is 14 less than 3 times the number.

9. Find the three consecutive numbers whose sum is 15.

10. The sum of three consecutive odd numbers is 33. Find the numbers.

11. The sum of four consecutive even numbers is 44. Find the numbers.

12. Divide $880 between A and B so that A shall receive $50 less than twice as much as B.

13. Divide 120 into two parts such that 7 times one part equals 8 times the other part.

14. Find a number such that 15 times the number is 10 times as great as the sum of the number and 4.

15. What price does a dealer pay for 6 dozen lead pencils if he sells them for 5¢ each and makes a profit of 90¢?

16. Find the number such that, if you add 3 and multiply the sum by 5, the result is 1 greater than if you add 5 and multiply by 3.

17. A bag contains an equal number of dollars, half dollars, quarters, dimes, and nickels. If the amount contained in the bag is $47.50, how many coins of each kind are there?

18. If 20 is added to a number the result will be 3 times as great as if 4 is subtracted from it. Find the number.

19. My neighbor's orchard contains 8 more trees than mine and together they contain 34 trees. How many trees does each orchard contain?

20. The sum of three numbers is 32. The second exceeds the smallest by 2 and the largest is 2 less than twice the smallest. Find the numbers.

21. The tens' digit of a number is 3 times the units' digit and the number exceeds 7 times the sum of its digits by 9. What is the number? (See problem 4.)

22. Two towns are 60 miles apart. A starts from one and walks $3\frac{1}{2}$ miles an hour toward the other town until he meets B who has started from the other town at the same time and is driving an automobile at 16 miles an hour. After how long will they meet? How far will each have gone?

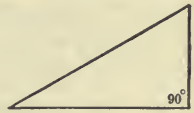
23. If in the last problem A had started at 8 o'clock and B at 10 o'clock, at what time would they have met?

24. What time is it if the number of hours past noon equals $\frac{1}{3}$ of the number of hours to midnight?

25. A man left half his property to his wife, one fifth to his daughter and the remainder, $6000, to his son. How much property did he leave?

26. The deposits in a bank during 2 days amounted to $21,000. The deposits on the second day were $\frac{1}{3}$ larger than on the first day. Find the deposits for each day.

27. The Washington Monument is 73 feet higher than the Great Pyramid in Egypt and the sum of their heights is 1037 feet. Find the height of each.



28. The sum of the three angles of any triangle is 180°. If in a right-angled triangle (having one angle 90°) one acute angle is twice as large as the other, how large is each angle?

29. How large is each angle in a right-angled triangle if one acute angle is 10° less than twice as large as the other?

30. How large is each angle in a triangle if the second angle is 10° larger than the smallest and the largest angle is equal to the sum of the other two? (See problem 28.)

31. An Iowa produce dealer ships eggs to the city of New York. The expense of shipping and selling the eggs is $\frac{2}{3}$ of the original cost of the eggs. If the eggs are sold for 25¢ a dozen, how much does the produce dealer pay for them?

32. A retail dealer received $\frac{1}{2}\frac{1}{5}$ more from the sale of a beef than he paid the packer. How much did he pay the packer if he received $ 84.20?

33. The cost of shipping wheat from Kansas to Philadelphia was $\frac{1}{3}\frac{1}{6}$ of the price paid to the Kansas farmer. How much a bushel did the farmer receive if the shipper received 1.17\frac{1}{2}$ a bushel in Philadelphia?

34. If the price of wheat in Kansas is $\frac{3}{4}$ of the price delivered in Liverpool, and the Kansas farmer receives 90¢ a bushel, what is the price of wheat in Liverpool?

35. The perimeter of a triangle is 60 centimeters. The second side is twice as long as the shortest and the longest side is 6 centimeters less than the sum of the other two sides. Find the length of each side.

36. A rectangular tennis court is 20 feet more than twice as long as it is wide and the distance around the court is 220 feet. Find the length and the width of the court.

37. The height of one of the big trees in California is 43 feet more than twice the distance around it at a point six feet from the ground. The sum of its height and girth is 466 feet. Find the height and the girth.

38. The largest package that can be sent by parcel post must not exceed 72 inches in length and girth combined.

What is the largest box with square ends that can be sent, if the box is twice as long as it is wide?

HINT. If $x =$ the number of inches in width, $4x =$ the number of inches in girth.

39. A star is added to the flag of the United States for each new state. There is one bar on the flag for each of the original colonies. What is the number of states and of original colonies if the number of stars is 9 more than three times the number of bars, and if the number of stars plus the number of bars is 61?

40. The cost of a cable message from New York to London is 25¢ a word. The rate from San Francisco to London is 1¢ more than 3 times the difference between the rates from New York and San Francisco to London. What is the rate from San Francisco?

HINT. Let $x =$ number of cents per word from San Francisco.
Then $x = 3(x - 25) + 1$. (Why?)

41. The total railway mileage of Ohio, Indiana, and Illinois in 1911 was approximately 28,000 miles. The mileage of Ohio exceeded that of Indiana by 2000 miles, and Illinois had 4000 miles less than the other two states together. Find the mileage of each state.

42. The steel bridge from New York to Long Island is the longest single arch in the world. The length of the arch exceeds twice the height of Washington Monument by 7 feet, and the sum of the length of the arch and the height of the monument is 1672 feet. How high is the monument and how long is the bridge?

43. The annual precipitation (rainfall and snow) of Michigan is 4 times that of Nevada and is $\frac{2}{3}$ as great as that of Washington. The sum of the numbers of inches in the three states is 93.5 inches. Find the number of inches for each state.

44. A mark (a German coin) is worth $4\frac{1}{2}$¢ more than a franc (a French coin). How much is each one worth in our money if two francs and three marks are worth $ 1.10?

45. The distance from the earth to the moon is about $\frac{10}{37}$ of the diameter of the sun, and the sum of the distance to the moon and the diameter of the sun is 1,128,000 miles. Find the distance to the moon and the diameter of the sun.

46. A man bought 200 acres of land for $ 15,200. For some of it he paid $ 70 per acre and for the rest he paid $ 85 per acre. How much did he buy at each price?

SOLUTION. Let $x =$ the number of acres at $ 70.

Hence $200 - x =$ the number of acres at $ 85.

Also $70x =$ the number of dollars for 1st part,

and $85(200 - x) =$ the number of dollars for 2d part.

Then $70x + 85(200 - x) = 15{,}200$. (By the conditions of the problem.)

Let the student complete the solution.

47. A grocer bought 70 pounds of coffee for $ 19.20. Part of it cost 24¢ a pound and the rest cost 30¢ a pound. How many pounds of each kind did he buy?

48. A grocer wishes to mix coffee that he sells at 28 cents a pound with other coffee that he sells at 35 cents, to get a blend that he can sell at 30 cents a pound. How many pounds of each should he take to get 70 pounds of the mixture?

49. How many pounds each of 50¢ tea and 75¢ tea should be mixed to get 20 pounds worth 60¢ a pound?

50. How many pounds each of white Dutch clover seed worth 40¢ a pound and blue grass seed worth 20¢ a pound, should be used to make 100 pounds of a lawn grass mixture worth 25¢ a pound?

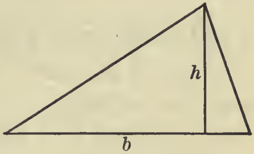
51. A man loaned $ 2000, part at 6 % and part at 4 %. The interest on each part was the same. How much was loaned at each rate?

52. A man loaned $ 1000, part at 5 % and part at 6 %. His interest was $ 57. How much was loaned at each rate?

RULES AND FORMULAS

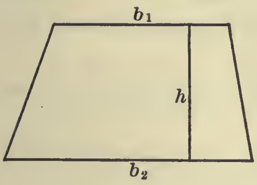
184. A rule can often be more easily remembered if expressed in algebraic language by means of a formula.

Thus, $i = prt$ (where i = the interest, p = the principal, r = the rate, and t = the time in years) is a formula by means of which the interest on a sum of money can be found when the principal, the rate, and the time are given.



$A = \frac{1}{2}bh$ is a formula by means of which the area, A, of a triangle can be found when the base, b, and the altitude, h, are given.

185. Translating Rules into Formulas. The area of a trapezoid equals the sum of the parallel sides multiplied by $\frac{1}{2}$ the altitude. The formula



$$A = \frac{1}{2}h(b_1 + b_2)$$

is a short way of writing the rule, where A represents the area, h the altitude, and b_1 and b_2 (read b sub one and b sub two) represent the two parallel sides of the trapezoid.

ORAL EXERCISE

186. *Express each of the following rules as a formula:*

1. The area, A, of a circle is equal to π (3.1416) times the square of the radius, r.

2. The area, A, of a rectangle is equal to the product of its two dimensions, a and b.

3. The diagonal, d, of a square is equal to one of its sides, s, multiplied by $\sqrt{2}$.

4. The distance, d, that a train goes is equal to the product of the rate per hour, r, multiplied by the number of hours (t).

5. The profit, p, is equal to the selling price, s, minus the cost, c.

6. The rate per cent of profit, r, is equal to the quotient of the selling price, s, minus the cost, c, divided by the cost.

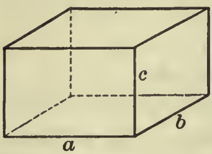
187. Translating Formulas into Rules. The formula for finding the area of a rectangle is $A = ab$, where A is the area, and a and b are its two dimensions. Hence this formula is an abbreviation for the rule: **The area of a rectangle is equal to the product of its two dimensions.**

EXERCISE

188. *Express each of the following formulas as rules:*

1. $C = 2\pi r$, where C represents the circumference of a circle and r its radius.

2. $V = a \cdot b \cdot c$, a formula for the volume of a rectangular parallelepiped whose dimensions are a, b, and c.



3. $rt = d$, $d \div r = t$ and $d \div t = r$, where d, r, and t represent distance, rate, and time respectively.

4. $A = \pi r^2$, where A represents the area of a circle and r its radius.

5. $V = \frac{1}{3} h\pi r^2$ where $V =$ volume of a cone, $h =$ altitude and $r =$ radius of base.

189. The Use of Formulas. In using a formula the problem may be merely that of evaluating an expression, or it may involve the solution of an equation.

1. Find the area of a triangle whose base is 6 inches long and whose altitude is 5 inches.

SOLUTION. Substituting 6 and 5 for b and h respectively in the formula $A = \frac{1}{2} bh$, we have $A = \frac{1}{2} \cdot 6 \cdot 5 = 15$.

$\therefore$ the area of the triangle is 15 sq. in.

2. Find the altitude of a triangle whose area is 20 square inches and whose base is 10 inches long.

SOLUTION. $20 = \frac{1}{2} \cdot 10 \cdot h$ or $20 = 5h$.

$\therefore h = 4$.

The altitude of the triangle is 4 inches.

EXERCISE

190. 1. Find the area of a triangle whose base is 9 inches and whose altitude is 7 inches.

2. Find the base of a triangle whose altitude is 5 inches and whose area is 18 square inches.

In problems in simple interest, if p represents the principal, r the rate of interest, t the time expressed in years, i the interest, and a the amount (principal plus interest), we have the following formulas:

$$(1)\ i = prt,\quad (2)\ a = p + i \text{ or } a = p + prt.$$

3. What is the interest on \$900 at 6% for 2 years?

4. What principal will produce \$288 interest in 4 years at 6%?

HINT: Substituting in $i = prt$, $288 = p \times .06 \times 4$, or $288 = .24p$. Solve.

5. How long will it take \$1200 at 5% to produce \$210?

6. At what rate will \$1800 produce \$252 interest in 2 years?

7. What principal will amount to \$1220 in 4 years at $5\frac{1}{2}$%?

HINT. Substituting in $a = p + prt$, $1220 = p + .22p$. Solve.

8. What principal at 5% will yield an annual income of \$350?

9. Using the formula of § 185 find the area of a trapezoid whose parallel sides are 12 inches and 15 inches and whose altitude is 8 inches.

10. Using the same formula, find the altitude of a trapezoid knowing that the parallel sides are 8 feet and 4 feet long and that the area is 50 square feet.

HINT. Substitute the numbers given for the proper letters of the formula and solve the resulting equation for h.

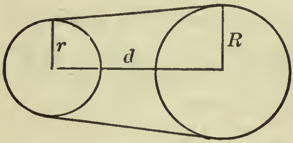
11. In a trapezoid $A = 72$ square inches, $b_1 = 17$ inches, $h = 2\frac{2}{5}$ inches; find b_2.

To find the perimeter of a rectangle, we have the formula:

$$p = 2(a + b).$$

12. Find p, when $a = 12$ inches, and $b = 7$ inches.

13. Find a when $p = 72$ feet and $b = 20$ feet.



A formula for the approximate length, l, of an open belt passing around two pulleys, as in the figure, is given by the equation $l = 2d + 3\frac{1}{4}(R + r)$, where d is the distance between the centers of the pulleys and R and r are the radii of the pulleys.

14. Find the length of the belt when the pulleys have radii of 2 feet and 1 foot respectively, and the distance between their centers is 7 feet.

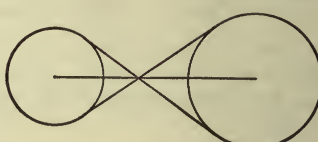
15. Find d, when $l = 27\frac{3}{8}$ feet, $R = 2$ feet, $r = 1\frac{1}{2}$ feet.

16. Find R if $l = 80.5$ feet, $d = 24$ feet, $r = 4$ feet.

17. $l = 12$ feet 3 inches, $R = 10$ inches, $r = 6$ inches, find d.

The formula for a crossed belt is

$$l = 2d + 3\frac{3}{8}(R + r).$$



18. Find the length of a crossed belt when the centers of the pulleys are 8 feet apart and their radii are 1.5 feet and .9 foot.

19. Find d if the length of the crossed belt is 26 feet and the radii of the pulleys are 1.8 feet and 1.2 feet.

REVIEW EXERCISE

191. 1. Define algebraic expression, monomial, polynomial, binomial, trinomial. Illustrate each.

2. Define and contrast factor and term; degree and power; exponent and coefficient.

3. What are the four principles used in solving equations?

4. What is the base in $(-3)^2$? in -3^2? Compare the values of 3^2 and $(-3)^2$; of 3^3 and $(-3)^3$; $(-3)^2$ and -3^2.

5. If a series of numbers are connected by the signs $+$, $-$, $\times$, $\div$, in what order must the operations be performed? $3+2\cdot 5-3(-3)=?$

6. Give the rules for adding algebraic expressions. How can results be tested?

7. What kind of expression is obtained by adding two like monomials? two unlike monomials?

8. What is the rule for subtraction? How may results be tested?

9. Give the rule for finding the sign of a product. What sign has the product in $(-1)^3\cdot(-2)^5\cdot(-7)$?

10. State the law of exponents in multiplication. Without using the law of exponents explain why $a^2\cdot a^3=a^5$.

11. Give the rules for multiplying, (*a*) two monomials; (*b*) a polynomial by a monomial; (*c*) two polynomials. How can you test the correctness of the product?

12. Give the rules for dividing, (*a*) one monomial by another; (*b*) a polynomial by a monomial; (*c*) one polynomial by another. How can you test the correctness of the quotient?

13. In multiplication, two factors are given and the product is required. In division, which two of these three numbers are given and which is required?

14. How may the subtrahend and the difference be combined to get the minuend? How may the divisor and the quotient be combined to get the dividend?

15. If the minuend is positive and the subtrahend is negative, what is the sign of the difference?

16. What is the sign of the sum of two negative numbers? of the difference? of the product? of the quotient?

17. If the product and the multiplier have the same sign, what is the sign of the multiplicand? If they have opposite signs, what is the sign of the multiplicand?

18. If the dividend and the divisor have the same sign, what is the sign of the quotient? If they have opposite signs, what is the sign of the quotient?

19. In the expression $a - 3m + 4p - 7b + n - 15$, inclose the third and the fourth terms in a parenthesis preceded by the minus sign, then inclose this parenthesis and the term immediately preceding and the one immediately following it in brackets preceded by the minus sign, leaving the final expression of the same value as the original polynomial.

20. How can you test the correctness of the factors of an algebraic expression?

21. What is the difference in meaning between $3x$ and x^3? Illustrate when $x = 5$.

22. What is the difference in meaning between the square of the difference of two numbers and the difference of the squares of the same numbers? Illustrate when the numbers are a and b.

23. Why do we change signs when removing a parenthesis preceded by the minus sign?

24. Give the rule for squaring a binomial.

25. When is a binomial the product of the sum and the difference of two numbers?

26. What must be added to $x^2 + 4x$ to make it an exact square? What must be added to $x^2 + 6x + 4$?

27. Is the product changed if an even number of its factors have their signs changed? Compare the value of $(a - b)^2$ with $(b - a)^2$.

28. State the rule for cubing a binomial.

29. Define equation; identical equation; conditional equation.

30. What is the root of an equation? Are any of the numbers, 1, 2, -3, 5 roots of $x^2 - 2x - 15 = 0$?

31. Describe briefly the steps used in solving an equation. What is meant by transposing? What principles are used in transposing?

32. What important difference is there between the equations $(x-1)(x+1) = x^2 - 1$ and $x^2 - 1 = 0$.

33. Subtract $1 - x + 2x^2$ from x^3. Subtract the same expression from 0.

34. Divide $x^2 - 7x + k$ by $x - 2$, giving quotient and remainder. How long should such divisions be continued? For what value of k will this division be exact?

In examples 35 to 45, $A = a^2 + 3ab - 4b^2$, $B = a^3 + 4a^2b - ab^2 - 4b^3$, $C = a + 4b$, $D = a^3 + 64b^3$.

Perform the indicated operations:

35. $B - (Ab + D)$.

36. $D \div C$.

37. $B - AC$.

38. $Aa - D + Cb^2$.

39. The minuend is B and the difference is D; find the subtrahend.

40. The divisor is C, the quotient is A, and the remainder is $16b^3$; find the dividend.

41. Find the value of B when $a = -2$ and $b = -3$.

42. Find the value of B when $a = b$.

43. Multiply B by C and verify the result by using $a = 1$, $b = 2$.

44. A is quotient, C is divisor; find dividend.

45. $C^2 = ?$ $D^2 = ?$

46. *Expand by type forms:*

(a) $(3z^3 - 4x)^2$.

(b) $(x + 2y)^3$.

(c) $(8 - 25x^3)(8 + 25x^3)$.

(d) $(4x^2 + 5y^2)^2$.

(e) $(4xy - 3xz)^2$.

(f) $(3b + 4x)^2$.

47. (a) $12a^2 + 24a^4 = 2a$ (?).
(b) $9m^2 + 16p^2 - 24mp = (3m - 4p)$ (?).
(c) $8x^3 - 12x^2y + 6xy^2 - y^3 = (? - ?)^3$.
(d) $9x^4y^4 - 36m^2n^2 = (3x^2y^2 - 6mn)$ (?).

48. Simplify $2a(a-h)^2 - (a^2 - 3ah)a - (a-h)(a-3h)a$.

49. Simplify $(8x^3 - 12x^2y + 6xy^2 - y^3) \div (2x-y)^2 + (y-2x)$.

50. Divide $[3x^2(x+a)^2 + (x+a)]$ by $(x+a)$.

51. If you add to a number $\frac{1}{3}$ of it and 7, the result is 27. What is the number?

Solve the following equations:

52. $3.5x + 9.3 = 1.25 + 10.3$.

53. $25 - 6(3x - 15) = 5$.

54. $3 - 3(2x + 4) = 6 - 4(2x + 3)$.

55. $2(x-3) - 3(1-2x) = 3(2-x) - 2(5-3x)$.

56. $(x+5)^2 - (x^2 + 95) = 0$.

57. $(6x+4)(8x-5) - (4x+12)(12x-21) = 0$.

58. $(x+12)(x-12) - (x+8)^2 = 0$.

59. Using x as the unknown number, write equations whose solutions will answer the following questions:
(a) What number added to 23.7 gives 14.81?
(b) What number subtracted from 12.84 gives 14.81?
(c) What number multiplied by 98 gives 12.25?
(d) What number multiplied by $\frac{1}{8}$ gives 12.25?
(e) To what number must $\frac{4}{3}$ be added if the result is to be equal to that obtained by multiplying the number by $\frac{4}{3}$?

60. Solve the equations of 59.

61. Divide $a^4 - b^4$ by $a - b$.

62. What must be added to $x^4 - 3x^3 - x + 5$ to produce $x^3 - x - 1$?

63. Solve $(4x-1)(x+3) - 4x^2 - (-10x + 3) + 6 = 0$.

64. $(a^5 - 48 - 17a^3 + 52a + 12a^2) \div (a - 2 + a^2) = ?$

65. Find $(x+1)^3 - (x-1)^3$ when $x = -\frac{1}{2}$.

66. Simplify $a - [3a - b - 2(b - a) + 3(a - 2b)]$.

67. Divide x^3 by $x + 1$.

68. State in algebraic symbols the type forms of multiplication given as special products.

69. Find the quotients:

(a) $[3x + 3y + a(x + y)] \div (x + y)$.
(b) $(1 - 9a^2c^6) \div (1 - 3ac^3)$.
(c) $(1 - 9xy + 8x^2y^2) \div (1 - 8xy)$.

70. Electric light bills are paid at the rate of 14¢ each for the first few units used and 4¢ each for the remainder. A bill for 35 units was \$2. How many units at each price were paid for?

71. Think of a number, double it, add 13, subtract 5, divide by 2. Show that the final result will always be 4 greater than the number you first thought of.

72. Think of a number, multiply it by 3, add 6, divide by 3, subtract the original number. Show that the result will always be 2.

73. Divide $x^3 - 10x + 17$ by $x - a$ until the remainder does not contain x. Compare the remainder with the dividend.

74. Divide $x^3 - 5$ by $x - a$ until the remainder does not contain x and compare as in example 73.

75. Divide $x^3 - 5$ by $x - 2$.

VIII. FACTORING

192. If two or more algebraic expressions are multiplied together, the result is their **product**, and the expressions multiplied are **factors** of the product.

Thus, $3 \times 5 = 15$. $\therefore$ 3 and 5 are factors of 15, also $m(x + y) = mx + my$. Here $mx + my$ is the product of which m and $(x + y)$ are the factors.

NOTE. Unless otherwise stated, expressions containing fractions or indicated roots are not considered as factors. Thus, although $3 = 5 \times \frac{3}{5}$, or $\sqrt{3} \times \sqrt{3}$, we shall not in this chapter consider these expressions as factors of 3.

193. Prime Number. A number which has no integral factors except itself and 1 is a **prime number.**

Thus, 7, 23, $a + b$, $a^2 + 3b^2$ are prime numbers.

Prime numbers used as factors are **prime factors.**

Thus, a and $a + b$ are the prime factors of $a^2 + ab$.

194. The student will recall that division is the process of finding one of two factors when their product and the other factor are given. In factoring it is required to find both factors when only the product is given. Thus factoring, like division, is an inverse of multiplication.

In arithmetic we learned a multiplication table and could factor all products that occur in the table from memory. For example, $42 = 6 \times 7$.

Corresponding to this we have in algebra some type forms of multiplication (Chapter V), and we shall be able to factor the corresponding products from memory.

Thus, $x^2 - y^2 = (x + y)(x - y)$,
and $a^2 + 2ab + b^2 = (a + b)^2$.

Success in this kind of factoring depends upon ability to recognize these type products.

ORAL EXERCISES

195. *Factor the following:*

1. $m^2 - n^2$.
2. $p^2 - q^2$.
3. $c^2 - d^2$.
4. $h^2 - k^2$.
5. $a^2 - 4$.
6. $4x^2 - 9$.
7. $x^2 + 2xy + y^2$.
8. $m^2 - 2mp + p^2$.
9. $h^2 + 2hk + k^2$.
10. $a^2 + 2a + 1$.
11. $p^2 - 2pq + q^2$.
12. $c^2 - 2c + 1$.
13. $a^2 + 4a + 4$.
14. $x^2 + 6x + 9$.
15. $m^2 - 8m + 16$.
16. $s^2 - 4st + 4t^2$.
17. $h^2 + 10h + 25$.
18. $a^2b^2 + 2ab + 1$.

196. When we try to factor products not found in the multiplication table in arithmetic, we generally look for an exact divisor, following certain rules regarding divisors.

Thus, 195 is clearly divisible by 5. If we divide, we get a quotient 39. Therefore $195 = 5 \times 39$.

Similarly, in an algebraic expression, if we can find a divisor, we can factor the expression.

For example, $3a^2 + 6ab$ is clearly divisible by $3a$, and the quotient is $a + 2b$, hence the factors of $3a^2 + 6ab$ are $3a$ and $a + 2b$.

Also $a(x + y) - b(x + y) = (x + y)(a - b)$.

We proceed to classify some of the simpler types of factoring.

197. Case I. Factors of Monomials. Square Root.

The factors of monomials are generally evident.

If the two factors of a product are equal, either of them is the **square root** of the product. The **radical sign** ($\sqrt{\ }$) is used to indicate that the square root of a number is to be taken.

Thus, $\sqrt{9x^4} = \sqrt{3x^2 \cdot 3x^2} = 3x^2$.

ORAL EXERCISE

198. *Factor the following:*

1. $x^3 = x\,(?)$.
2. $3\,x^2y = xy\,(?) = 3\,y\,(?)$.
3. $a^5b^3 = ab\,(?) = a^2b\,(?)$.
4. $72\,x^2y^2 = 9\,x\,(?) = 8\,xy\,(?)$.
5. $a^5 = a^2\,(?) = a^4\,(?)$.
6. $a^{n+2} = a^n\,(?) = a^{n+1}\,(?)$.
7. $x^{2n} = x^n\,(?) = x^{n-1}\,(?)$.
8. $m^5n^x = m^2n\,(?)$.
9. $6\,e^{x+4} = 2\,e^3\,(?) = 2\,e^{x+2}\,(?)$.
10. $39\,a^2b^3c^4 = abc\,(?) = 13\,a^2b^2c^2\,(?)$.

Find the indicated roots:

11. $\sqrt{4}$.
12. $\sqrt{a^2}$.
13. $\sqrt{4\,a^2}$.
14. $\sqrt{9\,a^2b^4}$.
15. $\sqrt{25\,x^4y^6}$.
16. $\sqrt{36\,m^2n^4}$.
17. $\sqrt{81\,x^6y^{12}}$.
18. $\sqrt{3^2 \cdot 5^2 \cdot x^4y^6}$.
19. $\sqrt{49\,a^{10}b^{14}}$.
20. $\sqrt{121\,m^6n^{12}r^{18}}$.
21. $\sqrt{169\,x^6y^8}$.
22. $\sqrt{144\,a^4b^8}$.
23. $\sqrt{\frac{1}{36}}$.
24. $\sqrt{\frac{4}{49}}$.
25. $\sqrt{\dfrac{a^2b^2}{n^2}}$.
26. $\sqrt{\dfrac{625\,a^2b^4c^6}{225\,m^8n^6}}$.
27. $\sqrt{\frac{1}{4}\,a^2b^4}$.
28. $\sqrt{\dfrac{a^{10}b^{16}}{x^2y^4}}$.
29. $\sqrt{\dfrac{1}{81\,a^4b^{10}}}$.
30. $\sqrt{\dfrac{49\,a^2}{3^2}}$.

199. Case II. Type Form $ab + ac$, — Polynomials with the Same Monomial Factor in Each Term.

1. Factor $4\,a + 6\,b - 10\,c$.

SOLUTION. $4\,a + 6\,b - 10\,c = 2(2\,a + 3\,b - 5\,c)$.

2. Factor $2\,x^3y + 6\,x^2y^2 - 8\,xy^3$.

SOLUTION. $2\,x^3y + 6\,x^2y^2 - 8\,xy^3 = 2\,xy(x^2 + 3\,xy - 4\,y^2)$.

3. Has $x^2 + xy + y^2$ a monomial factor?

4. How may results in factoring be verified?

200. To factor polynomials of the form $ab + ac$:

1. Find the greatest monomial factor of every term of the polynomial.

2. Divide the polynomial by this monomial.

3. The factors will be the monomial and the quotient obtained by dividing.

EXERCISE

201. *Factor:*

1. $cy + dy$.
2. $mp + np$.
3. $rs + ps$.
4. $hk + mk$.
5. $2p + 2r$.
6. $4x + 8y$.
7. $6r - 12r$.
8. $-5p + 15q$.
9. $-7a - 21a$.
10. $-pr - qr$.
11. $2ax + 6ax$.
12. $5pq - 10pq$.
13. $3ax - 6bx$.
14. $-6ck - 18bk$.
15. $3a^2 + 4ab$.
16. $3x^2 + 6x$.
17. $4x^2 - 8x$.
18. $px + qx^2$.
19. $35 - 14a^2$.
20. $42 - 28p^2$.
21. $48a + 6a^2$.
22. $a^3 - 3a^2 + 7a$.
23. $3x^3 - 21x^2 - 15x$.
24. $27x^2 - 3xy + 15y^2$.
25. $ab^2 + a^2b + a^2b^2$.
26. $ax + bx - x$.
27. $a^n + a^{n+1}$.
28. $a^3 + a^2b - ab^2$.
29. $x^2 - xy - xy^2$.
30. $2x^3y - 6x^2y^2 - 8x^4y^3$.
31. $5m^6 - 2m^4n + 10m^3n$.
32. $a^2 - ab + ac - a$.
33. $x^2 - xy - 6x + 6xz$.
34. $ac - bc + ac^2d - bdc$.
35. $ax - bx - axy + x$.
36. $x^3 - x - x^2y - xy^2$.
37. $a^3 + 4a^2 + 3a$.

ORAL EXERCISE

202. 1. State in algebraic symbols and in words the rule for squaring the sum or the difference of two numbers. (See § 130.)

Find the indicated squares:

2. $(x + y)^2$.
3. $(x - y)^2$.
4. $(m + n)^2$.
5. $(p - q)^2$.
6. $(h + k)^2$.
7. $(a - 2)^2$.
8. $(m + 5)^2$.
9. $(7 - r)^2$.
10. $(d - 9)^2$.
11. $(2x + 3)^2$.
12. $(5p - 6)^2$.
13. $(2a - 3b)^2$.
14. $(-3 + 2m)^2$.
15. $(2m - 3)^2$.
16. $(-x - 2y)^2$.

Square:

17. $(u - 10w)^2$. 20. $-(x + y)^2$. 23. $(-4m + 3a^2)^2$.
18. $(2mn - 5p)^2$. 21. $-(2 - 3a)^2$. 24. $(-m^2n - p^2q)^2$.
19. $(4x^2 + 5y)^2$. 22. $(-5a^3 - 2b^4)^2$. 25. $(-a^2 - a)^2$.

203. *If a trinomial contains two terms that are perfect squares, and if the absolute value of the other term is twice the product of their square roots, the trinomial is the square of a binomial.*

NOTE. To make $a^2 + b^2$ a perfect square we add $2ab$ (twice the product of the square roots of a^2 and b^2). To make $a^2 + 2ab$ a perfect square we add b^2 (the square of the quotient $2ab \div 2a$). To make $16p^2 + 25q^2$ a perfect square we add $2 \times 4p \times 5q = 40pq$. $16p^2 + 40pq + 25q^2 = (4p + 5q)^2$. Also to make $4h^2 + 12hk$ a perfect square we add $[12hk \div (2 \times 2h)]^2 = (3k)^2 = 9k^2$. $4h^2 + 12hk + 9k^2 = (2h + 3k)^2$.

EXERCISE

204. *Which of the following are squares of binomials?*

1. $a^2 + 4a + 4$.
2. $a^2 - 4a - 4$.
3. $x^2y^2 - 2xy + 1$.
4. $x^2 - x + \frac{1}{4}$.
5. $m^2 + mn + n^2$.
6. $2a + a^2 + 1$.
7. $1 - 6c + 9c$.
8. $p^2 + 2pq - q^2$.

9. How many negative signs may there be in the square of a binomial?

10. Compare the square of $a - b$ with the square of $b - a$. Compare the square of $a - b$ with the square of $a + b$.

11. What term must be supplied in each of the following in order to make the trinomial the square of a binomial? Of what binomial is the resulting trinomial the square?

(a) $x^2 + (\quad) + 16$.
(b) $4a^2 + (\quad) + 9b^2$.
(c) $25a^2 + 10a + (\quad)$.
(d) $(\quad) + 8x + 16$.
(e) $(\quad) - 8x + 4$.
(f) $a^2b^2 - 6abc + (\quad)$.
(g) $16m^2 - (\quad) + 25n^2$.
(h) $36p^4 + 24p^2 + (\quad)$.
(i) $(\quad) + 16a^2b + b^2$.
(j) $49m^2n^2 + (\quad) + 25p^2$.

205. Case III. Type Forms $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, — the Square of a Binomial.

Factor $a^2 + 4a + 4$.

Solution. The first term is the square of a, the last term is the square of 2, and the middle term is twice the product of a and 2.

$$\therefore a^2 + 4a + 4 = (a + 2)(a + 2) \text{ or } (a + 2)^2.$$

206. To factor a trinomial that is the square of a binomial:

1. Arrange the trinomial in order of the powers of some letter.

2. Extract the square roots of the first and last terms and connect the results with the sign of the middle term. The square of this binomial equals the trinomial.

In algebraic symbols this rule may be written:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$\text{or } a^2 - 2ab + b^2 = (a - b)^2.$$

Examples

1. Factor $4x^2y^2 - 12xyz + 9z^2$.

Solution. The arrangement is in order of powers of x.
Step 2 of the rule gives $4x^2y^2 - 12xyz + 9z^2 = (2xy - 3z)^2$.

2. Factor $9x^4 + 4x^2 + 12x^3$.

Solution. $9x^4 + 12x^3 + 4x^2 = (3x^2 + 2x)^2$.

EXERCISE

207. *Factor the following:*

1. $x^2 - 20x + 100$.
2. $m^2 + 6m + 9$.
3. $4a^2 + 4a + 1$.
4. $b^2 - 10bc + 25c^2$.
5. $x^4 + 2x^2 + 1$.
6. $a^8 - 40a^4 + 400$.
7. $x^2y^2 + 4xy + 4$.
8. $a^6 + 6a^3 + 9$.
9. $x^2 + 12x + 36$.
10. $b^4 - 4b^2 + 4$.
11. $a^2x^2 - 12ax + 36$.
12. $b^4 - 4b^2c + 4c^2$.
13. $9x^2 + 6x + 1$.
14. $m^2 + 14mn^3 + 49n^6$.
15. $r^2s^2 - 10rst + 25t^2$.
16. $a^2 + 4c^2 - 4ac$.

Factor:

17. $2ax + a^2 + x^2$.

18. $49p^2 - 28pq + 4q^2$.

19. $x^2b^2 + a^2y^2 - 2abxy$.

20. $1 - 20x + 100x^2$.

21. $9 - 12a + 4a^2$.

22. $25a^2 + 4a^2c^2 - 20a^2c$.

23. $81a^2b^4 - 18ab^2 + 1$.

24. $81a + 18a^2 + a^3$.
(HINT. First apply Case II.)

25. $49a^2b^2c + 28abc + 4c$.

26. $9a^4b^2 + 4c^2d^4 - 12a^2bcd^2$.

27. $8a^3 - 16a^2 + 8a$.

28. $a^2x^4 + 4ax^2 + 4$.

29. $9x^2 - 42x + 49$.

30. $3a^2x + 6ax^2 + 3x^3$.

31. $m^2 + 2mn + n^2 + 2(m + n) + 1$.

32. $4x^2 + 4xy + y^2 + 2(2x + y)z + z^2$.

208. State in algebraic symbols and in words the rule for multiplying the sum of two numbers by the difference of the same two numbers. See § 132.

ORAL EXERCISE

209. *Find the products:*

1. $(x + 2a)(x - 2a)$.
2. $(xy - z^2)(xy + z^2)$.
3. $(6ax - 9a^2)(6ax + 9a^2)$.
4. $[(a + b) + c][(a + b) - c]$.
5. $[x - (a + b)][x + (a + b)]$.
6. $(12a^2 - b^2)(12a^2 + b^2)$.

Find the quotients:

7. $(x^2 - y^2) \div (x - y)$.
8. $(a^2 - 9) \div (a + 3)$.
9. $(9r^2 - 1) \div (1 + 3r)$.
10. $(25 - 16a^2) \div (5 - 4a)$.
11. $(a^4 - 16) \div (a^2 - 4)$.
12. $[(a + b)^2 - c^2] \div [(a + b) - c]$.

What binomial will exactly divide each of the following? What is the quotient?

13. $x^2 - 16$.
14. $1 - 4a^2$.
15. $144r^2 - 121s^2$.
16. $a^4 - b^4$.

Factor the following:

17. $g^2 - h^2$.
18. $a^2 - 4$.
19. $4 - 9c^2$.
20. $x^2 - y^2z^2$.
21. $1 - 81x^4$.
22. $x^6 - y^6$.

210. Case IV. Type Form $a^2 - b^2$, — the Difference of Two Squares.

211. To factor the difference of two squares:

1. Find the square roots of the squares.

2. Use the sum of the square roots for one factor and their difference for the other factor.

In algebraic symbols this may be written:

$$a^2 - b^2 = (a + b)(a - b).$$

EXAMPLES

1. $16a^2 - 9 = (4a + 3)(4a - 3)$.
2. $x^4 - y^2 = (x^2 + y)(x^2 - y)$.
3. $(a - b)^2 - 9c^2 = (a - b + 3c)(a - b - 3c)$.

EXERCISE

212. *Factor the following:*

1. $9a^2 - 49$.
2. $x^4 - 4x^2$.
3. $25a^2b^2 - c^4$.
4. $a^4 - 49$.
5. $16c^4 - 25d^8$.
6. $a^2y^2 - b^2x^2$.
7. $x^4 - y^2$.
8. $9 - 16a^2b^4$.
9. $144 - 81$.
10. $64 - x^4$.
11. $25a^2b^4 - c^6$.
12. $121x^4 - 144y^4$.
13. $16a^4 - 1$.
14. $25a^6 - 16b^4$.
15. $x^4 - x^2$.
16. $a^8 - b^8$. (Four factors.)
17. $9a^2b^4 - 25c^4d^6$.
18. $x^2 - 100x^8y^2z^4$.
19. $1 - 400x^4$.
20. $9 - a^2$.

Factor:

21. $3 - 27x^2$.

22. $6m^2 - 24$.

23. $169 - z^4$.

24. $16 - a^4b^4$. (Three factors.)

213. Case IV, *a*. Sometimes polynomials of four or six terms can be written as the difference of two squares.

1. Factor $m^2 + 2mn + n^2 - x^2$.

SOLUTION. $m^2 + 2mn + n^2 - x^2 = (m+n)^2 - x^2$
$= (m+n+x)(m+n-x)$.

2. Factor $a^2 - x^2 + 2xy - y^2$.

SOLUTION. $a^2 - x^2 + 2xy - y^2 = a^2 - (x^2 - 2xy + y^2)$
$= (a+x-y)(a-x+y)$.

3. Factor $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.

SOLUTION. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$
$= (a^2 + 2ab + b^2) - (c^2 - 2dc + d^2)$
$= (a+b)^2 - (c-d)^2$.
$= (a+b+c-d)(a+b-c+d)$.

Care must be taken not to make mistakes when inserting or removing parentheses.

EXERCISE

214. *Factor the following:*

1. $c^2 - 2cd + d^2 - 4$.
2. $a^2b^2 - a^2 + 2ab - b^2$.
3. $x^2 - a^2 + 2ab - b^2$.
4. $a^2 - 2ab + b^2 - c^2$.
5. $1 - x^2 - 2xy - y^2$.
6. $9 - a^2 - b^2 + 2ab$.
7. $a^4 + 2a^2b^2 + b^4 - a^2b^2$.
8. $x^2 + 6x + 9 - 1$.
9. $2x^2 + 4x + 2 - 8$.
10. $25 - a^2 - b^2 - 2ab$.
11. $2 - 2(a-b)^2$.
12. $3m^2 - 6mn + 3n^2 - 12$.
13. $(p-q)^2 - (p+q)^2$.
14. $x^2 + y^2 - a^2 - b^2 - 2xy + 2ab$.
15. $x^2 - 2xy + y^2 - m^2 + 2mn - n^2$.
16. $4x^2 - 12ax - c^2 - k^2 + 9a^2 - 2ck$.
17. $4x^2y^2 - (x^2 + y^2 - z^2)^2$. (Four factors.)

215. **Case IV, *b*.** **Expressions of the form $a^2x^4 + kx^2 + b^2$, can sometimes be factored by a method known as "completing the square." By this method these expressions are put into the form of the difference of two squares and factored accordingly.**

1. Factor $9a^4 + 8a^2 + 4$.

SOLUTION. $9a^4 + 8a^2 + 4 = 9a^4 + 12a^2 + 4 - 4a^2$. (Why?)
$= (3a^2 + 2)^2 - 4a^2$
$= (3a^2 + 2 + 2a)(3a^2 + 2 - 2a)$
or $(3a^2 + 2a + 2)(3a^2 - 2a + 2)$.

2. Factor $x^4 + x^2y^2 + y^4$.

SUGGESTION. $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^2 - x^2y^2$.
Let the student complete the solution.

3. Factor $a^4 + 4$.

SOLUTION. $a^4 + 4 = a^4 + 4a^2 + 4 - 4a^2$.
$= (a^2 + 2 + 2a)(a^2 + 2 - 2a)$.
or $(a^2 + 2a + 2)(a^2 - 2a + 2)$.

216. It is clear that factoring by this method will depend upon our ability to change the given expression into a trinomial square by adding a monomial that is itself a perfect square. This monomial is then subtracted, thus leaving the original expression unchanged in value, but written in the form of the difference of two squares.

EXERCISE

217. *Factor the following:*

1. $a^4 - 6a^2 + 1$.
2. $4x^4 + 3x^2 + 1$.
3. $x^4 - 3x^2 + 1$.
4. $x^4 - 23x^2 + 1$.
5. $x^4 - 11x^2y^2 + y^4$.
6. $1 + 2a^2b^2 + 9a^4b^4$.
7. $x^4 + x^2y^4 + y^8$.
8. $1 + a^2 + a^4$.
9. $16a^4 + 4a^2x^2 + x^4$.
10. $a^4 + 9b^4 - 3a^2b^2$.
11. $r^4 - 14r^2 + 25$.
12. $r^4 - 15r^2 + 25$.
13. $r^4 + r^2 + 25$.
14. $4t^4 - 21t^2s^2 + s^4$.

Factor:

15. $p^4 + 5p^2 + 9.$

16. $p^4 + 2p^2 + 9.$

17. $49x^4 + 25y^4 + 66x^2y^2.$

18. $49x^4 + 25y^4 - 74x^2y^2.$

19. $x^4 + 4y^4.$

20. $a^4 + 64.$

21. $a^4b^4 + 8a^2b^2c^2 + 36c^4.$

22. $16a^4 - 28a^2b^2 + 9b^4.$

23. $16a^4 + 20a^2b^2 + 9b^4.$

24. $x^4 - 10x^2 + 9.$

25. $169 - 127b^2 + 9b^4.$

26. $169 + 69b^2 + 9b^4.$

27. $a^{4n} + a^{2n} + 1.$

28. $49x^{4n} + 10x^{2n} + 1.$

29. Consider $x^4 - 5x^2 + 4$ in each of the following ways:

(*a*) $x^4 - 5x^2 + 4 = x^4 - 4x^2 + 4 - x^2 =$ etc.

(*b*) $x^4 - 5x^2 + 4 = x^4 + 4x^2 + 4 - 9x^2 =$ etc.

(*c*) $x^4 - 5x^2 + 4 = x^4 - 5x^2 + \frac{25}{4} - \frac{9}{4} =$ etc.

REVIEW EXERCISE

218. *Factor the following:*

1. $a^2x + ax^2 - a^2x^2.$

2. $4a^2 - 4a + 1.$

3. $25x^2y^2 - 130xyz + 169z^2.$

4. $2x^2 + 20x + 50.$

5. $3m^2 - 3n^2.$

6. $2x^4 + 8.$

7. $10x - 40x^3.$

8. $3x^2 + 12xy + 12y^2 - 3.$

9. $4a^2 - 4ac + c^2.$

10. $(2x + 1)^2 - (2x - 1)^2.$

11. $x^4 + 5x^2y^2 + 9y^4.$

12. $x^2 - 16x + 64.$

13. $x^2 - 144.$

14. $81a^4 - 16b^4.$

15. $25m^4n^2x^8 - 20m^2nx^4yz^2 + 4y^2z^4.$

16. $1 - 20x + 100x^2.$

17. $(x + 2)^2 + 2(x + 2) + 1.$

18. $m^2 + 2mn + n^2 - 2(m + n)p + p^2.$

19. $(a - 4)^2 - 4(a - 4) + 4.$

20. $(2x + 3y)^2 - (2x - 3y)^2.$

21. $m^2 - n^2 + p^2 - q^2 - 2mp + 2nq.$

22. $a^{16} - b^{16}$ into five factors.

23. $(a^2 + 9b^2 - c^2)^2 - 36a^2b^2$ into 4 factors.

24. $8m^2n^2 - 2(m^2 + n^2 - p^2)^2$.

25. $4ab - b^2 + c^2 - 4a^2 + 9d^2 + 6cd$.

26. $4r^2s^2 - 16r^2 + t^2 - 4rst$.

27. $18(a^2m + bm)^2 - 32(a^2m - bm)^2$.

28. $27a^2 - 90a + 75 - 12b^2 - 12b - 3$.

29. $a^{2n} - 1$.

30. $x^{2r} - 4x^r + 4$.

31. $x^{5r} + 6x^{3r} + 9x^r$.

32. $-81 + a^2 + 2ab + b^2$.

219. Case V. Grouping for a Polynomial Divisor.

1. Given the expression $a(x + y) + b(x + y)$.

Solution. It is clear that this expression can be divided by $x + y$. The division can be performed mentally giving as a quotient $a + b$.

$$\therefore a(x + y) + b(x + y) = (x + y)(a + b).$$

2. Given $ab - 2br - 2as + 4rs$.

Solution. If we group the first two terms and also the last two terms, and remove from each group a monomial factor, we discover a binomial divisor.

$$\begin{aligned} ab - 2br - 2as + 4rs &= (ab - 2br) - (2as - 4rs) \\ &= b(a - 2r) - 2s(a - 2r) \\ &= (a - 2r)(b - 2s). \end{aligned}$$

3. Factor $a^2 - b^2 + c(a - b)$.

Solution. Here the binomial divisor is evidently $a - b$.

$$\therefore a^2 - b^2 + c(a - b) = (a - b)(a + b + c).$$

4. What binomial will divide $x^2 - y^2 - (x + y)^2$? What is the quotient? What are the factors?

5. Factor $(x + y)^2 - 9 + 4(x + y - 3)$.

Solution. $(x + y)^2 - 9 - 4(x + y - 3)$

$$\begin{aligned} &= (x + y + 3)(x + y - 3) - 4(x + y - 3) \\ &= (x + y - 3)(x + y + 3 - 4) \\ &= (x + y - 3)(x + y - 1). \end{aligned}$$

220. **To factor by grouping:**

Arrange the terms into groups and factor each group separately by any of the preceding methods. If the same polynomial factor occurs in each group, make it one of the factors of the expression, and divide by it to get the other factor.

Success in factoring by this method requires great care in inserting and removing parentheses.

The student is warned against thinking that an expression is factored when some group of its terms is factored.

Thus, $a^2 - b^2 + c^2 = (a + b)(a - b) + c^2$, but the expression $a^2 - b^2 + c^2$, is not factored and cannot be factored.

Factoring by grouping is frequently used in factoring polynomials of four terms. If such a polynomial is the product of two binomial factors, when it is properly grouped and the monomial factors are separated from each group, one of the binomials will appear as an exact divisor of the expression.

EXERCISE

221. *Factor the following:*

1. $ab + ac + bd + cd$.
2. $a^2 + 2ab + 3ac + 6bc$.
3. $4ab + 4ac - bd - cd$.
4. $5ab - 3b + 5ac - 3c$.
5. $3a + 3 - pa - p$.
6. $xy + 3x + y + 3$.
7. $pq - pr + qr - r^2$.
8. $1 - x + xy - x^2y$.
9. $m^2n - p^2mn + mx - p^2x$.
10. $bx + by + bz + cx + cy + cz$.
11. $a^2(1 - c) - b^2(c - 1)$.
12. $a^3x + ab^2x - a^2by - b^3y$.
13. $2ax - 2bx - 2ay + 2by$.
14. $5wv - 5w + v - 1$.
15. $(x + y)^2 + (x + y)$.
16. $(a + b)^2 + 2(a^2 - b^2)$.
17. $(a - b) - 2(a^2 - b^2)$.

18. $(a+2)(a^2-9)-(a+2)(a+3)-a-3.$

19. $x^2(m-n)+2ax(m-n)+a^2(m-n).$

20. $(a-2b)^2-4-5(a-2b+2).$

21. $3(x-y)^2-ay+ax.$

22. $3(x^2-y^2)-(y-x)+2(x+2y).$

HINT. First collect like terms.

23. $28x^3-12x^2-112x+48.$

24. $p^3+3p^2-5p-15.$

25. $m^4+5m^3-m^2-5m.$

26. $c^2-4d^2+3ac^2+12acd+12ad^2.$

HINT. Group first two and last three.

27. $(m+2)(n+3)-(n+3)(p+2).$

28. $3(m+n)^3-5(m+n)^2+m+n.$

29. $n^2(2m-1)-2n(2m-1)+(2m-1).$

30. $p^2+p+q+pq.$

31. $a^3+a^2-6a-6.$

32. $ax+bx+cx+a+b+c.$

33. $ac-5bc+a-5b-6c-6.$

34. $x^2-(a+b)x+ab.$

HINT. Remove parenthesis.

35. $x^2+(a-b)x-ab.$

36. $y^2-(a-2)y-2a.$

37. $x^2+(7-y)x-7y.$

38. $p^2-(a^2-a)p-a^3.$

39. $4r^2+2(d-c)r-cd.$

40. $a^4+(b-4)a^2-4b.$

222. Case VI. Type Form $x^2 + bx + c$, — the Product of Two Binomials having a Common Term.

$$\begin{array}{l} x+2 \\ \underline{x+5} \\ x^2+2x \\ \underline{\quad\quad 5x+10} \\ x^2+7x+10 \end{array} \qquad \begin{array}{l} x+p \\ \underline{x+q} \\ x^2+px \\ \underline{\quad\quad qx+pq} \\ x^2+(p+q)x+pq. \end{array}$$

It is readily seen that $x + 2$ and $x + 5$ are the factors of $x^2 + 7x + 10$, and $x + p$ and $x + q$ are the factors of $x^2 + (p + q)x + pq$. In factoring a trinomial of the type form $x^2 + bx + c$ (sometimes called a **quadratic trinomial**) the first term of each factor is x and the sum of the second terms of the factors is b and their product is c. (See § 134.)

ORAL EXERCISE

223. *Multiply the following:*

1. $(x+3)(x+4)$.
2. $(x-1)(x+3)$.
3. $(a+2)(a+5)$.
4. $(x-2)(x+3)$.
5. $(a^2+7)(a^2-9)$.
6. $(xy-5)(xy+7)$.
7. $(ab+1)(ab+3)$.
8. $(m^2+3)(m^2-7)$.
9. $(m^2-3)(m^2+7)$.
10. $(x+a)(x+b)$.

11. Find two numbers whose sum is 5 and whose product is 6. Factor $x^2 + 5x + 6$.

12. Find two numbers whose sum is -5 and whose product is 6. Factor $x^2 - 5x + 6$.

13. Find two numbers whose sum is -3 and whose product is -10. Factor $a^2 - 3a - 10$.

14. Find two numbers whose product is -6 and whose sum is -1. Factor $x^2 - x - 6$.

15. Find two numbers whose product is $6a^2$ and whose sum is $5a$. Factor $x^2 + 5ax + 6a^2$.

224. To factor a trinomial of the form $x^2 + bx + c$.

1. Find two numbers whose product is c and whose sum is b, the coefficient of x with its proper sign.

2. Write for the factors two binomials, the first term of each being x and the second terms the numbers found.

Notice that when c is negative the second terms of the two binomial factors have unlike signs. When c is positive the second terms of the factors have the same sign as the middle term.

EXAMPLES

1. Factor $x^2 - 4x - 5$.

SOLUTION. The two factors of -5 whose sum is -4 are -5 and $+1$.

$$\therefore x^2 - 4x - 5 = (x-5)(x+1).$$

A variation of this type which will not cause any difficulty is seen in the following:

2. Factor $a^2 + 3ab - 18b^2$.

SOLUTION. The two factors of $-18b^2$ whose sum is $+3b$ are $6b$ and $-3b$.

$$\therefore a^2 + 3ab - 18b^2 = (a+6b)(a-3b).$$

3. Factor $a^2b^2 - abc - 20c^2$.

SOLUTION. The two factors of $-20c^2$ whose sum is $-c$ are $-5c$ and $4c$.

$$\therefore a^2b^2 - abc - 20c^2 = (ab-5c)(ab+4c).$$

EXERCISE

225. *Factor the following:*

1. $x^2 - 7x + 10$.
2. $x^2 - 7x - 30$.
3. $x^2 - 7x + 12$.
4. $a^2 + 5a - 14$.
5. $a^2 - 5a - 14$.
6. $m^2 + 5m + 6$.
7. $b^2 - 9b + 20$.
8. $x^2 + x - 42$.
9. $y^2 + 3y - 18$.
10. $b^2 + 3b + 2$.
11. $c^2 - 3cd + 2d^2$.
12. $q^2 + 9q + 20$.
13. $x^2 - 11xy + 30y^2$.
14. $x^2 + (m+n)x + mn$.
15. $r^2 + ar + br + ab$.
16. $x^2 + (a+b+c)x + ac + bc$.
17. $x^2 + (a+b+c)x + ab + ac$.
18. $x^2 - 27x + 182$.
19. $x^2 - 28x + 195$.
20. $a^2 - 29a + 210$.
21. $m^2 - 45m + 164$.
22. $a^2 - 4ab - 12b^2$.

Factor:

23. $s^2 - 3s - 18$.

24. $p^2 - 5px + 6x^2$.

25. $m^2n^2 + 15mnp + 50p^2$.

26. $x^2 + 2xy - 35y^2$.

27. $t^2 + 2t + 1 + 5(t+1) + 6$.

28. $7m^2 - 14mn + 7n^2 - 91(m-n) + 84$.

29. $3a^3 + 30a^2 - 288a$.

30. $x^2 - \frac{2}{15}x - \frac{1}{15}$.

31. $ap^2 - (3q-2)ap - 6aq$.

32. $r^2 - 7rs - 18s^2$.

33. $x^4 + 4x^2 - 45$.

34. $a^4 - 5a^2 + 4$.

35. $a^4 - 4a^2 + 4$.

36. $b^4 + 4b^2c - 21c^2$.

37. $a^{2n} - a^n - 2$.

38. $x^{2n} + x^n - 2$.

39. $x^{2n} - 2x^n - 15$.

40. $a^3 - 5a^2x - 24ax^2$.

Case VII. Type Form $ax^2 + bx + c$, — the General Quadratic Trinomial.

226. This type differs from the last type in that the coefficient of x^2 is not positive 1. The expression is factored by changing the trinomial into a polynomial of four terms and then grouping.

Factor $6x^2 + 19x + 15$.

SOLUTION.
$$\begin{aligned} 6x^2 + 19x + 15 &= \mathbf{6x^2 + 10x + 9x + 15} \\ &= 2x(3x+5) + 3(3x+5) \\ &= (3x+5)(2x+3). \end{aligned}$$

The problem here is to change the original trinomial into the form in black-faced type. The numbers 10 and 9 which replace 19 as the coefficient of x are two factors of 90, and 90 is the product of 6 times 15.

227. **To factor a general quadratic trinomial $ax^2 + bx + c$:**

1. Find two numbers whose product is $a \times c$, and whose sum is b.

2. Replace bx by two terms having these numbers as the coefficients of x.

3. Factor by grouping as in Case V.

Examples

1. Factor $6x^2 + 7x - 3$.

Solution. Here $a = 6$, $b = 7$, $c = -3$. The two factors of $a \cdot c$, that is of $6 \cdot (-3) = -18$, whose sum is 7, are -2 and 9. Then we write

$$\begin{aligned} 6x^2 + 7x - 3 &= 6x^2 - 2x + 9x - 3. \\ &= 2x(3x - 1) + 3(3x - 1) \\ &= (3x - 1)(2x + 3). \end{aligned}$$

2. Factor $3a^2 - 11ab + 6b^2$.

Solution. Two factors of $3 \cdot 6b^2 = 18b^2$, whose sum is $-11b$ are $-2b$ and $-9b$.

$$\begin{aligned} 3a^2 - 11ab + 6b^2 &= 3a^2 - 2ab - 9ab + 6b^2 \\ &= a(3a - 2b) - 3b(3a - 2b) \\ &= (3a - 2b)(a - 3b). \end{aligned}$$

EXERCISE

228. *Factor the following:*

1. $2x^2 + 3x - 2$.
2. $2x^2 - 7x - 15$.
3. $6x^2 - x - 15$.
4. $6m^2 - 5m - 25$.
5. $14p^2 - 39p - 35$.
6. $10v^2 - 29v - 21$.
7. $2a^2 - 4a + 2$.
8. $12t^2 + 7t - 12$.
9. $6s^2 - s - 12$.
10. $2b^2 - 6b - 8$.
11. $4 + 4y - 15y^2$.
12. $2x^2 - 5x - 3$.
13. $7q^2 - 20q - 3$.
14. $2m^2 + 9m - 5$.
15. $6n^2 - 5n - 6$.
16. $9a^2 - 9a - 10$.
17. $12x^2 - 31x - 15$.
18. $36y^2 + 7xy - 15x^2$.
19. $8x^2 - 38x + 35$.
20. $(m + n)^2 - 11(m + n) - 26$.
21. $15x^2 + 29x - 14$.
22. $12n^2 - 31n - 15$.
23. $(p^2 + p)^2 - 14(p^2 + p) + 24$.
24. $6a^2 - ab - 35b^2$.
25. $(a + b)^2 - 3(a + b) - 54$.
26. $5x^2 + 10xy + 5y^2 + 20(x + y) - 105$.
27. $a^2 - (p + 1)qa^2 + pq^2a^2$.
28. $10a^{2n} + 31a^n + 15$.

REVIEW EXERCISE

229. *Factor the following:*

1. $a-1+b(a-1)$.
2. $m^2(1-n)-p^2(n-1)$.
3. $16x^2y^2-20xy$.
4. $3p+3q-ap-aq$.
5. $10t^3-7t^2-12t$.
6. $8a-14ax-15ax^2$.
7. $6x^2y+46xy-16y$.
8. $16x^2-9y^2$.
9. $2x^2-8$.
10. $2x^2+6x+4$.
11. $3-27s^2$.
12. $2m^2-20m+50$.
13. $6-9n+3n^2$.
14. $5x^2+30x+40$.
15. $2(q+1)^2-8$.
16. $1+y-2x-2xy$.
17. $2xy-2x+y-1$.
18. $2m-3n+2m^2-3mn$.
19. $2x^2+4xy+2y^2+2x+2y$.
20. $x^2+\frac{2}{3}x-\frac{5}{3}$.
21. $(ax-by)-(ax-by)^2$.
22. $(x^2-5x)^2-(x^2-5x)-20$.
23. $a^2-b^2-4a^2+8ab-4b^2$.
24. $am+bn+cp+bm+cn+ap+cm+an+bp$.
25. $7(a^2-b^2)-7(a+b)^2+7(a+b)$.
26. $(a^2-b^2)(a+b)-2(a^2-b^2)+(a-b)$.
27. $6(s^2-t^2)(s-t)+42(s^2-t^2)+72s+72t$.
28. $r(p-q)^2-4r(p-q)-12r$.

230. Case VIII. Type Forms a^3+b^3 and a^3-b^3, — Sum or Difference of the Cubes of Two Numbers.

By actual division we obtain the following:

1. $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.
2. $(a^3-b^3)\div(a-b)=a^2+ab+b^2$.

From 1 we may state the following:

The sum of the cubes of two numbers is always divisible by the sum of the numbers. The quotient is the square of the first number minus the product of the two numbers plus the square of the second number.

Let the student make the corresponding statement for 2.

ORAL EXERCISE

231. *Divide the following as indicated in* § 230:

1. $(p^3 - q^3) \div (p - q)$.
2. $(x^3 + y^3) \div (x + y)$.
3. $(8\,m^3 - 27\,n^3) \div (2\,m - 3\,n)$.
4. $(64\,a^3 + 125\,b^3) \div (4\,a + 5\,b)$.
5. $(27\,s^3 - t^3) \div (3\,s - t)$.
6. $(8\,p^3q^3 + r^3) \div (2\,pq + r)$.
7. $(\frac{1}{8}x^3 + y^3) \div (\frac{1}{2}x + y)$.
8. $(125\,d^3 + 64\,h^3) \div (5\,d + 4\,h)$.
9. $(1 - 27\,x^6) \div (1 - 3\,x^2)$.
10. $(m^6 + n^6) \div (m^2 + n^2)$.
11. $(125 - a^6) \div (5 - a^2)$.
12. $(x^6 - y^9) \div (x^2 - y^3)$.

232. From § 230 we may make rules for factoring the sum or the difference of the cubes of two numbers. These rules in algebraic symbols are:

1. $\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$.
2. $\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$.

In words we have from 1:

The sum of the cubes of two numbers equals the sum of the numbers multiplied by the square of the first number minus the product of the two numbers plus the square of the second number.

Let the student make the corresponding statement for the difference of the cubes of two numbers.

Example

Factor $x^3y^3 - 27$.

Solution. $x^3y^3 - 27$ may be written $(xy)^3 - 3^3$. It therefore comes under (2) and we have

$$(xy)^3 - 3^3 = (xy - 3)(x^2y^2 + 3\,xy + 9).$$

In this example, what replaces a of the type form? What replaces b?

EXERCISE

233. *Factor the following:*

1. a^3b^3+8.
2. a^3+27.
3. m^3-1.
4. $m^6+x^6=(m^2)^3+(x^2)^3$.
5. $125-a^3b^3$.
6. $1+v^3$.
7. $a^3b^3-m^3n^3$.
8. $64\,b^3x^3+27\,c^3y^3$.
9. v^3-u^3.
10. $p^3-\frac{1}{125}$.
11. $\frac{64}{27}-125\,m^3n^3$.
12. $32\,a^3-108\,b^3$.
13. $t^2m^3-t^2p^3$.
14. b^6-64.

15. $m^6-n^6=(m^3)^2-(n^3)^2$ or $(m^2)^3-(n^2)^3$. Factor both ways. Which way is to be preferred?

16. $1+6x+6x^2+x^3=(1+x^3)+6x(1+x)$
$=(1+x)[(1-x+x^2)+6x]$ etc.

17. a^3+5a^2+5a+1.
18. $a^3+3a^2x+3ax^2+x^3$.
19. $(a+b)^3+c^3$.
20. $(a-b)^3-c^3$.
21. $a^3-(b-c)^3$.
22. $(x^2+y^2)^3+8$.
23. $(1-2a)^3+1$.
24. $a^4-a(x+y)^3$.
25. $(x+y)^4-(x+y)$.
26. m^9-a^6.
27. $64\,x^6-1$.
28. $a^3-3a^2b+3ab^2-b^3$.
29. $a^{3n}-b^{3n}$.
30. $a^{3n}+b^{3n}$.

SUMMARY OF FACTORING

234. The following summary of the type forms in factoring is inserted here for convenience and review:

I. Factors of monomials. Square root.

II. Terms having a common monomial factor.

$$ab+ac=a(b+c).$$

III. The square of a binomial. $a^2+2ab+b^2=(a+b)^2$,
$a^2-2ab+b^2=(a-b)^2$.

IV. The difference of the squares of two numbers.

$$a^2 - b^2 = (a + b)(a - b).$$

IV *a*. Polynomials written as the difference of two squares.

$$(a^2 + 2\,ab + b^2) - c^2 = (a + b + c)(a + b - c).$$

IV *b*. Completing the square.

$$x^4 + x^2y^2 + y^4 = x^4 + 2\,x^2y^2 + y^4 - x^2y^2 = (x^2 + xy + y^2)(x^2 - xy + y^2).$$

V. Grouping terms. $a(x+y) + b(x+y) = (x+y)(a+b).$

VI. The quadratic trinomial of the form $x^2 + bx + c$.

VII. The general quadratic trinomial. $ax^2 + bx + c$.

VIII. The sum or the difference of two cubes. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$

Let the student translate each algebraic formula into verbal language.

235. The student finds it comparatively easy to factor algebraic expressions when they are classified under proper type forms, but in actual practice he is left to his own devices to determine to what type form a given expression belongs. Hence he usually meets with difficulty in factoring unclassified expressions.

In factoring miscellaneous exercises the student will find the following suggestions useful:

1. Carefully study the type forms and determine which one applies to the given expression.

2. If there is a monomial factor, write the expression as the product of the monomial factor and a polynomial. (II.)

3. If the polynomial is a binomial, note whether it is the difference of two squares or the sum or the difference of two cubes. (IV, VIII.)

4. If the polynomial is a trinomial, use the proper type form for factoring a trinomial. (III, IV *b*, VI, VII.)

5. If there are more than three terms, first try to factor by grouping the terms. It is also possible that the terms can be arranged as the difference of two squares. (IV, IV *a*.)

6. Continue the process of factoring till all the factors are prime.

7. Check, either by multiplying the factors together, or by substituting definite values for the letters both in the indicated product of the factors and in the original expression. In the latter case the two results should be equal.

REVIEW EXERCISE

236. *Factor into prime factors:*

1. $a^3b^2 + b^2$. (3 factors.)
2. $x^4 - 2x^2 + 1$. (4 factors.)
3. $54x^3y^5z^9 - 42x^5z^5 - 24xy^7z^7$.
4. $512x^6y^3 - 27z^9$.
5. $a^4x^2 - 2a^2b^3x + b^6$.
6. $1 - 14x^3y + 49x^6y^2$.
7. $2x^7y - 3x^6y^2 + 5x^3y^5$.
8. $z^4 + 22z^2 + 169$.
9. $a^6b^2 - a^2b^3x^3 - a^4x^4 + bx^7$.
10. $a^2 + 21a + 108$.
11. $729 - x^6y^{12}$. (4 factors.)
12. $b^2 + 12b + 35$.
13. $x^4 + x^3 + x + 1$.
14. $a^2 + 2ab - 15b^2$.
15. $81a^2 - 16(2a - 3x)^2$.
16. $(2a - 3b)^2 - 4b^2$.
17. $21a^8b^9 - 28a^5b^6 + 35a^3b^7c$.
18. $(a + b)^2 + 3(a + b) - 4$.
19. $ax - bx + cx + ay - by + cy$.
20. $2ax - 5ay + a - 2bx + 5by - b$.
21. $(4a - 5b)(5c - 2d) - (a + 4b)(5c - 2d)$.
22. $ax - a + x - 1$.
23. $a^2 - b^2 + 2bc - c^2$.
24. $x^2 - (c + 5)x + 5c$.
25. $x^2 + (a - b)x - ab$.
26. $x^2 - (n - 3)x - 3n$.
27. $64 + 27b^3$.
28. $125x^3 - 8y^6$.
29. $x^4y^4 - 81m^4$.
30. $x^4 + 2x^2y^2 - 15y^4$.
31. $3x^2 + 8x + 5$.
32. $2a^2 + 13a + 15$.
33. $8x^4 - x^2 - 9$.
34. $12a^2 - 17ab + 6b^2$.
35. $ab^2c + bcx + aby + xy$.
36. $-x^2 + 2x - 1$.
37. $a^4 + 4b^4$.
38. $a^6 - b^6$.
39. $a^8 - b^8$.
40. $a^{12} - b^{12}$.
41. $8x^4 - x^2y^2 - 9y^4$.
42. $12x^2 + 2xy - 30y^2$.
43. $m^4 + 4m^2p^2 + 100p^4$.

44. $2r^2 + 16rs + 32s^2$.

45. $(a+b)^2 - a^2 + b^2$.

46. $(x-y)^2 - x^2z + y^2z$.

47. $5x^3y - 23x^2y^2 + 12xy^3$.

48. $ab^2c + 2bcx - aby - 2xy$.

49. $(a+b)^2 + (a^2 - b^2) + a + b$.

50. $64x^2 + 81y^2 - 144xy$.

51. $18a^4 + 72m^4$. (3 factors.)

52. $8a^3 + 64x^3$.

53. $4a^2 - 12ab + 9b^2 - 9$.

54. $18a^2b^2 + 32x^4 + 48abx^2$.

55. $125x^6 - y^9$.

56. $4 - 12ab - 4a^2 - 9b^2$.

57. $a - a^7$.

58. $(a+b)^3 - (a^2-b^2)(a+b) + a(a+b)^2$.

59. $(x+y)^4 - (x+y)^3$.

60. $36a^4b^2c^2 - 24ab^5c^2 + 72a^3b^3c$.

61. $30x^3y^3 - 45x^2y^4 + 60xy^5$.

62. $(a-b)^2 - (a-b)$.

63. $z^2 - z - 156$.

64. $12abx^2 + 48a^2b^2xy + 48a^3b^3y^2$.

65. $6a^6 - 10a^4x - 18a^2x^2 + 30x^3$.

66. $3x^3 + 3x^2 - 36x$.

67. $9x^2 - 4y^2 + 4yz - z^2$.

68. $(a+3b)^2 - 9(b-c)^2$.

69. $a^2(1-c) - 4b^2(c-1)$.

70. $ay(x-m) - ax(y-m)$. (First expand.)

71. $ay(x-m) - ax(m-x)$.

72. $a^2x + ab^2x - aby - b^3y$.

73. $3(m+n)^3 - 4(m+n)^2 + m + n$.

74. $4(a-b)^3 + 12a(a-b)^2 - 6(a-b)a^2$.

75. $2n(2m-1) - 3n^2(2m-1) + 5(2m-1)$.

76. $2ax + 3bx + 4cx - 2ay - 3by - 4cy$.

77. $2ax - 3bx + 4cx + 2ay - 3by + 4cy$.

Factor:

78. $2\,ax - 3\,bx + 4\,cx - 2\,ay + 3\,by - 4\,cy.$

79. $27\,ax^3 + 8\,a^4y^3.$

80. $a^2b^3 - b^3 - a^2 + 1.$

81. $(a-b)(x-y)-(a-y)(x-b).$ Expand.

82. $(a+b)^2 - 4 - 2(a+b-2).$

83. $a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2.$

84. Factor $x^6 - y^6$ into two factors in two different ways:

85. $a^6 + b^6.$

86. $a^{10} - b^{10}.$ (2 factors.)

87. $a^{12} - b^{12}.$

88. $a^{2n} - 1.$

89. $a^{2n} - b^{2n}.$

90. $a^{2n} - a^2.$

91. $a^{2n+1} - a.$

92. $a^{2n} + 2\,a^n + 1.$

93. $a^{4n} - 1.$

94. $a^{n+1} + 2\,a^nb + a + 2\,b.$

95. $(a-b)^{m+1} - (a-b)^m.$

96. $x^{3n+1} - x.$

97. $x^{2n} - y^{2n} - x^n - y^n.$

98. $x^{2n} - y^{2n} + x^n + y^n.$

99. $a^{18} - 1.$

100. $a^{32} - 1.$

THE SOLUTION OF EQUATIONS BY FACTORING

237. A **root** of an equation has been defined as a value of the unknown quantity that **satisfies** the equation.

Is 2 a root of $x^2 - 5\,x + 6 = 0$? is 1 a root? is 6 a root? is 3 a root?

238. It is sometimes possible to write an equation in such a form that its roots are evident.

1. Consider the equation, $x^2 - 7\,x + 10 = 0.$

Factoring the first member, we have the same equation in another form.

$$(x-5)(x-2) = 0.$$

If the product of two or more factors is zero, one of the factors must be zero. Therefore this equation is satisfied, if

$x - 5 = 0$; that is, if $x = 5$,

or if $x - 2 = 0$; that is, if $x = 2$.

Therefore, 5 and 2 are roots of this equation. The roots may be verified by putting 5 and 2 for x in the equation.

2. Solve $x^3 = 4x$.

Make the second member zero by transposing $4x$.

$$x^3 - 4x = 0.$$

$$x(x-2)(x+2) = 0. \quad \text{(Factoring.)}$$

The second equation is satisfied if any one of the three linear equations,

$$x = 0,\ x - 2 = 0,\ x + 2 = 0,$$

is satisfied. (Why?)

This gives as solutions of the given equation $x = 0$, 2, or -2.

Verify by substituting 0, 2, and -2 for x in the original equation, $x^3 = 4x$.

We can solve equations by factoring if, when the second member is zero, we can factor the first member into factors of the first degree with respect to the unknown number.

239. To solve an equation by factoring:

1. Write the equation with the second member zero and the first member arranged in descending powers of the unknown number.

2. Factor the first member into linear factors with respect to the unknown number.

3. Put each factor equal to zero and solve the linear equations obtained.

EXAMPLES

1. Solve $2x(x-1) = 3x - 2$.

SOLUTION.

$$2x^2 - 2x = 3x - 2. \quad \text{(Multiplying.)}$$

$$2x^2 - 5x + 2 = 0. \quad \text{(Transposing.)}$$

$$(2x-1)(x-2) = 0. \quad \text{(Factoring.)}$$

$$2x - 1 = 0 \text{ or } x = \tfrac{1}{2}.$$

$$x - 2 = 0 \text{ or } x = 2.$$

Let the student verify the roots.

2. Solve $9(x-1) = (x+4)(x-1)$.

SOLUTION.

$$9(x-1) - (x+4)(x-1) = 0. \quad \text{(Transposing, without multiplying.)}$$

$$(x-1)(9-x-4) = 0. \quad \text{(Factoring.)}$$

$$(x-1)(5-x) = 0.$$

$$x - 1 = 0 \text{ or } x = 1.$$

$$5 - x = 0 \text{ or } x = 5.$$

Verify by substituting the roots in the equation.

EXERCISE

240. *Solve the following equations:*

1. $(x-5)(x-4)=0.$
2. $(x+\frac{1}{2})(7x-1)=0.$
3. $(2x-5)(x-3)=0.$
4. $x(3x-7)=0.$
5. $(4-x)(5x+1)=0.$
6. $x^2-7x=-10.$
7. $x^2-6x=7.$
8. $x^2=13x-42.$
9. $(x-1)(x-2)=12.$
10. $9x^2-16=0.$
11. $4x^2-4x+1=0.$
12. $y(y-6)=7y-42.$
13. $(y-11)(y-12)=2.$
14. $(r+6)(r-4)-(2+r)(2-r)=56.$
15. $(x-1)^2+(x+1)^2=29-(2x+3)^2.$
16. $2x^2-5x=3.$
17. $(x-2)^2-(x+2)^2+7x=0.$
18. $(3x-5)(3x+5)-(x-1)^2=10.$
19. $(x+1)^3-3x(x-1)=x^3+1.$
20. $(x+2)^3-2(x+2)^2=0.$
21. $x^2-ax-bx+ab=0.$
22. $x^2-4a^2-4a-1=0.$
23. $x^3+x^2=x+1.$
24. $(x-2)^2+25=10(x-2).$
25. $(x-7)(2x+5)=(3x-1)(x-7).$
26. $(x-3)(4x-5)=x^2-9.$
27. $x^2-9=8x.$
28. $(2x-\frac{9}{3})\cdot 4=(2x-\frac{9}{3})(5x-11).$
29. $(x-2)=(x-2)(x-3).$
30. $(x-1)(x-2)(x-3)+6=0.$ (Find one root only.)

SOLUTION OF PROBLEMS

241. 1. The larger of two numbers exceeds the smaller by 5, and their product is 84. Find the numbers.

SOLUTION. Let $x =$ the smaller number.
Hence $x + 5 =$ the larger number,
and $x(x + 5) =$ their product.
Then $x(x + 5) = 84$. (By the conditions.)
Hence $x^2 + 5x - 84 = 0$, (Why?)
or $(x + 12)(x - 7) = 0$. (Why?)
$\therefore x + 12 = 0$ and $x - 7 = 0$.
$\therefore x = -12$ or 7,
and $x + 5 = -7$ or 12.

The pairs of numbers that satisfy the conditions of the problems are -12 for the smaller and -7 for the larger, or 7 for the smaller and 12 for the larger.

To check the answers they should be put into the original problem and not into the equation. (Why?)

2. The length of a rectangular figure is 5 inches more than its width, and its area is 84 square inches. Find its dimensions.

SOLUTION. Let $x =$ the number of inches wide.
Hence $x + 5 =$ the number of inches long,
and $x(x + 5) =$ the number of square inches in area.
Then $x(x + 5) = 84$.

From this point the solution is exactly like that of the last problem. The answers -12 and 7 as values of x have to be considered in connection with the problem. The answer -12 cannot represent the number of inches in the width of a rectangle and is to be rejected in this problem. $x = 7$ is evidently the answer to be used. This will make the dimensions of the rectangle 7 inches and 12 inches.

3. The product of two consecutive numbers is 72. Make the equation and solve for the numbers.

4. The product of two consecutive even numbers is 80. Make and solve the equation to find the numbers.

5. The sum of two numbers is 19 and their product is 84. Find the numbers.

HINT. Let the numbers be represented by x and $19 - x$.

6. One of two numbers is twice as large as the other and their sum is 14. Find the numbers.

HINT. Let x and $2x$ represent the numbers. The equation is of first degree.

7. One of two numbers is twice the other and their product is 242. Find the numbers.

8. A rug is twice as long as it is wide. It contains $4\frac{1}{2}$ square yards of material. Find its dimensions.

9. The perimeter of a rectangle is 40 inches and its area is 91 square inches. Find the dimensions.

HINT. If the perimeter is 40 inches, the sum of the length and the width is 20 inches.

10. The side of one square is 4 inches more than that of another and the sum of their areas is 136 square inches. Find the side of each square.

11. If this page is 6 centimeters longer than it is wide, and its area is 216 square centimeters, find the dimensions.

12. The quotient exceeds the divisor by 8 and the dividend equals three times the sum of the divisor and the quotient. Find the divisor, the quotient, and the dividend.

13. The sum of the squares of two consecutive numbers exceeds 5 times the sum of the numbers by 6. Find the numbers.

14. A rectangle is 3 inches longer than it is wide. If both dimensions are increased by 2 inches, the area is 28 square inches. Find the original dimensions.

IX. HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

HIGHEST COMMON FACTOR

242. Rational Term. A term is **rational** if, when reduced to its simplest form, it contains no indicated roots.

Thus, $3\,ac^2$, $3\,a^2b$, and $\sqrt{4}$ are rational terms. $7\,a\sqrt{b}$ is not rational with respect to b but it is rational with respect to 7 and a. Is $\sqrt{27}$ rational? $\sqrt[3]{27}$?

243. Integral Term. A term is **integral** with respect to any set of numbers or letters if none of the numbers or letters appear in the denominator.

Thus, $\frac{7}{3}\,ab$ is integral with respect to 7, a, and b but not with respect to 3. $4\,mn^3pq^2$ is integral with respect to all letters and numbers involved. $\dfrac{2\,ab}{c}$ is not integral with respect to c.

244. A **polynomial** is **rational** and **integral** if all its terms are rational and integral.

Thus, $3\,ax + 2\,by$ is rational and integral.

$3\,ax + \dfrac{2\,b}{y}$ is rational but is not integral.

$3\,ax + \sqrt{2\,by}$ is integral but is not rational.

An expression may be rational and integral with respect to some particular letter involved. The three examples just given are all rational and integral with respect to x.

Also $x^2 + \dfrac{b}{a} \cdot x + \dfrac{c}{a}$ is rational and is integral with respect to x, b, and c, but is not integral with respect to a.

245. Degree. The **degree** of a **rational integral monomial** is the sum of the exponents of its literal factors.

Thus, $2a^2x$ and a^3 are of third degree; $4ab^2c$ is of fourth degree. Of what degree is $7x^3$? $3ax$? $2a^2x^2$?

We are sometimes concerned with the degree of a monomial with respect to some particular letter.

Thus, $3a^2x$ is of the second degree with respect to a. It is of the first degree with respect to x.

246. The **degree** of a **rational integral polynomial** is the same as that of its term of highest degree.

Thus, $3x^3 + 2x^2y^2$ is of the fourth degree. Of what degree is $ax^2 + bx + c$? of what degree with respect to x? with respect to a? b? c?

247. The student should note that *degree* and *power* are not the same. The power of a term may or may not be the same as the degree of the term.

Thus, $3x^2y^2$ and a^4 are both of the fourth degree, but $3x^2y^2$ is not a fourth power. Also $(a^2 + 2)^2$ is a second power, but is a fourth degree expression.

ORAL EXERCISE

248. *Give the degree of each of the following:*

1. $3a^2by$.
2. $4abcx$.
3. $2x^2y^2$.
4. $5axy$.
5. $ax^2 + bx + c$.
6. $axy + by^3$.
7. $(x + y)^2$.
8. $(x + y)^3$.
9. $(x + 1)^3 + a(x + 1)^2 + b(x + 1)$.

Of what degree is each of the above with respect to x? with respect to x and y?

249. Common Factor. If the same factor occurs in two or more algebraic expressions, it is a **common factor** of the expressions.

Thus, x is a common factor of $7x$ and $3xy$; and $2a$ is a common factor of $2a$, $4a$, and $6a^3$.

250. Two or more expressions may have several common factors.

Thus, $35x^3y^2$, $21x^2y^3$ and $42x^3y^3$ have what common factors of the first degree? of the second degree? of the third degree? of the fourth degree? Can you find a common factor of these expressions of higher degree than the fourth?

251. The **highest common factor** (H. C. F.) of two or more monomials is the greatest common divisor of their numerical coefficients multiplied by their highest degree literal common factor.

Thus, $7x^2y^2$ is the H. C. F. of $35x^3y^2$, $21x^2y^3$, and $42x^3y^3$.

252. The H. C. F. in algebra corresponds to the greatest common divisor (G. C. D.) in arithmetic. The G. C. D. is the *largest number* that will exactly divide two or more numbers; the H. C. F. is the *highest degree* algebraic expression that will divide two or more expressions.

We may find the G. C. D. of 12, 18, 24 by factoring thus:

$12 = 2^2 \cdot 3$, $18 = 2 \cdot 3^2$, $24 = 2^3 \cdot 3$.

Therefore the G.C.D. of 12, 18, and 24 is $2 \cdot 3 = 6$.

Similarly, we may find the H. C. F. of two or more algebraic expressions.

Find the H. C. F. of $12a^2bc$, $18a^3b^2c^2$, $24a^3c$.

SOLUTION.

$$12a^2bc = 2^2 \cdot 3 \cdot a^2bc.$$
$$18a^3b^2c^2 = 2 \cdot 3^2 \cdot a^3b^2c^2.$$
$$24a^3c = 2^3 \cdot 3 \cdot a^3c.$$

The H. C. F. is the G. C. D. of the numerical coefficients, 6, multiplied by their highest degree literal common factor a^2c; that is, the H. C. F. is $6a^2c$.

253. **To find the H. C. F. of two or more algebraic expressions, multiply together the lowest powers of all the prime factors common to all the expressions.**

In the case of monomials the H. C. F. is seen by inspection. If any of the expressions are polynomials, factor them into prime factors.

EXERCISE

254. *Find the H. C. F. in each of the following:*

1. $3a^2b,\ 6a^2b^2,\ 9ab^2.$
2. $4a^2x^2,\ 6a^3x^3,\ 12ax^2.$
3. $14a^2b^3x^2,\ 98a^3b^2x^4,\ 105a^4b^4x^5.$
4. $45a^6,\ 18a^5b^7,\ 108a^4b^{12}.$
5. $13x^4y^5,\ 52x^3y^6,\ 169x^3y^5.$
6. $3x^2y^3,\ 9x^4y^6,\ 12x^3y^3,\ 15x^2y^5.$
7. $4x^3y^2,\ 16x^4y^3,\ 64x^3y^4.$
8. $98x^2z^4,\ 180x^3z^6,\ 300x^4z^5.$
9. $15a^2bx^2y^3,\ 45b^3y^4,\ 90a^4b^4x^4.$
10. $14(a+b)^2(a-b),\ 10(a+b).$
11. $a^2b-b^3,\ a^2b-2ab^2+b^3,\ a^4b-ab^4.$

SOLUTION.

$$a^2b-b^3=b(a+b)(a-b).$$
$$a^2b-2ab^2+b^3=b(a-b)^2.$$
$$a^4b-ab^4=ab(a-b)(a^2+ab+b^2).$$
$$\therefore \text{H. C. F.}=b(a-b).$$

12. $24a^2x^2+36a^3x^3,\ 9ax-12a^2x^2.$
13. $3ax^3+4bx^4,\ ax^5-12bx^6.$
14. $4a^2b^2x^2-8abx^3,\ 8a^2bx^4-12abx^2.$
15. $18a^2b^4x^5-72x^8,\ 12abx^4.$
16. $2x^2-17x+36,\ 4x^2-12x-27.$
17. $(a+b)^2-c^2,\ a^2-(b+c)^2.$
18. $9x^4-16y^4,\ 9a^2x^2+12a^2y^2.$
19. $4x^2+12xy+9y^2,\ 16x+24y.$
20. $(a+b)^3,\ a^2+2ab+b^2,\ a^2-b^2.$
21. $48x^4-12y^4,\ 20x^3-10xy^2.$
22. $a^4-b^4,\ a^3-b^3,\ a^2-b^2.$
23. $x^2-5x+6,\ 3x^2-6x,\ x^2-6x+8.$
24. $3x^2-x-2,\ 6x^2+13x+6,\ 6x^2-5x-6.$

25. $2a^2b + 2ab^2 - 2abc$, $3bc^2 - 3b^2c - 3abc$.

26. $a^2 + b^2 - c^2 + 2ab$, $a^2 - b^2 + c^2 + 2ac$.

27. $a^2 - b^2 - ac + bc$, $ab + ac + b^2 - c^2$.

28. $mx - m - x + 1$, $m^2 - 2m + 1$.

29. $2ab - 3ac - 2b + 3c$, $3ab - 2ac - 3b + 2c$.

30. $x^2 - x - 20$, $x^2 + x - 30$, $x^2 - 25$.

LOWEST COMMON MULTIPLE

255. A product is a **multiple** of any of its factors.

Thus, $3x^2y$ is a multiple of x; of xy; of $3x$; etc.

256. A **common multiple** of two or more expressions is a multiple of each of them.

Thus, $6x^2y^3$ is a common multiple of $3x$, $2y$ and xy.

Two or more expressions have always an infinite number of common multiples.

Thus, $3x$, $2y$, and xy have as common multiples $6xy$, $6x^2y^2$, $6x^2y$, $12xy$, etc., indefinitely. Can you find a common multiple of these three monomials of lower degree than the second?

257. The **lowest common multiple** (L. C. M.) of two or more monomials is the arithmetical least common multiple of their numerical coefficients multiplied by their lowest degree literal common multiple.

258. In arithmetic the least common multiple of two or more numbers is the smallest number which may be exactly divided by each of them. In algebra the L. C. M. of two or more expressions is the lowest degree expression which may be exactly divided by each of them.

259. To find the L. C. M. of two or more algebraic expressions, multiply together the highest powers of all the different prime factors in the expressions.

The L. C. M. of monomials is seen by inspection. If the expressions are polynomials, first factor them into prime factors.

1. Find the L. C. M. of $9\,b^3c$, $12\,ac^2$, $4\,abc^3$.

SOLUTION. $9\,b^3c = 3^2b^3c$, $12\,ac^2 = 3 \cdot 2^2ac^2$, $4\,abc^3 = 2^2abc^3$.
$\therefore$ L. C. M. $= 3^2 \cdot 2^2ab^3c^3$ or $36\,ab^3c^3$.

2. Find the L. C. M. of $a^2 - 3\,a + 2$, $a^2 - 1$, $a^2 - 4\,a + 4$.

SOLUTION.
$$a^2 - 3\,a + 2 = (a - 1)(a - 2).$$
$$a^2 - 1 = (a + 1)(a - 1).$$
$$a^2 - 4\,a + 4 = (a - 2)^2.$$
$\therefore$ L. C. M. is $(a - 1)(a + 1)(a - 2)^2$.

EXERCISE

260. *Find the L. C. M. in each of the following, leaving the results, in the case of polynomials, in factored form:*

1. $3\,ab$, $4\,a^2bc$, $6\,ab^2c$.
2. 12, 18, $24\,x$.
3. $2\,a$, $3\,b$, $5\,c$.
4. $2\,a$, $3\,a$, $5\,a$.
5. $x^2 + 1$, $2\,x - 2$, $x^2 - 1$.
6. $m + n$, $(m - n)^2$, $m^2 - n^2$.
7. $a^2 - 6\,ab + 9\,b^2$, $a^2 - 9\,b^2$.
8. $3\,x + 6$, $6\,x^2 - 24$, $2\,x - 4$.
9. $3 - 3\,a^2$, $5 - 5\,a$, $1 + a$.
10. $x^3 - 1$, $(x - 1)^2$.
11. $a^3 - a$, $a^2 - 2\,a + 1$, $2\,a^2 - 5\,a + 3$.
12. $2\,a^2 - 5\,a + 3$, $4\,a^2 - 13\,a + 3$, $8\,a^2 - 6\,a + 1$.
13. What is the L. C. M. of two expressions that have no common factor?
14. $2\,x^2 + x - 1$, $x^2 - x - 2$, $2\,x^2 - 5\,x + 2$.
15. $2(2\,x + 5)$, $3\,x + 6$, $2\,x^2 + 9\,x + 10$.
16. $x^2 + 3\,x + 2$, $x^2 + 4\,x + 3$, $x^2 + 5\,x + 6$.
17. $a^2 - 3\,ab + 9\,b^2$, $a^3 + 27\,b^3$, $a + 3\,b$.
18. $6 - a - a^2$, $2 - 3\,a + a^2$, $1 - a$.
19. $xy - 2\,y^2$, $xy - y^2$, $x^2 - 3\,xy + 2\,y^2$.
20. $2 - 2\,x - a + ax$, $3 - 3\,x - b + bx$.
21. $3\,a^2 - 5\,ax + 2\,x^2$, $4\,a^2 - 9\,ax + 5\,x^2$.

X. FRACTIONS

261. An **algebraic fraction** is an indicated division.

Thus, $\frac{a}{b}$ (read the fraction, a divided by b) is the indicated quotient of a divided by b.

The **numerator** of the fraction is the **dividend** and the **denominator** is the **divisor**.

Terms of a Fraction. The numerator and the denominator are the **terms of a fraction.** The denominator of a fraction cannot be 0 since dividing by 0 has no meaning in the ordinary sense of division.

The topics studied under fractions in algebra agree closely with those of arithmetic, and the methods are similar.

EXERCISE

262. 1. Reduce $\frac{12}{15}$ to lowest terms. Also $\frac{35}{40}$.

2. Change the improper fraction $\frac{12}{5}$ to a mixed number. Give the rule.

3. Change $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{8}$ to equivalent fractions having the least common denominator. Give the rule.

4. State the rule for adding arithmetical fractions.

5. $1\frac{2}{3} + 3\frac{1}{4} + 3\frac{7}{8} = ?$

6. $75\frac{7}{8} - 12\frac{3}{4} = ?$

7. Multiply $2\frac{1}{2} \div \frac{3}{4}$ by $\frac{1}{2}$ of $\frac{3}{4} \times \frac{5}{6}$.

Note. For order of operations see § 55.

8. Find the value of $\frac{2}{3} \times \frac{8}{9} \div 2\frac{6}{7} + 5\frac{1}{2} \times \frac{9}{17}$.

9. $\dfrac{3\frac{2}{17} \times 8\frac{1}{16}}{4\frac{2}{13} \times 2\frac{1}{16}} = ?$

10. Evaluate $\frac{1}{6} \div \frac{2}{3} \times \frac{3}{4} + 16 \times \frac{1}{2} \times 8 - \frac{3}{7} \times \frac{5}{8}$.

11. What change, if any, is made in the value of a fraction when:

(*a*) The numerator is multiplied by an integer?

(*b*) The numerator is divided by an integer?

(*c*) The denominator is multiplied by an integer?

(*d*) The denominator is divided by an integer?

(*e*) Both terms of a fraction are multiplied by the same number?

(*f*) Both terms of the fraction are divided by the same number? Explain and illustrate each answer.

12. Give at sight answers to the following:

(*a*) $\frac{1}{2} \div 2$. (*c*) $\frac{2}{3} \div 2$. (*e*) $.5 \div \frac{1}{2}$.

(*b*) $\frac{3}{4} \div .5$. (*d*) $\frac{4}{5} \times 7$. (*f*) $.5 \times \frac{1}{2}$.

REDUCTION OF FRACTIONS

263. The principles of arithmetical fractions suggested in example 11 of the last article will be assumed to hold in algebraic fractions. In particular we now assume the following principle:

If the numerator and the denominator of a fraction are divided by the same number, the value of the fraction is not changed.

264. Lowest Terms of a Fraction. A fraction is in its **lowest terms** if the numerator and denominator have no common factor.

265. To reduce a fraction to its lowest terms divide both the numerator and the denominator by all their common factors.

266. Cancellation. Much time may be saved in solving problems involving fractions by canceling factors common to both numerator and denominator if any are present. The student should do this at every stage in the solution of a problem, always factoring and canceling whenever possible and never multiplying or dividing until all possible factors have been removed by cancellation.

1. $\dfrac{24}{36}=\dfrac{\cancel{2}\cdot\cancel{2}\cdot 2\cdot\cancel{3}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{3}\cdot 3}=\dfrac{2}{3}.$

2. $\dfrac{\overset{3\,a}{\cancel{21}\cancel{a^3}\cancel{b^2}\cancel{c}}}{\underset{5\qquad c}{\cancel{35}\cancel{a^2}\cancel{b^2}\cancel{c^2}}}=\dfrac{3\,a}{5\,c}.$

3. $\dfrac{a^2-b^2}{2\,a^5-2\,ab^4}=\dfrac{\cancel{(a+b)}\cancel{(a-b)}}{2\,a(a^2+b^2)\cancel{(a+b)}\cancel{(a-b)}}=\dfrac{1}{2\,a(a^2+b^2)}.$

267. In canceling a factor from the numerator and the denominator of a fraction the quotient 1 is not generally written. It is important to remember, however, that if all factors of either the numerator or the denominator are canceled that term of the fraction becomes 1. If the quotient is -1 it should be written.

Thus, $\dfrac{a-b}{3\,a^2-3\,b^2}=\dfrac{\cancel{a-b}}{3\cancel{(a-b)}(a+b)}=\dfrac{1}{3(a+b)}.$

Also, $\dfrac{b-a}{a^2-b^2}=\dfrac{\overset{-1}{\cancel{b-a}}}{(a+b)\cancel{(a-b)}}=\dfrac{-1}{a+b}.$

ORAL EXERCISE

268. *Reduce each fraction to its lowest terms:*

1. $\dfrac{x}{xy}$.
2. $\dfrac{a^2}{a^4}$.
3. $\dfrac{ab}{b}$.
4. $\dfrac{2\,xy}{4}$.
5. $\dfrac{3\,c^2d}{6\,cd}$.
6. $\dfrac{10\,pq^2}{5\,pq}$.
7. $\dfrac{rs}{r^2s}$.
8. $\dfrac{15\,cd}{3\,c}$.
9. $\dfrac{36\,h^2k^2}{9\,hk}$.
10. $\dfrac{72\,mn^3}{8\,mn^2}$.
11. $\dfrac{-11\,xy}{33\,x}$.
12. $\dfrac{14\,a}{-ab}$.
13. $\dfrac{-abc}{-a^2c}$.
14. $\dfrac{9\,u^2v^2w^2}{3\,uvw^3}$.
15. $\dfrac{7\,pq^2v^2}{21\,qv^2}$.
16. $\dfrac{-xy^2z}{y^4}$.
17. $\dfrac{-a^2bd^2}{4\,ad}$.
18. $\dfrac{102\,klm}{51\,k^2m^2}$.

EXERCISE

269 *Reduce to lowest terms:*

1. $\frac{3x^2y^2}{15x^3y^3}$.
2. $\frac{147a^3x^2}{49a^2x^3}$.
3. $\frac{27a^2b^3c^4}{3abc}$.
4. $\frac{17}{51a^4b^5}$.
5. $\frac{35a^4b^2c^4}{42a^5bc^5}$.
6. $\frac{39rst}{65r^2s^3t^4}$.
7. $\frac{(-2abc)^3}{8ab^2c^3}$.
8. $\frac{(15m^2n^2p)(7p^2n)}{(14p^3)(5m^2q)}$.
9. $\frac{a^2+ab}{a+a^2}$.
10. $\frac{x^2-1}{(x-1)^2}$.
11. $\frac{x^2-y^2}{xz-yz}$.
12. $\frac{x^3-2x^2}{3x^2-12}$.
13. $\frac{8a^2b-16ab^2}{12a^2x-48b^2x}$.
14. $\frac{x^2-1}{(x+1)^2}$.
15. $\frac{a^2-6a}{a^2-7a+6}$.
16. $\frac{3a^2b-9ab^2}{a^3-7a^2b+12ab^2}$.
17. $\frac{x^2-16x-17}{x^2-22x+85}$.
18. $\frac{a^3b-ab^3}{a^2b^3-a^4b}$.
19. $\frac{(5a-7)^2}{50a^2-98}$.
20. $\frac{x^3+x^2y}{x^2+2xy+y^2}$.
21. $\frac{7x^2+3}{245x^5+210x^3+45x}$.
22. $\frac{a^4-b^4}{a^2+2ab+b^2}$.
23. $\frac{x-xy+z-zy}{1-3y+3y^2-y^3}$.
24. $\frac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$.
25. $\frac{x^2+3x+2}{x^2+6x+5}$.
26. $\frac{a^{2n}+2a^n+1}{a^{2n}-1}$.

27. Common factors may be canceled from both terms of a fraction. May common terms or factors of common terms be canceled in the same way? Is $\frac{2(x+y)}{2+(y+z)}=\frac{x+y}{y+z}$? Is $\frac{a^2-(x-y)^2}{a^2+(p-q^2)}=\frac{-(x+y)^2}{(p-q)^2}$?

270. Algebraic Signs in Fractions. There are three signs to be considered when determining the value of a fraction; namely, the sign of the **numerator**, of the **denominator**, and of the **fraction** itself.

Thus, $\frac{12}{-3} = -4$ but $-\frac{12}{-3} = 4$. Also $\frac{-12}{-3} = 4$.

271. Given the fraction $-\frac{x-2}{(x-4)(x-6)}$. Find its value when (*a*) $x = 0$; (*b*) $x = 1$; (*c*) $x = 3$; (*d*) $x = 5$; (*e*) $x = 7$; (*f*) What sign has the answer when x is equal to or greater than 7? when x is negative? when x is less than 2?

272. Changes in the Signs of the Numerator and the Denominator of a Fraction. By the definition of a fraction and the rules of division we have:

(1) $\frac{4}{2} = 2$.

(2) $\frac{-4}{-2} = 2$.

(3) $\frac{-4}{2} = -2$.

(4) $\frac{4}{-2} = -2$.

(5) $-\frac{-4}{2} = -(-2) = 2$.

(6) $-\frac{4}{-2} = -(-2) = 2$.

By comparing (1) with (2), and (3) with (4), we have the following principle illustrated:

1. If the signs of both numerator and denominator of a fraction are changed, the value of the fraction is not changed.

A similar comparison of (1) with (5) and (6), and (2) with (5) and (6) will illustrate the principle:

2. If the sign of the numerator or of the denominator and the sign before the fraction are changed, the value of the fraction remains unchanged.

We thus have: $\frac{4}{2} = \frac{-4}{-2} = -\frac{-4}{2} = -\frac{4}{-2} = 2$,

and $-\frac{4}{2} = -\frac{-4}{-2} = \frac{-4}{2} = \frac{4}{-2} = -2$.

In algebraic symbols, from 1 and 2 we have:

$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b},$$

$$\text{and } -\frac{a}{b}=-\frac{-a}{-b}=\frac{-a}{b}=\frac{a}{-b}.$$

273. The student should note carefully that changing the signs of an *odd number of factors* in the numerator or in the denominator of the fraction will change the sign of that term of the fraction in which the factors occur, and, therefore, by § 272, 2 the sign before the fraction must be changed.

If one factor in the numerator and one in the denominator have their signs changed, which principle applies? What is the effect on the value of the fraction if the signs of an *even number of factors* in the numerator or in the denominator are changed?

EXAMPLES

1. $\frac{x-1}{2-x}=-\frac{x-1}{x-2}$. (Why?)

2. $\frac{a-b}{(b-c)(c-a)}=\frac{b-a}{(c-b)(c-a)}=\frac{(b-a)}{(b-c)(a-c)}=$ etc.

Let the student make other changes in the signs of this fraction.

Would it be possible to change the signs of $\frac{a-b}{(b-c)(c-a)}$ so as to make $b+c$ a factor of the denominator?

EXERCISE

274. 1. Show by multiplication that

$$(x-1)(x-2)=(1-x)(2-x).$$

2. Show that $(x-1)(x-2)=-(x-2)(1-x)$.

3. Compare $(x-1)(x-2)(x-3)$ with $(1-x)(2-x)(3-x)$.

4. Make fractions equivalent to each of the following having the sign of the denominator changed:

(a) $-\frac{a}{-3}$. (d) $\frac{-b}{(-c)^2}$. (g) $-\frac{m}{(-a)(-b)}$.

(b) $-\frac{7}{-b}$. (e) $\frac{a}{-(-b)^2}$. (h) $\frac{m}{(-a)(-b)^2}$.

(c) $\frac{b}{(-c)^3}$. (f) $-\frac{a}{-(-b)^3}$. (i) $\frac{-ax^2}{(-b)^2y}$.

5. Give a fraction equivalent to each fraction in example 4 with the signs of its numerator and denominator positive.

6. If a, b, and c are all positive numbers, what algebraic sign has the number represented by each fraction?

(a) $\frac{ab}{c}$. (d) $-\frac{(-a)^3}{b+c}$.

(b) $-\frac{a}{b(-c)}$. (e) $\frac{(a-b)^2}{-c}$.

(c) $\frac{(-a)^2}{(-b)c}$. (f) $-\frac{(-a)(-b)(-c)}{(-a-b-c)}$.

7. What change is necessary to make the denominator the same in each of the following pairs of fractions?

(a) $\frac{1}{a-b}, \frac{1}{b-a}$. (b) $\frac{1}{(x-y)(a-b)}, \frac{1}{(x-y)(b-a)}$.

(c) $\frac{1}{(x-y)(y+z)}, \frac{1}{(y-x)(z+y)}$.

(d) $\frac{a-x}{(a-2)(a+2)}, \frac{a+x}{(2-a)(2+a)}$.

8. Arrange the denominators of the following fractions in descending powers of x with the sign of the first term in each denominator positive.

(a) $\frac{1}{(x-2)(3+x)}$. (c) $\frac{x-1}{2-3x-x^2}$. (e) $-\frac{x-2}{(3-x)^3}$.

(b) $\frac{a-b}{(1-x)(1+x)}$. (d) $-\frac{x-2}{(3-x)^2}$. (f) $\frac{m-n}{(2-x)^2(x+2)}$.

275. An **algebraic improper fraction** is one whose numerator is of the same degree in some letter as the denominator, or of higher degree (§ 245).

Thus, $\frac{a^2+2}{a^2-a-1}$ and $\frac{a^3}{a^2-1}$ are improper fractions.

276. A **mixed expression** is an expression containing both integral and fractional parts.

Thus, $a^2+\frac{b}{a}$ is a mixed expression.

Any integral expression may be written in fractional form by supplying the denominator 1.

Thus, $a+x=\frac{a+x}{1}$.

277. Improper fractions may be reduced to integral or mixed expressions in the same way as arithmetical improper fractions, that is, by dividing the numerator by the denominator.

Thus, $\frac{x^2y^2}{xy}=xy$ and $\frac{c-4}{c}=1-\frac{4}{c}$.

ORAL EXERCISE

278. *Reduce the following improper fractions to integral or mixed expressions:*

1. $\frac{15}{4}$.
2. $\frac{16x}{4}$.
3. $\frac{12x^2}{3x}$.
4. $\frac{22a^3}{11a}$.
5. $\frac{s^2-t^2}{s-t}$.
6. $\frac{ax+3}{a}$.
7. $\frac{h+k}{h}$.
8. $\frac{m^2-4}{-m}$.
9. $\frac{c-d}{c}$.
10. $\frac{c-d}{d}$.
11. $\frac{abc-d}{ab}$.
12. $\frac{a^2-ab^2}{a-b}$.

279. When the numerator and the denominator are both polynomials, the problem is similar to that of an inexact division (§ 160). The process will be understood by studying an example.

Change $\frac{x^2+3x-17}{x-2}$ to a mixed expression.

$$\begin{array}{r|l} x^2+3x-17 & x-2 \\ \underline{x^2-2x} & x+5 \\ 5x-17 & \\ \underline{5x-10} & \\ -\ 7 & \end{array}$$

$$\therefore \frac{x^2+3x-17}{x-2} = x+5+\frac{-7}{x-2} \quad (1)$$

$$= x+5-\frac{7}{x-2} \quad (2)$$

Why is the form (2) of the same value as (1)?

The division should be continued until the remainder is of lower degree than the divisor.

EXERCISE

280. *Reduce the following improper fractions to integral or mixed expressions:*

1. $\frac{x^2+1}{x-1}$.
2. $\frac{x^2-1}{x-1}$.
3. $\frac{x^2+x-13}{x+1}$.
4. $\frac{3x^2+7x}{x+1}$.
5. $\frac{(x^2-x)^2}{-x}$.
6. $\frac{a^2+b^2}{a+b}$.
7. $\frac{a^3+b^3}{a+b}$.
8. $\frac{a^3+b^3}{a-b}$.
9. $\frac{x^5-1}{x-1}$.
10. $\frac{6xy-7y^2}{xy}$.
11. $\frac{x^2-3x^3+7}{x^2-3}$.
12. $\frac{x^4+x^2y^2+2y^4}{x^2+xy+y^2}$.
13. $\frac{27x^3+27x^2y+9xy^2+y^3}{3x-y}$.
14. $\frac{a^3-6a^2b+12ab^2+8b^3}{a^2+4ab+4b^2}$.

Reduce to integral or mixed expressions:

15. $\frac{6a^3+10b^3}{a+2b}$.

16. $\frac{3x^2+5x-8}{x^2+x+1}$.

17. $\frac{x+1}{x-1}$.

18. $\frac{x}{x-1}$.

19. $\frac{x^2}{x-1}$.

20. $\frac{x^3}{x-1}$.

281. To add arithmetical fractions, we must change them to fractions having a common denominator.

In reducing fractions to a common denominator we use the following principle:

If the numerator and the denominator of a fraction are multiplied by the same number (not zero), the value of the fraction is not changed.

ORAL EXERCISE

282. *Change each of the following fractions to an equivalent fraction whose numerator or denominator is as indicated:*

1. $\frac{1}{2x}=\frac{(\ \ )}{14x}$.

2. $\frac{a}{b}=\frac{(\ \ )}{b^2}$.

3. $\frac{2x}{5y}=\frac{(\ \ )}{15xy}$.

4. $\frac{n}{2n^2x}=\frac{3mnx}{(\ \ )}$.

5. $ax=\frac{(\ \ )}{a^2x^2}$.

6. $\frac{a-1}{a+1}=\frac{(\ \ )}{a^2-1}$.

7. $\frac{a-1}{a+1}=\frac{a^2-1}{(\ \ )}$.

8. $\frac{1}{a}+\frac{1}{b}=\frac{b}{(\ \ )}+\frac{a}{(\ \ )}$.

9. $a+\frac{1}{a}=\frac{(\ \ )}{a}+\frac{1}{a}$.

10. $3+\frac{a}{b+c}=\frac{(\ \ )}{b+c}+\frac{a}{b+c}$.

11. $\frac{1}{a+b}+\frac{1}{a-b}=\frac{(\ \ )}{a^2-b^2}+\frac{(\ \ )}{a^2-b^2}$.

12. $\frac{1}{(-a)^3}+\frac{1}{ab}=\frac{(\ \ )}{a^3b}+\frac{(\ \ )}{a^3b}$.

13. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{(\ \ )}{abc}+\frac{(\ \ )}{abc}+\frac{(\ \ )}{abc}$.

14. What principle is involved in all these examples?

283. The **lowest common denominator** (L. C. D.) of two or more fractions is the lowest common multiple of their denominators.

Thus, the L. C. D. of $\frac{1}{a(a-b)}$ and $\frac{1}{b(a+b)}$ is $ab(a^2-b^2)$.

To change these fractions to a common denominator, both terms of the first fraction, $\frac{1}{a(a-b)}$, must be multiplied by $b(a+b)$ and of the second fraction, $\frac{1}{b(a+b)}$ by $a(a-b)$.

Thus, $\frac{1}{a(a-b)}=\frac{b(a+b)}{ab(a-b)(a+b)}$ and $\frac{1}{b(a+b)}=\frac{a(a-b)}{ab(a-b)(a+b)}$.

284. To change two or more fractions to equivalent fractions with a common denominator:

1. **Factor the denominators into prime factors.**
2. **Find the lowest common denominator (L. C. D.) of the fractions.**
3. **Multiply the numerator and the denominator of each fraction by all the factors of the common denominator except those factors that are in its own denominator.**

The multiplication in part 3 of the rule is generally indicated in the denominator and performed in the numerator.

Why does step 3 of the rule not change the value of the fraction?

Example

Change $\frac{1}{a^2-ab}$ and $\frac{1}{ab-b^2}$ to equivalent fractions having the L. C. D.

Solution.

$$a^2-ab=a(a-b).$$
$$ab-b^2=b(a-b).$$
$$\therefore \text{L. C. D.}=ab(a-b).$$

(The L. C. D. is the L. C. M. of the denominators.)

$$\frac{1}{a^2-ab}=\frac{1}{a(a-b)}=\frac{b}{ab(a-b)}.$$
$$\frac{1}{ab-b^2}=\frac{1}{b(a-b)}=\frac{a}{ab(a-b)}.$$

EXERCISE

285. *Change the fractions in each example to equivalent fractions having the L. C. D.*

1. $\frac{1}{a^2b}, \frac{1}{ab^2}.$

2. $\frac{a}{bc}, \frac{b}{ca}, \frac{c}{ab}.$

3. $\frac{x}{(-y)^2}, \frac{-x}{(-y)^3}.$

4. $\frac{1}{a^2-ab}, \frac{1}{ab+b^2}.$

5. $\frac{a}{x-y}, \frac{b}{x^2-y^2}, \frac{c}{y+x}.$

6. $\frac{a}{x^2-y^2}, \frac{b}{y-x}.$

HINT. In example 6 change the denominator of the second fraction to $x - y$. Explain.

7. $\frac{-1}{x^2+4x+3}, \frac{1}{x^2-1}, \frac{1}{x+1}.$

8. $\frac{1}{a^3-b^3}, \frac{1}{a^2+ab+b^2}, \frac{1}{a^2-b^2}.$

9. $\frac{x+y}{(y-z)(z-x)}, \frac{y+z}{(z-x)(x-y)}, \frac{z+x}{(z-y)(y-x)}.$

SOLUTION. First change the signs in these fractions, to avoid the repetition of a factor with opposite signs.

$$\frac{x+y}{(y-z)(z-x)}, \frac{y+z}{(z-x)(x-y)}, \frac{z+x}{(y-z)(x-y)}.$$

Why is the change made in the last fraction permissible?

$$\text{L. C. D.} = (x-y)(y-z)(z-x):$$

$$\frac{x+y}{(y-z)(z-x)} = \frac{x^2-y^2}{(x-y)(y-z)(z-x)},$$

$$\frac{y+z}{(z-x)(x-y)} = \frac{y^2-z^2}{(x-y)(y-z)(z-x)},$$

$$\frac{z+x}{(y-z)(x-y)} = \frac{z^2-x^2}{(x-y)(y-z)(z-x)}.$$

10. $\frac{1}{a^2-6a+9}, \frac{1}{9-a^2}, \frac{1}{a-3}.$

11. $\frac{1}{x^2-x-12}, \frac{1}{x^2+8x+15}, \frac{1}{x^2+x-20}.$

12. $\dfrac{3a+b}{6a^2-ab-5b^2}, \dfrac{a-b}{18a^2+21ab+5b^2}.$

13. $\dfrac{a+x}{a-x}, \dfrac{a-x}{a+x}, \dfrac{a^2+x^2}{a^2-x^2}, \dfrac{4ax}{a^2+x^2}.$

14. $\dfrac{5a}{6b}, \dfrac{16a^2-17ab}{12ab-6b^2}, \dfrac{b}{b-2a}.$

15. $\dfrac{3}{2a-3}, \dfrac{2}{3+2a}, \dfrac{2a+15}{4a^2+9}.$

16. $\dfrac{b-a}{c-2b}, \dfrac{2ab+c^2}{bc-2b^2-ac+2a^2}.$

ADDITION AND SUBTRACTION OF FRACTIONS

286. In arithmetic, only the same kind of units, or the same parts of units, can be added. Hence, if two or more fractions are to be added, unless they already have a common denominator, it is necessary to reduce them to equivalent fractions having a common denominator, before they can be added. Their sum is then found by adding the numerators of the fractions and dividing the result by the common denominator.

Thus, $\frac{1}{7}+\frac{2}{7}+\frac{3}{7}=\frac{1+2+3}{7}=\frac{6}{7}$

and $\frac{1}{3}+\frac{5}{12}+\frac{7}{15}=\frac{20}{60}+\frac{25}{60}+\frac{28}{60}=\frac{20+25+28}{60}=\frac{73}{60}=1\frac{13}{60}.$

287. The same principle applies when the difference of two fractions is to be found. The difference of two fractions having the same denominators is the difference of their numerators divided by their common denominator.

Thus, $\frac{7}{8}-\frac{5}{8}=\frac{7-5}{8}=\frac{2}{8}=\frac{1}{4},$

and $\frac{5}{12}-\frac{7}{18}=\frac{15}{36}-\frac{14}{36}=\frac{1}{36}.$

288. The sum or the difference of algebraic fractions can be found in the same way.

$$\text{1. } \frac{a}{m}+\frac{b}{m}+\frac{c}{m}=\frac{a+b+c}{m}.$$

$$\text{2. Add: } \frac{m}{2\,an^2}+\frac{n}{2\,bm^2}+\frac{p}{abmn}.$$

SOLUTION. L. C. D. $= 2\,abm^2n^2$.

$$\frac{m}{2\,an^2}=\frac{bm^2\cdot m}{2\,abm^2n^2}=\frac{bm^3}{2\,abm^2n^2},$$

$$\frac{n}{2\,bm^2}=\frac{an^2\cdot n}{2\,abm^2n^2}=\frac{an^3}{2\,abm^2n^2},$$

$$\frac{p}{abmn}=\frac{2\,mn\cdot p}{2\,abm^2n^2}=\frac{2\,mnp}{2\,abm^2n^2}.$$

$$\frac{m}{2\,an^2}+\frac{n}{2\,bm^2}+\frac{p}{abmn}=\frac{bm^3+an^3+2\,mnp}{2\,abm^2n^2}.$$

$$\text{3. } \frac{a}{a-b}-\frac{b}{a+b}.$$

SOLUTION.

L. C. D. $= (a-b)(a+b)$ or a^2-b^2.

$$\frac{a}{a-b}-\frac{b}{a+b}=\frac{a(a+b)}{a^2-b^2}-\frac{b(a-b)}{a^2-b^2}=\frac{(a^2+ab)-(ab-b^2)}{a^2-b^2}$$

$$=\frac{a^2+ab-ab+b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}.$$

CHECK. Put $a=2$, $b=1$.

Then $\frac{2}{2-1}-\frac{1}{2+1}=\frac{2^2+1^2}{2^2-1^2}$; or $2-\frac{1}{3}=\frac{5}{3}$; or $\frac{5}{3}=\frac{5}{3}$.

Why do we not put both a and $b=1$?

289. To add or subtract fractions:

1. Reduce the fractions, if necessary, to equivalent fractions having the L. C. D.

2. For the numerator of the result, write the numerators (in parentheses if they are polynomials), joined by the signs between the fractions; and for the denominator of the result write the L. C. D.

3. Remove the parentheses in the numerator, and collect the terms.

4. Reduce the result to its lowest terms.

EXAMPLE

$$\frac{1}{a-b}-\frac{1}{a+b}+\frac{a}{a^2-b^2}=?$$

SOLUTION. L. C. D. $=(a+b)(a-b)$.

$$\frac{1}{a-b}-\frac{1}{a+b}+\frac{a}{a^2-b^2}=\frac{a+b}{(a+b)(a-b)}-\frac{a-b}{(a+b)(a-b)}+\frac{a}{a^2-b^2}$$

$$=\frac{(a+b)-(a-b)+a}{(a+b)(a-b)}$$

$$=\frac{a+2b}{(a+b)(a-b)}.$$

CHECK. Put $a=2$, $b=1$.

Then, $\frac{1}{2-1}-\frac{1}{2+1}+\frac{2}{4-1}=\frac{2+2}{3\cdot 1}$;

or $1-\frac{1}{3}+\frac{2}{3}=\frac{4}{3}$; or $\frac{4}{3}=\frac{4}{3}$.

290. The fraction line may be thought of as having the same effect on the numerator of the reduced fraction as a parenthesis. In writing the numerators over the L. C. D. the parenthesis is used to indicate that the whole numerator is to be added or subtracted.

ORAL EXERCISE

291. *Perform the indicated operations:*

1. $\frac{x}{2}+\frac{x}{3}$.
2. $\frac{a}{2}+\frac{a}{5}$.
3. $\frac{x}{y}+\frac{5}{y}$.
4. $\frac{a}{b^2}+\frac{c}{b}$.
5. $\frac{1}{a+b}+\frac{1}{a-b}$.
6. $\frac{x}{2}-\frac{x}{3}$.
7. $\frac{a}{2}-\frac{a}{5}$.
8. $\frac{x}{y}-\frac{5}{y}$.
9. $\frac{a}{b^2}-\frac{c}{b^3}$.
10. $\frac{1}{a+b}-\frac{1}{a-b}$.

EXERCISE

292. *Perform the indicated operations:*

1. $\frac{a+2b}{4}+\frac{a-b}{2}$.

2. $\frac{2cd}{ab}-\frac{c}{a}$.

3. $\frac{1}{a+b}-\frac{2}{a-b}$.

4. $\frac{a-2b}{5}-\frac{2a-5b}{5}$.

5. $\frac{x}{4}-\frac{x-4}{3}+\frac{x-5}{6}$.

6. $\frac{2x+y}{3}+\frac{x+2y}{3}$.

7. $\frac{7}{12}x+\frac{10}{21}x-\frac{1}{7}x-\frac{1}{4}x$.

8. $\frac{2x}{15}+\frac{5y}{12}+\frac{x}{5}-\frac{3y}{4}$.

9. $\frac{5a}{x^2}-\frac{2b}{xy}+\frac{3}{y}$.

10. $\frac{4a^2}{9xy}+\frac{7a^2}{12xy}-\frac{17a^2}{18xy}$.

11. $\frac{1}{a(x+a)}+\frac{1}{x(x+a)}$.

12. $\frac{3a+b}{a-b}-\frac{2a+b}{a+b}$.

13. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

14. $\frac{5}{4x-4}-\frac{7}{6x+6}$.

15. $\frac{1}{1+x}+\frac{1}{1-x}-\frac{2}{1+x^2}$.

16. $\frac{8}{2x-3}+\frac{5}{3-2x}-\frac{3x-4}{2x^2-x-3}$.

17. $\frac{x-4}{2x-1}-\frac{3x-5}{x+2}+\frac{5x^2+9x+14}{2x^2+3x-2}$.

18. $\frac{1}{x-1}+\frac{4}{x-1}-\frac{8}{1+x}+\frac{3x+7}{x^2-1}$.

19. $\frac{1}{x+2a}+\frac{1}{x-2a}-\frac{8a^2}{x^3-4a^2x}$.

20. $\frac{3q}{p}-\frac{p^2+q^2}{pq}+\frac{p^2r+q^2r-3}{pqr}$.

293. By supplying the denominator 1, we may include the reduction of mixed expressions to fractional forms under the addition and subtraction of fractions.

EXERCISE

294. *Perform the indicated operations:*

1. $m+\dfrac{m^2}{m-1}=\dfrac{m}{1}+\dfrac{m^2}{m-1}$

$$=\frac{m(m-1)}{m-1}+\frac{m^2}{m-1}$$

$$=\frac{m^2-m+m^2}{m-1}$$

$$=\frac{2m^2-m}{m-1}.$$

2. $3+\dfrac{b^2}{a^2-b^2}+\dfrac{a}{a+b}.$

4. $\dfrac{p^2-q^2}{p-q}-(p-q).$

3. $\dfrac{2b^2}{a^2-b^2}-2+\dfrac{b}{a-b}.$

5. $\dfrac{2a+1}{a+1}+\dfrac{3a-1}{1-a}+\dfrac{a(a+3)}{a^2-1}.$

HINT. Change the second fraction to an equivalent fraction whose denominator is $a-1$.

6. $\dfrac{a}{a^2-b^2}+\dfrac{1}{b-a}.$

7. $\dfrac{4}{2x+x^2}-\dfrac{4-x^2}{2x}-\dfrac{x^2}{4+2x}.$

8. $\dfrac{2x^2+2x-19}{x+3}-\dfrac{2x^2-25}{3+x}.$

9. $\dfrac{(p-q)^2}{4pq}+\dfrac{(p+q)^2}{4pq}-\dfrac{p}{2q}-\dfrac{q}{2p}.$

10. $\left(3a-\dfrac{2a+3}{2}\right)-\left(a-\dfrac{a-4}{3}\right)-2a.$

HINT. Remove parentheses and collect terms before changing to a common denominator.

11. $\dfrac{3x+2y}{3x-3y}-\dfrac{2x-4y}{4x-4y}+\dfrac{x+12y}{6y-6x}.$

12. $\dfrac{3x+4y}{3x+y}+\dfrac{2x-5y}{3y+9x}-\dfrac{6x+5y}{12x+4y}.$

13. $\dfrac{x-2y}{x-6y}-\dfrac{2}{3}-\dfrac{6y-2x}{3x-18y}.$

Perform the indicated operations:

14. $\frac{3a^2+b^2}{4a^2-4b^2}+\frac{2a+b}{6b-6a}-\frac{a-2b}{6a+6b}$.

15. $\frac{3x}{4y-4x}+\frac{x^2+5xy}{4x^2-4y^2}-\frac{2x^2+xy}{2xy+2y^2}$.

16. $\frac{a^2-3ab}{2a^2+b^2}-\frac{6a^3+a^2b}{2a^2b+b^3}+\frac{3a^2}{ab-b^2}$.

17. $\frac{5x}{6y}-\frac{10x^2-17xy}{12xy+6y^2}+\frac{y}{2x-y}$.

18. $\frac{9x^2+2xy}{12xy-6y^2}-\frac{3x^2}{4xy+2y^2}-\frac{11x^2+xy}{12x^2-3y^2}$.

19. $\frac{x}{2x-y}-\frac{4xy}{4x^2+4xy-3y^2}$.

20. $21a+11b-\frac{(7a+6b)^2}{4b}$.

21. $\frac{a^2}{ax+x^2}+\frac{x^2}{a^2-ax}-\frac{a^2}{x^2-a^2}$.

22. $\frac{y}{y+5x}+\frac{50x^2}{25x^2-y^2}+\frac{5x}{y-5x}$.

23 $\frac{1}{(a+x)^2}+\frac{1}{a^2-x^2}-\frac{1}{(x-a)^2}$.

24. $\frac{5a-2}{24a-6}-\frac{3a+1}{36a+9}-\frac{2a^2-5a+7}{48a^2-3}$.

25. $\frac{1}{2x}+\frac{1}{8-4x}-\frac{1}{8+4x}-\frac{1}{x^2-4}$.

26. $\frac{1}{2x+x^2}-\frac{1}{2x}+\frac{1}{2x+4}-\frac{1}{x-2}$.

27. $\frac{x^2+xy+y^2}{x^4+4y^4}-\frac{1}{x^2+2xy+2y^2}$.

28. $\frac{x+1}{x^2+x+1}+\frac{x-1}{x^2-x+1}+\frac{2}{x^4+x^2+1}$.

29. $\dfrac{a^2-2a+3}{a^3+1}+\dfrac{a-3}{a^2-a+1}-\dfrac{1}{a+1}$.

30. $\dfrac{2x}{6a^2+a-2}-\dfrac{x}{3a^2-a-2}$.

31. $\dfrac{7x+3}{2x^2-8}+\dfrac{3}{x-2}-\dfrac{4}{2-x}$.

32. $\dfrac{2}{a-b}-\dfrac{2}{a+b}-\dfrac{3b}{a^2+ab-2b^2}$.

33. $\dfrac{x^2-5}{x^2-12x+27}-\dfrac{x-2}{x^2-9}-3$.

34. $\dfrac{1+3x}{5+7x}-\dfrac{9-11x}{5-7x}-14\dfrac{(2x-3)^2}{25-49x^2}$.

35. $\dfrac{x}{2x+2y}-\dfrac{1}{2}\left(\dfrac{7x}{y-x}-\dfrac{5x^2+2xy}{x^2-y^2}+\dfrac{2xy}{xy+y^2}\right)$.

36. $1-\left(\dfrac{x-y}{x+y}-\dfrac{2y^2}{x^2-y^2}-\dfrac{2xy}{x^2+y^2}\right)$.

37. $\dfrac{5}{x^2+3x+2}+\dfrac{5}{x^2+5x+6}+\dfrac{5}{x^2+4x+3}$.

38. $\dfrac{1}{2x^2-7x+6}+\dfrac{1}{2x^2-x-6}+\dfrac{5}{9-4x^2}$.

39. $\dfrac{-2}{(m-1)^2}+\dfrac{m+1}{m^2-2m+1}+\dfrac{1}{1-m}$.

40. $\dfrac{a^2}{ab+b^2}+\dfrac{b^2}{ab+a^2}-\dfrac{a^2+b^2}{ab}$.

41. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

42. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$.

43. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

44. $\dfrac{a+b}{(c-a)(c-b)}+\dfrac{b+c}{(a-b)(a-c)}-\dfrac{c+a}{(b-c)(b-a)}$.

MULTIPLICATION OF FRACTIONS

295. The product of two or more arithmetical fractions is the product of their numerators divided by the product of their denominators. The result should be reduced to lowest terms.

Thus, 1. $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$. 2. $\frac{\cancel{3}}{\cancel{4}} \times \frac{\overset{3}{\cancel{12}}}{\underset{7}{\cancel{21}}} = \frac{3}{7}$.

296. The product of two or more algebraic fractions can be found in the same way.

$$1.\quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

To shorten the work, we usually cancel all factors common to the numerators and denominators before multiplying.

$$2.\quad \frac{\cancel{2}\,\overset{a}{\cancel{a^2}}}{\cancel{3}\,\cancel{b}c} \times \frac{\overset{4}{\cancel{12}}\,\cancel{b}x}{\underset{5}{\cancel{10}}\,\cancel{a}c} = \frac{4\,ax}{5\,c^2}.$$

$$3.\quad \frac{a^2 - x^2}{a + b} \cdot \frac{a^2 - b^2}{ax + x^2} \cdot \left(a + \frac{ax}{a - x}\right)$$

$$= \frac{\cancel{(a - x)}\cancel{(a + x)}}{\cancel{a + b}} \cdot \frac{(a - b)\cancel{(a + b)}}{x\cancel{(a + x)}} \cdot \frac{a^2}{\cancel{a - x}}$$

$$= \frac{a^2(a - b)}{x}.$$

ORAL EXERCISE

297. *Multiply the following:*

1. $\frac{2}{3} \cdot 2$.
2. $\frac{2}{3} \cdot \frac{1}{2}$.
3. $\frac{2}{3} \cdot 3$.
4. $\frac{2}{3}(-\frac{1}{3})$.
5. $\frac{a}{b} \cdot c$.
6. $\frac{a^2}{b^2} \cdot \frac{-c}{b}$.
7. $\left(-\frac{a^2}{b^2}\right)\left(-\frac{b}{a}\right)$.
8. $\frac{a - b}{a + b} \cdot \frac{a^2 - b^2}{2}$.
9. $\frac{a + b}{a - b} \cdot \frac{1}{a^2 - b^2}$.
10. $\left(-\frac{1}{a}\right) \cdot \left(\frac{a^2}{-2}\right)$.

11. $\left(-\frac{1}{a}\right)^2 \cdot (-a).$

12. $\left(\frac{a-b}{a+b}\right)^2 \cdot \frac{a+b}{a^2-b^2}.$

13. $\left(\frac{a}{b}-\frac{b}{c}\right)^2.$

14. $\frac{2a^8x^7}{3b^8} \cdot \frac{3b^4}{4c^4x}.$

15. What is the numerator of the product when all the factors of the numerator are canceled?

16. What is the nature of the product when all the factors of the denominator are canceled?

17. What is the product if all factors of both numerator and denominator are canceled?

298. **To multiply expressions some or all of which contain fractions:**

1. Reduce all integral and mixed expressions to the fractional form.

2. Factor all polynomials that can be factored in the numerators and denominators.

3. Cancel all factors common to the numerators and denominators.

4. Multiply together the remaining factors in the numerators for the numerator of the result; and the remaining factors in the denominator for the denominator of the result.

EXAMPLES

1. $$\frac{\cancel{5}\,\cancel{a^2}\cancel{b}}{\cancel{3}\,\cancel{c}\cancel{d}} \cdot \frac{\cancel{4}\,\cancel{b^2}\cancel{c}}{\underset{2}{\cancel{10}}\,\cancel{a^2}} \cdot \frac{\overset{3}{\cancel{9}}\,c^2\cancel{d}}{\underset{4}{\cancel{16}}\,\cancel{b^3}} = \frac{3c^2}{8}.$$

2. $$\left(\frac{1}{x^2}-1\right)\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\left(\frac{x^3}{x^3-1}\right)$$

$$=\frac{1-x^2}{x^2}\cdot\frac{x^2+x+1}{x^2}\cdot\frac{x^3}{x^3-1}$$ (Changing to fractional form.)

$$=\frac{(1+x)\overset{-1}{\cancel{(1-x)}}}{\underset{x}{\cancel{x^2}}}\cdot\frac{\cancel{x^2+x+1}}{\cancel{x^2}}\cdot\frac{\overset{\cancel{x}}{\cancel{x^3}}}{\cancel{(x-1)}\cancel{(x^2+x+1)}}$$ (Factoring and canceling.)

$$=\frac{-1-x}{x} \text{ or } -\frac{1+x}{x}.$$

EXERCISE

299. *Perform the indicated operations:*

1. $\frac{4}{3} \times \frac{5}{7} \times \frac{3}{10}$.

2. $7\frac{5}{8} \times \frac{8}{61}$.

3. $(\frac{3}{4} + \frac{1}{8})(\frac{1}{14} + \frac{5}{42})$.

4. $5\,mn \cdot \dfrac{4\,pq}{5\,mn}$.

5. $13\,m^2n^2 \cdot \dfrac{7\,p^2m^2}{8\,n^2}$.

6. $\dfrac{a^2}{bc} \cdot \dfrac{b^2}{ac} \cdot \dfrac{c^2}{ab}$.

7. $\left(-\dfrac{2\,a}{3\,b}\right)\left(-\dfrac{9\,b}{4\,a}\right)\left(\dfrac{a}{b}\right)$.

8. $\dfrac{a^2b}{x^2y} \cdot \dfrac{b^2c}{y^3z} \cdot \dfrac{x^2y^2z}{abc}$.

9. $\dfrac{(3\,a^4b)^2}{(5\,cd)^3} \cdot \dfrac{(5\,c)^4}{(6\,a)^5} \cdot \dfrac{(4\,b)^6}{40\,a^3b^5c}$.

10. $\left(-\dfrac{3\,a^2x^2}{4\,bc^3}\right)^2 \cdot \left(-\dfrac{2\,a^2c^4}{b^2x}\right)^3$.

11. $\dfrac{a+b}{a-b} \times \left(-\dfrac{a^2-b^2}{a^2+2\,ab+b^2}\right)$.

12. $\dfrac{u^2+uv+v^2}{u^2-v^2} \cdot \dfrac{(u-v)^2}{u^2-uv+v^2}$.

13. $\dfrac{2\,ab}{4\,a^2+12\,ab+9\,b^2} \cdot \dfrac{2\,a+3\,b}{a-b}$.

14. $\dfrac{x+5}{x^2-x-12} \cdot \dfrac{x^2+8\,x+15}{x-4}$.

15. $\left(\dfrac{3}{2}a - \dfrac{2}{5}b\right) \cdot \dfrac{10}{15\,a-4\,b}$.

16. $\left(\dfrac{25}{4}a^2 - b^2\right) \cdot \dfrac{2\,x}{5\,a-2\,b}$.

17. $\dfrac{a^3-b^3}{(a+b)^2} \cdot \dfrac{1}{2} \cdot \dfrac{a+b}{a^2-b^2}$.

18. $\dfrac{2}{3} \cdot \dfrac{m^2-p^2}{(m-p)^2} \cdot \dfrac{m^3-p^3}{m+p}$.

19. $\left(\dfrac{1}{4}a - \dfrac{1}{2}b\right)^2 \cdot \dfrac{1}{a-2\,b}$.

20. $\dfrac{1}{a^2-b^2} \cdot (a-b)$.

21. $\left(\dfrac{a}{b} - 1\right)\left(\dfrac{b}{a} + 1\right)$.

22. $\dfrac{1}{x-2\,y}\left(\dfrac{1}{2\,y} - \dfrac{1}{x}\right)$.

23. $\left(\dfrac{a}{4} - \dfrac{b}{5}\right)\left(\dfrac{a}{4} + \dfrac{b}{5}\right)$.

24. $\left(\dfrac{x}{y} - \dfrac{y}{x}\right)^2 + 2$.

25. $\dfrac{1}{x-y} \cdot \left(\dfrac{1}{x} - \dfrac{1}{y}\right)$.

26. $\dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)\left(\dfrac{a}{a-b}\right)$.

27. $\left(\dfrac{1}{2}a - 3\right)\left(\dfrac{2}{a^2-12\,a+36}\right)$.

28. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2} \cdot \dfrac{a^3+b^3}{a^3-b^3}$.

29. $\left(\frac{x^2+y^2}{y}-x\right)\frac{y}{x-y}\cdot\frac{x^2-y^2}{x^3+y^3}.$

30. $\left(\frac{x+y}{x}+\frac{x-y}{y}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$

31. $\left(\frac{3}{2}x-y\right)^2\cdot\left(\frac{1}{9x^2-4y^2}\right).$

32. $\left(\frac{3x}{2y}-\frac{2y}{3x}\right)\left(-\frac{xy^2-x^2y}{3x^2+xy-2y^2}\right).$

33. $\left(\frac{1}{a}+\frac{1}{b}\right)\frac{x^2-y^2}{a+b}\times\left(-\frac{b}{x-y}\right).$

34. $\left(x-\frac{y^2}{x}\right)\left(y+\frac{x^2}{y}\right)\times\left(\frac{y}{x^4-y^4}\right).$

35. $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\left(\frac{1}{x+y}\right)^2.$

36. $\frac{a^4-b^4}{a^3-b^3}\cdot\frac{a^2-b^2}{a^3+b^3}\cdot\frac{a^6-b^6}{a^2+b^2}.$

37. $\frac{a^2-(b-c)^2}{c^2-(b-a)^2}\cdot\frac{(a-b)^2-c^2}{(b-c)^2-a^2}.$

38. $\left(\frac{-3}{a-b}\right)^2\left(\frac{b^2-a^2}{-27}\right).$

39. $\left(-\frac{1}{2}\right)^2\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)\left(x-\frac{1}{x}\right)^2.$

40. $\frac{2x^2-8x+6}{x^2-5x+4}\cdot\frac{x^2-9x+20}{x^2-10x+21}\cdot\frac{x^2-7x}{2x^2-7x}.$

41. $\frac{4x}{(x^2-1)^2}\left(x-\frac{1}{x}\right)^2.$

42. $\frac{9x^2-6x}{4x^2-8x+3}\cdot\frac{2x^2+3x-9}{6x^2-7x+2}\cdot\frac{2x^2+13x-7}{2x^2+6x}.$

43. $\left(2a-\frac{b}{3}\right)^3\left(\frac{3}{6a-b}\right)^2.$

44. $\frac{1-x^2}{1+y}\cdot\frac{y^2-1}{x+x^2}\cdot\left(1+\frac{x}{1-x}\right).$

DIVISION OF FRACTIONS

300. The **reciprocal** of a number is 1 divided by the number.

Thus, $\frac{1}{2}$ is the reciprocal of 2; $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$, for $1 \div \frac{2}{3} = \frac{3}{2}$. The reciprocal of a fraction is evidently the fraction inverted.

301. **Division** has been defined as the process of finding one of two factors when their product and the other factor are given.

Thus, $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$, hence $\frac{15}{28} \div \frac{5}{7} = \frac{3}{4}$ and $\frac{15}{28} \div \frac{3}{4} = \frac{5}{7}$.

1. Divide $\frac{15}{28}$ by $\frac{5}{7}$.

   1. Let $\frac{15}{28} \div \frac{5}{7} = q$. (A quotient.)
   2. Then $\frac{15}{28} = \frac{5}{7} \times q$. (By definition of division.)
   3. $\therefore \frac{7}{5} \times \frac{15}{28} = \frac{7}{5} \times \frac{5}{7} \cdot q = q$. (Why?)
   4. $\frac{15}{28} \div \frac{5}{7} = \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{28}}} \times \frac{\cancel{7}}{\cancel{5}} = \frac{3}{4}$.

   (From 1 and 3, since each $= q$.)

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

   1. Let $\frac{a}{b} \div \frac{c}{d} = q$.
   2. Then $\frac{a}{b} = \frac{c}{d} \times q$. (Why?)
   3. $\therefore \frac{d}{c} \times \frac{a}{b} = \frac{d}{c} \times \frac{c}{d} \times q = q$. (Why?)
   4. $\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b} = \frac{ad}{bc}$.

   (From 1 and 3.)

302. Step 4 in each of the examples of § 301 gives us the usual rule:

To divide one fraction by another, multiply the dividend by the reciprocal of the divisor.

In case the dividend or the divisor is an integral or a mixed expression, or is the sum of two or more fractions, it must be changed to a single fraction before the rule is applied.

This method of division may be used also in dividing a fraction by an integer, since any integer can be written in the form of a fraction by supplying 1 as a denominator.

ORAL EXERCISE

303. *Find the quotients:*

1. $\frac{1}{2} \div \frac{1}{3}$.
2. $\frac{1}{2} \div 2$.
3. $(-2) \div \frac{1}{2}$.
4. $\frac{a}{b} \div \frac{b}{a}$.
5. $\frac{a}{b} \div b$.
6. $\frac{a^2}{b^2} \div \frac{ac}{bd}$.
7. $\left(-\frac{b}{2}\right) \div \left(-\frac{c}{3}\right)$.
8. $\left(-\frac{x}{y}\right)^2 \div \frac{x^2}{y^2}$.
9. $\left(-\frac{x}{y}\right)^3 \div \left(\frac{y}{x}\right)^2$.
10. $(-a)^2 \div \frac{1}{2a}$.
11. $1 \div \frac{a}{b}$.
12. $1 \div \left(-\frac{1}{x^2}\right)$.
13. $\frac{b}{c} \div c$.
14. $ab \div \frac{1}{ab}$.
15. $\frac{2bc}{3a} \div 5c$.
16. $a \div \frac{1}{b}$.
17. $abc \div \frac{ab}{cd}$.
18. $1 \div \frac{m}{n}$.
19. $\frac{15ab}{cd} \div \frac{1}{cd}$.
20. What is the reciprocal of $\frac{a}{b}$? of $\frac{a+b}{x}$? of x^2?
21. Is $\frac{b}{a} + \frac{d}{c}$ the reciprocal of $\frac{a}{b} + \frac{c}{d}$? If not, give the correct reciprocal.

EXERCISE

304. 1. What is the reciprocal of $\frac{a}{b}+1$? of $a+b$? of $x^2-\frac{1}{x}$?

2. Find the reciprocal of $3\frac{1}{5}$; of $a+\frac{1}{a}$.

3. Find the reciprocal of $\frac{a}{x}+\frac{b}{y}$; of $a+2+\frac{1}{a}$.

Find the quotients:

4. $3\frac{1}{5}\div 1\frac{7}{9}$.

5. $\frac{a}{3}\div\frac{a^2}{2}$.

6. $\frac{2a^2}{6b^3}\div\frac{18a^3}{b^4}$.

7. $\frac{2a^2}{bc}\div\frac{3ab}{c^2}$.

8. $\frac{a^3}{b^3}\times\frac{b^3}{c^3}\div\frac{a^3}{c^3}$.

9. $\frac{27x^2y}{25z^3}\div\frac{12x^3}{5yz^2}$.

10. $\frac{1}{3}$ of $\frac{x^3}{y^3}\div 3x^2$.

11. $\left(\frac{4abc}{9efg}\cdot\frac{3e}{4a}\right)\div\left(\frac{b}{2f}\cdot\frac{c}{g}\right)$.

12. Divide $\frac{x^2-y^2}{xy}$ by the reciprocal of $\frac{xy}{x+y}$.

13. The product is $\frac{3b}{4c}$ and one of the factors is $\frac{5a^2b}{8c^2d}$. Find the other factor.

14. The quotient is $\frac{2a}{3b}$ and the dividend is $\frac{14a^2}{15bc}$. What is the divisor?

15. $\left(a-\frac{1}{a}\right)\div\left(1-\frac{1}{a^2}\right)$.

16. $\frac{a^2-b^2}{a^3+b^3}\div\frac{a-b}{(a+b)^2}$.

17. $\frac{x^2-2x-3}{x^2-4x}\div\frac{x-3}{x^2-16}$.

18. $(\frac{4}{9}a-\frac{1}{3}b)\div(4a-3b)$.

19. $(x^2-6x-7)\div(x-7)$.

20. $(a^2-a+\frac{1}{4})\div\left(\frac{a}{5}-\frac{1}{10}\right)$.

21. $4-(-\frac{1}{9})-5\div(-\frac{5}{6})-7\div(-\frac{14}{15})$.

22. $\frac{3x+5}{6x-9}\div\frac{9x^2+30x+25}{4x^2-9}$.

23. $\left(a-\frac{25b^2}{a}\right)\div\left(1+\frac{5b}{a}\right).$

24. $\left(\frac{3a-b}{6}-\frac{5a-3b}{12}\right)\div(a+b).$

25. $\left(\frac{2a}{3b}-\frac{3b}{2a}\right)\div\left(\frac{4a^2+9b^2}{12ab}-1\right).$

NOTE. In a series of multiplications and divisions, the divisions may be changed to multiplications by inverting the fractions immediately preceded by the sign of division.

Thus, $$\frac{a}{b}\div\frac{c}{d}\times\frac{e}{f}=\frac{a}{b}\cdot\frac{d}{c}\cdot\frac{e}{f}=\frac{ade}{bcf}.$$

26. $\frac{3a^2b}{5cd^2}\div\frac{2a}{5d}\times\frac{6c}{a}.$

27. $\left(x^2+\frac{1}{x^2}+2\right)\div\left(x+\frac{1}{x}\right).$

28. By what must $\frac{x^2-3x+2}{x^2-7x+12}$ be divided to get as quotient $\frac{x-2}{x-4}$?

29. By what must $\frac{x^2-5x+6}{2x^2+3x-2}$ be multiplied to get as product $\frac{x-2}{x+2}$?

30. What is the dividend if both divisor and quotient are $\frac{x^2+2}{x-1}$?

31. $\frac{a-3b}{a^2-2ab-15b^2}\div\frac{a+3b}{a^2-8ab+15b^2}.$

32. $\frac{1}{x^2-4x+4}\div\frac{1}{8-x^3}.$

33. $\left(x-3+\frac{5x}{2x-6}\right)\cdot\frac{3x}{2}\div\left(2x-1+\frac{15}{x-3}\right).$

34. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\div\left(\frac{a}{b}-\frac{b}{c}+\frac{c}{a}\right).$

35. $\frac{a+b}{a-b}\div\frac{a^2+2ab+b^2}{a^2-b^2}.$

36. $\left(\frac{x^2y^4}{3}-23y^2+\frac{12}{x^2}\right)\div\left(\frac{2}{x^2y}-\frac{3}{x}+\frac{y}{3}\right).$

37. Find the reciprocal of $\frac{m}{n}-\frac{p}{q}.$

COMPLEX FRACTIONS

305. A fraction whose numerator, or denominator, or both, contain fractions or mixed expressions, is a **complex fraction**.

Thus, $\dfrac{\frac{2}{3}}{\frac{5}{7}}$, $\dfrac{a+\frac{1}{a}}{b}$ and $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ are complex fractions.

306. Inasmuch as a fraction is an indicated division, a complex fraction may evidently be simplified by dividing its numerator by its denominator. In the above examples we simplify as follows:

1. $\dfrac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}.$

2. $$\frac{a+\frac{1}{a}}{b} = \frac{a^2+1}{a} \div b = \frac{a^2+1}{a} \cdot \frac{1}{b} = \frac{a^2+1}{ab}.$$

3. $$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}} = \frac{b+a}{ab} \div \frac{b-a}{ab} = \frac{b+a}{ab} \cdot \frac{ab}{b-a} = \frac{b+a}{b-a}.$$

307. To simplify a complex fraction, first change the numerator and the denominator each to a single fraction and then multiply the numerator by the reciprocal of the denominator.

EXAMPLE

$$\frac{\frac{1}{x^2}-\frac{2}{x}+1}{x+1-\frac{2}{x}} = \frac{\frac{1-2x+x^2}{x^2}}{\frac{x^2+x-2}{x}}$$

$$= \frac{\overset{x-1}{\cancel{(x-1)^2}}}{\underset{x}{\cancel{x^2}}} \times \frac{\cancel{x}}{\cancel{(x-1)}(x+2)} = \frac{x-1}{x(x+2)}.$$

Care should be taken in such complex fractions as $\frac{\frac{a}{b}}{c}$ to indicate which line separates the numerator from the denominator. This may be done by using a heavier line.

$$\frac{\frac{a}{b}}{c}=\frac{a}{bc},\text{ while }\frac{a}{\frac{b}{c}}=\frac{ac}{b}.$$

EXERCISE

308. *Simplify the following:*

1. $\dfrac{1-\frac{1}{3}}{2+\frac{1}{2}}$.

2. $\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$.

3. $\dfrac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}$.

4. $1+\dfrac{x}{1+x+\frac{2x^2}{1-x}}$.

5. $\dfrac{1+\frac{1}{1-a}}{1-\frac{1}{1+a}}$.

6. $\dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$.

7. $\dfrac{\frac{a}{1+a}+\frac{1-a}{a}}{\frac{a}{1+a}-\frac{1-a}{a}}$.

8. $\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{2(a^2+b^2)}$.

9. $\dfrac{\frac{ab-2cd}{p^2-q^2}}{\frac{a^2b^2-4c^2d^2}{p^3-q^3}}$.

10. $\dfrac{\frac{x^2+xy+y^2}{\frac{2}{3}x-y}}{\frac{x^3-y^3}{4x^2-9y^2}}$.

11. $\dfrac{1+p-\frac{3p^2}{1-p}}{\frac{2}{p^2}+\frac{1}{p^3}}$.

12. $\dfrac{1+\frac{2mn}{m^2+n^2}}{\frac{m+n}{3}}$.

13. $\dfrac{\frac{1}{y^2}-\frac{2}{xy}-\frac{3}{x^2}}{\frac{1}{y}-\frac{3}{x}}$.

Simplify the following:

14. $\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{2}{a+b}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\dfrac{a}{b}+2+\dfrac{b}{a}}.$

15. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}.$

16. Find the value of the fraction in exercise 15, when $a=2x$.

17. $\dfrac{\dfrac{1}{p-q}+\dfrac{1}{p+q}}{\dfrac{1}{p+q}-\dfrac{1}{p-q}}.$

18. $\dfrac{\dfrac{a}{b}+\dfrac{x}{y}}{1-\dfrac{a}{b}\cdot\dfrac{x}{y}}.$

19. $\dfrac{3}{4+\dfrac{1}{5+\frac{1}{2}}}.$

20. $\dfrac{a}{b+\dfrac{1}{c+\dfrac{1}{d}}}.$

21. $\dfrac{\left(\dfrac{a}{b}\right)^3-1}{\left(\dfrac{a}{b}\right)^2-1}.$

22. $\dfrac{2a-\dfrac{5b}{a}+\dfrac{3b^2}{a^3}}{1-\dfrac{3b}{2a^2}}.$

23. $\dfrac{\dfrac{x^2}{y}-\dfrac{3y}{2}-\dfrac{y^3}{x^2}}{\dfrac{x}{y}-\dfrac{2y}{x}}.$

24. $\dfrac{\dfrac{x+2}{2}+\dfrac{2}{x+2}}{\dfrac{1}{x}+\dfrac{1}{2}}.$

25. $\dfrac{\dfrac{2xy}{3}-\dfrac{3y^2}{4}}{\dfrac{5x}{6}-\dfrac{8y}{9}}\div\dfrac{\dfrac{2}{3y}-\dfrac{3}{4x}}{\dfrac{5}{6yz}-\dfrac{8}{9xz}}.$

XI. EQUATIONS CONTAINING FRACTIONS

309. To solve an equation containing fractions, we first change the equation to one not containing fractions. This process is called **clearing of fractions.** The process will be understood by an example.

Solve $\frac{1}{7x} + \frac{3}{x} = \frac{5}{2x} + \frac{9}{28}$.

SOLUTION. $$\frac{1}{7x} + \frac{3}{x} = \frac{5}{2x} + \frac{9}{28}.$$

Multiply both members of this equation by $28x$, the L.C.D. of the fractions. The resulting equation will not contain any fractions; that is, it will be cleared of fractions.

$$\overset{4}{\cancel{28}}\cancel{x}\cdot\frac{1}{\cancel{7}\cancel{x}} + 28\cancel{x}\cdot\frac{3}{\cancel{x}} = \overset{14}{\cancel{28}}\cancel{x}\cdot\frac{5}{\cancel{2}\cancel{x}} + \cancel{28}x\cdot\frac{9}{\cancel{28}}. \quad \text{(See § 13, (c).)}$$

$$4 + 84 = 70 + 9x. \quad \text{(Why?)}$$
$$-9x = 70 - 4 - 84. \quad \text{(Why?)}$$
$$-9x = -18. \quad \text{(Why?)}$$
$$x = 2. \quad \text{(Why?)}$$

CHECK. Substitute 2 for x.

$$\tfrac{1}{14} + \tfrac{3}{2} = \tfrac{5}{4} + \tfrac{9}{28}.$$
$$\tfrac{22}{14} = \tfrac{44}{28} \text{ or } \tfrac{22}{14}.$$

In practice the student will generally find it possible to omit the second equation, canceling mentally.

310. To solve fractional equations:

1. Clear the equation of fractions by multiplying every term of both members by the L. C. D. of the fractions.

2. Solve the resulting integral equation in the usual manner.

ORAL EXERCISE

311. 1. Solve $\frac{2}{x}=7$; $\frac{2}{x}+\frac{1}{x}=4$; $\frac{3}{x}=\frac{12}{x}-4$.

2. Solve $\frac{3}{x}-\frac{2}{x}=5$; $\frac{4}{3x}=8$; $\frac{1}{2}+\frac{1}{4}=\frac{1}{x}$.

3. How do you multiply a fraction by an integer?

4. Give in order the principles used in solving the equation in § 309.

5. Explain transposing.

6. How do you find the L.C.D. of several fractions?

7. Why does multiplying the several fractions of an equation by their lowest common denominator clear the equation of fractions?

EXERCISE

312. *Solve the following equations:*

1. $\frac{x}{2}+\frac{x}{3}=\frac{5}{6}$.
2. $\frac{5}{m}+3=18$.
3. $\frac{1}{3p}=7$.
4. $\frac{1}{2}x+\frac{1}{3}x=10$.
5. $\frac{1}{2x}=.1$.
6. $\frac{x}{2}-\frac{4x}{9}=5$.
7. $\frac{10}{t}=7$.
8. $\frac{2c}{5}-\frac{3c}{2}=\frac{c}{2}-32$.
9. $\frac{1}{2}+\frac{1}{3}=\frac{1}{x}$.
10. $2\frac{1}{2}x=5$.
11. $\frac{40}{-a}=-5$.
12. $3x-\frac{2}{3}x=6x-22$.
13. $\frac{9}{x}=12+\frac{1}{x}$.
14. $\frac{a}{2}-\frac{a}{3}+\frac{a}{4}-\frac{a}{12}=32$.
15. $\frac{3}{y}=15-\frac{1}{y}$.
16. $.5y-2.25y=3.5$.

SOLUTION
$$.5y-2.25y=3.5.$$
$$-1.75y=3.5.$$
$$y=-2.$$

17. $\frac{8}{-x}+5=13.$

18. $.1x-.1=.5x-5.1.$

19. $\frac{13}{x}-5=21.$

20. $4.5x-11.5=35+1.4x.$

21. $-\frac{12}{r}+13=25.$

22. $.25m+3.85=m-.4m.$

313. The following examples will show the arrangement of the work in the solution of fractional equations:

1. Solve $\frac{x-2}{5}-\frac{x-3}{4}=\frac{x-7}{2}.$

SOLUTION. L. C. D. = 20.

$4(x-2)-5(x-3)=10(x-7)$, (Multiplying both members of the equation by the L. C. D.)

$4x-8-5x+15=10x-70.$ (Why?)

$4x-5x-10x=-70+8-15.$ (Why?)

$-11x=-77.$ (Why?)

$x=7.$ (Why?)

CHECK. $\frac{7-2}{5}-\frac{7-3}{4}=\frac{7-7}{2}$ or $1-1=0.$

2. Solve $\frac{8-x}{6}+\frac{3x-5}{3}=\frac{x+6}{2}-\frac{x}{3}.$

SOLUTION. L. C. D. = 6.

$(8-x)+2(3x-5)=3(x+6)-2x.$ (Why?)

$8-x+6x-10=3x+18-2x.$

$-x+6x-3x+2x=18-8+10.$

$4x=20.$

$x=5.$

CHECK. $\frac{8-5}{6}+\frac{15-5}{3}=\frac{5+6}{2}-\frac{5}{3}$, or $\frac{3}{6}+\frac{10}{3}=\frac{11}{2}-\frac{5}{3}$, or $\frac{23}{6}=\frac{23}{6}.$

NOTE. Treat the numerator of a fraction exactly as if written in a parenthesis, as is done in the second step of each example. Do this in every case and use great care in removing the parentheses. See §§ 100 and 102.

EXERCISE

314. *Solve the following equations:*

1. $\frac{2x}{3} - \frac{5x}{9} = 1.$ 2. $\frac{4}{x} = \frac{5}{x} - 1.$ 3. $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 1.$

NOTE. It will generally shorten the work of solving an equation if all fractions having the same denominator are combined before the equation is cleared of fractions.

4. $\frac{4}{5x} - \frac{2}{5x} = \frac{1}{10}.$ 5. $\frac{42}{x} - \frac{1}{x} + \frac{1}{3x} = 20\frac{2}{3}.$

6. $\frac{1}{9x} + \frac{1}{12x} + \frac{1}{8x} + \frac{1}{24x} - \frac{13}{72} = 0.$

7. $\frac{1}{8}x + \frac{1}{6}x - 6 = 1.$

8. $\frac{8(2+5r)}{9} = \frac{9r+2}{2}.$

9. $8x - \frac{x}{4} = \frac{x}{10} + 153.$

10. $\frac{7}{6}x - (\frac{1}{4} + \frac{17}{28}) = \frac{11}{36}(3x+1).$

11. $\frac{1}{2}(5x+1) - \frac{1}{3}(4x+5) = \frac{1}{4}(3x-1) - \frac{1}{20}(6x+4).$

12. $30(2-t) - \frac{t}{2} = \frac{1-5t}{16} - \frac{7}{16}.$

13. $\frac{7x-5}{16} - \frac{1+x}{2} + \frac{x+3}{6} = \frac{7}{9}(x-3).$

14. $\frac{7}{x} + \frac{1}{3} = \frac{23-x}{3x} + \frac{7}{12} - \frac{1}{4x}.$

15. $\frac{8}{9} \cdot \frac{x-4}{11} = \frac{-x}{3} - \left(\frac{2-16x}{33} - \frac{2-x}{9}\right).$

16. $\left(14\frac{2}{7} + \frac{3x+1}{28} - \frac{x+5}{14}\right)7 = 9x + 19.$

17. $(x - \frac{1}{2})^2 - (x + \frac{1}{2})^2 = x.$

18. $\frac{1}{4}x - \left(\frac{x}{4} - \frac{4}{3}\right)^2 = \frac{17}{9} - \frac{x^2}{16}.$

19. $2x - \left(x - \frac{x-1}{3}\right) = \frac{5x}{4}.$

20. $3x - \frac{3x-19}{2} - 8 = \frac{23-x}{4} + \frac{5x-38}{3} + 10.$

21. $\frac{4x-1}{3}=\frac{3x+5}{4}-\left(\frac{x-4}{6}-\frac{3}{4}\right).$

22. $\frac{x-3}{7}-\frac{x-25}{5}=7-\frac{2+x}{5}.$

23. $\frac{7x-2}{3}-\frac{4}{3}(x+3)=\frac{3}{2}(x+2)-6.$

24. $1-\left(\frac{3x-1}{4}+\frac{2x+1}{3}\right)=\frac{5-2x}{3}-\frac{7x-1}{8}.$

25. $3x-\frac{2x+5}{7}=16-\frac{7x+19}{2}-\frac{2x+1}{3}.$

26. $\frac{16}{15}(x-4)=\frac{1}{6}(3x+5).$

27. $\frac{5x-.4}{.3}+\frac{1.3-3x}{2}=\frac{1.8-8x}{1.2}.$

28. $12-2x-\frac{2-4x}{3}=\frac{3x-4}{2}-\frac{x-6}{4}.$

29. $\frac{x-5}{18}-\left(\frac{x-4}{20}-\frac{x-12}{3}-\frac{x+2}{24}\right)=0.$

30. $8-\frac{2}{3}(10-x)+\frac{4}{3}(15-x)=\frac{2}{3}(13-x)-\frac{15-x}{x}.$

315. Literal Equations. An equation in which some of the numbers that are regarded as known numbers are expressed in literal notation is a **literal equation.**

1. Solve $\frac{a}{x}-1=\frac{b}{x}-9.$

SOLUTION.
$$\frac{a}{x}-1=\frac{b}{x}-9.$$
$$\text{L. C. D.}=x.$$
$$a-x=b-9x.$$
$$8x=b-a.$$
$$x=\frac{b-a}{8}.$$

2. Solve $\frac{x}{a}-\frac{1}{3}(9a-3x)-\frac{a+x}{2a}=\frac{4a-x}{a}.$

SOLUTION. $$\frac{x}{a}-3a+x-\frac{a+x}{2a}=\frac{4a-x}{a}.$$

$$\text{L. C. D.} = 2a.$$

$$2x-6a^2+2ax-(a+x)=2(4a-x).$$

$$2x-6a^2+2ax-a-x=8a-2x.$$

$$3x+2ax=6a^2+9a.$$

$$(2a+3)x=3a(2a+3).$$

$$x=3a.$$

EXERCISE

316. *Solve the following literal equations, regarding x as the unknown number in each:*

1. $\frac{a}{x}-\frac{b}{x}=c.$

2. $\frac{x}{a}-\frac{x}{b}=c.$

3. $x-\frac{x}{a}=b.$

4. $\frac{a-bx}{c}+b=\frac{bc-x}{c}.$

5. $\frac{x-a}{a}-m=\frac{x-b}{b}-n.$

6. $a-\frac{b+x}{b}=b-\frac{a+x}{a}.$

7. $\frac{x}{a}-b=\frac{x}{b}-a.$

8. $\frac{a+b}{x}-c=d-\frac{a-b}{x}.$

9. $\frac{x-a}{bc}+\frac{x-b}{ac}+\frac{x-c}{ab}=0.$

10. $\frac{x+ab}{c}+\frac{x+ac}{b}+\frac{x+bc}{a}=0.$

11. $a\left(m-\frac{x}{n}\right)=b\left(n-\frac{x}{m}\right).$

12. $\frac{a-bx}{bc}+\frac{b-cx}{ac}+\frac{c-ax}{ab}=0.$

13. $\frac{ax+b}{x}\cdot\frac{d}{a}=\frac{b}{a}\cdot\frac{cx+d}{x}.$

14. $\frac{a}{3b}(2x+1)-\frac{1}{5b}(5ax-4b)=\frac{4}{5}.$

15. $\frac{a-bm}{mx}-\frac{c-bn}{nx}=1.$

16. $\left(\frac{1}{a}-x\right)(a+x)-\left(\frac{1}{a}+x\right)(a-x)=0.$

17. $\frac{a+1}{x}+\frac{b-1}{x}=(a+x)+(b-x).$

18. $\frac{3b(x-a)}{5a}+\frac{x-b^2}{15b}+\frac{b(4a+cx)}{6a}=0.$

19. $\frac{ax}{b}-\frac{b-x}{2c}-\frac{a(x-b)}{3d}=a.$

20. $\frac{a(b-x)}{bx}+\frac{b(c-a)}{cx}=\frac{a+b}{x}-\left(\frac{b}{c}+\frac{a}{b}\right).$

In equations 21 to 34 solve for each letter involved in terms of the others.

21. $\frac{1}{D}-\frac{1}{d}=\frac{1}{f}.$

SOLUTION. $\text{L. C. D.} = Ddf.$

$df-Df=Dd.$

Solving for f, $(d-D)f=Dd,$

$\therefore f=\frac{Dd}{d-D}.$

Solving for D, $(-d-f)D=-df,$

$\therefore D=\frac{df}{d+f}.$

Solving for d, $(f-D)d=Df,$

or $d=\frac{Df}{f-D}.$

22. $3x-5y+7z=\frac{4}{9}(x-y+3z).$

23. $a-4=(p+3q)(a+2).$

24. $S=\frac{n}{2}(a+l).$

25. $l=a+(n-1)d.$

26. $\frac{x}{a}+\frac{y}{b}=1.$

27. $y=mx+c.$

28. $lx+my=1.$

29. $Ax+By+C=0.$

30. $\frac{1}{10}-\frac{1}{p}=\frac{1}{x}.$

Solve the following equations for each letter involved:

31. $A = \frac{a(b+c)}{z}$.

32. $c = \frac{Kab}{b-a}$.

33. $s = \frac{c-a}{a-b}$.

34. $T = \frac{1}{a} + t$.

317. If an equation contains fractions with polynomial denominators, find the L. C. D., and proceed as in the preceding problems.

1. Solve the equation $\frac{3}{x^2-9} + \frac{1}{x+3} = \frac{2}{3-x}$

Arrange the denominators in descending powers of x and factor them to find the L. C. D.

SOLUTION.
$$\frac{3}{x^2-9} + \frac{1}{x+3} = \frac{-2}{x-3}.$$
$$\text{L. C. D.} = (x+3)(x-3).$$

Multiply every term of both members of the equation by the L. C. D. to clear of fractions.

$$(x+3)(x-3)\frac{3}{(x+3)(x-3)} + (x+3)(x-3)\frac{1}{x+3} = (x+3)(x-3)\frac{-2}{x-3}.$$
$$3 + (x-3) = -2(x+3).$$
$$3 + x - 3 = -2x - 6.$$
$$x + 2x = -3 + 3 - 6.$$
$$3x = -6.$$
$$x = -2.$$

CHECK.
$$\frac{3}{4-9} + \frac{1}{-2+3} = \frac{2}{3+2}, \text{ or } \frac{2}{5} = \frac{2}{5}.$$

2. Solve $\frac{1-2z}{3-4z} - \frac{5-6z}{7-8z} = \frac{8(1-3z^2)}{3(21-52z+32z^2)}$.

SOLUTION.
$$\frac{1-2z}{3-4z} - \frac{5-6z}{7-8z} = \frac{8(1-3z^2)}{3(3-4z)(7-8z)}.$$
$$\text{The L. C. D.} = 3(3-4z)(7-8z).$$
$$3(1-2z)(7-8z) - 3(5-6z)(3-4z) = 8(1-3z^2). \quad \text{(Why?)}$$
$$21 - 66z + 48z^2 - 45 + 114z - 72z^2 = 8 - 24z^2.$$

$$-66z + 114z = 8 - 21 + 45.$$
$$48z = 32.$$
$$z = \tfrac{2}{3}.$$

CHECK. $$\frac{1-\frac{4}{3}}{3-\frac{8}{3}} - \frac{5-\frac{12}{3}}{7-\frac{16}{3}} = \frac{8(1-\frac{4}{3})}{3(21-\frac{104}{3}+\frac{128}{9})} \text{ or } -\frac{8}{5} = -\frac{8}{5}.$$

EXERCISE

318. *Solve the following equations:*

1. $\dfrac{1}{x+2} + \dfrac{7}{3x+6} = \dfrac{2}{3}.$

2. $\dfrac{9}{2x+2} - \dfrac{7}{3x+3} = \dfrac{13}{12}.$

3. $\dfrac{7}{8x+2} - \dfrac{11}{20x+5} = 13.$

4. $\dfrac{15}{1-3x} - \dfrac{13}{1-3x} = 4.$

5. $\dfrac{5}{x+1} - \dfrac{3}{2x+2} = \dfrac{5}{2}.$

6. $\dfrac{9x+7}{2} - \left(x - \dfrac{x-2}{7}\right) = 36.$

7. $\dfrac{x+3}{2} - \dfrac{x-2}{3} = \dfrac{3x-5}{12} + \dfrac{1}{4}.$

8. $\dfrac{60-x}{14} - \dfrac{5x-5}{7} = 6 - \dfrac{24-3x}{4}.$

9. $2x - 1 + \dfrac{7x-2}{3} - \dfrac{3x+4}{5} = \dfrac{7x-4}{5} + \dfrac{5x+1}{3}.$

10. $\dfrac{3}{x+1} - \dfrac{x+1}{x-1} = \dfrac{x^2}{1-x^2}.$

11. $\dfrac{6x+7}{15} - \dfrac{2x-2}{7x-6} = \dfrac{2x+1}{5}.$

HINT. If some of the denominators are monomials, it is best to clear the equation of the monomial denominators first and then collect terms before clearing the equation of the polynomial denominators. In exercise 11 proceed as follows:

$$6x + 7 - \frac{15(2x-2)}{7x-6} = 6x + 3. \quad \text{(Multiplying by 15.)}$$
$$4 - \frac{30x - 30}{7x-6} = 0. \quad \text{(Collecting terms after transposing.)}$$
$$4(7x-6) - (30x - 30) = 0. \quad \text{(Multiplying by } 7x - 6.\text{)}$$
$$28x - 24 - 30x + 30 = 0.$$
$$-2x = -6.$$
$$x = 3.$$

Solve the following equations:

12. $\frac{55}{3x}+\frac{79-2x}{60-2x}=\frac{x+3}{x}$.

13. $\frac{5}{3x}+\frac{10-7x}{6-7x}=\frac{13+15x}{15x}$.

14. $\frac{8x+5}{14}-\frac{3-7x}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$.

15. $\frac{8x+37}{18}-\frac{7x-29}{5x-12}=\frac{4x+12}{9}$.

16. $\frac{y-5}{y+5}-\frac{y+5}{y-5}=-\frac{21y}{25-y^2}$.

17. $\frac{5}{2-x}+\frac{3}{2+x}-\frac{1}{x^2-4}=0$.

18. $\frac{5}{4}\cdot\frac{1}{x+4}-\frac{3}{4}\cdot\frac{1}{x+2}=\frac{1}{2}\cdot\frac{1}{x+6}$.

19. $\frac{2}{5}x-\frac{3x-3}{x+1}=3-\frac{1-4x}{10}$.

20. $\frac{7}{x-1}-\frac{3}{1+x}+\frac{18}{1-x^2}=0$.

21. $\frac{3}{x-1}+\frac{x-4}{x-3}=1$.

22. $\frac{5x^2-7}{5}=\left(x-\frac{7}{5}\right)^2$.

23. $\frac{1}{x-3}+\frac{2}{x+3}+\frac{3}{x^2-9}=0$.

24. $\frac{1}{x+2}+\frac{1}{x}=\frac{x+1}{x(x+2)}$.

25. $\frac{7x+26}{x+21}-\frac{17+4x}{21}=\frac{10-x}{3}+\frac{13+x}{7}$.

26. $\frac{1-x}{3}+\frac{3-x}{5}=\frac{6x+5}{8x-15}-\frac{1+8x}{15}$.

27. $1-\frac{2x+1}{3x-15}+\frac{x-11}{2x-10}=0$.

28. $\frac{3x-5}{5x-5}+\frac{5x-1}{7x-7}+\frac{x-4}{x-1}=2$.

29. $\frac{8x}{6x+2}=2-\left(\frac{7x}{15x+5}+\frac{x}{3x+1}\right)$.

30. $2\cdot\frac{4x+1}{x-2}-\frac{1}{3}\cdot\frac{2x-1}{x-2}+\frac{3x+2}{5x-10}=0.$

31. $\frac{4-2x}{3}-\frac{4}{6x-3}=\frac{3}{2x-1}-\frac{4x^2}{6x-3}.$

32. $\frac{3}{4}\cdot\frac{4x-5}{3x-7}=\frac{5}{7}\cdot\frac{7x-3}{5x-4}.$

33. $\frac{x+1}{x-1}(3x-11)=3(x-3).$

34. $6(x-6)=\frac{3x-14}{x-4}(2x-11).$

35. $\frac{3x-1}{3x+1}+\frac{3x+1}{x-1}+\frac{3x^2-67}{3x^2-2x-1}=5.$

36. $\frac{2x-3}{x-1}+\frac{3x+5}{x-2}+\frac{x^2-11}{x^2-3x+2}=6.$

37. $\frac{2x}{x-2}-\frac{2x^2+7}{x^2-3x+2}=\frac{2}{x-1}.$

38. $\frac{4x+5}{2x+6}-\frac{2(x-2)}{x-3}=\frac{-7}{6x^2-54}.$

39. $\frac{x}{3x-6}-\frac{3}{2x-4}+\frac{2x+2}{3x^2-6x}-\frac{1}{3}=0.$

40. $\frac{2x+1}{2x-16}-\frac{2x-1}{2x+12}=\frac{9x+17}{x^2-2x-48}.$

41. $\frac{x-2a}{a+b}-\frac{x+2b}{2a+2b}=\frac{3x-3a}{2b}.$

42. $\frac{a^2-4bx}{a^2+4b}+\frac{b^2-ax}{b^2+a}=2.$

43. $\frac{2x+a}{x+3a}+\frac{3x^2-22a^2}{x^2-9a^2}=5.$

44. $\frac{2a-x}{a-5}-\frac{5+x}{3}=\frac{5a+x}{a+2}-\frac{x+6}{2}.$

45. $\frac{1}{a+b}+\frac{a+b}{x}=\frac{1}{a-b}+\frac{a-b}{x}.$

Solve the following equations:

46. $\frac{a-x}{b}+\frac{b-x}{a}+2=0.$

47. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{b^2-a^2}{b^2-bx}.$

48. $\frac{4.5(x-2b)}{3x-3b}=\frac{3}{2}+\frac{b^2-5bx}{6x^2-6bx}.$

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

319. 1. What number added to both terms of the fraction $\frac{2}{5}$ will give a fraction whose value is $\frac{8}{9}$?

SOLUTION. Let $x =$ the required number.

Then $\frac{2+x}{5+x}=\frac{8}{9}$. (By the conditions.)

$\therefore 18+9x=40+8x.$

$\therefore x=22$, the required number.

CHECK. $\frac{2+22}{5+22}=\frac{24}{27}=\frac{8}{9}.$

2. The numerator of a fraction exceeds the denominator by 20, and if 7 is added to both terms of the fraction, the value of the resulting fraction is 3. Find the original fraction.

3. What number added to both terms of the fraction $\frac{3}{7}$ will double the value of the fraction?

4. The sum of the numerator and the denominator of a fraction is 20. If the numerator is multiplied by 2 and the denominator diminished by 3, the resulting fraction is equal to $\frac{7}{5}$. What is the original fraction?

5. The difference between two numbers is 16, and the quotient of the larger divided by the smaller is $2\frac{1}{7}$. What are the numbers?

6. $\frac{5}{6}$ of what number exceeds $\frac{4}{5}$ of the same number by 1?

7. In a division the dividend exceeded the divisor by 52, the quotient was 6, and the remainder was 8. Find the dividend and the divisor.

8. Divide 72 into two parts such that $\frac{2}{3}$ of one part shall exceed $\frac{1}{4}$ the other part by 26.

9. A man made a journey of 40 miles in $4\frac{2}{5}$ hours. Part of the way he traveled in an automobile at 20 miles an hour and the remaining distance he walked at the rate of 4 miles an hour. How far did he ride?

SOLUTION. Let $x =$ number of miles he rode.

Hence $40 - x =$ number of miles he walked.

Also $\frac{x}{20} =$ number of hours he rode,

and $\frac{40 - x}{4} =$ number of hours he walked.

Then $\frac{x}{20} + \frac{40 - x}{4} = 4\frac{2}{5}$. (By the conditions.)

Solve the equation.

10. A vessel that ordinarily goes 16 miles an hour is obliged to slacken to half speed during a part of a trip of 130 miles, thereby requiring 10 hours to make the trip. For how long a distance was it traveling under reduced speed?

11. If one man can do a piece of work in 8 days and another man can do the same work in 6 days, how long will it take both men working together?

SOLUTION. Let $x =$ number of days for both.

Hence $\frac{1}{x} =$ the part of the work both can do in one day.

Then $\frac{1}{8} + \frac{1}{6} = \frac{1}{x}$.

Let the student explain the equation and solve it.

12. A can do a piece of work in 5 days; B works only half as fast as A. How long will it take both working together?

13 A can do a piece of work in 12 days, but with B's help he can do it in 8 days. How long would it take B if he worked alone?

14. A tank has two inlet pipes. One can fill it in 40 minutes and the other in 60 minutes. How long will it take if both are running at the same time?

15. A tank has two inlet pipes numbered 1 and 2, and two discharge pipes, 3 and 4, with the following capacities: 1 running alone can fill the tank in 60 minutes; 2 alone can fill it in 80 minutes; 3 alone can empty it in 72 minutes, and 4 can empty it in 40 minutes.

(*a*) Beginning with the tank empty, how long will it take 1 and 2 to fill it?

(*b*) Beginning with the tank full, how long will it take 3 and 4 to empty it?

(*c*) Beginning with the tank full and all pipes flowing, how long will it take to empty it?

(*d*) Beginning with the tank empty, how long will it take to fill it if 1, 2, and 3 are flowing?

(*e*) Beginning with the tank half full, will it be filled or emptied, and after how long, if 2, 3, and 4 are flowing?

16. What amount of money drawing simple interest at 5 % will amount to $ 287.50 in 3 years?

Solution. Let $x =$ number of dollars on interest.

Hence $\frac{5x}{100} =$ number of dollars of interest per year,

and $\frac{15x}{100} =$ number of dollars of interest in 3 years.

Then $x + \frac{15x}{100} = 287.50.$

$\therefore 100x + 15x = 28750,$

or $115x = 28750.$

$\therefore x = 250.$

Therefore the original principal was $ 250.

17. What was the face of a note drawing 4 % simple interest if it took \$132.50 to settle the note 18 months after it was given?

18. A man loaned \$ 800 in two parts, one part yielding 5 % per annum and the other part yielding 6 %. The interest amounted to \$ 44.50 per year on the two notes. How was the money divided?

19. A man received \$ 665 for an automobile, which was 30 % below its original cost. How much did it cost?

20. How much water must be added to 80 pounds of a 5 per cent salt solution to obtain a 4 per cent solution?

Solution. Evidently it will require the addition of water to change the solution from 5 per cent salt to 4 per cent salt. The *amount* of the salt is, therefore, the same in both solutions, and we may use this fact as the basis of an equation.

Let $x =$ number of pounds of water to be added.

Hence $80 + x =$ number of pounds of salt and water in the new solution,

and $\frac{4}{100}(80 + x) =$ number of pounds of salt in the new solution.

Also $\frac{5}{100} \cdot 80 =$ number of pounds of salt in first solution.

Then $\frac{4}{100}(80 + x) = \frac{5}{100} \cdot 80$. (Since there was the same *amount* of salt in both solutions.)

$\therefore\ 4(80 + x) = 5 \cdot 80$, (Clearing the last equation of fractions.)

or $320 + 4x = 400$.

$\therefore\ 4x = 80$.

$\therefore\ x = 20$, the number of pounds of water required.

21. How much salt must be added to 80 pounds of a 5 % salt solution to change it to a 10 % salt solution?

Solution. Let $x =$ number of pounds of salt added.

Hence $80 + x =$ number of pounds of salt and water in new solution,

and $\frac{90}{100}(80 + x) =$ number of pounds of water in new solution.

Also $\frac{95}{100} \cdot 80 =$ number of pounds of water in original solution.

Then $\frac{90}{100}(80 + x) = \frac{95}{100} \cdot 80$.

$\therefore\ 90(80 + x) = 95 \cdot 80$,

or $7200 + 90x = 7600$.

$\therefore\ 90x = 400$.

$\therefore\ x = 4\frac{4}{9}$, the number of pounds of salt required.

Query. Why is it not sufficient merely to double the amount of salt in order to double the strength of the solution?

22. How much salt must be added to 100 pounds of a 10 % salt solution to change it to a 12 per cent solution?

23. How much water must be added to change 100 pounds of 10 % salt solution to a 4 % salt solution?

24. How much water must be added to each ounce of a 90 % alcohol solution to reduce it to a 60 % solution?

25. A merchant marked an article $8 and gave 20 % discount. Another merchant marked the same article at a higher price but gave $33\frac{1}{3}$ % discount. Find the marking price of the second merchant, if the discounted price was the same for both.

26. If 100 pounds of sea water contain 2.6 pounds of salt, how much fresh water must be added to make a new solution 30 pounds of which shall contain .6 of a pound of salt?

27. The sum of two numbers is 70. If 14 is subtracted from one of them and added to the other, the quotient of the numbers is inverted. What are the numbers?

28. The population of a city increased each year 5 % of the population of the preceding year. It now has 194,481 inhabitants. What was the population 3 years ago?

29. Find two numbers whose sum is s and whose quotient is $\frac{a}{b}$.

30. Divide the number 144 into two parts such that one part shall be $\frac{2}{3}$ of the other.

31. The numerator of a fraction is 35 less than its denominator. If both the numerator and the denominator are increased by 2, the fraction is equal to $\frac{2}{9}$. Find the fraction.

32. The cost per ounce of gold in December 1914 was about 41 times that of silver. Find the cost per ounce of each if 8.5 oz. of silver and $\frac{1}{2}$ oz. of gold together cost $14.50.

33. A watch chain weighing $\frac{7}{8}$ oz. is made of platinum and gold. How much of each metal is in the chain if the gold is

worth \$20 an ounce and the platinum is worth \$48 an ounce and the total value of the metal in the chain is \$22.75?

34. A man invests \$4500, part at 6% and part at 5%. The total income from the two investments is \$245. Find the amount invested at each rate.

35. A certain sum of money is invested in a 6% mortgage and \$500 more than this sum is invested in 4% bonds. If the incomes from the two investments are the same, how much is invested in each?

36. An estate of \$12,000 is divided among three heirs. The first receives $\frac{2}{3}$ as much as the second and the third receives \$400 more than the second. How much does each get?

37. A man can paint a house in 6 days; his son can paint it in 16 days. How many days would it take both working together?

38. A football team wins a game by 14 points and the losing team scores 4 less than half as many points as the winning team. What is the score?

39. The pressure of water at a depth of d feet on each square inch is given in pounds by the formula P (pressure) $=\frac{62.5}{144}d$. If the pressure of the air at the surface is 14 pounds per square inch, at what depth will it be 10 times as great?

40. It is 1024 miles from Chicago to Denver. A train that usually averages 32 miles an hour is delayed 2 hours by an accident, but by running 12 miles an hour faster just makes up the lost time. How far did it run at each rate?

41. A dairyman wishes to mix milk containing 5% butter fat with cream containing 30% butter fat to get a mixture containing 20% butter fat. How much of each should be taken to get 10 quarts of the mixture?

42. Any volume of aluminum weighs $\frac{2}{7}$ as much as the same volume of cast iron. When $\frac{1}{3}$ of the cast iron of a gasoline engine is replaced by aluminum parts of the same size, the weight of the engine is 320 pounds. What was the original weight of cast iron?

REVIEW OF FRACTIONS AND FRACTIONAL EQUATIONS

320. 1. What is the rule for adding fractions?

2. How do we "clear an equation of fractions"?

3. What principle is involved in "clearing an equation of fractions"?

4. How is a fraction multiplied by an integer?

5. Solve $\frac{1}{x^2-4}+\frac{2}{x+2}+\frac{1}{x-2}=0.$

6. Simplify $\frac{1}{x^2-4}+\frac{2}{x+2}+\frac{1}{x-2}.$

NOTE. The student should note that example 6 is not an equation, and that he is not to clear fractions.

7. Why do you have trouble if you try to solve the equation

$$\frac{1}{x+2}+\frac{1}{x}=\frac{2(x+1)}{x(x+2)}?$$

8. Solve $\left(5+\frac{x}{2}\right)\left(5-\frac{x}{2}\right)+\frac{x^2}{4}=x+12.$

9. Simplify $\left(5+\frac{x}{2}\right)^2+\left(5-\frac{x}{2}\right)^2+\frac{x^4}{4}.$

10. Solve $\frac{9}{x-9}-\frac{5}{x-5}-\frac{4}{x-4}=0.$

11. Simplify $\frac{3}{x-1}-\frac{5}{x-2}+\frac{3}{x-3}.$

12. Solve $\frac{a}{x}-\frac{b}{c}=\frac{c}{x}-\frac{b}{a}.$

13. Solve for t, $\frac{ab}{t} = bc + \frac{1}{t}$.

14. What number must be added to $\frac{4}{3}$ to get the same result that would be obtained by multiplying it by $\frac{4}{3}$?

15. $\frac{5}{x-17} + \frac{3}{x-19} - \frac{8}{x-18} = 0.$

16. $\frac{x-8}{x-3} + \frac{x-3}{x-5} + \frac{x-9}{x-7} = \frac{x-1}{x-3} + \frac{x-13}{x-5} + \frac{x-6}{x-7}.$

NOTE. First transpose the fractions and combine each pair having the same denominator.

17. $\frac{x+2}{x+7} + \frac{x+7}{x+5} + \frac{x+1}{x+3} = \frac{x+9}{x+7} + \frac{x-3}{x+5} + \frac{x+4}{x+3}.$

18. $\frac{a^2 - b^2}{a^3 - b^3} \div \frac{a^2 + 2ab + b^2}{a^2 + ab + b^2}.$

19. $\frac{x^2 - 7x + 12}{x^2 - x} \cdot \frac{x^3 - 1}{x^2 - 4x} \div \frac{x^2 + x + 1}{x^2}.$

20. Reduce to lowest terms $\frac{3x^3 + x^2 + x - 2}{x^3 - 1}$.

21. Reduce $\frac{a^4 - a^3 - a + 1}{a^4 + a^3 - a - 1}$.

22. $\frac{5+3x}{2-x} - \frac{5-3x}{2+x} + \frac{48-2x}{x^2-4}.$

23. Solve $x - \frac{x-2}{3} = 5\frac{3}{4} - \frac{x+10}{5} + \frac{x}{4}$.

24. Simplify $\frac{31}{12x^2 + x - 20} - \frac{25}{12x^2 + 25x + 12}$.

25. Solve $1.2x - .05 = .07x + .3x + 16.55.$

26. $\frac{\frac{a}{b+c}}{1 + \frac{b+c}{a+b}} \times \frac{\frac{c}{a+b}}{1 + \frac{c+a}{b+c}}.$

27. Prove that $\left(\frac{x^2+1}{x^2-1}-1\right)\div\left(\frac{2x}{x-1}-\frac{2x}{2-x}\right)$

$$=\frac{x-2}{x(x+1)(2x-3)}.$$

28. Show that $\frac{x^2+xy+y^2}{\frac{2}{3}x-y}\div\left(\frac{2x+3y}{x^2-y^2}\div\frac{4x^2-9y^2}{x^3-y^3}\right)=3(x+y).$

29. Show that $x=\frac{9}{19}$ satisfies the equation $\frac{12}{x}+\frac{10}{x-1}=6\frac{1}{3}.$

30. $\left(\frac{y^2}{x^2}+1+\frac{x^2}{y^2}\right)\left(\frac{x}{y}-\frac{y}{x}\right)=?$

31. $\dfrac{5-\dfrac{2-\dfrac{1-x}{3}}{4}}{3}=1.$ Solve.

32. Simplify $\frac{x^4+x^3-x-1}{1-y^2}\cdot\frac{y^2-1}{x^2-x}\cdot\left[1-\dfrac{1}{1-\dfrac{1}{x}}\right].$

(Princeton.)

33. Simplify $\left(\frac{m^2}{n^2}+1+\frac{n^2}{m^2}\right)\div\left(\frac{m^2+n^2}{mn}-1\right).$

(Sheffield Scientific School.)

34. Simplify $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{a}{b}-1}\cdot\dfrac{a^2-b^2}{a^3+b^3}.$ (Yale.)

35. Simplify $\left\{\frac{a^6+c^6}{a^3c^3}-\left[\frac{a}{c}+\frac{c}{a}\right]\right\}\div\left[\frac{a^2}{c^2}-\frac{c^2}{a^2}\right].$

Verify the result by using $a=2$, $c=1$ in the original fraction and in the answer. (Yale.)

36. Simplify $\dfrac{\frac{3}{2}}{x}-\left[\frac{x-1}{x}+\frac{1}{2}\left\{\frac{x-1}{x+1}-\frac{(x-2)(x-3)}{x(x+1)}\right\}\right].$

(Princeton.)

37. Simplify $\left(a^2+\frac{b^4}{a^2-b^2}\right)(a^2+b^2)\div\left(\frac{a}{a+b}+\frac{b}{a-b}\right)$.

(Sheffield Scientific School.)

38. Solve $\frac{2(x-a)}{b}+\frac{3(x-b)}{a}=5$.

39. Simplify $x+1+\frac{x^2}{x^2-1}-\frac{x^2}{x+1}+\frac{5x-4}{1-x^2}$.

40. Solve $\frac{7}{2b}+\frac{1}{6}=\frac{23-b}{6b}+\frac{7}{24}-\frac{1}{8b}$.

41. Show that $(100x+10y+z)\div 3=33x+3y+\frac{x+y+z}{3}$,

and from this equation show that if the sum of the digits of a number of three figures is divisible by 3, the number itself is divisible by 3. Show in the same way that any number of four figures is divisible by 3 if the sum of its digits is divisible by 3.

42. Show similarly that if the sum of the digits of a number is divisible by 9, the number itself is divisible by 9.

43. Any number ending in 5 can be written as $10a+5$, where a is the tens' figure. $(10a+5)^2=100a^2+100a+25=100\cdot a(a+1)+25$. From this we may get the squares of numbers of two figures ending in 5 by multiplying the first figure by 1 more than itself and writing the product before 25. Thus, $65^2=4225$. $(6\times 7=42.)$ Square all numbers of two figures that end in 5.

44. $(a+1)^2=a^2+2a+1$. The square of $a+1$ exceeds the square of a by $2a+1$. This means that the square of 21, $20+1$, exceeds the square of 20 by $2\cdot 20+1$, and therefore $21^2=441$. Square 31, 41, 51, 61, etc.

XII. RATIO AND PROPORTION

321. Ratio. The quotient of one number divided by another number of the same kind is their **ratio**. The former number is the **antecedent** and the latter is the **consequent**.

The ratio is usually written in the form of a fraction and its terms bear the same relation to each other as the numerator and the denominator of a fraction.

Thus, $\frac{\$10}{\$5}$ represents the ratio of \$10 to \$5. The value of this ratio is $\frac{10}{5}$, or 2. $\frac{a}{b}$ represents the ratio of a to b. It is usually read, the ratio of a to b or a divided by b. The above ratios are also sometimes written \$10 : \$5, and $a : b$. The colon is used here as a sign of division.

The value of a ratio is always an abstract number. (Why?)

ORAL EXERCISE

322. *Read the following ratios and give their values:*

1. $\frac{\$6}{\$8}$.
2. $\frac{2a^2}{3ab}$.
3. $\frac{\$15}{\$6}$.
4. $\frac{\$48}{50¢}$.
5. 7 men : 21 men.
6. $\frac{mx}{my}$.
7. $\frac{1}{4} : \frac{1}{2}$.
8. $\frac{x}{y} : \frac{y}{x}$.
9. $\frac{ma}{na}$.
10. $2\frac{1}{2} : .75$.
11. $\frac{3 \text{ ft.}}{10 \text{ in.}}$.
12. 2 yd. : 2 ft.
13. If the ratio of x to 3 is equal to 5, what is the value of x?

HINT. $\frac{x}{3} = 5$. Solve.

14. If the ratio of x to $\frac{1}{2}$ is equal to 2, what is the value of x?

15. What number bears to 5 the ratio .3? $\left(\text{Solve } \frac{x}{5} = .3.\right)$

16. Can you express a ratio between $12 and 4 ft.? 4 bu. and 2 qt.? 1 rd. and 1 in.? 10 sq. in. and 2 cu. in.?

Simplify the following ratios by treating them as fractions and reducing them to their lowest terms:

17. $(m^2 - n^2) : (m + n)$.

18. $x^3 - y^3 : x - y$.

19. Which ratio is the greater, $\frac{5}{7}$ or $\frac{7}{9}$? $\frac{3}{7}$ or $\frac{5}{11}$?

323. Proportion. An equality of two ratios is a **proportion.**

Thus, $\frac{12}{9} = \frac{36}{27}$ is a proportion. Also $\frac{a}{b} = \frac{c}{d}$ is a proportion, if a and b are the same kind of numbers, and c and d are also the same kind of numbers. This proportion is read, the ratio of a to b equals the ratio of c to d. The proportion is also sometimes written $a : b = c : d$, or $a : b :: c : d$. These proportions may be read, a is to b as c is to d. The fractional form is, however, much more commonly used.

EXERCISE

324. 1. What value must be given to d, if $a = 1$, $b = 2$, $c = 3$, in the proportion $\frac{a}{b} = \frac{c}{d}$?

2. What is the value of d if $a = 2$, $b = 3$, $c = 4$?

3. $a^2 - b^2 : a - b = ?$

4. Divide 60 into two parts that are in the ratio of 2 to 3.

HINT. Let x and $60 - x$ be the two numbers.

325. Terms of a proportion. The four numbers, a, b, c, and d are the *terms* of the proportion $a : b = c : d$. The first and fourth terms, a and d, are the **extremes**, and the second and third terms, b and c, are the **means.** The first and third terms, a and c, are the **antecedents**, and the second and fourth terms, b and d, are the **consequents.**

In the proportion $\frac{x}{x-4}=\frac{2}{3}$, name the extremes, the means, the antecedents, the consequents.

326. Fourth Proportional, Third Proportional, and Mean Proportional. The fourth term, d, of the proportion $\frac{a}{b}=\frac{c}{d}$ is the **fourth proportional** to the other three terms taken in the order a, b, c.

In the proportion $\frac{a}{b}=\frac{b}{c}$, where the means are equal, c is a **third proportional** to a and b, and b is the **mean proportional** between a and c.

ORAL EXERCISE

327. *In the following proportions name the extremes, the means, the antecedents, the consequents, the fourth proportionals, the mean proportionals, and the third proportionals.*

1. $\frac{2}{3}=\frac{4}{6}$.
2. $\frac{2}{3}=\frac{3}{4.5}$.
3. $a:b=c:d$.
4. $\frac{a}{b}=\frac{b}{c}$.
5. $m:p=q:s$.
6. $x:y=y:z$.

328. A proportion may be treated as an ordinary fractional equation. The unknown number may be in any term of the proportion.

Solve the proportion $\frac{3}{7}=\frac{5}{x}$ for x.

SOLUTION.

$$\frac{3}{7}=\frac{5}{x}.$$

$$3x=35.$$

$$x=11\tfrac{2}{3}.$$

CHECK. Substitute $11\frac{2}{3}$ for x in the proportion.

EXERCISE

329. *Solve for x in each proportion:*

1. $\frac{3}{x}=\frac{12}{16}$. 2. $\frac{51}{15}=\frac{68}{x}$.

3. $6.3 : x = 13\frac{1}{2} : 20$. (Write in fractional form.)

4. $\frac{20}{95}=\frac{x}{57}$. 5. $\frac{8\,ab}{x}=\frac{bc}{\frac{7}{4}ac}$.

6. Find the fourth proportional to (*a*) 3, 4, 6 : $\left(\frac{3}{4}=\frac{6}{x}\right)$; (*b*) 2, $4\frac{1}{2}$, $9\frac{1}{3}$; (*c*) *a*, *b*, *c*.

7. Find the third proportional to (*a*) 9 and 6; (*b*) a^2-b^2 and $a-b$; (*c*) *a* and *b*.

8. Divide 120 into two parts which are in the ratio of 2 to 3.

HINT. Let x and $120 - x$ represent the two parts. Why?

9. Divide 182 into two parts whose ratio equals $\frac{6}{7}$.

10. What number added to both terms of the ratio $\frac{1}{8}$ will give a ratio whose value is $\frac{2}{3}$?

11. Find a mean proportional between 2 and 8.

SOLUTION. The equation is $\frac{2}{x}=\frac{x}{8}$.

$$x^2 = 16.$$

$$x^2 - 16 = 0, \text{ or } (x-4)(x+4) = 0. \qquad (\S\,239.)$$

$$x = 4 \text{ or } -4.$$

12. Find a mean proportional between:

(*a*) 2 and 18. (*c*) $\frac{ab}{x}$ and $\frac{bx}{a}$.

(*b*) 3 and 27. (*d*) $\frac{(a+b)^2}{p-q}$ and $p-q$.

13. Divide $180 between two men so that their shares will be in the ratio of 13 to 5.

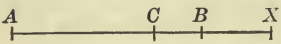
HINT. See example 8, or let $13\,x$ and $5\,x$ represent the two shares.

14. Divide \$180 among three men so that their shares shall bear to each other the relation 2 : 3 : 5.

Hint. This notation means that the first man's share is to the second man's share as 2 is to 3. Also the first man's share is to the third man's share as 2 is to 5. The shares may be represented by $2x$, $3x$, and $5x$.

15. Solve for x, $\frac{x-7}{x+3}=\frac{5}{6}$.

16. Solve for y, $y-7 : y-3 = y-11 : y-9$.



17. In the figure $AC = 9$ inches, $CB = 3$ inches, and $BX = x$. Find x, if $AC : CB = AX : BX$.

PROPERTIES OF PROPORTIONS

330. Consider the proportion $\frac{2}{3}=\frac{8}{12}$. Cleared of fractions this gives $2 \cdot 12 = 3 \cdot 8$. This illustrates the following important property of any proportion:

I. If four numbers are in proportion, the product of the means is equal to the product of the extremes.

Proof. Let a, b, c, and d be four numbers in proportion.

$$\text{Then } \frac{a}{b}=\frac{c}{d}.$$

$$\therefore a \cdot d = b \cdot c. \quad \text{(Clearing of fractions.)}$$

The last equation states that the product of the means in any proportion equals the product of the extremes. This is a test of the correctness of a proportion, or of the equality of two ratios.

Find the value of x in:

1. $2 : x = 3 : 6.$
$3x = 12.$
$x = 4.$

2. $x : 4 = 3 : 6.$
$6x = 12.$
$x = 2.$

3. $a : b = c : x$
$ax = bc.$
$x = \frac{bc}{a}.$

EXERCISE

331. *Find the value of x in each of the proportions* 1 to 10.

1. $8 : x = 24 : 3.$
2. $9 : 81 = x : 243.$
3. $18 : 7.2 = .4 : x.$
4. $a : b = x : c.$
5. $x : a = b : c.$
6. $x + 9 : 8 = x : 3.$
7. $\frac{1}{a} : x = b : ab.$
8. $\frac{ab^2}{c} : \frac{a^2b}{5c^3} = x : \frac{1}{10bc}.$
9. $a - x : a + x = 3 : 7.$
10. $x : 1.5 = 1\frac{5}{7} : 1.8.$

State which of the proportions 11 *to* 16 *are correct and which are incorrect.*

11. $5 : 6 = 15 : 18.$
12. $2 : 3 = 5 : 8.$
13. $3 : 5 = 77 : 112.$
14. $5 : 7 = 10 : 11.$
15. $(x + y) : (x - y) = (x^2 + 2xy + y^2) : (x^2 - y^2).$
16. $\frac{5m + 3}{10m + 9} = \frac{5m - 3}{10m - 9}.$
17. What is a fourth proportional?

Find the fourth proportional to each of the sets of three numbers in 18 *to* 23.

18. 5, 6, 10.
19. 8, 7, 5.
20. $m, n, p.$
21. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$
22. $\frac{m-n}{m+n}, \frac{(m-n)^2}{(m+n)^2}, \frac{m}{n}.$
23. $\frac{a^2-b^2}{a^2+b^2}, 1 + \frac{a}{b}, 1 - \frac{b}{a}.$
24. What is a third proportional?

Find the third proportional to each of the sets of two numbers in 25 *to* 29.

25. 9, 6; 16, 12.
26. $(a - b)^2, a^2 - b^2.$
27. $\frac{p^2 - q^2}{r}, \frac{p - q}{r}.$
28. $\frac{m^2}{l^2 - m^2}, \frac{lm - m^2}{(l + m)^2}.$
29. $\frac{p}{(p + m)^2}, \frac{2p^2}{m^3 + p^3}.$

332. Consider the proportion, $\frac{2}{4}=\frac{4}{8}$. Clearing of fractions gives $4^2=2\cdot 8$ or $4=\sqrt{2\cdot 8}$. This example illustrates the following property:

II. A mean proportional between two numbers is equal to the square root of their product.

Proof. Let a, b, and c be such numbers that

$$\frac{a}{b}=\frac{b}{c}.$$

$b^2 = ac.$ (Clearing of fractions.)

$\therefore b = \sqrt{ac}.$ (Extracting the square root of both members.)

Find the mean proportional between 3 and 12.

Solution.

$$3 : x = x : 12.$$
$$x^2 = 36.$$
$$x = \sqrt{36}, \text{ or } 6.$$

This may be verified by noting that $3 : 6 = 6 : 12$ is a true proportion. (Why?)

EXERCISE

333. *Find the mean proportional between each pair of numbers:*

1. 25 and 36.
2. 9 and 81.
3. $4a$ and ab^2.
4. $5a^2$ and $5b^2$.
5. $9a$ and $4ab^2$.
6. $3a^2b^2$ and $12c^2$.
7. $\frac{12a^3x^3}{5b^2z}$ and $\frac{3ax}{5z}$.
8. $\frac{5}{m^2+10m+25}$ and $\frac{(m+5)^2}{125}$.
9. Find a third proportional to 3 and 5.
10. Find a third proportional to x^2-y^2 and $x-y$.
11. $5ab$ is a mean proportional between $15a^2$ and what other number?
12. $3x$ is a mean proportional between 18 and what number?

334. From such an equation as $3 \cdot 8 = 4 \cdot 6$, we may form proportions by a proper arrangement of the numbers.

Thus, $\frac{3}{4} = \frac{6}{8}$, $\frac{3}{6} = \frac{4}{8}$, $\frac{8}{4} = \frac{6}{3}$, etc.

Can a proportion be made from the numbers involved in the equation $4 \cdot 10 = 5 \cdot 8$?

III. If the product of two numbers is equal to the product of two other numbers, the factors of either product may be made the means and the factors of the other product the extremes of a proportion.

PROOF. Let $ad = bc$.

Dividing both members of this equation by bd, we have

$$\frac{a}{b} = \frac{c}{d}.$$

Form proportions from the equation $pq = xy$.

SOLUTION.

$$\frac{pq}{qy} = \frac{xy}{qy}.$$

$$\therefore \frac{p}{y} = \frac{x}{q}, \text{or } p : y = x : q.$$

$$\text{Also } \frac{pq}{px} = \frac{xy}{px}.$$

$$\therefore \frac{q}{x} = \frac{y}{p}, \text{ or } q : x = y : p.$$

Let the student form proportions by dividing both members of $pq = xy$, (1) by py, (2) by qx.

In writing a proportion from two equal products, if any one factor of either of the products is written as first term in a proportion, *the other factor of that product becomes in every case the last term.*

EXERCISE

335. 1. Form proportions from $ad = bc$ by dividing both members by cd; by ac; by ab.

2. Form a proportion from $2x = 3y$.

SUGGESTION. Divide both members of the equation by $2y$. Could a proportion be formed by dividing by $3x$? by 6?

3. Form a proportion from $5u = 7w$.

4. Form a proportion from $x^2 = 2ab$.

5. Form a proportion from $x^2 - y^2 = a^2 - b^2$.

6. Can the numbers 2, 9, 3, and 7 be arranged as the terms of a proportion? Explain. Can 6, 8, 4 and 12 be so arranged? Why?

7. Write a proportion from $a = bc$.

8. What is the ratio of x to y in $12x = 30y$?

9. What is the ratio of x to y in $3x - 2y = x + y$?

10. Find the ratio of a to b in

$$\frac{2a - 3b}{b} = \frac{2c - 3d}{d}.$$

336. IV. If four numbers are in proportion, they are in proportion by inversion; that is, the second term is to the first as the fourth is to the third.

PROOF. Let $\frac{a}{b} = \frac{c}{d}$.

$ad = bc$. (Why?)

$\therefore \frac{b}{a} = \frac{d}{c}$. (Dividing by ac.)

Transform $\frac{2}{6} = \frac{3}{9}$ by inversion.

SOLUTION. $\frac{2}{6} = \frac{3}{9}$.

$\frac{6}{2} = \frac{9}{3}$. (Why?)

Let the student test the correctness of this last proportion.

337. If we interchange the means of the proportion $\frac{2}{3} = \frac{4}{6}$, we get $\frac{2}{4} = \frac{3}{6}$, which is another proportion. This transformation is always possible, and is stated as follows:

V. If four numbers are in proportion, they are in proportion by alternation; that is, the first term is to the third term as the second is to the fourth.

PROOF. Let $\frac{a}{b}=\frac{c}{d}$.

$$ad = bc. \quad \text{(Why?)}$$

$$\therefore \frac{a}{c}=\frac{b}{d}. \quad \text{(Why?)}$$

Transform $\frac{4}{5}=\frac{8}{10}$ by alternation.

SOLUTION. $\frac{4}{5}=\frac{8}{10}$.

$$\frac{4}{8}=\frac{5}{10}. \quad \text{(Why?)}$$

ORAL EXERCISE

338. *Transform the proportions* 1 *to* 4 *by inversion. Transform them by alternation.*

1. $2:3=6:9$.
2. $x:y=a:b$.
3. $3:-2=-9:6$.
4. $a:2a=b:2b$.
5. Can the proportion \$5 : \$10 = 2 men : 4 men, be transformed by alternation? Explain.
6. Can § 330, I, be applied to the proportion in the last example? Explain.

339. Given the proportion $\frac{4}{5}=\frac{8}{10}$. From this we may make another proportion as follows: $\frac{4+5}{5}=\frac{8+10}{10}$ or $\frac{9}{5}=\frac{18}{10}$. In general this may be stated as follows:

VI. If four numbers are in proportion, they are in proportion by composition; that is, the sum of the first two terms is to the second as the sum of the last two terms is to the fourth. Or the sum of the first two terms is to the first as the sum of the last two terms is to the third.

PROOF. Let $\frac{a}{b}=\frac{c}{d}$.

$$\frac{a}{b}+1=\frac{c}{d}+1. \quad \text{(Why?)}$$

$$\therefore \frac{a+b}{b}=\frac{c+d}{d}. \quad \text{(Why?)}$$

To prove $\frac{a+b}{a}=\frac{c+d}{c}$, transform the proportion $\frac{a}{b}=\frac{c}{d}$ by inversion and then proceed as before. Let the student do this.

Transform by composition $\frac{3}{4}=\frac{6}{8}$.

SOLUTION. $\frac{3+4}{4}=\frac{6+8}{8}$ or $\frac{7}{4}=\frac{14}{8}$.

EXERCISE

340. 1. Given $\frac{x}{y}=\frac{m}{n}$, prove that $\frac{x+m}{m}=\frac{y+n}{n}$.

HINT. Apply first V and then VI.

2. Transform by composition $\frac{x-5}{5}=\frac{1}{2}$.

3. Solve the equation $\frac{x-2}{5-x}=\frac{x-1}{3-x}$.

4. Solve the equation in example 3, first transforming by composition.

5. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+c}{c}=\frac{b+d}{d}$. (Use V and VI.)

6. If $\frac{m}{n}=\frac{x}{y}$, prove that $\frac{m+n}{x+y}=\frac{n}{y}$. (Use VI and V.)

341. Given the proportion $\frac{3}{4}=\frac{9}{12}$. From this we may make a proportion $\frac{3-4}{4}=\frac{9-12}{12}$ or $\frac{-1}{4}=\frac{-3}{12}$. We may also write $\frac{4-3}{4}=\frac{12-9}{12}$; that is, $\frac{1}{4}=\frac{3}{12}$.

In general this may be stated as follows:

VII. If four numbers are in proportion, they are in proportion by division; that is, the difference between the first two terms is to the second term as the difference between the last two terms is to the fourth. Or the difference between the first two terms is to the first as the difference between the last two terms is to the third.

PROOF. Let $\frac{a}{b}=\frac{c}{d}$.

$$\therefore \frac{a}{b}-1=\frac{c}{d}-1.$$

Let the student complete the proof.

Transform $\frac{5}{2}=\frac{10}{4}$ by division.

SOLUTION.
$$\frac{5}{2}=\frac{10}{4}.$$

$$\frac{5-2}{2}=\frac{10-4}{4}, \text{ or } \frac{3}{2}=\frac{6}{4}.$$

EXERCISE

342. 1. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a-b}{a}=\frac{c-d}{c}$. (Apply IV and proceed as above.)

2. Apply the transformation by division to $\frac{a+b}{b}=\frac{c+d}{d}$.

3. Apply the transformation by composition to $\frac{a-b}{b}=\frac{c-d}{d}$.

4. If $\frac{m+n}{n}=\frac{12}{5}$, find the value of $\frac{m}{n}$.

5. Solve $\frac{x+1}{1}=\frac{3}{7}$. (Apply VII.)

343. The last two transformations are sometimes referred to as transforming a proportion by *addition* instead of by *composition*, and by *subtraction* instead of by *division*.

344. A combination of the two preceding transformations may be made.

Thus, $\frac{3}{5}=\frac{9}{15}$, and $\frac{3+5}{3-5}=\frac{9+15}{9-15}$ or $\frac{8}{-2}=\frac{24}{-6}$.

This illustrates the following property of a proportion:

VIII. If four numbers are in proportion, they are in proportion by composition and division; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

PROOF. Let $\frac{a}{b}=\frac{c}{d}$.

$$\therefore \frac{a+b}{b}=\frac{c+d}{d}. \quad \text{(Why?)}$$

$$\text{Also } \frac{a-b}{b}=\frac{c-d}{d}. \quad \text{(Why?)}$$

Dividing the last two equations member by member, we have

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Transform $\frac{5}{6}=\frac{10}{12}$ by composition and division.

SOLUTION.

$$\frac{5}{6}=\frac{10}{12}.$$

$$\frac{5+6}{5-6}=\frac{10+12}{10-12},$$

or

$$\frac{11}{-1}=\frac{22}{-2}.$$

EXERCISE

345. 1. Transform $4:2=12:6$ by composition and division.

2. Transform $\frac{a+2}{a-2}=\frac{m+3}{m-3}$ by composition and division.

3. $a:b=c+x:c-x$. Solve for x, using § 330, I.

4. Solve the equation in 3, using § 344, VIII.

5. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

6. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+c}{a-c}=\frac{b+d}{b-d}$. (Alternation and composition and division.)

346. If several fractions are equal to each other, the sum of their numerators divided by the sum of their denominators equals any one of the fractions.

Thus, $\frac{2}{3}=\frac{6}{9}=\frac{8}{12}=\frac{14}{21}$ and $\frac{2+6+8+14}{3+9+12+21}$ or $\frac{30}{45}$ is equal to any one of these fractions.

This property of equal fractions may be stated thus:

IX. In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

PROOF. Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{x}{y}$.

Also let each ratio equal k.

$\frac{a}{b}=k$, from which $a=bk$. (Why?)

$\frac{c}{d}=k$, from which $c=dk$.

$\frac{e}{f}=k$, from which $e=fk$.

$\frac{x}{y}=k$, from which $x=yk$.

Then $a+c+e+x=k(b+d+f+y)$.

$$\therefore \frac{a+c+e+x}{b+d+f+y}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{x}{y}.$$

Thus, $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{1+2+3+4}{2+4+6+8}$ or $\frac{10}{20}$.

EXERCISE

347. 1. Apply IX to $\frac{1}{3}=\frac{2}{6}=\frac{4}{12}=\frac{5}{15}$.

2. Apply IX to $\frac{a}{b}=\frac{c}{d}$.

3. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{2}{3}$, what is the value of $\frac{a+c+e}{b+d+f}$?

4. If $\frac{a}{b}=\frac{m}{n}=\frac{x}{y}$, show that $\frac{a-m+x}{b-n+y}=\frac{a}{b}$.

HINT. $\frac{m}{n}$ may be replaced by $\frac{-m}{-n}$. (Why?) Then apply IX.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{r}{s}$, show that $\frac{2a+3c+4r}{2b+3d+4s}=\frac{a}{b}$.

HINT. $\frac{a}{b}=\frac{2a}{2b}$.

6. If $\frac{x}{a+b-c}=\frac{y}{a-b+c}=\frac{z}{b+c-a}$, prove that each one of these fractions is equal to $\frac{x+y+z}{a+b+c}$.

348. $\frac{1}{3}=\frac{2}{6}$, also $\frac{1^2}{3^2}=\frac{2^2}{6^2}$ or $\frac{1}{9}=\frac{4}{36}$. From $\frac{2}{3}=\frac{4}{6}$ we may get $\frac{4}{9}=\frac{16}{36}$ by squaring both members of the equation.

These examples illustrate the following property:

X. If four numbers are in proportion, the squares (or any like powers) of these numbers are in proportion.

PROOF. Let $\frac{a}{b}=\frac{c}{d}$.

Squaring both members of this equation, we have

$$\left(\frac{a}{b}\right)^2=\left(\frac{c}{d}\right)^2.$$

$$\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}.$$

The proof for other like powers is similar.

Thus, $\frac{2}{5}=\frac{4}{10}$. How does it follow that $\frac{4}{25}=\frac{16}{100}$?

EXERCISE

349. 1. If $\frac{m}{n}=\frac{x}{y}$, prove that $\frac{m^2}{x^2}=\frac{n^2}{y^2}$.

2. If $\frac{a}{b}=\frac{r}{s}$, show that $\frac{a^2+r^2}{b^2+s^2}=\frac{a^2}{b^2}$. (X and IX.)

3. In the proof of X we produced the equation $\frac{a^2}{b^2}=\frac{c^2}{d^2}$ from $\frac{a}{b}=\frac{c}{d}$. Is $\frac{a^2}{b^2}=\frac{a}{b}$? Explain.

SUMMARY OF THE PROPERTIES OF PROPORTIONS

350. Following are statements, in algebraic symbols, of the properties of proportions:

I. If $a : b = c : d$, then $ad = bc$.

II. If $a : b = b : c$, then $b = \sqrt{ac}$.

III. If $ad = bc$, then $a : b = c : d$ etc.

IV. If $a : b = c : d$, then $b : a = d : c$.

V. If $a : b = c : d$, then $a : c = b : d$.

VI. If $a : b = c : d$, then $a + b : b = c + d : d$.
or $a + b : a = c + d : c$.

VII. If $a : b = c : d$, then $a - b : b = c - d : d$.
or $a - b : a = c - d : c$.

VIII. If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

IX. If $a : b = c : d = e : f$, then $a + c + e : b + d + f = a : b$.

X. If $a : b = c : d$, then $a^2 : b^2 = c^2 : d^2$, or $a^n : b^n = c^n : d^n$.

EXERCISE

351. 1. What is meant by transforming a proportion by inversion? by alternation? by composition? by division? by composition and division?

2. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{2a}{5b} = \frac{2c}{5d}$; also that $\frac{a}{2b} = \frac{c}{2d}$.

3. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{2a + 5b}{2a - 5b} = \frac{2c + 5d}{2c - 5d}$.

4. Apply I to see if $13 : 17 = 19 : 24$.

5. Given $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a + 2b}{a} = \frac{c + 2d}{c}$.

6. Find a mean proportional between $\frac{(ab)^2}{p - q}$ and $\frac{p^2 - q^2}{p + q}$.

7. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that $\frac{a + c}{b + d} = \frac{e}{f}$; also that $\frac{a + e}{b + f} = \frac{e}{f}$.

8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a+c}{b+d}=\frac{a+e}{b+f}=\frac{c+e}{d+f}$.

9. (*a*) Find a third proportional to $\frac{3c}{7b}$ and $\frac{c}{b}$.

(*b*) Find a fourth proportional to a^3-b^3, a^2-b^2, $a-b$.

10. If $\frac{a}{b}=\frac{c}{d}$ and $\frac{a'}{b'}=\frac{c'}{d'}$, show that $\frac{aa'}{bb'}=\frac{cc'}{dd'}$.

11. Transform so that x shall occur only once, $\frac{a}{b}=\frac{c+x}{x}$.

SOLUTION.
$$\frac{a}{b}=\frac{c+x}{x}.$$
$$\frac{a-b}{b}=\frac{c+x-x}{x} \text{ or } \frac{a-b}{b}=\frac{c}{x}.$$

12. Transform so that x shall occur only once:

(*a*) $\frac{x+2}{x}=\frac{4}{3}$.

(*b*) $\frac{a}{b}=\frac{c+x}{c-x}$.

(*c*) $\frac{a}{b+x}=\frac{c}{b-x}$.

(*d*) $\frac{a}{b}=\frac{x}{x-c}$.

13. A cement block is to be made of Portland cement, sand, and gravel in the proportions 1 : 2 : 3. How much of each is there in a block that weighs 300 pounds?

14. The unequal sides of a rectangle are in the ratio of 3 to 5. Find the dimensions if the perimeter is 20 feet.

15. If $\frac{m+n}{m-n}=\frac{5}{3}$, find the value of $\frac{m}{n}$.

16. What number added to each of the numbers 1, 3, 19, and 27 will give numbers that form a proportion?

17. A and B do a piece of work for \$38. A works 5 days of 8 hours each and B works 4 days of 9 hours each. How should the money be divided?

18. The angles of a certain triangle are in the ratio 1 : 2 : 3. Their sum equals 180°. How large is each?

19. The sides of a triangle are in the ratio $3:4:5$. The perimeter is 100 inches. How long is each side?

20. If $m+n:m-n=x+y:x-y$, show that

$$x^2+m^2:x^2-m^2=y^2+n^2:y^2-n^2.$$

For what value of x does each set of numbers form a true proportion if taken in the order given?

21. $3, 4, 5, x$.

22. $2, 3, x+1, x+2$.

23. $x+1, x+2, x+4, x+8$.

24. $m+n, x-1, m-n, x+1$.

25. $5, 6, 3+x, 4+x$.

26. $15+x, 20+x, 1, 6$.

27. $3+x, 4+x, 25, 32$.

28. $x, 121-x, 5, 6$.

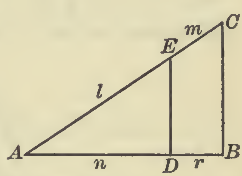
29. Find the ratio of x to y if $\dfrac{4x-3y}{2x+5y}=3$.

30. Show that four consecutive numbers cannot form a proportion.

HINT. Let n, $n+1$, $n+2$, $n+3$ represent the numbers.

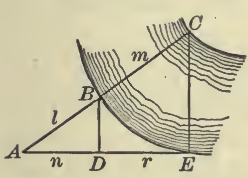
31. Brass consists of 2 parts of copper to 1 part of zinc. How many pounds of each are there in 9 pounds of brass?

32. Gunmetal consists of 9 parts of copper to 1 part of tin. How many pounds of each are there in 20 pounds of gunmetal?



33. It is proved in geometry that if a line is parallel to one side of a triangle, it divides the other two sides into parts that are in proportion. By actual measurement in the figure show that $l:m=n:r$.

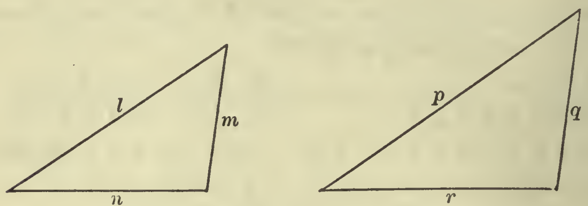
34. If $l=5$ inches, $m=2$ inches, and $n=4$ inches, find r.



35. If $l=8$ inches, $m=3$ inches, and $r=2.5$ inches, find n.

36. To measure the width of a river, BC, a triangle was laid out as shown in the figure, with BD parallel to EC. By actual measurement AB was found to be 96 feet, AD was 76 feet, and DE was 102 feet. Find BC.

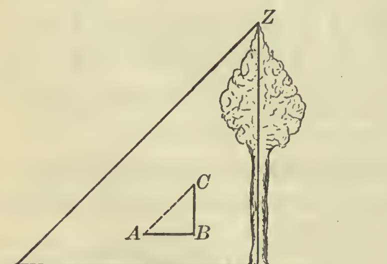
37. Similar triangles are triangles that have the same shape. It is stated in geometry that their corresponding sides are in proportion. Thus in the two triangles $l : p = m : q$; also $n : r = m : q$. Write another proportion involving the sides of the triangles.



38. If $l = 5$ inches, $m = 3.8$ inches, and $p = 7$ inches, find the length of q. Also find the length of n if $r = 8$ inches.

39. Of two similar triangles (see example 37) the sides of one are 5 inches, 8 inches, 10 inches, and the sides of the other are $7\frac{1}{2}$ inches, 12 inches, 15 inches. Show that their perimeters are in the same ratio as two corresponding sides.

40. In the figure, XY is the length of the shadow of the tree, YZ is the height of the tree; BC is a stick set in the ground and AB the length of its shadow. If $AB = 6$ feet, $BC = 4\frac{1}{2}$ feet, and $XY = 42$ feet, find the height of the tree.



HINT. The triangles are similar. (See example 37.)

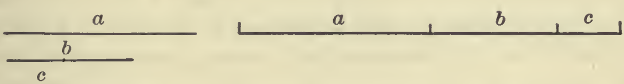
41. The Woolworth building (city of New York), the highest office building in the world, casts a shadow 625 feet long at the same time that a boy 4.8 feet high casts a shadow 4 feet long. How high is the building?

XIII. GRAPHS [1]

352. The student has seen in his general reading many different graphical, or pictorial, methods of representing data. A series of straight lines can be used to show the relative values of the grain crops, manufactured products, or the wealth of different countries. Pictures of soldiers of different sizes may represent, pictorially, the relative strength of the armies of different nations. In a similar way the strength of navies may be represented by ships.

353. 1. Determine by construction a line representing the sum of three given lines a, b, c.

Place the three lines, a, b, and c end to end and the total length is the sum required.



2. Represent on a scale of $\frac{1}{4}$ inch to a mile a distance of 10 miles.

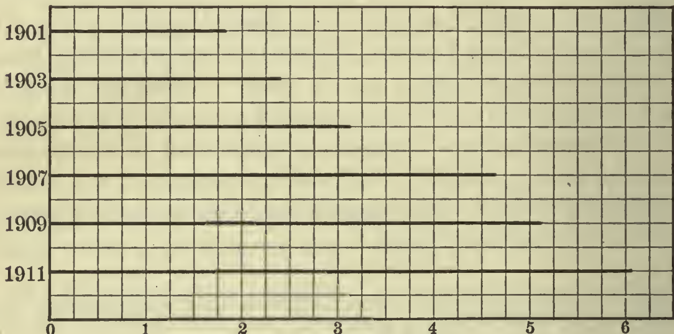
The line AB is 10 units long, each unit being $\frac{1}{4}$ inch. AB therefore represents a distance of 10 miles on a scale of $\frac{1}{4}$ inch to a mile.

3. Represent a distance of 25 miles on a scale of $\frac{1}{8}$ inch to a mile.

[1] The chapter on graphs may be omitted, if desired, without interrupting the sequence of the work.

4. The beet sugar produced in the United States from 1901 to 1911 expressed in tons was as follows: 1901, 184,000; 1903, 240,000; 1905, 313,000; 1907, 464,000; 1909, 512,000; 1911, 606,000.

By using a distance of $\frac{1}{2}$ inch to represent 100,000 tons these facts may be represented graphically as follows:



100,000 Tons

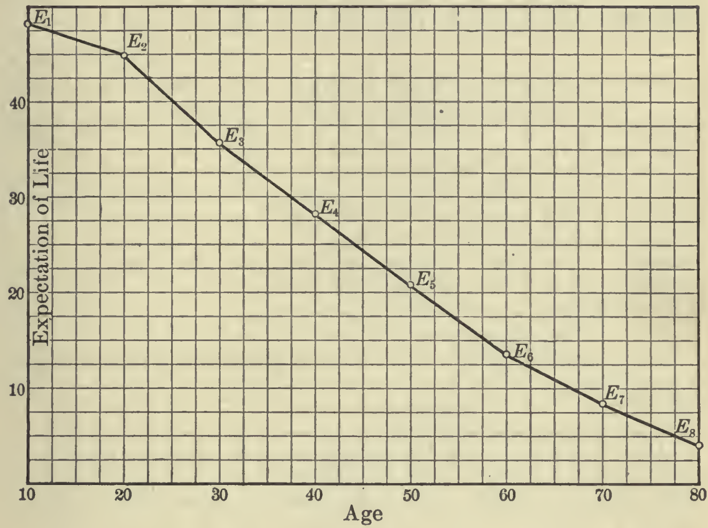
5. In the following table used by life insurance companies the premium charged a person at the age of 30 is computed on the basis that he is expected to live 35.33 years. Illustrate graphically the expectation of life for ages from 10 yr. to 80 yr.

Age	Expectation of Life
10	48.72
20	45.5
30	35.33
40	28.18
50	20.91
60	14.1
70	8.48
80	4.39

Draw two straight lines perpendicular to each other. Measure off on each equal spaces representing age and expectation of life, allowing 4 spaces for each 10 years. The expectation of life for 10 years, namely, 48.72 years, is shown at E_1. In the same way E_2, E_3, etc. may be located, showing the expectation of life at 20 yr., 30 yr., etc. of age. A continuous curve drawn through these points is the expectation of life curve for ages from 10 yr. to 80 yr. This curve shows at a glance the expectation of life for all ages from 10 yr. to 80 yr.

It is possible to estimate from this curve the expectation of life for ages not given in the table.

Determine from the figure the expectation of life at the ages 15, 25, 35, 55, 75.



6. The following table shows the annual premium per $ 1000 at different ages for life insurance.

Age	21	25	30	35	40	45	50	55	60
Premium	$ 18.40	$ 20.14	$ 22.85	$ 26.35	$ 30.94	$ 37.08	$ 45.45	$ 56.93	$ 72.83

Construct a curve showing the relation between the age and the premium. Measure the ages along the horizontal line and the premium on the vertical line.

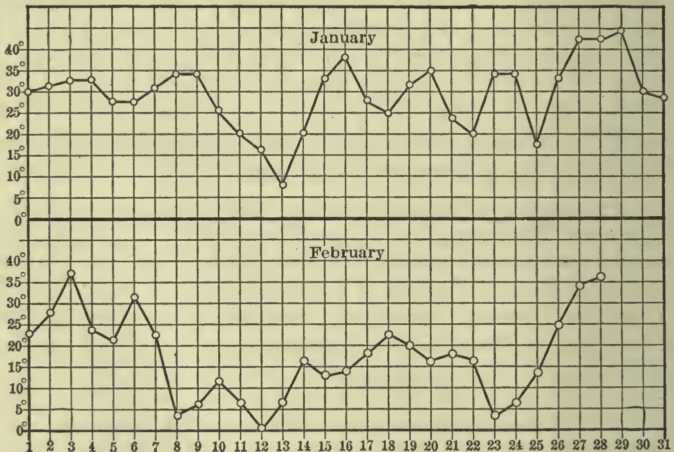
Note. The pupil should use cross-section paper for this work.

From the curve estimate the premium for a person at the age of 28, 37, 42, 54.

7. The following temperatures were taken from the weather reports at a certain city for January and February.

Day	1	2	3	4	5	6	7	8	9	10	11
January	30°	31°	32°	32°	26°	26°	31°	34°	34°	25°	20°
February	24°	28°	36°	24°	22°	32°	23°	4°	6°	11°	7°
Day	12	13	14	15	16	17	18	19	20	21	22
January	16°	9°	20°	34°	38°	28°	26°	32°	36°	24°	20°
February	0°	6°	16°	13°	14°	18°	22°	20°	16°	18°	17°
Day	23	24	25	26	27	28	29	30	31		
January	34°	34°	18°	33°	42°	42°	44°	30°	28°		
February	4°	7°	14°	25°	34°	36°					

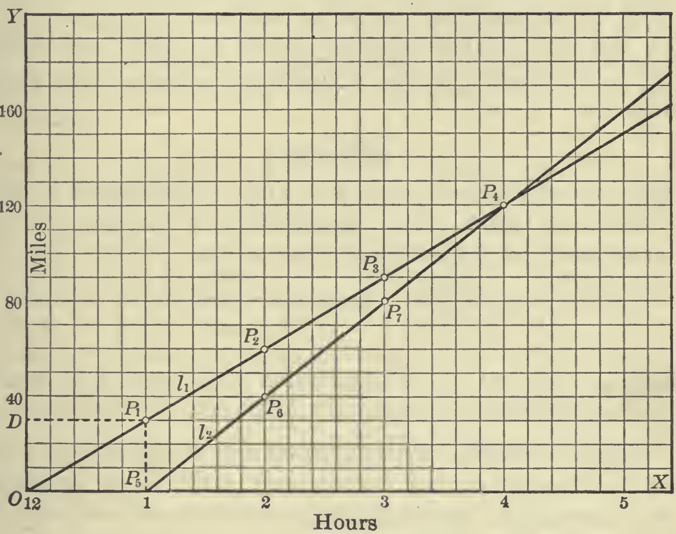
This temperature record is shown graphically in the following figure:



Observe that time, or dates, are represented on the horizontal line using 1 space for one day; the temperatures are measured in the direction of the vertical line, using 1 space for 5°.

8. Two trains leave Chicago going east on parallel lines. One starts at noon and runs at the average rate of 30 miles an hour; the other starts at 1 o'clock and runs 40 miles an hour. How far from Chicago, and at what time, will the fast train overtake the slow train?

Let the spaces on OY represent the number of miles traveled as indicated, and the spaces on OX represent the time.



At 1 o'clock the slow train will have run 30 miles. Measure the 30 miles along OY as OD. Measure the time, 1 hour, along OX. Draw the rectangle ODP_1P_5. P_1, by its distance from OX, represents the distance traveled, and, by its distance from OY, represents the time. Similarly P_2 represents the distance and the time after 2 hours; P_3, after 3 hours, etc. The points O, P_1, P_2, $P_3 \cdots$ lie in a straight line. Draw this line and call it l_1. If any point is taken on this line, it will be found that a distance and the corresponding time can be read at once from the figure. In a similar way draw l_2 through $P_5P_6P_7P_4$, representing the progress of the fast train.

It is evident that the intersection of the lines l_1 and l_2 will indicate the time of the day and the distance traveled when the distances are equal; that is, when the fast train overtakes the slow train. From the figure it appears that this occurs at 4 o'clock when the trains are 120 miles east of Chicago.

Determine from the figure how far the fast train is behind the slow train at 3 o'clock. When will the fast train pass the point where the slow train was at 2 o'clock?

354. In representing statistics and data graphically, first look over the numbers involved so as to choose convenient units. In general, if the numbers are large, select small units.

EXERCISE

355. 1. If a person saves 10¢ a day and deposits it in a savings bank which pays 3 % interest, the balances, to the nearest dollar, at the end of certain years are as follows:

Year	1	2	3	5	8	10	14	17	20
Balance	$ 37	$ 75	$ 115	$ 197	$ 330	$ 425	$ 635	$ 809	$ 999

Using two spaces on the horizontal line OX for one year, and four spaces on OY to represent $ 100, draw a smooth curve through the points located from the table and estimate the balances for the years omitted.

2. The table below gives the expense and receipts of a certain newspaper for various numbers of copies.

Numbers of copies	1000	2000	3000	4000
Expense in dollars	425	550	675	800
Receipts in dollars	300	495	690	885

Construct a graph showing the relation between the number of copies produced and the expenses. On the same diagram show the relation between the receipts and the number of copies. From the diagram estimate as nearly as possible the smallest number of copies that can be produced to make the paper pay. Use 1 inch on the horizontal line for 1000 copies, and $\frac{1}{4}$ inch on the vertical line to represent $100.

3. The following table shows the distances in miles of certain railway stations from Chicago, and the time of two trains, one to and one from Chicago. If each run is to be made at a constant speed, show graphically the progress of each train.

GOING WEST	MILES				MILES	GOING EAST
9 : 00 A.M.	0		Chicago		284	5 : 00
12 : 40 P.M.	127	arrive	Bloomington	leave	157	1 : 30
12 : 45		leave		arrive		1 : 25 P.M.
2 : 25	185	arrive	Springfield	leave	99	12 : 00 M.
2 : 35		leave		arrive		11 : 55
4 : 45	258		Alton		26	10 : 00
5 : 45	284		St. Louis		0	9 : 00 A.M.

At what point do the trains pass and how far is each from Chicago?

Let the horizontal line represent the time, using one half inch for one hour and one inch on the vertical line for 100 miles.

4. The following table gives the length of the circumferences of a circle for given radii:

Radius	0	1	2	4	6	8	10
Circumference	0	6.28	12.56	25.12	37.7	50.24	62.8

Measure the circumferences along the vertical axis, using one space for 2 units. Use two spaces for 1 unit in measuring radii on the horizontal axis. Locate all the points tabulated and draw a smooth curve through them. (*a*) Estimate from the figure the circumference of a circle with radius $2\frac{1}{2}$ units. (*b*) What is the approximate radius of a circle whose circumference is 44 units?

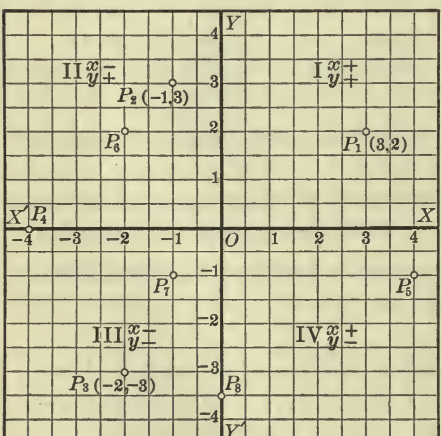
5. The following table shows the areas of circles for certain radii:

Radius	0	1	2	3	4	5
Area	0	3.14	12.56	28.26	50.24	78.5

Locate the points, using the same units as in the last example. Draw a smooth curve through the points.

(*a*) Estimate the area of a circle with radius $2\frac{1}{2}$ inches; 6 inches. (*b*) Estimate the radius of a circle with an area of 40 square inches; of 70 square inches.

356. Axes and Coördinates. If we draw two straight lines at right angles to each other as in the figure, we divide a plane surface into four quadrants. The lines are the **axes**. The horizontal line XX' is the *X-axis*, and the vertical line YY' is the *Y-axis*. The quadrants are the *first quadrant*, the *second quadrant*, the *third quadrant*, and the *fourth quadrant*, as indicated by the Roman notation. We name the spaces along these axes as shown in the figure.



If we select any point in the plane, as P_1, we can describe its position completely by telling how far it is to the right of the *Y-axis* and how far it is above the *X-axis*. These distances are, for the point P_1, 3 and 2 respectively, and they are the **coördinates** of P_1. The coördinate measured in the direction of the *X-axis* is the **abscissa**, usually designated by x, and the coördinate measured in the direction of the *Y-axis* is the **ordinate**, designated by y. The coördinates of a point are written in the form (x, y), the abscissa always being written first, followed by the ordinate. The signs + and − indicate the direction to be measured.

EXERCISE

357. 1. The coördinates of the point P_2 are $(-1, 3)$; of P_3, $(-2, -3)$. What are the coördinates of P_4? of P_5? of the intersection of the axes?

2. What are the coördinates of P_6? of P_7? of P_8?

3. Locate the points $(-1, 1)$, $(-3, 0)$, $(0, -3)$, $(0, 0)$.

4. Where are all the points which have abscissa 1?

5. Where are all the points which have ordinate -2?

6. Give the signs of the coördinates for each quadrant.

7. How many points may have 3 and 4 as the absolute values of the coördinates?

8. Locate the points $(3, 4)$, $(3, 2)$, $(3, 0)$, $(3, -1)$, $(3, -4)$. Draw a line through these points. What kind of line does this give?

9. If the abscissa is zero, where must the point be located?

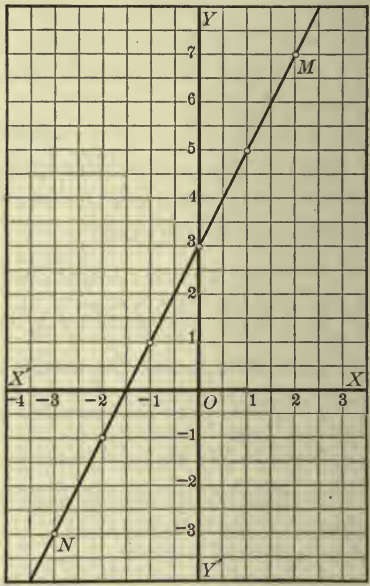
10. If the ordinate is zero, where must the point be located?

358. Function. When one quantity depends upon another for its value, the first quantity is a **function** of the second.

If a train travels at a uniform rate, the distance traveled is a function of the time. The cost of 10 yards of cloth is a function of the price per yard. The area of a square is a function of its side; the area of a circle is a function of its radius. The algebraic expression $2x+3$ is a function of x. In the equation $y=2x+3$, y is a function of x.

359. Graph of a Function. The values of x and y in the equation $y=2x+3$ may be pictured by means of a graph. Tabulating sets of values of x and y that satisfy the equation (allowing two squares for each unit) we have the following:

x	y
0	3
1	5
2	7
-1	1
-2	-1
-3	-3



When all the pairs of values of x and y are used as coördinates we have a series of points that appear to lie in a straight line. If fractional values of x are taken, other points between these will be found. The line MN, if extended indefinitely in both directions, is the **graph** of the function of x, $2x+3$, or of the equation $y=2x+3$. By this we mean that:

1. Every pair of values of x and y that satisfies the equation will, if used as coördinates of a point, give a point on this line MN.

2. The coördinates of any point on this line satisfy the equation.

EXERCISE

360. *Answer questions* 1 *to* 4 *by referring to the figure of* § 359, *and verify the answers by seeing if they satisfy the equation* $y = 2x + 3$.

1. What is the value of x for $y = 0$? for $y = -4$?

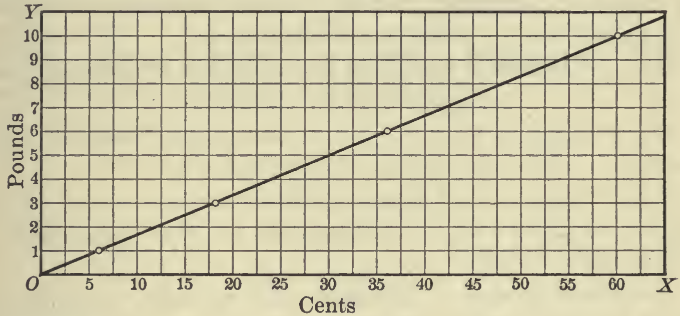
2. What is the value of y for $x = 2\frac{1}{2}$? for $x = -\frac{1}{2}$?

3. Does the point $(-3.5, -4)$ lie on the line MN?

4. Do the values $x = \frac{1}{2}$, $y = 4$ satisfy the equation?

5. When sugar is 6¢ a pound, the cost, c, is a function of the number of pounds, p. The equation connecting them is $c = 6p$.

Construct the graph for finding the cost, using the axis OX for the cost and OY for the weight.



SOLUTION

p	c
0	0
1	6
3	18
6	36
10	60

NOTE. When no negative numbers are to be used, the points are all in the first quadrant and the bottom line and the line at the left side may be used as axes.

Determine from the figure the cost of 5 pounds of sugar; 9 pounds. How many pounds can be bought for 48¢? for 55¢?

6. Construct a graph to show the cost of eggs at 28¢ a dozen. Extend the graph to 8 dozen and estimate from it the cost of $2\frac{1}{2}$ dozen; of 5 dozen.

Use 4 spaces for 1 dozen eggs on the X-axis, and 1 space for 7¢ on the Y-axis.

7. A train travels uniformly 45 miles an hour. Construct a graph and determine the distance it covers in 12 minutes, and the time it takes to go 24 miles.

HINT. The equation is $d = 45t$ where d represents the distance in miles and t the time in hours. Use 1 space for six minutes on the X-axis and 1 space for 5 miles on the Y-axis.

8. To change Fahrenheit temperatures to centigrade, the equation $C = \frac{5}{9}(F - 32)$ is used. In this equation F represents the number of degrees Fahrenheit and C the same temperature measured by a centigrade thermometer.

Thus, 50° Fahrenheit is changed into centigrade by substituting 50 for F.

$$C = \tfrac{5}{9}(50 - 32) = \tfrac{5}{9} \cdot 18 = 10.$$

Plot the graph for $C = \frac{5}{9}(F - 32)$ for the following Fahrenheit temperatures: $-10°$, $-20°$, 32°, 40°, 50°, 60°, 70°, 80°, 90°.

What temperature Fahrenheit will correspond to 20° centigrade? to 25°? 15° F. corresponds to what temperature centigrade? 72° F.?

9. Knowing that 1 kilogram = 2.2 pounds, construct a graph that will enable you to convert pounds into kilograms or kilograms into pounds. The equation is $K = 2.2\,p$. From this graph determine the number of kilograms in 11 pounds; in 14.3 pounds; in 22 pounds. Determine the number of pounds in 3 kilograms; in 5 kilograms; in 8 kilograms.

10. Given that 1 inch = 2.54 centimeters, construct a graph by means of which inches can be converted into centimeters and centimeters into inches. The equation is $i = 2.54\,c$.

11. At noon a boy begins to walk along a road at 4 miles an hour, and at 2 P.M. a cyclist rides after him at 10 miles an hour. Show in a graph the distance traveled in any time by the boy and by the cyclist and use the graph to find when the cyclist overtakes the boy. (See example 8, § 353.)

12. A newsboy sells papers at 1 cent each and makes $\frac{1}{3}$ cent profit on each paper. Represent graphically his sales and profits up to 50 sales.

HINT. For locating points, use numbers of sales that are multiples of 3, as 6, 12, 18, etc. The equation is $p = \frac{1}{3} s$.

13. Another newsboy sells papers at 1 cent each and gets $\frac{1}{2}$ cent profit on each paper. He has 6 cents carfare to pay. Represent on the same axes and to the same scale as used for the last exercise the sales and profits up to 50 sales.

HINT. The equation is $p = \frac{1}{2} s - 6$.

For what number of sales will the profits of the two boys be the same?

When will the first boy make more than the second? When will the second make more than the first?

XIV. LINEAR SIMULTANEOUS EQUATIONS WITH TWO UNKNOWN NUMBERS

361. Consider the equation

$$x + y = 5, \qquad (1)$$

where both x and y are unknown numbers. There is an indefinite number of pairs of values of x and y that satisfy the equation.

Thus $x = 1$, $y = 4$ is a solution, since $1 + 4 = 5$. Also $x = 2$, $y = 3$ is a solution, since $2 + 3 = 5$, and $x = -4$, $y = 9$ is a solution, since $-4 + 9 = 5$.

Tabulating some of the values of x and y that satisfy the equation, we have the following:

x	y	$x + y$
1	4	5
2	3	5
3	2	5
4	1	5
5	0	5
6	− 1	5
− 1	6	5
− 2	7	5

This tabulation could be continued indefinitely in both positive and negative numbers, and also in fractions. This means that there is an indefinitely large number of pairs of values of x and y which satisfy the equation. The equation is therefore **indeterminate.**

Tabulating the values of $x - y = 3, \qquad (2)$

we have the following:

x	y	$x - y$
1	− 2	3
2	− 1	3
3	0	3
4	1	3
6	3	3
0	− 3	3
− 1	− 4	3
− 2	− 5	3
− 3	6	3

This equation is **indeterminate.** It is seen, however, that the set of values, $x = 4$, $y = 1$, occurs in both tables. That is, $x = 4$, $y = 1$ will satisfy both equations. Thus the two equations considered together become a **determinate system,** since they determine a definite set of values of x and y; that is, $x = 4$, $y = 1$. The two equations, however, are each satisfied by sets of values of the unknown numbers which do not satisfy the other. They are therefore **independent equations.**

362. Independent Equations. Two or more equations containing two or more unknown numbers, and expressing different relations between the unknowns, are **independent equations.**

Thus, $x + y = 5$ and $x - y = 3$ are independent. (Why?) $x - y = 3$ and $2x - 2y = 6$ are not independent since the second can be reduced to the first by dividing both members by 2. Any solution of one is a solution of the other.

363. Simultaneous Equations. Two or more independent equations containing two or more unknowns which are satisfied by the same set of values of the unknowns are **simultaneous equations.**

Thus, $x + y = 5$ and $x - y = 3$ are simultaneous equations. (Why?)

364. Principles used in Solving Simultaneous Equations.

(*a*) If equal numbers are added to equal numbers, the resulting numbers are equal.

(*b*) If equal numbers are subtracted from equal numbers, the resulting numbers are equal.

ILLUSTRATION.

$$3 + 5 = 7 + 1 \qquad (1)$$

$$2 + 3 = 5 \qquad (2)$$

$$3 + 5 + 2 + 3 = 7 + 1 + 5. \quad \text{(Adding equations (1) and (2).)}$$

Let the student subtract (2) from (1) and note the result.

365. Tabulating values to find a set common to a system of simultaneous equations is too long a process. The following method is much shorter.

Solve the system of equations

$$x + y = 5,$$

$$x - y = 3.$$

SOLUTION.

$$2x = 8. \quad \text{(Adding the equations.)}$$

$$x = 4. \quad \text{(Why?)}$$

Substituting this value of x in the first equation, we have

$$4 + y = 5$$

$$y = 1. \quad \text{(Why?)}$$

By adding the equations we get rid of one of the unknowns; this process is known as elimination.

366. Elimination. The process of combining a system of equations so that one of the unknown numbers disappears is called **elimination.**

ORAL EXERCISE

367. Eliminate x in the following exercise and solve for the remaining letter:

1. $x + y = 10$,
   $x - 3y = 2$.
2. $2x + y = 5$,
   $2x - y = 1$.
3. $y - x = 7$,
   $y + x = 9$.
4. $x - 2y = 3$,
   $y - x = 5$.
5. $x + z = 9$,
   $x = 7$.
6. $x - z = 3$,
   $x - 2z = 1$.
7. $x - w = 5$,
   $x + w = 4$.
8. $x - y = a$,
   $x + y = b$.
9. $2x + y = 7$,
   $2x = 0$.
10. $2x + 3y = 10$,
    $3y - 2x = -4$.
11. $t + 3x = 4$,
    $t - 3x = -2$.
12. $2x - 2a = 9$,
    $2x - a = 8$.

368. Without attempting a complete discussion it may be stated that, in order to solve a system of linear equations with two or more unknowns, three conditions are necessary:

1. **There must be as many equations as there are unknown numbers.**

2. **The equations must be simultaneous; that is, there must be a set of values of the unknowns that will satisfy all the equations.**

3. **The equations must be independent; that is, they must express different relations between the unknown numbers.**

369.[1] The preceding discussion and definitions may be made clear by the use of the graphs of the equations.

[1] Section 369 may be omitted, if desired.

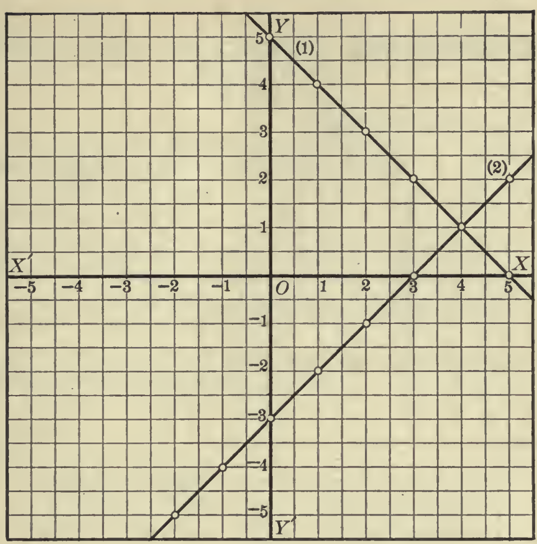


FIG. 1

Tabulating values for the equations $x + y = 5$ and $x - y = 3$, we obtain the following:

$x + y = 5$ (1)

x	y
0	5
1	4
2	3
3	2
4	1
5	0

$x - y = 3$ (2)

x	y
0	− 3
1	− 2
2	− 1
3	0
4	1
5	2
− 1	− 4
− 2	− 5

If we locate the points referred to the same set of axes and draw the lines through them, we get two intersecting lines.

The line (1) represents graphically equation (1) and the coördinates of every point on line (1) satisfy equation (1). Line (2) represents equation (2) and the coordinates of every point on line (2) satisfy equation (2). The values of x and y at the intersection of (1) and (2), or the coördinates of the point of intersection of the two lines, must satisfy both equations. Therefore $x = 4$, $y = 1$ is the solution of the two equations.

Consider the equations

$$x + y = 3, \quad (1)$$

$$2x + 2y = 8, \quad (2)$$

and tabulate values for both.

$x + y = 3$ (1)

x	y
0	3
1	2
2	1
3	0
4	− 1
− 1	4
− 2	5

$2x + 2y = 8$ (2)

x	y
0	4
1	3
2	2
3	1
4	0
5	− 1
− 1	5

If we locate these points and draw the lines on the same axes, we have figure 2.

The two lines which represent equations (1) and (2) are parallel, and hence have no intersection. Therefore there is no set of values of x and y that will satisfy both equations. In other words, the equations are not simultaneous.

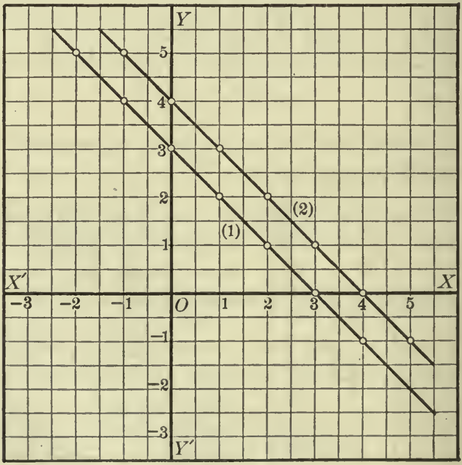


Fig. 2

Figure 1 represents the graph of a system of two **simultaneous independent equations** and shows their solution. Figure 2 represents a pair of equations that are independent, but are not simultaneous.

ELIMINATION BY ADDITION AND SUBTRACTION

Examples

370. **1.** Solve the system of equations

$$2x + y = 7, \qquad (1)$$

$$3x - y = 3. \qquad (2)$$

$5x = 10.$ (Adding equations (1) and (2).)

$x = 2.$

$4 + y = 7.$ (Substituting the value of x in (1).)

$y = 3.$

Check. $2 \cdot 2 + 3 = 7.$ (Substituting 2 for x and 3 for y in both equations.)

$3 \cdot 2 - 3 = 3.$

2. Solve the system

$$2x + y = 25, \qquad (1)$$

$$3x - 2y = 6. \qquad (2)$$

$4x + 2y = 50.$ (Multiplying equation (1) by 2. Why?) (3)

$7x = 56.$ (Adding equations (2) and (3).)

$x = 8.$

$16 + y = 25.$ (Substituting in (1).)

$y = 9.$

Check. As in example 1.

3. Solve the system

$$12x + 7y = 7, \qquad (1)$$

$$8x + 9y = 4. \qquad (2)$$

$24x + 14y = 14.$ (Multiplying equation (1) by 2.) (3)

$24x + 27y = 12.$ (Multiplying equation (2) by 3.) (4)

$-13y = 2.$ (Subtracting equation (4) from equation (3).)

$y = -\frac{2}{13}.$

$8x - \frac{18}{13} = 4.$ (Substituting in equation (2).)

$8x = \frac{70}{13}.$

$x = \frac{35}{52}.$

Check. $12 \cdot \frac{35}{52} + 7 \cdot (-\frac{2}{13}) = \frac{105}{13} - \frac{14}{13} = \frac{91}{13} = 7.$

$8 \cdot \frac{35}{52} + 9 \cdot (-\frac{2}{13}) = \frac{70}{13} - \frac{18}{13} = \frac{52}{13} = 4.$

371. The examples of the last article should be carefully studied. They illustrate the method of solving a system of simultaneous equations by addition and subtraction. A description of these solutions gives us the rule.

To solve a system of linear simultaneous equations with two unknown numbers:

1. Multiply one or both equations by such numbers as will make the coefficients of one of the unknowns the same in both equations.

2. Add or subtract the resulting equations to eliminate that one of the unknowns whose coefficients are numerically equal.

3. Solve the resulting linear equation.

4. Substitute the value of the unknown already found in one of the original equations, and solve the resulting equation for the other unknown.

372. Before applying the rule the equations are usually put into the form $ax + by = c$; that is, the term containing x is written first, followed by the term containing y, and the known term or terms are in the second member of the equation.

In determining which unknown to eliminate, study the equations with reference to the coefficients of the unknown.

EXERCISE

373. *Solve the following systems:*

1. $3x - 2y = 4,$
   $x + 2y = 4.$
2. $2x - 3y = -1,$
   $x + 4y = 16.$
3. $2t - 3u = -1,$
   $t - u = 1.$
4. $x + 5y = 3,$
   $4x - 2y = 1.$
5. $\frac{1}{2}m + \frac{2}{3}n = \frac{7}{6},$
   $m - n = \frac{1}{6}.$
6. $2x + 3y = 31,$
   $3x - y = 8.$
7. $4r + 5s = 40,$
   $6r - 7s = 2.$
8. $2x_1 + x_2 = 7,$
   $-2x_1 + 3x_2 = 13.$
9. $7x' - 3x'' = 15,$
   $5x' - 6x'' = 27.$
10. $8a + 17b = 42,$
    $2a + 19b = 40.$

11. $28\,m + n = 33,$
$-21\,m + 11\,n = 34.$

12. $12\,r_1 - 5\,r_2 = 64,$
$8\,r_1 + 3\,r_2 = 68.$

13. $7\,x - 12\,y = 115,$
$2\,x + 5\,y = 16.$

14. $8\,x + 3\,y = 37,$
$8\,y - 3\,x = 50.$

15. $8\,s + 21\,t = 649,$
$14\,s - 9\,t = 541.$

16. $2\,x - 11\,y = -95,$
$x - 3\,y = 0.$

17. $12\,x - 5\,y = 24,$
$3\,x + 10\,y = 6.$

18. $7\,v - 15\,w = -45,$
$8\,v + 5\,w = 15.$

19. $4\,x - 3\,y = 1,$
$7\,x = 3.5.$

20. $\frac{1}{8}\,x + 3\,y = 25,$
$8\,x + \frac{1}{8}\,y = 65.$

21. $\frac{1}{x} + \frac{2}{y} = 10,$ (1)
$\frac{4}{x} + \frac{3}{y} = 20.$ (2)

Solution. $\frac{4}{x} + \frac{8}{y} = 40.$ (From equation (1).) (3)

$\frac{5}{y} = 20.$ (Subtracting equation (2) from equation (3).)

$20\,y = 5.$ (Why?)

$y = \frac{1}{4}.$ (Why?)

$\frac{1}{x} + 8 = 10.$ (Substituting in equation (1).)

$\frac{1}{x} = 2.$

$x = \frac{1}{2}.$ (Why?)

22. $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{6}.$

23. $\frac{3}{x} + \frac{8}{y} = 3,$
$\frac{15}{x} - \frac{4}{y} = 4.$

24. $\frac{1}{2\,x} - \frac{1}{3\,y} = \frac{1}{4},$
$\frac{1}{3\,x} + \frac{1}{4\,y} = \frac{1}{2}.$

25. $\frac{5}{x} - 8\,y = \frac{7}{6},$
$\frac{3}{x} + 2\,y = \frac{2}{15}.$

Solve the following systems:

26. $2x - 3y = 5b - a,$ (1)
$3x - 2y = a + 5b.$ (2)

SOLUTION.
$6x - 9y = 15b - 3a.$ (Eq. (1) × 3.) (3)
$6x - 4y = 10b + 2a.$ (Eq. (2) × 2.) (4)
$-5y = 5b - 5a.$ (Eq. (3) − Eq. (4).)
$y = a - b.$ (Why?)
$2x - 3(a - b) = 5b - a.$ (Why?)
$2x - 3a + 3b = 5b - a.$ (Why?)
$2x = 2a + 2b.$ (Why?)
$x = a + b.$ (Why?)

27. $2x + y = 2a,$
$2x - y = 2b.$

28. $5x - 2y = 5a - 2b,$
$3x + 4y = 3a + 4b.$

29. $x + y = \frac{1}{2}(5a + b),$
$x - y = \frac{1}{2}(a + 5b).$

30. $2x + 3y = 10a - 2b + 3c,$
$x - 2y = -2a - b - 2c.$

31. $5x + 3y = 4a + b,$
$3x + 5y = 4a - b.$

32. $x + y = 10a - 3b,$
$2x - y = 2a + 3b.$

33. $ax + by = a,$
$\frac{a}{b}x - y = \frac{a}{b}.$

34. $10 + 7y + 4 = 0,$
$6x + 5y + 2 = 0.$

35. $7x - 3y = 27,$
$x : y = 6 : 5.$

36. $\frac{x}{3} + \frac{5}{y} = 4\frac{1}{3},$
$\frac{x}{6} + \frac{10}{y} = 2\frac{2}{3}.$

37. $ax + by = 2a,$
$a^2x - b^2y = a^2 + b^2.$

38. $x + my = -1,$
$y = n(x + 1).$

39. $x + 1 = ay,$
$y - bx = b.$

40. $3ax + by = a^2 + 1,$
$6x - 2b^2y = 2a - 2b.$

41. $3x + 2y = 8a - 7b,$
$ax + by = 2a^2 - 2b^2.$

ELIMINATION BY SUBSTITUTION

374. Principle of Substitution. Any number may be substituted for its equal.

375. **1.** Solve the system $2x + y = 25,$ (1)

$$3x - 2y = 6. \quad (2)$$

SOLUTION. $y = 25 - 2x.$ (From equation (1).) (3)

$3x - 2(25 - 2x) = 6.$ (Substituting in (2).)

$3x - 50 + 4x = 6.$

$7x = 56.$

$x = 8.$

$y = 25 - 2 \cdot 8 = 9.$ (From (3).)

Compare this with example 2, § 370.

2. Solve the system $x + y = 15,$ (1)

$$x : y = 2 : 3. \quad (2)$$

SOLUTION. $3x = 2y.$ (From (2).) (Why?) (3)

$x = \frac{2}{3}y.$ (4)

$\frac{2}{3}y + y = 15.$ (Substituting in (1).)

$\frac{5}{3}y = 15.$

$y = 9.$

$x = 6.$ (Substituting in (4).)

Let the student check mentally.

3. Solve the system $3x - 4y = 8,$ (1)

$$4x + 3y = -6. \quad (2)$$

SOLUTION. $x = \frac{8 + 4y}{3}.$ (From (1).) (3)

$4\left(\frac{8 + 4y}{3}\right) + 3y = -6.$ (Substituting in (2).)

$\frac{32 + 16y}{3} + 3y = -6.$

$32 + 16y + 9y = -18.$

$25y = -50.$

$y = -2.$

$x = \frac{8 + 4 \cdot (-2)}{3} = 0.$ (From 3).)

Check mentally.

376. **To solve a system of simultaneous equations by substitution:**

1. Find the value of either of the unknowns in terms of the other unknown and known numbers from one of the equations.

2. Substitute the value of this unknown for the same unknown in the other equation.

3. Solve the resulting equation.

4. Substitute the value of the unknown that has been found in one of the preceding equations to find the other unknown.

377. The method of solving simultaneous equations by substitution is especially convenient when one of the unknowns can readily be expressed in terms of the other. It is also much used in later work and should be well understood.

EXERCISE

378. *Solve the following systems by the method of substitution:*

1. $2x - 11y = -95,$
   $x - 3y = 0.$

2. $5x - 2y = 21,$
   $x : y = 5 : 2.$

3. $\frac{3}{4}c - 2d = 1,$
   $\frac{1}{3}c - d = 0.$

4. $x = 3y - 19,$
   $y = 3x - 23.$

5. $2w - \frac{5}{3}y = 4,$
   $3w - \frac{7}{2}y = 0.$

6. $5u - 4.9v = 1,$
   $3u - 2.9v = 1.$

7. $3x + 16y = 5,$
   $28y - 5x = 19.$

8. $\frac{3}{2}s + t = \frac{5}{2},$
   $s + t = \frac{1}{2}.$

9. $7x - y - 6a = 12b,$
   $x = y.$

10. $ax + by = c,$
    $x = 2y.$

11. $x - 2y = 69,$
    $2x + y = 78.$

12. $m + \dfrac{15}{n} = 13,$
    $m + \dfrac{60}{n} = 16.$

13. $x - \frac{2}{3}(y + 1) = 3,$
    $\frac{1}{3}(x - 1) - \frac{1}{2}y = 4\frac{1}{2}.$

14. $\dfrac{7 - 2x}{5 - 3y} = \dfrac{3}{2},$
    $y - x = 4.$

15. $\dfrac{x-3}{y+2}=\dfrac{2}{3}$,
$\dfrac{x+1}{y-2}=\dfrac{3}{2}$.

16. $x+4:y+1=2:1$,
$x+2:y-1=3:1$.

17. $ax+y=m$,
$x-y=n$.

18. $x+my=a$,
$x-ny=b$.

19. $x:y=a:b$,
$x+1:y+1=c:d$.

20. $\dfrac{m}{x}+\dfrac{1}{y}=p$,
$\dfrac{n}{x}+\dfrac{1}{y}=q$.

21. $\dfrac{x+m}{y-n}=\dfrac{p}{q}$,
$qx+py=s$.

22. $\dfrac{x}{m}+\dfrac{y}{n}=p$,
$\dfrac{r}{x}=\dfrac{q}{y}$.

23. $\dfrac{t-3}{u}=\dfrac{p}{q}$,
$\dfrac{3}{t}=\dfrac{2}{u}$.

24. $\dfrac{1}{x}+\dfrac{1}{y}=m$,
$\dfrac{1}{x}-\dfrac{1}{y}=n$.

379. Most of the equations thus far solved have been given in the form $ax+by=c$. In the following exercises the equations should be simplified before applying the rule.

EXERCISE

380. *Solve by either method:*

1. $4(3x-5)-2(y-x)=2$,
$2(5x-y)-3y=5$.

2. $\dfrac{7}{2x-y}=\dfrac{1}{x-y}$,
$2(x-y)=y-3$.

3. $4(x-3y)=8$,
$(x+y):(x-2y)=3:1$.

Solve by either method:

4. $x + y = 10,$
$$\frac{3x + 5y}{4} - \frac{5x + 3y}{11} = \frac{x + 5y}{11} \cdot \frac{5}{2}.$$

5. $5x - (3y - \frac{1}{2}) = .75,$
$4 + x - 2(y - \frac{1}{3}) = 0.$

6. $a : (b + y) = b : (3a + x),$
$ax + 2by = b^2.$

7. $x : y = 3 : 4,$
$x - 1 : y + 2 = 1 : 2.$

8. $x + 1 : y + 1 : x + y = 3 : 4 : 5.$

9. $x - 5 : y + 9 : x + y + 9 = 1 : 2 : 8.$

10. $\dfrac{2x + y - 1}{3x + 2y + 11} = \dfrac{1}{2}, \quad \dfrac{5x - 3y + 4}{6x - 3y + 3} = \dfrac{3}{4}.$

11. $bcx = cy - 2b, \quad b^2y + \dfrac{a(c^3 - b^3)}{bc} = \dfrac{2b^3}{c} + c^3x.$

12. $(a - b)x + (a + b)y = a + b, \quad \dfrac{x}{a + b} - \dfrac{y}{a - b} = \dfrac{1}{a + b}.$

13. $\dfrac{x + 1}{y} = a, \quad \dfrac{y + 1}{x} = b.$

14. $x + y = \dfrac{2(a^2 + b^2)}{a^2 - b^2}, \quad x - y = \dfrac{4ab}{a^2 - b^2}.$

(Do not clear of fractions.)

15. $x + y = \dfrac{a^2 + b^2}{a^2 - b^2}, \quad 2x + 3y = \dfrac{2a^2 + ab + 3b^2}{a^2 - b^2}.$

16. $(a + b)x + (a - b)y = 2(a^2 + b^2),$
$(a - b)x + (a + b)y = 2(a^2 - b^2).$

(First add the equations, then subtract them.)

17. $\frac{x+y+1}{x-y+1}=\frac{a+1}{a-1}$, $\frac{x+y+1}{x-y-1}=\frac{b+1}{b-1}$.

(Use composition and division on both equations.)

18. $(x+5)(y+7)=(x+1)(y-9)+112$,
$2x+10=3y+1$.

19. $2.60x-.41y-4.28+2.50x=0$,
$.50x+3.6y+3.23+.5y=11.93$.

20. $\frac{x+1}{3}-\frac{y+2}{4}=\frac{2(x-y)}{5}$,
$\frac{x-3}{4}-\frac{y-3}{3}=2y-x$.

21. $\frac{3x-2y}{5}+\frac{5x-3y}{3}=x+1$,
$\frac{2x-3y}{3}+\frac{4x-3y}{2}=y+1$.

22. $\frac{4}{5}(x-y)-\frac{1}{10}x-\frac{1}{20}y=14$,
$\frac{5}{6}(x-14)-\frac{7}{12}(y+12)=-2$.

23. $\frac{10x-2y+22}{56}-\frac{7x-3y}{14}=\frac{x+y-1}{8}$,
$\frac{x+y}{9}-\frac{2x+y+7}{18}=7x-2y$.

24. $2(x+y-c)=2(x-c)+x+3y-c$,
$x+7y=15c$.

25. $(x-1)(5y-3)=3(3x+1)+5xy$,
$(x-1)(4y+3)=3(7y-1)+4xy$.

26. $\frac{x}{a+b}+\frac{y}{a-b}=a+b$,
$\frac{x}{a}+\frac{y}{b}=2a$.

Solve by either method:

27. $(a+c)x-(a-c)y=2\,ab,$
$(a+b)y-(a-b)x=2\,ac.$

(Add the equations and simplify.)

28. $\frac{5\,x-y}{11}=\frac{8\,x-5\,y}{4}=y-3.$

(Put each of the first two equal to the last expression.)

29. $\frac{x}{a}=\frac{y}{b}=x+y+c.$

30. $\frac{7-6\,r}{10\,t-19}=\frac{4-3\,r}{5\,t-11},$
$\frac{6\,r-10\,t-17}{3\,r-5\,t+2}=\frac{4\,r-14\,t-5}{2\,r-7\,t+12}.$

31. $\frac{1}{c}:\frac{1}{d}=13:11,$
$\frac{5\,c+3}{2}=\frac{7\,d-4}{3}.$

32. $.75\,x+.8\,y=21,$
$\frac{x}{4}=\frac{y}{5}.$

33. $\frac{3}{x}+\frac{8}{y}=3,$
$\frac{15}{x}-\frac{4}{y}=4.$

PROBLEMS WITH TWO UNKNOWN NUMBERS

381. Many of the problems previously solved with one unknown number could have been solved with two unknowns. The problems following are to be solved using two unknown numbers.

1. Find two numbers whose sum is 40 and whose difference is 22.

SOLUTION. Let x = the greater number,
and y = the smaller number.
Then $x+y=40,$
and $x-y=22.$
$2\,x=62.$
$x=31.$
$y=9.$

2. The sum of two numbers is 21, and two times the first exceeds 3 times the second by 2. Find the numbers.

3. Find two numbers whose sum equals 95 and whose ratio is 2 : 3.

4. Two numbers are in the ratio of 4 to 7. If four is added to each of the numbers, the results are in the ratio of 2 to 3. Find the numbers.

5. The sum of the reciprocals of two numbers is $\frac{13}{21}$, and the ratio of the numbers is $\frac{7}{6}$. Find the numbers.

6. If the first of two numbers is multiplied by 3 and the second by 8, the sum of the products is 310; if the first is divided by 3 and the second by 8, the sum of the quotients is 10. What are the numbers?

7. The sum of two numbers is 350. If the first is divided by the second, the quotient is 8 and the remainder is 8. What are the numbers?

8. Find two numbers whose difference and quotient are each equal to 10.

9. Solve the last problem using a in place of 10.

10. A and B together have $120. If A would give B one third of his money, they would have equal amounts. How much has each?

SOLUTION. Let $x =$ number of dollars A has,
and $y =$ number of dollars B has.
Then $x + y = 120$. (By the first condition.)
Also $\frac{2}{3}x =$ number of dollars A has after giving B $\frac{1}{3}$ of his money,
and $y + \frac{1}{3}x =$ number of dollars B has after receiving $\frac{1}{3}$ of A's money.
Then $\frac{2}{3}x = y + \frac{1}{3}x$. (By the second condition.)
Solve the equations and verify the answers by putting them in the original problem.

11. A and B have a certain amount of money. If A had $15 more, he would have as much as B. If B had $15 more, he would have twice as much as A. How much has each?

12. If B gives A \$ 5 they will have equal amounts of money; but if A gives B \$ 5, B will have twice as much as A. How much has each?

13. A resolution was carried by a plurality of 20 votes. On reconsideration $\frac{1}{4}$ of those voting for it changed their votes and it was lost by 12 votes. How many voted each way the first time?

14. A dealer has two kinds of coffee, worth 30 cents and 40 cents a pound respectively. How many pounds of each must he take to make a mixture of 70 pounds, worth 36 cents a pound?

SOLUTION. Let $x =$ number of pounds 30-cent coffee,
and $y =$ number of pounds 40-cent coffee.
Then $x + y = 70$,
and $.30x + .40y = 25.20$. $(70 \times \$ .36 = \$ 25.20.)$

Let the student explain these equations and solve them.

15. A dozen oranges and 6 pineapples cost \$ 1.30. Six oranges and 2 pineapples cost \$ .50. Find the cost of an orange and of a pineapple.

16. A ruble is a Russian coin and a mark is a German coin. Two rubles and 3 marks are worth \$ 1.75 in our money; also a ruble is worth $3\frac{1}{2}$ ¢ more than two marks. Find the value of each in our money.

17. A man invested \$ 5000, a part at 5 % and the remainder at 6 %. The interest amounted to \$ 265 annually. How much was on interest at each rate?

SUGGESTION. One equation is $.05x + .06y = 265$. Let the student make the other.

18. 69 quarters and dimes are worth \$ 9.45. How many of each are there?

19. 40 stamps, some one-cent and the rest two-cent stamps, cost 65 ¢. How many of each were bought?

20. The units' digit of a number of two figures exceeds the tens' digit by 1; the number divided by the sum of its digits is equal to 5. Find the number.

SOLUTION. Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then $x + 1 = y$. (Why?)
Also $10x + y =$ the number. (Why?)

$$\text{Then } \frac{10x + y}{x + y} = 5.$$

$$10x + y = 5x + 5y.$$

$$\therefore 5x - 4y = 0.$$

$$\therefore 5x - 4(x + 1) = 0.$$

$$\therefore x = 4,$$

and $y = x + 1 = 5$.

Therefore the number is 45.

21. If the digits of a number of two figures are interchanged, the number obtained is $\frac{8}{3}$ of the original number. The units' digit exceeds the tens' digit by 5. Find the number.

22. If the digits of a two-figure number are interchanged, the resulting number multiplied by 2 exceeds the original number by 1. If the number is divided by the sum of its digits, the result is 7.3. Find the number.

23. What two-figure number, the sum of whose digits is 10, has the property that if its digits are interchanged, the number is diminished by 36?

24. The value of a fraction is $\frac{2}{3}$. If the numerator and the denominator are both diminished by 18, the value is $\frac{1}{3}$. What is the fraction?

SOLUTION. Let $n =$ the numerator,
and $d =$ the denominator.

$$\text{Then } \frac{n}{d} = \frac{2}{3},$$

$$\text{and } \frac{n - 18}{d - 18} = \frac{1}{3}.$$

Let the student solve. The answer is $\frac{24}{36}$.

25. The value of a ratio is 5. If 5 is subtracted from the antecedent and 5 is added to the consequent, the value is $2\frac{1}{2}$. Find the antecedent and the consequent.

26. A sum of money was divided equally among a number of people. Had there been 3 more people each would have received \$1 less. Had there been 5 less each would have received \$3 more. How many people were there and how much did each receive?

SOLUTION. Let x = number of people.
and y = number of dollars each received.
Hence xy = number of dollars divided.

$$\text{Then } (x+3)(y-1) = xy,$$
$$\text{and } (x-5)(y+3) = xy.$$

Let the student solve these equations. The number of people was 15.

27. A and B can do a piece of work in $4\frac{4}{5}$ days. After A and B work together for 2 days, B can finish the work in 7 days. How long will it take each?

SUGGESTION. If A can do the work in x days, what part of it can he do in one day?

28. It is known that gold loses $\frac{1}{19}$ of its weight when weighed in water, and silver loses $\frac{1}{10}$ of its weight. The gold and silver crown of King Hiero of Syracuse weighed 20 pounds and lost $1\frac{1}{4}$ pounds in water. How much of it was gold and how much was silver?

29. Two boys run on a circular track which is 90 yards around. If they run in opposite directions, starting at the same time, they meet in 5 seconds; but if they run in the same direction, the faster will overtake the slower in 45 seconds. How many yards a second can each boy run? How long would it take each to run 100 yards? At what point of the track do they meet when they run in opposite directions? At what point are they together when they run in same direction?

HINT. The equations are $5x+5y=90$ and $45x-45y=90$. (Explain.)

30. A and B have \$ 120 between them. If A spends one third of his money and B spends \$ 10, they will have only \$ 85. How much has each?

31. The area of a rectangle is unchanged if the length is diminished by 4 inches and the width is increased by 4 inches. It is increased by 40 square inches if both dimensions are increased by 2 inches. Find the length and the breadth.

32. A man walks 3 miles an hour up hill and $4\frac{1}{2}$ miles an hour down hill. In walking from A to B on a road no part of which is level, he requires $6\frac{1}{2}$ hours; but to walk from B to A he requires only 6 hours. How much of the road from A to B is up hill and how much is down hill?

33. In making the run between two ports a boat averages 14 miles an hour. On a certain trip it runs at its usual rate for 5 hours and then, on account of a fog, is obliged to proceed at one half of its regular speed, arriving in port 4 hours late. What is the distance between the two ports?

34. The circumference of a circle contains 360°. Find the number of degrees in each of two arcs into which it is divided if their difference is 240°.

35. Divide \$ 10,000 between two persons so that one of them shall receive $\frac{5}{11}$ as much as the other.

36. A farmer bought 10 cows and sold 16 sheep, having to pay out \$ 446 in excess of what he received. The next day he bought 3 cows and sold 12 sheep at the same price, paying the difference of \$ 72. Find the cost of one cow and one sheep.

37. The expression $ax^2 + bx - 30$ is equal to 330 when $x = 5$ and is equal to 64 when $x = 12$. Find the values of a and b.

38. A fraction is equal to $\frac{3}{7}$ when 10 is added to its numerator, and to $\frac{1}{3}$ when 4 is subtracted from its denominator. Find the value of the fraction.

39. The numerator of a fraction is 4 less than the denominator. If 16 is subtracted from the numerator, or if 36 is added to the denominator, the resulting fractions will be equal. Find the value of the fraction.

40. The law of a machine is given by $E = aR + b$, when E = efficiency of the machine and R = resistance due to friction. When $E = 4.2$, $R = 10$, and when $E = 7.34$, $R = 20$. Find the values of a and b.

41. The expression $mx - y + b$ is equal to 22, when $x = 5$ and $y = 2$. It is equal to 12 when $x = -3$ and $y = -4$. Find the values of m and b.

42. The expression $\frac{x}{a} + \frac{y}{b} - 1$ is equal to 5 when $x = 2$ and $y = 3$. It is equal to 4 when $x = 3$ and $y = 2$. Find the values of a and b.

43. If $x : y = 11 : 15$, find what values of x and y will satisfy the equation $5x + 7y = 32$.

44. The sum of the angles of a triangle is 180°. How many degrees are there in each of the acute angles of a right-angled triangle, if one of the acute angles is three times the other?

45. If $\frac{3x + 4y}{4x - 3y} = 3$, find the ratio of x to y.

46. What value of x and y will make the two expressions $2x + 5y - 23$ and $4x - 3y$ each have the value zero?

47. The price of admission to a moving picture show is 5¢ for children and 10¢ for adults. 435 tickets are sold and the receipts are $30. How many children and how many adults attend the show?

48. Brass is an alloy of copper and zinc. If copper is 16¢ a pound and zinc is 5¢ a pound, how many pounds of each must be used to make 300 pounds of brass that is worth $37?

49. Bell metal is an alloy of copper and tin. The value of the material in a bell weighing 400 pounds is \$82.56. If copper is 18¢ a pound and tin is 30¢ a pound, how many pounds of each metal are there in the bell?

50. Nickel-steel for automobile construction is an alloy of steel and nickel. The value of the material in 3825 pounds is \$74.50. If steel is 1¢ a pound and nickel is 30¢ a pound, how many pounds of each metal are there in the 3825 pounds?

THREE UNKNOWN NUMBERS

Examples

382. **1.** Solve the system $x + 5y + 6z = 29,$ (1)

$$10x + y + 2z = 18, \quad (2)$$

$$5x + 9y + 3z = 32. \quad (3)$$

$30x + 3y + 6z = 54.$ (Multiplying equation (2) by 3.) (4)

$-29x + 2y = -25.$ (Subtracting equation (4) from equation (1).) (5)

$10x + 18y + 6z = 64.$ (Multiplying equation (3) by 2.) (6)

$9x + 13y = 35.$ (Subtracting equation (1) from equation (6).) (7)

$-377x + 26y = -325.$ (Multiplying equation (5) by 13.) (8)

$18x + 26y = 70.$ (Multiplying equation (7) by 2.) (9)

$-395x = -395.$ (Subtracting equation (9) from equation (8).)

$x = 1.$

$y = 2.$ (Substituting $x=1$ in equation (5) or (7).)

$z = 3.$ (Substituting $x = 1$, $y = 2$ in equations (1), (2), or (3).)

Check. $1 + 5 \cdot 2 + 6 \cdot 3 = 29.$

$10 \cdot 1 + 2 + 2 \cdot 3 = 18.$

$5 \cdot 1 + 9 \cdot 2 + 3 \cdot 3 = 32.$

2. Solve the system $5u + 3v + 2w = 217,$ (1)

$$5u - 3v = 39, \quad (2)$$

$$3v - 2w = 20. \quad (3)$$

$6v + 2w = 178.$ (Subtracting equation (2) from equation (1).) (4)

$9v = 198.$ (Adding equations (3) and (4).)

$v = 22.$

$5u = 105.$ (Substituting $v = 22$ in (2).)

$u = 21.$

$w = 23.$ (Substituting $v = 22$ in (3).)

Let the student check the results.

383. **To solve a system of linear simultaneous equations with three unknowns:**

1. Transpose, if necessary, so that the unknowns will all be on the left-hand side of the equations and collect like terms.

2. Examine the equations to see what unknown can be most easily eliminated.

3. Eliminate that unknown, using two of the equations.

4. Eliminate the same unknown, using the third equation and one of the others.

5. Solve the resulting system for the two remaining unknowns.

6. Substitute the values of these two unknowns in one of the original equations to find the third unknown.

384. In the solution of a system of simultaneous equations involving three unknowns the following suggestions will be found useful:

1. If one of the three equations contains but two of the three unknowns, eliminate the third unknown from the other equations. There will then be two equations containing two unknowns which can be solved, according to § 371. (See example 2, § 382.)

2. In general, eliminate first the unknown that can be most easily eliminated. If the coefficients of this unknown are not the same in two of the equations, make them the same by using the smallest multipliers possible.

3. Addition or subtraction is usually the simplest method of elimination.

4. Study the model solutions and the suggestions given for special methods of shortening the work.

385. To solve a system of four equations with four unknowns, eliminate one of the unknowns by using the four equations in pairs three times, thus deriving three equations with three unknowns. Continue with these equations as indicated in § 383.

EXERCISE

386. 1. How can x be eliminated from the system

$$\begin{aligned} x + y + z &= 12, \\ 4x + 3y + 5z &= 49, \\ 5x - 2y + z &= 12? \end{aligned}$$

2. How can y be eliminated from the above system? How can z be eliminated?

3. Which letter will be most easily eliminated?

4. Which unknown should be eliminated first in solving the system

$$\begin{aligned} 2x - 3y + 7z &= 6, \\ 3x + 4y + 11z &= 18, \\ y - 3z &= -2? \end{aligned}$$

5. Solve the system of equations given in example 1.

6. Solve the system of equations given in example 4.

7. Can you find a definite solution of a system of two equations containing three unknowns?

Solve the following:

8. $2x + 5y - 3z = 13,$
$6x - 3y + 4z = 16,$
$5x + 3y - 6z = 15.$

9. $3x - y + z = 7,$
$x + 2y - 4z = -8,$
$2x - 2y + z = 2.$

10. $2x + 3y = 12,$
$3x + 2z = 11,$
$3y + 4z = 10.$

11. $2u - 7v = 9,$
$u + 4v = 12,$
$u + v + 2w = 14.$

Solve the following:

12. $x + 2y - .7z = 21,$
$3x + .2y - z = 24,$
$.9x + .7y - 2z = 27.$

13. $\frac{1}{5}x - \frac{1}{2}y = 0,$
$\frac{1}{3}x - \frac{1}{2}z = 1,$
$\frac{1}{2}z - \frac{1}{3}y = 2.$

14. $p + q + r = 36,$
$4p = 3q,$
$2p = 3r.$

(Try substitution.)

15. $x + y + z = 100,$
$y = .7x - 4,$
$z = .3x + 4.$

16. $x + y + z = 26,$
$x : z = 11 : 7,$
$y : z = 14 : 9.$

17. $r + s + t = 99,$
$r : s : t = 5 : 3 : 1.$

18. $x + y + z = s,$
$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$

19. $x + y = c,$
$y + z = a,$
$z + x = b.$

20. $2x + y + z = a,$
$x + 2y + z = a,$
$x + y + 2z = a.$

(Add all the equations and divide by 4.)

21. $\frac{x}{7} = \frac{y}{10} = \frac{z}{5},$
$2x + 3y = 88.$

22. $\frac{l}{3} + \frac{m}{6} + \frac{n}{9} + 1 = 0,$
$\frac{l}{6} + \frac{m}{9} + \frac{n}{12} + 1 = 0,$
$\frac{l}{9} + \frac{m}{12} + \frac{n}{15} + 1 = 0.$

23. $\frac{1}{x} + \frac{1}{y} = a,$
$\frac{1}{x} + \frac{1}{z} = b,$
$\frac{1}{y} + \frac{1}{z} = c.$

24. $a + b + c = m,$
$b + c + d = n,$
$c + d + a = v,$
$d + a + b = q.$

(First add all equations, then divide by 3.)

25. $x + y + z + w = a,$
$x - y + z - w = b,$
$x + y - z - w = c,$
$x - y - z + w = d.$

26. $\frac{2}{x} - \frac{5}{3y} + z = 3\frac{4}{27},$
$\frac{1}{4x} + \frac{1}{y} + 2z = 6\frac{11}{72},$
$\frac{5}{6x} - \frac{1}{y} + 4z = 12\frac{1}{36}.$

27. $x + 2y = 9,$
$3y + 4z = 14,$
$7z + u = 5,$
$2u + 5z = 8.$

28. $x + 2y - z = 4.6,$
$y + 2z - x = 10.1,$
$z + 2x - y = 5.7.$

29. $3x + 2y + 3z = 110,$
$5x + y = 4z,$
$2x + z = 3y.$

30. $x = y + \frac{1}{2},$
$z = x + \frac{17}{4},$
$y = 2z - 10.$

31. $x + 3y = \frac{18}{5},$
$x + 5y = 3z,$
$10y - 3z + 2 = 0.$

32. $3(x + 2y) - z = 2,$
$x + 2y = \frac{1}{2}z,$
$x + y + z = 5\frac{1}{6}.$

33. $9x - y = 1,$
$\frac{x + 2z}{3} = \frac{2}{3},$
$y - \frac{z}{6} = 17.$

34. $\frac{8x + y - z}{3} - \frac{8x + 3y - 4}{6} = 2x + y - z,$
$\frac{2x + y + 3z}{4} - \frac{2x - y + z}{2} = 2(x + y - 3) + z,$
$2x + y + z = 6.$

35. $a + 2b + 3c = 32,\ 2a + 3b + c = 42,\ 3a + b + 2c = 40.$

36. $x + y + z = 3,\ 2x + 4y + 8z = 13,\ 3x + 9y + 27z = 34.$

37. $x + y + 2z = 34,\ x + 2y + z = 33,\ 2x + y + z = 32.$

38. $x = 2\frac{1}{3}y - 6,\ y = 3\frac{1}{2}z - 1,\ z = 1\frac{1}{4}x - 8.$

PROBLEMS SOLVED WITH THREE UNKNOWNS

387. 1. The sum of three numbers is 100. If the first is divided by the second, the quotient is 5 and the remainder 1. The second divided by the third gives the same result. What are the numbers?

2. Find three numbers whose sum is 999, and which are to each other as 2 : 3 : 4. Solve, using three unknowns.

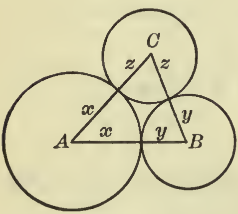
3. Solve problem 2 using only one unknown.

4. A number equals the sum of two other numbers. The largest number diminished by 2 equals three times the smallest. The largest increased by 2 equals twice the result of diminishing the middle number by 2. Find the numbers.

5. Find three numbers such that if the sum of each two is diminished by the other the results are respectively 0, 4, and 8.

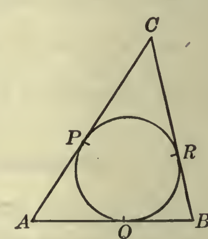
6. Three men, A, B, and C, working together can do a piece of work in $5\frac{1}{3}$ days. A and B together can do the work in $6\frac{6}{7}$ days. A and C can do it in $9\frac{3}{5}$ days. How long will it take each one working alone?

SUGGESTION. One equation is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{5\frac{1}{3}}$.



7. Suppose they all work together and receive $96 for doing the work, how much should each one receive?

8. In the figure the circles touch each other. The sides of the triangle are $AB = 7$ inches, $AC = 7$ inches, $BC = 5$ inches. Find the radii of the circles.



9. In the figure $AQ = AP$, $BQ = BR$, $CR = CP$. Find AP, BQ, and CR, knowing that $AB = 6$ inches, $BC = 8$ inches, $CA = 9$ inches.

10. The expression $ax^3 + bx^2 + cx + 5$ is equal to 10 when $x = 1$, to 15 when $x = 2$, and to 20 when $x = 3$. Find the values of a, b, and c.

HINT. Substitute 1, 2, and 3 successively for x. The resulting equations are

$$a + b + c + 5 = 10.$$
$$8a + 4b + 2c + 5 = 15.$$
$$27a + 9b + 3c + 5 = 20.$$

11. Find the values of l, m, and n in the equation $\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=1$, if it is satisfied by $x=1$, $y=2$, $z=3$; $x=2$, $y=-1$, $z=3$, and $x=-3$, $y=2$, $z=1$.

12. The expression $ax+by+cz=6$ is satisfied by $x=1$, $y=2$, $z=3$; $x=2$, $y=3$, $z=1$; $x=3$, $y=2$, $z=1$. Find the values of a, b, and c.

13. The sum of the angles of a triangle is 180°. Find the number of degrees in each of the angles of a triangle if the sum of the first and second angles and also the sum of the first and third angles is 100°.

14. Find the number of degrees in each angle of a triangle, if the first angle is twice the second and three times the third in value.

15. The sum of the angles of any quadrilateral is 360°. Find the number of degrees in each angle of a quadrilateral, if the sum of the first and second angles is 160°, the sum of the first and third angles is 210°, and the sum of the first, third, and fourth angles is 220°.

16. Find the number of degrees in each angle of a quadrilateral, if the sum of two opposite angles is 200° and their difference is 46°, and the difference of the other two opposite angles is 30°.

17. Three boys have together 88 cents; the first two have 50 cents, while the first and third have 62 cents. How much has each boy?

18. Hard phosphor bronze for machine bearings contains equal amounts of phosphor tin and antimony, with 9 times as much copper as of both other metals. How much of each metal is there in 120 pounds of bronze?

19. Bronze medals are usually an alloy of copper, tin, and zinc. The value of 100 pounds of this material is \$18.57, and the tin costs 16 times as much as the zinc. How many pounds of each metal are there if copper is 18¢, tin 30¢, and zinc 5¢ a pound?

20. An aluminum alloy for crank cases in automobiles is made of aluminum, copper, and tin, and is worth about \$24.50 a hundred pounds. The aluminum costs 20¢ a pound, copper 20¢ a pound, and tin 35¢ a pound. The value of the tin in 100 pounds is $8\frac{1}{2}$ times that of the copper. Find the number of pounds of each in 100 pounds.

21. German silver is made of equal parts of copper, zinc, and nickel, and is worth about 25¢ a pound. The value of the nickel is 9 times that of the zinc and 2 pounds of copper and 1 pound of zinc are worth as much as 1 pound of nickel. Find the value of 1 pound of each.

XV. SQUARE ROOT

388. Power. Square Root. A **power** of a number is the product that arises from using the number one or more times as a factor. The second power is the square of the number, and the number itself is the **square root** of its second power.

Thus, 9, m^2, $a^2 + 2\,ab + b^2$ are the squares of 3, m, and $a + b$, respectively, and 3, m, and $a + b$ are the square roots of 9, m^2, and $a^2 + 2\,ab + b^2$ respectively.

389. Since the square root of a number is one of the two equal factors of a perfect second power, numbers that are not exact squares, strictly speaking, have no square roots. They do, however, have approximate square roots and these approximate square roots can be found to any required degree of accuracy.

390. $5^2 = 25$ and $(-5)^2 = 25$. Therefore 25 has two square roots. Compare a^2 and $(-a)^2 = a^2$.

Thus it appears that any number has two square roots.

1. $\sqrt{4} = 2$ and -2.

For convenience, we write $\sqrt{4} = \pm 2$, and read "the square roots of four are positive and negative two."

2. $\sqrt{a^2} = \pm a$.

3. $\sqrt{4\,x^4y^2} = \pm 2\,x^2y$.

ORAL EXERCISE

391. *Find the square roots of the following numbers:*

1. 81.	4. $.25\,a^4$.	7. $\frac{9}{121}p^4q^{10}$.
2. 196.	5. $x^4y^2z^8$.	8. $.01\,y^2z^8$.
3. $25\,a^2$.	6. $144\,m^6$.	9. $\frac{1}{25}\,a^4b^{14}$.

Find the square roots of the following numbers:

10. $4a^6y^4$. **12.** $2.25x^6$. **14.** $9h^{12}y^6$.

11. $.0144a^2$. **13.** $1.21a^4b^4$. **15.** $\frac{16}{25}a^4b^6$.

392. Square Roots of Monomials by Factoring. The square roots of a perfect square can be found by inspection if the number can readily be factored into prime factors.

1. Find the square roots of 2916.

SOLUTION.
```
2)2916
2)1458
3) 729
3) 243
    81
```
$\therefore \sqrt{2916} = \sqrt{2^2 \cdot 3^2 \cdot 9^2} = \pm 2 \cdot 3 \cdot 9 = \pm 54.$

2. Find the square roots of 5184.

SOLUTION. $5184 = 3^4 \cdot 2^6.$

$$\therefore \sqrt{5184} = \pm(3^2 \cdot 2^3) = \pm 72.$$

3. Similarly

$$\sqrt{441a^2b^4} = \sqrt{3^2 \cdot 7^2 \cdot a^2 \cdot b^4} = \pm 3 \cdot 7 \cdot a \cdot b^2 = \pm 21ab^2.$$

EXERCISE

393. *Find the square roots by factoring:*

1. 1225.
2. 2916.
3. 2401.
4. 4761.
5. 7744.
6. 5184.
7. $784a^2b^2$.
8. $1764a^4b^6$.
9. $15625x^4$.
10. $98.01a^6$.
11. $23.04x^2y^2z^2$.
12. $.0841x^2$.
13. $43.56a^6$.
14. $.0004x^2$.
15. $.0121x^4y^4$.
16. $a^2 + 2ab + b^2$.
17. $a^2 - 2a + 1$.
18. $4(a^2 - 2a + 1)$.

SQUARE ROOT OF POLYNOMIALS

394. Since $a^2 + 2ab + b^2 = (a + b)^2$,

$$\therefore \sqrt{a^2 + 2ab + b^2} = a + b.$$

In order to find the square root of an algebraic expression when it is not so evident, as in this case, a systematic method must be followed.

1. Find the square root of $a^2 + 2ab + b^2$.

	$a^2 + 2ab + b^2$	$a + b$
Subtract a^2	a^2	
Trial divisor, $2a$	$2ab + b^2$	
Complete divisor, $2a + b$		
Multiply by b,	$2ab + b^2$	
Subtract $2ab + b^2$		

1. The square root of a^2 is a, the first term of the root.

2. Subtract a^2, giving the remainder $2ab + b^2$.

3. Since $2ab$ is obtained by taking twice the product of the first term of the binomial $a + b$ by the second term, reversing the process and dividing $2ab$ by $2a$, or twice the part of the root already found, gives the second term, b, of the root. *Hence $2a$ is used as a trial divisor.*

4. Add b to $2a$ to form the complete divisor, $2a + b$.

5. Multiply $2a + b$ by b and subtract.

The above process can easily be extended to extract the square root of any polynomial.

395. **To extract the square root of a polynomial:**

1. Arrange the terms in ascending or descending powers of some letter.

2. Take the positive square root of the first term of the polynomial as the first term of the root and subtract its square from the given polynomial.

3. Take twice the part of root already found for the first trial divisor and divide the first term of the remainder by it. Take the quotient as the second term of the root.

4. Add the quotient just found to the trial divisor to form the complete divisor. Multiply the complete divisor by the last term of the root and subtract the product from the first remainder.

5. If the second remainder is not zero, take twice the part of the root already found for a second trial divisor, and divide the first term of the remainder by the first term of the trial divisor to find the next term in the root. Add this term to the trial divisor to form the second complete divisor and proceed as before until the remainder is zero, or, if the polynomial is not an exact square, until the required number of terms in the root has been found.

Examples

1. Extract the square root of $9x^4 + 6x^3y - 29x^2y^2 - 10xy^3 + 25y^4$.

The square root of $9x^4$ is $3x^2$.	$9x^4 + 6x^3y - 29x^2y^2 - 10xy^3 + 25y^4 \mid 3x^2 + xy - 5y^2$
Subtract $(3x^2)^2$.	$9x^4$
Trial divisor, $6x^2$.	$6x^3y - 29x^2y^2$
Complete divisor, $6x^2 + xy$.	
Multiply by xy.	$6x^3y + x^2y^2$
Subtract $6x^3y + x^2y^2$.	$-30x^2y^2 - 10xy^3 + 25y^4$
New trial divisor, $6x^2 + 2xy$.	
New complete divisor, $6x^2 + 2xy - 5y^2$.	
Multiply by $-5y^2$.	$-30x^2y^2 - 10xy^3 + 25y^4$
Subtract $-30x^2y^2 - 10xy^3 + 25y^4$.	

Therefore the square root of the given expression is $3x^2 + xy - 5y^2$. $-(3x^2 + xy - 5y^2)$ is also a square root of the expression. (Why?)

2. Extract the square root of $4x^6 - 12x^5 + 13x^4 - 14x^3 + 13x^2 - 4x + 4$.

In practice we usually abbreviate the work somewhat, as in the following:

$$
\begin{array}{r|l}
 & 4x^6 - 12x^5 + 13x^4 - 14x^3 + 13x^2 - 4x + 4 \;\big|\; 2x^3 - 3x^2 + x - 2 \\
 & 4x^6 \\
4x^3 - 3x^2 & -12x^5 + 13x^4 \\
 & -12x^5 + 9x^4 \\
4x^3 - 6x^2 + x & 4x^4 - 14x^3 + 13x^2 \\
 & 4x^4 - 6x^3 + x^2 \\
4x^3 - 6x^2 + 2x - 2 & -8x^3 + 12x^2 - 4x + 4 \\
 & -8x^3 + 12x^2 - 4x + 4
\end{array}
$$

EXERCISE

396. 1. How can a square root be checked?

Find the square roots of the following expressions:

2. $m^4 - 2m^2n^2 + n^4$.
3. $4x^4 - 12x^2y^3 + 9y^6$.
4. $25a^6 + 4b^{14} + 20a^3b^7$.
5. $36x^2 - 36xz + 9z^2$.
6. $9a^2x^2 - 24ax + 16$.
7. $4 - 12z + 9z^2$.
8. $\frac{1}{4}a^8 + \frac{4}{9}b^2 - \frac{2}{3}a^4b$.
9. $\frac{4}{25}m^{10} + \frac{4}{7}m^5n + \frac{25}{49}n^2$.
10. $\dfrac{a^2}{16} - \dfrac{a^5}{10} + \dfrac{a^8}{25}$.

11. $\frac{x^2}{9}+\frac{xy}{6}+\frac{y^2}{16}$.

12. $49x^4+121y^4-154x^2y^2$.

13. $25a^2+60abx+36b^2x^2$.

14. $49a^6-42a^5+79a^4-30a^3+25a^2$.

15. $9a^2+24a^3+46a^4+40a^5+25a^6$.

16. $9a^2-24a^3-14a^4+40a^5+25a^6$.

17. $4-12a+25a^2-24a^3+16a^4$.

18. $16a^6-16a^5+20a^4+4a+1$.

19. $a^2-4ab-4ac+8bc+4b^2+4c^2$.

20. $a^2+6ax+6ay+18xy+9y^2+9x^2$.

21. $4x^2-12xy+16xz-24yz+9y^2+16z^2$.

22. $16a^2-40ab+24ac-30bc+25b^2+9c^2$.

23. $16a^2-24a^3+25a^4-20a^5+10a^6-4a^7+a^8$.

24. $4x^2+9y^2+16z^2+25v^2-12xy+16xz-20xv-24yz+30yv-40zv$.

25. $\frac{a^2}{9}+\frac{4ab}{15}+\frac{4b^2}{25}+\frac{ac}{2}+\frac{3bc}{5}+\frac{9c^2}{16}$.

SQUARE ROOT OF ARITHMETICAL NUMBERS

ORAL EXERCISE

397. *Following are the squares of some numbers :*

$1^2=1$.	$10^2=100$.	$100^2=10,000$.	$1000^2=1,000,000$.
$3^2=9$.	$11^2=121$.	$101^2=10,201$.	$1001^2=1,002,001$.
$9^2=81$.	$99^2=9801$.	$999^2=998,001$.	$9999^2=99,980,001$.

By comparing these numbers and their squares, answer the following:

1. How many figures are there in the square of a number of one figure? of two figures? of three figures? of n figures?

2. How many figures are there in the square root of 121? of 1521? of 12,100?

3. Use the facts brought out in examples 1 and 2 to explain why we point numbers off into periods of two figures each when extracting square roots.

4. How many figures are there in the square root of a number of 5 figures? of 6 figures? of 7 figures? of $2n$ figures? of $2n-1$ figures? of $2n+1$ figures?

398. The whole process of extracting the square root of an arithmetical number is shown in finding the square root of $a^2+2ab+b^2$ (§ 394).

Since $37^2=(30+7)^2=30^2+2\cdot 30\cdot 7+7^2=1369$, we can, by reversing the process, find the square root of 1369. Arranging as in § 394, we have

$$\begin{array}{lr|l} & 30^2+2\cdot 30\cdot 7+7^2 & 30+7 \\ & 30^2 & \\ \hline \text{Trial divisor, } 2\cdot 30 & 2\cdot 30\cdot 7+7^2 & \\ \text{Complete divisor, } 2\cdot 30+7 & 2\cdot 30\cdot 7+7^2 & \end{array}$$

The square root of the first term is 30. After subtracting 30^2, the remainder is $2\cdot 30\cdot 7+7^2$.

The trial divisor, $2\cdot 30$, is contained in the first part of the remainder 7 times. Then the complete divisor $2\cdot 30+7$ is formed. The complete divisor is contained exactly 7 times in the remainder.

$\therefore$ 30 + 7, or 37, is the square root of $30^2+2\cdot 30\cdot 7+7^2$, or 1369. -37 is also a square root of 1369.

399. This is equivalent to the following:

$$\begin{array}{lr|r|l} & & 13'69 & 37 \\ & 30^2= & 9\,00 & \\ \text{Trial divisor,} & 2\cdot 30=60 & 4\,69 & \\ \text{Complete divisor,} & 2\cdot 30+7=67 & 4\,69 & \end{array}$$

First separate 1369 into periods of two figures each. (Why?) Since 9 is the largest perfect square in 13, the square root of 1369 lies between 30 and 40. Therefore 3 is the first figure of the root. Subtracting 900 and using $2\cdot 30$ as a trial divisor (why?), we find that the next figure of the root is 7. Completing the divisor by adding 7 (why?), we find that it is contained in the remainder exactly 7 times.

400. To find the square root of a number:

1. Separate the number into periods of two figures each beginning at the decimal point.

2. Write the positive square root of the largest perfect square in the left-hand period as the first figure of the root.

3. Subtract the square of the first figure of the root from the left-hand period and annex the second period of the number.

4. Form a trial divisor by doubling the part of the root already found and annexing one cipher.

5. Divide the remainder by this trial divisor.

6. Write the quotient as the next figure of the root and add the quotient to the trial divisor for a complete divisor.

7. Multiply the complete divisor by the last figure obtained in the root and subtract the result from the last remainder.

8. If there are more than two periods in the number, annex the next period to the remainder and repeat steps 4 to 7 until there is no remainder, or in case the number is not a perfect square, until the required number of decimal places is obtained.

Examples

401. **1.** Find the square root of 4719.69.

```
        47'19.'69 | 68.7
        36        |
128   | 11 19.
      | 10 24.
136.7    | 95.69
         | 95.69
```

The first trial divisor is 120, and the complete divisor is 128.

The second trial divisor is 136.0, and the complete divisor is 136.7.

Therefore $\sqrt{4719.69} = \pm 68.7$.

2. Find the square root of 41209.

```
      4'12'09 | 203
      4       |
403 |  12 09
    |  12 09
```

Here the first trial divisor, 40, is larger than the remainder 12. Put a zero in the root and bring down another period. The trial divisor now becomes 400, and the next figure in the root is 3. The square roots are $\pm$ 203.

3. Find the square root of 2 to three decimal places.

```
        2.'00'00'00 | 1.414+
        1
24     |100
       | 96
281     |400
        |281
2824    |11900
        |11296
           604
```

After pointing off into periods, the decimal point may be neglected. How will the number of *figures* to the left of the decimal point in the answer compare with the number of *periods* to the left of the decimal point in the number?

EXERCISE

402. 1. In extracting the square root of a number, why do we separate the number into periods of two figures each?

2. Will the division of the remainder by the trial divisor *always* give the next figure of the root? Explain your answer.

3. Square the result in example 3, § 401, and add the remainder; that is, $1.414^2 + .000604$.

Extract the square roots of the following:

4. 4096.	13. 119025.	22. 101062809.
5. 6241.	14. .093025.	23. .00917764.
6. 161.29.	15. .007569.	24. 1400.2564.
7. 2.3716.	16. .098596.	25. .00762129.
8. 61504.	17. 12.8881.	26. .0009979281.
9. 1108.89.	18. 11669056.	27. 100020001.
10. 277729.	19. 6504.4225.	28. 29495761.
11. 13456.	20. .83064996.	29. 64128064.
12. 30276.	21. 95121009.	30. 44105040144.

Find the square roots of the following, to two decimal places:

31. 2.2.	35. 7.	39. 3.666.
32. 3.	36. 8.	40. 27.1917.
33. 5.	37. 3.1416.	41. 391.
34. 6.	38. 210.	42. 10.004.

43. 40.003. **45.** $\frac{17}{8}$. **47.** 80.

44. $\frac{13}{2} = 6.5$. **46.** $5\frac{1}{2}$. **48.** 82.

By reducing to a decimal find the square roots to three decimal places in examples 49 to 53:

49. $\frac{2}{3}$. **50.** $2\frac{3}{4}$. **51.** $5\frac{1}{6}$. **52.** $16\frac{2}{3}$. **53.** $\frac{1}{2}$.

By first making the denominator a perfect square find the square roots in examples 54 to 61 to three decimal places:

54. $\frac{5}{6}$.

SUGGESTION. $\sqrt{\frac{5}{6}} = \sqrt{\frac{30}{36}} = \frac{\sqrt{30}}{6} = \frac{5.477}{6} = .913$. This result is correct to three decimal places.

55. $\frac{7}{11}$. **56.** $\frac{5}{8}$. **57.** $\frac{3}{5}$. **58.** $\frac{11}{18}$. **59.** $\frac{17}{32}$.

60. $\frac{\sqrt{4+\sqrt{2}}}{2}$. **61.** $\sqrt{\frac{2+\sqrt{2.5}}{2}}$.

62. To find the approximate square root of $\frac{1}{2}$, why is it better to use $\frac{2}{4}$ than $\frac{1}{2}$?

403. It is sometimes desired to find the square root of an algebraic expression that is not a perfect square correct to a certain number of terms. The process does not differ from that used when the expression is a perfect square.

Find the square root of $1 + x$, in the ascending powers of x to three terms.

$$\begin{array}{l|l|l}
 & 1 + x & 1 + \frac{1}{2}x - \frac{1}{8}x^2 \cdots \\
 & 1 & \\
\hline
2 + \frac{1}{2}x & x & \\
 & x + \frac{1}{4}x^2 & \\
\hline
2 + x - \frac{1}{8}x^2 & -\frac{1}{4}x^2 & \\
 & -\frac{1}{4}x^2 - \frac{1}{8}x^3 + \frac{1}{64}x^4 & \\
\hline
 & \qquad -\frac{1}{8}x^3 - \frac{1}{64}x^4 &
\end{array}$$

To check, square the result and add the last remainder.

EXERCISE

404. *Find the square roots of :*

1. $1 - x$ to three terms and check.
2. $1 - x^2$ to three terms and check.
3. $4 + x^2$ to four terms.
4. $1 + x + x^2$ to three terms and check.
5. $a^2 + x$ to three terms.
6. $1 + 4x^2$ to three terms.
7. $1 + x - x^2$ to three terms.
8. $13x^2 - 3x^3 + 4x^4 - 12x + 4$ to three terms and check.
9. $a^6 + 4a^5b - 2a^4b^2 - 12a^3b^3 + 9a^2b^4$.
10. $16x^6 - 24x^5y + 65x^4y^2 - 42x^3y^3 + 49x^2y^4$.
11. $a^2 + 4ab - 2ac + 4b^2 - 4bc + c^2$.
12. $4a^4 - 12a^3b^2 - 4a^2b^3 + 9a^2b^4 + 6ab^5 + b^6$.
13. $a^4 - 2a^3 + 3a^2 - 2a + 1$.
14. $1 + x^2 - 2x^4 + x^6 + 2x - 2x^3$.
15. $2mn + p^2 + 2np + n^2 + 2mp + m^2$.
16. $1 + 4y^2 + x^2 - 4y + 2x - 4xy$.
17. $4 + 13a^2 + 9a^4 - 4a - 6a^3$.
18. $18x^2 + x^4 + 1 - 8x^3 - 8x$.
19. $a^2b^2 + 2a^2b + a^2 - 2ab^2 - 2ab + b^2$.
20. $\frac{m^4}{4} + \frac{n^6}{9} + \frac{m^2p^4}{4} - \frac{m^2n^3}{3} + \frac{p^8}{16} - \frac{n^3p^4}{6}$.
21. $\frac{x^2}{y^2} - \frac{4xz}{uy} + \frac{4z^2}{u^2} + \frac{6qx}{vy} + \frac{9q^2}{v^2} - \frac{12qz}{uv}$.
22. $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2} + 1\right) + \frac{4y^2}{x^2}\left(\frac{y^2}{x^2} + 1\right) + 3$.

Find the square root of each of the following polynomials to three terms:

23. $1 + x^2$.

24. $1 - 4a^2$.

25. $a^2 + x^2$.

26. $1 + x + x^2 + x^3 + x^4$.

27. $1 - x + x^2 - x^3 + x^4$.

28. $1 + x^2 + x^4 + x^6 + x^8$.

Find the fourth root of:

29. $\frac{1}{3}$ to two decimal places.

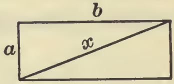
SUGGESTION. Take the square root of the square root.

30. $x^4 + 4x^3 + 6x^2 + 4x + 1$.

31. 14641.

32. 3 to two decimal places.

33. In the figure $x^2 = a^2 + b^2$. Find the value of x to two decimal places when $a = 3$ inches and $b = 7$ inches.



34. As in example 33 find the value of x when $a = b = 10$.

35. Find t correct to one decimal place, if $t = \sqrt{\frac{2d}{g}}$, when $d = 100$ and $g = 32$.

36. Find s in $s = \sqrt{1 - c^2}$, if $c = \frac{1}{2}\sqrt{3}$.

37. Find s in example 36 to two decimal places if $c = \frac{1}{2}$.

38. Find T to two decimal places in $T = \sqrt{s(s-a)(s-b)(s-c)}$, where $a = 6$, $b = 7$, $c = 9$, and $s = \frac{1}{2}(a + b + c)$.

39. Find T in example 38 if $a = b = c = 8$.

40. Would the value of $\sqrt{5}$ to three decimal places give the value of $10\sqrt{5}$ correct to three decimal places? Would it give the value of $\frac{\sqrt{5}}{10}$ correct to three decimal places?

41. Find y if $y = 20\sqrt{1 + \sqrt{2} + \sqrt{3}}$. Get the answer correct to two decimal places.

XVI. QUADRATIC EQUATIONS

405. Quadratic Equations. A **quadratic equation**, or an equation of the second degree containing one unknown number, is an equation that, when reduced to its simplest integral form, contains the second power of the unknown number, and no higher power than the second.

Thus, $2x^2 + 3x = 7$, $x^2 - 5 = 0$, $3y^2 - 5y = 0$ are all quadratic equations, but $(x-1)(x+2) = x^2 + 7$ is not a quadratic equation, for, when reduced to its simplest form, it becomes $x - 9 = 0$, a linear equation.

406. Absolute Term. The term, or group of terms, not containing the unknown number is the **absolute term** of the equation.

Thus, in the equation $3x^2 + 5x - 7 = 0$, -7 is the absolute term.

407. Incomplete Quadratic. If a quadratic equation does not contain a term of the first degree in the unknown number, as, $x^2 - 5 = 0$, or if the absolute term is 0, as $3x^2 - 5x = 0$, it is an **incomplete quadratic equation.**

408. Complete Quadratic. If a quadratic equation contains a term of the second degree in the unknown number, a term of the first degree in the unknown number, and an absolute term, the equation is a **complete quadratic equation.**

Thus, $2x^2 + 3x = 7$ is a complete quadratic equation.

INCOMPLETE QUADRATIC EQUATIONS

409. Solution of the Quadratic Equation Lacking the Term Containing the Unknown of the First Degree.

1. Solve $x^2 - 25 = 0$.

SOLUTION.
$$x^2 - 25 = 0.$$
$$x^2 = 25.$$
$$x = \pm 5.$$

2. Solve $x^2 - 7 = 0$.

SOLUTION.
$$x^2 - 7 = 0.$$
$$x^2 = 7.$$
$$x = \sqrt{7} = \pm 2.65+.$$

410. To solve a quadratic equation in which the first degree term in x is lacking:

1. Clear of fractions, expand, transpose, and reduce the equation to the form $x^2 = k$.

2. Extract the square root of both members, using the double sign before the root of the second member.

3. If the second member is not a perfect square, find its approximate value to any required number of decimal places.

EXAMPLES

1. Solve $(3x + 1.5)(3x - 1.5) = 54$.

SOLUTION.
$$(3x + 1.5)(3x - 1.5) = 54.$$
$$9x^2 - 2.25 = 54. \quad \text{(Expanding.)}$$
$$9x^2 = 56.25. \quad \text{(Transposing and collecting.)}$$
$$x^2 = \frac{56.25}{9}.$$
$$x = \pm\frac{7.5}{3} = \pm 2.5. \quad \text{(Extracting the square roots of both members.)}$$

CHECK. Substitute the answers in the original equation. Thus,
$$(3 \times 2.5 + 1.5)(3 \times 2.5 - 1.5) = 9 \times 6 = 54.$$
Let the student verify the negative answer.

2. Find correct to two decimal places the roots of
$$\frac{x - 10}{6} = \frac{7}{x + 10}.$$

SOLUTION.
$$x^2 - 100 = 42. \quad \text{(Clearing of fractions.)}$$
$$x^2 = 142.$$
$$x = \pm 11.916. \quad \text{(Extracting square root.)}$$
The closest approximation to two decimal places is $x = \pm 11.92$.

411. The verification of approximate answers may become tedious. Approximate verifications will generally serve to detect large errors in answers. In the above example, 12 is a close approximation to the answer. Putting 12 for x in both members we should have $\frac{1}{3}$ in the first member and $\frac{7}{22}$ in the second. These values are not greatly different, and the answer is probably correct. The most satisfactory verification in this case is to go carefully over the work again.

EXERCISE

412. *Solve the following equations:*

1. $x^2 - 169 = 0.$
2. $\dfrac{x^2}{40} = 40.$
3. $13x^2 - 19 = 7x^2 + 5.$
4. $\frac{2}{3}x^2 = \frac{27}{32}.$
5. $x^2 = 6\frac{1}{4}.$
6. $x^2 - a^2 = 2a + 1.$
7. $x^2 + 4 = 13.$
8. $x^2 = 30276.$
9. $x^2 - \frac{1}{2} = 8\frac{1}{2}.$
10. $3x^2 = 7 - x^2.$
11. $\dfrac{z^2}{3} = 27.$
12. $3x^2 = 210.25.$
13. $(2x + 7)(5x - 9) + (2x - 7)(5x + 9) = 1874.$
14. $\dfrac{2m - 1}{m - 2} = \dfrac{m - 5}{3m - 2}.$
15. $\dfrac{25 + x}{9 + x} = \dfrac{13 + x}{47 - x}.$
16. $\dfrac{4}{x + 3} - \dfrac{4}{x - 3} = -\dfrac{1}{3}.$
17. $\dfrac{3x^2}{4} - \dfrac{15x^2 + 8}{6} = 2x^2 - 3.$

413. Solution of the Incomplete Quadratic with Absolute Term Lacking.

Solve $3x^2 - 5x = 0.$ (See § 239.)

SOLUTION. $(3x - 5)x = 0.$ (Factoring the left member.)

$3x - 5 = 0$ or $x = 0.$ (§ 238.)

$x = \frac{5}{3}$ or $0.$

414. **To solve a quadratic equation in which the absolute term is lacking:**

1. Clear of fractions, expand, transpose, and simplify until the equation is in the form $ax^2 + bx = 0$.

2. Solve the equation by factoring.

EXAMPLES

1. Solve $x + 2 = \dfrac{7x - 4}{x - 2}$.

SOLUTION.

$$x^2 - 4 = 7x - 4. \quad \text{(Clearing of fractions.)}$$
$$x^2 - 7x = 0. \quad \text{(Simplifying.)}$$
$$x(x - 7) = 0.$$
$$x = 0 \text{ or } x - 7 = 0. \quad \text{(Why?)}$$
$$x = 0 \text{ or } x = 7.$$

Check the answers mentally.

2. Solve $(3x - 5)^2 - (2x - 3)^2 = 16$.

SOLUTION.

$$9x^2 - 30x + 25 - 4x^2 + 12x - 9 = 16. \quad \text{(Expanding.)}$$
$$5x^2 - 18x = 0. \quad \text{(Transposing and collecting.)}$$
$$x(5x - 18) = 0. \quad \text{(Factoring.)}$$
$$x = 0 \text{ or } 5x - 18 = 0. \quad \text{(Why?)}$$
$$x = 0 \text{ or } \tfrac{18}{5}.$$

EXERCISE

415. *Solve:*

1. $x^2 - 9x = 0$.
2. $3x^2 + 4x = 0$.
3. $5m^2 - 3m = 0$.
4. $ax^2 + bx = 0$.
5. $(3x - 7)^2 - (5x - 3)^2 = 40$.
6. $(x + 5)^2 + (x - 3)^2 = 34$.
7. $2(x + 3)^2 - (x - 3)^2 = 9$.
8. $(x - a)^2 + 2(x + a) = a^2 + 2a$.
9. $(x - 5)^2 - (2x - 3)^2 = 16$.
10. $(m - 1)(m + 1) = 2m - 1$.
11. $\dfrac{x^2}{3} - \dfrac{5x}{2} = 0$.
12. $\dfrac{2p^2}{5} + 2 + \dfrac{6p}{7} = 2$.
13. $y + 10 = \dfrac{6y - 100}{y - 10}$.

Solve:

14. $\frac{2}{z+1}=1-\frac{1}{z-1}$.

15. $\frac{x+3}{x+5}+\frac{3x-2}{x-5}=1$.

16. $\frac{2x-1}{2x+1}+\frac{2x+3}{x+2}=\frac{1}{2}$.

17. $\frac{x-1}{x-2}-\frac{x-3}{x-4}=-\frac{1}{4}$.

COMPLETE QUADRATIC EQUATIONS

416. The complete quadratic equation has been solved by factoring. The student should carefully review §§ 237 to 239.

EXERCISE

417. *Solve the following by factoring:*

1. $x^2-3x+2=0$.
2. $x^2+5x+6=0$.
3. $2t^2+t-3=0$.
4. $m^2-2m-24=0$.
5. $\frac{x}{2}+\frac{10}{6x}=\frac{11}{6}$.
6. $3x^2-5x=10+2x^2-2x$.
7. $(x+5)^2=2(x+3)^2-17$.
8. $x(x-1)=380$.
9. $(2p-8)^2=4(3p+25)+12$.
10. $\frac{z+2}{z+3}=\frac{36}{(z+3)^2}-1$.
11. $y(y+5)=3(-y-5)$.
12. $\frac{x^2}{6}=\frac{x}{3}-\frac{x}{2}+1$.
13. $(2m)^2-5(2m)-6=0$.
14. $4(r^2-1)+r+1=0$.
15. $5(x^2-4)-(x-2)=0$.
16. $s^2-8=7s-14$.
17. $3w^2+w=10$.
18. $(5x-2)(6x^2-x-2)=0$.
19. $x^2-5x+4=0$.
20. What is a root of an equation?

418. **Completing the Square.** In § 204 we learned that any one of the three terms of a perfect trinomial square can be supplied if we know the other two terms.

ORAL EXERCISE

419. *In each of the following supply the proper number in the parenthesis to make a perfect trinomial square, and find the square root of the trinomial:*

1. $x^2 + 2ax + (\ \ )$.
2. $4x^2 + 4x + (\ \ )$.
3. $4x^4 + 4x^2 + (\ \ )$.
4. $4x^4 + (\ \ ) + 4x^2$.
5. $9 + 6x + (\ \ )$.
6. $(\ \ ) + 6x + 1$.
7. $x^2 + 2x + (\ \ )$.
8. $x^2 + 6x + (\ \ )$.
9. $x^2 - 8x + (\ \ )$.
10. $x^2 + x + (\ \ )$.
11. $x^2 - 3x + (\ \ )$.
12. $x^2 - \frac{x}{2} + (\ \ )$.
13. $x^2 - \frac{x}{a} + (\ \ )$.
14. State a rule for completing the square in expressions of the form $x^2 + px$.

420. The *p*-form. Every quadratic equation with one unknown number can be written in the form $ax^2 + bx + c = 0$. This can be further simplified by dividing through by a, giving $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. By putting $\frac{b}{a} = p$ and $-\frac{c}{a} = q$, the equation assumes the form $x^2 + px = q$. For convenience we shall call this the p-form. It requires that the coefficient of x^2 be $+1$, and that the absolute term be in the second member of the equation. p and q may be any positive or negative numbers, integers, fractions, monomials, or polynomials.

EXAMPLES

Examples 1 to 5 below are in the p-form.

1. $x^2 - 7x = 10$; $p = -7$, and $q = 10$.
2. $x^2 + \frac{x}{2} = -9$; $p = \frac{1}{2}$, and $q = -9$.
3. $x^2 = 90$; $p = 0$, and $q = 90$.
4. $x^2 + (a + b)x = 0$; $p = a + b$ and $q = 0$.
5. $x^2 - \frac{x}{a} = (b - c)^2$; $p = -\frac{1}{a}$, and $q = b^2 - 2bc + c^2$.

EXERCISE

421. *Change the following into the p-form, and determine the value of p and q for each:*

1. $x^2 + bx + c = 0.$
2. $2x^2 + 15.9 = 13.6x.$
3. $(x - 7)(x - 5) = 0.$
4. $(x - 1)^2 = a(x^2 - 1).$
5. $c(a - x)^2 + (x - b)^2 = a^2 + b^2.$
6. $\dfrac{21 - x}{4 - x} - \dfrac{x}{4} + 1 = 0.$

422. The Solution of the Complete Quadratic Equation by Completing the Square. The solution of a quadratic equation by factoring fails when the factors cannot be found. The method about to be given will solve in all cases.

1. Solve the equation $x^2 + 9x = 10$.

SOLUTION. $x^2 + 9x = 10.$

$x^2 + 9x + \frac{81}{4} = \frac{121}{4}.$ (Adding $\frac{81}{4}$ to both members.)

$x + \frac{9}{2} = \pm \frac{11}{2}.$ (Extracting square roots.)

$x = 1$ or $-10.$

CHECK. $1^2 + 9 \cdot 1 = 10.$

$(-10)^2 + 9(-10) = 100 - 90 = 10.$

How do you determine that $\frac{81}{4}$ is to be added? Why do you add it to *both* members? Why do you use the double sign in the second member? Why do you not use the double sign in both members?

The solution of example 1 illustrates the method of solving complete quadratic equations by "completing the square." This equation was in the p-form at first. The steps required to reduce any quadratic equation to the p-form are already familiar to the student.

2. Solve, getting the answers correct to two decimal places, $x^2+(x+2)^2=180$.

SOLUTION.
$$x^2+x^2+4x+4=180. \quad \text{(Why ?)}$$
$$2x^2+4x=176. \quad \text{(Why ?)}$$
$$x^2+2x=88. \quad \text{(Why ?)}$$
$$x^2+2x+1=89. \quad \text{(Why ?)}$$
$$x+1=\pm 9.43^+. \quad \text{(Why ?)}$$
$$x=8.43^+ \text{ or } -10.43^+.$$

These roots can be obtained to any required degree of accuracy by finding the square root of 89 correct to more decimal figures.

423. To solve a complete quadratic equation:

1. Reduce the equation to the *p*-form.

2. Complete the square of the first member by adding to both members the square of one half the coefficient of *x*.

3. Extract the square root of each member of the equation and solve the resulting linear equations.

EXAMPLES

1. Solve $6x^2=x+15$.

SOLUTION.
$$6x^2-x=15. \quad \text{(Why ?)}$$
$$x^2-\tfrac{1}{6}x=\tfrac{5}{2}. \quad \text{(Why ?)}$$
$$x^2-\tfrac{1}{6}x+\tfrac{1}{144}=\tfrac{5}{2}+\tfrac{1}{144}=\tfrac{361}{144}. \quad \text{(Why ?)}$$
$$x-\tfrac{1}{12}=\pm\tfrac{19}{12}. \quad \text{(Why ?)}$$
$$x=\tfrac{1}{12}\pm\tfrac{19}{12}=\tfrac{5}{3} \text{ or } -\tfrac{3}{2}.$$

CHECK.
$$6\cdot(\tfrac{5}{3})^2=\tfrac{5}{3}+15 \text{ or } \tfrac{50}{3}=\tfrac{50}{3}.$$

Let the student check the other root.

2. Solve $x^2+ax=ac+cx$.

SOLUTION.
$$x^2+ax-cx=ac.$$
$$x^2+(a-c)x=ac.$$
$$x^2+(a-c)x+\frac{(a-c)^2}{4}=ac+\frac{(a-c)^2}{4} \text{ or } \frac{a^2+2ac+c^2}{4}.$$
$$x+\frac{a-c}{2}=\pm\frac{a+c}{2}.$$
$$x=-\frac{a-c}{2}\pm\frac{a+c}{2}.$$
$$x=c \text{ or } -a.$$

Let the student check mentally.

EXERCISE

424. *Solve the following equations by completing the square, finding all roots correct to two decimal places:*

1. $x^2 + 2x = 3$.
2. $x^2 - 10x = 200$.
3. $t^2 + t = 12$.
4. $x^2 - x = 12$.
5. $x^2 + 3x = 10$.
6. $u^2 + 3u = 108$.
7. $x^2 + 17x = 30$.
8. $x^2 - 8x + 15 = 0$.
9. $a^2 - 40a + 111 = 0$.
10. $x^2 - 2.4x + .8 = 0$.
11. $x^2 - 2ax = 3a^2$.
12. $c^2 + 2c = -1$.
13. $x^2 - 3bx + \frac{9}{4}b^2 = 0$.
14. $x^2 - 32x = 32$.
15. $2v^2 + 3v = 108$.
16. $3x^2 - 5x = 2$.
17. $6x^2 + 1 = 5x$.
18. $5m - m^2 = -50$.
19. $15x^2 + 8x = 3.75$.
20. $9x^2 + 17x = 310$.
21. $\frac{1}{6}x^2 + \frac{1}{72}x - \frac{1}{12} = 0$.
22. $b(7 - b) = 6$.
23. $\frac{5}{4}x^2 - 11x - 15 = 0$.
24. $(2x - 15)(3x + 8) = -154$.
25. $8x^2 + 2x - 15 = 0$.
26. $20x^2 + 2x - 7 = 0$.
27. $6(x^2 + 1) = 13x$.
28. $3s^2 - 16 = 7s$.
29. $x^2 + \frac{1}{2}x = 2$.
30. $x^2 + 6.51 = 5.2x$.
31. $y^2 + .2y - .15 = 0$.
32. $x^2 + bx - 2b^2 = 0$.
33. $(x - 7)(x - 5) = 40$.
34. $z + \frac{1}{z} = \frac{5}{2}$.
35. $x^2 + 22(x + 5) = 0$.
36. $(4x - 1)(x + 1) = 75$.
37. $p(p - 6) = 7p - 42$.
38. $4x^2 + (x - 1)^2 - 3x = 31$.
39. $q(q - 2) = 67$.
40. $7 \cdot x + \frac{1}{4} = \frac{4x + 7}{16x}$.
41. $x(x + 1) = \frac{103}{36}$.
42. $\frac{3x}{4}\left(x - \frac{4}{3}\right) = \frac{19}{24}$.
43. $x^2 - 8x - 14 = 0$.
44. $(x + 4)(x + 5) = 2(x + 2)(x + 4)$.
45. $(3 - 2x)(1 - 3x)(2 - x) = x(1 - 6x)(x - 2)$.
46. $(x + 6)(x - 4) + (x + 2)(x - 2) = 56$.

47. $(x-1)^2+(x+1)+(2x+3)^2=29.$

48. $4c^2-3c=31-(c-1)^2.$

49. $(4-d)(5d+1)-d(4-d)=0.$

50. $(7-x)x=\left(1-\frac{x}{7}\right)(3x+8).$

51. $x(x+1)(x+3)-(x+\frac{1}{3})(x+\frac{7}{3})(x+\frac{1}{2})=0.$

52. $3x(x+1)-(7+2x)=0.$

53. $118z-2\frac{1}{2}z^2=20.$

54. $x^2+ab=x(a+b).$

55. $(x-3b)(x+2b)=6b^2.$

56. $(2y-e)(2e-y)+(5y+2e)e=0.$

57. $r(r+1.25)=.75(r+1.25).$

58. $(r+3)^2+(r+5)^2=514.$

59. $L^2=1-L.$

60. $\frac{1}{12}x^2+\frac{7}{144}x-\frac{1}{12}=0.$

61. $x(7x-1)+\frac{4}{3}x-\frac{20(x+3)+8}{2}=0.$

62. $\frac{4x-7}{5}+\frac{2x+3}{9}=\frac{23}{45}x^2-\frac{94}{45}.$

63. $\frac{2x(2x-5)}{2x-1}-\frac{2}{2x-1}-3=0.$

64. $\frac{8x}{x+2}-6-\frac{20}{3x}=0.$

65. $\left(\frac{x}{2}+1\right)^2-\frac{3}{2}x=3.$

66. $\frac{20x}{3}+\frac{1}{x}=\frac{19}{3}.$

67. $\frac{x+11}{x+3}-\frac{2x+1}{x+5}=0.$

68. $\frac{48}{z+3}-\frac{165}{z+10}+5=0.$

69. $x:2(x-3)=x-3:x-1.$

70. $10:Z=Z:10-Z.$

71. $\frac{3}{2(x^2-1)}-\frac{1}{4(x+1)}=\frac{1}{8}.$

72. $\frac{4x}{x-1}-\frac{x+3}{x}=4.$

73. $\frac{x^2-x+3}{x^2-4x+5}=\frac{x+5}{x-1}.$

Solve the following equations by completing the square, finding all roots correct to two decimal places:

74. $\dfrac{3y-2}{y}-\dfrac{2(y-2)}{y+1}-2=0.$ 75. $\dfrac{1}{x+1}+\dfrac{1}{x+2}-\dfrac{1}{x+3}=0.$

76. $\dfrac{x-5}{x+3}+\dfrac{x+8}{x-3}+\dfrac{80}{9-x^2}=\dfrac{1}{2}.$

77. $\dfrac{2x+7}{2x-3}+\dfrac{3x-2}{x+1}=5.$

425. 1. In solving any problem by means of an algebraic equation, the student should first carefully *read* the problem so that he can correctly translate the verbal language into the algebraic language of the equation.

2. He should then solve the equation in the most direct way possible.

3. He should check and interpret the results of the solution. It should be noted that the conditions of the problem, with all their restrictions, cannot always be translated into an algebraic equation, so that the solution of the equation may give roots that do not satisfy the conditions of the problem. See example 4 following.

PROBLEMS

426. 1. The area of a circle is πR^2 where $\pi = 3.1416$ and R is the radius. Find the radius of the circle whose area is 78.54 square inches.

2. The area of a circle is 100 square inches. Find the radius correct to two decimal places.

3. Find two consecutive integers if the sum of their squares is 25.

SOLUTION. Let $x =$ the smaller number.

Hence $x + 1 =$ the larger number. (Why?)

Then $x^2 + (x + 1)^2 = 25$, (By the conditions.)

or $2x^2 + 2x + 1 = 25$. (Why?)

$\therefore x^2 + x = 12$. (Why?)

$$x^2 + x + \tfrac{1}{4} = \tfrac{49}{4}.\quad \text{(Why?)}$$
$$x + \tfrac{1}{2} = \pm \tfrac{7}{2}.$$
$$x = 3 \text{ or } -4 = \text{the smaller number,}$$
$$\text{and } x + 1 = 4 \text{ or } -3 = \text{the larger number.}$$

The answers are 3 and 4, or -4 and -3, either pair of numbers satisfying the conditions.

4. The square upon the longest side of a right-angled triangle is equal to the sum of the squares upon the other two sides. In a certain right-angled triangle one of the sides about the right angle is 1 inch longer than the other and the hypotenuse is 5 inches long. Find the two sides about the right angle.

Solution. Let x = number of inches in one of the sides.

Hence $x + 1$ = number of inches in other.

Then $x^2 + (x+1)^2 = 25$. (Why?)

The solution from this point on is exactly the same as in problem 3, but the negative answers that were satisfactory in problem 3 have to be rejected. The sides of the triangle are 3 inches and 4 inches.

Note. In the solution of applied problems, careful attention must be given to the interpretation of the answers obtained. Sometimes one, sometimes both answers satisfy the conditions of the problem. It may happen that neither of the answers will satisfy the conditions. (Why?)

5. By solving as in problem 3, find out if there are two consecutive integers the sum of whose squares is 32.

6. Separate 360 into two factors whose difference is 9.

(This problem can be solved by any one of the three equations (*a*) $x(x-9) = 360$; (*b*) $x(x+9) = 360$; (*c*) $x - \frac{360}{x} = 9$. Explain and solve each equation.)

7. The sum of a number and its reciprocal is $\frac{5}{2}$. What is the number? Do both roots of the equation satisfy the conditions?

8. The area of a rectangle is 720 square inches. The difference of its two unequal sides is 12 inches. Find the dimensions.

9. How long is each side of a square if the diagonal is 10 inches long? (See problem 4.)

10. The two unequal sides of a rectangle are in the ratio of 5 to 12, and the diagonal is 6.5 inches long. Find the dimensions.

SUGGESTION. Let the number of inches in the sides be $5x$ and $12x$ and see problem 4.

11. The area of a rectangle is 2400 square inches. The ratio of its two unequal sides is 5 to 12. Find its dimensions.

12. The sum of the areas of two squares is 233 square inches; the sum of their sides is 21 inches. Find the side of each square.

13. In the accompanying figure the shaded area is equal to 21.46 square inches. Find the radius of the circle. (The side of the square equals twice the radius of the circle, and the difference in their areas is the shaded part. See also the first problem of this set.)

14. Find two numbers, one of which is double the other, such that the sum of their squares exceeds the sum of the numbers by 68.

15. Find two numbers, one of which is double the other, if the square of their sum exceeds the sum of their squares by 100.

16. Find two consecutive numbers if the sum of their squares exceeds the product of the numbers by 43.

17. If 18 is divided by a certain number, the quotient is greater by $1\frac{1}{2}$ than if the divisor were increased by 2. Find the first divisor.

18. Find two consecutive even numbers the sum of whose reciprocals is $\frac{9}{40}$.

19. A train makes a run of 280 miles in 1 hour and 45 minutes less time than another train whose rate is 8 miles an hour less. Find the rate of each train.

SUGGESTION. Remember that distance $\div$ rate $=$ time.

20. A woman buys cloth for \$8. Had she paid 40¢ more per yard she would have received one yard less for the same amount. How much per yard did the cloth cost?

21. A man bought a flock of sheep for \$75. If he had paid the same sum for a flock containing 3 more sheep, they would have cost \$1.25 less per head. How many did he buy, and at what price per head?

22. $S = \frac{1}{2}gt^2 + v_0t$ is a formula much used in physics. Find t when $S = 520$, $g = 32$, and $v_0 = 24$.

23. Find the value of t in $S = \frac{1}{2}gt^2 + v_0t$ when $S = 100$, $g = 32$, and $v_0 = 0$.

24. $m : n = x^2 : (a - x)^2$ is a relation used in the study of light. Find the value of x when $m = 4$, $n = 3$, and $a = 150$ cm.

25. A rope 100 feet long is stretched around four posts set at the corners of a rectangle whose area is 576 square feet. Find the dimensions of the rectangle.

26. The sum of the two unequal sides of a rectangle is 20 feet and the diagonal is 16 feet long. Find the lengths of the sides correct to 2 decimal places.

27. A farmer bought some sheep for \$134.40. If each sheep had cost him 80¢ less, he could have bought 3 more for the same amount. How many sheep did he buy?

28. A traveler made a journey of 630 miles. He would have required 4 days less to make the journey had he gone 10 miles farther each day. How many days did the journey require, and how many miles did he travel each day?

29. A traveler made a journey of 630 miles. He would have required 4 days more to make the journey had he traveled 10 miles less each day. How many days did the journey require, and how far did he travel each day?

30. Solve $V = \frac{1}{3}h(S^2 + s^2 + Ss)$ for S where $V = 252$, $h = 12$, and $s = 3$.

31. The sides of a triangle are 18 inches, 16 inches, and 9 inches. By how much may the sides be equally shortened so that they may form the sides of a right-angled triangle?

32. Divide a straight line 8 inches long into two segments such that double the square on one segment shall equal the rectangle whose base and altitude are respectively the whole line and the other segment.

33. Solve the equation $ax^2 + bx + c = 0$ when $a = 5$, $b = 20$, $c = 16$.

REVIEW QUESTIONS

427. **1.** What is a quadratic equation?

2. Illustrate each of the three forms of quadratic equations.

3. What is a complete quadratic equation? an incomplete quadratic?

4. Give the rules for solving incomplete quadratics.

5. In what form must an equation be written if it is to be solved by factoring?

6. Give at sight six roots of the equation $(x^2 + 2x)(x^2 - 1)(x^2 - 5x + 6) = 0$. Can you give at sight any roots of $(x^2 + 2x)(x^2 - 1) = 37$? (Explain.)

7. What is the p-form of the quadratic equation? How is the quadratic in one unknown reduced to the p-form? Why is the p-form used when solving by completing the square?

8. Reduce $7x^2 - 3x + 2 = 5(3 - x)$ to the p-form and give the value of the absolute term when in the p-form.

9. Given the equation $x^2 + 2x + 1 = 9$. In solving this equation the next step gives $x + 1 = \pm 3$. Why is it not $\pm(x + 1) = \pm 3$?

10. Can you solve a quadratic equation that lacks the absolute term by completing the square?

XVII. SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

428. One equation of the first degree and the other of the second degree.

1. Of what degree in x is $ax^2 + bx + c = 0$? of what degree in a? (See §§ 245, 246.) Of what degree in x and y is $2x + y = 10$? Of what degree in x and y is $3xy = 1$?

2. What is the principle of substitution? (See § 374.)

3. Explain the solution of simultaneous equations of the first degree by the method of substitution.

429. Solve the simultaneous quadratic system,

$$x + y = 6,$$
$$x^2 + 3y = 16.$$

$y = 6 - x$. (From the first equation.)

$x^2 + 3(6 - x) = 16$. (Substituting.)

$$x^2 - 3x = -2.$$
$$x^2 - 3x + \tfrac{9}{4} = \tfrac{1}{4}.$$
$$x - \tfrac{3}{2} = \pm \tfrac{1}{2}.$$
$$x = 2 \text{ or } 1.$$

$y = 4$ or 5. (Substituting in the first equation.)

There are two sets of answers. $x = 2$, $y = 4$ will satisfy both equations. Also $x = 1$, $y = 5$ will satisfy both equations.

430. To solve a system of simultaneous equations when one equation is of the first degree and the other of the second degree:

1. Find the value of one of the unknown numbers in terms of the other unknown and known numbers from the first degree equation.

2. Substitute the value of the unknown thus found in the second degree equation and solve the resulting quadratic.

3. Substitute each value of the unknown already found in the original linear equation and solve for the other unknown.

4. Arrange the answers in pairs as found.

EXERCISE

431. *Solve the following systems of simultaneous quadratic equations. Find results involving decimals correct to two decimal places:*

1. $x - y = 2,$
$x^2 + xy = 40.$ (Why is it better, in this example, to substitute $x - 2$ than to use $x = y + 2$?)

2. $2x^2 - y^2 = 7,$
$2x - y = 3.$

3. $3x - y = 5,$
$xy - x = 0.$

4. $u^2 + v^2 = 40,$
$u = 3v.$

5. $5x^2 + y = 3xy,$
$2x - y = 0.$

6. $(x + y)(x - 2y) = 7,$
$x - y = 3.$

7. $xy = 135,$
$\frac{x}{y} = \frac{3}{5}.$

8. $x^2 - y^2 = 240,$
$x - y = 6.$

9. $x + y = 37,$
$x^2 + y^2 = 949.$

10. $m^2 + n^2 = 130,$
$m + n : m - n = 8 : 1.$

11. $x^2 + y^2 + xy = 147,$
$x + y = 13.$

12. $15(x^2 - y^2) = 16xy,$
$x - y = 2.$

13. $x + y = 15,$
$x^2 + y^2 = 150.$

14. $p^2 + q^2 = 25,$
$3p + 4q = 24.$

15. $x^2 + 2xy - y^2 = 7(x - y),$
$2x - y = 5.$

16. $r : s = 9 : 4,$
$r : 12 = 12 : s.$

17. $\frac{x^2 + y + 1}{y^2 + x + 1} = \frac{3}{2},$
$x - y = 1.$

18. $xy = 360,$
$x - y = 9.$

19. $a = 2b,$
$(a + b)^2 - (a^2 + b^2) = 100.$

20. $\frac{x^2 + y^2}{x + y} = \frac{x + y + 2}{3} = \frac{5}{3}.$

21. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$ (Regard $\frac{1}{x}$ and $\frac{1}{y}$ the unknowns.)
$\frac{1}{x} + \frac{1}{y} = 5.$

22. $y - z = 8,$
$yz = 240.$

23. $\frac{x+y}{x-y} = 12,$
$x^2 - y^2 = 48.$

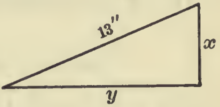
24. $\frac{1}{a} + \frac{1}{b} = \frac{9}{20},$
$\frac{1}{a^2} + \frac{1}{b^2} = \frac{41}{400}.$

432. Many of the problems in § 426 could have been solved by using two unknown numbers instead of one. In general, the student will find it easier to state such problems algebraically by using two unknowns than by using one unknown.

433. Problems Involving Simultaneous Quadratics.

1. The difference of two numbers is 4 and the sum of their squares is 106. Find the numbers.

The equations required are evidently $x - y = 4$, $x^2 + y^2 = 106$. Let the student solve the system.



2. The sum of two sides about the right angle in a right-angled triangle is 17 inches, and the hypotenuse is 13 inches long. Find the sides about the right angle.

Solution. Let x = the number of inches in one of the sides, and y = the number of inches in the other side.

Then $x + y = 17$, (By the first condition.)
and $x^2 + y^2 = 169$ (By the second condition.)

$$x = 17 - y.$$
$$(17 - y)^2 + y^2 = 169.$$
$$289 - 34y + y^2 + y^2 = 169. \quad \text{(Why?)}$$
$$y^2 - 17y = -60. \quad \text{(Why?)}$$
$$y^2 - 17y + \tfrac{289}{4} = \tfrac{49}{4}. \quad \text{(Why?)}$$
$$y - \tfrac{17}{2} = \pm \tfrac{7}{2}.$$
$$\therefore y = 12 \text{ or } 5,$$
$$\text{and } x = 5 \text{ or } 12. \quad \text{(Why?)}$$

Therefore the sides about the right angle are 12 inches and 5 inches.

3. The sum of two numbers is 21 and their product is 68. What are the numbers?

4. The perimeter of a rectangle is 27 feet and the area is 44 square feet. What are the dimensions?

5. The perimeter of a rectangle is 34 inches, and the diagonal is 13 inches. What are the dimensions?

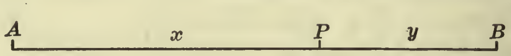
6. Two fields of unequal size are both square. Their total area is 50 acres and it takes $1\frac{1}{2}$ miles of fence to inclose them. Find the dimensions of the fields.

7. The sum of the areas of two circles is 13,273.26 square yards and the sum of the radii is 79 yards. Find the lengths of the radii.

8. The product of the sum and the difference of two numbers is a and the quotient of the sum divided by the difference is b. Find the two numbers.

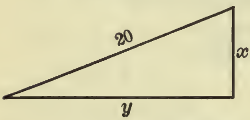
9. The area of a rectangle is 1224 square feet and the unequal sides are in the ratio of 3 to 5. Find the dimensions.

10. A line AB, 10 inches long, is divided at P into two



parts, x and y, so that x is a mean proportional between AB and y. Find the lengths of x and y.

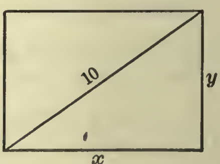
11. In a right-angled triangle the hypotenuse is 20 inches long and the sum of the other sides is 28 inches. Find the other sides.



12. The hypotenuse of a right triangle is 10 inches and the perimeter is 24 inches. Find the length of the two sides about the right angle.

13. The area of a right-angled triangle equals one half the product of the sides about the right angle. If the area of a right-angled triangle is 30 square inches and the sum of the sides about the right angle is 20 inches, find the length of these sides correct to two decimal places.

14. The perimeter of a rectangle is 26 inches and one of the diagonals is 10 inches long. Find the lengths of the sides.



GENERAL REVIEW

434. **1.** Factor (*a*) $x^2 - 6\,ax - 9\,b^2 - 18\,ab$.

(*b*) $24\,x^2 + 6\,xy - 18\,y^2$. (Princeton.)

2. Find the L. C. M. and the H. C. F. of
$(x^3 + a^3)(x^2 + a^2)$, $(x^2 + ax + a^2)(3\,x - a)$, $3\,x^2 + 2\,ax - a^2$.
(Harvard.)

3. Simplify $\left(\frac{1}{x+1} + \frac{1}{x-1}\right) \div \left(\frac{1}{x-1} - \frac{1}{x+1}\right)$. (Regents.)

4. A number multiplied by 17 is increased by 1056. What is the number?

5. In 1912 a father's age was three times that of his son who was born in 1890. When will the son's age equal one half the father's age?

6. Solve the system by addition or subtraction:

$$\frac{x-3}{y-4} = \frac{6}{7},\quad \frac{x+5}{y+1} = \frac{7}{6}.$$

7. Solve by substitution:

$$9\,x = 13\,y,\quad \frac{x}{y} - \frac{y}{5} = \frac{2}{35}.$$

8. Divide $x^4 - 3\,x^3 - 36\,x^2 - 71\,x + k$ by $x^2 - 8\,x - 3$.

9. For what value of k in example 8 will the division be exact?

10. A and B start from the same place, A traveling due north and B due west. B travels one mile an hour faster than A and at the end of 3 hours they are 15 miles apart. What is the rate of each?

11. Resolve into factors:

(*a*) $\frac{x^2}{y^2} - 3\frac{y^2}{x^2} + 2$. (*b*) $x^6 - y^6$. (*c*) $9 - 6\,c + c^2$.

12. Simplify $\frac{3}{x+1} - \frac{x+1}{x-1} - \frac{x^2}{1-x^2}$.

13. Solve $\frac{3}{x+1} - \frac{x+1}{x-1} - \frac{x^2}{1-x^2} = 0$.

14. Solve $\frac{1}{2x-1}-\frac{2}{x+2}-\frac{3}{2x+2}+\frac{1}{2x^2+3x-2}=0.$

15. Simplify $\frac{x-4}{2x-1}-\frac{3x-5}{x+2}+\frac{5x^2+9x+14}{2x^2+3x-2}.$

16. Find the value of S in $S=\frac{1}{2}gt^2+v_0t$ when $t=3, g=32,$ and $v_0=0.$

17. Find g in example 16 if $S=277.6$, $t=4$, $v_0=5$.

18. Find t in example 16, if $S=450$, $g=32$, $v_0=10$.

19. Solve $3x^2-7x-2=0$, finding the values of x correct to two decimal figures.

20. Divide $x^4+x^3+ax^2+bx-3$ by x^2+2x-3, and find what values a and b must have in order that there shall be no remainder.

21. When $\frac{a}{b}=\frac{c}{d}$, prove $\frac{2a-3b}{3b}=\frac{2c-3d}{3d}.$

22. Solve $5x-\frac{3}{x}\frac{(x-1)}{3}=\frac{3(x-5)}{2}.$

23. Find $\sqrt{8.1}$ to 3 decimal figures.

24. A room is one yard longer than it is wide. At $1.75 a square yard a carpet for the floor costs $52.50. Find the dimensions of the room.

25. Solve $ax-by=0$, $x-y=c$.

26. Factor
(a) $27x^3-64$;
(b) $16a-25ab^2$;
(c) $16x^2+25y^2+40xy$;
(d) x^6+y^6;
(e) $x-1+x^3-x^2$. (Regents.)

27. Simplify

$$\frac{\frac{a^2+b^2}{b}-a}{\frac{1}{b}-\frac{1}{a}}\times\frac{a^2-b^2}{a^3+b^3}\times\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\times\left(\frac{a}{a+b}+\frac{b}{a-b}\right).$$

(Regents.)

28. Find the square root of $5x^2 - 23x^4 + 12x + 8x^5 - 22x^3 + 16x^6 + 4$.

29. Solve the system $x + y + z = 4$, $2x + 3y - z = 1$, $3x - y + 2z = 1$.

30. Factor (*a*) $x^2 - 4ax - 4b^2 + 8ab$;

(*b*) $(a + b)(c^2 - d^2) - (a^2 - b^2)(c - d)$.

31. Simplify $\dfrac{x - \dfrac{1}{x^2}}{x + \dfrac{1}{x} - 2} \div \dfrac{\left(x + \dfrac{1}{x}\right)^2 - 1}{\left(1 - \dfrac{1}{x}\right)\left(x - 1 + \dfrac{1}{x}\right)}$.

32. Solve the system $x + y + z = 1$, $mx + y + z = 0$, $4x + 4y - 3z = 0$.

33. For what value of m will the value of x and z be the same in example 32?

34. A train makes a run of 120 miles. A second train starts one hour later and traveling 6 miles an hour faster reaches the end of the same run 20 minutes later than the first train. Find the time of the run for each train.

35. Solve $(x - 1)(x - 2) = 15$.

36. There are ten numbers in a series as follows: $x, x + y, x + 2y \cdots x + 9y$. The product of the first and last is 70, and the sum of all the numbers is 95. Find the numbers.

37. A chauffeur engages to accomplish a journey of 100 miles in a specified time. After he has traveled 50 miles at a rate that will just enable him to keep his engagement, his car is delayed 20 minutes. By driving the remaining distance 5 miles an hour faster, he reaches his destination on time. Find the original rate. (Sheffield.)

38. Simplify $\dfrac{\dfrac{1}{3x - 2} - \dfrac{1}{3x + 2}}{9 - \dfrac{4}{x^2}}$.

39. Solve $3bx - 7(x + b) + ac - cx = 0$.

40. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{4a^2-5b^2}{5b^2}=\frac{4c^2-5d^2}{5d^2}$.

41. If $y=\frac{a+2t}{b-at}$, find the value of t in terms of the other letters. (Princeton.)

42. Find the L.C.M. of $6x^2-5x-6$, $3+x-2x^2$, $2x^3-3x^2-2x+3$.

43. Factor into linear factors $4a^2b^2-(a^2+b^2-c^2)^2$.
(Princeton.)

44. Factor (*a*) $32a^3b^3-4b^6$; (*b*) $x^2+2xy-a^2-2ay$.

45. Solve $x-y=4$, $\frac{1}{y}-\frac{1}{x}=\frac{4}{117}$.

46. How much water must be added to 80 pounds of a 5 % salt solution to obtain a 4 % solution? (Yale.)

47. If $m=\frac{1}{a+1}$, $n=\frac{2}{a+2}$, $p=\frac{3}{a+3}$; find the value of $\frac{m}{1-m}+\frac{n}{1-n}+\frac{p}{1-p}$. (Univ. of Penn.)

48. Simplify $\frac{a}{b^2}-\cfrac{a}{b^2+\cfrac{cd}{a-\frac{c}{b}}}$. (Harvard.)

49. Evaluate $a-\{5b-[a-(3a-3b)+2c-3(a-2b-c)]\}$, if $a=-3$, $b=4$, $c=-5$. (Yale.)

50. A train from Chicago to Denver running at an average rate of 40 miles an hour makes the journey in $6\frac{2}{5}$ hours shorter time than one that runs 32 miles an hour. Find the distance from Chicago to Denver.

51. The rates of the trains remaining as in the last problem, the faster of two trains from New York to Chicago makes the run in 6 hours less time than the slower train. Find the distance between New York and Chicago.

52. Find by factoring the H. C. F. and the L. C. M. of $x^2 + a^2 - b^2 + 2ax$, $x^2 - a^2 + b^2 + 2bx$ and $x^2 - a^2 - b^2 - 2ab$. (Harvard.)

53. A and B each shoot 30 arrows at a target. B makes twice as many hits as A and A makes three times as many misses as B. Find the number of hits and misses of each. (Univ. of California.)

54. I have $6 in dimes, quarters, and half dollars, there being 33 coins in all. The number of dimes and quarters together is ten times the number of half dollars. How many coins of each kind are there?

55. Write by inspection all the roots of

$$(x^2 + 2x)(x^2 - 3x + 2)(x - 10) = 0.$$

56. Solve $x^2 - 1.6x - .23 = 0$, obtaining the values of the roots correct to three significant figures. (Harvard.)

57. Solve $\frac{x+1}{3x+2} = \frac{2x-3}{3x-2} - 1 - \frac{36}{4-9x^2}$. (Princeton.)

58. A train running 30 miles an hour requires 21 minutes longer to go a certain distance than does a train running 36 miles an hour. What is the distance? (Cornell.)

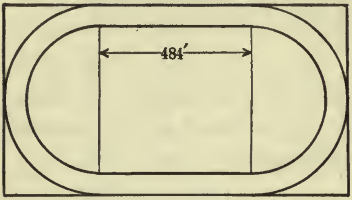
59. A physician having 100 cubic centimeters of a 6 % solution of a certain medicine wishes to dilute it to a $3\frac{1}{2}$ % solution. How much water must he add? (A 6 % solution contains 6 % of medicine and 94 % of water.) (Case.)

60. Solve $2x + 5y = 85$, $2y + 5z = 103$, $2z + 5y = 57$. (Vassar.)

61. Find the values of k that will satisfy the equation

$$k^2 - a^2 - 8k - 4a + 12 = 0.$$

62. A workman receives $3.60 for his regular day's work and double pay for overtime. In a certain day he received $5.20 for 11 hours' work. How much of the time was overtime?



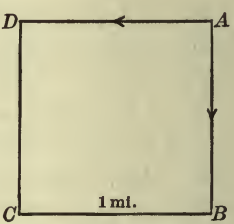
63. A half mile race track is to be laid out with semicircular ends in a rectangular field. If each of the straight sides is 484 feet long, what must be the radius of the semicircular ends of the track. (Use $\pi = 3\frac{1}{7}$.)

64. The horse power (H. P.) of a gasoline engine is given approximately by the formula $\text{H. P.} = \frac{D^2 N}{2.5}$, where D is the diameter of the cylinder in inches and N is the number of cylinders. State this formula as a rule.

65. Using the formula of example 63, find the H. P. of a two cylinder motor boat engine if the diameter of each cylinder is 5 inches.

66. What is the approximate diameter of each cylinder of a six cylinder 40 H. P. automobile engine?

67. Two men start from the same corner A, going in the directions indicated around a field 1 mile square. The man going along AB walks 4 miles an hour, and the other man goes 3 miles an hour. Where and after how long will they meet?



XVIII. EXPONENTS

435. What is an exponent? (See § 64.)

Up to the present time only positive integers have been used as exponents, and for positive integral exponents we have developed the laws for multiplication and division.

$$a^m \cdot a^n = a^{m+n}.$$ **Multiplication Law** (§ 114).

$$a^m \div a^n = a^{m-n}.$$ **Division Law** (§ 147).

436. Laws of Exponents for Involution.

1. To find a power of a power:

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2$$ (By the definition of exponent.)

$$= a^6.$$ (By the law of exponents in multiplication.)

$$\therefore (a^2)^3 = a^{2\times 3}.$$

Also $(a^4)^3 = a^4 \cdot a^4 \cdot a^4 = a^{12}$.

$$\therefore (a^4)^3 = a^{4\times 3}.$$

In general, $(a^m)^n = a^{mn}$. **Power of a Power.**

2. To find a power of a product:

$$(ab)^3 = ab \cdot ab \cdot ab$$ (Definition of Exponent.)

$$= a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot$$ (§ 73.)

$$= a^3b^3.$$

Also $(abc)^2 = abc \cdot abc = a \cdot a \cdot b \cdot b \cdot c \cdot c = a^2b^2c^2$.

In general, $(ab)^n = a^nb^n$. **Power of a Product.**

An integral exponent of a product can be distributed to the factors of the product.

3. To find a power of a quotient:

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^3}{b^3}. \quad \text{(Why?)}$$

In general, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. **Power of a Quotient.**

An integral exponent of an indicated division can be distributed to the dividend and the divisor.

ORAL EXERCISE

437. *Perform the operations indicated:*

1. $a^5 \cdot a^4$.
2. $x^{10} \cdot x^7$.
3. $y^4 \cdot y^8$.
4. $m^2 \cdot m^a$.
5. $b^c \cdot b^6$.
6. $a^{2n} \cdot a^n \cdot a$.
7. $z^{2x} \cdot z^{5x} \cdot z$.
8. $a^{b-1}a$.
9. $c^{n-2}c^2$.
10. $d^{n+1} \cdot d^{n-1}$.
11. $x^{2y+1}x^{y-1}$.
12. $m^{p+2} \cdot m^{p-3}$.
13. $t^{2v+1} \cdot t^{1-v}$.
14. $x^{10} \div x^6$.
15. $a^{11} \div a^7$.
16. $m^5 \div m^2$.
17. $c^x \div c^2$.
18. $d^{2x} \div d^{x-1}$.
19. $b^{c+5} \div b^4$.
20. $y^{2x+1} \div y^{x-1}$.
21. $a^{2n-3} \div a^{n+1}$.
22. $(a^2)^5$.
23. $(c^5)^4$.
24. $(x^7)^9$.
25. $\left(-\frac{a^2}{b^3}\right)^4$.
26. $\left(-\frac{x^y}{y^x}\right)^3$.
27. $\left(\frac{p^r}{q^t}\right)^s$.
28. $(x^n y^p)^q$.
29. $(t^{n-1} \cdot s^{n-2})^n$.
30. $(a^2b^3)^4$.

438. According to the definition of an exponent (§ 64), such expressions as $a^{\frac{2}{3}}$, a^0, a^{-5} have no meaning, since it is impossible to use a two thirds times, or zero times, or -5 times, as a factor. It is convenient, however, to use fractions, zero, and negative numbers as exponents and to define them in such a way that the laws for positive integral exponents shall hold for these exponents.

439. Fractional Exponents. Assuming the law of multiplication to hold for fractional exponents, we have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$.

$\therefore a^{\frac{1}{2}} = \sqrt{a}$, since $a^{\frac{1}{2}}$ is one of the two equal factors of a.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$

$\therefore a^{\frac{1}{3}} = \sqrt[3]{a}.$ (Why?)

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}.$

Again, $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2.$

$\therefore a^{\frac{2}{3}} = \sqrt[3]{a^2}.$ (Why?)

Similarly, $a^{\frac{3}{5}} \cdot a^{\frac{3}{5}} \cdot a^{\frac{3}{5}} \cdot a^{\frac{3}{5}} \cdot a^{\frac{3}{5}} = a^{\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}} = a^3.$

$\therefore a^{\frac{3}{5}} = \sqrt[5]{a^3}.$ (Why?)

In general, $a^{\frac{p}{q}} = \sqrt[q]{a^p}.$

440. Stated in words we have:

The numerator of a fractional exponent indicates a power and the denominator indicates a root.

Thus, $2\,a^{\frac{2}{3}} = 2\sqrt[3]{a^2}$; $3\,a^{\frac{1}{p}}b^{\frac{q}{2}} = 3\sqrt[p]{a}\sqrt{b^q}$; $8^{\frac{5}{3}} = \sqrt[3]{8^5}$ or $(\sqrt[3]{8})^5 = 32.$

EXERCISE

441. *Write with radical signs, noting carefully what is the base for each fractional exponent:*

1. $a^{\frac{3}{2}}.$
2. $a^{\frac{7}{3}}.$
3. $2\,x^{\frac{1}{2}}.$
4. $(3\,a)^{\frac{1}{2}}b^{\frac{2}{3}}.$
5. $\left(\frac{a^2b}{3\,x}\right)^{\frac{2}{5}}.$
6. $a + b^{\frac{1}{2}}.$
7. $(a+b)^{\frac{1}{2}}.$
8. $a + 2\,b^{\frac{1}{2}}.$
9. $3\,r^{\frac{p}{q}}.$
10. $5\,r^{\frac{p}{q}}s^{\frac{n}{2}}.$
11. $5\,x^{\frac{p}{2}}(a^m)^{\frac{p}{q}}.$
12. $2\,a^{\frac{1}{3}} + 3\,b^{\frac{2}{3}}.$

Write with fractional exponents:

13. $\sqrt[3]{a^2}.$
14. $\sqrt{a^3}.$
15. $3\sqrt[3]{a^2}.$
16. $\sqrt{2\,ab}.$
17. $\sqrt[3]{a+b}.$
18. $a + \sqrt[3]{b}.$
19. $4\sqrt[3]{(a+b)^2}.$
20. $3\sqrt{x} \cdot \sqrt[p]{a^r}.$
21. $\sqrt{a^2 - b^2}.$
22. $2\sqrt{a} + \sqrt{2\,a}.$
23. $\sqrt[n]{a} + \sqrt{a}.$
24. $\sqrt[p]{x^m} + \sqrt[m]{x^p}.$

Find the values of the following:

25. $16^{\frac{3}{2}}$. (Extract the root first.)

26. $4^{\frac{3}{2}}$; $8^{\frac{2}{3}}$.

27. $27^{\frac{2}{3}}$; $9^{\frac{3}{2}}$.

28. $(-125)^{\frac{4}{3}}$.

29. $27^{\frac{2}{3}} \cdot 27^{\frac{1}{3}}$.

30. $9^{\frac{3}{2}} \cdot 9^{\frac{1}{2}}$.

31. $\frac{1}{5} \cdot 25^{\frac{3}{2}}$.

32. $25^{\frac{1}{2}} + 216^{\frac{2}{3}}$.

33. $243^{\frac{3}{5}} - 256^{\frac{3}{4}}$.

34. $(-8)^{\frac{2}{3}} \cdot (-8)^{\frac{1}{3}}$.

35. $18769^{\frac{1}{2}}$.

36. $1.5^{\frac{1}{2}}$ to two decimal places.

37. Apply the third law of exponents to $(a^{\frac{2}{3}})^3$. What does the result suggest as to the meaning of $a^{\frac{2}{3}}$?

442. Zero Exponent. If we assume law 1 to hold when $n = 0$, we shall have:

$$a^m \cdot a^0 = a^{m+0} = a^m,$$

$$\text{or } a^m \cdot a^0 = a^m.$$

Dividing by a^m, $\quad a^0 = a^m \div a^m = 1.$

$$\therefore a^0 = 1.$$

443. Stated in words we have:

Any base with the exponent zero is equal to unity.

Thus, $\quad a^0 = 2^0 = 100^0 = (x + y)^0 = 1.$

444. Negative Exponent. If we assume law 1 to hold when n is a negative number, we may write:

$$a^{-3} \cdot a^3 = a^{-3+3} = a^0 = 1,$$

$$\text{or } a^{-3} \cdot a^3 = 1.$$

Dividing by a^3, $\quad a^{-3} = \frac{1}{a^3}.$

In general, $\quad a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1,$

$$\text{or } a^{-n} \cdot a^n = 1.$$

Dividing by a^n, $\quad a^{-n} = \frac{1}{a^n}.$

445. The last equation is the definition of a negative exponent in algebraic symbols. Stated in words we have:

Any base affected by a negative exponent is equal to 1 divided by that base with a positive exponent of the same absolute value.

EXAMPLES

1. $3x^0 = 3 \cdot 1 = 3.$
2. $3x^{-1} = 3 \cdot \frac{1}{x} = \frac{3}{x}.$
3. $(xy)^0 = 1.$
4. $a^0 + x^0 = 1 + 1 = 2.$
5. $a^{-2} + b^{-2} = \frac{1}{a^2} + \frac{1}{b^2} = \frac{b^2 + a^2}{a^2b^2}.$

ORAL EXERCISE

446. *Simplify by using the definitions of exponents and reducing the results when possible:*

1. $a^4 \cdot a^0.$
2. $4(a + b)^0.$
3. $[4(a + b)]^0.$
4. $4(a^0 + b^0).$
5. $(a^0)^n.$
6. $1^n \cdot (-1)^3.$
7. $(x - y)^0 \cdot 5^{-1}.$
8. $(25^{\frac{1}{2}} + 8^{\frac{1}{3}})^{-1}.$
9. $3a^{-1};\ 3^{-1}a.$
10. $9 \cdot 3^{-2};\ 9 + 3^{-2}.$
11. $9 \div 3^{-2};\ 9^{-1} + 3^{-2}.$
12. $(-\frac{1}{3})^{-3};\ (-\frac{1}{3})^{-2}.$
13. $9^2 \cdot 3^{-5}.$
14. $3^{-2} \cdot 2^{-3};\ (-3)^{-2};\ -3^{-2}.$
15. $\sqrt{25} \cdot 5^{-1}.$
16. $(a^0 + b^0 + c^0)^{-2}.$
17. $64 \cdot 2^{-6}.$
18. $4^{-2} \div 2^{-4}.$
19. $64^{-1} \cdot 2^6.$
20. $a^0 \cdot a^{-1} \cdot a^{-2}.$
21. $a^0 + \frac{1}{a^0}.$

EXERCISE

447. *Simplify as much as possible:*

1. $(25^{\frac{1}{2}} + 8^{\frac{1}{3}})^{-2}.$
2. $8^{-2} - 32^{\frac{2}{5}}.$
3. $2^2 + 2^{-1} \cdot 4^{\frac{3}{2}}.$
4. $9 \cdot 4.5^{-1}.$
5. $5^{-2} + (-\frac{1}{2})^3.$
6. $25 \div 5^{-1}.$

Simplify as much as possible:

7. $a^3 \div (-a)^2$; $(-a)^3 \div a^2$.

8. $100^{\frac{3}{2}} + 100^{\frac{1}{2}} + 100^{-\frac{1}{2}}$.

9. $3^{-2} + 27^{\frac{2}{3}}$.

10. $3^{-2} + 3 \cdot 9^{\frac{3}{2}}$.

11. $3 \cdot 3^{-1} + 4 \div 4^{-1}$.

12. $(2^0 + 3^0)^3$; $(2^3 + 3^3)^0$.

13. $\sqrt[3]{8} \cdot 8^{\frac{1}{3}}$; $2^{-2} + 8^{-\frac{2}{3}}$.

14. $16^{\frac{3}{4}} + 2^{-2} \cdot 8^{\frac{2}{3}}$.

448. Negative Exponents in Fractions.

1. $\dfrac{ab^{-1}}{c} = \dfrac{a}{c} \cdot b^{-1} = \dfrac{a}{c} \cdot \dfrac{1}{b} = \dfrac{a}{bc}$. (Explain each step.)

2. $\dfrac{ab}{2c} = \dfrac{ab}{2} \cdot \dfrac{1}{c} = \dfrac{abc^{-1}}{2}$. (Explain.)

3. $\dfrac{a^{-1}b}{cd^{-1}} = a^{-1} \cdot b \cdot \dfrac{1}{c} \cdot \dfrac{1}{d^{-1}} = \dfrac{1}{a} \cdot b \cdot \dfrac{1}{c} \cdot d = \dfrac{bd}{ac}$.

4. $\left(\dfrac{a}{b}\right)^{-2} = \dfrac{1}{\left(\dfrac{a}{b}\right)^2} = \dfrac{1}{\dfrac{a^2}{b^2}} = \dfrac{b^2}{a^2} = \left(\dfrac{b}{a}\right)^2$. (Explain.)

449. These examples illustrate the following principles:

1. Any factor of the numerator of a fraction may be made a factor of the denominator, or any factor of the denominator may be made a factor of the numerator, if the sign of its exponent is changed.

2. Any fraction affected by a negative exponent is equal to the reciprocal of that fraction with the sign of its exponent changed.

The student should carefully note that *factors, not terms*, can be changed from the numerator to the denominator or from the denominator to the numerator by changing the sign of the exponent.

EXERCISE

450. *In the following examples make the exponents positive and simplify the expressions as much as possible:*

1. $2x^{-3}$.

2. $2a^{-3}x$.

3. $\dfrac{a}{x^{-4}} + \dfrac{a^2}{x^{-2}}$.

4. $a^{-1}b^{-1}$.

5. $a^{-1} + b^{-1}$.

6. $(a + b)^{-1}$.

7. $\dfrac{a^{-1}+b^{-1}}{x^{-1}+y^{-1}}$.

8. $(-3)^3 \cdot 3^{-3}$.

9. $\dfrac{a^2}{b^2}+\dfrac{b^{-2}}{a^{-2}}$.

10. $\dfrac{ab^{-4}}{a^{-4}b}$.

11. $(\frac{2}{3})^{-2}$.

12. $(-\frac{1}{2})^{-3}$.

13. $\left(\dfrac{1}{a}\right)^{-1} \cdot a^{-1}$.

14. $\dfrac{2\,a^{-1}b^{-1}}{a^{-2}-b^{-2}}$.

Write examples 15 *to* 26 *in integral form, using negative exponents when necessary:*

15. $\dfrac{1}{a}$.

16. $\dfrac{1}{a^2}$.

17. $\left(\dfrac{1}{a^2}\right)^2$.

18. $\dfrac{a^m}{a^{-m}}$.

19. $\left(\dfrac{b}{a}\right)^x$.

20. $\dfrac{x}{-z^3}$.

21. $\dfrac{a^m}{xy^2}$.

22. $\dfrac{1}{2^{-1}}$.

23. $\dfrac{1}{x^2}-\dfrac{1}{y^2}$.

24. $\dfrac{1}{x+y}$.

25. $\dfrac{x+y}{x-y}$.

26. $\dfrac{x^{-1}+y^{-1}}{x^{-1}y^{-1}}$.

Find the numerical values in examples 27 *to* 34:

27. $\dfrac{1}{5^0} \cdot \dfrac{5}{(-4)^{-3}}$.

28. $\dfrac{1}{4^{-2}} \cdot 2^{-4}$.

29. $\dfrac{2}{3^{-2}} \cdot 2^{-1}$.

30. $100 \cdot 5^{-3}$.

31. $1000 \cdot 5^{-2}$.

32. $48 \cdot 10^{-5}$.

33. $5 + 17 \cdot 10^{-3}$.

34. $2135 \cdot 10^{-7}$.

35. Write with positive exponents and simplify the result: $\dfrac{a^{-3}+b^{-3}}{a^{-2}-b^{-2}}$.

SOLUTION.
$$\frac{\dfrac{1}{a^3}+\dfrac{1}{b^3}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}=\frac{\dfrac{b^3+a^3}{a^3b^3}}{\dfrac{b^2-a^2}{a^2b^2}}=\frac{b^3+a^3}{a^3b^3}\cdot\frac{a^2b^2}{b^2-a^2}$$
$$=\frac{b^2-ba+a^2}{ab(b-a)}.$$

Write examples 36 to 43 with positive exponents, and simplify the results:

36. $\dfrac{a - a^{-1}}{a + a^{-1}}$.

37. $\left(\dfrac{a}{b}\right)^{-1} + \dfrac{a^{-1}}{b^{-1}}$.

38. $\dfrac{a^{-3}b^{-m}}{x^{-4}y^{-n}}$.

39. $\dfrac{3a^{-1}}{3 + a^{-1}}$.

40. $\dfrac{2x^{-1} - b}{2a - b^{-1}}$.

41. $\dfrac{a^{-1} + b^{-1}}{a^{-2} + b^{-2}}$.

42. $\dfrac{a^{-2} + b^{-2}}{a^{-4}b^{-4}} \cdot (b^2 + a^2)^{-1}$.

43. $2(a + b)^{-1} + 2(a - b)^{-1}$.

451. We shall *assume* that all the laws of exponents that have been established for positive integral exponents hold for the other exponents that have been defined. For convenience, we repeat here the four definitions and the five laws of exponents in algebraic symbols.

Definitions of Exponents

1. $a^n = a \cdot a \cdot a \cdots$ **to** n **factors when** n **is a positive integer.**
2. $a^{\frac{p}{q}} = \sqrt[q]{a^p}$. **Fractional Exponent.**
3. $a^0 = 1$. **Zero Exponent.**
4. $a^{-n} = \dfrac{1}{a^n}$. **Negative Exponent.**

Laws of Exponents

1. $a^m \cdot a^n = a^{m+n}$. **Multiplication Law.**
2. $a^m \div a^n = a^{m-n}$. **Division Law.**
3. $(a^m)^n = a^{mn}$. **Power of a Power.**
4. $(ab)^n = a^n b^n$. **Power of a Product.**
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$. **Power of a Quotient.**

The definitions of exponents and the laws of exponents should be thoroughly committed to memory.

ORAL EXERCISE

452. *Apply law* 1 *to each of the following:*

1. $a^5 \cdot a^{-3}$.
2. $a^5 \cdot a^{-5}$.
3. $y^m \cdot y^0$.
4. $x^m \cdot x^{-n}$.
5. $r^m \cdot r$.
6. $r^{m+3} \cdot r^{m-3}$.
7. $b^{n+2} \cdot b^{2-n}$.
8. $a^{n-1} \cdot a \cdot a^n$.
9. $a^{n-1} \cdot a^{n-1}$.
10. $a^{-3} \cdot b^{-2} \cdot b^5 \cdot a^3$.
11. $(a+b)(a+b)^{-1}$.
12. $(a+b)^2(a+b)^{-1}$.
13. $(a+b)^{-2}(a+b)^0$.
14. $(x+y)^{n+1}(x+y)^{2-n}$.
15. $5\,ax^{-6} \cdot 5^{-2}abx^7$.
16. $(a-x)^{-3}(x-a)^{-2}$.
17. $(a-x)^{-4}(x-a)^5$.
18. $\sqrt{a} \cdot \sqrt[3]{a} \cdot a^{\frac{1}{6}}$.
19. $\sqrt[3]{a^{-1}} \cdot a^{-\frac{2}{3}}$.
20. $d^{\frac{4}{5}} \cdot d^{\frac{1}{10}}\sqrt{d}$.

Apply law 2 *to examples* 21 *to* 41:

21. $a^8 \div a^3$.
22. $a^8 \div a^{-3}$.
23. $a^{-5} \div a^5$.
24. $a^5 \div a^{-5}$.
25. $b^{n-3} \div b^3$.
26. $a^{-2}b^3 \div a^{-3}b^2$.
27. $a^{-3} \div (-a)^{-3}$.
28. $a^{-3} \div (-a^2)$.
29. $a^{-3} \div (-a^2)$.
30. $2^{n-3} \div 2^{n-4}$.
31. $2\,x^{-n} \div x^{n-2}$.
32. $3\,a^0b^n \div (-b^{n-2})$.
33. $(x-y)^{-1} \div (x-y)^{-2}$.
34. $(x-y)^3 \div (y-x)$.
35. $a^{\frac{2}{3}} \div a^{-\frac{1}{3}}$.
36. $a^{\frac{3}{2}} \div a^{\frac{2}{3}}$.
37. $a \div 2\,a^{-\frac{2}{5}}$.
38. $(2\,x)^0 \div (2\,x)^{-\frac{1}{2}}$.
39. $3\,abc \div a^{\frac{1}{2}}b^{-1}c^0$.
40. $(a+b) \div (a+b)^{-1}$.
41. $25\,a^{-\frac{2}{3}} \div 5^{-1}a^{\frac{1}{3}}$.

Apply law 3 *to examples* 42 *to* 47:

42. $(a^{-2})^{-3}$; $(-a^3)^2$.
43. $(-a^2)^3$; $(a^2)^{-3}$.
44. $(a^{-3})^{-4}$; $(a^3)^{-4}$.
45. $(x^4)^0$; $(x^0)^{-1}$.
46. $(-b^{-2})^{-2}$; $(-b^{-2})^{-3}$.
47. $(r^{m-n})^{\frac{1}{m^2-n^2}}$.

Apply law 3 to examples 48 to 61:

48. $(a^{x+y})^{x-y}$.

49. $[(2a)^{-3}]^{-1}$.

50. $[(x+y)^{-\frac{1}{2}}]^2$.

51. $(x^{n-1})^{n+1}$.

52. $(2^6)^{\frac{2}{3}}$; $[(-2)^{-6}]^{-\frac{2}{3}}$.

53. $(a^2)^n$; $(a^n)^3$.

54. $(x^{n-1})^2$.

55. $(m^{x-y})^{x+y}$.

56. $\sqrt{xyz} \div (xyz)^{\frac{1}{3}}$.

57. $[(-a)^4]^3$.

58. $[(a^{-m})^{-r}]^p$.

59. $\left[\left(-\dfrac{a}{b}\right)^3\right]^{\frac{1}{4}}$.

60. $(a^{n-1})^n$.

61. $(2^{-2})^{-3}$.

Apply laws 4 and 5 in the following:

62. $(2a^2b^3)^4$.

63. $(-ab)^2$; $(ab)^{n-2}$.

64. $(-ab)^3$; $(-ab)^4$; $(-ab)^5$.

65. $(x^ny)^n$.

66. $(3ab^{n-2})^3$.

67. $\left(\dfrac{a^2b^3}{2c}\right)^3$.

68. $c^{2r}\left(\dfrac{c^n}{c^r}\right)^2$.

69. $\left(\dfrac{a^{-2}b^0}{x^{-4}y^{-1}}\right)^{-3}$.

70. $[(x-y)(x+y)^{-1}]^{-1}$.

71. $\left(\dfrac{5}{x}\right)^{-2}$; $\dfrac{m}{x}\cdot\left(\dfrac{m}{x}\right)^{-1}$.

72. $3\left(\dfrac{a}{b}\right)^{-1}\cdot\dfrac{b}{a}$.

73. $[(x-y)(y-x)^{-1}]^3$.

74. $[(x-y)^2(y-x)^{-2}]^3$.

75. $(\frac{1}{3})^{-3}(\frac{1}{3})$.

76. $[2(x-y)^{-1}]^{-2}$.

77. $\left(\dfrac{81}{a^2+2ab+b^2}\right)^{\frac{1}{2}}$.

78. $\sqrt{\dfrac{a^2+2ab+b^2}{4}}$.

79. $(6^0\cdot 25^2\cdot a^4)^{\frac{1}{4}}$.

80. $(8^{-1}\cdot a^{-3}b^6)^{-\frac{1}{3}}$.

81. $\left(\dfrac{a^0+b^0}{4}\right)^2$.

EXERCISE

453. 1. State the four definitions of exponents in words.

2. State the five laws of exponents in words.

In the following examples use definitions and laws to simplify:

3. $9^0\cdot 9^{-1}\cdot 9^{\frac{1}{2}}\cdot 9^{-\frac{1}{2}}$.

4. $(9^{-2})^{-1}$.

5. $(-64)^{-\frac{2}{3}}$.

6. $(64)^{-\frac{3}{2}}$.

7. $.008^{-\frac{2}{3}}$.

8. $5 \cdot 2^{-2}$.

9. $(\frac{1}{3})^{-2}$.

10. $7 \cdot 7^{-1}$; $7 \div 7^{-1}$.

11. $.1^{-2}$; $(-.1)^2$.

12. $(a^0 + b^0)^2$; $(a^0b^0)^2$.

13. $(a^2 + b^2)^0$.

14. $4(a-b)^0$.

15. $(\frac{1}{3})^{-4} - 16^0$.

16. $(-1)^0 + (-1)^2$.

17. $(-1)^0 + (-1)^2 + (-1)^3$.

18. $\dfrac{2}{9(a^0 + b^0 + c^0)^{-2}}$.

19. $2a^{-\frac{3}{4}} \div a^{-1}$.

20. $a^{-\frac{3}{4}} \div 2a^{-1}$.

21. $c^{-8} \div c^{-5}$.

22. $x^5 \div x^{-5}$.

23. $a\left(\dfrac{a}{b}\right)^{-1}$.

24. $x^{2-n} \div x^n$.

25. $x^{n-2} \div x^n$.

26. $(a^{-1} + b^{-1}) \div ab$.

27. $\dfrac{1}{x} \cdot \left(\dfrac{m}{x}\right)^{-1}$.

28. $a^{-1} \div a^0$.

29. $b^8 \div b^{8-x}$.

30. $\dfrac{a^{m-1}b^{n-1}}{a^m b^n}$.

31. $(a^{\frac{3}{4}}b^{\frac{3}{2}})^{\frac{1}{3}}$.

32. $9^2 \cdot 3^{-5}$.

33. $(-x)^3 \div (-x)^2$.

34. $[(\frac{1}{3})^{-3}]^{-1}$.

35. $1 \div 4^{-2}$.

36. $1 \div (-3)^{-3}$.

37. $.2^{-1}$; $.5^{-2}$; 1.5^{-2}.

38. $5 \div 2^{-1}$; $8 \cdot 10^{-5}$.

39. $(a^{-\frac{3}{4}})^{-\frac{4}{5}}$.

40. $\dfrac{1}{(-3)^{-2}}$; $\dfrac{1}{(-3)^{-3}}$.

41. $\left(\dfrac{a}{x}\right)^{-n} \cdot \dfrac{x}{a}$.

42. $(a^{-2})^{-3}$; $(a^{-3})^2$; $(a^{-3})^{-1}$.

43. $(-a^2)^{-5}$; $(-a^5)^{-2}$; $(-a^{-5})^{-2}$.

44. $(-a^3)^{-4}$; $(-a^{-3})^4$; $(-a^{-4})^{-3}$.

45. $(-a^3)^{-2n}$; $(-a^{2n})^{-3}$; $(-a^{-2n})^{-3}$.

46. $(-a^{2n-1})^2$.

47. $\left(\dfrac{a^{-2}b^3}{x^{-1}y^4}\right)^{-2} = \dfrac{a^4b^{-6}}{x^2y^{-8}} = ?$

48. $\left(\dfrac{a^{-3}b^3}{x^3y^{-2}}\right)^3$.

49. $a^{\frac{3}{5}}b^{\frac{1}{2}} \div a^{-\frac{1}{5}}b^{-\frac{1}{4}}$.

50. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

51. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$.

In the following examples use definitions and laws to simplify:

52. $(a^{\frac{1}{3}} + b^{-\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{-\frac{1}{3}} + b^{-\frac{2}{3}})$.

53. $\sqrt{a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b}$.

54. $(a^{\frac{1}{2}} + a^{-\frac{1}{2}})^2 - (a^{\frac{1}{2}} - a^{-\frac{1}{2}})^2$.

55. $(a^{\frac{1}{2}} - 2)(a^{\frac{1}{2}} - 3)$.

56. $(a - x)(a^2 - x^2)^{-1}$.

57. $(a^2b^{-\frac{1}{2}}c^{\frac{2}{5}})^{-10}$.

58. $\sqrt{a^4x^{-12}} = (a^4x^{-12})^{\frac{1}{2}}$, etc.

59. $\sqrt[9]{\dfrac{x^3}{y^3}}$.

60. $\sqrt[6]{x^3y^{-2}}$.

61. $(\sqrt{x^my^{-2}})^2$.

62. $\sqrt[3]{\sqrt{a}}$.

63. $\sqrt{\sqrt{\sqrt{5}}}$.

64. $\sqrt[r]{\sqrt[p]{\sqrt[q]{a^{rp}}}}$.

65. $\sqrt[3]{a^{12}x^3y^{-12}}$.

66. $\sqrt{(-\frac{1}{2})^2x^4y^8}$.

67. $\sqrt{(-\frac{1}{2})^{-2}x^{-4}y^8}$.

68. $\sqrt{x}\sqrt[3]{x}$.

69. $2\sqrt{a^3}\sqrt[3]{a^4}$.

70. $2 \cdot 4^{\frac{3}{2}} \cdot 8^{-1}$.

71. $3 \div \sqrt[4]{3}$.

72. $\sqrt{3} \div \sqrt[3]{3}$.

73. Simplify $\dfrac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(a\sqrt{x})^5(-x\sqrt{a})^7$.

74. Simplify

$$(\sqrt{2xy})^6 + 5\cdot(\sqrt{2xy})^5(-\sqrt{2xy}) + 10(\sqrt{2xy})^4(-\sqrt{2xy})^2.$$

75. Simplify $\left(\dfrac{b^{-1}\sqrt{c}}{a\sqrt{a}}\right)^4 \cdot \left(\dfrac{2\sqrt{2a}}{\sqrt[3]{c}}\right)^3$.

76. Show that $2^{n+1} - 2^n = 2^n$. (Factor.)

77. Show that $2^n + 2^{n+2} = 5 \cdot 2^n$.

78. Show that $5^n + 5^{n+1} = 6 \cdot 5^n$.

79. Is $3 \cdot 3^2 = 9^2$? Is $3 \cdot 3^n = 9^n$?

80. Show that $2^n \cdot 4^n = 2^{3n}$.

81. Show that $\dfrac{1^0 + 2^{-1}}{1^0 - 2^{-1}} = 3$.

82. $(a^0 + a^{-1} + a^{-2})a^2$.

83. Compare the value 2^{3^4} with $(2^3)^4$ without actually performing the indicated operations.

454. The fundamental operations are performed upon expressions involving fractional, zero, and negative exponents in the same way as when the exponents are positive integers.

1. Multiply $3x + x^{-1} + 2$ by $3x + x^{-1} - 2$.

$$\begin{array}{l} 3x + 2 + x^{-1} \\ \underline{3x - 2 + x^{-1}} \\ 9x^2 + 6x + 3 \\ \quad - 6x - 4 - 2x^{-1} \\ \qquad\qquad 3 + 2x^{-1} + x^{-2} \\ \hline 9x^2 \qquad + 2 \qquad + x^{-2} \end{array}$$

Arranging in descending powers of x, the absolute term being considered as having x^0 for its literal part, we have the work as indicated at the left.

2. Divide $a^{-1} + 8$ by $a^{-\frac{2}{3}} - 2a^{-\frac{1}{3}} + 4$.

$$\begin{array}{l|l} a^{-1} + 8 & a^{-\frac{2}{3}} - 2a^{-\frac{1}{3}} + 4 \\ \cline{2-2} \underline{a^{-1} - 2a^{-\frac{2}{3}} + 4a^{-\frac{1}{3}}} & a^{-\frac{1}{3}} + 2 \\ \qquad 2a^{-\frac{2}{3}} - 4a^{-\frac{1}{3}} + 8 & \\ \qquad \underline{2a^{-\frac{2}{3}} - 4a^{-\frac{1}{3}} + 8} & \end{array}$$

3. Find the square root of $x^{\frac{1}{2}} - 4x^{\frac{1}{4}}y^{-\frac{1}{2}} + 4y^{-1}$.

$$\begin{array}{r|l} & x^{\frac{1}{2}} - 4x^{\frac{1}{4}}y^{-\frac{1}{2}} + 4y^{-1} \;\; \underline{|\, x^{\frac{1}{4}} - 2y^{-\frac{1}{2}}} \\ & x^{\frac{1}{2}} \\ \cline{2-2} 2x^{\frac{1}{4}} - 2y^{-\frac{1}{2}} & -4x^{\frac{1}{4}}y^{-\frac{1}{2}} + 4y^{-1} \\ & \underline{-4x^{\frac{1}{4}}y^{-\frac{1}{2}} + 4y^{-1}} \end{array}$$

In arranging in descending powers of a letter, where should the term not containing the letter of arrangment stand? Arrange $x^{-2} + x^2 + 2$ in descending powers of x.

EXERCISE

455. *Perform the operations indicated:*

1. $(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1)(x^{\frac{1}{2}} - x^{\frac{1}{4}} + 1)$.
2. $(x^2y^{-2} + 1 + x^{-2}y^2)(xy^{-1} - x^{-1}y)$.
3. $(x^3 + x^{-1})(x - x^{-3})$.

Perform the operations indicated:

4. $(4\,a^{\frac{2}{3}} - 6\,a^{\frac{1}{3}} + 9)(2\,a^{\frac{1}{3}} + 3)$.

5. $(2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1)(2^{\frac{1}{3}} + 1)$.

6. $(a^{\frac{1}{3}} - a^{-\frac{1}{3}})^3$.

7. $(x^{-1} - y^{-1} + z^{-1})^2 - (x^{-1} + y^{-1} - z^{-1})^2$.

8. $\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)(a^{\frac{1}{2}}b^{-\frac{1}{2}} - a^{-\frac{1}{2}}b^{\frac{1}{2}})$.

9. Arrange in descending powers of x and multiply $(-x^0 + x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^0 + x^{\frac{1}{2}} + x^{-\frac{1}{2}})$.

10. Arrange and multiply $(x^{-4} + 1 - 2\,x^{-2})(x^{-2} + x^{-4})$.

11. Arrange and divide $8\,a + a^{-2} + 6\,a^{-1} + 12$ by $2\,a^{-1} + a^{-2}$.

12. $(r^{-4} - 7\,r^{-2} - 30) \div (r^{-2} + 3)$.

13. $(a^{-2} - 35 - 2\,a^{-1}) \div (a^{-1} - 7)$.

14. $(30\,a^{-1} - 53\,a^{-\frac{1}{2}} + 8) \div (6\,a^{-\frac{1}{2}} - 1)$.

15. $(3\,x - 10\,x^{\frac{1}{2}} + 3) \div (3\,x^{\frac{1}{2}} - 1)$.

16. $(1 + 8\,a^{-1} + 15\,a^{-2}) \div (1 + 3\,a^{-1})$.

17. $(a^{\frac{5}{3}} + 12\,a^{\frac{2}{3}} - 48 + 52\,a^{\frac{1}{3}} - 17\,a) \div (a^{\frac{1}{3}} - 2 + a^{\frac{2}{3}})$.

18. $(m^{\frac{2}{5}} - 36 - 21\,m^{-\frac{2}{5}} - 3\,m^{\frac{1}{5}} - 71\,m^{-\frac{1}{5}}) \div (1 - 3\,m^{-\frac{2}{5}} - 8\,m^{-\frac{1}{5}})$.

19. $(x + x^{\frac{1}{2}} + 1) \div (x^{\frac{1}{2}} - x^{\frac{1}{4}} + 1)$.

20. $(a^{-1} + 27) \div (a^{-\frac{2}{3}} - 3\,a^{-\frac{1}{3}} + 9)$.

21. $(x^{-\frac{3}{2}} + y^{-\frac{3}{2}}) \div (x^{-\frac{1}{2}} + y^{-\frac{1}{2}})$.

22. Find the square root of $a^{\frac{1}{2}} + 12\,a^{\frac{1}{4}} + 36$.

23. $\sqrt{x^{-2} - 2\,x^{-1}y + y^2}$.

24. $(x^{-1} - 22\,x^{-\frac{1}{2}}y^{-\frac{1}{4}} + 121\,y^{-\frac{1}{2}})^{\frac{1}{2}}$.

25. $\sqrt{2 + a^2x^{-2} + a^{-2}x^2}$.

26. $(x^{-4} - 6\,x^{-3} + 13\,x^{-2} - 12\,x^{-1} + 4)^{\frac{1}{2}}$.

27. $(2\,a + 2\,a^{-1} + 3 + a^2 + a^{-2})^{\frac{1}{2}}$.

28. $(a^{-2} + 6\,a^{-1} + 9)^{-\frac{1}{2}}$.

Expand by applying type forms, examples 29 *to* 36 :

29. $(a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})$.

30. $(x^{\frac{1}{2}} - y^{-\frac{1}{2}})^2$.

31. $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$.

32. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^3$.

33. $(a^{\frac{1}{2}} + 2\,a^{\frac{1}{3}})^2$.

34. $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$.

35. $(a^{\frac{1}{2}} - 2)(a^{\frac{1}{2}} + 7)$.

36. $(a^{-1} + 3)(a^{-1} - 5)$.

37. Factor $a - 3\,a^{\frac{1}{2}} + 2$; $2\,x^{-2} + 3\,x^{-1} - 2$.

38. Factor $a - b$ into two factors, one of which is $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

HINT. $a - b = (a^{\frac{1}{2}})^2 - (b^{\frac{1}{2}})^2$.

39. Factor $a - b$ into two factors, one of which is $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

40. Factor (*a*) $x - 20\,x^{\frac{1}{2}} + 100$.

(*b*) $a - 4\sqrt{a} - 5$.

(*c*) $a^2c^{\frac{1}{2}} + ac + a^{\frac{3}{2}}c^{\frac{3}{2}} + a^{\frac{1}{2}}c^2$.

(*d*) $x - a^{\frac{1}{2}}x^{\frac{1}{2}} - b^{\frac{1}{2}}x^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}}$.

(*e*) $2\,a + 5\,a^{\frac{1}{2}} + 3$.

41. $a^{\frac{1}{3}} \cdot a^{-\frac{3}{4}} \cdot \sqrt[3]{a^4}\,\sqrt[12]{a} \cdot a^{-1}$.

42. $\left(\frac{a}{b}\right)^{-2}\left(\frac{b}{c}\right)^{-3}\left(\frac{c}{a}\right)^{-4}$.

43. $(a^2b^3)^{-4} \div (a^{-3}b^{-4})^3$.

44. $(a^{-2} - b^{-2}) \div \left(\frac{1}{a} - \frac{1}{b}\right)$.

45. $(\frac{5}{2})^{2p} \cdot (\frac{4}{5})^{2p} \cdot (\frac{1}{2})^{2p}$.

HINT. $4^{2p} = 2^{4p}$. (Why?)

46. $\left(\frac{x^{m+n}}{x^n}\right)^m \div \left(\frac{x^n}{x^{m+n}}\right)^{n-m}$.

47. $2^n(2^{n-1})^n \div (2^{n+1} \cdot 2^{n-1})$.

48. Show that $(2^{n+4} - 2.2^{n+1}) \cdot 2^{-n-2} = 3$.

49. $(xyz)^{x+y+z} \div (x^{y+z}y^{x+z}z^{x+y})$.

50. $(e^x + e^{-x})^2 - 2$.

51. $\left(\frac{1}{a + 2\,a^{-1}}\right)^{-1}$.

52. $\left[x^{\frac{2}{3}}\left(\frac{x^{\frac{1}{2}}}{\sqrt[3]{x}}\right)^{\frac{5}{2}}\right]^{\frac{3}{26}}$.

53. $\left(\frac{x^{p+q}}{x^q}\right)^p\left(\frac{x^{q-p}}{x^p}\right)^{p-q}$.

54. $x^{-1}\left(\frac{-3\,x^{\frac{2}{3}}}{\sqrt[3]{x^{-\frac{1}{2}}}}\right)^2$.

55. $(a^0)^{-\frac{4}{5}}(a^{-\frac{1}{3}})^{-\frac{3}{2}}$.

56. $(a^{-2}b^3c^{-5}d^{\frac{1}{3}})^{-3}$.

57. $2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}}\sqrt[8]{4}$.

58. $2 \cdot 5^{-4} \cdot 5^{-3}4^{-1} \cdot 10^6$.

Perform the operations indicated:

59. $(x+y)^{-3}(x+y)^{m+3}$.

60. $(x+y)^{\frac{a}{b}} \cdot (x+y)^{\frac{b}{a}}$.

61. $\left[\left(\dfrac{a^3b^{\frac{1}{3}}}{cd^5}\right)^{-1}\right]^{-2}$.

62. What is the sign of the answer in $(-3)^{2m}$? in $(-3)^{2m+1}$? (In each case m is any integer.)

63. $\dfrac{27(x^0+y^0+z^0)^{-3}}{(9x^{-3}y^5c^{-6})^{-\frac{1}{2}}}$.

64. $a^{\frac{1}{2}}y^{\frac{1}{3}}\left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2 \div \dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

65. $\left(\dfrac{x^{-2}y}{x^{\frac{3}{2}}y^{-\frac{5}{2}}}\right)^{\frac{2}{7}} \div \left(\dfrac{xy^{-\frac{1}{2}}}{y^{-1}\sqrt{x}}\right)^{-2}$.

66. $\left(\dfrac{4a^5\sqrt[4]{x^5}}{y^{-7}\sqrt[3]{b^4}}\right)^{-\frac{1}{2}}$.

67. $\sqrt[3]{\dfrac{(27x^3y^{-7}z^{-9})^{-1}}{4(x^0+y^0)^{-2}}}$.

68. $\left(\dfrac{y^{-1}\sqrt{x}}{y^{-1}\sqrt{x}}\right)^{-2} \div \sqrt[7]{\dfrac{x^3y^{-5}}{x^{-4}y^2}}$.

69. $\dfrac{a^{-1}b^{-1}}{a^{-3}+b^{-3}}$.

70. $\dfrac{a^{-1}+2b^{-1}}{a^{-3}+8b^{-3}}$.

71. Divide $x^{-6}-3x^{-4}y^{-2}+3x^{-2}y^{-4}-y^{-6}$ by $x^{-2}-y^{-2}$.

72. Divide $x^{-3}+x^{-2}y^{-1}+x^{-1}y^{-2}+y^{-3}$ by $x^{-1}+y^{-1}$.

73. Find the value of $\sqrt{a^2-b^2}$ when $a=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$ and $b=x^{\frac{1}{2}}-x^{-\frac{1}{2}}$.

74. Show that $\{\sqrt[5]{2^2}(\sqrt[7]{3})^3(6^{-\frac{1}{70}})^{29}\}^{35}=(\frac{3}{2})^{\frac{1}{2}}$.

XIX. RADICALS

CLASSIFICATION OF NUMBERS AND DEFINITIONS

456. Real and Imaginary Numbers. The numbers of algebra are divided into two classes, **real** and **imaginary.**

Real Numbers. Real numbers include all positive and negative integers, positive and negative fractions, and all indicated roots except even roots of negative numbers.

Imaginary Numbers. Even roots of negative numbers are **imaginary numbers.**

Thus, $5, -7, \frac{1}{3}, \sqrt[3]{-3}, \sqrt{5-\sqrt{23}}$ are real numbers.

$\sqrt{-3}, \sqrt{5-\sqrt{28}}$ are imaginary numbers.

457. Rational and Irrational Numbers. Real numbers are divided into two classes, **rational** and **irrational numbers.**

Any integer, or a number that can be expressed as the quotient of two integers, is a **rational number.**

Thus, $3, .25, 5\frac{1}{2}, .333 \cdots (= \frac{1}{3}), \sqrt{9}, \sqrt[3]{-27}$ are rational numbers.

All other *real* numbers are **irrational numbers.**

Thus, $\sqrt{5}, \sqrt{9+\sqrt{4}}$ are irrational.

The irrational numbers, so far as we shall be concerned with them in elementary algebra, are indicated roots that can be obtained only approximately.

458. Radical. The indicated root of a number is a **radical.**

Thus, $\sqrt{2}, \sqrt{9}, \sqrt[3]{a+b}, \sqrt{-4}$ are **radicals.**

A radical may be a rational number, an irrational number, or an imaginary number.

Thus, $\sqrt{4}$ is rational, $\sqrt{5}$ is irrational, and $\sqrt{-4}$ is imaginary.

Radical Expression. An expression that contains a radical is a **radical expression.**

Thus, $\sqrt{a}$, $3+\sqrt{2}$, $(a+\sqrt{b})^2$ are radical expressions.

459. Order of Radicals. Indicated square roots are radicals of the **second order**; indicated cube roots are radicals of the **third order**, etc.

Thus, $\sqrt{3}$ is of the **second order**, $\sqrt[3]{a}$ is of the **third order**, $\sqrt[4]{a^2+b^3}$ is of the **fourth order.**

460. Index of a Radical. The index of a radical is the number placed to the left and above the radical sign to indicate the order of the radical. The index of a square root is omitted.

Radicand. The expression under the radical sign is the **radicand.**

Thus, in $2\sqrt[5]{3a^3b}$, $3a^3b$ is the radicand, and 5 is the index of the radical.

461. Surd. An irrational number which is the indicated root of a rational number is a **surd.**

Thus, $\sqrt{2}$, $\sqrt[3]{a}$ are surds, but $\sqrt{9}$ and $\sqrt{2+\sqrt{3}}$ are not surds.

Quadratic Surd. A surd of the second order is a **quadratic surd.**

462. Principal Root. It has been seen that $\sqrt{4}=\pm 2$. From this we should infer that $x+\sqrt{4}=x\pm 2$. However, in dealing with radicals and expressions containing radicals it is customary to use only the positive root.

Thus, $x+\sqrt{4}=x+2$, and $x-\sqrt{4}=x-2$.

The positive square root of a number is its **principal square root.**

463. Principle 1. The square root of a product equals the product of the square roots of its factors.

In symbols, $\sqrt{ab} = \sqrt{a}\sqrt{b}.$

This principle follows immediately from the fourth law of exponents.

Thus, $\sqrt{ab} = (ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}} = \sqrt{a} \cdot \sqrt{b}.$

Principle 2. The square root of the quotient of two numbers equals the quotient of their square roots.

In symbols, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$

This follows from the fifth law of exponents.

Thus, $\sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{\sqrt{a}}{\sqrt{b}}.$

These two principles may be stated for any root.

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

REDUCTION OF RADICALS

464. Case I. To remove a factor from under the radical sign.

$$\sqrt{a^2b} = \sqrt{a^2}\sqrt{b} = a\sqrt{b}. \quad (\S\ 463)$$
$$\sqrt[3]{a^3b} = \sqrt[3]{a^3}\sqrt[3]{b} = a\sqrt[3]{b}. \quad \text{(Why?)}$$

465. If any factor of the radicand is a perfect power of the same degree as the radical index, it may be removed from under the radical sign by extracting the required root of the factor and multiplying the result by the coefficient of the radical.

Examples

1. $6\sqrt[3]{54} = 6\sqrt[3]{27 \cdot 2} = 18\sqrt[3]{2}.$
2. $2\sqrt{2\,a^5b^3} = 2\sqrt{(a^4b^2)2\,ab} = 2\,a^2b\sqrt{2\,ab}.$
3. $\sqrt{(a^2 - b^2)(a + b)} = \sqrt{(a + b)^2(a - b)} = (a + b)\sqrt{a - b}.$

EXERCISE

466. *Whenever possible, remove factors from under the radical sign:*

1. $\sqrt{18}$.
2. $\sqrt{4ab^2}$.
3. $\sqrt[3]{9a^4b^2c}$.
4. $\sqrt{a^3b^3}$.
5. $\sqrt{a^3+b^3}$.
6. $\frac{4}{3}\sqrt{27b^5}$.
7. $\sqrt{a^2+a^2b^2}$.
8. $\sqrt{5x^3-20x^2+20x}$.
9. $\sqrt[3]{6a^2b^2}$.
10. $\sqrt[3]{8x^8}$.
11. $a\sqrt{a^{2n}b}$.
12. $a\sqrt[n]{a^{n+1}}$.
13. $2\sqrt{(a^2-b^2)(a-b)}$.
14. $\sqrt{a^2-b^2}$.
15. $\sqrt{m^2-2mn+n^2}$.
16. $\sqrt{p^{2a+1}}$.
17. $\sqrt[m]{(x-y)^{2m}}$.
18. $\sqrt{m^3-n^3)(m-n)}$.
19. $\sqrt[3]{a^6+n}$.
20. $\sqrt[3]{2(a^3-3a^2b+3ab^2-b^3)}$.
21. $\sqrt[3]{(x-y)^m}$.
22. $\sqrt[3]{(a-b)^2(a^2-b^2)}$.
23. $\sqrt{(a^3-b^3)(a-b)^2}$.
24. $\sqrt{m^{2x+1}}$.
25. $b\sqrt{a^6b^{8+a}}$.
26. $\sqrt{(x^2-y^2)^3}$.
27. $\sqrt[n]{x^{2n+1}}$.
28. $\sqrt[3]{(x^3-3x^2+3x-1)z^2}$.
29. $\sqrt{a^4b^{4+x}}$.
30. $\sqrt{a^4+b^4}$.

When the radicand is negative and the root index is an odd number, the negative sign should always be removed from under the radical sign.

Thus, $\sqrt[3]{-16}=\sqrt[3]{-8\cdot 2}=-2\sqrt[3]{2}$; $\sqrt[3]{-a}=\sqrt[3]{(-1)^3a}=-1\sqrt[3]{a}$ or $-\sqrt[3]{a}$.

31. $\sqrt[3]{-32}$.
32. $5a-\sqrt[3]{-a^4}$.
33. $2x+\sqrt[5]{-x^6}$.
34. $\sqrt[5]{-m^6n}$.
35. $\sqrt[5]{-a^6-a^7}$.
36. $\sqrt[3]{a^3-a^6}$.
37. $\sqrt[5]{-160}$.
38. $\sqrt[3]{-500}$.
39. $\sqrt[3]{-m^9n^7}$.
40. $\sqrt[3]{12a^3}+\sqrt[3]{-8a^3}$.
41. $\sqrt[3]{x^4}-\sqrt[3]{-1}$.

467. Case II. To change a radical whose radicand is a fraction to an equivalent radical expression whose radicand is integral.

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{ab}{b^2}}=\frac{\sqrt{ab}}{b}, \text{ or } \frac{1}{b}\sqrt{ab}. \quad (\S\ 463.)$$

468. If the radicand is a fraction:

1. Multiply both terms of the fraction by the smallest number that will make the denominator a perfect power of the same degree as the radical index.

2. Factor the new radicand in such a way that the denominator may be removed under Case I, or under Principle 2, § 463.

EXAMPLES

1. $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{1}{9}\cdot 6}=\frac{1}{3}\sqrt{6}.$

2. $\sqrt[3]{-\frac{27}{4}}=\sqrt[3]{-\frac{54}{8}}=\sqrt[3]{(-\frac{27}{8})\cdot 2}=-\frac{3}{2}\sqrt[3]{2}.$

3. $\frac{2x}{3}\sqrt[3]{\frac{5a}{4x^4}}=\frac{2x}{3}\sqrt[3]{\frac{10ax^2}{8x^6}}=\frac{2x}{3}\sqrt[3]{\frac{1}{8x^6}\cdot 10ax^2}$

$$=\frac{2x}{3}\cdot\frac{1}{2x^2}\sqrt[3]{10ax^2}$$

$$=\frac{1}{3x}\sqrt[3]{10ax^2}.$$

4. $\sqrt{a^{-1}}=\sqrt{\frac{1}{a}}=$ etc. Let the student complete the solution.

EXERCISE

469. *Change to equivalent radical expressions having integral radicands:*

1. $8\sqrt{\frac{7a}{16x^2}}.$
2. $x\sqrt{\frac{5a^3}{6x^5}}.$
3. $66\sqrt{5\frac{1}{3}}.$
4. $\sqrt[3]{-\frac{3}{4}}.$
5. $\sqrt{\frac{5}{32}}.$
6. $ab\sqrt{\frac{a^4+b^5}{a^3b^5}}.$
7. $\frac{3}{a^2}\sqrt{\frac{13ax^{-1}}{18}}.$
8. $-8\sqrt[3]{-\frac{5a^{-1}}{36}}.$
9. $a\sqrt{a^{-1}+b^{-1}}.$
10. $\sqrt{\frac{a^4}{x^5}-\frac{a^5}{x^6}}.$
11. $\sqrt[n]{\frac{a^{n+1}}{b}}.$
12. $\sqrt{\frac{a^{n-2}}{b}}.$

Change to equivalent radical expressions having integral radicands:

13. $\sqrt[3]{\dfrac{a}{m^2}}$.
14. $\sqrt{\dfrac{3x^2}{6x^5}}$.
15. $\sqrt[m]{\dfrac{c}{c^{m-1}}}$.
16. $\sqrt[p]{\dfrac{x^a}{x^p}}$.
17. $\sqrt{\dfrac{x^2-1}{x-1}}$.
18. $\sqrt{\dfrac{a-b}{a^2-b^2}}$.
19. $\sqrt[3]{\dfrac{(m+n)^2}{m^2n^2}}$.
20. $\sqrt{\dfrac{x-y}{x\cdot y}}$.
21. $\sqrt{\dfrac{abx}{a+b}}$.
22. $\sqrt{\dfrac{a^2-b^2}{ab}}$.
23. $\sqrt{\dfrac{b+c}{b-c}}$.
24. $\sqrt{\dfrac{b-c}{b+c}}$.
25. $\sqrt[3]{\dfrac{a+b}{(a-b)^2}}$.
26. $\sqrt{\dfrac{p^2qr^3}{(p+q)^3}}$.
27. $\sqrt{\dfrac{ab+ac}{(b+c)^3}}$.

470. Case III. To reduce a radical to an equivalent radical with a smaller radical index.

1. $\sqrt[8]{a^2}=(a^2)^{\frac{1}{8}}=a^{\frac{1}{4}}=\sqrt[4]{a}$.

Let the student state the definitions and laws on which this reduction is based.

2. $\sqrt[6]{a^2b^4}=\sqrt[6]{(ab^2)^2}=\sqrt[3]{ab^2}$.

3. $\sqrt[6]{8a^3x^3}=\sqrt{2ax}$. Cancel the factor 3 from the radical index and from the exponents of the factors of the radicand. (Why?)

It is clear from these examples that this reduction depends upon the definitions and the laws of exponents, and that it is possible because the radical index and the exponents of all the factors of the radicand have a common factor.

Thus, in example 2, the index, 6, and the exponents of the factors of the radicand, 2 and 4, have a common factor 2.

471. If the radical index and the exponents of all the factors of the radicand have a common factor, that factor may be canceled from the index and from all the exponents.

EXERCISE

472. *Reduce the order of the following wherever possible:*

1. $\sqrt[4]{a^2b^2}$.
2. $\sqrt[4]{a^2+b^2}$.
3. $\sqrt[4]{(a+b)^2}$.
4. $\sqrt[6]{64\,a^3}$.
5. $\sqrt[6]{64\,a^2}$.
6. $\sqrt[3]{12\,x^6y^9}$.
7. $\sqrt[4]{4\,a^2}$.
8. $\sqrt[4]{9\,a^2b^6}$.
9. $\sqrt[6]{9\,a^2b^6}$.
10. $\sqrt[9]{27\,m^6n^3}$.
11. $\sqrt[6]{16+9}$.
12. $\sqrt{a^2+a^6}$.
13. $\sqrt[6]{125}$.
14. $\sqrt[2n]{a^n}$; $\sqrt[n]{a^{2n}}$.
15. $\sqrt[10]{32\,a^{45}}$.
16. $\sqrt[6]{a^3b^2}$.
17. $\sqrt[12]{a^3+a^4}$.
18. $\sqrt[4]{a^2\cdot(a^2)^3}$.
19. $\sqrt[9]{27\,a^{3n}}$.
20. $\sqrt[8]{256}$.
21. $\sqrt[12]{625}$.
22. $\sqrt[6]{343}$.
23. $\sqrt[3n]{c^{-2n}\cdot c^{14n}}$.
24. $\sqrt{x^n\cdot x^{n+1}}$.

473. Simplest Form of a Radical. A radical is in its **simplest form if:**

1. The radicand is integral.

2. The radicand is as small as possible; that is, contains no factor that can be removed from under the radical sign.

3. The radical index is as small as possible.

474. Corresponding to the three parts of § 473 radicals may be simplified as follows:

1. If the radicand is a fraction, simplify by Case II.

2. If the radicand contains a factor that may be removed, simplify by Case I.

3. If the radical index can be reduced, simplify by Case III.

475. Real numbers occur in five different radical forms.

1. Radicals that are rational numbers, $\sqrt{16}=4$.

2. Radicals in their simplest form, $\sqrt{7}$, $\sqrt[3]{4}$.

3. Radicals with fractional radicands, $\sqrt{\frac{7}{11}}=\frac{1}{11}\sqrt{77}$.

4. Radicals of which a factor of the radicand can be removed from under the radical sign, $\sqrt{28}=2\sqrt{7}$.

5. Radicals of which the radical index may be reduced, $\sqrt[4]{a^2}=\sqrt{a}$.

EXERCISE

476. *In the following list of radical expressions, each of the five forms given in § 475 is included. Examine each, telling to which class it belongs, and simplify all that are not simple.*

1. $\sqrt{a^3}$.

2. $\sqrt{a^4}$.

3. $\sqrt[4]{a^2}$.

4. $\sqrt{a^2b^3}$.

5. $\sqrt{a^{-1}}$.

6. $(a^4b)^{\frac{1}{3}}$.

7. $\sqrt[3]{27a^3x}$.

8. $2\sqrt{\frac{10}{27}}$.

9. $(\frac{5}{7})^{\frac{1}{2}}$.

10. $\sqrt{(a-b)^3}$.

11. $\frac{1}{a}\sqrt{\frac{11}{12}a^2}$.

12. $\sqrt{\frac{1}{4}+\frac{1}{9}}$.

13. $\sqrt{3^{-1}-4^{-1}}$.

14. $\sqrt{432}$.

15. $\sqrt[3]{a^3(1-b)^{-3}}$.

16. $\sqrt[3]{-648}$.

17. $\sqrt{ax^2-bx^2}$.

18. $ab\sqrt{\frac{a^2+b^2}{a^2b^2}}$.

19. $\sqrt{(a^2-a^2b^2)}$.

20. $\frac{10}{3}\sqrt{1-.01}$.

21. $\frac{3}{4}\sqrt{5\frac{1}{3}}$.

22. $\sqrt{9\,x^3y^8z^{10}}$.

23. $\sqrt{7\,x^4y^9z^{11}}$.

24. $\sqrt{a^4+b^6}$.

25. $\sqrt{a^4+a^6}$.

26. $20\,b^3\sqrt{\frac{31\,a^3}{50\,b^3}}$.

27. $\frac{5}{a}\sqrt{1-25^{-1}}$.

28. $\sqrt{\frac{(x^0+b^0+c^0)^2}{3}}$.

29. $\sqrt{16^{-\frac{1}{2}}}$.

30. $\sqrt{\frac{2^{n+2}}{2^n}}$.

31. $\sqrt[4]{\frac{9}{625}}$.

32. $\sqrt{1-\left(\frac{x}{3}\right)^2}$.

33. $(\frac{4}{125})^{-\frac{1}{2}}$.

34. $\sqrt[n]{x^{n+3}}$.

35. $\sqrt[n]{5\,x^{2n+1}}$.

36. $\sqrt[n]{x^{n-1}}$.

37. $\sqrt{x^{2n+2}}$.

38. $\sqrt[m]{a^{2m}c^m}$.

39. $\sqrt[n]{x^{2n-1}y^{4n}}$.

40. $\sqrt[m]{(x^2-y^2)^{2m}}$.

41. $\sqrt[12n]{m^{36}}$.

42. $\sqrt[xy]{a^{24x}}$.

43. Is the diagonal of a square whose side is 5 inches a rational or an irrational number? Express the length of the diagonal in its simplest radical form.

44. Is the diagonal of a rectangle whose sides are 3 inches and 4 inches rational or irrational?

45. Which of the radicals of examples 1 to 42 are rational numbers?

46. Is $\sqrt{5+\sqrt{4}}$ a surd?

477. Case IV. To change a radical to an equivalent radical of a higher order.

1. $\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}$. Let the student explain.

2. $\sqrt{ab^3} = (ab^3)^{\frac{1}{2}} = (ab^3)^{\frac{5}{10}} = \sqrt[10]{(ab^3)^5} = \sqrt[10]{a^5b^{15}}$.

478. From these examples we have the rule:

To change a radical to an equivalent radical of higher order, multiply the radical index and the exponents of all the factors of the radicand by the required multiplier. (Compare with Case III.)

Examples

1. Change $\sqrt{3}$ to a 6th order radical.

The radical index must be multiplied by 3 to make a 6th order radical.

$$\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}.$$

2. Change $3\sqrt{5\,ax^3}$ to equivalent radicals of the 4th and the 6th orders.

$$3\sqrt{5\,ax^3} = 3\sqrt[4]{25\,a^2x^6} = 3\sqrt[6]{125\,a^3x^9}.$$

EXERCISE

479. *Change the following radicals to equivalent radicals as indicated:*

1. Change $\sqrt{5}$ to 4th order; to 6th order.
2. Change $\sqrt[3]{3\,ab^2}$ to 6th order; to 9th order.
3. Change $\sqrt{\frac{a^2}{3}}$ to 6th order; to 8th order.
4. Change $\sqrt[5]{2}$ to 15th order.
5. Change $\sqrt[4]{a^2}$ to 8th order; to 2d order.
6. Change 5 to a radical of 2d order.
7. Change $\sqrt{3}$ and $\sqrt[3]{5}$ to radicals of 6th order.
8. Change the radicals in 7 to 12th order.

Change the following radicals to equivalent radicals as indicated:

9. Can $\sqrt{3}$ and $\sqrt[3]{5}$ both be changed to radicals of lower order than the 6th?

10. Change $\sqrt{11}$ and $\sqrt[3]{5}$ to radicals having the lowest common radical index.

SOLUTION. $\sqrt{11} = 11^{\frac{1}{2}} = 11^{\frac{3}{6}} = \sqrt[6]{1331}.$

$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{25}.$

11. Reduce $\sqrt{3}$ and $\sqrt[3]{5}$ to radicals having the lowest common radical index.

12. Write $\sqrt{xy}$, $\sqrt[3]{x^2y^2}$, $\sqrt[4]{x^3y^3}$ with the lowest common radical index.

13. Reduce $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{4}$ to surds with the lowest common radical index. Also $2\sqrt[3]{2}$, $\sqrt[6]{4}$, $\sqrt[9]{8}$.

14. Reduce to lowest common radical index $\sqrt{a-b}$ and $\sqrt[3]{a+b}$.

15. Reduce to radicals of the same order $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $5^{\frac{1}{5}}$.

16. Which is greater, $\sqrt{2}$ or $\sqrt[3]{3}$?

SOLUTION. $\sqrt{2} = \sqrt[6]{8}.$

$\sqrt[3]{3} = \sqrt[6]{9}.$

$\therefore$ $\sqrt[3]{3}$ is greater.

Which is greater:

17. $\sqrt{5}$ or $\sqrt[3]{11}$?

18. $\sqrt{5}$ or $\sqrt[3]{12}$?

19. $\sqrt{\frac{1}{2}}$ or $\sqrt[3]{\frac{1}{2}}$?

20. $\sqrt[3]{25}$ or $\sqrt{11}$?

21. $(\frac{2}{3})^{\frac{2}{3}}$ or $(\frac{3}{4})^{\frac{3}{4}}$?

Arrange in order of magnitude:

22. $\sqrt{2}$, $\sqrt[4]{6}$, $\sqrt[3]{2\frac{1}{2}}$.

23. $\sqrt{5}$, $\sqrt[3]{10}$, $\sqrt[5]{15}$.

24. $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$.

25. $\sqrt{5}$, $\sqrt[3]{10}$, $\sqrt[4]{20}$.

26. How do we compare the values of radicals of different order?

480. Case V. To introduce a coefficient under the radical sign.

$$a\sqrt{b}=\sqrt{a^2}\sqrt{b}=\sqrt{a^2b}. \quad (\S\ 463.)$$

To introduce a coefficient under the radical sign, raise it to a power corresponding to the radical index and multiply it by the radicand.

EXAMPLE

$2\sqrt[3]{5}=\sqrt[3]{2^3\cdot 5}=\sqrt[3]{40}.$

EXERCISE

481. *Introduce the coefficients under the radical sign:*

1. $3\sqrt{a}.$
2. $a\sqrt{2}.$
3. $2\,a^2\sqrt[3]{a^{-6}}.$
4. $\frac{1}{2}\cdot 2^{\frac{1}{2}}.$
5. $5\,a\sqrt{a-b}.$
6. $(a-b)\sqrt{a+b}.$
7. $\frac{12}{5}\sqrt{(\frac{1}{9}+\frac{1}{16})a}.$
8. $4\sqrt{\frac{1}{2}}.$
9. Which is greater, $2\sqrt{3}$ or $3\sqrt{2}$?
10. Which is greater, $2\sqrt{2}$ or $3\sqrt[3]{3}$?
11. How does the transformation of radicals in Case V compare with the transformation in Case I?

ADDITION AND SUBTRACTION OF RADICALS

482. Similar Radicals. Radicals which, when reduced to their simplest form, have the same index and the same radicand are **similar radicals.**

$\sqrt{3\,a}$, $2\sqrt{3\,a}$, and $(a+b)\sqrt{3\,a}$ are similar radicals. Also $\sqrt{3}$, $\sqrt{27}$, $\sqrt{75}$ are similar radicals, since $\sqrt{27}=3\sqrt{3}$ and $\sqrt{75}=5\sqrt{3}$.

Show that $\sqrt{18}$, $\sqrt[4]{4}$, and $(50\,a^2)^{\frac{1}{2}}$ are similar radicals.

483. Similar radicals are added and subtracted with reference to their common radical part as the unit of addition.

1. $2\sqrt{3}+5\sqrt{3}=7\sqrt{3}.$

2. $2\sqrt{12}+\sqrt{3a^2}-\frac{2}{3}\sqrt{3}=4\sqrt{3}+a\sqrt{3}-\frac{2}{3}\sqrt{3}$
$=(\frac{10}{3}+a)\sqrt{3}.$

3. $3\sqrt{2}+\sqrt{18}-\sqrt{24}+\sqrt{\frac{2}{3}}$
$=3\sqrt{2}+3\sqrt{2}-2\sqrt{6}+\frac{1}{3}\sqrt{6}=6\sqrt{2}-\frac{5}{3}\sqrt{6}.$

To add or subtract radicals:

1. Reduce all radicals to their simplest forms.

2. Add or subtract the coefficients of similar radicals, and join dissimilar radicals with their respective signs.

EXERCISE

484. *Simplify and combine as much as possible the following radical expressions:*

1. $\sqrt{2}+\sqrt{18}+\sqrt{50}.$
3. $\sqrt{50}+\sqrt{72}-\sqrt{8}.$
2. $\sqrt{8}+\sqrt{32}-\sqrt{72}.$
4. $\sqrt{128}-\sqrt{32}-\sqrt{18}-\sqrt{2}.$
5. $\sqrt{32}+\sqrt{50}+\sqrt{72}+\sqrt{25}.$
6. $\sqrt{63}+\sqrt{700}-\sqrt{175}+\sqrt{28}.$
7. $\sqrt{147}-\sqrt{192}+\sqrt{108}-\sqrt{125}.$
8. $2\sqrt{54}-\frac{1}{2}\sqrt{96}-3\sqrt{75}-\sqrt[4]{4}.$
9. $-5\sqrt{675}+3\sqrt{27}+2\sqrt{12}-4\sqrt{432}.$
10. $12^{\frac{1}{2}}+75^{\frac{1}{2}}-108^{\frac{1}{2}}+48^{\frac{1}{2}}.$
11. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{2}.$
12. $2\sqrt[3]{16}+\sqrt[3]{250}+4\sqrt[3]{128}-2\sqrt[3]{54}.$
13. $\sqrt[3]{24}+\sqrt[3]{375}-\sqrt[3]{1029}.$
14. $\sqrt{2}+\sqrt[3]{16}+\sqrt{50}+\sqrt[3]{2}-a\sqrt{2}.$
15. $\sqrt[3]{128}+\sqrt{72}-\sqrt{50}-\sqrt[3]{54}.$
16. $\sqrt{80}+3\sqrt[3]{1029}+4\sqrt[3]{81}+2\sqrt{32}.$
17. $4\sqrt{8}-\sqrt[3]{875}-3\sqrt{18}+2\sqrt[3]{189}.$
18. $\sqrt{120}-\sqrt{\frac{5}{6}}+\frac{1}{2}\sqrt{\frac{6}{5}}.$
19. $\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{8}}+\sqrt{\frac{1}{32}}-\sqrt{27}+\sqrt{9}.$
20. $\frac{1}{2}\sqrt{\frac{5}{8}}+\frac{1}{5}\sqrt{40}-3\sqrt{\frac{5}{2}}-\sqrt{\frac{125}{8}}.$

21. $\frac{\sqrt{2}}{2}-\sqrt{\frac{1}{2}}-\frac{1}{2}\sqrt{2}-\sqrt{\frac{4}{5}}+\sqrt[3]{2}.$

22. $\sqrt{84}+\sqrt{\frac{3}{7}}+\sqrt[3]{\frac{1}{5}}+\frac{1}{15}\sqrt[3]{200}.$

23. $2\sqrt{a^3}+3\sqrt{ab^2}.$

24. $3\sqrt{63\,ab^3}-\sqrt{112\,a^3b^3}.$

25. $9\sqrt{3\,x^5z}+24\,x\sqrt{3\,xz^3}+16\sqrt{3\,xz^5}.$

26. $\sqrt[3]{135}-3\sqrt[3]{-40}+5\sqrt[3]{-320}-7\sqrt[3]{-625}.$

27. $3\,b^2\sqrt{a^3c}+\frac{2}{c}\sqrt{a^5c^3}-c^4\sqrt{\frac{ac}{b^2}}.$

28. $\sqrt[6]{a^4x^8}-\sqrt[3]{27\,a^5x}-\sqrt[3]{-125\,a^2x}.$

29. $\sqrt{x}+3\sqrt{2\,x}-2\sqrt{3\,x}+2\,x^{\frac{1}{2}}-(8\,x)^{\frac{1}{2}}+\sqrt{12\,x}.$

30. $2\,a^{\frac{1}{2}}-\sqrt{x^2a}+\sqrt{(x-1)^2a}.$

31. $4\sqrt{3\,a}-7\sqrt{12\,a^2}+5\sqrt{48\,a}+6\sqrt{27\,a^2}-5\sqrt{75\,a}.$

32. $7\cdot 24^{\frac{1}{3}}-5\cdot(-81)^{\frac{1}{3}}+5\sqrt[3]{-192}+2\sqrt[3]{-375}.$

33. $5\sqrt{16}+5\sqrt[3]{-54}-6\sqrt[3]{-128}+7\sqrt[3]{-250}+2\sqrt[3]{432}.$

34. $2\sqrt{\frac{5}{3}}+\sqrt{\frac{60}{4}}-\frac{\sqrt{15}}{3}+\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{15}}.$

35. $2\sqrt[3]{\frac{1}{2}}-\sqrt{18}+\frac{1}{3}\sqrt[4]{4}-\sqrt{\frac{25}{18}}+\frac{7}{4}\sqrt[4]{4}.$

36. $\sqrt{(a+b)^2x}+\sqrt{(a-b)^2x}-2\sqrt{a^2x}.$

37. $\sqrt{4+4\,x^2}+\sqrt{9+9\,x^2}+\sqrt{a^2+a^2x^2}-5\sqrt{1+x^2}.$

38. $\sqrt{\frac{3\,x^3+30\,x^2+75\,x}{18}}-\sqrt{\frac{3\,x^3-6\,x^2+3\,x}{2}}.$

39. $3\sqrt{125\,m^3n^2}+n\sqrt{20\,m^3}-\sqrt{500\,m^3n^2}-m\sqrt{45\,mn^2}.$

40. Find in the simplest radical form the sum of the six diagonals of three squares whose sides are respectively 1 inch, 2 inches, and 3 inches.

MULTIPLICATION AND DIVISION OF RADICALS

485. To multiply or divide monomial radical expressions.

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{ab} \text{ and } \sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}. \quad (\S\ 463.)$$

Also $\sqrt{a}\div\sqrt{b}=\sqrt{\dfrac{a}{b}}$ and $\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\dfrac{a}{b}}$. (§ 463.)

Examples

1. $\sqrt{7a}\cdot\sqrt{14a}=\sqrt{98a^2}=7a\sqrt{2}.$
2. $3\sqrt[3]{a^2b}\cdot\sqrt[3]{ab^4}=3\sqrt[3]{a^3b^5}=3ab\sqrt[3]{b^2}.$
3. $5\sqrt{5a}\div 2\sqrt{3b}=\dfrac{5}{2}\sqrt{\dfrac{5a}{3b}}=\dfrac{5}{6b}\sqrt{15ab}.$

486. To multiply or divide two monomial radicals of the same order:

1. Make the product, or the quotient, of the coefficients of the given radicals the coefficient of the result.

2. Make the order of the result the same as that of the given radicals.

3 Make the product, or the quotient, of the radicands the radicand of the result.

4. Simplify the resulting radical expression.

Note. If the radicals are not of the same order, reduce them to equivalent radicals having the same radical index. (§ 478.)

Examples

1. Multiply $3\sqrt{a}$ by $-2\sqrt[3]{5a^2}$.

Solution. $3\sqrt{a}\cdot(-2\sqrt[3]{5a^2})=-6\sqrt[6]{a^3}\sqrt[6]{25a^4}$ (Why?)

$=-6\sqrt[6]{25a^7}$ (Why?)

$=-6a\sqrt[6]{25a}.$

2. Divide $3\sqrt{x}$ by $-21\sqrt[3]{4b^2}$.

Solution. $3\sqrt{x}\div(-21\sqrt[3]{4b^2})=-\dfrac{1}{7}\sqrt[6]{\dfrac{x^3}{16b^4}}$ (Why?)

$=-\dfrac{1}{14b}\sqrt[6]{4b^2x^3}.$ (Why?)

EXERCISE

487. *Perform the indicated operations, giving all results in the simplest form:*

1. $\sqrt{3}\cdot\sqrt{6}$.
2. $\sqrt{3}\cdot\sqrt{12}$.
3. $\sqrt[3]{2}\cdot\sqrt[3]{4}$.
4. $\sqrt{14}\div\sqrt{35}$.
5. $\sqrt{5x}\div\sqrt{x}$.
6. $\sqrt[3]{-5}\cdot\sqrt[3]{50}$.
7. $\sqrt{3}\cdot 5\sqrt{8}$.
8. $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{6}$.
9. $7^{\frac{1}{2}}\cdot 42^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}$.
10. $2^{\frac{1}{3}}\cdot 4^{\frac{2}{3}}$. $(4^{\frac{2}{3}}=(4^2)^{\frac{1}{3}}.)$
11. $(\sqrt{d^3})^2$.
12. $\sqrt[3]{2d^2}\cdot\sqrt[3]{4d}$.
13. $\sqrt[3]{9x}\cdot\sqrt[3]{9x^2}$.
14. $\sqrt[3]{25y^2}\cdot\sqrt[3]{-50y^5}$.
15. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{5}{6}}$.
16. $\sqrt{\frac{7}{40}}\cdot\sqrt{\frac{21}{10}}$.
17. $\sqrt{\frac{24}{35}}\cdot\sqrt{\frac{10}{21}}$.
18. $\sqrt{12}\div\sqrt{6}$.
19. $\sqrt{54}\div\sqrt{3}$.
20. $5\sqrt{7}\div 2\sqrt{5}$.
21. $2\sqrt{5}\div 5\sqrt{2}$.
22. $\sqrt{ab}\div\sqrt{bx}$.
23. $\sqrt{a^2b}\div 2\sqrt{ab^2}$.
24. $2\div\sqrt{2}$. $(2=\sqrt{4}.)$
25. $a\div\sqrt[3]{a}$.
26. $\sqrt{\frac{2a}{3b}}\cdot\sqrt{\frac{b}{a}}\cdot$
27. $\sqrt{\frac{5a}{6b}}\cdot\sqrt{\frac{10a}{3b}}$.
28. $\sqrt{a+b}\sqrt{a-b}$.
29. $(\sqrt{a-b})^2$.
30. $(\sqrt{3}\cdot\sqrt{4}\cdot\sqrt{5})\div(\sqrt{10}\cdot\sqrt{6})$.
31. $\sqrt[3]{4}\div\sqrt{2}$.
32. $\sqrt[3]{\frac{2}{3}}\cdot\sqrt{6}$.
33. $\sqrt[3]{\frac{4}{3}}\cdot\sqrt{\frac{3}{2}}$.
34. $\sqrt{.5}\cdot\sqrt[3]{\frac{3}{10}}$.
35. $\sqrt{\frac{3}{8}}\cdot\sqrt{\frac{4}{3}}$.
36. $(\sqrt{2ax^2})^2$.
37. $\sqrt{2}\cdot\sqrt[3]{3}\cdot\sqrt[4]{4}$.
38. $\sqrt[6]{\frac{1}{3}}\cdot\sqrt[4]{3}$.
39. $\sqrt{\frac{4}{3}}\cdot\sqrt[4]{6}$.
40. $(2\sqrt[3]{-5})^2$.
41. $\sqrt{a^2-b^2}\div c\sqrt{a+b}$.
42. $\sqrt{ab^2-b^2c}\div\sqrt{a-c}$.
43. $\sqrt[3]{\frac{a^3b^2}{dx^5}}\div\frac{1}{c}\sqrt{\frac{ab}{dx}}$.
44. $\frac{ac}{b^3d^3}\sqrt[3]{\frac{bcd}{e}}\sqrt[6]{\frac{b^{10}d^7e}{a^2c^5}}$.

Perform the indicated operations, giving all results in the simplest form:

45. $ab \div \sqrt{\frac{a}{x}} + \frac{a}{x} \div \sqrt{ax}.$

46. $\sqrt{10} \cdot \sqrt{15} + \sqrt{7} \cdot \sqrt{42}.$

47. $\sqrt{2} \cdot \sqrt[3]{4} + \sqrt[3]{2} \cdot \sqrt{4}.$

48. $\sqrt[3]{7} \cdot \sqrt[3]{-49} + \sqrt{49}.$

49. $(3\sqrt{2})^3 \div \sqrt{2}.$

50. $\frac{1}{2}\sqrt{\frac{8x}{3y^2}} \cdot \frac{3}{4}\sqrt{\frac{3y^3}{2x^3}}.$

51. $5\sqrt{\frac{a^2}{b}} \cdot \frac{1}{2}\sqrt{\frac{4a^2}{b}}.$

52. $\sqrt{\frac{a^2}{m}} \cdot \sqrt[3]{\frac{8a}{m^4}}.$

53. $\sqrt[3]{\frac{2x^4}{25y^5}} \cdot 5\sqrt[3]{\frac{4x^5}{5y}}.$

54. $\sqrt[3]{\frac{4a}{x}}\sqrt{\frac{x^3}{2a^3}}.$

488. To multiply radical expressions when one or both factors are polynomials.

The method and the form of the work are the same as in the multiplication of polynomials in Chapter V.

Multiply each term of the multiplicand by each term of the multiplier, simplify all results, and combine the terms as much as possible.

EXAMPLES

1. $(3\sqrt{2} + \sqrt{12} - 2\sqrt[3]{5})\sqrt{2} = 6 + 2\sqrt{6} - 2\sqrt[6]{200}.$

2. Multiply $(3 + \sqrt{5} + 2\sqrt{6})$ by $(\sqrt{5} + \sqrt{6})$.

$$\begin{array}{r} 3 + \sqrt{5} + 2\sqrt{6} \qquad\qquad \\ \sqrt{5} + \sqrt{6} \qquad\qquad\qquad \\ \hline 3\sqrt{5} + 5 + 2\sqrt{30} \qquad\qquad \\ 12 + \sqrt{30} + 3\sqrt{6} \\ \hline 3\sqrt{5} + 17 + 3\sqrt{30} + 3\sqrt{6} \end{array}$$

3. Expand $(\sqrt{2} + \sqrt[3]{3})^2$.

In this exercise multiply by type form.

$$(\sqrt{2} + \sqrt[3]{2})^2 = (\sqrt{2})^2 + 2\sqrt{2}\sqrt[3]{3} + (\sqrt[3]{3})^2.$$
$$= 2 + 2\sqrt[6]{72} + \sqrt[3]{9}.$$

EXERCISE

489. *Multiply the following:*

1. $(2\sqrt{3}+3\sqrt{4}+4\sqrt{6})\sqrt{3}.$

2. $(4\sqrt{5}+2\sqrt{6}+\sqrt{20})4\sqrt{5}.$

3. $(3\sqrt{18}+4\sqrt{12}+5\sqrt{50}+\sqrt{27})2\sqrt{3}.$

4. $(4\sqrt{25}+3\sqrt{75}+2\sqrt{18})2\sqrt{3}.$

5. $(\sqrt{6}+\sqrt[3]{2}-2\sqrt[4]{5})\sqrt{3}.$

6. $\left(\sqrt{a}+\sqrt{\frac{1}{a}}\right)\sqrt{a}.$

7. $\left(a^2\sqrt{\frac{b}{a}}+b^2\sqrt{\frac{a}{b}}-\sqrt{ab}\right)\sqrt{ab}.$

8. $\left(a\sqrt{\frac{1}{a}}+\frac{1}{a}\sqrt{a}\right)a^{\frac{1}{2}}.$

9. $(3\sqrt[3]{4}+4\sqrt[3]{5}+2\sqrt[3]{6})\sqrt[3]{10}.$

10. $(2\sqrt{\tfrac{1}{2}}+\sqrt[3]{\tfrac{1}{8}}-2\sqrt{3}+4\sqrt[3]{27})\sqrt{\tfrac{1}{3}}.$

11. $\left(\sqrt{5a^3x}-\sqrt{20ax^3}+\sqrt{\frac{9a}{5x}}\right)\sqrt{5ax}.$

12. $\left(\frac{2\sqrt{a}}{3\sqrt[3]{b}}-\sqrt[3]{a}-\frac{1}{6}\sqrt[6]{ab^2}\right)\sqrt[3]{\frac{a^3}{b^2}}.$

13. $(9a^2-6\sqrt{a^3x}+15\sqrt[3]{ax^3})3\sqrt[6]{a^5x}.$

14. $(\sqrt{a^2-x^2}+\sqrt{(a-x)^{-1}})\sqrt{a-x}.$

15. $(7+2\sqrt{6})(9-5\sqrt{6}).$

16. $(5\sqrt{14}+3\sqrt{5})(7\sqrt{14}-2\sqrt{5}).$

17. $(\sqrt{12}-2\sqrt{7})(2+\sqrt{21}).$

18. $(2\sqrt{7}-5\sqrt{6})\left(\frac{3\sqrt{7}}{2}-2\sqrt{6}\right).$

19. $(2\sqrt{2}-\sqrt{8})(3\sqrt{2}+5\sqrt{8}).$

Multiply the following:

20. $(2\sqrt{5}+3\sqrt{2}-8\sqrt{6})(2+5\sqrt{2}-3\sqrt{12})$.
21. $(5\sqrt{8}+3\sqrt{18}+5\sqrt{27})(9\sqrt{6}-12\sqrt{18})$.
22. $(\sqrt[3]{2}-\sqrt[3]{3})(\sqrt[3]{4}+\sqrt[3]{9})$.

Apply type forms to multiply examples 23 to 32, and to others when convenient.

23. $(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})$.
24. $(\sqrt{15}-\sqrt{14})(\sqrt{15}+\sqrt{14})$.
25. $(\sqrt{10}+\sqrt{6})^2$; $(\sqrt{2}-1)^2$.
26. $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})\sqrt[3]{16}$.
27. $(\sqrt{2}-1)^3$.
28. $(\sqrt{a}+\sqrt{a-b})(\sqrt{a}-\sqrt{a-b})$.
29. $(\sqrt{a-b}+\sqrt{a+b})^2$.
30. $(6\sqrt{5}-5\sqrt{3})^2$.
31. $(3\sqrt{2}-5)(3\sqrt{2}+7)$.
32. $(\sqrt{2}+1)(\sqrt{2}+3)$.
33. $(\sqrt{2}-1)(2+\sqrt{2}+1)$.
34. $\sqrt{3}\cdot\sqrt{2}(\sqrt{24}+\frac{1}{2}\sqrt{96}+\sqrt{486})$.
35. $\sqrt{5-\sqrt{2}}\sqrt{5+\sqrt{2}}$.
36. $\sqrt{4+\sqrt{7}}\sqrt{4-\sqrt{7}}$.
37. $\sqrt[3]{8-\sqrt{10}}\sqrt[3]{8+\sqrt{10}}$.
38. $\sqrt[3]{\frac{11-\sqrt{13}}{2}}\sqrt[3]{\frac{11+\sqrt{13}}{2}}$.
39. $(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})(\sqrt[3]{x}+\sqrt[3]{y})$.
40. $(\sqrt{3}-1)^4$.
41. $(-2\sqrt{5}-\sqrt{3})^2$.
42. Find the value of x^2+x-1 when $x=-1+\sqrt{5}$.
43. Find the value of $2x^2-5x+2$ when $x=\frac{3+\sqrt{17}}{4}$.
44. Is $-5+\sqrt{10}$ a root of $4x^2+20x+15=0$?
45. Find a mean proportional between $7-\sqrt{13}$ and $7+\sqrt{13}$.
46. Find a fourth proportional to 3, $2+\sqrt{5}$, $2-\sqrt{5}$.

47. Find a third proportional to 2 and $\sqrt{2}+\sqrt{\frac{1}{2}}$.

48. Solve $2:(\sqrt{5}-1)=(\frac{1}{2}\sqrt{5}-\frac{1}{2}):x$.

49. $\sqrt{a^2-1}\sqrt{\dfrac{a-1}{a+1}}$.

50. $\sqrt{25y^6-9}\sqrt{\dfrac{5y^3-3}{5y^3+3}}$.

51. $\sqrt{\dfrac{1}{y}-\dfrac{a}{b}}\cdot\sqrt{\dfrac{1}{y}+\dfrac{a}{b}}$.

52. $\sqrt{\dfrac{4}{x}-\dfrac{x}{4}}\cdot\sqrt{\dfrac{4}{x}+\dfrac{x}{4}}$.

53. $\dfrac{a+1}{a-2}\sqrt{\dfrac{3a^3}{2b^5}}\cdot\dfrac{a^2-4}{a^2-1}\sqrt{\dfrac{3a}{2b^3}}$.

54. $(a^2\sqrt{x}-a\sqrt{x^3}-\frac{1}{2}\sqrt{x^5})(4\sqrt{x}-\dfrac{6}{a}\sqrt{x^3})$.

55. $(m^2\sqrt{ab^3}-m\sqrt{a^3b})^2$.

56. $(a\sqrt{y}+b\sqrt{y^3})^2$.

57. $(3\sqrt{x^5}-2\sqrt{x^3})^2$.

58. $\dfrac{a+3}{2x-1}\sqrt{\dfrac{5m}{4n^2}}\cdot\dfrac{4x^2-1}{9-a^2}\sqrt{\dfrac{64n^6}{125m^3}}$.

59. $\dfrac{5+2a}{2m-n}\sqrt{\dfrac{m^2-b^2}{y+x}}\cdot\dfrac{4m^2-n^2}{4a^2-25}\sqrt{\dfrac{x^2-y^2}{b+m}}$.

60. $\sqrt{a\sqrt[3]{a^2}}\cdot\sqrt{\sqrt[3]{a}}$.

RATIONALIZING FACTORS

ORAL EXERCISE

490. *Multiply the following:*

1. $\sqrt{3}\cdot\sqrt{3}$.
2. $\sqrt[3]{4}\cdot\sqrt{2}$.
3. $\sqrt[3]{a}\cdot\sqrt[3]{a^2}$.
4. $\sqrt{a^3}\cdot\sqrt{a}$.
5. $(2-\sqrt{5})(2+\sqrt{5})$.
6. $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$.
7. $(2\sqrt{7}+3\sqrt{5})(2\sqrt{7}-3\sqrt{5})$.
8. $(a\sqrt{b}+x\sqrt{y})(a\sqrt{b}-x\sqrt{y})$.
9. $(\sqrt{b}-1)(\sqrt{b}+1)$.
10. $(2\sqrt{x}-3\sqrt{y})(2\sqrt{x}+3\sqrt{y})$.

491. The product in each example of § 490 is a rational number.

Rationalizing Factor. If the product of two irrational factors is a rational expression, either of the factors is a **rationalizing factor** of the other.

If two binomial quadratic surds of the form $a\sqrt{b}+x\sqrt{y}$ differ in one sign only, either is the rationalizing factor of the other, for $(a\sqrt{b}+x\sqrt{y})(a\sqrt{b}-x\sqrt{y})=a^2b-x^2y$, a rational expression.

492. Two binomial quadratic surds that differ only in one of their signs are **conjugate quadratic surds.**

Thus, $2-\sqrt{3}$ and $2+\sqrt{3}$ are conjugate quadratic surds. What is their product?

DIVISION OF POLYNOMIAL RADICAL EXPRESSIONS

493. In the ordinary division of a polynomial by a monomial we have,

$$(12\,a^2-15\,ab)\div 3\,a=\frac{12\,a^2}{3\,a}-\frac{15\,ab}{3\,a}=4\,a-5\,b.$$

Similarly, $$(2\sqrt{10}-3\sqrt{3})\div\sqrt{2}=\frac{2\sqrt{10}}{\sqrt{2}}-\frac{3\sqrt{3}}{\sqrt{2}}$$
$$=2\sqrt{5}-\tfrac{3}{2}\sqrt{6}.$$

494. To divide a polynomial radical expression by a monomial, divide each term of the polynomial by the monomial and simplify the result.

EXERCISE

495. *Find the quotients:*

1. $\sqrt{15}\div 2\sqrt{3}$.
2. $4\sqrt{12}\div 2\sqrt{3}$.
3. $(\sqrt{6}+4\sqrt{18}-8\sqrt{2})\div\sqrt{3}$.
4. $(\sqrt{72}+\sqrt{3}-4)\div\sqrt{8}$.

5. $(3\sqrt{15}-\sqrt{20}+\sqrt{10}-7)\div 2\sqrt{5}$.

6. $(2\sqrt{32}+3\sqrt{2}+4)\div 4\sqrt{8}$.

7. $(\sqrt{8}+\sqrt[3]{12}+\sqrt[4]{2})\div 2\sqrt{2}$.

8. $(6+2\sqrt{3}-\sqrt[3]{18})\div\sqrt{6}$.

9. Solve $2:3-\sqrt{7}=4+\sqrt{28}:x$.

10. $(42\sqrt{5}-30\sqrt{3})\div 2\sqrt{15}$.

11. $\sqrt{\frac{3}{a^3}}\div\sqrt{\frac{6}{a}}$.

12. $\sqrt{a^2x}\div\sqrt{ax^5}$.

13. $(a-b)\div\sqrt{ab}$; $(a-b)\div\sqrt{a-b}$.

14. $(1-\sqrt{a})\div\sqrt{1-\sqrt{a}}$.

15. $(acy^2\sqrt[3]{z^2}-adz\sqrt[3]{y})\div a\sqrt[3]{yz^2}$.

ORAL EXERCISE

496. *What is the smallest rationalizing factor of each of the following?*

1. $\sqrt{a}$.

2. $\sqrt[3]{a}$.

3. $\sqrt[3]{a^2}$.

4. $\sqrt{a^3}$.

5. $2^{\frac{1}{2}}$.

6. $2^{\frac{1}{3}}$.

7. $2^{\frac{1}{4}}$.

8. $2^{\frac{3}{4}}$.

9. $3\sqrt{2ax^2}$.

10. $\sqrt{a-b}$.

11. $\sqrt{a^3-a^2b}$.

12. $2\sqrt{7}$.

13. $2\sqrt[3]{21}$.

14. $\sqrt[5]{a^2}$.

15. $n\sqrt{m^3}$.

16. $3\sqrt[3]{\frac{2a^2b}{c^2}}$.

17. $1-\sqrt{2}$.

18. $3a-\sqrt{b}$.

19. $2\sqrt{x}+3\sqrt{y}$.

20. $\sqrt{a}-\sqrt{b}$.

21. $4-3\sqrt{d^7}$.

22. $c\sqrt{u}-d\sqrt{v}$.

23. According to what type form of multiplication should two conjugate quadratic surds be multiplied?

24. Can you give more than one rationalizing factor for a given radical expression? Try with $\sqrt{a^3b}$; $\sqrt{12}$.

RATIONALIZING DENOMINATORS

497. Division by radicals may be performed by a method known as "Rationalizing Denominators." When the divisor is a binomial or a polynomial, this is the only practical method.

Study the following examples:

1. $2 \div \sqrt{2} = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}.$ Explain.

2. $(5-\sqrt{3}) \div (5+\sqrt{3}) = \dfrac{5-\sqrt{3}}{5+\sqrt{3}} = \dfrac{(5-\sqrt{3})(5-\sqrt{3})}{(5+\sqrt{3})(5-\sqrt{3})}$
$= \dfrac{28-10\sqrt{3}}{22} = \dfrac{14-5\sqrt{3}}{11}.$

3. $3 \div 2\sqrt[3]{4} = \dfrac{3}{2\sqrt[3]{4}} = \dfrac{3\sqrt[3]{2}}{2\sqrt[3]{4}\sqrt[3]{2}} = \dfrac{3\sqrt[3]{2}}{4}.$

498. The preceding examples illustrate the rule.

To divide when the divisor is a radical expression:

1. Indicate the division in fractional form.

2. Multiply the numerator and the denominator by the smallest factor that will produce a rational denominator.

3. Reduce the resulting fraction to its lowest terms.

EXERCISE

499. *Find the quotients:*

1. $3 \div \sqrt{6}.$
2. $8 \div 2\sqrt{2}.$
3. $\sqrt{6} \div \sqrt{3}.$
4. $3 \div 2\sqrt{3}.$
5. $9 \div \sqrt{3}.$
6. $5 \div 2\sqrt[3]{4}.$
7. $3 \div \sqrt[3]{2}.$
8. $2 \div \sqrt[3]{9}.$
9. $5\sqrt[3]{12} \div 4\sqrt[3]{24}.$
10. $8 \div 3\sqrt[3]{5}.$
11. $(1+2\sqrt{2}) \div \sqrt{3}.$
12. $(2+3\sqrt{3}) \div \sqrt{5}.$

13. $(3+4\sqrt{3})\div\sqrt{6}$.

14. $(6+3\sqrt{6})\div\sqrt{6}$.

15. $(8\sqrt{3}-2\sqrt{5})\div 3\sqrt{2}$.

16. $3\div(2+\sqrt{2})$.

17. $4\div(3+\sqrt{3})$.

18. $(3+\sqrt{7})\div(3-\sqrt{7})$.

19. $(4-\sqrt{8})\div(\sqrt{3}+\sqrt{5})$.

20. $2\sqrt{3}\div(3-\sqrt{3})$.

21. $(3\sqrt{7}+4\sqrt{6})\div(2\sqrt{7}-3\sqrt{6})$.

22. $(\sqrt{2}+\sqrt{3})\div(3\sqrt{2}+2\sqrt{3})$.

23. $(\sqrt{2}+\sqrt{3}+\sqrt{5})\div(\sqrt{2}-\sqrt{3})$.

24. $(\sqrt[3]{3}-\sqrt[3]{2})\div\sqrt{3}$.

25. $(8+5\sqrt{12})\div 3\sqrt{72}$.

26. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})\div(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.

27. $(a^{\frac{1}{2}}b^{\frac{3}{2}}+a^{\frac{3}{2}}b^{\frac{1}{2}})\div a^{\frac{1}{2}}b^{\frac{1}{2}}$.

28. $1\div(\sqrt{12}-\sqrt{3})$.

29. $3\div(\sqrt{2}+\sqrt{3}-2)$.

SOLUTION. $$\frac{3(\sqrt{2}+\sqrt{3}+2)}{(\sqrt{2}+\sqrt{3}-2)(\sqrt{2}+\sqrt{3}+2)}=\frac{3(\sqrt{2}+\sqrt{3}+2)}{1+2\sqrt{6}}$$
$$=\frac{3(\sqrt{2}+\sqrt{3}+2)(1-2\sqrt{6})}{(1+2\sqrt{6})(1-2\sqrt{6})}=\frac{6-15\sqrt{2}-9\sqrt{3}-12\sqrt{6}}{-23}.$$

Two rationalizing factors are required when the divisor, in its simplest form, is a trinomial quadratic surd expression.

30. $(2+\sqrt{3})\div(\sqrt{2}-\sqrt{3}+\sqrt{5})$.

31. $(1-\sqrt{3})\div(1+\sqrt{2}+\sqrt{3})$.

32. $5\div\sqrt{4+\sqrt{3}}$.

SOLUTION. $$5\div\sqrt{4+\sqrt{3}}=\frac{5}{\sqrt{4+\sqrt{3}}}=\frac{5\sqrt{4-\sqrt{3}}}{\sqrt{4+\sqrt{3}}\cdot\sqrt{4-\sqrt{3}}}$$
$$=\frac{5\sqrt{4-\sqrt{3}}}{\sqrt{13}}=\frac{5\sqrt{13}\sqrt{4-\sqrt{3}}}{13}=\frac{5\sqrt{52-13\sqrt{3}}}{13}.$$

33. $\sqrt{3}\div\sqrt{4+\sqrt{3}}$.

34. $12\div\sqrt{6-\sqrt{11}}$.

35. $12\div\sqrt{6-\sqrt{10}}$.

36. $\sqrt{5}\div\sqrt{\sqrt{6}+\sqrt{2}}$.

Find the quotients:

37. $2 \div (a - \sqrt{a^2 - 4})$.

38. $(\sqrt{a+b} + \sqrt{a-b}) \div (\sqrt{a+b} - \sqrt{a-b})$.

39. $(4a^2\sqrt[3]{1+x}) \div (3x\sqrt[3]{(1-x)^2})$.

40. $(\sqrt{2} + \sqrt{18} + \sqrt{50}) \div \sqrt{2}$.

Rationalize denominators in the following fractions:

41. $\dfrac{6}{\sqrt[5]{8}}$.

42. $\dfrac{15}{\sqrt[3]{12}}$.

43. $\dfrac{8\sqrt{3} - 2\sqrt{5}}{3\sqrt{2}}$.

44. $\dfrac{2\sqrt{2} + 3\sqrt{3}}{4\sqrt{3}}$.

45. $\dfrac{4\sqrt{2} + 3\sqrt{3} + 2\sqrt{5}}{2\sqrt{3}}$.

46. $\dfrac{\sqrt{5} + \sqrt{6} + \sqrt{7}}{3\sqrt{3}}$.

47. $\dfrac{4\sqrt[3]{4} + 5\sqrt[3]{12} + \sqrt{2}}{\sqrt[3]{2}}$.

48. $\dfrac{3 + \sqrt{7}}{4 + \sqrt{5}}$.

49. $\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{3} - \sqrt{5}}$.

50. $\dfrac{\sqrt{2} + \sqrt{3}}{3\sqrt{\frac{1}{3}} - 2\sqrt{\frac{1}{2}}}$.

51. $\dfrac{\sqrt{a} - \sqrt{b}}{c\sqrt{x} - d\sqrt{y}}$.

52. $\dfrac{2}{\sqrt{a^2 + b} + \sqrt{a^2 - b}}$.

53. $\dfrac{\sqrt{a^2 - b^2} - \sqrt{a^2 + b^2}}{\sqrt{a^2 - b^2} + \sqrt{a^2 + b^2}}$.

54. $\dfrac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{\sqrt{x^3 + 1} + \sqrt{1 - x^3}}$.

55. $\dfrac{2\sqrt{1 - b^2} - 3\sqrt{1 - c^2}}{\sqrt{1 - b^2} + \sqrt{1 - c^2}}$.

56. $\dfrac{x\sqrt{m^3 - n^2} + y\sqrt{m^3 - n^2}}{x\sqrt{m^3 - n^2} - y\sqrt{m^3 - n^2}}$.

500. To find the approximate value of a fraction having a radical in the denominator.

Divide 1 by $\sqrt{2}$.

This might be done in either of the following ways:

$$\frac{1}{\sqrt{2}} = \frac{1}{1.414\ldots} = .707\ldots$$

or $$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}=\frac{1.414\cdots}{2}=.707\cdots$$

The second method is much to be preferred. Why?

When approximate values of such quotients are desired, change them to a form having a rational denominator.

EXERCISE

501. *Find values of the following, correct to two decimal figures, knowing that* $\sqrt{2}=1.4142\cdots$, $\sqrt{3}=1.7321\cdots$, $\sqrt{5}=2.2361\cdots$, $\sqrt{6}=2.4495\cdots$.

1. $\frac{3}{\sqrt{3}}$.
2. $2\div(\sqrt{2}-1)$.
3. $3\sqrt{5}\div 2\sqrt{3}$.
4. $(3-\sqrt{5})\div(5-2\sqrt{5})$.
5. $(7-\sqrt{5})\div(3+\sqrt{5})$.
6. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
7. $\sqrt{6}\div(\sqrt{3}-\sqrt{2})$.
8. $\frac{a}{a+\sqrt{a}}$ when $a=2$.

Find the roots correct to three decimal places:

9. $2x=1+x\sqrt{3}$.
10. $x\sqrt{5}=12+x$.
11. $x\sqrt{a}-a=x\sqrt{b}-b$, when $a=2$, $b=3$.

INVOLUTION AND EVOLUTION OF RADICALS

502. Powers and Roots of monomial radical expressions can be found by using fractional exponents and the laws of exponents. Powers may be found also by the ordinary multiplication of radicals, as in § 486.

1. $(2\sqrt[3]{x^2})^2=(2\cdot x^{\frac{2}{3}})^2$ (Definition of fractional exponents.)

$=4\cdot x^{\frac{4}{3}}$ (Laws 4 and 3 of Exponents.)

$=4\sqrt[3]{x^4}$ (Why?)

$=4x\sqrt[3]{x}$. (Why?)

2. $\sqrt{2\sqrt[3]{x^2}} = \sqrt{\sqrt[3]{8x^2}}$

$= [(8x^2)^{\frac{1}{3}}]^{\frac{1}{2}}$ (Why?)

$= (8x^2)^{\frac{1}{6}}$ (Why?)

$= \sqrt[6]{8x^2}.$ (Why?)

Note that in taking a root of a radical expression, a coefficient must be introduced under the radical sign, but it is better not to do this in finding a power of such an expression.

EXERCISE

503. *Perform the indicated operations:*

1. $(\sqrt{x})^2$; $(\sqrt{x})^3$; $(\sqrt[3]{2})^2$.
2. $(\sqrt{a}\sqrt[3]{x})^2$; $(\sqrt{a}\sqrt[3]{x})^3$.
3. $(2\sqrt[3]{3})^2$; $(2\sqrt{3})^{\frac{1}{2}}$.
4. $(2\sqrt{3})^4$; $(2\sqrt{3})^{\frac{1}{4}}$; $(2\sqrt[3]{3})^{\frac{1}{4}}$.
5. $\sqrt{\sqrt{x}}$; $\sqrt[3]{\sqrt{x}}$; $\sqrt[3]{x\sqrt{x}}$.
6. $(\sqrt[3]{a^2})^{\frac{1}{2}}$; $(\sqrt[3]{a^2})^2$.
7. $\sqrt[5]{\sqrt[3]{5}}$; $\sqrt[5]{\sqrt[3]{5^{\frac{1}{2}}}}$.
8. $\sqrt[p]{\sqrt[q]{a}}$; $\sqrt{\sqrt[q]{a}}$.
9. $(\sqrt{3-\sqrt{2}})^2$.
10. $(\sqrt{5-\sqrt{2}})^4$.
11. $(\sqrt{(x+y)^3})^4$.
12. $(\sqrt[5]{(x-y)^2})^4$.
13. $(\sqrt[6]{(a^2-b^2)^4})^3$.
14. $(\sqrt[8]{(1-a^2)^6})^3$.
15. $(\sqrt[m]{(x^2-y^2)^{2m}})^n$.
16. $(\sqrt[x]{a^2}-\sqrt[y]{b})^2$.
17. $(\sqrt[3]{x^2-y^2})^6$.
18. $(-\sqrt[3]{a^2-b^2})^9$.

SQUARE ROOT OF A BINOMIAL QUADRATIC SURD

504. It is sometimes possible to express the square root of a binomial quadratic surd as a binomial surd. The process of finding the square root of a binomial surd is readily understood by reversing the direct process of squaring a binomial surd.

Thus, $$(\sqrt{3}+\sqrt{2})^2 = 3 + 2\sqrt{6} + 2 \quad (1)$$
$$= 5 + 2\sqrt{6}. \quad (2)$$

This may be compared with

$$(a+b)^2 = a^2 + 2\,ab + b^2. \quad (3)$$

The trinomials in (1) and (3) are both perfect trinomial squares, and the trinomial in (1) bears the same relation to $(\sqrt{3}+\sqrt{2})$ that the trinomial in (3) bears to $a+b$. From (3) we readily write

$$\sqrt{a^2 + 2\,ab + b^2} = a + b.$$

and from (1) we may write in the same way,

$$\sqrt{3 + 2\sqrt{6} + 2} = \sqrt{3} + \sqrt{2}.$$

If we are asked to find $\sqrt{5+2\sqrt{6}}$, the problem evidently reduces to that of changing $5 + 2\sqrt{6}$ into the trinomial $3 + 2\sqrt{6} + 2$. This may be done by finding two factors of 6 whose sum is 5. These factors are 3 and 2, and we write

$$\sqrt{5+2\sqrt{6}} = \sqrt{3 + 2\sqrt{6} + 2}$$
$$= \sqrt{3} + \sqrt{2}.$$

505. To find the square root of a binomial quadratic surd:

1. Change the binomial to the form $a \pm 2\sqrt{b}$.

2. Find two factors of b whose sum is a, and write these two factors as the first and third terms of a trinomial equal to the original binomial.

3. Extract the square root of this perfect trinomial square.

Strictly speaking $\sqrt{5+2\sqrt{6}}$ is $\pm(\sqrt{3}+\sqrt{2})$. If we are concerned with the positive root only, it will always be obtained by writing the larger factor of b as the first term of the trinomial.

EXAMPLES

1. Find $\sqrt{7+\sqrt{40}}$.

SOLUTION. $$\sqrt{7+\sqrt{40}} = \sqrt{7 + 2\sqrt{10}} \quad \text{(Step 1 of the rule.)}$$
$$= \sqrt{5 + 2\sqrt{5\cdot 2} + 2} \quad \text{(Step 2 of the rule.)}$$
$$= \sqrt{5} + \sqrt{2}. \quad \text{(Step 3 of the rule.)}$$

2. Find $\sqrt{19-4\sqrt{12}}$.

SOLUTION.
$$\sqrt{19-4\sqrt{12}}=\sqrt{19-2\sqrt{48}}$$
$$=\sqrt{16-2\sqrt{16\cdot 3}+3}$$
$$=4-\sqrt{3}.$$

3. Find $\sqrt{6+\sqrt{11}}$.

SOLUTION.
$$\sqrt{6+\sqrt{11}}=\sqrt{6+2\sqrt{\tfrac{11}{4}}}$$
$$=\sqrt{\tfrac{11}{2}+2\sqrt{\tfrac{11}{2}\cdot\tfrac{1}{2}}+\tfrac{1}{2}}$$
$$=\sqrt{\tfrac{11}{2}}+\sqrt{\tfrac{1}{2}}=\tfrac{1}{2}(\sqrt{22}+\sqrt{2}).$$

EXERCISE

506. *Find the square root of:*

1. $3+2\sqrt{2}$.
2. $(6-2\sqrt{5})$.
3. $7+2\sqrt{6}$.
4. $7\pm\sqrt{48}$.
5. $8-\sqrt{28}$.
6. $9-\sqrt{56}$.
7. $7-\sqrt{45}$.
8. $8-\sqrt{60}$.
9. $5+\sqrt{9}$.
10. $8-\sqrt{15}$.
11. $8+\sqrt{39}$.
12. $9-\sqrt{32}$.
13. $9+\sqrt{65}$.
14. $9+\sqrt{80}$.
15. $2+\sqrt{3}$.
16. $\frac{3}{2}+\sqrt{2}$.
17. $9+\sqrt{17}$.
18. $9+\sqrt{56}$.
19. $9-\sqrt{77}$.
20. $5+\sqrt{21}$.
21. $12\pm\sqrt{44}$.
22. $12\pm\sqrt{80}$.
23. $12\pm\sqrt{108}$.
24. $12\pm\sqrt{140}$.
25. $10\pm\sqrt{19}$.
26. $10\pm\sqrt{36}$.
27. $10+\sqrt{51}$.
28. $10\pm\sqrt{64}$.
29. $10\pm\sqrt{75}$.
30. $10\pm\sqrt{84}$.

31. Try to discover, by studying the form of these binomial surds, and the relation of the numbers, how to make examples like the above.

MISCELLANEOUS EXERCISES IN EXPONENTS AND RADICALS

507. 1. (*a*) $(x+x^{-1})^2$.
(*b*) $(x+x^{-1})(x-x^{-1})$.
(*c*) $(\sqrt{x}+\sqrt{y}+\sqrt{z})^2$.
(*d*) $(a^{\frac{1}{2}}+a^{\frac{3}{2}})(a^{\frac{1}{2}}-a^{\frac{3}{2}})$.

2. $4^{-\frac{1}{2}}+(\frac{1}{2})^{-2}+21^0+9^{\frac{3}{2}}$.

3. $4x^0+(4x)^0+4^{-\frac{1}{2}}x^0$.

4. Find the value of $\sqrt{x^2 - y^2}$ when $x = a^{\frac{1}{2}} + a^{-\frac{1}{2}}$ and $y = a^{\frac{1}{2}} - a^{-\frac{1}{2}}$.

5. Find the value of $(x^2 - y^2)^0$ when $x = 3$ and $y = 1$.

6. $5\sqrt{24} - \sqrt{54} + 3\sqrt{96}$.

7. $(a - b) \div (a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})$.

8. Find the square root of:

$$x^{-2} + y^{\frac{4}{3}} + 2x^{-\frac{3}{2}}y^{\frac{1}{3}} - 2x^{-\frac{1}{2}}y - x^{-1}y^{\frac{2}{3}}.$$ (Yale.)

9. Simplify, writing the result with rational denominator:

$$\frac{\left(a^{\frac{1}{2}} + \frac{1}{x^{-\frac{1}{2}}}\right)^2 - \left(\frac{1}{a^{-\frac{1}{2}}} - x^{\frac{1}{2}}\right)^2}{x + \sqrt{a^2 + x^2}}.$$ (Mass. Institute of Technology.)

10. (a) $[(a^{\frac{1}{2}} + b^{\frac{1}{2}})^{\frac{1}{2}} + (a^{\frac{1}{2}} - b^{\frac{1}{2}})^{\frac{1}{2}}]^2$;

(b) $[(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 + (a^{\frac{1}{2}} - b^{\frac{1}{2}})^2]^2$.

11. $\dfrac{x^{-\frac{1}{2}} + y^{-\frac{1}{2}}}{\sqrt{x} + \sqrt{y}} \cdot (x^{\frac{1}{2}}y^{-\frac{1}{2}})^3$. (Princeton.)

12. $2^{\frac{1}{2}} \cdot 2^{\frac{2}{3}} \div 54^{-\frac{1}{3}}$. (Yale.)

13. $-x^2(9 - x^2)^{-\frac{1}{2}} + \sqrt{9 - x^2} + \dfrac{3}{\sqrt{1 - \left(\frac{x}{3}\right)^2}}$. (Yale.)

14. Simplify $\sqrt[7]{x^2y^{12}} \div \left(\dfrac{1}{xy}\right)^{\frac{1}{7}} \cdot \left(\dfrac{y^2}{x^3}\right)^{\frac{2}{7}}$. (Princeton.)

15. $3\sqrt{\dfrac{5}{2}} + \sqrt{40} + \sqrt{\dfrac{2}{5}} - \dfrac{1}{\sqrt{10}}$. (Princeton.)

16. Find to 3 decimal places $\dfrac{4 - \sqrt{3}}{4 + \sqrt{3}}$.

17. $\sqrt{17 + 12\sqrt{2}}$.

18. $\sqrt{\dfrac{x + y}{x - y}} - \sqrt{\dfrac{x - y}{x + y}} + \dfrac{2x}{x^2 - y^2}\sqrt{x^2 - y^2}$. (Yale.)

19. $\left(\sqrt{\dfrac{64\,a^2b^6}{81\,m^6n^2}}+\sqrt[3]{-\dfrac{a^{-3}}{m^{-9}}}\right)^2$. Give answer with positive exponents.

20. Find a mean proportional between $\sqrt{6}-\sqrt{2}$ and $\sqrt{6}+\sqrt{2}$.

21. Find a fourth proportional to 1, $2+\sqrt{7}$, $2-\sqrt{7}$.

22. $\sqrt{x^{\frac{5}{3}}-4\,x^{\frac{4}{3}}+2\,x^{\frac{7}{6}}+4\,x-4\,x^{\frac{5}{6}}+x^{\frac{2}{3}}}$.

23. $(ab^{-2}c^2)^{\frac{1}{2}}(a^3b^2c^{-3})^{\frac{1}{3}}+\sqrt[3]{\dfrac{a^6}{b}}$.

24. Simplify (*a*) $\sqrt{14+6\sqrt{5}}$.

(*b*) $\frac{1}{2}\sqrt{1-x}+x(1-x)^{-\frac{1}{2}}$.

(*c*) $\dfrac{\sqrt{2}-2\sqrt{5}}{\sqrt{3}+\sqrt{5}}$. Give answer in simplest radical form.

(*d*) $3\sqrt{\frac{2}{5}}+2\sqrt{\frac{1}{10}}-4\sqrt{\frac{1}{4}}$. (Sheffield.)

25. $(\frac{3}{4})^{-1}-2^{-3}+(\frac{16}{9})^{\frac{1}{2}}+128^{-\frac{3}{7}}-(7\sqrt{5})^0$.

26. $6\sqrt{33\frac{1}{3}}-\sqrt{96}+\dfrac{2}{\sqrt{6}}-\dfrac{4}{3}\sqrt{\dfrac{27}{8}}+\dfrac{24}{\sqrt{6}}$.

27. Show that $4^{n-2}\cdot 8^{2-n}\cdot 2^n=4$.

28. Simplify $\left(\dfrac{c^0x^2}{a}\right)^{-\frac{3}{2}}\times\left(\dfrac{\sqrt[6]{a^3}+\sqrt{b^0}}{c^2x^{-1}}\right)^{-2}$. (Yale.)

29. Simplify $\dfrac{\sqrt{2}+2\sqrt{3}}{\sqrt{2}-\sqrt{12}}$, and compute the value correct to two decimal places.

30. Simplify (*a*) $\dfrac{x^{-1}y^0z^{-3}}{x^{-2}y^3z^2}$.

(*b*) $3\sqrt{\frac{2}{5}}+2\sqrt{\frac{1}{10}}-4\sqrt{\frac{1}{40}}$. (*c*) $\sqrt[3]{2\,x}\div\sqrt{8\,x^3}$.

(*d*) $\dfrac{2\,x\sqrt[3]{(1+4\,x)^2}-4\,x^2(1+4\,x)^{-\frac{1}{3}}}{(1+4\,x)^{\frac{4}{3}}}$. (Sheffield.)

31. Simplify:

(*a*) $(\frac{4}{25})^{\frac{1}{2}} \times (\frac{125}{8})^{\frac{1}{3}}$.

(*c*) $\sqrt{\frac{3a}{x}} + \sqrt{\frac{3x}{a}} - \sqrt{\frac{ax}{3}}$.

(*b*) $2(1-2x)^{-\frac{2}{3}} + 3\sqrt[3]{1-2x}$.

(*d*) $\sqrt[3]{\frac{1}{4}} \div \sqrt{\frac{1}{2}}$. (Sheffield.)

32. Simplify $\dfrac{1}{1-\sqrt{2x}} + \dfrac{1}{1+\sqrt{2x}} - \dfrac{2}{1-2x}$.

33. Simplify:

(*a*) $\sqrt{2} \div \sqrt[3]{4}$.

(*c*) $\dfrac{2\sqrt{3-2x} - x(3-2x)^{-\frac{1}{2}}}{x^2}$.

(*b*) $2\sqrt{\frac{y}{x}} - \sqrt{\frac{x}{y}} + \sqrt{2 + \frac{x^2+y^2}{xy}}$.

(*d*) $\dfrac{a^0b^{-1}c^{-3}}{a^{-2}b^0c^2}$.

34. Simplify $\dfrac{9a^2}{x} + 1 - \dfrac{6a\sqrt{a}}{x} - \left(\dfrac{3a - \sqrt{a}}{\sqrt{x}}\right)^2$.

35. $3\sqrt{28} - \sqrt{5}\sqrt{27} + \sqrt{60} - \frac{3}{4}\sqrt{112}$.

36. $\sqrt{\frac{7}{8}} \div \frac{7}{8} - \sqrt{\frac{9}{10}} \div \frac{9}{10} + 3\sqrt{14}$.

37. $\sqrt{\frac{1}{2}}\sqrt{\frac{1}{3}}\sqrt{\frac{1}{5}}\sqrt{\frac{1}{7}} + \sqrt{480} - \sqrt{13\frac{1}{8}}$.

38. $(\sqrt{3} + \sqrt{2} - 1)^2$.

39. $(\sqrt{2} - \sqrt[3]{2})^3$.

40. $\dfrac{15 + 6\sqrt{5}}{2 + \sqrt{5}} - \dfrac{7 - 2\sqrt{5}}{4 - \sqrt{5}}$.

41. $(\sqrt[3]{x^4} - 2 + x^{-\frac{4}{3}}) \div (\sqrt[3]{x^2} - x^{-\frac{2}{3}})$.

42. $\dfrac{20 + 30\sqrt{2}}{3 + \sqrt{3}} - \dfrac{5 - 2\sqrt{2}}{2 - \sqrt{3}}$.

43. Show that $\dfrac{\dfrac{2x-b}{b\sqrt{3}} + \dfrac{2b-x}{x\sqrt{3}}}{1 - \dfrac{2x-b}{b\sqrt{3}} \cdot \dfrac{2b-x}{x\sqrt{3}}} = \sqrt{3}$.

XX. RADICAL EQUATIONS

508. A **radical equation**, or an **irrational equation**, is an equation in which the unknown number is involved in a radicand.

Thus, $3\sqrt{x} = 5$, $\sqrt{x} + 3 = 5$ are radical equations, but $x\sqrt{3} = 5$ is not a radical equation. (Why?)

Is $x^{\frac{1}{2}} = 10$ a radical equation?

To solve a radical equation it is generally necessary to rationalize the equation. The following examples will illustrate the method:

1. $$\sqrt{3x} - 5 = 0.$$

SOLUTION.
$$\sqrt{3x} = 5. \quad \text{(Why.)}$$
$$3x = 25. \quad \text{(Squaring both members.)}$$
$$x = 8\tfrac{1}{3}.$$

CHECK. $$\sqrt{3 \cdot 8\tfrac{1}{3}} - 5 = \sqrt{25} - 5 = 0.$$

2. $$14 + \sqrt{2x} = 16.$$

SOLUTION.
$$\sqrt{2x} = 2.$$
$$2x = 4.$$
$$x = 2.$$

Check mentally.

3. $$\sqrt{x + 40} - 3 = 7 - \sqrt{x}.$$

SOLUTION.
$$\sqrt{x + 40} = 10 - \sqrt{x}. \quad \text{(Why?)}$$
$$x + 40 = 100 - 20\sqrt{x} + x.$$
$$20\sqrt{x} = 60.$$
$$\sqrt{x} = 3.$$
$$x = 9.$$

Check mentally.

4. (*a*) $x-4-\sqrt{x+16}=0$. (*b*) $x-4+\sqrt{x+16}=0$.

SOLUTION.

$$x-4=\sqrt{x+16}.$$
$$x^2-8x+16=x+16.$$
$$x^2-9x=0.$$
$$x(x-9)=0.$$
$$x=0 \text{ or } 9.$$

CHECK. When $x=0$.

$0-4-\sqrt{0+16}=-4-4=-8.$

$\therefore$ 0 is not a root.

CHECK. When $x=9$.

$9-4-\sqrt{9+16}=9-4-5=0.$

$\therefore$ 9 is a root.

(*b*)

$$x-4=-\sqrt{x+16}.$$
$$x^2-8x+16=x+16.$$
$$x^2-9x=0.$$
$$x(x-9)=0.$$
$$x=0 \text{ or } 9.$$

CHECK. When $x=0$.

$0-4+\sqrt{0+16}=-4+4=0.$

$\therefore$ 0 is a root.

CHECK. When $x=9$.

$9-4+\sqrt{9+16}=5+5=10.$

$\therefore$ 9 is not a root.

If the principal square root of the radical is taken, $x=9$ satisfies (*a*) but not (*b*). Also, $x=0$ satisfies (*b*) but not (*a*).

509. Extraneous Root. From example 4, (*a*) and (*b*) it is seen that in solving a radical equation, roots are sometimes found that do not satisfy the equation. Such roots are **extraneous roots.**

The student will notice that in step 2, under both (*a*) and (*b*) the equations are the same since the squares of $\sqrt{x+16}$ and $-\sqrt{x+16}$ are the same, $x+16$. It is here that the extraneous root is introduced.

510. To solve a radical equation:

1. Arrange the terms of the equation so that a radical is alone in one member.

2. Raise both members of the equation to a power corresponding to the order of the radical.

3. Solve the resulting linear or quadratic equation by the usual methods.

When more than one radical occurs in the equation it may be necessary to repeat steps 1 and 2 of the rule one or more times.

Example

Solve: $\sqrt{x+60}=2\sqrt{x+5}+\sqrt{x}$.

SOLUTION.
$$x+60=4x+20+4\sqrt{x^2+5x}+x.$$
$$40-4x=4\sqrt{x^2+5x}.$$
$$10-x=\sqrt{x^2+5x}.$$
$$100-20x+x^2=x^2+5x.$$
$$-25x=-100.$$
$$x=4.$$

Check mentally.

EXERCISE

511. *Solve the following radical equations:*

1. $\sqrt{x}=3$.
2. $(2x)^{\frac{1}{2}}=4$.
3. $\sqrt[3]{2x}=4$.
4. $2+\sqrt{x}=5$.
5. $\sqrt[3]{x^2}=4$.
6. $x^{\frac{2}{3}}=4$.
7. $\sqrt{z}=a+\sqrt{b}$.
8. $\sqrt{23w+52}-16=19$.
9. $8\sqrt{4x+5}=7\sqrt{7x-13}$.
10. $\sqrt{4x+17}+14=15+2\sqrt{x}$.
11. $\sqrt{49x+85}-12=-11+7\sqrt{x}$.
12. $\sqrt{p+9}+11=10+\sqrt{p}$.
13. $\sqrt{x+45}=9-\sqrt{x}$.
14. $\sqrt{(2x-1)(2x+3)}=2x-1$.
15. $\sqrt{32+x}=4+\sqrt{x}$.
16. $5\sqrt{x}-7=3\sqrt{x}-1$.
17. $\sqrt{x}+\sqrt{2x}=1$.
18. $\sqrt{x}+\sqrt{3x}=2$.
19. $2\sqrt{x}-\sqrt{2x}=2$.
20. $\dfrac{2\sqrt{x}+1}{3\sqrt{x}-2}=\dfrac{2\sqrt{x}+3}{3\sqrt{x}-5}$.
21. $\dfrac{\sqrt{x}+29}{\sqrt{x}+5}=\dfrac{\sqrt{x}+37}{\sqrt{x}+7}$.
22. $\dfrac{\sqrt{x}+4}{\sqrt{x}+2}=\dfrac{\sqrt{x}+8}{\sqrt{x}+5}$.

23. $\left(18-\sqrt[4]{10-\sqrt{3(x^2-3)}}\right)^{\frac{1}{4}}=2.$

24. $(2\sqrt{x}+3)(2\sqrt{x}-3)=2.$

25. $\sqrt{1+16\,x}+2\sqrt{14+4\,x}=11.$

26. $\dfrac{5\,x-9}{\sqrt{5\,x}-3}=\dfrac{\sqrt{5\,x}-3}{2}+5.$

27. $\sqrt{7\,x+2}=\dfrac{5\,x+6}{\sqrt{7\,x+2}}.$

28. $\sqrt{14-x}+\sqrt{11-x}=\dfrac{3}{\sqrt{11-x}}.$

29. $\sqrt[3]{x^3+12\,x^2}=x+4.$

30. $\sqrt{x^2+2\,x-14}=\sqrt{x^2-5}-1.$

31. $\sqrt{5\,x-4}=\sqrt{2\,x+1}+1.$

32. $\sqrt[3]{(x+2)(x-5)}=2.$

33. $2\sqrt{\dfrac{3\,x^2-1}{4}+\dfrac{2}{3}}=\sqrt{3}.$

34. $\sqrt{x-6}+\sqrt{x-1}=\sqrt{x-9}+\sqrt{x+6}.$

Solve, and determine whether any of the roots of equations 35 to 43 are extraneous:

35. $\sqrt{10+x}+\sqrt{10-x}=6.$

36. $\sqrt{10+x}-\sqrt{10-x}=6.$

37. $\sqrt{12\,x+109}=2\,x+3.$

38. $\sqrt{3\,x-5}+\sqrt{x+6}=0.$

39. $\sqrt{x+5}=x-1.$

40. $3\,x-4\sqrt{x-7}=2(x+2).$

41. $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=\sqrt{6}.$

42. $\sqrt{3+\sqrt[4]{\dfrac{1-a}{x}}+\sqrt{a}}=2.$

43. $\sqrt{x+3}+\sqrt{2\,x-3}=6.$

Solve:

44. $\dfrac{\sqrt{3x^2+4}-\sqrt{2x^2+1}}{\sqrt{3x^2+4}+\sqrt{2x^2+1}}=\dfrac{1}{7}.$

HINT. Use composition and division.

45. $x+4x\sqrt{4x+5}=(4x+1)\sqrt{4x+5}-2.$

46. $\dfrac{\sqrt{3x^2-1}+\sqrt{3-x^2}}{\sqrt{3x^2-1}-\sqrt{3-x^2}}=\dfrac{a}{b}.$

47. Find a number which added to its square root gives 56.

48. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x.$

49. $\dfrac{5n-1}{\sqrt{5n}+1}=1+\dfrac{\sqrt{5n}-1}{2}.$

50. $\dfrac{\sqrt{1-z}}{2-\sqrt{1+z}}=\dfrac{\sqrt{1+z}}{2+\sqrt{1-z}}.$

51. $\dfrac{1}{1+\sqrt{1-x}}+\dfrac{1}{1-\sqrt{1-x}}=\dfrac{2x}{9}.$

52. $\sqrt{2s-3}=s-3.$

53. $\sqrt{2x-5}+6=x+2.$

54. $\sqrt{2x+1}+2\sqrt{x}=\dfrac{21}{\sqrt{2x+1}}.$

55. $\dfrac{\sqrt{m}}{3-\sqrt{m}}+\dfrac{3-\sqrt{m}}{\sqrt{m}}=\dfrac{5}{2}.$

56. $\sqrt{3x^2-6x+\frac{8}{3}}=\sqrt{5x-2x^2-\frac{22}{9}}.$

57. $\sqrt{(x-1)(3x-6)}=x-2.$

58. $\dfrac{\sqrt{2p-1}}{\sqrt{p^2-9}}=\dfrac{3}{\sqrt{p+11}}.$

59. $\dfrac{\sqrt{10+x}}{\sqrt{10-x}}=\dfrac{\sqrt{x-2}}{\sqrt{10-x}}+1.$

XXI. IMAGINARY NUMBERS

512. Consider the equation $x^2 + 4 = 0$, or $x^2 = -4$.

This equation asks the question: What is the number whose square is -4? There is no rational or irrational number that will answer this question, for all real numbers are positive or negative, and their squares are positive numbers. Hence the square root of a negative number has no meaning. A similar difficulty arises if we attempt to solve the quadratic $x^2 + 2x + 2 = 0$.

In order, then, to make the solution of the quadratic equation general, that is, always possible, we require a different number from any we have previously studied.

If we solve the equation $x^2 + 4 = 0$, or $x^2 = -4$, by the method of § 410 we get $x = \pm\sqrt{-4}$.

In order that the result should represent the solution of the equation, $\pm\sqrt{-4}$ must be such a number that $(\pm\sqrt{-4})^2 = -4$.

We define $\pm\sqrt{-a}$ as such a number that

$$(\pm\sqrt{-a})^2 = -a.$$

513. Imaginary Number. An even root of a negative number is an **imaginary number.**

In the present chapter we shall deal only with the square root of the negative number.

514. The student must not fall into the error of thinking that $\sqrt{-a}\sqrt{-a} = \sqrt{(-a)(-a)} = a$, as would be the case in the multiplication of ordinary radicals. We are now dealing with a new kind of number which does not always obey the laws of radicals and which is *defined* as such a number that $(\pm\sqrt{-a})^2 = -a$. This number is wholly different from an ordinary square root. Thus, $\sqrt{9} = \pm 3$, and $\sqrt{7} = \pm 2.645 \cdots$ correct to three decimal places, but it is not possible to find exactly or approximately in *real* numbers the value of $\sqrt{-4}$.

We shall always deal with the imaginary number as the product of two factors, one real and the other the *imaginary unit*, $\sqrt{-1}$.

For example, $\sqrt{-4} = 2\sqrt{-1}$, $\sqrt{-a} = \sqrt{a}\sqrt{-1}$.

To further facilitate the work, we introduce the symbol i for the imaginary unit and write $\sqrt{-4} = 2\,i$ and $\sqrt{-a} = i\sqrt{a}$.

515. Powers of the Imaginary Unit.

By definition, $(\sqrt{-1})^2 = -1$ or $i^2 = -1$.

From this we get the following:

$i = i,$	$i^5 = i,$	$i^9 = ?$	$i^{13} = ?$
$i^2 = -1,$	$i^6 = -1,$	$i^{10} = ?$	$i^{14} = ?$
$i^3 = -i,$	$i^7 = -i,$	$i^{11} = ?$	$i^{15} = ?$
$i^4 = 1,$	$i^8 = 1,$	$i^{12} = ?$	$i^{16} = ?$

Any power of the imaginary unit may therefore be reduced to one of the four numbers, i, -1, $-i$, 1.

What is the value of $i^4 + i^6$? $i^7 + i^{11}$?

516. Complex Number. The sum of a real number and an imaginary number is a **complex number.**

Thus, $2 + i$, and $3 + 2\,i$ are *complex numbers.*

$a + bi$ is the **general form** of the complex number. bi is the **general form** of the pure imaginary. In either of these a and b may have any real values.

517. Operations with Imaginary Numbers.

All operations with imaginary numbers can be performed by first writing the numbers in the general form and then proceeding as with real numbers, using the symbol *i* as we should use any other letter. In case higher powers of *i* occur at any time in the course of the work, they should be reduced as indicated in § 515.

Do not leave *i* in any denominator; that is, rationalize the denominator.

EXAMPLES

518. 1. Reduction to General Form.

(*a*) $\sqrt{-b}=\sqrt{-1\cdot b}=\sqrt{-1}\sqrt{b}=i\sqrt{b}.$

(*b*) $3-\sqrt{-2}=3-i\sqrt{2}.$

(*c*) $2+\sqrt{-4}=2+2i.$

2. Addition and Subtraction of Imaginary Numbers.

(*a*) $\sqrt{-9}+\sqrt{-16}+\sqrt{-25}=3i+4i+5i=12i.$

(*b*) $\sqrt{-2}+\sqrt{-18}+\sqrt{-200}-\sqrt[3]{-54}=i\sqrt{2}+3i\sqrt{2}$
$+10i\sqrt{2}+3\sqrt[3]{2}=3\sqrt[3]{2}+14i\sqrt{2}.$

(*c*) $\sqrt{-44}+\sqrt{-99}-\sqrt{-176}+\sqrt{-275}=$
$2i\sqrt{11}+3i\sqrt{11}-4i\sqrt{11}+5i\sqrt{11}=6i\sqrt{11}.$

3. Multiplication of Imaginary Numbers.

(*a*) $i^5=i;\ i^{10}=-1.$

(*b*) $(1+i)(1-i)=1-i^2=1-(-1)=2.$ (Explain.)

(*c*) $\sqrt{-12}\sqrt{-3}=i\sqrt{12}\cdot i\sqrt{3}=i^2\sqrt{36}=6i^2=-6.$
(Explain.)

(*d*) $\sqrt{-3}\cdot\sqrt{27}=i\sqrt{3}\cdot\sqrt{27}=i\sqrt{81}=9i.$

4. Division of Imaginary Numbers.

(*a*) $\sqrt{-8}\div\sqrt{-2}=2i\sqrt{2}\div i\sqrt{2}=2.$

(*b*) $$\frac{\sqrt{75}}{2\sqrt{-3}}=\frac{5\sqrt{3}}{2i\sqrt{3}}=\frac{5}{2i}=\frac{5i}{2i^2}=-\frac{5i}{2}.$$

(*c*) $$\frac{\sqrt{2}}{2+\sqrt{-2}}=\frac{\sqrt{2}}{2+i\sqrt{2}}=\frac{\sqrt{2}(2-i\sqrt{2})}{(2+i\sqrt{2})(2-i\sqrt{2})}$$
$$=\frac{2\sqrt{2}-2i}{4-2i^2}=\frac{2\sqrt{2}-2i}{6}=\frac{\sqrt{2}-i}{3}.$$

EXERCISE

519. *Simplify the following expressions, according to the general rule given in § 517, and the illustrative examples of § 518.*

1. Write in general form:

(*a*) $3\sqrt{-3}$.

(*b*) $7-\sqrt{-5}$.

(*c*) $3+\sqrt{-4}$.

(*d*) $\sqrt{3}+\sqrt{-3}$.

(*e*) $5+\sqrt{-a^2}$.

(*f*) $\sqrt{-16}$.

2. $\sqrt{-9}+\sqrt{-16}-\sqrt{-36}+\sqrt{-81}$.
3. $\sqrt{-16}+\sqrt{-4}-\sqrt{-9}+\sqrt{-144}$.
4. $\sqrt{-4}+\sqrt{-9}+3i+\sqrt{16}$.
5. $\sqrt{-2}+\sqrt{-72}-\sqrt{-32}$.
6. $\sqrt{-63}+\sqrt{-700}-\sqrt[3]{-64}$.
7. $\sqrt{-45}-3\sqrt{20}+4\sqrt{-80}+\sqrt[3]{-125}$.
8. $\sqrt{-84}+\sqrt{-\frac{3}{7}}$.
9. $3b^2\sqrt{-a^3c}+\dfrac{2}{c}\sqrt{-a^5c^3}$.
10. $i+i^2+i^3+i^4$.
11. $(\sqrt{-2})^2$; $(\sqrt{-3})^4$; $(-\sqrt{-4})^2$.
12. $(\sqrt{-1})^{100}$; $(\sqrt{-1})^{101}$; $(\sqrt{-1})^{102}$.
13. $\sqrt{-2}\cdot\sqrt{-32}$.
14. $\sqrt{-3}\cdot\sqrt{-4}$.
15. $\sqrt{-5}\cdot\sqrt{-5}$.
16. $(-\sqrt{-5})(\sqrt{-125})$.
17. $(-\sqrt{-a})\sqrt{-b}$.
18. $\sqrt{-\frac{1}{5}}\cdot\sqrt{-5}\cdot\sqrt{25}$.
19. $\sqrt{-12}\cdot\sqrt{3}$.
20. $\sqrt{-25}\cdot\sqrt{4}$.
21. $(-\sqrt{-3})(\sqrt{-12})(\sqrt{-4})$.
22. $(-\sqrt{-\frac{1}{2}})(-\sqrt{-2})\cdot(\sqrt{-3})$.
23. $(\sqrt{-3}+\sqrt{7}+3\sqrt{-5})2\sqrt{-3}$.
24. $(-3\sqrt{-5}+4\sqrt{8}-3\sqrt{-7})(-4\sqrt{-3})$.
25. $(-1+\sqrt{-3})^2$.
26. $(-1+\sqrt{-3})^3$.

27. $(1+i)^3$; $(1+i)^4$.

28. $(4+3\sqrt{-2})(4-3\sqrt{-2})$.

29. $(\sqrt{12}+2\sqrt{-8})(\sqrt{12}-2\sqrt{-8})$.

30. $(\sqrt{-2}+\sqrt{-3})\div(\sqrt{-2}-\sqrt{-3})$.

31. $3\sqrt{-8}\div(-2\sqrt{-2})$; $6\sqrt{-48}\div 4\sqrt{-3}$.

32. $5\sqrt{28}\div 3\sqrt{-7}$; $\dfrac{\sqrt{64}}{-\sqrt{-16}}$.

33. $42\div(3-2\sqrt{-3})$.

34. $(1+i)\div(1-i)$.

35. $(1+\sqrt{-2}+\sqrt{3})\div(1+\sqrt{-3})$.

36. $42\div(2i\sqrt{3}+3i\sqrt{6})$.

37. Find the value of x^2+x+1 (a) when $x=\dfrac{-1+\sqrt{-3}}{2}$; (b) when $x=\dfrac{-1-\sqrt{-3}}{2}$.

38. Find the value of $x+\dfrac{1}{x}$ when $x=1+i$.

39. $i+i^2+i^3\cdots i^8$.

40. $i\cdot i^2\cdot i^3\dots i^8$.

41. $(x^2-c^2i)^2$.

42. $(m^3+xi)^2$.

43. $(4-\sqrt{-4})^2$.

44. $(7+\sqrt{-9})^2$.

45. $(-4+2bi)^2$.

46. $(3+5i)(4-7i)$.

47. $(\sqrt{5}+\sqrt{-6})(\sqrt{6}-\sqrt{-8})$.

48. $(\sqrt{n^2}-\sqrt{-n^2})^2$.

49. $(ai-bi)^3$.

50. $(a+bi)^3-(a-bi)^3$.

51. $\dfrac{2}{3+\sqrt{-2}}$.

52. $\dfrac{2i}{3+2i\sqrt{-6}}$.

53. $\dfrac{1-2i\sqrt{3}}{1+2i\sqrt{3}}$.

54. $\dfrac{1+i}{1-i^2}$.

55. $\dfrac{1}{1-i}+\dfrac{1}{1+i}$.

56. $\dfrac{1-i}{1+i}+\dfrac{1+i}{1-i}$.

Simplify the following expressions:

57. $\dfrac{3+2i}{3-2i}+\dfrac{3-2i}{3+2i}$.

58. $\dfrac{x-i\sqrt{y}}{x+i\sqrt{y}}$.

59. $\dfrac{3-5\sqrt{-8}}{3+5\sqrt{-8}}$.

60. $\dfrac{\sqrt{-x}+\sqrt{-y}}{\sqrt{-x}-\sqrt{-y}}$.

61. $\dfrac{63+16i}{4+3i}$.

62. $\dfrac{56+33i}{12-5i}$.

63. $\dfrac{1-20i\sqrt{5}}{7-2i\sqrt{5}}$.

64. $\dfrac{7-24i}{4-3i}$.

65. $\dfrac{5-29i\sqrt{5}}{7-3i\sqrt{5}}$.

66. $\dfrac{1+33i\sqrt{3}}{4+3i\sqrt{3}}$.

67. $\dfrac{\sqrt{3}+i\sqrt{2}}{\sqrt{3}-i\sqrt{2}}$.

68. $\dfrac{1}{(1+i)^2}+\dfrac{1}{(1-i)^2}$.

69. $\dfrac{1}{(1+i)^4}-\dfrac{1}{(1-i)^3}$.

70. $\dfrac{\sqrt{x-y}+\sqrt{y-x}}{\sqrt{x-y}-\sqrt{y-x}}$.

71. $\dfrac{\sqrt{x}+\sqrt{-y}}{\sqrt{x}-\sqrt{-y}}-\dfrac{\sqrt{y}+\sqrt{-x}}{\sqrt{y}-\sqrt{-x}}$.

Solve the following equations, writing imaginary answers in the general form:

72. $x^2+2x+2=0$.

SOLUTION.

$$x^2+2x=-2.$$
$$x^2+2x+1=-1$$
$$x+1=\pm i.$$
$$x=-1\pm i.$$

CHECK. $(-1\pm i)^2+2(-1\pm i)+2=\mp 2i-2\pm 2i+2=0.$

73. $x^2+4x+6=0$.

74. $x^2-4x+8=0$.

75. $x^2-2ax+4a^2=0$.

76. $x^2-4x+7=0$.

77. $2x^2+5x+4=0$.

78. $x^2+x+1=0$.

79. $3x^2+4=2x$.

80. $x^2+5=4x$.

81. $x^2+2x+4=0$.

82. $3x^2-10x+10=0$.

XXII. QUADRATIC EQUATIONS

(*Continued from Chapter XVI*)

520. Equations of the forms $x^2 = k$, and $ax^2 + bx = 0$.

1. What is a quadratic equation? (§ 405.)
2. What is an incomplete quadratic? (§ 407.)
3. What are the two forms of incomplete quadratics?

521. In the solution of the following examples, irrational answers may be left in the simplest radical form, and imaginary answers in the general form.

EXERCISE

1. Give the rule for solving the quadratic in which the first degree term is missing. (§ 410.)

Solve the following:

2. $x^2 = 10.24.$

3. $(x + \frac{1}{9})(x - \frac{1}{9}) = 0.$

4. $(x - 3)(x + 2) = 19 - x.$

5. $2x(x+3) - x = 5(x+1) - 3.$

6. $(x - 3)(x + 3) = 1.$

7. $\frac{3}{x} - \frac{1}{x} = \frac{5 + (x - 5)}{8}.$

8. Give the rule for solving the quadratic in the form $ax^2 + bx = 0.$ (§ 414.)

9. $6x^2 = 18x.$

10. $(3x + 1)^2 + (3x - 1) = 0.$

11. $3x^2 - x\sqrt{8} = 0.$

12. $x^2\sqrt{3} + 3x = 0.$

13. $x^2 + 9 = 0.$

14. $(\sqrt{5} - x)(\sqrt{5} + x) = 0.$

15. $(x + \sqrt{6})(x - \sqrt{6}) = -2.$

16. $\frac{5}{2x^2} - \frac{4}{3} = \frac{7}{4x^2}.$

17. $\frac{1 + x}{1 - x} - \frac{x + 25}{x - 25} = 0.$

18. $\frac{a + x}{a - x} - \frac{x + b}{x - b} = 0.$

19. $ax^2 = a^2(a + 4b) + 4ab^2.$

20. $4x^2 + x\sqrt{-1} = 0.$

Solve the following:

21. $x = -\frac{1}{x}$.

22. $(x+4)^2 = 2^3 \cdot x$.

23. $a^2(b^2 - x) = b^2(a - x)^2$.

24. $\frac{ax^2 - bx + c}{mx^2 - nx + p} = \frac{c}{p}$.

25. $\frac{x+2}{3x+4} - \frac{3x-4}{x-2} = 0$.

26. $(2x+7)(5x-9) + (2x-7)(5x+9) = 1874$.

27. $(1+x)(2+x)(3+x) + (1-x)(2-x)(3-x) = 120$.

28. $\frac{a-x}{1-ax} = \frac{1-bx}{b-x}$.

29. $\frac{1}{1+\sqrt{1-x}} + \frac{1}{1-\sqrt{1-x}} = \frac{2x}{9}$.

30. $\sqrt{x+4} - \sqrt{5x-24} = \frac{6}{\sqrt{x+4}}$.

31. $2\sqrt{5+2x} - \sqrt{13-6x} = \sqrt{37-6x}$.

32. If a quadratic equation lacks the absolute term, one root is zero. Why?

33. The roots of a quadratic in the form $x^2 = k$ are equal in absolute value but of opposite sign. Why?

COMPLETE QUADRATICS

522. Solution by Completing the Square.

What is a complete quadratic? (§ 408.)

We may solve complete quadratics by three different methods; namely, **by completing the square, by formula,** and **by factoring.**

1. Which of these methods have already been treated in Chapter XVI?
2. What is meant by the p-form of the quadratic? (§ 420.)
3. How is an equation reduced to the p-form?
4. Change $ax^2 + bx + c = 0$ to the p-form.
5. What must be added to complete the square?
   (*a*) $x^2 - 3x + (\quad)$; (*b*) $x^2 + x + (\quad)$; (*c*) $4x^2 + 5x + (\quad)$.

523. To solve a complete quadratic equation:

1. Reduce the equation to the p-form.

2. Complete the square of the first member by adding to both members the square of one half the coefficient of x.

3. Extract the square root of each member, using both roots in the second member of the equation, and solve the resulting linear equations.

Unless otherwise suggested, the irrational answers may be left in simplest radical form.

Example

$$\frac{x+1}{x-1}=\frac{2x-1}{x+1}-3.$$

Solution. L. C. D. $= x^2-1$.

$$x^2+2x+1=2x^2-3x+1-3x^2+3. \quad \text{(Why?)}$$
$$2x^2+5x=3. \quad \text{(Why?)}$$
$$x^2+\tfrac{5}{2}x=\tfrac{3}{2}. \quad \text{(The } p\text{-form.)}$$
$$x^2+\tfrac{5}{2}x+\tfrac{25}{16}=\tfrac{49}{16}. \quad \text{(Why?)}$$
$$x+\tfrac{5}{4}=\pm\tfrac{7}{4}.$$
$$\therefore x=\pm\tfrac{7}{4}-\tfrac{5}{4}=\tfrac{1}{2} \text{ or } -3.$$

EXERCISE

524. *Solve the following by completing the square:*

1. $2x^2-5x=3.$
2. $x^2-\frac{5}{6}x+\frac{1}{6}=0.$
3. $9x^2=x+\frac{2}{3}.$
4. $5y^2=39+2y.$
5. $\dfrac{z}{9-z}+\dfrac{9-z}{z}=2.5.$
6. $\dfrac{x-5}{x+1}=x-1.$
7. $\dfrac{x-1}{x-13}=\dfrac{1}{x}.$
8. $(x-5)(x-3)+x^2-15=0.$
9. $\left(\dfrac{3}{x}+5\right)(9x-1)=(3+5x)\cdot 4.$
10. $(x-p)(4x-5p)=x^2-p^2.$
11. $\dfrac{2x(2x-5)}{2x-1}+\dfrac{2}{1-2x}=3.$
12. $x^2-6x+4=0.$
13. $x^2-2ax+b=0.$

Solve the following by completing the square:

14. $ax^2 - 2bx = c$.

15. $2x^2 + 5x + 4 = 0$.

16. $x^2 + x + 1 = 0$.

17. Verify 16.

18. $2x^2 + 15.9 = 13.6x$.

19. $x^2 + 2x = 0$.

20. $2x^2 - .21x + .001 = 0$.

21. $\frac{5x-1}{9} + \frac{3x-1}{5} = \frac{2}{x} + x - 1$.

22. $\frac{7-x}{11-2x} - \frac{4x-5}{1-3x} = 2$.

23. $\frac{5+x}{3-x} - \frac{8-3x}{x} = \frac{2x}{x-2}$.

24. $\frac{6x+4}{5} - \frac{15-2x}{x-3} = \frac{7(x-1)}{5}$.

(Multiply through by 5 and transpose.)

25. $\frac{x+1}{9} + \frac{12}{x+4} = \frac{x-4}{4} + \frac{17}{6}$.

26. $\frac{5x-7}{9} + \frac{14}{2x-3} = x - 1$.

27. $\frac{\frac{3x}{2}+5}{\frac{x}{2}-7} + \frac{x}{2} = 1$.

28. $\frac{x+2}{5x(2x-1)} = \frac{3}{x} - \frac{16x}{-4x^2+1}$.

29. $\frac{x+2}{3x-2} + \frac{7x-2}{3x+2} + \frac{6x^2+9x+5}{4-9x^2} = 0$.

30. $\frac{x+a}{a} + \frac{x+b}{b} + \frac{x^2-ab}{ab} = 0$.

31. $\frac{x+a-b}{a} + \frac{x-a-2b}{b} + \frac{a+b-x}{x} = 0$.

32. $\frac{3x}{2} - \frac{3x-20}{18-2x} = 2 + \frac{3x^2-80}{2(x-1)}$.

33. $\sqrt{2x+1} - x = \frac{1}{2}$.

34. $\sqrt{10x-34} + 2\sqrt{x+4} = \sqrt{2(3x+35)}$.

35. $\frac{x}{2}+\sqrt{\frac{5x}{2}}=3.$

36. $\sqrt{x+2+\sqrt{2x^2+x}}=2.$

37. $\frac{1}{x-1}-\frac{1}{x-3}=\frac{1}{35}.$

38. $\frac{3}{10-x}-1=\frac{4}{x-7}.$

39. $(x-2)(3x+1)=10+(2x+1)(x-3).$

40. $\frac{5x-1}{9}+\frac{2}{x}=x-1-\frac{3x-1}{5}.$

In examples 41 to 44 *find the roots correct to two decimal places:*

41. $x^2+2x-2=0.$

42. $\frac{x+1}{x-3}+\frac{x+4}{x+2}=0.$

43. $x^2+3x-11=0.$

44. $x+10+\frac{1}{x}=0.$

525. Solution by Factoring. The equation $(x-a)(x-b)=0$ is satisfied when $x=a$ or $x=b$ and not otherwise.

For suppose $x=a$,

We then have $(a-a)(a-b)=0,$

or $0\cdot(a-b)=0.$

Therefore the equation is satisfied when $x=a$.

When $x=b$, we have

$$(b-a)(b-b)=0 \text{ or } (b-a)0=0$$

Therefore the equation is satisfied when $x=b$.

To prove that the equation is not satisfied when x has any other value:

Suppose $x=c$, where c is different in value from both a and b

$$\therefore (c-a)(c-b)=0.$$

We should now have the product of two factors, neither one of which is 0, equal to 0, and this would be absurd.

526. Section 525 furnishes us the basis of the factoring method of solving equations.

To solve an equation by factoring:

1. Write the equation in order of powers of the unknown number, and with the second member zero.

2. Factor the first member into linear factors if possible.

3. Put each factor equal to zero, and solve the resulting equations.

527. This method of solving equations applies equally well to equations of higher degree than the second if the factoring can be accomplished. Also, it is not necessary to factor into linear factors, for factors of the second degree with respect to the unknown may be put equal to zero and solved by the preceding methods of solving quadratics.

Example

Solve $(3x+2)(2x+3)=(x-3)(2x-4)$

Solution. $6x^2+13x+6=2x^2-10x+12$. (Why?)

$4x^2+23x-6=0$. (Why?)

$(4x-1)(x+6)=0$. (Why?)

$4x-1=0$ and $x+6=0$. (Why?)

$x=\frac{1}{4}$ or -6.

EXERCISE

528. Solve the first 30 examples of the following set orally. The student should review Cases VI and VII of Factoring.

Solve:

1. $(x-4)(x-5)=0$.
2. $(x+4)(x-7)=0$.
3. $(2x-5)(x-3)=0$.
4. $(x+\frac{1}{2})(7x-1)=0$.
5. $x^2-13x+30=0$.
6. $x^2+13x+30=0$.
7. $x^2-13x-30=0$.
8. $x^2+13x-30=0$.
9. $x^2+13x+40=0$.
10. $x^2+19x+90=0$.
11. $x^2-21x+20=0$.
12. $x^2-7x+12=0$.
13. $x^2-7x+6=0$.
14. $x^2+16x+48=0$.
15. $x^2-x-42=0$.
16. $x^2+x-42=0$.

17. $x^2 - 2x = 63.$

18. $x^2 + 2x = 35.$

19. $x^2 - 10x = 39.$

20. $x^2 - 10x = 0.$

21. $x^2 - 16 = 0.$

22. $ax^2 - x = 0.$

23. $(x - a)(x - a) = 0.$

24. $x^2 - 6x + 9 = 0.$

25. $x(x - 5) = 36.$

26. $x - 7 = \frac{30}{x}.$

27. $3x^2 - 2x - 1 = 0.$

28. $4x^2 - 4x + 1 = 0.$

29. $(x - 1.25)(x + .75) = 0.$

30. $x^2 - \frac{5}{6}x + \frac{1}{6} = 0.$

31. $\frac{5x+7}{9} + \frac{14}{2x+3} = x + 1.$

32. $\frac{3}{x-3} - \frac{2}{x-2} = 2.$

33. $\sqrt{\frac{19-z}{3}} = \frac{z-1}{3}.$

34. $\frac{1}{x-1} = \sqrt{\frac{2x-3}{x-1}}.$

35. $\sqrt{1+y} + \sqrt{1-y} = \sqrt{3-y}.$

36. $\sqrt{4-x} + \sqrt{1+x} = \sqrt{9+x}.$

37. $\frac{1}{(x-1)^2} - \frac{2x-3}{x-1} = 0.$

38. $\frac{a^2x^2}{f^2} - \frac{2ax}{g} + \frac{f^2}{g^2} = 0.$

39. $\frac{1}{x-3} + \frac{1}{x-2} = \frac{3}{2}.$

40. $\frac{3r-2}{r} - \frac{2r-4}{r+1} = 2.$

41. $x^2 + 3ax = abx + 3a^2b.$

42. $(x^2 - 9)(x + 3) = 0.$

43. $3ax + 12a^2\sqrt{x} = 15a^3.$

44. $\frac{x-1}{x+1} + \frac{x-2}{x} = \frac{1}{3}.$

45. $(x - a)(x + a) = 4 - a^2.$

46. $(x - a)(x - b) = ab - a - b + 1.$

47. $(x - 1)^2 = a(x^2 - 1).$

48. $a^2(b - x)^2 - b^2(a - x)^2 = 0.$

49. $a^2 - x^2 - (a - x)(b + c - x) = 0.$

50. $a^2(a - x)^2 = b^2(b - x)^2.$

51. $x^2 - (a + b)x + ab = 0.$

52. $x + \frac{1}{x} = a + \frac{1}{a}.$

53. $\frac{ax^2 + bx + c}{a_1x^2 + b_1x + c_1} = \frac{c}{c_1}.$

54. $(x - 7)(2x + 5) - (3x - 1)(x - 7) = 0.$

Solve:

55. $\left(4-\frac{1}{x}\right)(2x-3)=4x-1.$

56. $(7-x)x=\left(1-\frac{x}{7}\right)(3x+8).$

57. $\frac{3x}{2}-\frac{3x-20}{18-2x}=2+\frac{3x^2-8}{2x-2}.$

529. General Form. The general form of the quadratic equation is $ax^2+bx+c=0$, where a, b, and c are any numbers.

1. $3x^2-7x+5=0$ is in the general form. In this equation $a=3$, $b=-7$, $c=5$.

2. $(a^2+b^2)x^2+(a-b)x+3=0$. Here $a=a^2+b^2$, $b=a-b$, and $c=3$.

3. Change to the general form

$$(ax-b)(c-d)=(a-b)(cx-d)x.$$

$$acx-adx-bc+bd=acx^2-bcx^2-adx+bdx. \quad \text{(Expanding.)}$$
$$-acx^2+bcx^2+acx-bdx-bc+bd=0. \quad \text{(Transposing.)}$$
$$(bc-ac)x^2+(ac-bd)x-(bc-bd)=0. \quad \text{(Collecting.)}$$
$$\therefore a=bc-ac,\ b=(ac-bd),\ c=-(bc-bd).$$

530. Any quadratic equation may be changed to the general form $ax^2+bx+c=0$. The steps are, in general, clearing of fractions, expanding, transposing, and collecting terms.

EXERCISE

531. *Reduce each of the following equations to the general form $ax^2+bx+c=0$ and determine the values of a, b, and c, in each case:*

1. $9x^2+4x=325.$
2. $17x^2=418.$
3. $x^2-2ax+b=0.$
4. $x^2-ax=0.$
5. $x^2-x=2+\sqrt{2}.$
6. $x^2+1=\frac{1}{2}x.$
7. $ax^2-bx=c.$
8. $ax^2-(a^2+1)x+a=0.$
9. $(a-x)^2=0.$
10. $(x+a+b)^2=0.$
11. $abx^2-a^2x-b^2x=ab.$
12. $\sqrt{x+16}=x-3.$

532. Formula for Solving Quadratic Equations. We have seen that every quadratic equation can be reduced to the form $ax^2 + bx + c = 0$. The solution of this equation leads to a formula that can be used for solving any quadratic equation.

$$ax^2 + bx + c = 0.$$

$$ax^2 + bx = -c. \quad \text{(Why?)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \quad \text{(Why?)}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \quad \text{(Why?)}$$

$$= \frac{b^2 - 4ac}{4a^2}. \quad \text{(Why?)}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The last result gives the two roots of the equation whatever the values of a, b, and c may be. The roots of any quadratic equation can therefore be found by substituting the values of a, b, and c in any particular equation in the formula for the roots; that is, in

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

This formula should be carefully committed to memory.

533. To solve a quadratic equation by the formula:

1. Change the equation into the form $ax^2 + bx + c = 0$.

2. Determine the values of a, b, and c for the given equation.

3. Substitute the values of a, b, and c in the formula and simplify the results.

Examples

1. Solve $6x^2 + x = 15.$

Solution. $6x^2 + x - 15 = 0.$ (General form.)

$$a = 6,\ b = 1,\ c = -15.$$

$$x = \frac{-1 \pm \sqrt{1 - 4\cdot 6\cdot(-15)}}{2\cdot 6}$$

$$= \frac{-1 \pm 19}{12} = \frac{3}{2} \text{ or } -\frac{5}{3}.$$

Check. $6\left(\frac{3}{2}\right)^2 + \frac{3}{2} = \frac{27}{2} + \frac{3}{2} = 15.$

2. Solve $mx^2 - m^2x - x + m = 0.$

Solution. $mx^2 - (m^2 + 1)x + m = 0.$

$$a = m,\ b = -(m^2 + 1),\ c = m.$$

$$x = \frac{(m^2 + 1) \pm \sqrt{m^4 + 2m^2 + 1 - 4m^2}}{2m}$$

$$= \frac{(m^2 + 1) \pm (m^2 - 1)}{2m}$$

$$= m \text{ or } \frac{1}{m}.$$

3. Solve $x^2 - 7x - 30 = 0.$

Solution. $x = \frac{7 \pm \sqrt{49 + 120}}{2} = \frac{7 \pm 13}{2} = 10 \text{ or } -3.$

EXERCISE

534. *Solve by the formula:*

1. $x^2 - 6x - 7 = 0.$
2. $x^2 + 8x = 20.$
3. $x^2 - 6x - 16 = 0.$
4. $x(x + 5) = 84.$
5. $x^2 + 7x = 30.$
6. $x^2 - 13x + 42 = 0.$
7. $x^2 - 19x = 0.$
8. $x^2 + 4x = 1.$
9. $2x^2 + 5x + 4 = 0.$
10. $9x^2 + 5 = 12x.$
11. $36x^2 + 6x - 5 = 0.$
12. $x^2 + 8x + 16 = 0.$
13. $4x^2 - 4x + 1 = 0.$
14. $x^2 - 4 = 0.$
15. $x^2 + \frac{2}{3}x = 40.$
16. $3x^2 + 17x + 70 = 0.$
17. $x^2 + \frac{7}{20}x = \frac{3}{10}.$
18. $20x^2 - 2x = 6.$

19. $\frac{1}{12}x^2 - 2x + 12 = 0.$

20. $2x^2 - \frac{2}{3}x = \frac{1}{6}.$

21. $(2x + 1)^2 = 0.$

22. $x^2 + ax = b.$

23. $ax^2 + 2bx + c = 0.$

24. $ax^2 + bx = c.$

25. $x^2 - ax = 0.$

26. $(x - 1)^2 - ax^2 + a = 0.$

27. $(x - 6)(x - 5) + (x - 7)(x - 4) = 10.$

28. $(2x - 5)^2 - (x - 6)^2 = 80.$

29. $2x + \frac{1}{x} = 3.$

30. $\frac{x}{4} - \frac{21 - x}{4 - x} = 1.$

31. $\frac{x}{4} + \frac{25}{x} = 3.$

32. $\frac{x - 8}{x + 2} = \frac{x - 1}{2(x + 5)}.$

33. $(a - x)(x - b) + ab = 0.$

34. $(a - x)^2 + (x - b)^2 = a^2 + b^2.$

35. $x^2 - x\sqrt{3} + 1 = 0.$

36. Verify 35.

37. $\frac{x - 1}{x} - \frac{2x - 3}{x - 1} + \frac{x - 8}{x - 9} = 0.$

38. $x(a + b - x) = c(a + b - c).$

39. $(n - p)x^2 + (p - m)x + (m - n) = 0.$

40. $\frac{ax^2 - bx + c}{x^2 - x + 1} = c.$

41. $(a - x)^2 - (a - x)(x - b) - (x - b)^2 = (a - b)^2.$

42. $(a - x)^3 + (x - b)^3 = (a - b)^3.$

In the following equations find the roots correct to two decimal places:

43. $3x^2 - 2x = 40.$ 44. $x^2 - x - 1 = 0.$ 45. $x^2 - 30 = \frac{x}{3}.$

46. Given $S = \frac{1}{2}gt^2 + v_0t.$ When $S = 200$, $g = 32$, and $v_0 = 10$, find t.

47. $x^2 + 1.92x - 3.83 = 0.$

48. $x^2 + 3.14x + 2.45 = 0.$

49. $3x^2 - 4x - 10 = 0.$

50. $5(x^2 - 1) + 10x = 7x^2 - 15.$

THEORY OF QUADRATIC EQUATIONS

535. Relations between the Roots and the Coefficients of a Quadratic Equation.

Consider the equation

$$ax^2 + bx + c = 0.$$

If we let r_1 and r_2 represent the roots, we may write,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\text{and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a},$$

$$\text{and } r_1 r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{c}{a}.$$

The student should work out these results in detail.

Therefore we have:

The sum of the roots of a quadratic equation in the form $ax^2 + bx + c = 0$ is $-\frac{b}{a}$, and the product is $\frac{c}{a}$.

Examples

1. $3x^2 - 7x + 2 = 0.$

The sum of the roots is $\frac{7}{3}$ and the product is $\frac{2}{3}$.

2. $3x^2 - 3x = 7.$

$$3x^2 - 3x - 7 = 0. \quad \therefore r_1 + r_2 = 1, \text{ and } r_1 r_2 = -\tfrac{7}{3}.$$

EXERCISE

536. *Find, without solving, the sum and the product of the roots of the following:*

1. $x^2 - 21x + 20 = 0.$
2. $x^2 + 7x + 12 = 0.$
3. $x^2 + 16x + 48 = 0.$
4. $4x^2 - 8x - 3 = 0.$
5. $x^2 - \frac{1}{2}x = \frac{1}{2}.$
6. $3x^2 - x = 24.$

7. $7x^2 + x = 50.$

8. $2x^2 - 14x + 23 = 0.$

9. $5x^2 + 13 = 14x.$

10. $ax^2 - 2bx = c.$

11. $6x^2 + 7x = 3.$

12. $(x-3)(x-5) = 0.$

13. $3x^2 + 11 = 5x.$

14. $5x^2 = 12.$

15. $3x^2 = 5x.$

16. $x^2 + px = q.$

17. Show that, if an equation is in the form $x^2 + mx + n = 0$, $r_1 + r_2 = -m$, and $r_1 \cdot r_2 = n$.

18. What is the sum of the roots of an incomplete quadratic equation of the form $x^2 = k$?

19. One root of $x^2 + 4x - 45 = 0$ is 5. Determine the other in two ways, without using any of the usual methods of solving.

20. One root of $2x^2 - 7x - 15 = 0$ is 5. Find the other, in two ways, as in example 19.

537. Nature of the Roots of $ax^2 + bx + c = 0$. The roots of the equation $ax^2 + bx + c = 0$ have been found in § 532 to be

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Whether the roots r_1 and r_2 are real or imaginary (§ 456), and if real whether they are rational or irrational (§ 457), depends upon the expression $\sqrt{b^2 - 4ac}$. (Why?)

1. If $b^2 - 4ac = 0$, **the roots are real, rational, and equal.** (Why?)

2. If $b^2 - 4ac$ **is a negative number, the roots are imaginary.** (Why?)

3. If $b^2 - 4ac$ **is positive, the roots are real and unequal, and they are rational or irrational according as $b^2 - 4ac$ is or is not a perfect square.**

538. Discriminant. The expression $b^2 - 4ac$ is the **discriminant**, since, by means of it, we determine the **nature** of the roots.

1. Determine the nature of the roots of $x^2 + 5x - 6 = 0$.

SOLUTION. The discriminant is $25 - 4 \cdot 1 \cdot (-6) = 49$.

$\therefore$ The roots are *real*, *rational*, *unequal*.

Let the student determine the roots.

2. Determine the nature of the roots of $9x^2 + 5 = 12x$.

SOLUTION. $9x^2 - 12x + 5 = 0$.

The discriminant is -36. (Why?)

$\therefore$ The roots are imaginary.

3. For what value of k are the roots of $x^2 - 6x + k = 0$ equal to each other?

SOLUTION. The discriminant is $36 - 4k$.

If k has a value that satisfies the equation $36 - 4k = 0$, the roots will be equal. This gives $k = 9$.

Therefore $k = 9$ is the value required to make the roots equal to each other.

If $k < 9$, $36 - 4k$ is a positive number, and therefore the roots will be real.

If $k > 9$, $36 - 4k$ is negative and the roots will be imaginary.

Let the student determine the roots when $k = 9$; when $k = 10$; when $k = 8$.

EXERCISE

539. *Determine, without actually solving, the nature of the roots of the following equations:*

1. $x^2 - 7x + 10 = 0$.
2. $12x^2 - x - 1 = 0$.
3. $3x^2 - 12x + 5 = 0$.
4. $3x^2 - 8x + 7 = 0$.
5. $2x^2 - 5x - 9 = 0$.
6. $4x^2 - 13x + 3 = 0$.
7. $25x^2 - 10x + 1 = 0$.
8. $3x^2 + 2x + 5 = 0$.
9. $x^2 - 5x = 50$.
10. $x^2 - 5x + 50 = 0$.
11. $2x^2 - 7x + 30 = 0$.
12. $2x^2 - 7x - 30 = 0$.
13. $-7x^2 + 22x = 3$.
14. $2x^2 + 3 = 5x$.

15. $x^2-3x+k=0$ when $k=2\frac{1}{4}$. 17. Answer 15 when $k > 2\frac{1}{4}$.

16. Answer 15 when $k < 2\frac{1}{4}$. 18. $x+\frac{1}{x}=\frac{5}{2}$.

19. $x+\frac{1}{x}=k$, when k lies between 2 and -2 in value.

20. What value of c will give equal roots in $2x^2+4x+3c=0$?

21. Verify 20.

22. For what value of k is one root three times the other in $x^2-kx+75=3$?

SOLUTION. Here $r_1=3r_2$, or $\frac{k+\sqrt{k^2-300}}{2}=3\,\frac{k-\sqrt{k^2-300}}{2}$. Solve for k.

23. For what value of k does one root of $x^2-kx+40=0$ exceed the other by 3?

24. Find k if $r_1=7r_2$ in $x^2-kx+63=0$.

25. Find k if $r_1=2r_2$ in $4x^2-9x+k=0$. Verify.

540. To form an Equation with Given Roots.

We have seen (§ 525) that $(x-a)(x-b)=0$ has the roots a and b and no other roots. Similarly $(x-a)(x-b)(x-c)=0$ has the roots a, b, and c and no other roots. Thus, we can make an equation with any required roots.

1. Make an equation whose roots are 2 and 3.

SOLUTION. $(x-2)(x-3)=0$, or $x^2-5x+6=0$, is the required equation.

2. Form an equation whose roots are 2 and -5.

SOLUTION. $(x-2)(x+5)=x^2+3x-10=0$. Explain the factor $x+5$.

3. Form an equation whose roots are $\frac{2}{3}$ and $\frac{3}{5}$.

The result is indicated by the equation $(x-\frac{2}{3})(x-\frac{3}{5})=0$. For convenience, multiply by 15 in the form of the two factors, $3\cdot 5$; thus,

$$3(x-\tfrac{2}{3})5(x-\tfrac{3}{5})=0.$$
$$(3x-2)(5x-3)=0 \text{ or } 15x^2-19x+6=0.$$

4. Form an equation whose roots are $1 \pm \sqrt{2}$.

SOLUTION. $(x-1-\sqrt{2})(x-1+\sqrt{2})=0$, or $x^2-2x-1=0$.

5. Form an equation whose roots are 1, -1, 2.

SOLUTION. $(x-1)(x+1)(x-2)=0$, or $x^3-2x^2-x+2=0$.

EXERCISE

541. *Make equations whose roots are as indicated:*

1. 2, 3.
2. 4, 5.
3. 7, -1.
4. 0, 6.
5. -3, -2.
6. $-a$, $-b$.
7. 2, 2.
8. ± 2.
9. a, $2a$.
10. b, $-2b$.
11. $\pm\sqrt{3}$.
12. $1 \pm \sqrt{5}$.
13. $1 \pm i$, 1.
14. 2, $1 \pm 2\sqrt{3}$.
15. $\pm i$, 2.
16. $2+\sqrt{2}$, $3-\sqrt{2}$.

EQUATIONS IN THE FORM OF QUADRATICS

542. Quadratic Form. An equation is in the **form of a quadratic** if it contains two powers of the unknown, one of which is the square of the other.

$x^4-5x^2+4=0$, $x+x^{\frac{1}{2}}-6=0$, and $x^{-\frac{2}{3}}+x^{-\frac{1}{3}}-6=0$ are examples of quadratic forms. These equations may be solved for x^2, $x^{\frac{1}{2}}$, and $x^{-\frac{1}{3}}$ respectively, and the results so found can then be solved for x.

An equation may be in quadratic form with respect to some polynomial containing x.

$(x^2-2)+\sqrt{x^2-2}-6=0$ is such an equation. It may be solved for $\sqrt{x^2-2}$ just as $z^2+z-6=0$ may be solved for z.

The method of solving such equations will be understood by examples.

1. $x^4-5x^2+4=0$.

$(x^2-4)(x^2-1)=0$. (Solving for x^2 by factoring,)

$x^2=4$ or 1.

$x=\pm 2$ or ± 1.

2. $x + x^{\frac{1}{2}} - 6 = 0.$

$$x^{\frac{1}{2}} = \frac{-1 \pm \sqrt{1+24}}{2} = -3 \text{ or } 2. \quad \text{(Solving by the formula for } x^{\frac{1}{2}}.)$$

$$x = 9 \text{ or } 4.$$

This is a radical equation; 4 is a root and 9 is an extraneous root.

3. $x^{-\frac{2}{3}} + x^{-\frac{1}{3}} - 6 = 0.$

$$x^{-\frac{1}{3}} = -3 \text{ or } 2. \quad \text{(Solving as a quadratic for } x^{-\frac{1}{3}}.)$$

$$\frac{1}{x^{\frac{1}{3}}} = -3 \text{ or } 2. \quad \text{(Why?)}$$

$$x^{\frac{1}{3}} = -\tfrac{1}{3} \text{ or } \tfrac{1}{2}. \quad \text{(Why?)}$$

$$x = -\tfrac{1}{27} \text{ or } \tfrac{1}{8}.$$

This is a radical equation, and both roots satisfy the equation.

4. $(x^2 - 2) + \sqrt{x^2 - 2} - 6 = 0.$

Here we regard $\sqrt{x^2 - 2}$ as the unknown,

$$\sqrt{x^2-2} = \frac{-1 \pm \sqrt{1+24}}{2} = -3 \text{ or } 2. \quad \text{(By formula.)}$$

$$x^2 - 2 = 9 \text{ or } 4.$$

$$x^2 = 11 \text{ or } 6.$$

$$x = \pm\sqrt{11} \text{ or } \pm\sqrt{6}.$$

Do the roots satisfy the equation?

In the solution of such an equation as example 4, it is sometimes convenient to substitute a single letter for the expression we are regarding as the unknown. For example, we might have put z for $\sqrt{x^2 - 2}$ and then we should have

$$z^2 + z - 6 = 0.$$

$$z = \frac{-1 \pm \sqrt{25}}{2} = -3 \text{ or } 2.$$

$$\sqrt{x^2 - 2} = -3 \text{ or } 2, \text{ etc.}$$

5. $\dfrac{a-x}{x-b}+\dfrac{x-b}{a-x}=\dfrac{a^2+b^2}{ab}.$

Let $\dfrac{a-x}{x-b}=z.$

Then the equation takes the form

$$z+\frac{1}{z}=\frac{a^2+b^2}{ab}.$$

$$abz^2-(a^2+b^2)z+ab=0.$$

$$(az-b)(bz-a)=0.$$

whence $z=\dfrac{b}{a}$ or $\dfrac{a}{b}.$

$$\frac{a-x}{x-b}=\frac{a}{b}. \qquad \text{Also } \frac{a-x}{x-b}=\frac{b}{a},$$

$$x=\frac{2ab}{a+b}. \qquad \text{(Explain.)} \qquad x=\frac{a^2+b^2}{a+b}.$$

EXERCISE

543. *Solve the following equations as quadratics, substituting a single letter for an expression containing the unknown when desirable:*

1. $x^4-13x^2+36=0.$
2. $x^2(x^2-90)+729=0.$
3. $x-7\sqrt{x}+12=0.$
4. $x-4=3x^{\frac{1}{2}}.$
5. $\dfrac{x^2}{5}+\dfrac{5}{x^2}=\dfrac{26}{5}.$
6. $x^6-9x^3+8=0.$
7. $\dfrac{2}{x^2+3}+\dfrac{5}{x^2}=2.$
8. $x^4-21x^2=100.$
9. $(x^2-10)(x^2-3)=78.$
10. $(x^2-5)^2+(x^2-1)^2=40.$
11. $x^{-6}-7x^{-3}=8.$
12. $9x^{\frac{1}{4}}=9+2x^{\frac{1}{2}}.$
13. $2\sqrt[4]{x^3}+\sqrt{x^3}-3=0.$
14. $2(\sqrt{x}-3)^2-3=\sqrt{x}.$
15. $x^{\frac{3}{2}}+8x^{\frac{1}{2}}=9x.$
16. $2x^{\frac{1}{3}}-3x^{\frac{2}{3}}+x=0.$
17. $(x^2+2x)^2+3(x^2+2x)=10.$
18. $\left(x+\dfrac{1}{x}\right)^2-8\left(x+\dfrac{1}{x}\right)+7=0.$
19. $\dfrac{x^2+3}{x^2-3}+2\dfrac{x^2-3}{x^2+3}+3=0.$

20. $60 - 4\sqrt{x^2 + x + 6} = x^2 + x + 6.$

Let $z = \sqrt{x^2 + x + 6}$.

21. $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11.$

HINT. This equation may be written

$$(x^2 - 2x + 5) + 6\sqrt{x^2 - 2x + 5} = 16.$$

22. $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}.$

23. $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3.$

24. $(2x^2 - 3x + 1)^2 = 22x^2 - 33x + 1.$

25. $\left(x + \frac{1}{x}\right)^2 + 4x + \frac{4}{x} = 12.$

26. $\frac{x^2 + x + 5}{x^2 + x - 2} + \frac{x^2 + x - 5}{x^2 + x - 4} = 10.$

Put $x^2 + x = z$.

27. $2(x + 3)(x + 4) = (x^2 + 7x)(x^2 + 7x - 3).$

28. $x^2 = 8\sqrt{x^2 + 16} - 32.$

29. $x^2 + \sqrt{5x + x^2} = 42 - 5x.$

30. $x^2 - 9x - 9\sqrt{x^2 - 9x - 11} = -9.$

31. $\frac{3}{x + \sqrt{5 - x^2}} - \frac{3}{x - \sqrt{5 - x^2}} = 4.$

32. $2x^{10} = 3x^6 - x^8.$

33. $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} + 3 = 0.$

34. $(x^2 - 2)^{\frac{2}{3}} - 4(x^2 - 2)^{\frac{1}{3}} = 5.$

35. $9x^{-4} + 4x^{-2} = 5.$

36. $x^2 - 4\sqrt{x^2 - 2} = -1.$

37. $\sqrt{2x} - 7x = -52.$

38. $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24.$

Solve the following equations as quadratics:

39. $\left(\frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-1}\right)^2+2\left(\frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-1}\right)=15.$

40. $\left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)=15.$

PROBLEMS LEADING TO QUADRATIC EQUATIONS

544. 1. The sum of two numbers is 17 and their product is 42; find the numbers.

2. The sum of two numbers is 17 and the sum of their squares is 185; find the numbers.

3. Find the sides of a rectangle, knowing that its perimeter is 52 feet and its area is 160 square feet.

4. In a right-angled triangle the measures of the two sides about the right angle and the hypotenuse are three consecutive integers. Find them.

5. Same as problem 4, if the sides are consecutive even numbers.

6. The sum of the two sides about the right angle of a right-angled triangle is 21 inches, and the hypotenuse is 16 inches. Find the sides correct to two decimal places.

7. A certain rectangle contains 216 square feet. If both dimensions are increased by 2, the area is increased by 64 square feet. Find the dimensions of the rectangle.

8. The sum of the roots of a quadratic equation is 3 and their product is $1\frac{5}{9}$. Find the roots.

9. Same as problem 8, if the sum of the roots is 2 and their product is 4.

10. Two numbers differ by 2.1, and the square of their sum is 25. Find the numbers.

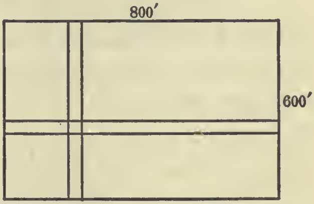
11. Determine the positive value of x correct to two decimal places for the equation $x^2+y^2=36$, knowing that $5y=27$.

12. Determine the positive value of b to two decimal places, in $a^2 + b^2 = c^2$, if $a = 2.1$ and $c = 4.3$.

13. Determine the larger root of $x^2 = .100 - .200\ x$ correct to three significant figures. (Harvard.)

14. A rectangular box is 3 inches deep, and is 2 inches longer than it is wide. Find its length and breadth, if its volume is 105 cubic inches.

15. A rectangular plot of land is 600 feet by 800 feet. It is divided into four rectangular blocks by two streets of equal width running through it. Find the width of the streets if together they cover an area of 67,500 square feet.



16. What is the width of the streets in problem 15 if together they cover one fourth the area of the plot?

17. How wide a strip must be cut around the outside of a lawn 60 feet by 80 feet so that the strip cut may contain half the plot?

18. A rectangular tin box is 2 inches deep, and is 2 inches longer than it is wide. Find the length and breadth if it requires 88 square inches of tin to make the box, including the cover, making no allowance for waste.

19. If from a certain number the square root of half that number is subtracted, the result is 25; find the number. (Regents.)

20. The numerator of a fraction exceeds the denominator by 2. If both terms of the fraction are increased by 2, the value of the fraction is diminished by $\frac{1}{6}$. Find the fraction.

21. The units' digit of a number exceeds the tens' digit by 1. The product of the digits equals $\frac{4}{9}$ of the number. What is the number?

22. In an automobile race of 462 miles the winning car runs 2 miles an hour faster than the losing car and wins the race by $\frac{1}{2}$ hour. What is the winner's rate and what is the time?

23. A broker buys a certain number of shares of stock for \$960. Later the price falls \$20 a share and he finds that he might have bought 4 shares more for the same money. How many shares did he buy?

24. If q is the area of a rectangle and p is the perimeter, show that $x^2 - \frac{p}{2}x + q = 0$ is an equation for finding the dimensions.

25. A man buys apples for \$12. If the price had been 20¢ less per bushel, he could have bought 5 bushels more for the same money. Find the number of bushels bought and the price per bushel.

26. How wide a strip must be plowed around a field 60 rods long and 40 rods wide to have the field half plowed?

27. A stream flows at the rate of 4 miles an hour. A man can row up the stream 10 miles and back to the starting point in 6 hours. Find the rate at which the man would row in still water.

NOTE. It is to be assumed in this problem that the rate at which the man rows upstream is equal to his rate in still water minus the rate of the current, and that in going downstream his rate is that of his rowing in still water plus the rate of the stream.

28. A motor boat goes 12 miles up a river and returns to the starting point in $4\frac{1}{5}$ hours. Find the rate of the current if the boat can run 7 miles an hour in still water.

29. A rectangle is 6 inches by 10 inches. It is to be doubled in area by equal additions to the length and the width. Find to two decimal places the increase in the dimensions.

30. Solve 29 if the area is to be made four times as great.

31. A company owns two factories that together can make 252 automobiles in 12 days. Working alone one factory requires 7 days longer than the other to make this number. Find the number of days for each factory. (Yale.)

32. If the product of three consecutive numbers is divided by each in turn, the sum of the quotients is 191. Find the numbers.

33. A man having bought an article, sells it for $21. He loses as many per cent as he gave in dollars for the article. How much did it cost him? (Yale.)

34. Find the price of eggs when if two less were given for 30 ¢ the price would be 2 ¢ per dozen higher. (Amherst.)

35. If a ball is thrown vertically upwards with a velocity v_0, the distance in feet to which it will rise in t seconds is given by the formula $d = v_0t - \frac{1}{2}gt^2$. $(g = 32.)$ Solve this equation for t when $v_0 = 200$, and $d = 300$.

REVIEW OF EQUATIONS

545. Equations may be classified as to their degree into three groups:

1. **Linear Equations.** (Simple or first degree equations)
2. **Quadratic Equations.** (Second degree equations)
3. **Higher Degree Equations.**

546. Equations may be classified as to form into three groups:

1. **Rational Integral Equations.**
2. **Rational Fractional Equations.**
3. **Irrational Equations.**

547. In solving an equation we begin by simplifying as much as possible, including such steps as expanding, clearing of fractions, rationalizing, transposing, and collecting terms. All these steps aim toward some particular *form*. The *form* will depend upon the kind of equation we are solving and, in the case of quadratic equations, the method we intend to use in its solution.

Exceptions to the general directions just given include radical equations solved as quadratic forms without rationalizing, and fractional equations solved by substitution.

Skill in selecting the best methods of solving equations, and in discovering methods of simplifying the work will be gained by experience and by a conscious effort on the part of the student to achieve such ends.

Higher degree equations, if they can be solved at all, by the methods of elementary algebra, must come under "Quadratic Forms" or under the factoring method.

REVIEW QUESTIONS

548. 1. What is a simple equation? a quadratic equation?

2. What is a rational integral equation? an irrational equation?

3. State, in full, the three different methods of solving quadratics.

4. To what form do we reduce the quadratic when we solve by "completing the square"? by formula? by factoring?

5. Can the formula be used to solve an incomplete quadratic?

6. What do we mean by the "nature of the roots" of a quadratic?

7. How do we determine the nature of the roots of an equation without solving the equation? the sum of the roots? the product?

8. How do we form an equation with given roots?

9. What do we mean by the extraneous roots of a radical equation?

10. How many roots has a quadratic equation?

11. Knowing that 3 and -5 are the roots of $x^2+2x-15=0$, can you at once write the factors of $x^2+2x-15$?

12. Translate into verbal language the condition that $ax^2 + bx + c = 0$ may have equal roots; the condition that $x^2 + px = q$ may have real roots.

13. Do both roots of a quadratic equation necessarily satisfy the conditions of the problem from which the equation may be derived? How can such results be checked?

REVIEW EXERCISE

549. *Solve:*

1. $\dfrac{2-5x}{5x+1}+\dfrac{7+x}{3-2x}=\dfrac{148-5x^2}{3+13x-10x^2}-2.$

2. $\dfrac{x}{9-x}+\dfrac{9-x}{x}=25.$

3. $\sqrt{\dfrac{20}{x^2}+9}-\sqrt{\dfrac{20}{x^2}-9}=3.$

4. $(x+11)^{\frac{1}{2}}+3(x+11)^{\frac{1}{4}}=4.$

5. $\dfrac{1}{x+a}-\dfrac{x+a}{x-a}=\dfrac{x^2}{a^2-x^2}.$

6. $\sqrt{x+18}+\sqrt{x-18}=6.$

7. $\dfrac{b-a}{x-b}-\dfrac{a-2b}{x+b}+\dfrac{3x(a-b)}{x^2-b^2}=0.$

8. $x^2+12x=2\sqrt{x^2+12x-4}+67.$

9. $y^{\frac{1}{2}}-\dfrac{1}{y^{\frac{1}{2}}}=1\frac{1}{2}.$

10. Give nature, sum, and products of the roots of the following:

(*a*) $5x^2-7x+2=0.$ (*d*) $2x^2-6x=-m$, when $m>4\frac{1}{2}$.

(*b*) $x^2-5=4x.$ (*e*) $5-3x^2+7x=0.$

(*c*) $x^2+2=0.$ (*f*) $3x^2-3x+\frac{3}{4}=0.$

11. Form equations whose roots are:

(*a*) 7, 3.
(*b*) $\frac{1}{2}, -\frac{1}{2}$.
(*c*) $3, \frac{1}{3}$.
(*d*) $a - b, b - a$.
(*e*) $3 \pm \sqrt{2}$.
(*f*) $3 \pm i$.

12. Solve $7x + \sqrt{x^2 - 17x + 4} = 2x^2 - 27x + 5$.

13. $\dfrac{x + \sqrt{12a - x}}{x - \sqrt{12a - x}} = \dfrac{\sqrt{a} + 1}{\sqrt{a} - 1}$.

14. For what value of m are the roots equal in

$$3x^2 - 5x + m = 0?$$

15. What change is made in the roots of the complete quadratic by changing the sign of b?

16. $x + 1 - \dfrac{x^2}{1 - x^2} = \dfrac{x^2}{x + 1} + \dfrac{5x - 4}{x^2 - 1}$.

17. In two years the population of a city increased from 6400 to 8100; the rate per cent of increase during the first year was equal to the rate per cent of increase during the second year. What was this rate?

18. In the equation $x^2 + y^2 = 1$, $y = \frac{1}{3}\sqrt{3}$; find x.

19. Solve, getting the answers correct to two decimal places,

$$3x^2 - x - \sqrt{2} = 0.$$

20. Solve $x^2 + x + 3x\sqrt{3} + 4 = 0$. (Yale.)
One answer is $1 - \sqrt{3}$.

21. A pedestrian having 18 miles to go to keep an appointment finds that at his present rate he will be half an hour late. If he quickens his pace by half a mile an hour, he will arrive on time. At what rate is he walking?

22. A room is two yards longer than it is wide and the floor contains 24 square yards. Find the dimensions of the room.

Solve:

23. $\dfrac{12-x}{x-5}+\dfrac{37-3x}{25-x^2}=1.$

24. $\dfrac{x}{2(x+2)}+\dfrac{3x-2}{4-x^2}=\frac{1}{2}.$

25. $\sqrt{5x+1}-\sqrt{3x}=1.$

26. $\sqrt{2x-1}+\sqrt{2x+6}=7.$

27. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$

28. $\sqrt{3x+1}+\sqrt{5x+4}=3.$

29. $\sqrt{5x+10}-\sqrt{5x}=2.$

30. $6x^{\frac{2}{3}}-11x^{\frac{1}{3}}-2=0.$

31. $\sqrt{3x-2}+6=5\sqrt[4]{3x-2}.$

32. $\dfrac{3}{2x-1}-\dfrac{2}{\sqrt{2x-1}}-5=0.$

33. Find a number such that one half its square shall exceed the square of one half the number by one half the number.

34. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}=\dfrac{a}{b}.$ (Apply composition and division.)

35. $\dfrac{x-\sqrt{ax}}{a+\sqrt{ax}}+\dfrac{a-\sqrt{ax}}{x+\sqrt{ax}}=\dfrac{x-a}{a}.$ (Clear of fractions and factor.)

XXIII. QUADRATIC EQUATIONS (*Continued*)

SIMULTANEOUS EQUATIONS

550. The **degree** of an equation is determined with respect to the letters that are regarded as unknowns. For example, $ax + b = 0$ is of the first degree with respect to x as unknown. (§§ 245, 246.)

$s = \frac{1}{2}gt^2$ is of the second degree with respect to t as unknown.

$ax + by = c$ is of the first degree with respect to x and y.

$3xy + x + y = 5$ is of the second degree, regarding x and y as unknowns.

551. There is no general method for solving simultaneous quadratic systems since, in general, the elimination of one of the unknowns gives rise to an equation of higher degree that cannot be solved by methods of elementary algebra. We shall consider some special forms of quadratic systems.

552. Case I. One Equation of the First Degree and the other of the Second. (Review of Chapter XVII.)

A system of equations involving two unknowns, one equation linear and the other quadratic, can always be solved, since the elimination of one unknown by substitution from the first degree equation into the second degree equation gives rise to a quadratic. (See § 430.)

Solve the system $(x + y)(x - 2y) = 7.$ (1)

$x - y = 3.$ (2)

Solution. $x = y + 3.$ (From (2).)

$(y + 3 + y)(y + 3 - 2y) = 7.$ (Substituting in (1).)

$(2y + 3)(-y + 3) = 7.$

$-2y^2 + 3y + 9 = 7.$

$$2y^2 - 3y - 2 = 0.$$

$$y = \frac{3 \pm \sqrt{9+16}}{4} = 2 \text{ and } -\frac{1}{2}.$$

When $y = 2,\ x = 2 + 3 = 5.$

When $y = -\frac{1}{2},\ x = -\frac{1}{2} + 3 = \frac{5}{2}.$

$$x = 5,\ \tfrac{5}{2}.$$
$$y = 2,\ -\tfrac{1}{2}.$$

Check both sets of roots.

EXERCISE

553. *Solve the following systems, grouping the answers properly at the end of the solution. Leave irrational answers in simplest radical form. Verify one set of answers in examples* 1 *to* 6:

1. $x + y = 13,$ $xy = 36.$

2. $x + y = 10,$ $x^2 + y^2 = 58.$

3. $xy - 5x = 1,$ $7x - y = 1.$

4. $x^2 + 4xy = 57,$ $x + y = 7.$

5. $3x + 5y = 35,$ $x^2 + 2y^2 = xy + 8x - y + 13.$

6. $\dfrac{5x - y}{4} = \dfrac{7}{4x + 3y},$ $3x - 2y = 1.$

7. $\dfrac{r}{4} + \dfrac{s}{5} = 3,$ $(r + s)^2 = 200 - r.$

8. $3m - n = 5,$ $mn - m = 0.$

9. $x^2 + 3xy + y^2 = 7,$ $2x = 1.$

10. $x^2 + y^2 = 130,$ $\dfrac{x + y}{x - y} = 8.$

11. $ab = 147,$ $a : b = 3 : 1.$

12. $\dfrac{1}{x} + \dfrac{1}{y} = 5,$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 13.$

    HINT. Use $\dfrac{1}{x}$ and $\dfrac{1}{y}$ as unknowns.

13. $x + y = 29,$ $\sqrt{x} + \sqrt{y} = 7.$

    HINT. Use $\sqrt{x}$ and $\sqrt{y}$ as unknowns.

14. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{2},$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{5}{4}.$

15. $x + y = a,$ $x^2 + y^2 = 2xy.$

16. $p^2 = 10,$ $3p = 5q.$

17. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{10}{3},$ $4x - 7y = 5.$

Solve the following systems:

NOTE. It frequently happens that other systems can be reduced so as to consist of one equation of the first degree and one of the second. In example 18 eliminate y^2 by adding the equations.

18. $x + y + 2y^2 = 11,$
$3x - 2y - 2y^2 = -9.$

19. $xy - x = 12,$
$xy + 3y = 35.$

20. $(u + 1)(v + 2) = 28,$
$(u + 3)(v + 4) = 54.$

21. $3(x^2 - y^2) = 2x + 17,$
$x^2 - y^2 + x + y = 18.$

22. $4xy - 5x^2 - 2x + y = 12,$
$x(4y - 5x) = 13 - x + y.$

NOTE. The method of elimination by substitution will sometimes solve a system when one equation is of the third degree, and the other of the first degree.

23. $x^3 + y^3 = 9,$
$x + y = 3.$

24. $x^3 - y^3 = 19,$
$x - y = 1.$

25. $v^3 - u^3 = 1304,$
$v - u = 8.$

26. $t^3 + r^3 = 65,$
$t + r = 5.$

27. $x^2 + y^2 + xy = 67,$
$x + y = 9.$

554. Case II. Homogeneous Equations of the Second Degree. An algebraic expression, or an equation, in which all the terms are of the same degree is **homogeneous**.

The homogeneity of a literal equation is determined with respect to the unknown numbers.

Thus, $a^3 + 3a^2b$ is a homogeneous expression.

$ax^2 + 5xy = 0$ is a homogeneous equation in x and y.

$x^2 + 5xy + x + 3y = 10$ is not homogeneous. (Why not?)

$x^2 + 3xy + y^2 = 5$, is homogeneous, except with respect to the absolute term.

555. It is always possible to solve a simultaneous system when both equations are of the second degree and one of them is homogeneous.

Solve
$$2x^2 + xy - 6y^2 = 0,$$
$$3x^2 - 4xy = 3.$$

SOLUTION. $(2x - 3y)(x + 2y) = 0$. (Factoring the first equation.)
The first equation is therefore equivalent to the two equations, $2x - 3y = 0$ and $x + 2y = 0$. We now form two systems, using each of these first degree equations with $3x^2 - 4xy = 3$.

System *A*. $x + 2y = 0$.
$$3x^2 - 4xy = 3.$$
$$y = -\frac{x}{2}.$$
$$3x^2 - 4x\left(-\frac{x}{2}\right) = 3,$$
$$\text{or } 3x^2 + 2x^2 = 3.$$
$$x^2 = \tfrac{3}{5}.$$
$$x = \pm\frac{\sqrt{15}}{5},$$
$$\text{and } y = -\frac{x}{2} = \mp\frac{\sqrt{15}}{10}.$$

System *B*. $2x - 3y = 0$.
$$3x^2 - 4xy = 3.$$
$$y = \frac{2x}{3}.$$
$$3x^2 - 4x \cdot \frac{2x}{3} = 3,$$
$$\text{or } 3x^2 - \frac{8x^2}{3} = 3.$$
$$x^2 = 9.$$
$$x = \pm 3,$$
$$\text{and } y = \frac{2x}{3} = \pm 2.$$

$$x = \pm 3,\ \pm\frac{\sqrt{15}}{5}.$$
$$y = \pm 2,\ \mp\frac{\sqrt{15}}{10}.$$

When signs are paired in this way it is understood that the top signs are to be used together and the bottom signs together.

Thus,
$$x = 3,\ -3,\ \frac{\sqrt{15}}{5},\ \frac{-\sqrt{15}}{5}.$$
$$y = 2,\ -2,\ \frac{-\sqrt{15}}{10},\ \frac{\sqrt{15}}{10}.$$

Let the student check the answers, substituting them in the original equation.

556. **To solve a system of two second degree simultaneous equations when one is homogeneous:**

1. Write the homogeneous equation with its second member zero, and factor the first member.

2. Put each factor equal to zero (why?) and use each of the two linear equations obtained with the other one of the two original equations as in Case I.

Solve the system $3x^2 - 8xy + 4y^2 = 0,$
$$x^2 + y^2 + 13(x - y) = 0.$$

SOLUTION. $(3x - 2y)(x - 2y) = 0.$ (Factoring the first equation.)

System A. $3x - 2y = 0,$
$$x^2 + y^2 + 13(x - y) = 0.$$
$$x = \frac{2y}{3}.$$
$$\left(\frac{2y}{3}\right)^2 + y^2 + 13\left(\frac{2y}{3} - y\right) = 0,$$
$$\text{or } \frac{13y^2}{9} - \frac{13y}{3} = 0.$$
$$y^2 - 3y = 0.$$
$$y = 0 \text{ or } 3,$$
$$\text{and } x = \tfrac{2}{3}y = 0 \text{ or } 2.$$

System B. $x - 2y = 0,$
$$x^2 + y^2 + 13(x - y) = 0.$$
$$x = 2y.$$
$$(2y)^2 + y^2 + 13(2y - y) = 0,$$
$$\text{or } 5y^2 + 13y = 0.$$
$$y(5y + 13) = 0.$$
$$y = 0 \text{ or } -\tfrac{13}{5},$$
$$\text{and } x = 2y = 0 \text{ or } -\tfrac{26}{5}.$$

$$x = 0,\ 2,\ 0,\ -\tfrac{26}{5},$$
$$\text{and } y = 0,\ 3,\ 0,\ -\tfrac{13}{5}.$$

557. If both equations are of the second degree and homogeneous except with respect to the absolute term, the absolute term may be eliminated and the method of the last article applied.

$$x^2 - 3xy = 4,$$
$$3x^2 + xy - 2y^2 = 50.$$

SOLUTION. $25x^2 - 75xy = 100.$ (The first equation multiplied by 25.)
$6x^2 + 2xy - 4y^2 = 100.$ (The second equation multiplied by 2.)
$19x^2 - 77xy + 4y^2 = 0.$ (Subtracting.)
$(19x - y)(x - 4y) = 0.$

System A. $y = 19x,$
$$x^2 - 3xy = 4.$$
$$x^2 - 57x^2 = 4.$$
$$56x^2 = -4.$$
$$x^2 = -\tfrac{1}{14}.$$
$$x = \pm\frac{i}{14}\sqrt{14},$$
$$\text{and } y = 19x = \pm\frac{19i}{14}\sqrt{14}.$$

System B. $x = 4y,$
$$x^2 - 3xy = 4.$$
$$16y^2 - 12y^2 = 4.$$
$$4y^2 = 4.$$
$$y^2 = 1.$$
$$y = \pm 1,$$
$$\text{and } x = 4y = \pm 4.$$

The answers are $x = \pm 4,\ \pm\frac{i}{14}\sqrt{14}.$
$$y = \pm 1,\ \pm\frac{19i}{14}\sqrt{14}.$$

The student will note that the elimination of the absolute terms of the original equations is similar to the elimination of one of the unknown numbers in a system of linear simultaneous equations by the method of addition and subtraction.

558. Optional Method. When the equations are homogeneous and of the second degree, the substitution of $y = vx$ will always effect a solution. The solution of the last example by this method is shown below:

SOLUTION. $x^2 - 3xy = 4.$ (1)

$3x^2 + xy - 2y^2 = 50.$ (2)

Substitute vx for y in both equations.

$x^2 - 3vx^2 = 4$, whence $x^2 = \frac{4}{1 - 3v}$. (From (1).)

$3x^2 + vx^2 - 2v^2x^2 = 50$, whence $x^2 = \frac{50}{3 + v - 2v^2}$. (From (2).)

$$\frac{4}{1 - 3v} = \frac{50}{3 + v - 2v^2}. \quad \text{(Equating the two values of } x^2.)$$

$$12 + 4v - 8v^2 = 50 - 150v.$$

$$-8v^2 + 154v - 38 = 0.$$

$$4v^2 - 77v + 19 = 0.$$

$$v = \frac{77 \pm \sqrt{77^2 - 4 \cdot 4 \cdot 19}}{8}$$

$$= \frac{77 \pm 75}{8} = 19 \text{ or } \frac{1}{4}.$$

System *A*. $v = 19.$

$$x^2 = \frac{4}{1 - 3v}.$$

$$x^2 = \frac{4}{-56} = -\frac{1}{14}.$$

$$x = \pm \frac{i}{14}\sqrt{14},$$

and $y = vx = \pm \frac{19i}{14}\sqrt{14}.$

System *B*. $v = \frac{1}{4}.$

$$x^2 = \frac{4}{1 - 3v}.$$

$$x^2 = \frac{4}{\frac{1}{4}} = 16.$$

$$x = \pm 4,$$

and $y = vx = \frac{1}{4}(\pm 4) = \pm 1.$

Ans. $x = \pm \frac{i}{14}\sqrt{14},\ \pm 4,$

$$y = \pm \frac{19i}{14}\sqrt{14},\ \pm 1.$$

By comparing the two methods it will be seen that there is not much difference in the amount of work involved.

EXERCISE

559. *Solve the following systems. Verify one set of answers in each system in examples 1 to 6:*

1. $x^2 - 4y^2 = 0,$
$x^2 + 3xy + y = 24.$

2. $15x^2 - 16xy - 15y^2 = 0,$
$3x^2 + 5y^2 = 120.$

3. $3x^2 - 4xy - 7y^2 = 0,$
$5x = 4y^2 - 1.$

4. $(3x - y)(3x + y) = 35,$
$(x - 2y)(x - 5y) = 0.$

5. $x^2 - xy = 5,$
$xy + y^2 = 36.$

6. $x^2 + 3xy = 5,$
$xy - y^2 = 6.$

7. $x^2 + 5xy = 14,$
$y^2 + 6xy = 13.$

8. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2},$
$xy = 8.$

9. $\frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{5}{2},$
$x^2 + y^2 = 90.$

10. $2x^2 - 3xy + y^2 = 3.$
$x^2 + 2xy - 3y^2 = 5.$

11. $(x + y)^2 = 3x^2 - 2,$
$(x - y)^2 = 3y^2 - 11.$

12. $(2x+5y)(3x-5y) = 44,$
$x^2 - 6xy + 12y^2 = 3.$

13. $x^2 - 3xy = 5xy - 15y^2,$
$x^2 - 3xy = 10.$

14. $x^2 + z^2 = 661,$
$x^2 - z^2 = 589.$

15. $x^2 + xy + y^2 = 79,$
$x^2 - xy + y^2 = 37.$

16. $p^2 + pq + q^2 = 139,$
$5q^2 - 4pq = -75.$

17. $2x^2 - xy = 28,$
$x^2 + 2y^2 = 18.$

18. $(s - t)(s + t) = 40,$
$(3s + t)(3t + s) = 384.$

19. $x^2 + 3y^2 = 7,$
$7x^2 - 5xy = 18.$

20. $x^2 + xy = a,$
$y^2 + xy = b.$

21. $ax^2 + bxy = a,$
$by^2 + axy = b.$

22. $x^2 + 3y^2 = 7,$
$7x^2 - 5xy = 18.$

23. $3x^2 + 3xy + 2y^2 = 8,$
$x^2 - xy - 4y^2 = 2.$

24. $6x^2 + 5xy - 6y^2 = 0,$
$2x^2 - y^2 = -1.$

560. Case III. Symmetrical Equations. An equation is **symmetrical** if an interchange of the unknowns does not change the equation except in the order of its terms.

Thus, $x + 3xy + y = 10$ becomes, by interchanging x and y, $y + 3yx + x = 10$. Therefore the equation is symmetrical.

$x + y = 5$ becomes $y + x = 5$ and is therefore symmetrical.

Is $x^2 + y^2 = 5$ symmetrical? $xy + x = 10$? $x + 2y = 5$?

561. A system consisting of two symmetrical equations can generally be solved by combining the equations in such a way as to find values of $x + y$ and $x - y$.

Solve the system,

$$x^2 + y^2 = 17, \quad (1)$$

$$x + y = 5. \quad (2)$$

Solution. Here we have the value of $x + y$; we look for xy and thence $x - y$.

$x^2 + 2xy + y^2 = 25$. (Squaring equation (2)). (3)

$2xy = 8$. (Subtracting equation (1) from equation (3)). (4)

$x^2 - 2xy + y^2 = 9$. (Subtracting equation (4) from equation (1)).

$x - y = \pm 3$.

We now replace the original system by the two systems:

A. $x + y = 5.$	*B.* $x + y = 5.$	*Ans.* $x = 4, 1.$
$x - y = 3.$	$x - y = -3.$	$y = 1, 4.$
Whence $x = 4.$	Whence $x = 1.$	
$y = 1.$	$y = 4.$	

562. 1. The student should follow some such systematic arrangement of his work as is found in § 561.

2. He should carefully study the equations to determine what steps will lead to the desired forms. No general rules can be given since the method of procedure varies with the form of the equations.

3. It will usually be found helpful to find a value of xy, as in (4) § 561, and use this value in combination with one of the preceding equations to form an equation containing some power of $x + y$ or $x - y$.

4. From this equation values of $x+y$ and $x-y$ can be found.

5. It will be well also to divide the equations, member by member, when this is found possible. Thus, if $x^3 + y^3 = 15$ and $x + y = 3$, by dividing we get $x^2 - xy + y^2 = 5$.

563. Optional Method. Symmetrical equations can be solved also by the following method:

$$x^2 + y^2 = 17, \quad (1)$$
$$x + y = 5. \quad (2)$$

Substitute $u + v$ for x and $u - v$ for y.

Equation (1) becomes $(u + v)^2 + (u - v)^2 = 17$. (From (1).) (3)

$$2u^2 + 2v^2 = 17. \quad (4)$$
$$(u + v) + (u - v) = 5. \quad \text{(From (2).)}$$
$$2u = 5.$$
$$u = \tfrac{5}{2}.$$
$$2(\tfrac{5}{2})^2 + 2v^2 = 17. \quad \text{(From (4).)}$$
$$2v^2 = \tfrac{9}{2}.$$
$$v^2 = \tfrac{9}{4}.$$
$$v = \pm \tfrac{3}{2}.$$
$$x = u + v = \tfrac{5}{2} \pm \tfrac{3}{2} = 4 \text{ or } 1,$$
$$\text{and } y = u - v = \tfrac{5}{2} \mp \tfrac{3}{2} = 1 \text{ or } 4.$$

EXERCISE

564. *Solve the following systems, and verify one set of answers in examples* 1 *to* 5:

1. $x^2 + y^2 = 50$,
   $xy = 7$.

   Multiply the second equation by 2 and combine with the first equation to get values of $x^2 + 2xy + y^2$ and $x^2 - 2xy + y^2$.

2. $x^2 - xy + y^2 = 7$,
   $x + y = 4$.

3. $x^2 + xy + y^2 = 14$,
   $x - y = \sqrt{2}$.

4. $x^3 + y^3 = 98$,
   $x^2 - xy + y^2 = 49$.

5. $x^2 + y^2 = 269$,
   $x - y = 3$.

6. $x^2 + xy + y^2 = 6$,
   $x^2 - xy + y^2 = -6$.

7. $x^2 + y^2 + x + y = 146$,
   $xy = 63$.

8. $x^3 + y^3 = 28$,
   $x + y = 4$.

9. $x^3 - y^3 = 26$,
   $x - y = 2$.

10. $(x-y)^2-3(x-y)=4,$
$x^2+y^2+2\,xy=49.$

SOLUTION. From the first equation $x-y=4$ or -1. (Why?)
From the second equation $x+y=\pm 7$. (Why?)
We have, then, the four systems.

A. $x+y=7,$ $x-y=4.$ *B.* $x+y=7,$ $x-y=-1.$ *C.* $x+y=-7,$ $x-y=4.$ *D.* $x+y=-7,$ $x-y=-1.$

Let the student complete the solution.

11. $x^2+y^2-(x+y)-12=0,$
$xy-2(x+y)+8=0.$

12. $3(x^2+y^2)=8(x+y)-1,$
$xy=(x+y)+1.$

13. $x^2+y^2=xy=x+y.$

$x^2+2\,xy+y^2=x^2y^2.$ (From $x+y=xy$.)
But $x^2+y^2=xy.$
$2\,xy=x^2y^2-xy.$ (Subtracting.)

Let the student find the values of xy in the last equation and continue the solution.

14. $x^4+x^2y^2+y^4=21.$
$x^2+xy+y^2=7.$

15. $x^2+xy+y^2=91,$
$x+\sqrt{xy}+y=7.$

Divide the first equation by the second.

16. $x^3+y^3=-2\,xy,$
$x+y=-2.$

17. $x^3+y^3=280,$
$x^2-xy+y^2=28.$

18. $x^2+y^2+x+y=168,$
$\sqrt{xy}=6.$

19. $x^2+y^2+x+y=18,$
$2\,xy=12.$

20. $x=\frac{7}{3}\sqrt{x+y},$
$y=\frac{11}{3}\sqrt{x+y}.$

21. $p+pq+q=47,$
$p+q=12.$

22. $\frac{1}{x}+\frac{1}{y}=\frac{1}{2},$

$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{36}.$

23. $r^2+rs+s^2=217,$
$r+s=17.$

24. $x^4-y^4=609,$
$x^2+y^2=203.$

GENERAL SUGGESTIONS FOR THE SOLUTION OF SIMULTANEOUS QUADRATIC EQUATIONS

565. In solving a quadratic system, first determine under which one of the following three cases it occurs:

I. One equation of the first degree and the other of the second.

II. Homogeneous second degree equations.

III. Symmetrical systems.

If the system does not come under one of these three cases, try to derive another system, or other systems, from the given equations that will come under one of these three cases. At all times remember that the object is to *eliminate* one of the unknowns.

566. Among special devices may be mentioned the following:

1. The immediate elimination of one of the unknowns by addition, subtraction, or substitution.

2. The elimination of the second degree terms.

3. Finding a quadratic form in some *expression* containing the unknowns and solving for the value of this expression. If this *expression* is of the first degree in the unknown, the given system may be replaced by two systems under Case I.

4. Dividing the equations member by member.

EXERCISE

567. *Solve the following systems:*

1. $x : y = 2 : 3,$
   $x^2 + y^2 = 5(x + y) - 2.$

2. $(x + y)^2 + (x + y) = 12,$
   $3x^2 + y^2 = x + y + 4.$

3. $x^2 + y^2 = 34,$
   $x^2 - 2y^2 + 3x = -50.$

4. $x(y - 4) = 14,$
   $y(x + 1) = 33.$

5. $5x - 10y + (x - 2y)^2 = 6,$
   $xy + x + y = 7.$

6. $\frac{x^2}{y^2} + 5\frac{x}{y} = 14,$
   $x = y^2 + 1.$

7. $5\frac{x}{y} + 3\frac{y}{x} = 8,$
   $x^2 + y = x + 4.$

8. $4x^2 - 9xy + 5y^2 = 0,$
$7x^2 - 3xy = 3x + 2y - 1.$

9. $2x^2 - 3y^2 = 6,$
$3x^2 - 2y^2 = 19.$

10. $(x+y)(x-2y) = 7,$
$x - y = 3.$

11. $\dfrac{x+y}{1-xy} = a,$
$x = y.$

12. $\dfrac{x+y}{1-xy} = 1,$
$x = \dfrac{1}{\sqrt{3}}.$

13. $x^2 + y^2 + x + y = 18,$
$x^2 - y^2 + x - y = 6.$

14. $x^2 - y^2 = 40,$
$xy = 21.$

15. $3xy - 2(x+y) = 28,$
$2xy - 3(x+y) = 2.$

16. $(2x-y)^2 - 12(2x-y) = 189,$
$x^2 - 4xy + 4y^2 - 3x + 6y = 54.$

17. $x^2 + y^2 - 2(x-y) = 38,$
$xy + 3(x-y) = 25.$

18. $x\left(1 + \dfrac{x}{y}\right) = a,$
$y\left(1 + \dfrac{x}{y}\right) = b.$

19. $x\left(1 + \dfrac{x}{y}\right) = a,$
$y\left(1 + \dfrac{y}{x}\right) = b.$

20. $\dfrac{1}{x} + \dfrac{1}{y} = 5,$
$x - y = .3.$

21. $\sqrt{x-5} + \sqrt{y+2} = 5,$
$x + y = 16.$

22. $8(x-5)^2 - 3(y-7)^2 = 80,$
$4(x-5)^2 + 5(y-7)^2 = 144.$

23. $4x^2 - 9y^2 = 0,$
$4x^2 + 9y^2 = 8(x+y).$

24. $x^2 + y^2 = xy + 189,$
$60(x-y) = xy.$

25. $\dfrac{x+y}{1-xy} = 3,$
$\dfrac{x-y}{1+xy} = \dfrac{1}{3}.$

26. $x^2 - xy = 3,$
$xy - y^2 = 2.$

27. $2x + \sqrt{xy} = 10,$
$3y - 2\sqrt{xy} = -1.$

28. $y = x^2,$
$ay + bx + c = 0.$

29. $x^2 + y^2 + x + y = 36,$
$2(x^2 + y^2) + 3xy = 88.$

30. $x^2 + y^2 - xy = 7,$
$(x - y) + xy = 5.$

31. $x^2 + xy - a = 0,$
$y^2 + xy - b = 0.$

32. $x^2y + xy^2 = 30,$
$\frac{1}{x} + \frac{1}{y} = \frac{5}{6}.$

33. $x^2y^2 + xy - 2 = 0,$
$x + y = -1.$

34. If $x^2 - 2y = -\frac{1}{4}$ and $y = \sqrt{1 - x^2}$, show that y is equal to $\frac{1}{2}$ or $-\frac{5}{2}$.

35. $x + y + z = 2,$
$xy = -1,$
$xyz = -2.$

36. $xy = 12,$
$xz = 15,$
$yz = 20.$

37. $x + y + z = 6,$
$2x - y + z = 3,$
$x^2 + y^2 + z^2 = 14.$

38. $xy + xz = 80,$
$xy + yz = 98,$
$xz + yz = 108.$

39. $x + y + z = 37,$
$x^2 + y^2 + z^2 = 481,$
$y^2 = xz.$

40. $(x + 1)(y + 1) = 15,$
$(y + 1)(z + 1) = 35,$
$(z + 1)(x + 1) = 21.$

ELIMINATION

568. It often happens in the study of mathematics and physics that it is necessary to eliminate one or more unknown quantities, or variable quantities, and either solve the resulting equation for the other unknown or derive an equation containing it. Some one of the methods of the present chapter will generally accomplish the desired elimination.

$W = fs,\ f = ma,\ s = \frac{1}{2}at^2,\ v = at.$ Find W in terms of m and v; that is, eliminate f, s, a, and t.

SOLUTION. $W = ma \cdot \frac{1}{2}at^2$ Substituting for f and s.
$= \frac{1}{2}ma^2t^2.$ Why?
$\therefore W = \frac{1}{2}mv^2.$ Why?

EXERCISE

1. Given $v = gt$ and $s = \frac{1}{2} gt^2$; eliminate t and show that $v = \sqrt{2gs}$. (Physics.)

2. Given $v_1 = v_0 + kv_0t_1$ and $v_2 = v_0 + kv_0t_2$; eliminate v_0. (See § 566, 4.) (Physics.)

3. Solve the result in problem 2 for k.

4. Given $\frac{x+y}{1-xy} = 1$ and $y = \frac{2x}{1-x^2}$; eliminate y.

5. $S = \frac{n}{2}(a + l)$ and $l = a + (n-1)d$. Eliminate l, and express the value of S in terms of a, n, and d.

6.
$$\begin{aligned} wy + zx &= 1, \\ w &= 2x\sqrt{1-x^2}, \\ y &= \sqrt{1-x^2}, \\ z &= y^2 - x^2. \end{aligned}$$

Eliminate w, y, z.

The resulting equation in x should be satisfied by $x = \frac{1}{2}$.

7. Given $C = \frac{E}{R}$ and $C' = \frac{E}{R+r}$; eliminate E. Also find E in terms of C, C', and r.

8. If $v = \sqrt{\frac{k}{d}}$ and $k = \frac{Tg}{\pi R^2}$, find the value of v after eliminating k.

9. Given $C = \frac{E + E'}{R + R' + r}$ and $C' = \frac{E - E'}{R + R' + r}$; find E' after eliminating $R + R' + r$.

10. Given $v = u - gt$ and $s = ut - \frac{1}{2} gt^2$; eliminate t and solve the resulting equation for u.

11. Given $C = \frac{E}{R+r}$ and $C' = \frac{nE}{R+nr}$; eliminate r.

12. Eliminate x from the two equations $ax^2 + bx + c = 0$, and $2ax + b = 0$.

13. Given $F = \frac{IL}{d^2} m$ and $\frac{B}{m} = \frac{1}{d^2}$; show that $F = B \cdot I \cdot L$.

PROBLEMS LEADING TO SIMULTANEOUS QUADRATIC EQUATIONS

569. 1. The difference of the two sides about the right angle in a right-angled triangle is 2 inches. The hypotenuse is 10 inches long. Find the unknown sides.

2. The sum of two sides about the right angle in a right triangle is s and the hypotenuse is h. Express the values of the two sides about the right angle in terms of s and h.

3. The difference between the sides and a diagonal of a square is 3 inches. Find the side and diagonal to two decimal places.

4. The sum of the roots of a quadratic equation is 2, and their product is -1. Find the roots and make the equation.

5. If a polygon has n sides, it has $\frac{n(n-3)}{2}$ diagonals. Two polygons have together 18 sides, while the number of diagonals of the one is to the number of diagonals of the other as 4 to 7. How many sides has each?

6. Two polygons have together 12 sides and 19 diagonals. How many sides has each?

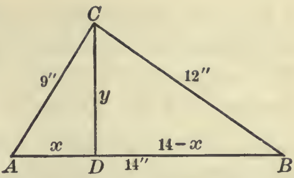
7. Two adjacent square plots of unequal sides are inclosed by a continuous fence. The total area of the fields is 5200 square rods and the length of the fence is 1 mile. How large is each plot? (Draw a figure.)

8. Besides zero there are two pairs of numbers such that their sum, their product, and the difference of their squares have the same value. Find these numbers.

9. In a proportion the sum of the means is 5 and the sum of the extremes is 7. The sum of the squares of all the terms is 50. Find the terms. (Use only 2 unknowns.)

10. Find two factors of p whose sum is s.

11. In the figure, CD forms a right angle with AB. The sides are 9 inches, 12 inches, and 14 inches as indicated. Find the length of x and y to one decimal place.



(Note that this would enable us to find the area of a triangle when we have given the three sides.)

12. A mean proportional between two numbers equals $\sqrt{10}$. The sum of the squares of the numbers is 29. Find the numbers.

13. The sum of the areas of two squares is 125 square inches. The sum of their four diagonals is $30\sqrt{2}$ inches. Find the sides of the squares.

14. Find the dimensions of a rectangular room, knowing that the floor has an area of 240 square feet; one side wall contains 180 square feet, and one end wall 108 square feet.

15. The sum of two numbers is one sixth of the difference of their squares, and the sum of the squares is 306. Find the numbers.

16. Divide 84 in two parts such that the sum of their squares is 3560.

17. If to the product of two numbers is added the greater number, we obtain 855; but if to the same product is added the smaller number, we obtain 828. Find the two numbers.

18. The sum of the squares of two numbers is 410. If the greater number is diminished by 4 and the smaller number is increased by 4, the sum of the squares is 394. Find the two numbers.

19. A number is formed of two figures of which the sum is 13. If 34 is added to the product of the two figures, the sum is equal to the number obtained by reversing the figures of the first number. Find the number.

20. The diagonal of a rectangle is 65 inches. If 9 inches are added to the width and 3 inches subtracted from the length of the rectangle, the diagonal remains the same. Find the dimensions of the rectangle.

21. The diagonal of a rectangle is 89 feet. If each side of the rectangle is diminished by 3 feet, the diagonal will be 85 feet. Find the length of each side.

22. The hypotenuse of a right-angled triangle is 35 feet. If the shorter side is diminished 5 feet and the longer side increased 2 feet, the hypotenuse will be 1 foot less. Find the two sides of the triangle.

23. Two square gardens have a total area of 2137 square yards. A rectangular lawn of which the dimensions are equal respectively to the sides of the two squares has an area of 1093 square yards less than that of the two gardens together. Find the sides of the two squares.

24. The sum of the areas of two circles is 13,273.26 square inches, and the sum of their radii is 79 inches. Find the two radii.

25. The sum of the surfaces of two spheres is 1000 square inches, and the sum of the radii is 12 inches. Find the two radii correct to two decimal places. (The surface of a sphere is $4\pi R^2$ square inches if R is the number of inches in radius.)

26. The sum of the surfaces of two spheres is 14,388.53 square inches, and the difference of the radii is 9 inches. Find the radii.

27. The sum of the volumes of two spheres is 14,778.0864 cubic inches and the sum of the radii is 20 inches. Find the radii correct to two decimal places. (The volume of a sphere equals $\frac{4}{3}\pi R^3$ cubic inches if R is the number of inches in the radius.)

28. Three numbers are such that if we take the product of them two at a time the results will be 240, 160, and 96. What are the numbers?

29. The sum of three sides of a right-angled triangle is 208 feet. The sum of the two sides about the right angle is 30 feet longer than the hypotenuse. Find the lengths of the three sides.

30. The product of the sum and the difference of two numbers is a; the quotient of the sum divided by the difference is b; find the two numbers.

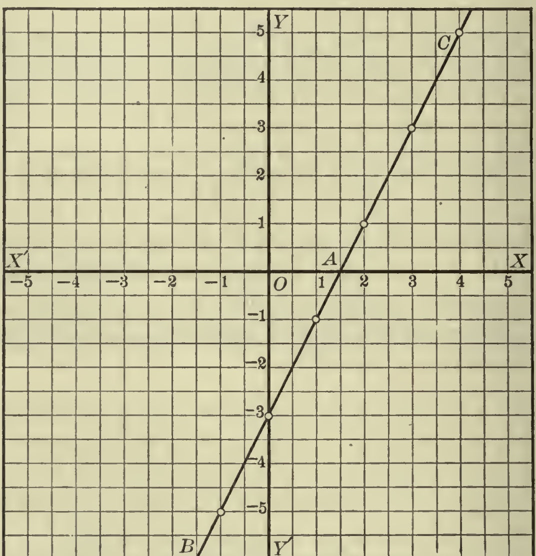
31. A rectangle whose dimensions are 6 inches and 10 inches is to be doubled in area by increasing the length and width by additions proportional to the present dimensions. Find the necessary addition to both dimensions.

32. Solve problem 31 if the area is to be made four times as great.

33. The dimensions of a rectangular piece of tin are in the ratio of 3 to 5. Two-inch squares are cut from the corners and the sides and ends are turned up to form a box. What were the original dimensions of the rectangle if the box holds 88 cubic inches?

XXIV. GRAPHICAL SOLUTION OF EQUATIONS[1]

570. Graphical Solution of Equations Containing One Unknown Number. In § 369 a system of two linear simultaneous equations was solved graphically. A linear equation in one unknown can be solved graphically. (Review §§ 356, 358, 359.)



571. Consider the equation $2x - 3 = 0$. For definite values of x, values of $2x - 3$ may be found and tabulated as follows:

x	$2x - 3$
-1	-5
0	-3
1	-1
2	1
3	3
4	5

Use the pairs of values of x and $2x - 3$ as the coördinates of points and draw the line BC through these points. This line is the graph of $2x - 3$.

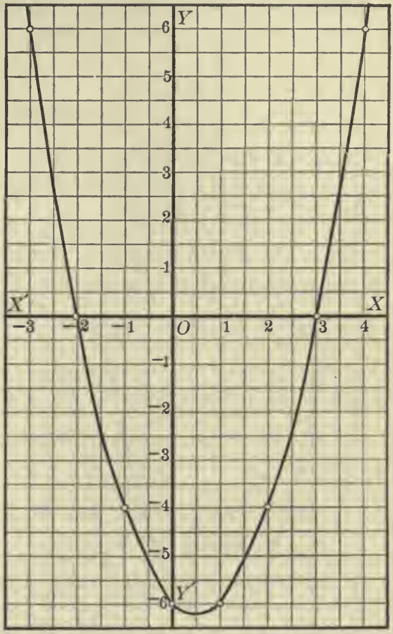
[1] This chapter may be omitted, if desired, without interrupting the sequence of the work.

This graph crosses the x-axis at the point A whose coördinates are $(\frac{3}{2}, 0)$; that is, at the point where the graph crosses the x-axis

$$2x - 3 = 0$$

and $x = \frac{3}{2}$.

The abscissa of the point of intersection of the graph of $2x - 3$ with the x-axis is the root of $2x - 3 = 0$.



572. Consider the second degree equation

$$x^2 - x - 6 = 0.$$

We shall tabulate values of x and the corresponding values of $x^2 - x - 6$, and use these numbers as the coördinates of points. This will give us the graph of $x^2 - x - 6$.

x	$x^2 - x - 6$
4	6
3	0
2	− 4
1	− 6
0	− 6
− 1	− 4
− 2	0
− 3	6

The intersections of this graph with the x-axis are points whose abscissas correspond to ordinates 0. Since the ordinates are the values of $x^2 - x - 6$, the abscissas whose ordinates are 0 must represent the values of x for which $x^2 - x - 6 = 0$. In the figure these values of x are 3 and − 2. These numbers are the roots of the equation $x^2 - x - 6 = 0$.

The result may be checked by solving this equation by one of the algebraic methods.

Thus, $x = \frac{1 \pm \sqrt{25}}{2} = 3$ and -2.

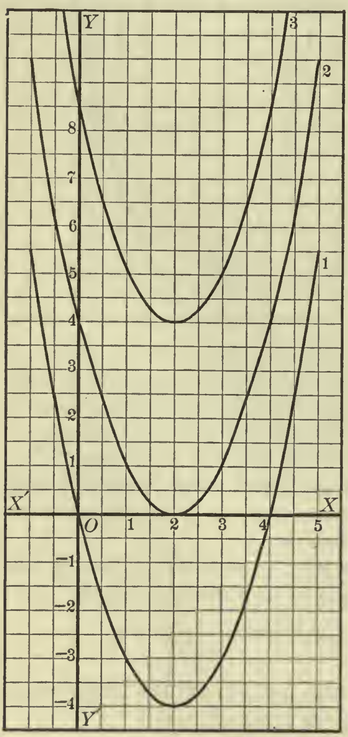
The graph just obtained is the graph of the expression $x^2 - x - 6$, or of the equation $y = x^2 - x - 6$.

EXERCISE

573. *Solve graphically as in § 572 the following equations:*

1. $x^2 - x - 8 = 0.$
2. $x^2 + x - 8 = 0.$
3. $x^2 + x - 5 = 0.$
4. $-x^2 - 2x + 7 = 0.$
5. $2x^2 + 3x - 19 = 0.$
6. $2x^2 - 3x - 10 = 0.$
7. $2x^2 + 2x - 10 = 0.$
8. $3x^2 - 6x + 2 = 0.$
9. $2x^2 + 5x - 8 = 0.$
10. $2x^2 - 5x + 2 = 0.$

574. The accompanying figure shows the graphs for a series of expressions of the form $x^2 - 4x + k$, each formed from the preceding by adding a positive number to the absolute term. The corresponding equations are as follows:



1. $x^2 - 4x = 0.$
2. $x^2 - 4x + 4 = 0.$
3. $x^2 - 4x + 8 = 0.$

The addition of a positive number to the absolute term increases by the number added the value of $x^2 - 4x + k$, that is, of the ordinate corresponding to any value of x. The points on the curve are all raised equally and therefore the graph for the new expression is of the same shape as that of the original expression, but it has a different position relative to the x-axis. In fact, the first two terms of the expression determine the shape of the graph, while the absolute term affects its position vertically.

In the figure it is seen that increasing the absolute term makes the intersections of the graph with the x-axis approach each other; therefore this change makes the roots approach each other in value. The second equation has equal roots; its graph is tangent to the x-axis. The third equation has imaginary roots; its graph does not touch the x-axis.

EXERCISE

575. 1. What is the graphical interpretation of the absolute term of a function of x? of the absence of an absolute term?

2. How will the graph for $3 - 5x - x^2$ differ from the graph for $x^2 + 5x - 3$?

3. How does the graph for $x - 3$ differ from the graph for $x - 1$?

4. Draw the graph for $y = x$; $y = x^2$.

5. Explain how the graph for $y = x^2$ can be used for finding squares, or square roots, of numbers.

6. Draw the graph for $y = x^3$.

7. Show that the points (1, 0) and (0, -1) are common to the graphs of $y = x - 1$, $y = x^2 - 1$, $y = x^3 - 1$, etc.

576. Consider the first degree simultaneous equations

$$2x + 3y = 6,$$
$$3x - 2y = 2.$$

Tabulating values of x and y for these equations, we have the following:

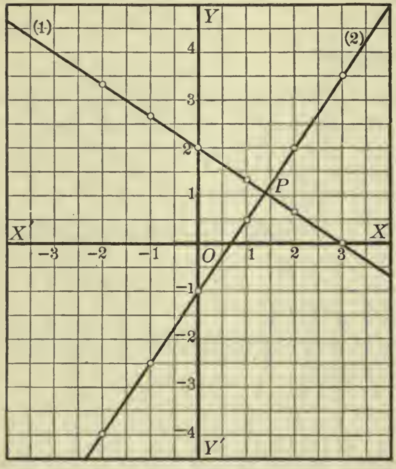
$2x + 3y = 6.$ (1)

x	y
0	2
1	$\frac{4}{3}$
2	$\frac{2}{3}$
3	0
-1	$\frac{8}{3}$
-2	$\frac{10}{3}$

$3x - 2y = 2.$ (2)

x	y
0	-1
1	$\frac{1}{2}$
2	2
3	$\frac{7}{2}$
-1	$-\frac{5}{2}$
-2	-4

Locating the points corresponding to the values of x and y as in §§ 356, 359, we get the graphs of these two equations. The intersection of these lines, P, has for abscissa approximately 1.4 and for ordinate 1.1. These values, $x \doteq 1.4$ and $y = 1.1$, are common to both lines, that is, to both equations.



The results agree closely with those obtained by solving the system by one of the usual methods.

$$2x + 3y = 6.$$
$$3x - 2y = 2.$$
$$6x + 9y = 18.$$
$$6x - 4y = 4.$$
$$13y = 14.$$
$$y = \tfrac{14}{13}, \text{ or } 1.07^+.$$
$$x = \tfrac{18}{13}, \text{ or } 1.38^+.$$

Since the graph of a linear equation is a straight line, and a straight line is determined by two of its points, it will not be necessary to tabulate as many values as were shown in this example. It will be well, however, for the student to find three or four sets of values, as the additional values serve as a check on the first two computed.

EXERCISE

577. *Solve graphically:*

1. $x + y = 5,$
   $4x - 3y = 6.$
2. $x + 2y = 5,$
   $2x - y = 0.$
3. $x + y = 0,$
   $3x - y = 2.$
4. $2x - y = 5,$
   $3x + 2y = -3.$
5. $x = 2y,$
   $x + 2y = 8.$
6. $5x + 4y = 22,$
   $3x + y = 9.$
7. $7x - 2y = 31,$
   $4x + 3y = -3.$
8. $6x + 11y = -28,$
   $5y - 18x = 8.$
9. $6x + 2y = -3,$
   $5x - 3y = -6.$
10. $4x + 15y = 7,$
    $14x + 6y = 9.$

578. Consider the second degree system:

$$x^2 - y = 4,$$
$$x + 2y = 3.$$

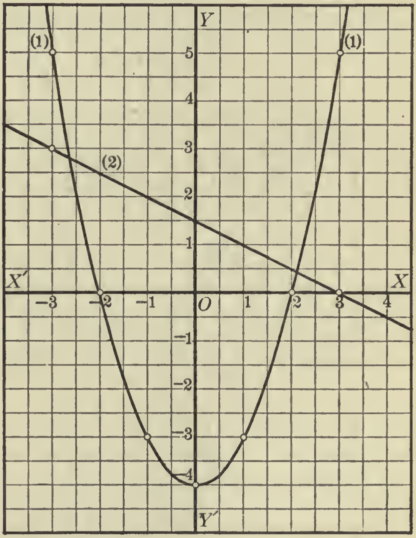
$x^2 - y = 4.$ (1)

x	y
0	−4
±1	−3
±2	0
±3	5

$x + 2y = 3.$ (2)

x	y
0	$\frac{3}{2}$
1	1
3	0
−3	3

Locating the points tabulated and drawing the graphs we have a curve for equation (1) and a straight line for equation (2). The intersections are points whose coördinates satisfy both equations and therefore give the roots of the system.



The roots are approximately

$x = 2.1, \ -2.7.$
$y = .4, \ 2.8.$

If solved by the usual method, we find

$x = 2.1^+, \ -2.6^+.$
$y = .45^+, \ 2.8^+.$

The student is not to understand that the graphical method of solving a system of simultaneous equations is to replace the algebraic method. The algebraic method is generally much shorter than the graphical method. However, the graphical method of representing equations plays a very important part in higher mathematics and in the applications of mathematics to problems of physics and engineering. It may also be noted that the algebraic methods do not always furnish the solutions of simultaneous quadratics (§ 551). The graphical method can generally be depended upon to give good approximations to the real roots in such cases.

Solve graphically: $x^2 + y^2 = 16,$
$x^2 - y^2 = 4.$

$x^2 + y^2 = 16.$ (1)

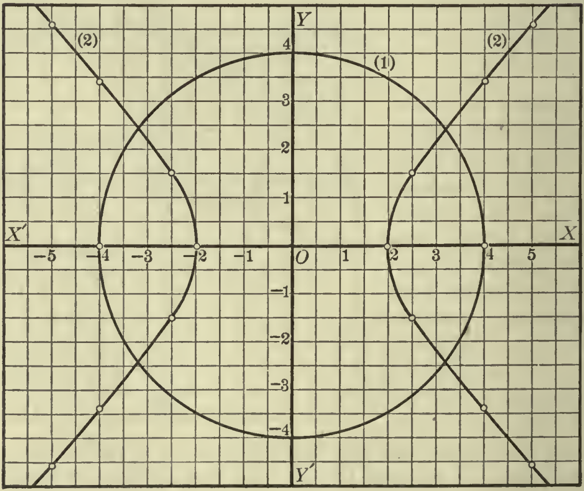
x	y
0	± 4
± 1	$\pm 3.87^+$
± 2	$\pm 3.46^+$
± 3	± 2.64
± 4	0
± 5	imag.

$x^2 - y^2 = 4.$ (2)

x	y
0	imag.
± 1	imag.
± 2	0
$\pm \frac{5}{2}$	$\pm \frac{3}{2}$
± 4	$\pm 3.46^+$
± 5	$\pm 4.58^+$

The intersections give the roots approximately as follows:

$$x = 3.1, \quad 3.1^-, \quad -3.1, \quad -3.1.$$
$$y = 2.5, \quad -2.5, \quad 2.5, \quad -2.5.$$

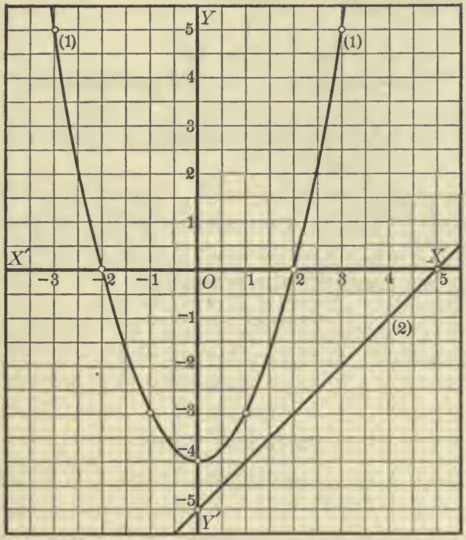


The algebraic solution gives

$$x = \sqrt{10},\ -\sqrt{10} \text{ or } 3.16^+,\ 3.16^+,\ -3.16^+,\ -3.16^+$$
$$y = \pm\sqrt{6},\ \pm\sqrt{6} \text{ or } 2.44^+,\ -2.44^+,\ 2.44^+,\ -2.44^+.$$

The curve for equation (2) is a hyperbola. The curve for a second degree equation in two unknown numbers is, in general, a circle, a parabola, an ellipse, or a hyperbola.

579. Imaginary roots cannot be found by this method. The presence of imaginary roots is indicated by a failure of the graphs to intersect. Thus, if we attempt to solve the system



$x^2 - y = 4$, (1), $x - y = 5$, (2), we shall find that the graphs have no common points. The graph of the first equation is shown in § 578. The second gives the line (2) as shown in the figure. The algebraic solution of this system gives

$$x = \frac{1 \pm \sqrt{-3}}{2}, \qquad y = \frac{-9 \pm \sqrt{-3}}{2}.$$

EXERCISE

580. *Solve graphically:*

1. $x^2 + y^2 = 9$, (Circle.)
$x - y = 0.$

2. $x^2 + y^2 = 9,$
$x + y = 0.$

3. $x^2 + 2y^2 - 2x = 15$, (Ellipse.)
$x + 2y = 1.$

4. $x^2 + y^2 = 4,$
$y = x - 2\sqrt{2}.$

Solve example 4 also algebraically.

5. $x^2 + 2y^2 - 2x = 15,$
$x^2 - 2y^2 = -7.$ (Hyperbola.)

6. $(x-1)^2 + (y-1)^2 = 6$, (Circle.)
$x - y = 0.$

7. $x^2 + xy + y^2 = 9,$
$x^2 + y^2 = 9.$

8. $x^2 + y = 7,$
$x + y^2 = 11.$

Try to solve example 8 algebraically.

XXV. THE PROGRESSIONS

ARITHMETICAL PROGRESSION

581. Series. A succession of terms formed according to some definite law is a **series.**

Thus, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ⋯ and a, a^2, a^3 ⋯ are series. What is the fourth term of each?

582. Arithmetical Progression. A series in which each term after the first is found by adding a constant quantity to the preceding term is an **arithmetical progression** (A.P.).

583. Common Difference. The constant number added is the **common difference.** The common difference is found by subtracting any term from the term immediately following it.

Thus, 1, 3, 5, 7 ⋯, and 12, 8, 4, 0, -4, -8 ⋯ are arithmetical progressions. In the first, 2 is the common difference and is added to each term to form the next; in the second, -4 is the common difference and is added to each term to form the next.

ORAL EXERCISE

584. *What is the common difference in each of the following series?*

1. 7, 11, 15, 19, ⋯.
2. 5, 8, 11, 14, ⋯.
3. $\frac{1}{2}$, $6\frac{1}{2}$, $12\frac{1}{2}$, $18\frac{1}{2}$, ⋯.
4. a, $a+d$, $a+2d$, $a+3d$, ⋯.
5. $a-x$, a, $a+x$, ⋯.
6. $a-3d$, $a-d$, $a+d$ ⋯.
7. a, b, $2b-a$, ⋯.
8. 1, a, $2a-1$, ⋯.

9. Form the next two terms in each of the series in examples 1 to 6.

585. Last Term. In the arithmetical progression let

a represent the first term,
d the common difference,
n the number of terms,
l the last term, and
s the sum of terms.

Then, if we examine the series,

First term	Second term	Third term	Fourth term	nth term
a,	$(a+d)$,	$(a+2d)$,	$(a+3d)$,	$\cdots a+(n-1)d$,

we notice that in any term the coefficient of d is one less than the number of the term in the series.

Hence in a series of n terms, the nth term being the last,

$$l = a + (n-1)d. \qquad (A)$$

1. Find the 7th term of the series $4, 2, 0, -2, \cdots$.

Solution. In this series $a=4$, $d=-2$, $n=7$.

$$\therefore l = 4 + 6 \cdot (-2) = -8.$$

2. Find the first 4 terms and the last term, when $a=2$, $d=\frac{2}{7}$, $n=8$.

Solution. The first four terms are $2, 2\frac{2}{7}, 2\frac{4}{7}, 2\frac{6}{7}$,

$$l = 2 + 7 \cdot \tfrac{2}{7} = 4.$$

586. Sum of the Terms. To find the sum of a number of terms in arithmetical progression:

Write the sum of the series in the usual order, (1), and in reverse order, (2), and add the two equal series.

$$\begin{array}{lll}
(1) & S = a & +(a+d)+(a+2d)+(a+3d)+\cdots+(l-d)+l \\
(2) & S = l & +(l-d)+(l-2d)+(l-3d)+\cdots+(a+d)+a \\
\hline
 & 2S = (a+l) & +(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)
\end{array}$$

$$= n(a+l). \quad \text{(Why?)}$$

$$\therefore S = \frac{n}{2}(a+l). \qquad (B)$$

Substituting the value $l = a + (n-1)d$ from (*A*), we can get a formula for the sum in terms of a, n, and d.

$$S = \frac{n}{2}[2a + (n-1)d]. \qquad (C)$$

1. Find the sum of the terms of the series, 2, 5, 8, 11, ⋯, to 12 terms.

Solution. $l = 2 + 11 \cdot 3 = 35.$

$$S = \frac{n}{2}(a + l) = \frac{12}{2}(2 + 35) = 222.$$

2. Find the sum of the series 4, 2, 0, – 2 ⋯, to 20 terms.

Solution. In the series $a = 4$, $d = -2$, and $n = 20$.
Hence, using formula (*C*), $S = \frac{20}{2}[2 \cdot 4 + (20-1)(-2)]$
$= 10(8 - 38) = -300.$

3. The first term of a series is 5, the last term is 161, and the sum of the series is 3320. Find the number of terms and the common difference.

Solution. Using formula (*B*), $3320 = \frac{n}{2}(5 + 161) = 83n.$

$\therefore n = 40.$

Using formula (*A*), $161 = 5 + 39d.$

$\therefore d = 4.$

587. An arithmetical progression can be completely determined if any two of its terms are known.

The 6th and 15th terms of an A. P. are 14 and 32, respectively. Find the 20th term.

Solution. $a + 5d = 14,$
and $a + 14d = 32.$
$\therefore d = 2$ and $a = 4.$
Hence the 20th term $= 4 + 19 \cdot 2 = 42.$

588. Arithmetical Mean. When several quantities are in A. P. the terms between the first and last terms are the **arithmetical means** between them.

The arithmetical mean between two numbers is equal to one half their sum.

PROOF. Let a and b be two numbers and A their arithmetical mean. Since a, A, b are in arithmetical progression,

$$b - A = A - a, \quad \text{(Why?)}$$

$$\text{or } A = \frac{a+b}{2}.$$

Any number of arithmetical means may be inserted between two numbers by means of formula (*A*).

Insert 10 arithmetical means between 10 and 72.

SOLUTION. In this case $a = 10$, $l = 72$, $n = 12$. (Why?)
Substituting in (*A*), $72 = 10 + 11\,d$.

$$\therefore d = \tfrac{62}{11} = 5\tfrac{7}{11}.$$

Therefore the series is 10, $15\frac{7}{11}$, $21\frac{3}{11}$, etc.

589. If any three of the five numbers a, d, n, l, and S are known, it is possible to find the other two from one or both of the formulas

$$l = a + (n-1)d. \qquad (A)$$

$$S = \frac{n}{2}(a + l). \qquad (B)$$

It will be noted that four of the five numbers involved in an A. P. are found in each formula.

In (*A*), we have a, n, d, l.

In (*B*), we have a, n, l, S.

In order to use the formulas in the solution of problems we need to know three of the five numbers. If the three given numbers, and the one required, are all found in one formula, the problem may be solved from that formula alone.

1. Given $a = 5$, $n = 7$, $l = 15$, find d.

SOLUTION. a, l, n and d are all in (*A*). From it we may write

$$15 = 5 + 6 \cdot d.$$

$$d = \tfrac{5}{3}.$$

2. From the data in example 1, find S.

SOLUTION. Substituting their values for a, n, and l in (B) we have

$$S = \frac{7}{2}(5 + 15) = 70.$$

3. Write the first four terms of the series from the data in example 1.

SOLUTION. First proceed as in the solution of example 1 for $d = \frac{5}{3}$. Then we have, for the series,

$$5, 6\tfrac{2}{3}, 8\tfrac{1}{3}, 10 \cdots.$$

590. If the three numbers given and the number required are not all found in either (A) or (B) alone, these formulas may be treated as a pair of simultaneous equations after the proper substitutions have been made.

Given $d = 2$, $l = 20$, $S = 108$; find a and n and the series.

SOLUTION. From (A), $20 = a + (n - 1)2$, or $22 = a + 2n$.

From (B), $108 = \frac{n}{2}(a + 20)$ or $216 = an + 20n$.

This gives us the simultaneous system $a + 2n = 22$, $an + 20n = 216$.

Solving,

$$a = 22 - 2n.$$
$$216 = (22 - 2n)n + 20n.$$
$$2n^2 - 42n + 216 = 0.$$
$$n^2 - 21n + 108 = 0.$$
$$(n - 9)(n - 12) = 0.$$
$$\therefore n = 9 \text{ or } 12.$$
$$\text{When } n = 9,\ a = 4.$$
$$\text{When } n = 12,\ a = -2.$$

The series is either $4, 6, 8, \cdots, 20$, or $-2, 0, 2, 4, \cdots, 20$.

EXERCISE

591. 1. Show that the three numbers, $x - y$, x, $x + y$ form an A. P. Similarly for $x - 3y$, $x - y$, $x + y$, $x + 3y$.

2. $a = -3$, $d = 2$, $n = 8$; find l and S.

3. $a = 3$, $d = 3$, $l = 15$; find n and S.

4. $a = 4$, $n = 12$, $l = 26$; find d and S.

5. $d = \frac{1}{2}$, $n = 3$, $l = 2$; find a and S.

6. $a = 15$, $d = -\frac{1}{2}$, $S = 137\frac{1}{2}$; find n and l.

7. $a = 4$, $n = 15$, $S = 270$; find d and l.

8. $d = 2$, $n = 15$, $S = 270$; find a and l.

9. $a = 10$, $l = 37$, $S = 235$; find d and n.

10. $d = -2$, $l = -24$, $S = -144$; find a and n.

11. $n = 13$, $l = 41$, $S = 299$; find a and d.

12. Find 4 arithmetical means between 5 and 18.

13. Insert 6 arithmetical means between -5 and 13.

14. The sum of three terms of an A.P. is 45; the sum of the squares of the terms is 773. Find the series.

NOTE. Use $x - y$, x, and $x + y$ to represent the series.

15. How many times does the clock strike in 12 hours?

16. Find the sum of the first 20 odd numbers.

17. Find the sum of the first 20 even numbers.

18. Show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

19. If you save 1¢ today, 2¢ tomorrow, 3¢ the next day, and so on, how many days will elapse before the total savings amount to $10?

20. The fifteenth and twenty-eighth terms of an A.P. are respectively 12 and 19. Find the first and the fiftieth terms.

21. Insert four arithmetical means between 9 and 11.

22. The sum of the first 8 terms of an A.P. is 64 and the sum of the first 18 terms is 324. Find the series.

23. The sum of the first 7 terms of an A.P. is 7 and the sum of the next 8 terms is 68. Find the series.

24. Between 6 and 10, there are 12 numbers so that the whole series of 14 numbers forms an A.P. What is the sum of the series?

25. The sum of the third and fifth terms of an A. P is 32, and the sum of the fourth and tenth terms is 50. Find the first term, and the sum of the first 20 terms.

26. Twenty potatoes are laid out in a straight line one yard apart. How far must a boy run to pick them up and bring them, one at a time, to a basket placed in the line and one yard from the first potato?

27. A freely falling body falls $\frac{1}{2}g$ feet the first second, $\frac{3}{2}g$ feet the second second, $\frac{5}{2}g$ feet the third second, and so on. How far will it fall in t seconds?

28. If $g = 32.16$ feet, through what distance does a body fall if it reaches the ground in 6 seconds? How far does it fall in the 6th second?

29. If a ball is dropped from the top of Washington Monument, 550 ft. high, how long does it take to reach the ground?

30. How long does it take the ball in problem 29 to get halfway to the ground?

31. In Italy 24-hour clocks are used. How many strokes does such a clock strike in a day?

32. In an A. P. of ten terms the product of the first and last terms is 70 and the sum of all is 95. Find the series.

33. How many numbers of 2 figures are divisible by 3? $(a = 12, l = 99, d = ?)$

34. Find the sum of all numbers of two figures each that are divisible by 3?

35. What is the sum of the first 50 multiples of 7?

36. The sum of n terms of the series 2, 5, 8, $\cdots$, is 950. Find n.

37. The sum of n terms of the series 87, 85, 83, $\cdots$, is the same as the sum of n terms of 3, 5, 7, $\cdots$. Find n.

GEOMETRICAL PROGRESSION

592. A **geometrical progression** (G. P.) is a series in which each term after the first is derived by multiplying the preceding term by a constant multiplier called the **ratio.**

Thus, 3, 6, 12, 24, $\cdots$, and 36, -6, 1, $-\frac{1}{6}$, $\cdots$, are geometrical progressions. The ratios are respectively 2 and $-\frac{1}{6}$.

593. Ratio. The **ratio** (r) is found by dividing any term by the term immediately preceding it.

ORAL EXERCISE

594. *What is the ratio in each of the following series?*

1. $2, 6, 18, 54, \cdots$.
2. $12, 6, 3, \frac{3}{2}, \cdots$.
3. $5, -10, 20, -40$.
4. a, ar, ar^2, ar^3.
5. $1, \sqrt{2}, 2, \cdots$.
6. $a, b, \frac{b^2}{a}, \cdots$.
7. $1, -1, 1, \cdots$.
8. $1, i, -1, \cdots$.
9. Form the next two terms in each of the series in examples 1 to 8.

595. Last Term. If a is the first term, l the last term, r the ratio, and n the number of terms, we have the following from the definitions:

1st term	2d term	3d term	4th term	5th term		nth term
a	ar	ar^2	ar^3	ar^4	$\cdots$	ar^{n-1}

By examining this series we notice that the exponent of r is always one less than the number of the term in the series.

Hence, in a series of n terms, the nth term being the last,

$$l = ar^{n-1}. \qquad (A)$$

Thus, the 8th term of 3, $\frac{3}{2}$, $\frac{3}{4}$, $\cdots$, is $3 \cdot (\frac{1}{2})^7 = \frac{3}{128}$, and the last term of 1, 5, 25 to 10 terms is $l = 1 \cdot 5^9 = 5^9$.

596. Sum of the Terms. To find the sum of the terms in a geometrical progression:

Write the sum of the series,

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \qquad (1)$$

$$(1) \times r, \qquad rS = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \qquad (2)$$

$$(1)-(2) \qquad S - rS = a \qquad\qquad - ar^n$$

$$\therefore S = \frac{a - ar^n}{1 - r} \text{ or } \frac{ar^n - a}{r - 1}. \qquad (B)$$

Since $ar^{n-1} = l$ or $ar^n = rl$, the formula may be written

$$S = \frac{a - rl}{1 - r} \text{ or } \frac{rl - a}{r - 1}, \qquad (C)$$

a formula that is sometimes useful.

Find the sum of 6, 3, $1\frac{1}{2}$, $\cdots$, to 10 terms.

Solution. $a = 6$, $r = \frac{1}{2}$, $n = 10$.

$\therefore$ from (B), we have $S = \dfrac{6 - 6(\frac{1}{2})^{10}}{1 - \frac{1}{2}} = \dfrac{3069}{256} = 11\frac{253}{256}$.

597. Geometrical Mean. If several quantities are in G. P., the terms between the first and last terms are the geometrical means between them.

The geometrical mean between two numbers is the square root of their product.

Proof. Let G be a geometrical mean between a and b.

$\dfrac{G}{a} = \dfrac{b}{G}$, for each fraction equals the ratio of the series.

$\therefore G^2 = ab$ and $G = \pm\sqrt{ab}$.

Find a geometrical mean between $\sqrt{8}$ and $\sqrt{2}$.

Solution. $G = \pm\sqrt{\sqrt{8} \cdot \sqrt{2}} = \pm 2$.

The student should notice that the geometrical mean and the mean proportional are the same. See § 326.

We may also insert several geometrical means between two given numbers.

Insert 4 geometrical means between 16 and $\frac{243}{2}$.

SOLUTION. Here $a = 16$, $n = 6$, (Why?) $l = \frac{243}{2}$.

From (*A*),

$$\frac{243}{2} = 16r^5.$$
$$r^5 = \frac{243}{32}.$$
$$r = \frac{3}{2}.$$

Let the student complete the solution.

598. Application of the Formulas. The formulas in geometrical progression to be remembered are:

$$l = ar^{n-1}. \qquad (A)$$

$$S = \frac{ar^n - a}{r - 1} \text{ or } \frac{a - ar^n}{1 - r}. \qquad (B)$$

$$S = \frac{rl - a}{r - 1} \text{ or } \frac{a - rl}{1 - r}. \qquad (C)$$

In (*A*), we have a, n, r, l.
In (*B*), we have a, n, r, S.
In (*C*), we have a, r, l, S.

599. The suggestions of § 589 hold here except that we may not be able to solve for n or r. In most cases the use of n as an unknown introduces equations of a type wholly new to the pupil, that is, with the unknown number an exponent. When r is unknown it may become necessary to extract roots higher than the second or third. Both these problems can be solved by inspection in some simple cases; for example, $2^n = 8$, $\therefore n = 3$; and $r^5 = 243$, $\therefore r = 3$.

Logarithms may also be used in such solutions.

1. Given $a = 1$, $l = 2$, $n = 4$; find r.

r, a, l, and n are all in formula (*A*).

Hence we may write $2 = 1 \cdot r^3$.

$$r^3 = 2.$$
$$r = \sqrt[3]{2}.$$

The series is $1, \sqrt[3]{2}, \sqrt[3]{4}, 2$.

2. From the data in example 1 find S.

Both formulas for S involve r. Using the result obtained in example 1, we may write from (*C*),

$$S = \frac{2\sqrt[3]{2} - 1}{\sqrt[3]{2} - 1}.$$

The result may be left in this form.

EXERCISE

600. 1. Find the 6th term of 1, 2, 4, 8, ⋯.

2. Find the sum of $1 + 2 + 4 \cdots$ to 6 terms.

3. Insert 3 geometrical means between 5 and 8.

4. Find the difference between the arithmetical mean and geometrical mean of 1 and 2.

5. The fourth term of a G. P. is 54 and the fifth term is 486. Find a and r.

6. Find the sum of the first five terms, when $a = 1$, $r = \frac{2}{3}$.

7. Find a fraction whose value is

$$1 + x + x^2 + x^3 + \cdots x^{15}.$$

8. Find the sum of the first 5 terms of $1, -\frac{1}{2}, \frac{1}{4}, \cdots$.

9. Find the geometrical mean between 14 and 686. Between 38 and 123 to two decimal places.

10. If $l = 128$, $r = 2$, $n = 7$, find a and S.

11. If $a = 9$, $l = 2304$, $r = 2$, find S and n.

12. If $a = 2$, $l = 1458$, $S = 2186$, find r and n.

INFINITE GEOMETRICAL SERIES — REPEATING DECIMALS

601. The student will recognize the identity

$$\tfrac{1}{3} = .3333 \cdots$$

This means that the repeating decimal .333 ⋯ approaches in value the fraction $\frac{1}{3}$.

It is evident also that the repeating decimal equals

$$.3+.03+.003+\cdots$$

This is a geometrical series with $a=.3$, $r=.1$ and n indefinitely large.

Using formula (B), we have

$$S=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}$$

$$=\frac{.3}{1-.1}-\frac{.3\times .1^n}{1-.1}$$

$$=\frac{1}{3}-\frac{.3\times .1^n}{.9}.$$

The second term, $\frac{.3}{.9}\times .1^n=\frac{1}{3}\times .1^n$, becomes smaller as n becomes larger. Thus, when $n=6$, $\frac{1}{3}\times .1^n=\frac{1}{3}$ of .000001. When n becomes infinitely large the term becomes so small that it may be neglected, and we have the sum of $.3+.03+.003+\cdots$ indefinitely $=\frac{1}{3}$.

602. Formula for an Infinite Geometrical Series. The result found in the last article may be generalized in the following statement:

When r is less than unity and the number of terms is infinite,

$$S=\frac{a}{1-r}.$$

Proof. $$S=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

When $r<1$ and n is infinitely large, r^n is smaller than any assignable number and therefore the term $\frac{ar^n}{1-r}$ may be neglected. This leaves

$$S=\frac{a}{1-r}.$$

1. Find the value of $1 + \frac{1}{2} + \frac{1}{4} \cdots$ to an infinite number of terms.

$$a = 1,\ r = \tfrac{1}{2},\ S = \frac{1}{1 - \frac{1}{2}} = 2.$$

2. Find the value of $3.2727 \cdots$.

Solution. Note that 3 is not part of the infinite geometrical series that follows it. First find the value of $.2727 \cdots = .27 + .0027 + \cdots$. Here

$$a = 27,\ r = .01.$$

$$S = \frac{.27}{1 - .01} = \frac{.27}{.99} = \frac{3}{11}.$$

$$\therefore 3.2727 \cdots = 3\tfrac{3}{11}.$$

EXERCISE

603. *Find l and S:*

1. When $a = 2$, $r = 2$, $n = 7$.
2. When $a = 5$, $r = 4$, $n = 9$.
3. When $a = 6$, $r = \frac{3}{4}$, $n = 6$.
4. $a = 40$, $r = \frac{3}{7}$, $n = \infty$; find S.

(∞ is the symbol for an infinitely large number.)

5. $a = 9$, $r = \frac{2}{3}$, $n = \infty$; find S.

6. If the first term of a geometrical series is a, and the second term is b, what is the ratio?

7. What is the sum of the first four terms of the series in example 6?

8. If $a > b$ and n is infinite, show that the value of S in example 6 is $\frac{a^2}{a - b}$.

9. Find the sum of the infinite series $\frac{1}{m+1} + \frac{1}{(m+1)^2} + \frac{1}{(m+1)^3} \cdots$, when $m > 0$.

10. What is the significance of making $m > 0$ in example 9? Find the series and the answer in 9 when $m = 2$.

11. What common fraction reduces to the repeating decimal $.777 \cdots$?

12. Find the fractional form for $3.25757 \cdots$.

Some of the problems that follow will require the use of the formulas of A. P., and some will be in G. P.

13. How many numbers of two figures each are exactly divisible by 7?

14. How many numbers of three figures each are multiples of 7?

15. How many numbers under 1000 are powers of 2?

16. What is the sum of all the three figure numbers that are multiples of 5?

17. In an A. P., $S = 20 + 13\sqrt{2}$, $n = 10$, $l = 2.6\sqrt{2}$; find a and d.

18. What kind of series is $-2, \sqrt{2} - 1, 2\sqrt{2}, \cdots$? Write the next two terms.

19. Given $a + \sqrt{2},\ 2a + 2,\ 3a + 2\sqrt{2},\ 4a + 4 \cdots$. Show that the sum of this series to 10 terms is $55a + 31(2 + \sqrt{2})$.

20. Indicate the sum of n terms of the series in example 19.

21. The first term of an infinite geometrical series is 3 and the second term is 2. Find the sum.

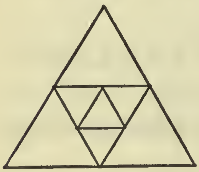
22. Find the 10th and 15th terms of $\frac{a-1}{a}, \frac{a-2}{a}, \frac{a-3}{a}, \cdots$.

(Williams College.)

23. Find the sum of 10 terms of $6, -4, \frac{8}{3} \cdots$ and the sum of 12 terms of $-5, -1, 3 \cdots$. (Williams College.)

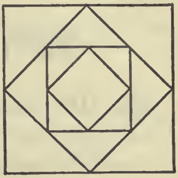
24. Determine whether $3\frac{3}{5}, 4\frac{4}{5}, 6\frac{2}{5} \cdots$ are in A. P. or G. P. and find the sum of the first 6 terms by the general formula.

25. Find the geometrical mean between $6 + \sqrt{2}$ and $6 - \sqrt{2}$. Find the arithmetical mean between the same numbers.



26. The side of an equilateral triangle is 10 inches. The midpoints of its sides are joined, forming another equilateral triangle, and this process is repeated indefinitely. Find the sum of all the lines.

27. Lines are drawn joining the middle points of the sides of a square, thus forming a second square, and the middle points of the sides of this square are joined. If this process is repeated indefinitely, find the sum of all the lines, if a side of the original square is 6 inches.



28. What number added to each of the numbers 1, 8, 22 will make a G. P.?

29. What distance will an elastic ball travel before coming to rest if it falls 20 feet and rebounds each time $\frac{2}{3}$ of the distance of its last fall indefinitely, that is, until it comes to rest?

30. The difference between two numbers is 48. Their arithmetical mean exceeds their geometrical mean by 18. Find the numbers. (Yale.)

31. Find the sum of n terms of

$$(x-y)+\left(\frac{y^2}{x}-\frac{y^3}{x^2}\right)+\left(\frac{y^4}{x^3}-\frac{y^5}{x^4}\right).$$

(Yale.)

XXVI. THE BINOMIAL FORMULA

604. By means of the **binomial formula** it is possible to raise a binomial to any required power without actually performing the multiplications.

The following is the binomial formula:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots.$$

The proof of this formula will be assumed, but the student should note that the following results obtained by using the formula agree with the results obtained by actual multiplication.

For $n = 2$,

$$(a+b)^2 = a^2 + 2\,a^{(2-1)}b + \frac{2\cdot(2-1)}{1\cdot 2}a^{2-2}b^2 = a^2 + 2\,ab + b^2.$$

For $n = 3$,

$$(a+b)^3 = a^3 + 3\,a^{(3-1)}b + \frac{3\cdot(3-1)}{1\cdot 2}a^{3-2}b^2 + \frac{3(3-1)(3-2)}{1\cdot 2\cdot 3}a^{3-3}b^3$$
$$= a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

Let the student verify the formula for $(a+b)^4$.

605. By observing the formula, we note the following points which may be used as a rule:

In the expansion of $(a+b)^n$:

1. The number of terms is $n+1$.

2. The first term is a^n.

3. The result is in descending powers of a and ascending powers of b, b appearing first in the second term.

4. Each coefficient is found from the preceding term by multiplying the coefficient of that term by the exponent of a and dividing the result by the exponent of b plus 1.

1. In the expansion of $a + b$ to a certain power one of the terms is $792\,a^5b^7$. What is the next term?

SOLUTION. Applying part 4 of the above rule, we have for the coefficient $\frac{792 \times 5}{8} = 495$.

Therefore the next term is $495\,a^4b^8$.

2. Expand $(a^3 + b^2)^6$ by the binomial formula.

SOLUTION.
$$(a^3+b^2)^6 = (a^3)^6 + 6(a^3)^5(b^2) + 15(a^3)^4(b^2)^2 + 20(a^3)^3(b^2)^3 + 15(a^3)^2(b^2)^4 + 6(a^3)(b^2)^5 + (b^2)^6$$
$$= a^{18} + 6\,a^{15}b^2 + 15\,a^{12}b^4 + 20\,a^9b^6 + 15\,a^6b^8 + 6\,a^3b^{10} + b^{12}.$$

3. Expand $(2\,x^{\frac{1}{2}} - y^{-3})^5$.

SOLUTION. Write this, $[(2\,x^{\frac{1}{2}}) + (-y^{-3})]^5$. Here a is $2\,x^{\frac{1}{2}}$ and b is $-y^{-3}$. Then
$$[(2\,x^{\frac{1}{2}}) + (-y^{-3})]^5 = (2\,x^{\frac{1}{2}})^5 + 5(2\,x^{\frac{1}{2}})^4(-y^{-3}) + 10(2\,x^{\frac{1}{2}})^3(-y^{-3})^2 + 10(2\,x^{\frac{1}{2}})^2(-y^{-3})^3 + 5(2\,x^{\frac{1}{2}})(-y^{-3})^4 + (-y^{-3})^5$$
$$= 32\,x^{\frac{5}{2}} - 80\,x^2y^{-3} + 80x^{\frac{3}{2}}y^{-6} - 40\,xy^{-9} + 10\,x^{\frac{1}{2}}y^{-12} - y^{-15}.$$

EXERCISE

606. 1. In expanding a binomial by the formula:

(1) How does the exponent of the first term compare with the power of the binomial?

(2) What is the exponent of the first term of the binomial in each term of the expansion after the first?

(3) In what term of the expansion does the second term of the binomial first appear?

(4) How does its exponent change from term to term?

(5) What is the coefficient of the first term of the expansion? of the second term?

(6) How is the coefficient of each term after the second formed?

(7) How many terms are there?

Expand by the binomial formula and simplify the terms:

2. $(a-b)^4$.
3. $(a+2b)^5$.
4. $(a^2-3b)^6$.
5. $(2x-\frac{1}{2}y^2)^5$.
6. $(\frac{1}{2}x^3+2y^{\frac{1}{2}})^7$.
7. $(x-2y)^7$.
8. $(3x+y)^8$.
9. $(5-2t^{\frac{1}{2}})^6$.
10. $(x^2-y)^7$.
11. $(\frac{3}{2}-\frac{2}{3}x)^7$.
12. $(\frac{1}{3}-3y)^6$.
13. $(\frac{1}{2}+2z^{\frac{1}{3}})^5$.
14. $(x+x^{-1})^8$.
15. $\left(\frac{a}{b}-\frac{b}{a}\right)^7$.
16. $\left(\frac{2a}{3b}-\frac{3b}{2a}\right)^6$.
17. $(1+\sqrt{-1})^4$.
18. $(1-i)^8$.
19. $(1+\sqrt{-3})^3$.
20. $(3+\sqrt{-5})^7$.
21. $(a+bi)^6-(a-bi)^6$.
22. $(1+\sqrt{x})^7-(1-\sqrt{x})^7$.

23. What are the signs of the terms in the expansion of $(a-b)^7$? of $(-a+b)^7$? of $(-a+b)^6$? of $(-a-b)^7$? of $(-a-b)^6$?

24. Expand $(2-3i)^5$.

25. Find the first 4 terms and the last term of $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^{20}$.

26. Find the first four terms and the last term of $(a+b)^{100}$.

27. How many terms are there in the expansion of $(a+b)^5$? of $(a+b)^6$? of $(a+b)^n$?

28. What are the exponents of a and b in the fourth term of $(a+b)^5$? in the fifth term of $(a+b)^5$? in the fourth term of $(a+b)^6$? in the fifth term of $(a+b)^6$?

29. What is the sum of the exponents of a and b in each term of the expansion of $(a+b)^5$? of $(a+b)^6$? of $(a+b)^n$?

607. Binomial Coefficients and Exponents. The coefficients in the expansion of $a+b$ to any power, and the exponents of a and b in this expansion, are called respectively the **binomial**

coefficients and the **binomial exponents** of the expansion. These terms are used to distinguish them from the reduced results in cases when a, for example, is represented by $2x^{\frac{1}{2}}$ and b by a similar expression. The expansions should always be made first in *binomial coefficients* and *exponents*. (Why?)

An interesting relation among the *binomial coefficients* of successive powers of a binomial is shown in the following scheme known as Pascal's Triangle:

$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	6	4	1		
$n=5$	1	5	10	10	5	1	
$n=6$	1	6	15	20	15	6	1

The numbers in the first line are the coefficients of $(a+b)$; in the second line of $(a+b)^2$; in the third line of $(a+b)^3$, etc.

Any number in this scheme equals the number directly over it plus the number at the left of the one over it. The coefficients of the 5th power of $a+b$ are found from the coefficients of the fourth power as follows:

$1+4=5$; write 5 under 4; $4+6=10$; write 10 under 6; etc.

Let the student write the coefficients for $n=7$ and $n=8$.

608. Any Required Term. Writing the binomial formula and numbering the terms, we may make a formula for any term.

$$\overset{\text{1st}}{(a+b)^n=a^n}+\overset{\text{2d}}{na^{n-1}b}+\overset{\text{3d}}{\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2}+\overset{\text{4th}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3}+\cdots$$

$$+\overset{\text{6th}}{\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{n-5}b^5}+\cdots$$

$$+\overset{\text{10th}}{\frac{n(n-1)(n-2)\cdots(n-8)}{1\cdot 2\cdot 3\cdots 9}a^{n-9}b^9}+\cdots.$$

A simple way to find any required term in the expansion of $(a+b)^n$ is to start with the exponent of b.

What is the exponent of b in the 2d term? in the 3d term? in the 4th term? How does it compare with the number of the term? *In the rth term it is* $r-1$. In a similar way the exponent of a is n minus the exponent of b; that is, $n-(r-1)$ or $n-r+1$. By further comparison of the coefficient of any term with the number of the term it will be seen that the rth term is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}b^{r-1}.$$

1. Find the 7th term of $(\frac{1}{2}a^2-2b^{\frac{1}{2}})^{12}$.

First write the binomial in the form $[(\frac{1}{2}a^2)+(-2b^{\frac{1}{2}})]^{12}$.

Applying the formula for the rth term when $r=7$ and $n=12$,

we have $\dfrac{\not{12}\cdot 11\cdot\not{10}\cdot\overset{3}{\not{9}}\cdot\overset{4}{\not{8}}\cdot 7}{1\cdot\not{2}\cdot\not{3}\cdot\not{4}\cdot\not{5}\cdot\not{6}}(\frac{1}{2}a^2)^6(-2b^{\frac{1}{2}})^6=924\,a^{12}b^3.$

2. Find the middle term of $(x-\sqrt{y})^{14}$.

How many terms are there in the expansion? What is the number of the middle term?

$$\frac{\not{14}\cdot 13\cdot\not{12}\cdot 11\cdot\overset{2}{\not{10}}\cdot\overset{3}{\not{9}}\cdot 8}{1\cdot\not{2}\cdot\not{3}\cdot\not{4}\cdot\not{5}\cdot\underset{2}{\not{6}}\cdot\not{7}}x^7(-\sqrt{y})^7=-3432\,x^7y^3\sqrt{y}.$$

EXERCISE

609. *Find only the terms asked for:*

1. The third term of $(a-b)^9$.
2. The third term of $(-a+x)^{10}$.
3. The third term of $(a+b)^{100}$.
4. The sixth term of $(x^2-x^{-1})^8$.
5. The seventh term of $(2x^{-\frac{1}{2}}-y^{-\frac{2}{7}})^6$.
6. The middle term of $\left(x+\dfrac{1}{x}\right)^{12}$.
7. The middle term of $(3x-\frac{1}{3})^{10}$.
8. The sixth term of $(1+x)^n$.

9. The two middle terms of $\left(x - \frac{1}{x}\right)^{11}$.

10. The term containing x^5 in $(a - x)^7$.

11. The term containing a^5 in $(a - x)^9$.

12. What term of what power of $a + b$ contains a^2b^{11}? a^3b^{11}?

13. Write the first four terms, and the last term of $(2a^2 - b^{\frac{1}{2}})^8$.

14. Find, in simplest radical form, the value of $(\sqrt{2} + \sqrt{3})^4$.

Write the first three terms, and the last term in the following:

15. $(2a^{\frac{1}{3}} - b^2)^7$.

16. $\left(\frac{x}{3} - 3\right)^8$.

17. $(1 - \frac{2}{3}x)^6$.

18. $(-\frac{2}{3}x + \frac{3}{2}y)^5$.

19. $\left(-\frac{x}{2} - 2y^{\frac{1}{2}}\right)^6$.

20. $(-\frac{4}{3}a - \frac{3}{2}b)^7$.

21. $(3 - 2x)^8$.

22. $(-3 + 2x)^8$.

23. Find the 7th and 8th terms of $(a + b)^{10}$.

24. Find the 4th term of $(a + b)^{11}$. The 4th from the last.

25. Write the next to the last term of $(3a^{\frac{1}{2}} - b^{\frac{1}{3}})^{10}$.

26. What is the exponent of x in the first term of $(x + x^{-1})^{12}$? in the second term? in the third term? Find the term that does not contain x.

27. Is there a term in $(x + x^{-1})^{11}$ that does not contain x?

28. Find the last three terms of $(\sqrt{2} - b^{\frac{1}{2}})^8$.

XXVII. VARIATION

610. In numbers that are related to each other through mathematical equations, some of the numbers may be changing in value, while others may have fixed values.

If a train travels at a *uniform rate* of r miles per hour, we may express the distance it has traveled after the lapse of any time by the equation, $d = rt$. In this equation d and t vary in value from moment to moment, but r is a constant, for by the conditions, the rate is uniform.

611. Variable and Constant. A number that is changing in value is a **variable**; a number whose value does not change is a **constant.**

The formulas of algebra, geometry, physics, and their practical applications, involve, in general, variables and constants.

In the illustration just given, $d = rt$, d and t are variables and r is a constant.

In $A = \pi R^2$, A and R are variables and π is a constant.

612. Direct Variation. If two variable numbers are so related to each other that through all their changes in value their ratio remains unchanged, one of these numbers **varies directly** as the other, or simply **varies as** the other.

613. Constant of Variation. The constant value of the ratio of the variable numbers in direct variation is the **constant of variation.**

We may write $d = rt$, when r is constant and d and t are variables, in the form $\frac{d}{t} = r$. Then by definition, we have "distance varies as the time."

The student must remember that d, t, and r are abstract numbers. They represent the numerical measures of concrete magnitudes; that is, d equals the *number of miles* traveled in t hours.

614. Notation. The symbol for variation is $\propto$. $a \propto b$ is read "a varies as b."

It is customary to use a letter with different subscripts to represent different values that a variable number has at different periods of its variation.

Thus, d_1, d_2, $d_3 \cdots$ (read d-sub one, etc.) are used to represent the distances traveled in the times t_1, t_2, $t_3 \cdots$, respectively.

615. In agreement with this notation and the definition of variation, we have, for uniform motion,

$$\frac{d_1}{t_1} = r,\ \frac{d_2}{t_2} = r,\ \frac{d_3}{t_3} = r, \text{ etc.}$$

where r is the constant of variation, in this case the uniform rate of motion.

It is evident that the constant of variation can be found in any particular case, if we know a set of corresponding values of the two variables.

Thus, if $d_1 = 140$ miles, and $t_1 = 4$ hours, $r = \frac{140}{4} = 35$, the number of miles per hour.

616. If a and b are two variable numbers, and $a \propto b$, then $a_1 : a_2 = b_1 : b_2$, where a_1 and b_1, a_2 and b_2 are sets of corresponding values of the variables.

Proof. We have given $a \propto b$.

$\therefore \dfrac{a_1}{b_1} = k$, and $\dfrac{a_2}{b_2} = k$, where k is the constant of variation.

$\therefore \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$. (Why?)

$\therefore \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$. (Why?)

617. On account of the possibility of expressing the relation of variation in the form of a proportion as just proved, it is common to speak of one of two variable numbers as proportional to the other. This means that one varies as the other.

This form of expressing the relation of variation is often used in geometry and in physics.

Thus, in geometry we say, "The areas of triangles having equal bases are proportional to their altitudes." In physics for uniform motion, we have, "The distance is proportional to the time."

EXERCISE

618. 1. If $x \propto y$, and $x = 10$, when $y = 5$, what is the constant of variation?

2. Using the data and answer of example 1; find y_1 if $x_1 = 50$.

3. The area of a circle varies as the square of the radius. Express this as a variation, using A and R for area and radius respectively.

4. $A_1 = 314.16$, $R_1 = 10$. Find the constant of variation in example 3.

5. Find R_2, if $A_2 = 100$.

6. The area of a triangle varies as the product of its base and altitude. What is the constant of variation? ($T = \frac{1}{2} bh$.)

7. From example 6 show that $T_1 : T_2 = b_1 \cdot h_1 : b_2 \cdot h_2$, where T, b, and h represent respectively the area, the base, and the altitude of a triangle.

8. For a freely falling body we have, in physics, the formula $S = \frac{1}{2} gt^2$, where g is a constant. Show that, for falling bodies, the distance is proportional to the square of the time.

9. From example 8 derive $S_1 : S_2 = t_1^2 : t_2^2$.

10. Given $S_1 = 64$ feet and $t_1 = 2$ seconds. Find g.

11. The weight of a sphere of given material varies (directly) as the cube of the radius. Two spheres of the same material have radii of 2 inches and 6 inches respectively. If the weight of the first is 6 pounds, what is the weight of the second? (Sheffield Scientific School.)

12. Find from example 11 the radius of a sphere that weighs 48 pounds.

13. The surface of a sphere varies as the square of the radius. Express this in the form of a variation. Express as a proportion.

14. If the surface of a sphere is 1256 square inches when the radius is 10 inches, what is the constant of variation?

15. If $x \propto y$ and $y \propto z$, show that $x \propto z$.

Let k, l, and m be, respectively, the constants of variation. Show that $kl = m$.

619. Inverse Variation. One number varies inversely as another if the ratio of the first to the reciprocal of the second is constant.

Thus, x *varies inversely as* y, if $x = k \cdot \frac{1}{y}$, or $xy = k$.

The inverse variation of x and y may be indicated by any one of the three expressions, $x \propto \frac{1}{y}$, $x = \frac{k}{y}$, or $xy = k$. Here k is the constant of variation.

Instead of "*varies inversely*" we sometimes say "*is inversely proportional to.*"

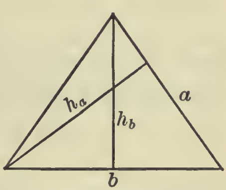
EXERCISE

620. 1. If x varies inversely as y, and x_1 and y_1 are respectively 9 and 2, what is the constant of variation?

2. Find y_2 in example 1, if $x_2 = 30$.

3. If x varies inversely as y, show that $x_1 : x_2 = y_2 : y_1$.

4. Show that any two altitudes of a triangle are inversely proportional to the sides upon which they are drawn, using the relation $T = \frac{1}{2}ah_a = \frac{1}{2}bh_b$.



5. If x varies inversely as y, show that for a multiplication of the value of y by any number we have a division of the value of x by the same number.

6. A train has a run of 240 miles. Show that the time required is inversely proportional to the rate of the train.

7. A person has a given sum of money with which to buy horses. Does the number of horses that he can buy vary directly or inversely as the price per horse?

8. A certain piece of work is to be done. Does the time required to do the work vary directly or inversely as the number of workmen employed?

9. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

10. The weight of a body varies inversely as the square of its distance from the center of the earth. If a body weighs 1 pound on the surface of the earth (4000 miles from the center), how much will it weigh 10,000 miles from the center?

621. Joint Variation. If one number varies as the product of two others it **varies jointly** as these two other numbers.

622. If x varies as y when z is constant, and x varies as z when y is constant, then x varies jointly as y and z.

PROOF. Let x_1, y_1, z_1 and x_2, y_2, z_2 be any two sets of corresponding values of the three variables.

Consider the variation of y and z as taking place separately, and let x change in value from x_1 to x' due to the change in y from y_1 to y_2, z remaining constant.

$$\therefore \frac{x_1}{x'} = \frac{y_1}{y_2}. \quad (\S\ 616.) \qquad (1)$$

Now let y remain constant and z change from z_1 to z_2, which will change x from the intermediate value x' to x_2.

$$\therefore \frac{x'}{x_2} = \frac{z_1}{z_2}. \qquad (2)$$

(1) × (2) gives

$$\frac{x_1}{x_2} = \frac{y_1 z_1}{y_2 z_2}. \qquad (3)$$

$$\therefore \frac{x_1}{y_1 z_1} = \frac{x_2}{y_2 z_2}. \qquad (4)$$

This last equation shows that the ratio of x to yz is the same for any two sets of corresponding values of x, y, and z.

That is, $$\frac{x}{yz} = k,$$

or $$x \propto yz.$$

623. If x varies directly as y and inversely as z, then $x \propto \frac{y}{z}$.

Let the student prove this.

It is evident that if $x \propto \frac{y}{z}$, we have an increase in x for an increase in y, but a decrease in x for an increase in z.

EXERCISE

624. 1. If x varies jointly as y and z, and $x_1 = 63$, $y_1 = 5$, $z_1 = 9$; find the constant of variation.

2. By using the constant of variation found in example 1, find y_2, if $x_2 = 72$ and $z_2 = 18$.

3. Given that $x \propto$ jointly as y and z, and $x_1 = 225$, $y_1 = 12$, $z_1 = 15$, $x_2 = 405$, $y_2 = .6$: find by proportion the value of z_2. (See equation 3 of § 622.)

4. The total area T of a right circular cylinder varies jointly as R and $R + H$, where R is the radius of the base and H is the altitude. When $R = 7$ inches and $H = 13$ inches, $T = 880$ square inches; find T, when $R = 5$ inches and $H = 10$ inches.

5. The weight of right circular cylinders of the same material varies jointly as the height and the square of the radius of the base. A steel cylinder weighing 22 pounds has a base with radius 1 inch and its altitude is 7 inches. Find the weight of another cylinder whose base has a radius of 2 inches and whose altitude is 14 inches.

6. The time required by a pendulum to make one vibration varies directly as the square root of the length. If a pendulum 100 centimeters long vibrates once in a second, find the time of one vibration of a pendulum 36 centimeters long. (Yale.)

XXVIII. LOGARITHMS

625. The processes of multiplication, division, involution, and evolution can be greatly abridged by the use of the laws of exponents. A system of computation by means of tables is based upon these laws.

626. By means of a table of powers of 2 we can perform the operations of multiplication, division, involution, and evolution upon powers of 2.

$2^0 = 1.$
$2^1 = 2.$
$2^2 = 4.$
$2^3 = 8.$
$2^4 = 16.$
$2^5 = 32.$
$2^6 = 64.$
$2^7 = 128.$
$2^8 = 256.$
$2^9 = 512.$
$2^{10} = 1024.$
$2^{11} = 2048.$
$2^{12} = 4096.$
$2^{13} = 8192.$
$2^{14} = 16384.$
$2^{15} = 32768.$
$2^{16} = 65536.$
$2^{17} = 131072.$
$2^{18} = 262144.$
$2^{19} = 524288.$
$2^{20} = 1048576.$

ORAL EXERCISE

627. 1. $32 \times 128 = ?$

SOLUTION. From the table $32 = 2^5$ and $128 = 2^7$.

$$\therefore 32 \times 128 = 2^5 \times 2^7 = 2^{5+7} = 2^{12} = 4096.$$

2. $16384^{\frac{2}{7}} = ?$

SOLUTION. From the table $16384 = 2^{14}$.

$$\therefore 16384^{\frac{2}{7}} = (2^{14})^{\frac{2}{7}} = 2^4 = 16.$$

3. $256 \times 8 = ?$

4. $64^2 = ?$

5. $\sqrt[5]{1024} = ?$

6. $\sqrt[3]{4096} = ?$

7. $(32768 \times 8192)^{\frac{1}{4}} = ?$

8. $\sqrt[4]{65536} = ?$

9. Divide 1048576 by 2048.

10. Divide 524288 by 512.

11. Divide 8192 by $\sqrt[5]{1024}$.

628. Logarithm. The **logarithm** of a number is the **exponent** of the power to which a fixed number called the base must be raised to produce the number.

Thus, in $2^{13} = 8192$, 13 is the logarithm of 8192 to the base 2. This, in the notation of logarithms, is written

$$\log_2 8192 = 13,$$

and is read, the logarithm of 8192 to the base 2 is 13.

Any expression of the form $a^b = c$ can be changed to logarithmic notation.

Thus, $a^b = c$ and $\log_a c = b$ according to the definition of logarithm, represent the same relation between a, b, and c.

EXERCISE

629. *Change the following into logarithmic notation:*

1. $2^3 = 8.$
2. $7^k = 14.$
3. $m^k = y.$
4. $64^{\frac{2}{3}} = 16.$
5. $64^x = 16.$
6. $3^{x+1} = 27.$
7. $9^{-\frac{1}{2}} = \frac{1}{3}.$
8. $8^x = 32.$
9. $x^5 = 32.$

Read the following, and change each from the logarithmic notation to the exponential form:

10. $\log_b a = c.$
11. $\log_a b = c.$
12. $\log_{64} 16 = \frac{2}{3}.$
13. $\log_a a^3 = 3.$
14. $\log_{10} 1 = 0.$
15. $\log_9 81 = 2.$
16. $\log_{\frac{1}{9}} 81 = -2.$
17. $\log_9 x = \frac{1}{2}.$
18. $\log_3 729 = 7.$
19. $\log_x 32 = 5.$

Find the value of x in each of the following:

20. Solve for x, $\log_{10} 100 = x.$

Solution. $10^x = 100.$
But $10^2 = 100.$
$\therefore 10^x = 10^2.$
$\therefore x = 2.$

21. $\log_2 32 = x.$
22. $\log_{32} 2 = x.$
23. $\log_a a^4 = x.$
24. $\log_2 1 = x.$
25. $\log_{.5} .125 = x.$
26. $\log_8 16 = x.$
27. $\log_{16} 8 = x.$
28. $\log_a a = x.$
29. $\log_a 1 = x.$
30. $\log_2 8 = x + 1.$
31. $\log_3 27 = x.$
32. $\log_{a^2} a^8 = x^2.$
33. $\log_2 x = 5.$

Solution. $2^5 = x.$
$\therefore x = 32.$

34. $\log_a 2x = -2.$
35. $\log_2 x = 8.$

630. Laws of Logarithms. The laws of logarithms for multiplication, division, involution, and evolution are exactly the same as the corresponding laws of exponents, as the student might anticipate, since logarithms are exponents.

1. Law of Multiplication. The logarithm of a product equals the sum of the logarithms of its factors.

In symbols, $\log_k ab = \log_k a + \log_k b.$

PROOF. Let $\log_k a = x$ and $\log_k b = y$.

$\therefore k^x = a$ and $k^y = b$. (Definition of logarithm.)

$\therefore k^{x+y} = ab$. (Why?)

$\therefore \log_k ab = x + y$, (Definition of logarithm.)

or $\log_k ab = \log_k a + \log_k b$. (Why?)

2. Law of Division. The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.

In symbols, $\log_k \frac{a}{b} = \log_k a - \log_k b.$

PROOF. Let $\log_k a = x$ and $\log_k b = y$.

$\therefore k^x = a$ and $k^y = b$. (Why?)

$\therefore k^{x-y} = \frac{a}{b}$. (Why?)

$\therefore \log_k \frac{a}{b} = x - y$, (Why?)

or $\log_k \frac{a}{b} = \log_k a - \log_k b$. (Why?)

3. Law of Powers. The logarithm of the power of a number equals the exponent of the power multiplied by the logarithm of the number.

In symbols, $\log_k a^n = n \cdot \log_k a.$

PROOF. Let $\log_k a = x$.

$\therefore k^x = a$. (Why?)

$\therefore k^{nx} = a^n$. (Why?)

$\therefore \log_k a^n = nx$, (Why?)

or $\log_k a^n = n \log_k a$. (Why?)

4. **Law of Roots.** **The logarithm of the root of a number equals the quotient of the logarithm of the number divided by the index of the root.**

In symbols, $\log_k \sqrt[n]{a} = \frac{1}{n} \log_k a.$

PROOF. Let $\log_k a = x.$

$$\therefore k^x = a. \quad \text{(Why?)}$$

$$\therefore k^{\frac{x}{n}} = \sqrt[n]{a}. \quad \text{(Why?)}$$

$$\therefore \log_k \sqrt[n]{a} = \frac{1}{n} \cdot x = \frac{1}{n} \cdot \log_k a.$$

631. According to these laws we may make such transformations as the following:

1. $\log \frac{ab}{c} = \log a + \log b - \log c.$ (Why?)

2. $\log (ab)^2 = 2(\log ab) = 2 \log a + 2 \log b.$ (Why?)

3. $\frac{1}{2} \log x - \frac{1}{3} \log y + \frac{1}{4} \log z = \log \frac{\sqrt{x}\sqrt[4]{z}}{\sqrt[3]{y}}.$ (Why?)

4. $3 \log a - 4 \log b = \log \frac{a^3}{b^4}.$ (Why?)

EXERCISE

632. *Using the laws of logarithms express examples* 1 *to* 9 *in terms of* $\log a$, $\log b$, $\log c$, *and* $\log x$ *as in examples* 1 *and* 2, § 631.

1. $\log 3a^2x.$
2. $\log \left(\frac{a}{b}\right)^2.$
3. $\log (a^2 \div b^2).$
4. $\log a\sqrt{\frac{b^2}{c}}.$
5. $\log a \sqrt[3]{b}.$
6. $\log \sqrt[4]{ab}.$
7. $\log a^3\sqrt[3]{a}.$
8. $\log a^3\sqrt{bxc^n}.$
9. $\log \sqrt{ax^2}.$

Express examples 10 *to* 14 *as the logarithm of a single term as in examples* 3 *and* 4, § 631.

10. $\log a + \log b - \log c$.

11. $\log a + \log b - 2 \log c$.

12. $3 \log a + 4 \log b$.

13. $\log a^2 + 2 \log b - 2 \log ab$.

14. $\log \frac{x}{y} + \log xy - \log x + \log y$.

15. $\log (x - y) + \log (x + y)$.

633. Common Logarithms. Any other base than 2 might have been used and a table similar to that of § 626 formed. In practice, logarithmic computations are made with the **common or Briggs system** of logarithms. In this system the base is 10.

Common logarithms are exponents, positive or negative, of powers of 10.

From the definition of common logarithms since

$$\begin{aligned}
10^4 &= 10{,}000, &\therefore \log_{10} 10{,}000 &= 4.\\
10^3 &= 1000, &\therefore \log_{10} 1000 &= 3.\\
10^2 &= 100, &\therefore \log_{10} 100 &= 2.\\
10^1 &= 10, &\therefore \log_{10} 10 &= 1.\\
10^0 &= 1, &\therefore \log_{10} 1 &= 0.\\
10^{-1} &= .1, &\therefore \log_{10} .1 &= -1.\\
10^{-2} &= .01, &\therefore \log_{10} .01 &= -2.\\
10^{-3} &= .001, &\therefore \log_{10} .001 &= -3.
\end{aligned}$$

Clearly the logarithms of numbers between 1000 and 10,000 lie between 3 and 4; similarly the logarithms of numbers between 100 and 1000 lie between 2 and 3, etc. Therefore the logarithms of most numbers will have an integral part and a decimal part.

634. Characteristic, Mantissa. The integral part of a logarithm is the **characteristic** and the decimal part is the **mantissa. The mantissa is always positive.**

Thus, $\log_{10} 20 = 1.3010$. The **characteristic** of $\log_{10} 20$ is 1 and the **mantissa** is .3010.

635. Mantissa Law. The mantissa depends only upon the sequence of the figures and is independent of the position of the decimal point.

ILLUSTRATION. We may find in the tables that

$$\begin{aligned} \log_{10} 625 &= 2.7959. \\ \log_{10} 62.5 &= \log_{10} 625 - \log_{10} 10 \text{ (By law 2)} \\ &= 2.7959 - 1 \\ &= 1.7959. \end{aligned}$$

Note that the mantissa is the same for log 625 and log 62.5.

PROOF. Moving the decimal point to the right or the left multiplies or divides the number by 10 or 100 or 1000, etc. Therefore the logarithm of the number will be increased or diminished by log 10, or log 100 or log 1000, etc. But the logarithms of 10, 100, 1000, etc. are integral numbers, and increasing or diminishing the logarithms by integers will not change the decimal part, the **mantissa**, of the logarithm.

636. Characteristic Law. The characteristic of a number greater than unity is one less than the number of figures to the left of the decimal point.

Consider 225.16. As it lies between 100 and 1000, its logarithm is between 2 and 3; that is, its logarithm is 2 + a fraction. Similarly for any number more than unity.

Find between what two powers of 10 each of the following lies: (*a*) 21; (*b*) 3.1; (*c*) 5437.1. What is the characteristic of the logarithm of each?

The characteristic of a number less than unity is negative, and is numerically equal to one more than the number of zeros preceding the first significant figure of the number.

The following will illustrate the laws that govern both characteristic and mantissa:

$$\begin{aligned} &\log_{10} 7235 &&= 3.8594. &&\text{(From tables.)} \\ &\log_{10} 723.5 = \log_{10} 7235 - \log_{10} 10 &&= 2.8594. &&\text{(Why?)} \\ &\log_{10} 72.35 = \log_{10} 723.5 - \log_{10} 10 &&= 1.8594. &&\text{(Why?)} \\ &\log_{10} 7.235 = \log_{10} 72.35 - \log_{10} 10 &&= 0.8594. &&\text{(Why?)} \\ &\log_{10} .7235 = \log_{10} 7.235 - \log_{10} 10 &&= .8594 - 1. &&\text{(Why?)} \end{aligned}$$

The last number is a negative number. We ordinarily write

$$\log_{10} .7235 = \bar{1}.8594.$$

This is to be understood to mean $-1 + .8594$. The *mantissa* is *positive.* In practice, to avoid a negative characteristic 10 is added and subtracted, thus,

$$\bar{1}.8594 = 9.8594 - 10.$$

$$\text{Also } \log_{10} .07235 = \log_{10} .7235 - \log_{10} 10$$
$$= \bar{2}.8594 = 8.8594 - 10,$$
$$\text{and } \log_{10} .007235 = \bar{3}.8594 = 7.8594 - 10.$$

EXERCISE

637. *What is the characteristic of the logarithm of each of the following numbers?*

1. 25.
2. .5.
3. .0004.
4. 1.732.
5. 101.
6. 99.
7. .9.
8. .99.
9. If $\log 247 = 2.3927$, what are the logarithms of 24.7? .0247? 2.47? 24,700? .247? .000247?
10. What are the logarithms of 3, 27, 81, 243, and $\frac{1}{3}$ in a system of which the base is 3?
11. What are the logarithms of 5, 25, 125, 625, 3125, $\frac{1}{5}$ in a system of which the base is 5?
12. What are the logarithms of 36, 216, 1296, in a system of which the base is -6? Why is a negative number not convenient as the base of a system of logarithms?

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, find:

13. $\log 6$.
14. $\log 9$.
15. $\log 12$.
16. $\log 15$.
17. $\log 3^4$.
18. $\log 6^2$.
19. $\log 7.5$.
20. $\log \frac{6}{5}$.
21. $\log 375$.
22. How many figures are there in 25^{30}? in 30^{25}?
23. How many zeros are there between the decimal point and the first significant figure of $(\frac{1}{2})^{100}$? $(\frac{1}{5})^{50}$?
24. How can you find $\log_{10} 5$ from $\log_{10} 2 = .3010$?

USE OF TABLES

638. In the tables on pp. 462 and 463 the mantissas are given correct to but four decimal places. By using these tables, results can generally be relied upon as correct to 3 figures and usually to 4. If a greater degree of accuracy is required, five-place or even seven-place tables must be used.

639. To find the logarithm of a given number:

Write the characteristic before looking in the tables for the mantissa. (§ 636.)

Find the mantissa in the tables.

(1) **When the number consists of not more than three figures:**

In the column N, at the left-hand side of the page, find the first two figures of the number. In the row N, at the top or bottom of the page, as convenient, find the third figure. The mantissa of the number will be found at the intersection of the row containing the first two figures and the column containing the third figure.

1. Find log 384.

The characteristic is 2 (Why?). In the column N find 38 and in row N find 4. The mantissa 5843 will be found at the intersection of the row 38 and column 4.

$\therefore \log 384 = 2.5843$.

2. What is log 3.84? log 38.4? log 0.0384?

(2) **When the number consists of more than three figures:**

Find as above the mantissa of the logarithm of the number consisting of the first three figures. To correct for the remaining figures **interpolate by assuming that, for differences small as compared with the numbers, the differences between numbers are proportional to the differences between their logarithms.** This statement is only approximately true, but its use leads to results accurate enough for ordinary computations.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Find log 3847.

$$\begin{aligned} &\text{Mantissa of log } 3850 = .5855. \\ &\text{Mantissa of log } \underline{3840} = \underline{.5843.} \\ &\qquad\qquad\qquad\quad 10 \quad 0.0012. \end{aligned}$$

$$\text{Mantissa of log } 3847 = .5843 + \tfrac{7}{10} \text{ of } 0.0012 = .5851.$$

The difference between 3840 and 3850 is 10; the difference between the mantissas of their logarithms (.5855 − .5843) is 0.0012. Assuming that each increase of 1 unit between 3840 and 3850 produces an increase of 1 tenth of the difference in the mantissas, the addition for 3847 will be 7 tenths of 0.0012 or 0.00084. .5843 + 0.00084 = .5851. Therefore, the mantissa of log 3847 = .5851.

EXERCISE

640. *Find the logarithms of:*

1. 1845. 2. 6.897. 3. 0.04253.

641. To find the number corresponding to a given logarithm:

The number corresponding to a logarithm is its **antilogarithm.**

The characteristic determines the position of the decimal point.

(1) **If the mantissa is found in the tables, the number is found at once.**

Find antilog 3.5877.

The mantissa is found at the intersection of row 38 and column 7.

$$\therefore \text{antilog } 3.5877 = 3870.$$

(2) **If the exact mantissa is not found in the tables,** the first three figures of the corresponding number can be found and to them can be annexed figures found by interpolation.

Find antilog 3.5882.

log 3880 = 3.5888	log required number = 3.5882
log 3870 = 3.5877	log 3870 = 3.5877
10 0.0011	log req.no. − log 3870 = 0.0005

$$3870 + (\tfrac{5}{11} \text{ of } 10) = 3874.54^{+}.$$

The two mantissas in the table nearest to the given mantissa are .5888 and .5877, differing by 0.0011. The corresponding numbers, since the characteristic is 3, are 3880 and 3870, differing by 10. The difference between the smaller mantissa 5877 and the required mantissa 5882 is 0.0005. Since an increase of 11 ten thousandths in mantissas corresponds to an increase of 10 in the numbers, an increase of 5 ten thousandths in mantissas may be assumed to correspond to an increase of $\frac{5}{11}$ of 10 in the numbers. Therefore the number is $3870 + (\frac{5}{11}$ of $10) = 3874.54^{+}$. The last two figures are uncertain.

EXERCISE

642. *Find the antilogarithms of:*

1. 2.9445.	3. $\bar{1}.6527$.	5. 1.9994.
2. $\bar{2}.4065$.	4. 3.7779.	6. 0.7320.

643. The **cologarithm** of a number is the logarithm of its reciprocal. The cologarithm of 100 equals the logarithm of $\frac{1}{100}$, that is, -2.

$$\text{Since } \log 1 = 0, \therefore \log\frac{1}{n} = \log 1 - \log n = 0 - \log n,$$

$$\text{therefore colog}\, n = -\log n.$$

As the cologarithm of a number equals the logarithm with its sign changed, adding the cologarithm will give the same result as subtracting the logarithm. The former is sometimes more convenient.

To avoid negative results it is often more convenient to add and subtract 10.

1. Find colog 47.3.

$$\begin{aligned} \log 1 &= 10.0000 - 10 \\ \log 47.3 &= \underline{1.6749} \\ \text{colog } 47.3 &= 8.3251 - 10 \end{aligned}$$

In subtracting 1.6749 or any other logarithm from 10, the result may be obtained mentally by subtracting the right-hand figure from 10 and all the others from 9.

2. Find the value of $\dfrac{452 \times 23}{5371 \times 29}$.

$$\log \frac{452 \times 23}{5371 \times 29} = \log 452 + \log 23 - \log 5371 - \log 29$$
$$= \log 452 + \log 23 + \text{colog } 5371 + \text{colog } 29.$$
$$\log 452 = 2.6551.$$
$$\log 23 = 1.3617.$$
$$\text{colog } 5371 = 6.2699 - 10.$$
$$\text{colog } 29 = 8.5376 - 10.$$
$$\text{Adding} \quad 8.8243 - 10.$$
$$\text{antilog } 8.8243 - 10 = 0.066728^{+}.$$

Therefore $$\frac{452 \times 23}{5371 \times 29} = 0.066728^{+}.$$

3. Find $50^{\frac{3}{4}}$.

$$\log 50^{\frac{3}{4}} = \tfrac{3}{4} \log 50.$$
$$\log 50 = 1.6990.$$
$$\tfrac{3}{4} \log 50 = \tfrac{3}{4} \text{ of } 1.6990 = 1.2742.$$
$$\text{antilog } 1.2742 = 18.80.$$
$$\therefore\ 50^{\frac{3}{4}} = 18.80.$$

644. Compound Interest. Problems in compound interest that involve long computations can readily be solved by means of logarithms.

To find the amount (A) at the end of n years of a given sum of money (P) invested at compound interest at a given rate (r):

The amount of P dollars at compound interest, at the end of the first year is, $A = P + rP = P(1 + r)$.

At the end of the second year,

$$A = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

At the end of the third year,

$$A = P(1 + r)^2 + rP(1 + r)^2 = P(1 + r)^3.$$

At the end of the nth year,

$$A = P(1 + r)^{n-1} + rP(1 + r)^{n-1} = P(1 + r)^n.$$

What will be the amount of $1500 for 12 years at 4%, the interest being compounded annually?

$$\begin{aligned} \text{Here } A &= 1500(1+.04)^{12} \\ \log A &= \log 1500 + 12 \log 1.04 \\ &= 3.1761 + 12 \times .0170 \\ &= 3.3801. \\ \therefore A &= \$2405.55. \end{aligned}$$

EXERCISE

645. **1.** Find from the tables the logarithm of each of the following numbers: (*a*) 74; (*b*) 129; (*c*) 2004; (*d*) 16.21; (*e*) 9.547; (*f*) .018; (*g*) .21; (*h*) $\frac{11}{13}$; (*i*) $\frac{41}{57}$; (*j*) $7\frac{8}{9}$; (*k*) $\frac{43}{154.2}$.

2. Find the logarithm of each of the following numbers: (*a*) 7^5; (*b*) 212^{14}; (*c*) 3.171^4; (*d*) 31.2; (*e*) 918.4; (*f*) .00084; (*g*) 42.5^3; (*h*) $.1871^3$; (*i*) .00427.

3. Find from the tables the numbers corresponding to the following logarithms: (*a*) .7412; (*b*) 2.9983; (*c*) .9060; (*d*) .7033; (*e*) $\bar{4}.9883$; (*f*) 1.0881; (*g*) 3.6538; (*h*) $\bar{3}.5051$.

4. Perform the following operations by means of logarithms: (*a*) 256×311; (*b*) 451×215; (*c*) $7643 \div 213$; (*d*) $972 \div 41$; (*e*) $158 \times \sqrt[5]{.39}$; (*f*) $7^4 \times 4^{11}$; (*g*) 615×5^3; (*h*) $61^3 \div 17^4$; (*i*) $19^3 \times 8^{10}$; (*j*) $17\sqrt{29}$; (*k*) $\frac{41 \cdot \sqrt{613}}{153}$; (*l*) $36^{13} \times (\frac{9}{11})^{\frac{5}{6}}$.

5. At birth a child has $500 placed in the bank for him, to accumulate at 4% compound interest till he is 21. What amount will he receive when he is 21?

6. The first Folio of Shakespeare, regarded as the most valuable book printed in the English language, was published in 1623. The original cost was £1 or approximately $5. The last copy offered for sale in 1912 brought $9000. One

would naturally think that the purchaser of this first Folio in 1623 made a fine investment. What would an original investment of $ 5 amount to in 1912 at 6 % compound interest?

7. If at the beginning of the year 1, one cent had been invested at 4 % compound interest, what would the amount be in 1915? What would be the radius of a sphere of gold that would represent the value of the investment in 1916, if a cubic foot of gold is worth $ 362,900?

8. If $\log 2 = .3010$ find the value of x in the equation $2^x = 10$.

9. Compute the value of $3^{\sqrt{2}}$ by means of logarithms.

(Harvard.)

10. About 300 years ago the Indians sold Manhattan Island to Peter Minuit for $ 24. Suppose this money had been put out at compound interest at 6 %, how much would it have amounted to at the present time?

11. According to the will of Benjamin Franklin, the cities of Boston and Philadelphia each received £ 1000 in July 1791 to be invested at 5 % compound interest for 100 years. In July 1891 the total amount of the fund in Boston was $ 391,168.68 and in Philadelphia $ 100,000. How much should have been realized by the terms of the will? (£ 1000 = $ 5000.)

12. A chain of letters is started for the purpose of aiding an old railroad man who is ill. Number 1 sends a letter to each of 5 friends, each of them in turn sends a letter to 5 friends, and so on. If the chain ends with letter number 50 and each person who receives a letter sends 10 cents, how much does the man receive?

XXIX. GENERAL REVIEW

646. 1. If $a = 3$, $b = 2$, $c = 1$, find the value of each of the following expressions:

(1) $2a^2 - b^2$; $2(a^2 - b^2)$; $(2a^2 - b^2)^2$; $2(a^2 - b^2)^2$.

(2) $abc - (a + b + c)$.

(3) $(a^2 + b^2)(a + b)(a - b)$.

(4) $[a^2 + (b - c)a - bc](b - c)$.

(5) $\sqrt{(a + b + c)\,abc}$.

2. Verify the following identities:

(1) $ab = \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2$.

(2) $pq = \dfrac{(p+q)^2 - (p-q)^2}{4}$.

(3) $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$.

(4) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

3. Solve the following problems by translating the verbal language of the problem into an equation with one unknown:

(1) In five years a boy will be double the age he was five years ago. How old is he?

(2) I have as many brothers as sisters said a boy. And I, said one of his sisters, have twice as many brothers as sisters. How many brothers and sisters were there?

(3) Can there be three consecutive integers such that their sum is three times the smallest?

(4) The sum of three consecutive numbers is three times the middle number. What are the three numbers? Does this problem lead to an equation or to an identity?

4. Express in algebraic language the following theorems:

(1) The product of two numbers is equal to the difference of the squares of their half sum and their half difference.

(2) Every integer that is a perfect square diminished by unity is equal to the product of the number that is one less than its square root by the number that is one more than its square root.

(3) The difference between the squares of two consecutive integers is an odd number, obtained by increasing by unity twice the smaller of the two numbers.

5. The lengths of the sides of a triangle are $a = 5$ inches, $b = 4$ inches, $c = 3$ inches. Indicate the semi-perimeter by $s = \frac{a+b+c}{2}$, and find the area, A, of the triangle, if $A = \sqrt{s(s-a)(s-b)(s-c)}$.

6. If $2s$ represents the perimeter of a triangle and a, b, c its sides, verify the following:

(1) $-a + b + c = 2(s - a)$.
(2) $a - b + c = 2(s - b)$.
(3) $a + b - c = 2(s - c)$.
(4) $a = 2s - (b + c)$.
(5) $b = 2s - (a + c)$.
(6) $c = 2s - (a + b)$.

7. Having given the trinomials $r + s + t$, $r + s - t$, $r - s + t$, $-r + s + t$, from the sum of the first three subtract the sum of the last three increased by the sum of the second and third.

8. If $A = (p + q) + (r + s)$, $B = (p + q) - (r + s)$, $C = (p - q) + (r - s)$, $D = (p - q) - (r - s)$, find the value of $A + B + C + D$ and of $A \times D$ by type multiplication.

9. Prove that $[m - (p + q) + r] - \{m - [(p + q) - r]\} + \{m - (p + r) + q\} = m - p + q - r$.

10. Apply the general formulas to the following:

(1) $(p^3 - q^3)^2$.
(2) $(4a^2 - 5b^2)^2$.
(3) $(2a - z)(2a + z)$.
(4) $(1 + x)(x - 1)(x^2 - 1)$.
(5) $(a + b - c)(a - b + c)$.
(6) $(a^2 + a + 1)(a^2 - a + 1)$.
(7) $(n+2)(n+3)(n^2+5n+6)$.
(8) 103^2.

11. Show that $(1 + x + x^2 + x^3 + x^4 + x^5)(1 - x) = 1 - x^6$. Show also that $(1 - x + x^2 - x^3 + x^4 - x^5)(1 + x) = 1 - x^6$.

12. Divide $3x^{4p} + 14x^{3p} + 9x^p + 2$ by $3x^{2p} - x^p + 2$.

13. Divide $x^{3m} - x^{3n}$ by $x^{2m} + x^{m+n} + x^{2n}$.

14. What value must the coefficient k have in order that $x^4 - 5x^3 + 9x^2 + kx + 2$ may be exactly divisible by $x^2 - 3x + 2$?

15. Apply the general formulas to the following:

(1) $(16 - x^4) \div (x^2 + 4)$. (2) $(1 - a^3) \div (a - 1)$.
(3) $(n^9 + 64a^6b^{15}) \div (4a^2b^5 + n^3)$.
(4) $(m^9n^9 + 1) \div (m^3n^3 + 1)$.
(5) $[x^3 - (a - b)^3] \div (x - a + b)$.
(6) $[(p - q)^2 - (r - s)^2] \div (p - q - r + s)$.

16. Divide $a^2(b + c) - b^2(c + a) + c^2(a + b) + abc$ by $a - b + c$.

17. Square and cube each of the following:

$2a + 1$; $x - 7$; $3x - 5y$; $ax^2 + by^2$; $x + \frac{1}{2}p$; $\frac{1}{2}m - \frac{1}{2}n$.

18. Square each of the following:

$a^3 - a^2 + a - 1$; $x - y + z + 1$; $ax^2 + bx + c$.

19. Verify the following identities:

(1) $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$.
(2) $(a^2 + b^2 + c^2)^2 = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2$.
(3) $[(n + 2)^2 - (n + 1)^2] - [(n + 1)^2 - n^2] = 2$.
(4) Show that the identity $(n + 1)^2 - n^2 = 2n + 1$ expresses that the difference between the squares of two consecutive integers is always an odd number.

20. Evaluate each of the following expressions:

(1) $2 \times 5 + 12 \div 4 - 7 + 6 - \dfrac{4 \times 7}{6}$.

(2) $6 \times 7 - 3^2 \times 5 + \sqrt[3]{8} \times 5 - 7$.

21. Extract the square roots of the following polynomials:

(1) $3a^2 - 2a + 1 - 2a^3 + a^4$.
(2) $16x^8 + 9y^8 - 30x^2y^6 + 49x^4y^4 - 40x^6y^2$.

22. Factor into prime factors:

(1) $7a^4 + 7a^2b^2 - 14a^3b$. (2) $m^2(a-b) + n^2(b-a)$.

(3) $7pqx^2 - 42pqx + 63pq - 7prx^2 + 42prx - 63pr$.

(4) $xy - x + y - 1$. (5) $xx' + xx'' + x^2 + x'x''$.

(6) $x^3 + y^3 - x^2y - xy^2$.

(7) $a^2b + b^2c + c^2a + a^2c + b^2a + c^2b + 3abc$.

(8) $7x^2 - 28x^4$.

(9) $-2uv - u^2 - v^2 - 2uw - 2vw - w^2$.

(10) $x^3(x^2-y^2) - y^3(x^2-y^2) - xy(x-y)^2(x+y)$.

(11) $(a^2+b^2)^2 - (a^2-b^2)^2$.

(12) $x(x-1) - (x-1)^2 + x^2 - 1$.

(13) $a^3 - x^3 + a^2x - ax^2 - a + x$.

(14) $xy^7 - x^7y$. (15) $x^8 + 10x^4 + 21$.

(16) $5a^2x^4 - 5a^4x^2 - 5a^2b^2x^2 + 5a^4b^2$.

(17) $x^4 - 4nx^3 + 6n^2x^2 - 4n^3x + n^4$.

(18) $3x^3 - 16x^2 - 3x + 16$. (22) $x^2 + 2ax - b^2 + a^2$.

(19) $x^2 - 7x + 10$. (23) $x^4 - 3a^2x^2 + a^4$.

(20) $3x^2 + 12x + 9$. (24) $x^6 - (a^3+b^3)x^3 + a^3b^3$.

(21) $9x^2 - 12x - 5$. (25) $ax^2 + (a+b)xy + by^2$.

(26) $4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2$.

23. Solve the following equations by factoring:

(1) $x^2 - (a+b)x + ab = 0$. (6) $x^3 - 2x^2 - 4x + 8 = 0$.

(2) $p^2 + 3p + 2 = 0$. (7) $3x^3 + 7x^2 = 3x + 7$.

(3) $t^3 - t^2 = 9t - 9$. (8) $v^3 + v^2 - v - 1 = 0$.

(4) $z^2 + z - 30 = 0$. (9) $s^4 + s^2 - 12 = 0$.

(5) $x^2 + \frac{1}{6}x - \frac{1}{6} = 0$. (10) $k^3 + k^2 = 0$.

(11) Find a number such that if 3 and 5 are subtracted from it in turn, the product of the two remainders is 120.

(12) Find two numbers such that their difference is 2 and the sum of their squares is 130.

24. Find the H. C. F. of $x^2 - 3x + 2$, $x^2 - 2x + 1$, $x^2 + x - 2$.

25. Find the H. C. F. of $x^2 + 2x + 1$, $x^4 - 10x^2 + 9$,
$x^3 + 2x^2 - 5x - 6$.

26. Find the L. C. M. of $x^2+(a+b)x+ab$ and $x^2+(a-b)x-ab$.

27. Find the L. C. M. of p^3+q^3, p^2-q^2, $p^2+2pq+q^2$.

28. Simplify the following fractions:

(1) $\dfrac{(x+a)^2-(b+c)^2}{(x+b)^2-(a+c)^2}$.

(2) $\dfrac{x^2-6x+5}{x^2-11x+10}$.

(3) $\dfrac{30x^2-18x-12}{16x^2+4x-20}$.

(4) $\dfrac{2x^3-7x^2+7x-2}{3x^3-10x^2+9x-2}$.

29. Reduce each of the following groups of fractions to groups having a common denominator:

(1) $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$.

(2) $\dfrac{m}{a+b}, \dfrac{n}{a-b}, \dfrac{p}{a^2-b^2}$.

(3) $\dfrac{p+1}{p^2-8p+7}, \dfrac{p-1}{p^2-6p-7}, \dfrac{p-7}{p^2-1}$.

30. Show that $\dfrac{x^2+8x+15}{x^2+7x+10}-\dfrac{x-1}{x+2}=\dfrac{4}{x+2}$.

31. Show that $\dfrac{1}{1-\dfrac{1}{a}}-1-\dfrac{1}{a(a-1)}=\dfrac{1}{a}$.

32. Show that $\dfrac{n+1}{2n+3}-\dfrac{n}{2n+1}=\dfrac{1}{(2n+1)(2n+3)}$.

33. Show that $\dfrac{a}{b}-\dfrac{a}{a+b}=\dfrac{a}{b}\times\dfrac{a}{a+b}$. From this relation find two numbers such that their product is equal to their difference.

34. Simplify the following:

(1) $\dfrac{a^2-9b^2}{c^2-4d^2}\times\dfrac{c-2d}{a-3b}$.

(2) $\left(1+\dfrac{x}{y}\right)\left(1-\dfrac{x}{y}\right)\left(1+\dfrac{x^2}{y^2}\right)$.

(3) $\left(\dfrac{m+n}{m-n}+1\right)\left(1-\dfrac{m-n}{m+n}\right)$.

(4) $\left(\frac{a}{b}-\frac{b}{a}\right)^2.$ (5) $\left(\frac{x}{y}+\frac{y}{x}\right)^2.$

(6) $\frac{x^2+3x+2}{x+3} \times \frac{x+2}{x^2+4x+3} \times \frac{x^2+6x+9}{x^2+4x+4}.$

(7) $\frac{1}{6a^2}+\frac{1}{3a}.$ (10) $a+\cfrac{1}{b+\cfrac{1}{c}}.$

(8) $\frac{x^2-y^2}{x^2+2xz+z^2} \div \frac{x+y}{x+z}.$ (11) $\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x}}}.$

(9) $\frac{a^2x^2-x^4}{a^3-x^3} \div \frac{ax^2+x^3}{a^2+ax+x^2}.$ (12) $\cfrac{ab}{a-\cfrac{ac}{b+c}}.$

35. If $y=\frac{1-z^2}{1+z^2}$ and $z=\frac{1-x}{1+x}$, express y in terms of x.

36. If $x+\frac{1}{x}=s$, show that $x^2+\frac{1}{x^2}=s^2-2$ and $x^3+\frac{1}{x^3}$ $=s^3-3s$.

37. Show that $\left\{\left[\left(\frac{1}{x^8-y^8} \div \frac{1}{x^4+y^4}\right) \div \frac{1}{x^2+y^2}\right] \div \frac{1}{x+y}\right\}$ $\div \frac{1}{x-y}=1.$

38. How much water must be added to 80 pounds of a 5 per cent salt solution to obtain a 4 per cent solution? (Yale.)

39. Simplify $\left[x+y-\cfrac{1}{x+y-\cfrac{xy}{x+y}}\right]\frac{x^3-y^3}{x^2-y^2}.$ (Cornell.)

40. What is the price of eggs when 2 less for 24 cents raises the price 2 cents a dozen? (Yale.)

41. What values of x will make the product $(x-a)(x-b)$ $(x-c)$ equal to zero?

42. Factor $x^3 + 10x^2 + 21x$ and indicate the values of x that will make the expression zero.

43. Simplify the expression:

$$\tfrac{1}{6}[x(x+1)(x+2) + x(x-1)(x-2)] + \tfrac{3}{2}(x-1)x(x+1).$$

44. Divide $(x^3-1)a^3 - (x^3+x^2-2)a^2 + (4x^2+3x+2)a - 3(x+1)$ by $(x-1)a^2 - (x-1)a + 3$.

45. Show that $\frac{1}{4}(x^2+y^2) + z^2 - \frac{1}{2}xy + xz - yz$ becomes $(y-z)^2$ or $(z-y)^2$ when $-x = y$.

46. By what transformation can $a(x-b)$ be put into the form $(a+b)x - (a+x)b$?

47. Solve the following equations:

(1) $2(x-1) = 6$.

(2) $13(12-z) = 14$.

(3) $3(x-5) + 8 = 17$.

(4) $5x + (7-2x) = 11$.

(5) $8(37-5x) = 4(3x-17)$.

(6) $28 + 2y - 16y - 6y - 12 + 2y = 0$.

(7) $9x + 22 - 2x = 193 - 22x - 84$.

(8) $5x - .3x = 4.5x + 2$.

(9) $.9x - 1.5x = x - 3.5$.

(10) $.25x + .943 = 1.9x - 6.812$.

(11) $.15x + 1.575 - .875x = .0625x$.

(12) $1.111 - .1111x = .3333$.

(13) $.5x + 2 - \frac{3}{4}x = .4x - 11$.

48. Solve the following equations:

(1) $x + 5x - b = 2a$.

(2) $3a + 2z - 4b = 5z - b$.

(3) $k(k + 3acx + 3) = kx + 3abk - k^2 - ackx$.

(4) $(x-2a)^2 + (x+2b)^2 = 2(x-2c)^2$.

(5) $x - \dfrac{mx}{n} = p$.

(6) $\dfrac{p+y}{p} + q = \dfrac{q+y}{q} + m$.

(7) $\dfrac{m}{2x} + p = \dfrac{n}{3x} + q$.

(8) $\dfrac{\dfrac{m+n}{x}}{\dfrac{1}{m}} = \dfrac{p}{q}$.

(9) $\dfrac{5x}{3a+b} - 2 = \dfrac{8b}{2a}$.

49. Form a proportion with the numbers 75, 18, 27, 50.

50. Knowing three terms of a proportion, how can the fourth be found?

51. Solve each of the following proportions for x:

(1) $\frac{75}{10}=\frac{57}{x}$.

(2) $\frac{p^2-q^2}{a+b}:\frac{(p+q)^2}{a^2-b^2}=\frac{a-b}{p+q}:x$.

(3) $3.15:x=6.75:20$.

(4) $x:3\frac{3}{4}=4\frac{1}{2}:11\frac{1}{4}$.

(5) $\frac{3a}{5b}:\frac{12a}{7c}=\frac{14c}{15b}:x$.

52. Form as many proportions as possible from each of the following equations:

(1) $xy=vt$.

(2) $m^2=rs$.

(3) $(a+b)^2=m^2-n^2$.

(4) $x^2=a^2-b^2$.

53. Find a fourth proportional to each of the following sets of numbers:

(1) 27, 90, 45. (2) p, q, r. (3) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

54. Find a mean proportional between each of the following:

(1) $\frac{ay}{b}, \frac{ab}{y}$. (2) $\frac{2(a^2-ab)}{35b}, \frac{-10a}{7(ab-b^2)}$.

55. Find a third proportional to each of the following:

(1) 8, 9. (2) $\frac{36a^2b^2}{(a^2-b^2)^2}, \frac{4(a^2-ab)}{b(a+b)^2}$.

56. If $\frac{a}{b}=\frac{c}{d}$, prove each of the following relations:

(1) $\frac{a\pm b}{b}=\frac{c\pm d}{d}$.

(2) $\frac{a\pm b}{a}=\frac{c\pm d}{c}$.

(3) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

(4) $\frac{a+c}{b+d}=\frac{a}{b}$.

57. Find the values of x and y in each of the following:

(1) $\frac{x}{y}=\frac{4\frac{1}{2}}{3\frac{3}{4}}$, when $x+y=9$.

(2) $\frac{x}{y}=\frac{7}{8}$, when $x+y=15$.

(3) $x:y=3.5:4$, when $x-y=2.5$.

58. Combine each of the following proportions so as to eliminate x and leave the new proportion in its simplest form:

(1) $\frac{c}{d}=\frac{b}{x}$, $\frac{d}{g}=\frac{x}{f}$.

(2) $\frac{l}{m}=\frac{n}{x}$, $\frac{x}{p}=\frac{v}{r}$.

(3) $\frac{a+b}{a-b}=\frac{x}{(c+d)^2}$, $\frac{a}{14b}:x=\frac{3c}{7b}:\frac{2c}{a}$.

59. The ratio of the sun's diameter to the earth's is 542 : 5; of the earth's to the moon's, 11 : 3. Find the ratio of the diameter of the sun to that of the moon.

60. The age of a father to that of his son is as 7 to 4. What is the age of each if the father is 24 years older than his son?

61. If in the composition of powder, the ratio of niter to carbon is 31 : 9 and of carbon to sulphur is 9 : 10, how much of each must be used to make 1200 pounds of powder?

62. Solve by two methods: $2x+y=11$, $3x-y=4$.

63. Solve the following systems of equations:

(1) $15x-7y=9$,
$9y-7x=13$.

(2) $4x+9y=106$,
$8x+17y=198$.

(3) $3y-4x-1=0$,
$18-3x=4y$.

(4) $5+4x=16y$,
$5x+28y=19$.

(5) $\frac{7}{6}x+\frac{5}{3}y=34$,
$\frac{7}{8}x+\frac{1}{8}y=12$.

(6) $2\frac{4}{9}y-\frac{5}{6}x=90$,
$2\frac{1}{2}x+\frac{2}{3}y=90$.

(7) $\frac{3x-5y}{2}+3=\frac{2x+y}{5}$,
$8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$.

(8) $.25x+3y=10$,
$4.5x-4y=6$.

(9) $25.9v-60.1u=1$,
$24.1v-55.9u=1$.

(10) $.2y+.25x=2(y-x)$,
$.8x-3.7y=-15.3$.

64. The formula for converting a temperature of F degrees Fahrenheit into its equivalent temperature C degrees centigrade is $C=\frac{5}{9}(F-32)$. Express F in terms of C, and compute F when $C=30°$; when $C=28°$.

65. Find two numbers such that their sum and difference are in the ratio $5:1$ and their sum to their product in the ratio $5:8$.

66. A servant is given \$2 to buy 10 pounds of sugar and 4 pounds of cheese and should have 60 cents left. She makes a mistake and buys 10 pounds of cheese and 4 pounds of sugar and lacks 24 cents. What is the price of cheese and of sugar?

67. Simplify the following:

(1) $a^{2m-n}a^{m+n}$.

(2) $(a+b)^m(a+b)^n$.

(3) $(p+q)^{a-b}(p+q)^b$.

(4) $(a^x+b^y)(a^x-b^y)$.

(5) $(m^x-n^y)^2$.

(6) $\dfrac{a}{b^x}+\dfrac{a}{b^{x-1}}$.

(7) $\dfrac{x}{m^{p+q}}+\dfrac{y}{m^{p+1}}-\dfrac{v}{m^p}$.

(8) $(-a)^3(-a)^5$.

(9) $\left(\dfrac{7-c}{m-5}\right)^4\left(\dfrac{5-m}{7-c}\right)^6$.

(10) $\left(\dfrac{8-a}{x-y}\right)^4\left(\dfrac{x-y}{a-8}\right)^3$.

(11) $\dfrac{4x^{p+1}}{5y^n}\times\dfrac{125y^{n-1}}{8x^p}$.

(12) $\left(\dfrac{a^2b^3}{c^4d^5}\right)^2$.

(13) $\dfrac{(a^6b^2)^2}{(a^5b^3)^4}$.

(14) $[(ax)^{3y+4z}]^{5y}$.

(15) $(x^{4p}-1)\div(x^p-1)$.

(16) $(x+y)^{a-b}:(x+y)^{b-a}$.

(17) $\dfrac{x^{4m+n}}{y^{2m-6n}}\times\dfrac{y^{6m+3n}}{x^{14n-13m}}$.

68. Perform the following operations and write the results so that each term shall have the integral form affected by negative exponents where necessary:

(1) $\left(\dfrac{1}{x^2}+\dfrac{a}{x}+a^2\right)\left(\dfrac{a}{x}-a^2+a^3x\right)$.

(2) $\left(\dfrac{2x^2}{a^2}-\dfrac{x}{a}-1-\dfrac{a}{x}-\dfrac{3a^2}{x^2}\right)\div\left(\dfrac{2a}{x}-\dfrac{3a^2}{x^2}\right)$.

69. Write the following expressions without using either the negative exponent, or the exponent zero, and simplify the results:

(1) x^0. (2) $\dfrac{1}{x^0}$. (3) u^{-0}. (4) $\dfrac{1}{m^{-0}}$. (5) a^5b^0. (6) $\dfrac{s^{-0}}{t^{-4}}$.

(7) m^0m^{-0}.

(8) $(a^0)^5(b^2)^{-2}$.

(9) $\dfrac{8a^{-5}}{a^2} \times \dfrac{(a^0+b^0)^{-3}}{a^{-10}}$.

(10) $\left(\dfrac{x}{y}\right)^0$.

(11) $(a^{2x-3})^{-2}$.

(12) $(m^{3x-5} - n^{5x-3})^{-x}$.

(13) $\left(\dfrac{a^{-1}+b^{-1}}{n^{-2}-m^{-2}}\right)^{-2}$.

(14) $\left(\dfrac{a^{-3x}+b^{-2y}}{a^{-6x}-b^{-4y}}\right)^{-3}$.

70. Verify the identity

$$\frac{a^2(b^{-1}-c^{-1})+b^2(c^{-1}-a^{-1})+c^2(a^{-1}-b^{-1})}{a(b^{-2}-c^{-2})+b(c^{-2}-a^{-2})+c(a^{-2}-b^{-2})} = \frac{a+b+c}{a^{-1}+b^{-1}+c^{-1}}.$$

71. Write the following expressions without using the radical sign or negative exponents:

(1) $\sqrt[3]{a^2}$. (2) $(\sqrt{a})^6$. (3) $\sqrt[4]{x^3}$. (4) $\sqrt{p^{a-b}}$.

(5) $\sqrt[3]{(a+b)^{m+3}}$. (6) $\sqrt[3]{a^3-3a^2b+3ab^2-b^3}$.

(7) $\sqrt{a^{-1}}$. (8) $\sqrt[p]{m^{-1}n^{-2}}$. (9) $\sqrt[m]{\dfrac{x}{y^2}}$.

(10) $\sqrt[b]{\sqrt[a]{a^m}}$. (11) $\sqrt[\frac{1}{2}]{a}$. (12) $\sqrt[9]{x^3}$.

(13) $\sqrt[m]{\dfrac{1}{x}}$. (14) $\sqrt{a^{-1}\sqrt[3]{x^2}}$.

72. Write the following expressions using only radical signs and positive integral exponents:

(1) $a^{\frac{1}{3}}$. (3) $e^{\frac{5}{6}}$. (5) $d^{\frac{p}{q}}$. (7) $c^{-\frac{3}{4}}$.

(2) $m^{\frac{3}{4}}$. (4) $y^{\frac{1}{m}}$. (6) $c^{\frac{m+2}{m+1}}$. (8) $e^{-\frac{m}{n}}$.

73. Simplify the following radical expressions:

(1) $2\sqrt{108a^4b^7}$.

(2) $\sqrt{7(14a-21b)}$.

(3) $\sqrt{(n^3-n)(n+1)}$.

(4) $\sqrt{(a^2+b^2)^2-(2ab)^2}$.

(5) $\sqrt{a^{2m+n}b^{2mn}c^{m+2n}}$.

(6) $\sqrt{\dfrac{a^2}{c^2}-\dfrac{2a}{c}+1}$.

(7) $\sqrt{98-7\sqrt{147}}$.

(8) $\sqrt{21+\sqrt{13+\sqrt{7+\sqrt{4}}}}$.

(9) $\sqrt{a\sqrt{a\sqrt{a^2}}}$.

(10) $\sqrt{bc-\dfrac{a^2bc}{(b+c)^2}}$.

(11) $4\sqrt{32}-5\sqrt{50}+3\sqrt{18}$.

(12) $(\sqrt{a}+3\sqrt{b})-(15\sqrt{a}-2\sqrt{b})+(4\sqrt{b}+7\sqrt{a})$.

(13) $\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}-\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$.

(14) $\sqrt{p}\times\sqrt{q}\times\dfrac{\sqrt{p}}{\sqrt{q}}$.

(15) $(\sqrt{p+q}-\sqrt{p-q})(\sqrt{p+q}+\sqrt{p-q})$.

(16) $\sqrt{n^2-n}\div\sqrt{n+1}$.

(17) $(x+y)\div\frac{1}{3}\sqrt{x^2-y^2}$.

(18) $\sqrt{\dfrac{a^2+a}{b^2+b}}\times\left(\dfrac{b^2-b}{a^2-a}\right)\times\sqrt{\dfrac{b^2-1}{a^2-1}}$.

74. Simplify the following imaginary expressions:

(1) $\sqrt{-50}+\sqrt{-25}+\sqrt{-18}+\sqrt{-1}-\sqrt{-4}+2\sqrt{-2}$.

(2) $\sqrt{-3}\sqrt{8}\sqrt{-6}$.

(3) $(x+\sqrt{-6})(x-\sqrt{-6})$.

(4) $\sqrt{1+\sqrt{-1}}\sqrt{1-\sqrt{-1}}$.

(5) $\sqrt{x-y}\sqrt{y-x}$.

(6) $\dfrac{\sqrt{10}}{\sqrt{-5}}$.

(7) $\dfrac{x+y}{x-y}\sqrt{\dfrac{-(x-y)^3}{(x+y)^3}}$.

(8) $(-1+\sqrt{-3})^3+(-1-\sqrt{-3})^3$.

(9) $\dfrac{36}{7+2\sqrt{-5}}$.

(10) $\dfrac{1-i}{1+i}+\dfrac{1+i}{1-i}$.

75. Solve the following equations:

(1) $\sqrt{5x}=20$.

(2) $\sqrt{x+9}=5\sqrt{x-3}$.

(3) $5-\sqrt{3y}=4$.

(4) $\sqrt{2v+8}=\sqrt{5v+2}$.

(5) $\sqrt[3]{10y-4}=\sqrt[3]{7y+11}$.

(6) $3\sqrt{16x+9}=12\sqrt{4x}-9$.

(7) $(x-3)^{\frac{1}{2}}+x^{\frac{1}{2}}=\dfrac{3}{(x-3)^{\frac{1}{2}}}$.

(8) $\dfrac{4}{x+\sqrt{4-x^2}}+\dfrac{4}{x-\sqrt{4-x^2}}=\dfrac{12}{7}$.

(9) $\dfrac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\dfrac{\sqrt{4x}+1}{\sqrt{4x}-1}$.

76. Solve the following quadratic equations by completing the square:

(1) $x^2-8x=-7$.

(2) $x^2+12=7x$.

(3) $x(x-1)=380.$

(4) $t^2+10t-56=0.$

(5) $250+2x^2=3x^2-15x.$

(6) $6x^2-\frac{7}{4}+4x=x+5x^2.$

(7) $(3x-2)^2=8(x+1)^2-100.$

(8) $(x-5)^2=4.$

(9) $3x^2+5x-42=0.$

(10) $2x^2-8=3x+12.$

(11) $5x(x-2)+2=-4-x(4x-5).$

(12) $3(z^2+2)^2-54=3z^4+z(5z-7).$

(13) $5(\frac{1}{2}+u)^2+4u=(3-u)^2+\frac{45}{4}.$

(14) $\frac{x-3}{x+4}+\frac{x-4}{2(x-1)}=\frac{1}{2}.$

(15) $v-3+\frac{v+6}{v-6}=2v-7.$

(16) $3m^2-6m=-\frac{5}{3}.$

(17) $5x^2-8x+3=0.$

(18) $(5x-2)(x+1)=(x-\frac{3}{5})5-5.$

(19) $(1.2-x)^2+(x+.8)^2=2(6x-.2)^2.$

(20) $\frac{s-2}{s-1}-\frac{s-3}{s+3}=\frac{s-4}{s-1}-\frac{7}{4}.$

77. Solve the following equations as quadratics:

(1) $x^6+4x^3=96.$

(2) $x^{\frac{10}{3}}-16x^{\frac{5}{3}}=512.$

(3) $ax^{11}+bx^9+cx^7=0.$

(4) $2x^3-6-x^{\frac{3}{2}}=0.$

(5) $\frac{1}{3}\sqrt{x^2+\frac{1}{2}x+8\frac{1}{2}}=\frac{1}{4}(63-2x^2-x).$

(6) $x^2-5x+2\sqrt{x^2-5x+3}=12.$

Hint. Add 3 to both members and treat x^2-5x+3 as the unknown.

(7) $2x^2-4x+3\sqrt{x^2-2x+6}=15.$

(8) $x^2+2\sqrt{x^2+6x}=24-6x.$

(9) $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

(10) $8+9\sqrt{(3x-1)(x-2)}=3x^2-7x.$

(11) $3x^2-7+3\sqrt{3x^2-16x+21}=16x.$

(12) $\sqrt{4x^2-7x-15}-\sqrt{x^2-3x}=\sqrt{x^2-9}.$

Hint. The factor $\sqrt{x-3}$ can be removed from each expression.

(13) $\sqrt{2x^2-9x+4}+3\sqrt{2x-1}=\sqrt{2x^2+21x-11}.$

(14) $x^2 + x + \frac{1}{x} + \frac{1}{x^2} = 4.$

HINT. $\left(x^2 + 2 + \frac{1}{x^2}\right) + x + \frac{1}{x} = 6$, or $\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 6 = 0.$

(15) $x^2 + x + \frac{2}{x} + \frac{4}{x^2} = 8.$

(16) $x^2 + x + \frac{1}{x} + \frac{1}{x^2} = 6\frac{3}{4}.$

(17) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0.$

HINT. Dividing by x^2, $12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0.$ Put $x + \frac{1}{x} = z$, then $x^2 + \frac{1}{x^2} = z^2 - 2.$

(18) $x^4 + x^3 - 4x^2 + x + 1 = 0.$

(19) $x^4 + 1 - 3(x^3 + x) = 2x^2.$

(20) $\sqrt{x^2 + x} + \frac{\sqrt{x - 1}}{\sqrt{x^3 - x}} = \frac{5}{2}.$

78. Solve the following systems of quadratic equations:

(1) $x^2 + y^2 = 10,$
$x - y = 2.$

(2) $x + y = 23,$
$x + xy = 144.$

(3) $x - y = 20,$
$x^2 - xy = 100.$

(4) $x^2 - 4y^2 = 9,$
$2x - y = 8.$

(5) $2v + 3u = 20,$
$3uv - u^2 = 38.$

(6) $x + y = a,$
$x^2 + y^2 = bxy.$

(7) $y - z = 1,$
$yz = 20,$
$u^2 + y^2 = 74.$

(8) $x^2y + xy^2 = 120,$
$x^3 + y^3 = 152.$

(9) $x^2 - y^2 = \frac{1}{4},$
$x + y = \frac{3}{2}.$

(10) $\frac{1}{x} + \frac{1}{xy} + \frac{1}{y} = 47,$
$\frac{1}{x} + \frac{1}{y} = 12.$

(11) $x + y + x^2 + y^2 = 1\frac{5}{8},$
$y - x + y^2 - x^2 = -1.$

(12) $x^3 - y^3 = 279,$
$x^2 + xy + y^2 = 93.$

(13) $\frac{1}{t^3} - \frac{1}{u^3} = 1304,$
$\frac{1}{t} - \frac{1}{u} = 8.$

(14) $x^3 + y^3 = 152,$
$x^2 - xy + y^2 = 19.$

(15) $2x^2 - 3xy + y^2 = 24,$
$3x^2 - 5xy + 2y^2 = 33.$

(16) $xy + x = 104,$
$xy - y = 84.$

(17) $5x + 2y = 29,$
$5xy = -105.$

(18) $3x^2 - y^2 = 83,$
$x + y = 15.$

(19) $x + y + 2\sqrt{x + y} = 8,$
$x^2 + xy = 8.$

(20) $xy = 80,$
$\frac{x}{y} = 5.$

(21) $2x - 5y = -18,$
$3xy = 264.$

(22) $x^2 - y^2 = 120,$
$x + y = 20.$

(23) $x^2 + y^2 = 250,$
$x - y = 22.$

(24) $x^2 + y^2 - x + y = 32,$
$2xy = 30.$

(25) $x^2y^2 + 3xy = 18,$
$x + y = 5.$

(26) $3x^2 - y^2 = 23,$
$2x^2 - xy = 12.$

(27) $x^4 + x^2y^2 + y^4 = 931,$
$x^2 + xy + y^2 = 19.$

79. If r_1 and r_2 are the roots of the quadratic equation $x^2 + px + q = 0$, show that $r_1 + r_2 = -p$ and $r_1r_2 = q$. Also show that $(x - r_1)(x - r_2) = 0$.

80. Form equations whose roots are the following:

(1) 3, 5. (2) $-2, 7$. (3) $-2, -\frac{1}{5}$. (4) 4.3, 2.5.
(5) 3, $\frac{1}{3}$.
(6) $4 + \sqrt{3},\ 4 - \sqrt{3}$,
(7) $\sqrt{7},\ \sqrt{3}$.
(8) $2 + \sqrt{-3},\ 2 - \sqrt{-3}$.
(9) 3, $\frac{1}{3}$, 4, $\frac{1}{4}$. (10) $1, -1, 3, -3$. (11) 3, 0, 1.

81. Show that the roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

82. (1) What determines the nature of the roots of a quadratic equation?

(2) When are the roots real?
(3) When are they imaginary?
(4) When are they real and unequal?
(5) When are they equal?

83. Determine whether the roots of the following equations are real or imaginary:

(1) $x^2 - x - 12 = 0.$

(2) $x^2 + 9x + 8 = 0.$

(3) $x^2 + 12x + 11 = 0.$

(4) $x^2 + 52x = 87.$

(5) $8x + 20 = x^2.$

(6) $16x - 63 = x^2.$

(7) $2x^2 - 18x + 65 = 0.$

(8) $2x + 5 = x^2.$

(9) $x^2 + 6x + 4 = 0.$

(10) $x^2 + 8x + 25 = 0.$

(11) $x^2 = 2 - 14x.$

(12) $x^2 - 31x + 246\frac{1}{2} = 0.$

84. Solve the following systems of quadratic equations:

(1) $x^2 - 3y^2 + 3x - 1 = 0,$
$3x - y + 13 = 0.$
(Yale.)

(2) $5x^2y^2 - 2 = 3xy,$
$x + 5y = 1.$
(Princeton.)

(3) $2x + y = 1,$
$5x^2 - y^2 = 2.$
(Princeton.)

(4) $\dfrac{x+y}{1-xy} = 2,$
$\dfrac{x-y}{1+xy} = \dfrac{1}{3}.$
(Princeton.)

(5) $x + y = 7,$
$(x-1)^2 + (y-2)^2 = 28.$
(Princeton.)

(6) $x^2 + xy = \frac{5}{12},$
$xy + y^2 = \frac{5}{18}.$
(Princeton.)

(7) $x - y - \sqrt{x-y} = 2,$
$x^3 - y^3 = 2044.$
(Yale.)

(8) $x^2 + y^2 = 13,$
$y^2 = 4(x-2).$
(Cornell.)

(9) $x^2 + y^2 = xy + 37,$
$x + y = xy - 17.$
(Columbia.)

85. Define arithmetical progression; common difference. What is meant by an increasing series? a decreasing series? If a certain term and the common difference are known, how can the preceding term be formed?

86. In any A. P. if a represents the first term, d the common difference, n the number of terms, l the last term, and s the sum of the terms, prove that $s = \frac{n}{2}(a + l) = \frac{n}{2}[2a + (n-1)d].$

87. Find l and s in the following series:

(1) 5, 8, 11, ⋯, to 12 terms.
(2) 1, 1.1, 1.2, ⋯, to 20 terms.
(3) $3n$, $5n$, $7n$, ⋯, to 36 terms.
(4) $5x$, $5x+3y$, $5x+6y$, ⋯, to 15 terms.

88. Given (1) $a=7$, $l=-3.5$, $n=36$, find d and s.
(2) $a=14.5$, $l=32$, $d=.7$, find n and s.
(3) $a=2$, $l=87$, $s=801$, find d and n.
(4) $a=-45$, $n=31$, $s=0$, find d and l.
(5) $l=11\frac{2}{3}$, $n=37$, $s=209\frac{2}{3}$, find a and d.
(6) $n=33$, $s=-33$, $d=-\frac{1}{3}$, find a and l.
(7) $a=9$, $d=4$, $s=624$, find n and l.
(8) $s=281\frac{1}{7}$, $d=\frac{3}{7}$, $l=15\frac{3}{7}$, find a and n.

89. If 142 and 149 are the last two terms of an A. P. and $n=22$, find a and s.

90. Insert 10 arithmetical means between 4 and 26.

91. Define geometrical progression. What is the ratio? If a certain term and the ratio are known, how can the preceding term be formed? the following term?

92. In any G. P. if a represents the first term, l the last term, n the number of terms, r the ratio, and s the sum of the terms, prove that

$$l=ar^{n-1} \text{ and } s=\frac{lr-a}{r-1}=\frac{a(r^n-1)}{r-1}=\frac{a(1-r^n)}{1-r}.$$

93. What does the last formula of example 92 become when the series becomes infinite $r<1$?

94. Find l and s in the following series:

(1) 4, 8, 16, ⋯, to 7 terms.
(2) 13, 1.3, .13, ⋯, to 7 terms.
(3) m^2, m^2n, m^2n^2, ⋯, to 10 terms.
(4) b, $\frac{b}{1-m}$, $\frac{b}{(1-m)^2}$, ⋯, to 9 terms.

95. Given (1) $a = 36$, $l = \frac{4}{9}$, $n = 5$, find r.
(2) $l = 128$, $r = 2$, $n = 7$, find a and s.
(3) $a = 3$, $l = 192\sqrt{2}$, $r = \sqrt{2}$, find s and n.
(4) $a = 10$, $l = \frac{5}{16}$, $s = 19\frac{11}{16}$, find r and n.

96. Find the two unknowns in each of the following:
(1) $a = 3$, $r = 3$, $s = 29{,}523$. (3) $l = 1296$, $r = 6$, $s = 1555$.
(2) $r = 2$, $n = 7$, $s = 635$. (4) $a = 18$, $n = 3$, $s = 1026$.

97. Find the sum of the series $9, -3, 1, -\frac{1}{3}, \cdots$, to infinity.

98. Find the sum to infinity of the series, $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \cdots$. Also find the sum of the positive terms. (Yale.)

99. Expand each of the following by the binomial formula:
(1) $(b + x)^4$. (2) $(a - x)^7$. (3) $(2x + 3a)^6$. (4) $(\sqrt{a} + x)^7$.

100. Find the 6th term in the expansion of $(3 + 2x^2)^9$ and the 7th term in the expansion of $(\frac{1}{2}a - x)^{17}$.

101. Expand $(2x^{\frac{1}{2}} - y^{\frac{1}{3}})^3$. (University of Michigan.)

102. Raise 98 to the 5th power by the Binomial Theorem. (Write $98 = 100 - 2$.) (Yale.)

103. Find the first three terms of $(1 + 2x)^8$ by the Binomial Theorem. (Sheffield Scientific School.)

104. Find the coefficient of x^3 and x^4 in the expansion of $(1 + 2x)^8$, using the Binomial Theorem.
(Sheffield Scientific School.)

105. Expand the expression $(x^{\frac{1}{2}} - x^{-\frac{1}{3}})^6$ and write the result in a form free from negative exponents. (Harvard.)

106. Define logarithm; base. In the equation $a^x = N$, what is x? what is a?

107. What are the logarithms of 3, 81, 243, 729 in a system of which the base is 3?

108. What are the logarithms of $\frac{1}{3}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$ in a system of which the base is 3?

109. Given $\log_{10} 3 = .477$; find $\log_3 10$, $\log_3 .1$, $\log_3 .01$.
(Sheffield Scientific School.)

110. Compute the value of x from the equation

$$x = \frac{(39.71)^3 \sqrt{13.16}}{(46.71)^4},$$

using logarithms. (Sheffield Scientific School.)

111. Perform the following operations, using logarithms:

(1) $9^5 \times 3^4$. (3) $\sqrt[9]{76245}$. (5) $15^4 \div 23^5$.

(2) $\sqrt[3]{4158}$. (4) $\frac{41^2\sqrt{613}}{153}$. (6) $\frac{318\sqrt[3]{79}}{14^5}$.

112. Solve the following equations:

(1) $2^x = 1024$. (3) $12^x = 20{,}737$.

(2) $10{,}000^x = 10$. (4) $31^{\frac{1}{x}} = 4$.

113. Write the roots of $(x^2 + 2x)(x^2 - 2x - 3)(x^2 - x + 1) = 0$.
(Sheffield Scientific School.)

114. Solve the equation $x^2 - 1.6x - .23 = 0$, obtaining the values of the roots correct to 3 significant figures. (Harvard.)

115. The distance s that a body falls from rest in t seconds is given by the formula $s = 16t^2$. A man drops a stone into a well and hears the splash after 3 seconds. If the velocity of sound in air is 1086 feet a second, what is the depth of the well? (Yale.)

116. A man spent \$539 for sheep. He kept 14 of the flock that he bought and sold the remainder at an advance of \$2 per head, gaining \$28 by the transaction. How many sheep did he buy and what was the cost of each? (Yale.)

117. Solve by factoring $x^3 + 30x = 11x^2$.
(Colorado School of Mines.)

118. Solve the equation $.03x^2 - 2.23x + 1.1075 = 0$.
(Colorado School of Mines.)

119. How many pairs of numbers will satisfy simultaneously the two equations $3x + 2y = 7$ and $x + y = 3$? Show by means of a graph that your answer is correct. What is meant by eliminating x in the above equations by substitution? by subtraction? (Colorado School of Mines.)

120. An automobile went 80 miles and back in 9 hours. The rate of speed returning was 4 miles per hour faster than the rate going. Find the rate each way. (Cornell.)

121. A goldsmith has two alloys of gold, the first being $\frac{3}{4}$ pure gold, the second $\frac{5}{12}$ pure gold. How much of each must he take to produce 100 ounces of an alloy which shall be $\frac{2}{3}$ pure gold? (Harvard.)

122. A man walked to a railway station at the rate of 4 miles an hour and traveled by train at the rate of 30 miles an hour, reaching his destination in 20 hours. If he had walked 3 miles an hour and ridden 35 miles an hour, he would have made the journey in 18 hours. What was the total distance traveled? (Mass. Institute of Technology.)

123. A page is to have a margin of 1 inch, and is to contain 35 square inches of printing. How large must the page be, if the length is to exceed the width by 2 inches?

(Mount Holyoke College.)

124. Factor the following expressions:

(*a*) $a^{\frac{3}{4}} - b^{\frac{3}{4}}$.

(*b*) $x^2y^2z^2 - x^2z - y^2z + 1$.

(*c*) $16(x + y)^4 - (2x - y)^4$. (Mount Holyoke College.)

125. If four quantities are in proportion and the second is a mean proportional between the third and the fourth, prove that the third will be a mean proportional between the first and the second. (Princeton.)

126. Solve $x^2 + y^2 - xy = 7$.

$x + y = 4$. (Smith College.)

127. The diagonal of a rectangle is 13 feet long. If each side were longer by 2 feet, the area would be increased by 38 square feet. Find the lengths of the sides. (Smith College.)

128. A field could be made into a square by diminishing the length by 10 feet and increasing the breadth by 5 feet, but the area would then be diminished by 210 square feet. Find the length and the breadth of the field. (Vassar College.)

129. Simplify $\dfrac{\sqrt{2}+2\sqrt{3}}{\sqrt{2}-\sqrt{12}}$, and compute the value of the fraction to two decimal places. (Yale.)

130. In going 7500 yards a front wheel of a wagon makes 1000 more revolutions than a rear one. If the wheels were each 1 yard greater in circumference, a front wheel would make 625 more revolutions than a rear one. Find the circumference of each. (Yale.)

131. In the expansion of $(2x-3x^{-1})^8$, find the term that does not contain x. (Princeton.)

COLLEGE ENTRANCE EXAMINATIONS

UNIVERSITY OF CALIFORNIA

Elementary Algebra

1. At a football game there were sixteen thousand persons. The number of women was six times the number of children and the number of men was three thousand less than twice the number of women. How many men, women, and children were there?

2. Solve: (*a*) $2x + 3 = 0$.
   (*b*) $3n + 2(n + 4) = 4n + 14$.

3. Factor: (*a*) $a^4 - 7x^2 + 12$.
   (*b*) $b^3 - 27$.
   (*c*) $(3x - 1)^2 - (x^2 + 4y^2 - 4xy)$.

4. The difference between two numbers is 14 and their product is 176. Find the numbers.

5. Solve for x: (*a*) $\frac{a}{b} = \frac{x}{c - x}$.
   (*b*) $x(x^2 - 4)(x^2 - 9) = 0$.

6. (*a*) Describe the method of locating a point on squared paper.
   (*b*) Construct a graph of (*a*) $2x = 5y + 10$, (*b*) $x = 5$.

7. Solve graphically: $\begin{cases} x + 2y = 4, \\ 2x + y = -1. \end{cases}$

CORNELL UNIVERSITY

Elementary Algebra

1. Multiply $1 + 2x - x^2 - \frac{1}{2}x^3$ by itself, and then find the value of the result if $1 - 2x = 3$.

2. What is the value of $x^3 + y^3$ if $x + y = 4$, and $2x^2 + 2y^2 = 17$?

3. (*a*) Add $\frac{1}{x}$, $\frac{1}{1-x}$, $\frac{3x-1}{(x+1)^2}$, and $\frac{-2}{x+1}$, and express the result as a fraction in its lowest terms.

(*b*) Rationalize the denominator of $\frac{\sqrt{x}-4\sqrt{x-2}}{2\sqrt{x}+3\sqrt{x-2}}$.

4. Find a root of $x^2 - x - 1 = 0$, and verify correctness of the result.

5. Solve:

$$\begin{aligned} 2x + 4y + 5z &= 19, \\ -3x + 5y + 7z &= 8, \\ 8x - 3y + 5z &= 23. \end{aligned}$$

6. A takes three hours longer than B to walk 30 miles; but if A doubles his pace he takes two hours less than B. Find the rate at which A and B each walk.

7. Find the time between three and four o'clock when the minute and hour hands are opposite each other.

Intermediate Algebra

1. Solve for x and check results:

$$\frac{2x^2}{x^2-1} + \frac{x}{x-1} = \frac{x}{x+1} + 3.$$

2. Solve and check:

$$\begin{aligned} x + y + 2\sqrt{x+y-1} &= 25, \\ x - y + 3\sqrt{x-y+1} &= 9. \end{aligned}$$

3. For what values of m will the roots of $2m^2 + x^2 - 2mx + 4x - 5m + 4 = 0$ be real and distinct?

4. Evaluate

$$\frac{(81)^{-\frac{3}{4}} + (-27)^{-\frac{4}{3}}}{3(9)^{-\frac{3}{2}} + (27)^{-1}}.$$

5. Find the greatest common divisor of $x^4 + x^3 - x^2 - x$ and $x^4 + 4x^3 + 3x^2 - 4x - 4$. Also find the least common multiple.

6. What is the sum of $1 + 3 + 5 + \cdots + (2n-1)$, n being a positive integer? What is the least odd integer such that the sum of all the positive odd integers up to and including it will exceed 45,370?

7. From a thread whose length is equal to the perimeter of a square, 1 yd. is cut off. The remainder equals the perimeter of a square whose area is $\frac{4}{9}$ that of the first. What was the original length of the thread?

PRINCETON UNIVERSITY

Algebra A I

1. Simplify
$$\frac{1}{1-\dfrac{1}{2+\dfrac{x^2+3}{3-x^2}}} \div (x-3).$$

2. Simplify $\{x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}(x^{-1}y^{\frac{2}{3}})\}^{\frac{3}{2}}$; $3\sqrt{\frac{5}{2}}+\sqrt{40}+\sqrt{\frac{2}{5}}-\sqrt{\frac{1}{10}}$.

3. Factor
$$x^2-4\,ax-4\,b^2+8\,ab;$$
$$a^2+cd-ab-bd+ac+ad;$$
$$(x+1)(6\,x^2-x)-15(x+1).$$

4. Find the H. C. F. of
$$x^4-2\,x^3-3\,x^2-2\,x-4 \text{ and } x^4-x^3-7\,x^2-2\,x+4.$$

5. Solve
$$x+y+t=1,$$
$$2\,x+y+3\,t=4,$$
$$3\,x+y+7\,t=13.$$

6. A gave B as much money as B had; then B gave A as much money as A had left; finally A gave B half as much as B then had left. A ends with \$4 and B with \$36; how much had each originally?

Algebra A II

1. Solve (*a*) $x-\dfrac{a}{b}=\dfrac{b}{a}-\dfrac{1}{x}$;

(*b*) $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}$.

2. Solve the following equations, pairing the corresponding values of x and y and testing one solution in each case:

(*a*) $x+y=4$, $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{5}{2}$;

(*b*) $2(x^2+y^2)+x+y=11$, $xy=1$.

3. Show that the series whose terms are the reciprocals of the terms of a G. P. is a G. P.

How many terms of the progression $\frac{5}{3}, \frac{23}{12}, \frac{13}{6} \cdots$ must be taken to make the sum $36\frac{1}{2}$?

4. A earned \$6, and B, who worked 4 days more than A, earned \$14. Had their wages per day been interchanged they would together have earned \$19. How many days did each work?

YALE UNIVERSITY—SHEFFIELD SCIENTIFIC SCHOOL

Elementary Algebra

(Omit one question in 1–3 and one in 7–9)

1. Solve
$$3-\frac{1}{x+2}-\frac{3}{4x-6}=\frac{5}{2x^2+x-6}.$$

2. Solve and verify $\sqrt{5-2x}+\sqrt{15-3x}=\sqrt{26-5x}.$

3. Draw the graphs of the equations:
$$y=x-1,$$
$$y=x^2-4x+5.$$

Solve them simultaneously and explain the relation between the graphs and the solutions.

4. Simplify (a) $\dfrac{x+2}{x^2+4x+3}-2\left(1-\dfrac{2}{1+x}\right)+\dfrac{x}{27+x^3}.$

(b) $$\frac{m-n-\dfrac{2n(m-n)}{m+n}}{\dfrac{m^2+n^2}{mn+n^2}-1}.$$

5. Simplify (a) $\sqrt[4]{\dfrac{3}{8}}\div\sqrt[3]{\dfrac{9}{16}}.$ (b) $\dfrac{7m^{-3}n^{\frac{1}{2}}}{m^{\frac{2}{3}}(m-n)^0}\div m^{-1}n^{-\frac{1}{2}}.$

6. Write the first three and last three terms of the expansion of

$(a^{-1}+2a^{\frac{1}{2}})^8$ and simplify the result.

7. A and B started in business at the same time. The first year A lost \$5000, but during the second year gained 25% on the amount left at the end of the first year. B started with $\frac{3}{4}$ as much money as A and gained 20% the first year, but lost \$2050 the second year. He then had the same amount as A. How much had each at first?

8. A 13-foot ladder leaning against a building lacks 3 feet of reaching a window, while a 17-foot ladder with its base placed 3 feet farther from the wall just reaches it. How high is the window from the ground and how far was the bottom of the first ladder from the wall?

9. Divide \$700 between A, B, C, and D, so that their shares may be in geometrical progression and the sum of A's and B's shares equal to \$252.

APPENDIX

REMAINDER THEOREM, FACTOR THEOREM, AND SYNTHETIC DIVISION

647. Remainder Theorem.

1. Divide $x^2 - 5x + 8$ by $x - a$.

$$
\begin{array}{l|l}
x^2 - 5x + 8 & x - a \\
\underline{x^2 - ax} & x + (a - 5) \\
(a-5)x + 8 & \\
\underline{(a-5)x - a^2 + 5a} & \\
\quad a^2 - 5a + 8 = \text{Rem.} &
\end{array}
$$

2. Divide $x^2 - 5x + 8$ by $x - 2$.

$$
\begin{array}{r|l}
x^2 - 5x + 8 & x - 2 \\
\underline{x^2 - 2x} & x - 3 \\
-3x + 8 & \\
\underline{-3x + 6} & \\
2 = 2^2 - 5 \cdot 2 + 8 = \text{Rem.} &
\end{array}
$$

Note that the remainder in the first division is the same as the dividend except that a has replaced x. The remainder when dividing by $x - 2$ is the result of substituting 2 for x in the dividend.

648. If a rational and integral expression in x is divided by $x - a$ until the remainder does not contain x, the remainder is the expression obtained by substituting a for x in the dividend.

Proof. Call the dividend D_x, and let D_a be the result of putting a for x in the dividend. Call the quotient Q_x, and let Q_a be the result of substituting a for x in the quotient and let the remainder be R.

$D_x = (x - a) Q_x + R$. (Dividend = divisor × quotient + remainder.)

This equation is an identity; that is, it is true for all values of x. Substitute a for x in both members.

$$\text{Then } D_a = (a - a) Q_a + R.$$

$$\therefore\ D_a = 0 \cdot Q_a + R \text{ or } D_a = R. \qquad \text{Q.E.D.}$$

The student should note that R is not changed when a is substituted for x, since R does not contain x.

EXAMPLES

1. Find the remainder when $x^3 - x + 7$ is divided by $x - 3$.

SOLUTION. Substitute 3 for x.

$$3^3 - 3 + 7 = 31, \text{ the remainder.}$$

2. Find the remainder when $x^3 - 5x^2 + 2$ is divided by $x + 2$.

SOLUTION. $x + 2 = x - (-2)$.

Substitute -2 for x.

$$(-2)^3 - 5(-2)^2 + 2 = -8 - 20 + 2 = -26, \text{ the remainder.}$$

EXERCISE

649. *Find the remainder when each expression is divided by the binomial opposite it:*

1.	$x^2 - 3x + 8$,	$x - 1$.	8.	$x^3 - 2x + 3$,	$x + 1$.
2.	$x^3 - 2x + 3$,	$x - 2$.	9.	$x^3 + 1$,	$x + 1$.
3.	$2x^2 - 5x + 4$,	$x - 5$.	10.	$x^5 + 1$,	$x + 1$.
4.	$x^3 - 2x + 1$,	$x - 1$.	11.	$x^7 + 1$,	$x + 1$.
5.	$3x^3 - 17x - 30$,	$x - 3$.	12.	$x^3 - 1$,	$x - 1$.
6.	$y^3 - 5y + 4$,	$y - 3$.	13.	$x^3 + x^2 + x + 1$,	$x + 2$.
7.	$z^5 - 2z^3 + 3$,	$z - 1$.	14.	$x^3 + x^2 + x + 1$,	$x + 1$.

650. **Factor Theorem.** The student will observe that in some of the examples in § 649 the remainder was 0, and, therefore, the divisor was a factor of the dividend.

651. If a rational integral expression containing x vanishes (becomes equal to 0) when a is put for x, then $x - a$ is a factor of the expression.

PROOF. If the result of substituting a for x is 0, the remainder when the expression is divided by $x - a$ is 0, and the divisor is a factor of the dividend. (§ 648.)

EXAMPLES

1. Show that $x - 1$ is a factor of $x^3 - 2x + 1$.

SOLUTION. Substituting 1 for x gives

$1^3 - 2 \cdot 1 + 1 = 0$; $\therefore$ $x - 1$ is a factor.

By dividing $x^3 - 2x + 1$ by $x - 1$ the other factor is found to be $x^2 + x - 1$. $\therefore$ $x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$.

2. Factor $x^3 - 5x - 12$.

The number to substitute for x must be found by trial. Try different factors of 12.

$$3^3 - 5 \cdot 3 - 12 = 0.$$
$$\therefore x - 3 \text{ is a factor.}$$
$$x^3 - 5x - 12 = (x - 3)(x^2 + 3x + 4).$$

EXERCISE

652. *Factor by the factor theorem:*

1. $x^3 - 7x + 6$.
2. $x^3 - 8$.
3. $x^2 - 5x + 6$.
4. $x^2 - 5x - 6$.
5. $x^3 - 2x^2 - 9$.
6. $x^3 - 2x - 56$.
7. $3x^3 + x^2 - 28$.
8. $2x^3 + x^2 - x - 2$.
9. $x^4 + 2x + 1$.

Solution. All terms are positive, so no positive number need be tried. (Why ?)

$$(-1)^4 + 2(-1) + 1 = 0.$$
$$\therefore x - (-1) \text{ or } x + 1 \text{ is a factor.}$$
$$x^4 + 2x + 1 = (x + 1)(\quad ? \quad).$$

10. $x^3 + 3x + 14$.
11. $x^3 - 6x^2 + 11x - 6$.
12. $x^3 - 6x^2 + 13x - 10$.
13. $2x^3 + 3x + 5$.
14. $5x^4 - 21x - 38$.
15. $7x^3 + 9x + 16$.

653. Synthetic Division. Division of polynomials containing x by divisors of the form $x - a$ can be performed very expeditiously by the method known as synthetic division.

$$\begin{array}{r|l}
x^3 - 6x^2 + 11x + 2 & x - 2 \\ \cline{2-2}
\underline{\cancel{x^3} - 2x^2 \qquad\qquad} & x^2 - 4x + 3 \\
-4x^2 + \cancel{11x} \quad & \\
\underline{\cancel{-4x^2} + \ \ 8x \qquad} & \\
3x + \cancel{2} & \\
\underline{\cancel{3x} - 6} & \\
8 &
\end{array}$$

The coefficients of the first term and of each partial remainder, except the last, are the coefficients of the terms of the quotient. The terms crossed off could as well as not be entirely omitted from the work. The

x's could all be omitted, since the orderly arrangement of the work would enable us to replace each x with the proper exponent. The subtractions that we made in the original division, $-2x^2$ from $-6x^2$; $8x$ from $11x$; -6 from 2, can be changed to additions by using 2 instead of -2 as a multiplier.

The work is thus reduced to the following:

$$\begin{array}{r|l} 1-6+11+2 & 2 \\ 2-\ 8+6 & \\ \hline 1-4+\ 3 & +8, \text{ remainder.} \end{array}$$

1. Write the coefficients of the dividend in order with their signs and write the second term of the divisor with its sign changed to the right.

2. Bring down the coefficient of the highest degree term, 1, as the coefficient of the first term of the quotient.

3. Multiply this number by 2 and write the product, 2, under the second term and add. This gives -4, the coefficient of the second term of the quotient.

4. Multiply -4 by 2 and write the product under 11 and add.

5. Repeat this process to the end of the polynomial.

6. The first three numbers on the last line are the coefficients of the quotient $x^2 - 4x + 3$, and the last number is the remainder.

If any power of x from the highest down to the absolute term is missing, a zero coefficient must be supplied in its place in the detached coefficients.

Examples

1. Divide $x^4 + x^2 + 8$ by $x - 1$.

Division:

$$\begin{array}{r|l} 1+0+1+0+8 & 1 \\ 1+1+2+2 & \\ \hline 1+1+2+2 & +10, \text{ remainder.} \end{array}$$

The quotient is $x^3 + x^2 + 2x + 2$ and the remainder is 10.

2. Divide $x^3 + 6x^2 - 16$ by $x + 2$.

Division:

$$\begin{array}{r|l} 1+6+0-16 & -2 \\ -2-8+16 & \\ \hline 1+4-8\quad & 0 \end{array}$$

The quotient is $x^2 + 4x - 8$ and the division is exact.

EXERCISE

654. *Divide the following by synthetic division :*

1. $(x^2 - 9x + 20) \div (x - 1)$.
2. $(x^2 - 9x + 20) \div (x - 2)$.
3. $(x^2 - 9x + 20) \div (x - 4)$.
4. $(x^2 - 9x + 20) \div (x - 5)$.
5. $(x^3 - 9x + 20) \div (x - 2)$.
6. $(2x^3 - 9x + 3) \div (x - 3)$.
7. $(3x^3 - 5x + 2) \div (x - 2)$.
8. $(x^3 + 2x + 1) \div (x + 1)$.
9. $(x^3 + 1) \div (x + 1)$.
10. $(x^3 - 1) \div (x - 1)$.
11. $(x^3 - 1) \div (x + 1)$.

12. Factor $x^4 - 5x^3 + 3x^2 + 15x - 18$.

SOLUTION. Substituting 2 for x makes the expression vanish. Therefore $x - 2$ is a factor. Divide by $x - 2$.

$$\begin{array}{r|l} 1 - 5 + 3 + 15 - 18 & 2 \\ 2 - 6 - \ \ 6 + 18 & \\ \hline 1 - 3 - 3 + \ \ 9 \quad & \underline{0} \end{array}$$

The quotient $x^3 - 3x^2 - 3x + 9$ vanishes for $x = 3$. Divide by $x - 3$.

$$\begin{array}{r|l} 1 - 3 - 3 + 9 & 3 \\ 3 + 0 - 9 & \\ \hline 1 + 0 - 3 \quad & \underline{0} \end{array}$$

The quotient is $x^2 - 3$.

$\therefore x^4 - 5x^3 + 3x^2 + 15x - 18 = (x - 2)(x - 3)(x^2 - 3)$.

Factor the following:

13. $x^3 - 10x + 3$.
14. $2x^3 + 5x^2 - 4$.
15. $x^3 - x + 24$.
16. $x^3 - 13x^2 + 49x - 45$.
17. $x^3 + x^2 + x - 3$.
18. $x^3 + x + 2$.
19. $x^3 - 3x - 322$.
20. $x^3 + 5x + 150$.

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