ELEMENTARY

ANALYTICAL GEOMETRY.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

CAMBRIDGE SCHOOL AND COLLEGE TEXT BOOKS.

ELEMENTARY

ANALYTICAL GEOMETRY

BY THE REV.

THOMAS G. VYVYAN, M.A.

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE, AND MATHEMATICAL MASTER OF THE CHARTERHOUSE.

NEW EDITION, WITH ALTERATIONS AND ADDITIONS.

CAMBRIDGE:

DEIGHTON, BELL, AND CO.

LONDON: GEORGE BELL AND SONS.

1875.

PREFACE.

THIS work is intended for beginners, that is for those who have acquired some little familiarity with algebraical working, and know some elementary trigonometry.

The method of interpreting geometry by analysis has been used throughout the first six chapters instead of the method, perhaps more logical, but certainly more difficult, of taking the equations, and interpreting them geometrically.

This edition differs from the preceding in several important respects.

It has been almost entirely re-written. New chapters have been added on focal properties of conics and on abridged notation and trilinear co-ordinates: the central conics are discussed together, and the chapter on the general equation has been enlarged.

I have to thank the Rev W. Allen Whitworth for his kind permission, of which I have freely availed myself, to make use of his *Modern Geometry* in writing the chapter on abridged notation and trilinear coordinates.

Many new examples have been added to this edition: the exercises in the middle of the chapters are easy applications of the book-work: the examples at the end of each chapter are generally more difficult: as in the first edition, many have, by Mr Walton's kind permission, been selected from his *Problems in Plane Coordinate Geometry:* a considerable number have been set in various examinations at Cambridge during the last six years.

A List of Formulæ, which it is hoped will be convenient for reference, will be found at the end of the book.

CHARTERHOUSE,
August, 1875.

CONTENTS.

CHAPTER I.

The Point.

CHAPTER II.

The Straight Line.

CHAPTER III.

The Circle.

CHAPTER IV.

Conic Sections. Focal Properties.

CHAPTER V.

The Parabola.

CHAPTER VI.

Central Conics.

CHAPTER VII.

General Equation.

CHAPTER VIII.

ABRIDGED NOTATION: TRILINEAR COORDINATES.

ERRATA.

Page 26, q. 4, *for* $(r, \theta-\pi)$ *read* $(-r, \theta-\pi)$.

„ 148, Art. 125, second line,

for $SP=ex_1+a$ read $SP=ex_1-a$.

„ 150, last line but one,

for $\dfrac{\cos^2\alpha}{a^2}+\dfrac{\sin^2\beta}{b^2}$ *read* $\dfrac{\cos^2\alpha}{a^2}+\dfrac{\sin^2\alpha}{b^2}$.

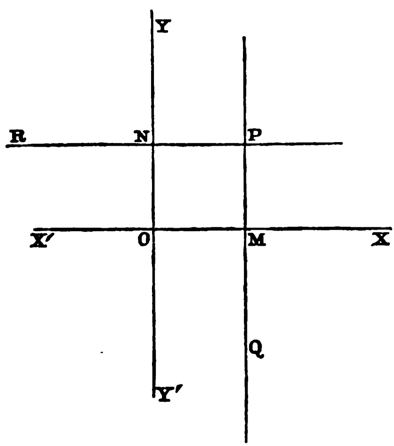
„ 188, q. 33, *for* $\frac{1}{2}$ *read* $\frac{1}{12}$.

ANALYTICAL GEOMETRY.

CHAPTER I. *The Point.*

1. IN Analytical Geometry we connect the sciences of space and number; we determine equations which represent certain well-known lines and curves, and deduce their geometrical properties from those equations; or, having given equations, we discuss the nature of the curves represented by them.

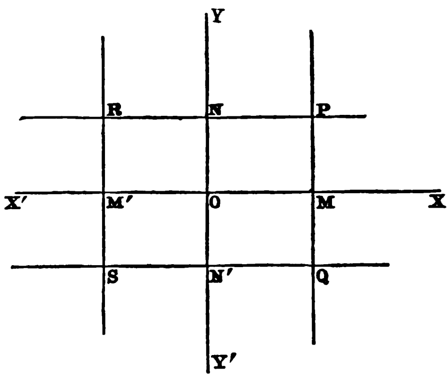
2. First it is necessary to shew how the relative distances of points in a plane may be represented by algebraical symbols.



Let O be any point in the plane: through O draw any straight line XOX' of unlimited length, and draw YOY', also of unlimited length, perpendicular to XOX': in OX take any point M, and through M draw a straight line parallel to YOY'; it is evident that all points in this line are at the same distance OM from YOY'.

Let us denote distance from YOY' by the symbol x; then all points in the line PMQ have the same value of x, and if we call that value a, the equation $x=a$ is true for all points on that line.

Similarly, if through any point N on YOY' we draw RNP parallel to XOX', all points in this line are at the



same distance from XOX'; and if we denote distance measured from XOX' by the symbol y, and $ON=b$, the equation $y=b$ is true for all points on RNP.

At the point P, at which these lines intersect, $x=a$, $y=b$.

3. If, however, we take points M', N' on OX', OY', such that $OM'=OM$, $ON'=ON$, and through M', N' draw straight lines parallel to YOY', XOX' respectively, it would seem that at each of the points P, Q, R, S, where these 4 lines intersect, $x=a$, $y=b$: it is necessary therefore to adopt some convention or rule to distinguish between these 4 points.

Let us suppose that all lines drawn in any definite direction are considered positive; then those drawn in the opposite direction will be negative.

Let all lines drawn to the right of YOY' parallel to XOX' be positive, then those to the left will be negative; thus if $OM = a$, $OM' = -a$.

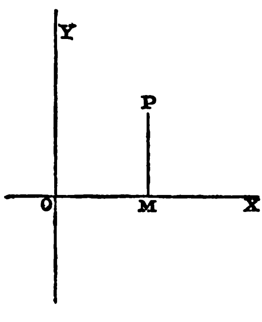
Similarly, let lines drawn parallel to YOY' above XOX' be positive, then those drawn below will be negative; thus if $ON = b$, $ON' = -b$.

If then, as before, x, y denote distance from YOY', XOX' respectively, then at P, $x = a$, $y = b$, at Q, $x = a$, $y = -b$, at R, $x = -a$, $y = b$, and at S, $x = -a$, $y = -b$.

4. *Axes, Coordinates, System of Coordinates.*

We can now explain what we mean when we speak of the coordinates of a point.

Let O be a fixed point in a plane XOY: XOX', YOY' two straight lines in that plane at right angles to each other, P any point in the plane.



Through P draw PM perpendicular to OX, cutting it in M; then it is evident that if OM, PM are known in magnitude and direction or sign, the position of P is completely determined.

OM is obviously equal to the perpendicular from P on OY.

OM, PM are called the coordinates of P, and are denoted by x, y respectively: O is called the origin, OX the axis of x, since x is measured along it, OY the axis of y, since y is measured parallel to it.

OM, PM are said to be the coordinates of P belonging to the system XOY.

The point P, whose coordinates are x, y, is often called the point (xy): thus, a point, for which $x=a$, $y=b$, is called the point (ab).

When the point P is not completely determined, its coordinates are denoted by the variables x, y; x', y'; X, Y; &c. When the position of a point is completely known, the coordinates are generally denoted by the letters a, b; h, k; or by x, y with suffixes such as x_1, y_1; x_2, y_2; &c.

Thus, if we want to determine the position of a point with reference to fixed points we shall use x, y for the coordinates of the *unknown* point, and a, b; h, k; x_1, y_1; &c., for the *known* coordinates of the fixed points.

A system of axes may be either rectangular or oblique; that is, the angle YOX may be either a right angle, or an oblique angle.

We shall, in future, always suppose the axes to be rectangular, unless the contrary is stated.

When the angle YOX is not a right angle, it is usually denoted by the symbol ω.

The student is now recommended to take a piece of paper, and draw two straight lines on it at right angles to each other, to measure off distances from the point of intersection along these lines equal to $\frac{1}{4}$ inch, and draw straight lines parallel to the original lines from these points, thus dividing the paper into small squares; he will then be able to do the following exercise.

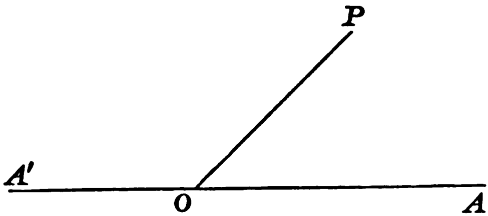
Ex. 1. 1. Let 1 represent $\frac{1}{4}$ inch; indicate by a figure the relative positions of the following points: $(3, 1)$, $(2, 2)$, $(4, 5)$, $(3, -1)$, $(-1, 3)$, $(-3, -1)$, $(0, 2)$, $(0, -3)$, $(4, 0)$, $(0, 4)$, $(-2, 0)$, $(0, 0)$.

2. Take any lengths OA, OB along the axes of x and y respectively, let $OA=a$, $OB=b$; determine the position of the points $(0, a)$, $(0, b)$, $(a, 0)$, (a, b), $(-a, b)$, $(2a, -3b)$, $(-3a, 0)$, $\left(\frac{a}{2}, -b\right)$.

5. *Polar Coordinates.*

There is another kind of system of coordinates which is often useful. Let O be, as before, a fixed point, AOA' a fixed straight line, P any point. Join OP.

It is evident that P is known in position if we know its distance from O, and the angle that distance makes with OA.



Thus, if we denote the distance OP by r, and the angle POA by θ, the position of P is determined if r and θ are known.

r, θ are called the polar coordinates of P; O is called the *pole*, OA the *initial line*, OP the *radius vector of P*.

As in Trigonometry, the angle POA is considered positive when measured in the direction opposite to that of the order of figures on a watch, negative when in that direction.

Ex. 2. Indicate by a figure the relative position of the following points, when a represents $\frac{1}{2}$ inch;

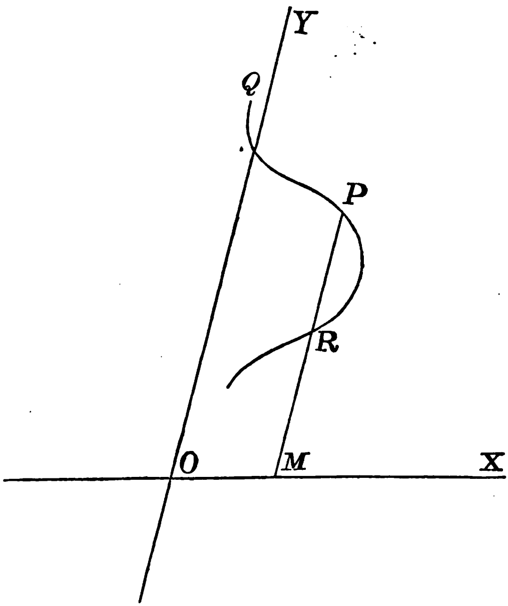
$$(a, 0),\ \left(a, \frac{\pi}{2}\right),\ (2a, 30^0),\ \left(a\cos\frac{\pi}{3}, \frac{\pi}{3}\right),\ \left(5a, \tan^{-1}\frac{4}{3}\right),$$

$$\left(-5a, \tan^{-1}\frac{3}{4}\right),\ (-a, 30^0),\ (a, 210^0).$$

6. *Equation to a curve; Locus of an Equation.*

We will now explain how curves may be represented by indeterminate equations between the variables x, y, or r, θ.

Let QPR be any curve, then as we pass along it from point to point, it is evident that we get at each point different values of x and y: that is, the values of x and y are not determined by saying that they belong to some point on the curve QPR.



If however we take a particular point P on the curve such that x is of determinate value, the value of y becomes determinate also. There may, it is true, be more than one such value of y, but however many there are, they are determined by the fact of determining x.

Hence for every curve there must be some determinate relation between x and y, which may be represented by an equation; this equation is called the equation to the curve.

The curve is said to be the *locus* of the points whose coordinates are connected by the equation, or, more shortly, the *locus of the equation*, because those pairs of values of x and y which simultaneously satisfy the equation, are the coordinates of any point on the curve.

We shall hereafter determine the equations which belong to certain well-known lines and curves, and discuss the geometrical meaning of the indeterminate equations of the first and second degrees, namely

$$Ax + By + C = 0,$$

and $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

7. First, let us consider the meaning of equations in which only one of the variables x and y is involved.

Suppose we have the equation $x = a$; since the distance of all points represented by this equation from the axis of y is constant, these points must lie on a straight line at that distance parallel to that axis: hence, $x = a$ is the equation to a straight line parallel to the axis of y.

Similarly, $y = b$ is the equation to a straight line parallel to the axis of x.

Next, suppose we have the quadratic equation,

$$x^2 + px + q = 0.$$

Let a_1, a_2, be the roots of this equation.

Then for every point on the locus of this equation, either $x = a_1$, or $x = a_2$, and therefore this point lies on one of the lines represented by the equations $x = a_1$, $x = a_2$. Similarly, since every equation involving one variable has as many roots as it has dimensions, it must represent a series of straight lines parallel to one of the axes, at distances from that axis equal to the different roots.

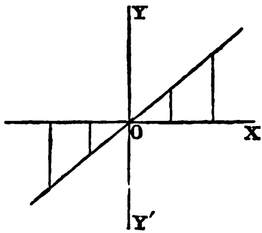
8. We can often discover the shape of the locus of any equation by giving a series of values such as -1, 0, 1, 2 &c. to x or y and tracing the corresponding values of y or x.

For this purpose it is necessary as before to take paper ruled in squares.

We will give some examples.

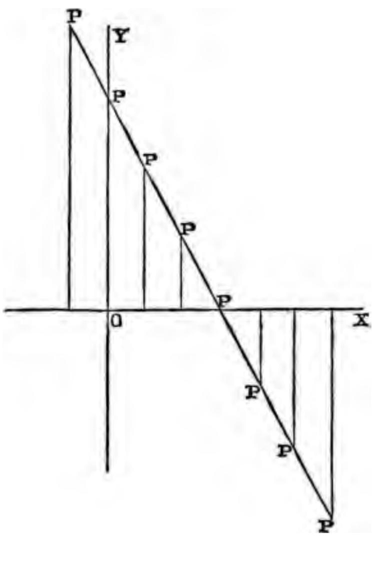
(1) $x=y$.

Give x the values -2, -1, 0, 1, 2 in succession, then we



get the same value of y, and observe that all the points so found lie on the line which bisects YOX.

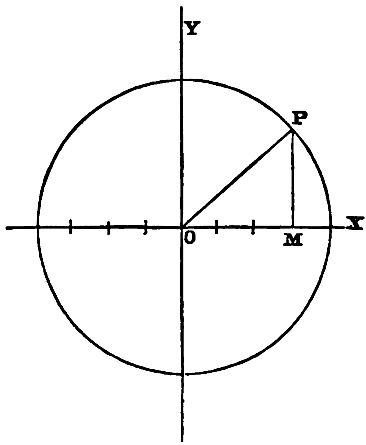
(2) $2x+y=6$.



Give x the values -1, 0, 1, 2, 3, 4, 5, 6, then we get for y, 8, 6, 4, 2, 0, -2, -4, -6. Hence by drawing lines as in the figure the points marked P lie on a straight line.

(3) $x^2+y^2=16$.

Here, if either x^2 or $y^2>16$, the other variable becomes



impossible: this shews that the whole figure lies within the square formed by the four lines $x=4$, $y=4$, $x=-4$, $y=-4$.

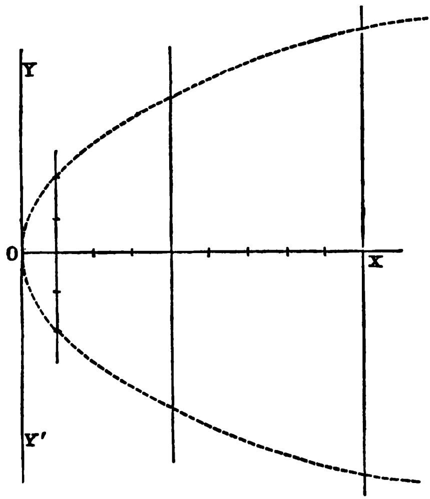
Again, if P be a point in the figure $OM^2+PM^2=OP^2$; $\therefore OP^2=16$, $OP=4$.

This shews that P must be a point on the circle whose centre is O and radius 4.

(4) $y^2=4x$.

Here if x is negative y is impossible, no part of the curve therefore lies to the left of YOY'.

For every value of x there are two equal and opposite values of y: the curve therefore is divided into two exactly equal parts by the axis of x.



Now give x the values 0, 1, 4, 9, 16, &c. in succession, then the values of y are

$$0, \ \pm 2, \ \pm 4, \ \pm 6, \ \pm 8, \ \&c.$$

As x increases y increases without limit.

The dotted line in the figure will therefore represent the curve.

Ex 3. 1. Give to x values from -4 to $+4$, and trace the following loci;

$$y=2x. \quad 2y+x=0, \quad x+y=6, \quad \frac{x}{4}+\frac{y}{2}=1,$$

$$x^2+y^2=4, \quad xy=1, \quad y^2=\frac{x}{4}, \quad \frac{x^2}{4}+y^2=1.$$

2. Give to y values from 0 to 6, and so trace the following loci;

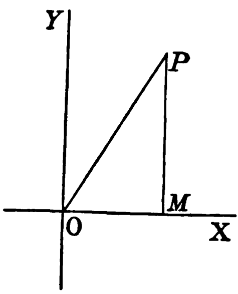
$$3y+4x=12, \quad x+y=0, \quad xy+4=0, \quad x^2-y^2=4.$$

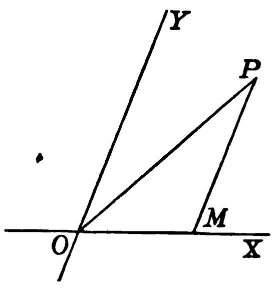
3. Give values to θ from 0 to 2π, and so trace the following, where $a=1$ inch;

$$r=a\cos\theta, \quad r\cos\theta=a, \quad r\cos\theta+2a=0.$$

9. Since indeterminate equations represent lines or curves, if we take two such equations, the resulting determinate values of x and y will represent the point or points of intersection of those curves: indeed, the whole science of analytical geometry treats of the equations to loci, and the points resulting from the intersection of two loci: thus the point (a, b) may be considered as the intersection of the loci of the two equations $x=a$, $y=b$, which, as we have already seen, represent straight lines parallel to the axes.

10. *To find the distance of any point from the origin in terms of the coordinates of that point.*





Let O be the origin, P the point whose distance OP is required, let the coordinates of P be x, y.

First, let the axes be rectangular,

then $$OP^2=OM^2+PM^2=x^2+y^2,$$

$$\therefore OP=(x^2+y^2)^{\frac{1}{2}}.$$

Secondly, let the axes be oblique;

then $$OP^2 = OM^2 + PM^2 - 2OM \cdot PM \cos OMP$$

$$= OM^2 + PM^2 + 2OM \cdot PM \cos YOM,$$

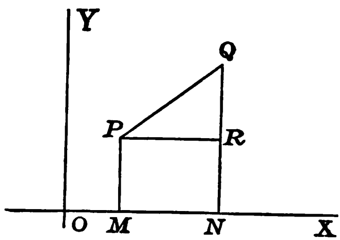
or $$OP = (x^2 + y^2 + 2xy \cos \omega)^{\frac{1}{2}}.$$

11. *To find the distance between two points.*

Let δ be the distance required.

Let P, Q be the points, and let their coordinates be x_1, y_1; x_2, y_2, respectively.

Draw PR parallel to the axis of x cutting QN in R. And, first, let the axes be rectangular.



Then $$PQ^2 = PR^2 + QR^2$$

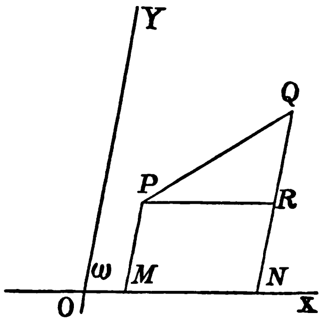
But $$PR = MN = ON - OM = x_2 - x_1,$$

$$QR = QN - PM = y_2 - y_1;$$

$$\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

or $$\delta = \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}^{\frac{1}{2}}.$$

Next, let the axes be inclined at an angle ω.



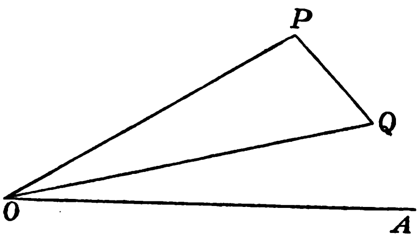
Then $PQ^2 = PR^2 + QR^2 - 2PR \cdot QR \cos PRQ,$

but $PRQ = ONQ = \pi - \omega;$

$$\therefore PQ^2 = PR^2 + QR^2 + 2PR \cdot QR \cos \omega,$$

or $\delta = \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1) \cos \omega\}^{\frac{1}{2}}.$

Next, let the coördinates be polar.

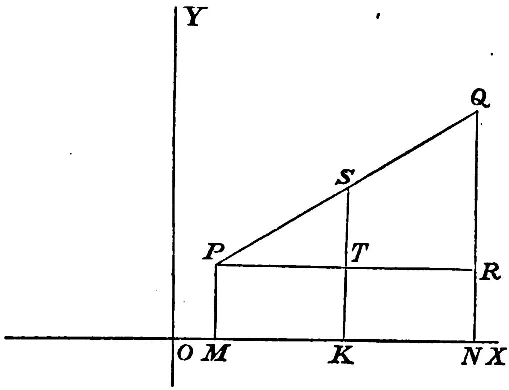


Let $OP = r_1,\ OQ = r_2,\ POA = \theta_1,\ QOA = \theta_2,$

then $\delta^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cos POQ$

$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 \sim \theta_2).$$

12. *To find the coordinates of the point which bisects the straight line joining two given points.*



Let S be the point required, OK, SK its coordinates; let SK cut PR in T.

Then $OK = OM + MK = OM + PT = OM + \frac{1}{2}PR$

$$= x_1 + \frac{x_2 - x_1}{2} = \frac{x_1 + x_2}{2}.$$

Similarly $\qquad SK = \frac{y_1 + y_2}{2}.$

Similarly we may find the coordinates of the point which divides the straight line joining two given points in a given ratio.

Let the given ratio be $m : n$.

Then, in the preceding figure,

$$PT : TR :: m : n;$$

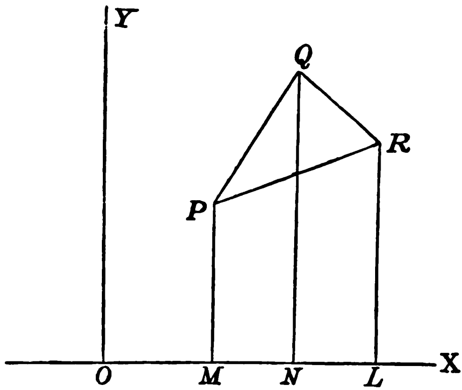
$$\therefore PT : PR :: m : m+n;$$

$$\therefore OK = x_1 + \frac{m}{m+n}(x_2 - x_1) = \frac{mx_2 + nx_1}{m+n}.$$

Similarly $$SK = \frac{my_2 + ny_1}{m+n}.$$

If $m+n=1$, we have $OK = mx_2 + nx_1$, a form which is often useful.

13. *To find the area of the triangle whose angular points are given.*



Let PQR be the triangle, and let the coordinates of P, Q, R be x_1y_1, x_2y_2, x_3y_3 respectively.

Then area $PQR = PQNM + RLNQ - PMLR$.

But $PQNM = \frac{1}{2}MN(PM + QN) = \frac{1}{2}(x_2 - x_1)(y_2 + y_1)$.

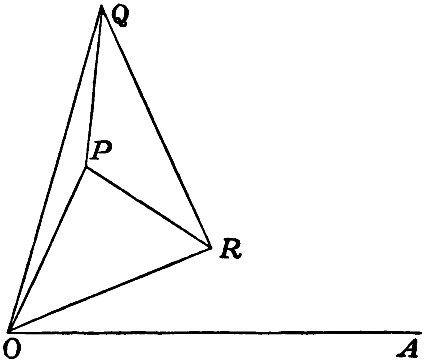
Similarly, $RLNQ = \frac{1}{2}(x_3 - x_2)(y_3 + y_2)$,

$$PMLR = \frac{1}{2}(x_3 - x_1)(y_3 + y_1);$$

$\therefore$ area required

$$= \frac{1}{2}\{(x_2 - x_1)(y_2 + y_1) + (x_3 - x_2)(y_3 + y_2) - (x_3 - x_1)(y_3 + y_1)\}$$
$$= \frac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1).$$

It is easy to see, that if the axes be inclined at an angle ω, the preceding expression must be multiplied by $\sin\omega$.



If the coordinates be polar,

let

$OP=r_1$, $OQ=r_2$, $OR=r_3$, $POA=\theta_1$, $QOA=\theta_2$, $ROA=\theta_3$;

then $\qquad \Delta PQR=\Delta QOR-\Delta QOP-\Delta POR$

$$=\tfrac{1}{2}\{r_2r_3\sin(\theta_2-\theta_3)-r_2r_1\sin(\theta_2-\theta_1)-r_1r_3\sin(\theta_1-\theta_3)\}$$

$$=\tfrac{1}{2}\{r_2r_3\sin(\theta_2-\theta_3)+r_1r_2\sin(\theta_1-\theta_2)+r_3r_1\sin(\theta_3-\theta_1)\}.$$

In these two expressions the coordinates must be taken in such an order as to make the whole expression positive.

Ex. 4.

1. Determine the points whose coordinates satisfy the following pairs of equations:

(i) $x+y=3$, $x-y=1$, (ii) $3x-4y=2$, $7x-9y=7$. (iii) $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$.

(iv) $x+y=7(x-y)$, $x^2+y^2=100$. (v) $xy+2=9y$, $xy+2=x$. (vi) $y^2=4ax$, $x=y$.

(vii) $x^2+y^2=5a^2$, $x^2=4ay$. (viii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $x^2+y^2=a^2$.

2. Determine the distances of the following points from the origin, the axes being (i) rectangular, (ii) inclined at an angle 60^0,

$$3a, 4a; \ -2b, b; \ a \sin \alpha, a \cos \alpha; \ a, -3a.$$

3. Determine the distances between the following pairs of points:

$$3, 4 \text{ and } 4, 3; \ -3, 4 \text{ and } 4, -3; \ 1, 1 \text{ and } -1, -1;$$

$$a, 0 \text{ and } 0, -a; \ h, k \text{ and } 2h, -3k;$$

$$a, b \text{ and } b, a; \ -3a, 2a \text{ and } -9a, -6a;$$

the axes being rectangular.

4. The axes being inclined at an angle ω, determine the distances between the following pairs of points:

$$a, 0 \text{ and } 0, a; \ 0, 0 \text{ and } a \cos\frac{\omega}{2}, \ a \sin\frac{\omega}{2};$$

$$1, -1 \text{ and } -1, 1; \ 0, 2 \text{ and } 3, 0; \ 0, 2 \text{ and } -3, 0.$$

5. Determine the distances between the following points, whose coordinates are polar:

$$a, \theta \text{ and } b, \phi; \ a, \theta \text{ and } a, -\theta; \ a, \theta \text{ and } -a, -\theta;$$

$$2a, 30^0 \text{ and } a, 60^0.$$

6. Determine the coordinates of the points of bisection of the lines joining the following pairs of points;

$$1, 1 \text{ and } 3, 3; \ -1, 1 \text{ and } 3, -5; \ 2, -2 \text{ and } -2, 2;$$

$$h, k \text{ and } 2h, -3k.$$

7. Determine the areas of the triangles whose angular points are

(i) $0, 0; \ 1, 2; \ 2, 1.$ (ii) $3, 4; \ -3, -4; \ 0, 4.$

(iii) $0, 0; \ x_1, y_1; \ x_2, y_2.$ (iv) $a, 0; \ -a, 0; \ 0, b.$

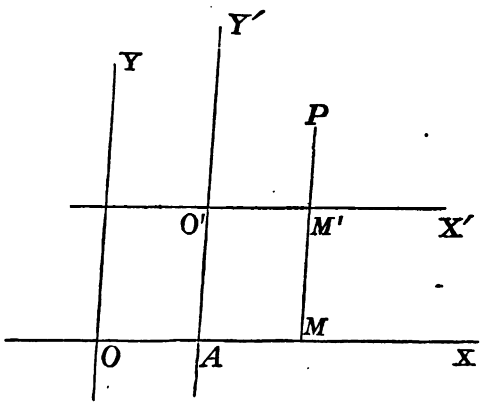
(v) the pole; $r, 0; \ r, \frac{\pi}{2}.$ (vi) $a, 0; \ 2a, \frac{\pi}{3}; \ 3a, \frac{2\pi}{3}.$

14. *Transformation of coordinates.*

Since the same point may be referred to different systems of coordinates, there must be definite relations connecting these various systems.

We will investigate those relations which are most practically useful, and then shew how to transform the coordinates of a point from any one system to any other.

15. *To change the origin from one point to another, the direction of the axes remaining unaltered.*



Let OX, OY be the old axes, $O'X'$, $O'Y'$ the new: let (x, y), (x', y') be the coordinates of the *same* point P referred to the old and new systems respectively: let h, k be the coordinates of the new origin referred to the old axes, and therefore

$$OA = h,\ O'A = k,\ OM = x,\ PM = y,\ O'M' = x',\ PM' = y'.$$

Then $$OM = OA + AM = OA + O'M',$$

$$x = x' + h,$$

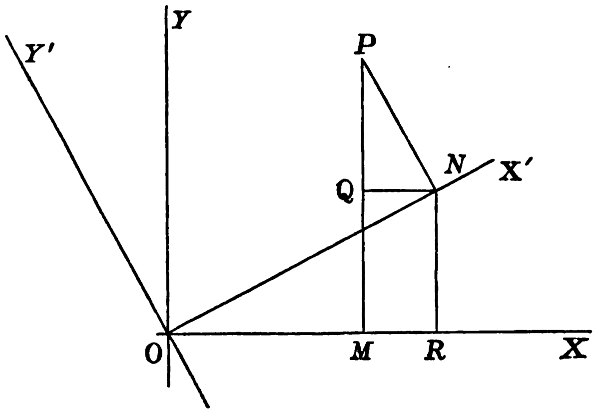
$$PM = MM' + PM' = O'A + PM',$$

$$y = y' + k.$$

Ex. Transform the equation $x^2+y^2=a^2$, by changing the origin to the point (α, β).

Here
$$x = x' + \alpha,$$
$$y = y' + \beta;$$
$$\therefore (x'+\alpha)^2 + (y'+\beta)^2 = a^2,$$
the equation required.

16. *To change the coordinates from one rectangular system to another, the origin being unaltered.*



Let P be the point; x, y, its coordinates referred to the original axes OX, OY; x', y', referred to the axes OX', OY';

then
$$OM = x,\ PM = y,\ ON = x',\ PN = y';$$
let
$$XOX' = YOY' = \theta.$$

From N draw NQ, NR perpendicular to PM, OX, respectively; then $NPQ = QNO = NOR = \theta$.

$$\therefore OM = OR - RM = OR - NQ = ON\cos\theta - PN\sin\theta,$$
or
$$x = x'\cos\theta - y'\sin\theta;$$

so $PM = MQ + QP = RN + QP = ON \sin\theta + PN \cos\theta$,

or $$y = x' \sin\theta + y' \cos\theta.$$

Ex. In the equation

$$x^2 - y^2 = a^2,$$

turn the axes through an angle -45^0.

The equation becomes

$$\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 = a^2,$$

or $$2x'y' = a^2.$$

17. *To transform an equation from one rectangular system to another, both the origin and the direction of the axes being changed.*

First transform the equation to axes through the new origin, parallel to the original axes; next turn these axes through the required angle.

Thus; if h, k be the coordinates of the new origin referred to the old axes, θ the angle between the original and final axes of x, we shall have

$$x = h + x' \cos\theta - y' \sin\theta,$$

$$y = k + x' \sin\theta + y' \cos\theta.$$

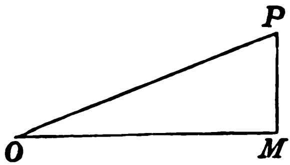
In all these transformations attention must be paid to the *sign* of θ.

The two steps in the transformations had better be taken separately.

18. *To transform an equation from rectangular to polar coordinates.*

Let the coordinates of P be x, y, referred to rectangular, and r, θ, referred to polar, coordinates.

(i) Let the origin of rectangular coordinates be the pole, and the axis of x the initial line.



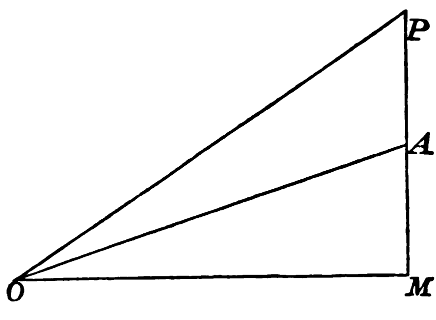
Then $$OM = OP \cos POM,$$
$$PM = OP \sin POM,$$
or $$x = r\cos\theta,\quad y = r\sin\theta.$$

Conversely, if we wish to transform from polar to rectangular coordinates,

$$r^2 = r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2,$$

$$\tan\theta = \frac{y}{x}.$$

(ii) Let the origin be the pole, and let the initial line make an angle a with the axis of x.

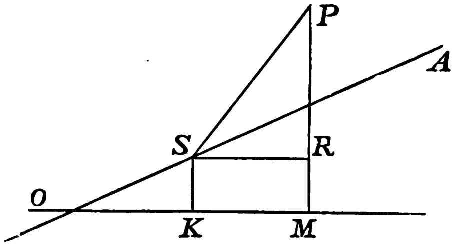


Then $OM = OP\cos POM = OP\cos(POA + AOM)$,

or $$x = r\cos(\theta + a).$$

Similarly $$y = r\sin(\theta + a).$$

(iii) Next let the coordinates of the pole be h, k, and let the initial line make an angle a with the axis of x.



Then $$OM = OK + KM = OK + SP \cos PSR,$$

or $$x = h + r\cos(\theta + a);$$

$$PM = SK + PR = SK + SP \sin PSR,$$

or $$y = k + r\sin(\theta + a).$$

Conversely, $$r^2 = (x-h)^2 + (y-k)^2,$$

$$\tan(\theta + a) = \frac{y-k}{x-h}.$$

The preceding transformations are the only ones that are generally useful: we will give a few examples, and a general theorem for transformation from any system to any other.

EXAMPLES.

(i) Transform the equation

$$x^2 + y^2 = a^2$$

to polar coordinates.

Here $$x = r\cos\theta,\ y = r\sin\theta;$$

$$\therefore\ r^2(\cos^2\theta + \sin^2\theta) = a^2,$$

or $$r = a.$$

(ii) Transform the equation

$$x\cos a + y\sin a = p$$

to polar coordinates, the coordinates of the pole being $p\cos\alpha$, $p\sin\alpha$, and the initial line being inclined to the axis of x at an angle α.

Here $$x = p\cos\alpha + r\cos(\theta+\alpha),$$
$$y = p\sin\alpha + r\sin(\theta+\alpha);$$

$$\therefore\ p\cos^2\alpha + r\cos(\theta+\alpha)\cos\alpha + p\sin^2\alpha + r\sin(\theta+\alpha)\sin\alpha = p,$$

or $$r\cos\theta = 0, \quad \therefore\ \theta = \frac{\pi}{2}.$$

(iii) Transform the equation

$$r = a\cos\theta$$

to rectangular coordinates, the origin coinciding with the pole, and the initial line with the axis of x.

Here $$r^2 = ar\cos\theta,$$

or $$x^2 + y^2 = ax.$$

Ex. 5.

1. Transform the following equations by changing the origin to the point (1, 1):

$$x+y=2,\ x+y+2=0,\ x=y,\ x^2+y^2=1,\ x^2-y^2=1,\ y^2=4x.$$

2. Transform the equations

$$x+y=c,\quad x^2-y^2=a^2,\quad (x+y-2a)^2=4xy$$

by turning the axes through an angle 45°.

3. Change the coordinates from rectangular to polar in the equations

$$x^2-y^2=a^2,\ x^2+y^2=a^2,\ y^2=4ax,\ x\cos\alpha+y\sin\alpha=a.$$

4. Change the coordinates from rectangular to polar in the equation $y^2=4ax$, the coordinates of the pole being a, 0.

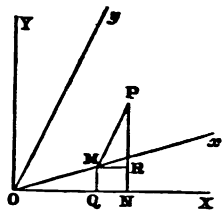
5. Change the coordinates from polar to rectangular in the equations

$$\theta=\frac{\pi}{3},\ r=c,\ \frac{l}{r}=\cos\theta+\sin\theta,\ r^2\cos 2\theta=a^2,\ r^2=a^2\cos 2\theta.$$

19. It is obvious that coordinates may be changed from any system to any other: we will now give equations from which any particular transformation may be effected.

We will suppose the two systems to have the same origin, since we have seen that any system may be transferred to parallel axes through the origin (h, k) by simply writing $x'+h$ for x, $y'+k$ for y.

Let Ox, Oy be any axes, P the point x, y, so that $OM=x$, $PM=y$.



Through O draw any straight lines OX, OY at right angles to each other; draw PN perpendicular to OX, and let $ON=X$, $PN=Y$.

Draw MQ, MR perpendicular to ON, PN, respectively: let $xOX=\alpha$, $yOX=\beta$.

Then

$$X=OQ+QN=OQ+MR=x\cos\alpha+y\cos\beta;$$

so $\quad Y=x\sin\alpha+y\sin\beta.$

Now if any other axes be drawn through O, making angles α', β' with OX, and x', y' be the coordinates of P referred to them, we shall have

$$X=x'\cos\alpha'+y'\cos\beta',\quad Y=x'\sin\alpha'+y'\sin\beta',$$

$$\therefore\ x\cos\alpha+y\cos\beta=x'\cos\alpha'+y'\cos\beta',$$

$$x\sin\alpha+y\sin\beta=x'\sin\alpha'+y'\sin\beta'.$$

By solving these simple simultaneous equations, we can get any pair of the quantities x, y, x', y' in terms of the other pair.

Thus if we have to transform from one pair of axes to another with the same origin, we shall always have

$$x=ax'+by',\quad y=a'x'+b'y',$$

where a, b, a', b', depend only on the angles the axes make with each other, and not on the position of the point P.

If the origin, as well as the direction of the axes, is changed, these equations will become

$$x = ax' + by' + c, \quad y = a'x' + b'y' + c'.$$

Hence we can prove the following important theorem.

20. *The degree of any equation cannot be altered by any transformation of coordinates.*

Let $lx^m y^n$ represent the highest term in any equation: let the axes be altered so that

$$x = ax' + by' + c, \quad y = a'x' + b'y' + c';$$

then $$lx^m y^n = l(ax' + by' + c)^m (a'x' + b'y' + c')^n.$$

Now there is no term in $(ax' + by' + c)^m$ of a higher degree in x', y', than the m^{th}, or in $(a'x' + b'y' + c')^n$ than the n^{th}: hence there is no term in their product of a higher degree than the $(m+n)^{th}$.

Hence the degree of an equation cannot be raised by transformation. Neither can it be lowered, for then it could be raised by transforming back again.

The general equation of the first degree $lx + my = d$, represents therefore one distinct class of lines, the general equation of the second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents another, and so on.

The rest of this work will consist of a discussion of the lines which give rise to these two equations.

EXAMPLES ON CHAPTER I.

1. A regular octagon is inscribed in a circle of radius a; determine the coordinates of its angular points, taking the centre of the circle as origin, and two diameters passing through four angular points as rectangular axes.

2. Tangents are drawn to the circle at the vertices of the octagon in the preceding question: determine the coordinates of the vertices of the octagon so formed.

3. A regular hexagon is described: if a be the length of a side, and two adjacent sides be taken as axes, determine the coordinates of the vertices.

4. Shew that the polar coordinates (r, θ), $(-r, \pi+\theta)$, $(r, \theta-\pi)$ all represent the same point.

5. ABC is a triangle, D, E the centres of the circumscribed and inscribed circles: find the coordinates of D, E: (1) B being the origin, BC axis of x, and the axes rectangular: (2) AB, AC being axes: (3) A being the pole, AB the initial line.

6. In the triangle ABC, straight lines are drawn from the angles bisecting the sides: find the coordinates of their point of intersection, taking two sides as axes.

7. Determine the angular points and area of the triangles whose vertices are (i) the intersections of the loci $x+y=0$, $x=y$, $y=3a$, (ii) the origin, and the intersections of the loci $x^2+y^2=4$, $x+y=2$.

8. If in the equation

$$x^2+y^2-2ax-2by+a^2+b^2-c^2=0$$

the origin be changed to the point (a, b), the equation becomes

$$x'^2+y'^2=c^2.$$

9. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$, by changing the origin to the point $\frac{a}{2}$, $\frac{b}{2}$, and turning the axes through an angle θ, such that $\tan\theta=-\frac{b}{a}$.

10. If (x, y), (x', y') be the coordinates of a point referred to rectangular and oblique systems with the same origin, and if the axes of the first system bisect the angles between those of the second, then

$$x=(x'+y')\cos\frac{\omega}{2},\quad y=(x'-y')\sin\frac{\omega}{2}.$$

11. If two oblique systems have the same origin and axis of x then $x = x' + y'\frac{\sin(\omega - \omega')}{\sin \omega}$, $y = y'\frac{\sin \omega'}{\sin \omega}$, ω, ω' being the angles between the axes respectively.

12. If the origin of oblique axes inclined at an angle ω be the pole and the axis of x the initial line, $x = r\frac{\sin(\omega - \theta)}{\sin \omega}$, $y = r\frac{\sin \theta}{\sin \omega}$.

13. From the preceding question obtain r and θ in terms of x and y.

CHAPTER II. *The Straight Line.*

21. A straight line may satisfy two geometrical conditions and no more.

Thus;

(1) It may pass through two fixed points:

(2) It may pass through one fixed point, and make a fixed angle with a fixed line:

(3) It may be at a given distance from a given point and that distance may make a fixed angle with a given line: that is, really, it may touch a fixed circle and be parallel to a fixed straight line.

This is of course a complicated case, but it is very useful.

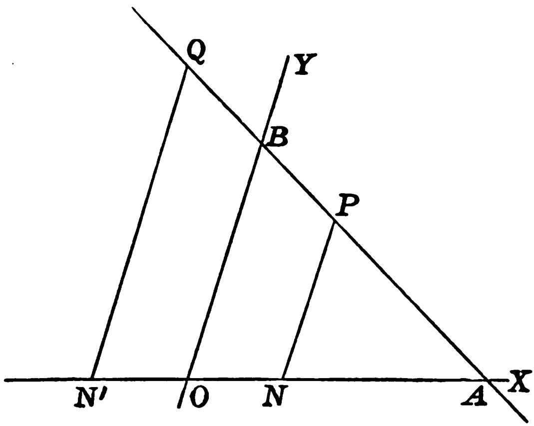
There are many other more complicated conditions, which a given straight line can satisfy, but they can generally be reduced to one of the foregoing.

We will investigate the preceding cases geometrically, shew that the equations so obtained can be reduced to one general form, that this form always represents some straight line, and hence obtain the equations to straight lines under various other conditions.

We have already seen (Art. 7) that the equations $x=a$, $y=b$, represent straight lines parallel to the axes of y and x respectively.

22. *To find the equation to the straight line which passes through two fixed points.*

First, let the points be on the axes.



Let the straight line cut the axis of x in the point A and that of y in the point B; let $OA = a$, $OB = b$; let the coordinates of P, any point in the line, be x and y.

Then, $\because PN$ is parallel to BO,

$$\therefore \frac{PN}{BO} = \frac{AN}{AO} = \frac{AO - ON}{AO},$$

or, $$\frac{y}{b} = \frac{a - x}{a} = 1 - \frac{x}{a},$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1.$$

This equation is therefore satisfied by every point in AB, and by no other points, since it is deduced from the geometrical relation $PN : OB :: AN : OA$, which is only true when P lies on AB; it is therefore the equation to AB.

It may easily be seen that this equation holds for all points on the line AB, or AB produced either way.

For let Q be such a point; then

$$\frac{QN'}{BO}=\frac{AN'}{AO}=1+\frac{ON'}{AO},$$

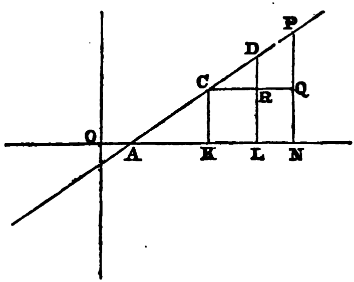
$$\therefore \frac{-ON'}{AO}+\frac{QN'}{BO}=1,$$

but if x, y, be the coordinates of Q, $x=-ON'$, $y=QN'$,

$$\therefore \frac{x}{a}+\frac{y}{b}=1, \text{ as before.}$$

Since this equation does not involve the angle AOB, it is true, whether the coordinates are rectangular or oblique.

23. Next, let the coordinates of the two points C, D be x_1, y_1; x_2, y_2 respectively.



As before, let P be a point on the line required, x, y its coordinates.

Then in the figure

$$OK=x_1,\ OL=x_2,\ ON=x,$$

$$CK=y_1,\ DL=y_2,\ PN=y.$$

Draw CQ parallel to OX, cutting DL, PN in R, Q respectively.

Then

$$CR = x_2 - x_1,\ DR = y_2 - y_1,\ CQ = x - x_1,\ PQ = y - y_1.$$

Then by similar triangles *CDR*, *CPQ*,

$$\frac{PQ}{DR} = \frac{CQ}{CR},$$

that is, $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$, the equation required.

This equation may be written in either of the forms

$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$$

or
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

In this equation since the magnitude of YOX is not involved, the axes may be either rectangular or oblique.

N.B. $\dfrac{y_2 - y_1}{x_2 - x_1}$, that is $\dfrac{DR}{CR}$, which is the coefficient of x, is, when the axes are rectangular, equal to the tangent of the angle which the line makes with the axis of x.

24. *To find the equation to a straight line which passes through a given point, and makes a given angle with the axis of x.*

Let ACP be the straight line, C the given point $(x_1 y_1)$, P the point (xy), CAX the given angle $= a$.

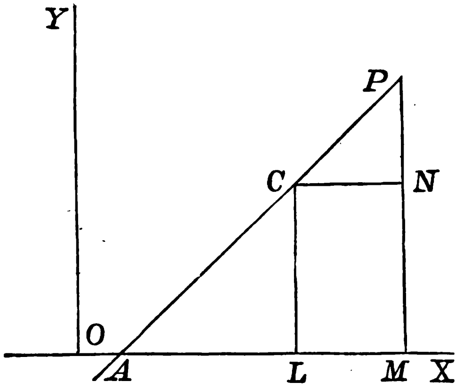
Then
$$\tan a = \frac{PN}{CN} = \frac{y - y_1}{x - x_1},$$

or
$$y - y_1 = (x - x_1)\tan a.$$

If $\tan a = n$, this equation becomes $y - y_1 = n(x - x_1)$.

If the coordinates be oblique, we still have

$$\frac{PN}{CN}=\frac{CL}{AL}=\frac{\sin a}{\sin(\omega-a)},$$



which is constant since ω and a are constant, and therefore

$$y-y_1=n(x-x_1)$$

still represents a straight line through the point $(x_1 y_1)$ which makes a fixed angle with the axis of x. Two particular cases are useful:

(1) Let the fixed point be the origin, then $x_1=0, y_1=0$. The equation to any straight line through the origin is therefore $y=nx$, where n may have any value.

(2) Let the fixed point be on the axis of y at a distance c from the origin, then the equation becomes

$$y-c=nx,$$

or $$y=nx+c.$$

25. The equation $y-y_1=(x-x_1)\tan a$ may be connected with polar coordinates and put in a very convenient form.

Let CP, the distance between the fixed point and any point on the line, be denoted by r;

$$\therefore CQ = r\cos a, \quad PQ = r\sin a,$$

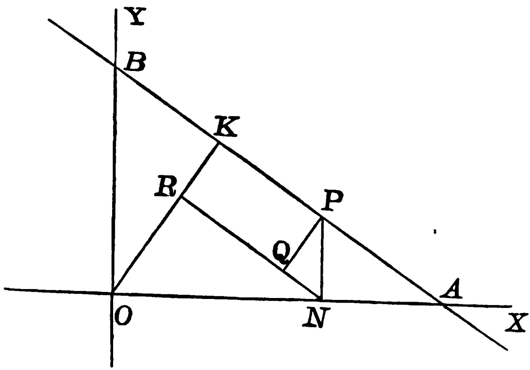
or

$$x - x_1 = r\cos a, \quad y - y_1 = r\sin a;$$

$$\therefore x = x_1 + r\cos a, \quad y = y_1 + r\sin a.$$

These equations are very useful when we want to determine the rectangle, square, &c. of lines through a fixed point, and know a curve on which the other end of the line lies.

26. *To find the equation to a straight line in terms of the perpendicular from the origin and the angles that perpendicular makes with the axes.*



Let APB be the straight line, OK the perpendicular from O, and let $OK = p$, $KOA = a$.

Draw NR parallel to AB, cutting OK in R, and draw PQ perpendicular to NR.

Then $$OR + RK = OK = p,$$

but $$OR = ON\cos a = x\cos a,$$

$$RK = PQ = PN\sin a = y\sin a,$$

since $$QNP = RON = a;$$

$$\therefore x \cos a + y \sin a = p,$$

the equation required.

If the axes be oblique, let $KOB = \beta$,

then $$NPQ = \beta, \quad PQ = y \cos \beta,$$

and the equation becomes

$$x \cos a + y \cos \beta = p.$$

N.B. $\cos a$ and $\cos \beta$ are often called the direction cosines of the line.

Ex. 6.

1. Interpret the equations:

$$\frac{x}{3} + \frac{y}{2} = 1, \quad \frac{x}{2} - \frac{y}{3} = 1, \quad \frac{x}{2} + \frac{y}{3} + 1 = 0, \quad \frac{y}{3} - \frac{x}{2} = 1.$$

2. Find the equations to the straight lines which pass through the following pairs of points: (a, b), (b, a): (h, k), $(-h, -k)$: (h, k), $(h, -k)$: $(k, -k)$, $(-k, -k)$: $(3, 4)$, $(1, 2)$: $(5, 6)$, $(0, -1)$: $(a \cos \theta, b \sin \theta)$, $(a \cos \phi, b \sin \phi)$.

3. Find the equation to the straight line which passes through the point $(-1, 2)$ and makes an angle 30^0 with the axis of x.

4. Find the equation to the straight line which passes through the point $(2, 2)$ and makes an angle of 45^0 with the axis of x.

5. Find the equation to the straight line when the perpendicular from the origin makes an angle $-a$ with the axis of x and its length is b.

27. The student will probably have observed that in each of the equations we have found to straight lines there is one term involving x, one involving y, and a constant term, but no terms in x^2, xy, or y^2: we shall now prove that every equation of this sort, such as

$$lx + my = d,$$

where l, m, d are constants, represents some straight line.

28. Conditions that two equations represent the same locus.

Suppose we have two equations, one that to a known line, another which we wish to interpret, we can find the conditions that the two equations shall represent the same locus.

In general, two equations taken together will give determinate values of x and y, and therefore represent points; if, however, the coefficient of every term in the second is the same multiple of the coefficient of the corresponding term of the first, we may divide by that constant multiple and so obtain that first equation.

The condition required is therefore that the ratio of the coefficients of x^2, xy, y^2, x, y, &c. and of the constant terms in the two equations should be the same.

In practice, it is best to divide each equation by the coefficient of some term or by the constant term; then the coefficients of the other terms must be identical.

This process is called equating coefficients.

Thus, the equation $lx+my=d$ may be written

$$\frac{lx}{d}+\frac{my}{d}=1.$$

Now the equation to the straight line which cuts off intercepts a, b from the axes is

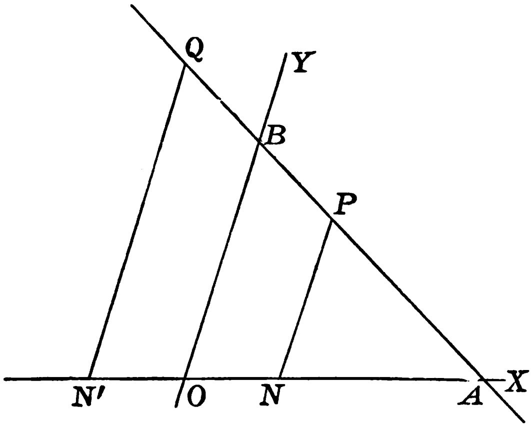
$$\frac{x}{a}+\frac{y}{b}=1 \text{ (Art. 22).}$$

Hence, these equations will represent the same line if

$$\frac{l}{d}=\frac{1}{a},\ \frac{m}{d}=\frac{1}{b},\ \text{or}\ \frac{d}{l}=a,\ \frac{d}{m}=b.$$

Now since we can always measure off lengths equal to $\frac{d}{l}$, $\frac{d}{m}$ along the axes, the equation $lx+my=d$ must represent the straight line which cuts off intercepts $\frac{d}{l}$, $\frac{d}{m}$ from the axes of x and y respectively.

29. This proposition is so important that we will prove it independently.



Let $lx+my=d$ be an equation, and P any point whose coordinates (x, y) satisfy the equation so that

$$l\,.\,ON+m\,.\,PN=d,$$

then P shall lie on a certain straight line.

Measure $OA=\frac{d}{l}$, $OB=\frac{d}{m}$, and join AP, BP.

Now $$PN=y=\frac{d-lx}{m},$$

and $$AN=OA-ON=\frac{d}{l}-x=\frac{d-lx}{l};$$

$$\therefore \frac{PN}{AN}=\frac{l}{m}=\frac{\frac{d}{m}}{\frac{d}{l}}=\frac{OB}{OA};$$

$\therefore$ P is a point on the straight line AB.

Since the magnitude of the angle BOA is not involved in this proof, the above equation will represent a straight line whether the coordinates be rectangular or oblique.

This equation is often written $Ax+By+C=0$.

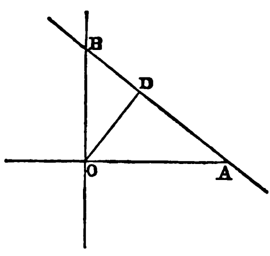
In this case the intercepts on the axes are $-\frac{C}{A}$, $-\frac{C}{B}$ respectively.

Since there are three constants in this equation, it would seem that a straight line could satisfy three conditions: this, however, is not the case, since we can divide by any one of the three without altering the equation, and then the straight line which it represents will be completely determined by the ratios thus obtained.

30. *To find the angle which the line*

$$lx+my=d$$

makes with the axis of x, *and the perpendicular on it from the origin.*



Let $OA=\frac{d}{l}$, $OB=\frac{d}{m}$, then, by the preceding Article, AB is the line required.

Let BAX, the angle which BA makes with the positive direction of the axis of x, be called θ.

Draw OD perpendicular to AB, and let $OD=\delta$.

(i) Let the axes be rectangular.

Then $$\tan\theta=-\tan BAO=-\frac{OB}{OA}=-\frac{\frac{d}{m}}{\frac{d}{l}}=-\frac{l}{m}.$$

Again $AB \cdot OD = OB \cdot OA$, since each of these rectangles is double the triangle OAB.

But $AB^2 = OA^2 + OB^2 = \frac{d^2}{l^2} + \frac{d^2}{m^2} = \frac{d^2}{l^2m^2}(l^2 + m^2)$;

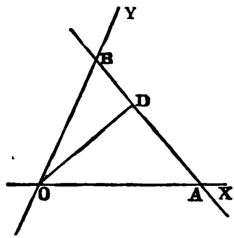
$$\therefore \frac{d\delta(l^2+m^2)^{\frac{1}{2}}}{lm} = \frac{d^2}{lm}, \qquad \delta = \frac{d}{(l^2+m^2)^{\frac{1}{2}}}.$$

We might have obtained these values by equating coefficients in the equations

$$lx + my = d, \quad y = x \tan\theta + b,$$
$$x \cos a + y \sin a = \delta.$$

(ii). Let the axes be oblique, so that $XOY = \omega$.

Make the same construction as before.



Then $\qquad \sin BAO = \sin(\pi - \theta) = \sin\theta,$

and $\qquad \sin OBA = \sin(\theta - \omega)$; ....Euc. i. 32.

$$\therefore \frac{\sin\theta}{\sin(\theta-\omega)} = \frac{OB}{OA} = \frac{l}{m},$$

or, $m \sin\theta - l \sin(\theta - \omega) = 0$;

therefore, expanding $\sin(\theta - \omega)$,

$$(m - l\cos\omega)\sin\theta = -l\sin\omega\cos\theta;$$

$\therefore$ dividing by $\cos\theta$, $\tan\theta = \frac{l\sin\omega}{l\cos\omega - m}$.

Again, $AB = \{OA^2 + OB^2 - 2OA \cdot OB\cos\omega\}^{\frac{1}{2}}$

$$= \frac{d}{lm}(l^2 + m^2 - 2lm\cos\omega)^{\frac{1}{2}};$$

and $AB\,.\,OD = OA\,.\,OB \sin\omega = \dfrac{d^2}{lm}\sin\omega$;

$$\therefore\ \delta = \frac{d\sin\omega}{(l^2+m^2-2lm\cos\omega)^{\frac{1}{2}}}.$$

In these expressions, if we put $\omega = \dfrac{\pi}{2}$, we get the values of θ and δ previously found, as manifestly ought to be the case.

31. We are now in a position to interpret any simple equation, and to draw the straight line which it represents.

Put $y=0$ in the equation, the corresponding value of x will give the point at which the line cuts the axis of x.

Put $x=0$, we obtain the point at which it cuts the axis of y.

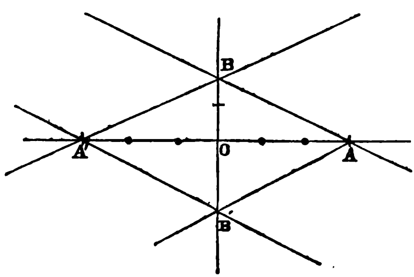
Join these points, and we obtain the line required.

We may observe that if the coefficients of x and y have the same sign, the line makes an obtuse angle with the axis of x, if different, an acute.

Examples. Interpret the equations:

$$x+2y=3;\qquad x+2y+3=0;$$
$$x-2y=3;\qquad x-2y+3=0.$$

In the first equation put $y=0$, $\therefore x=3$; then put $x=0$, $\therefore\ y=\dfrac{3}{2}$.



Now measure $OA=3$ along the axis of x, and $OB=\frac{3}{2}$ along the axis of y. AB is the line required.

To interpret $x+2y+3=0$: this line passes through the points $x=-3$, $y=0$; $x=0$, $y=-\frac{3}{2}$; measure $\therefore OA'=OA$ and $OB'=OB$ along the negative parts of the axes. $A'B'$ is the line required.

So the third and fourth equations represent AB', $A'B$ respectively.

To determine the angles which these lines make with the axes, divide the coefficient of x by that of y and change the sign; then $-\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ are the tangents of these angles.

The perpendiculars from the origin are

$$\frac{3}{\sqrt{5}},\ \frac{-3}{\sqrt{5}},\ \frac{3}{\sqrt{5}},\ \frac{-3}{\sqrt{5}}.$$

Ex. 7.

1. Draw the straight lines represented by the following equations, where 1 represents a line ½ an inch in length.

$$x=2,\ x+3=0,\ y=1,\ x=y,\ 2x+3y=0,\ \frac{x}{2}+\frac{y}{3}=1,$$

$$4x-3y=1,\ 4x+3y=1,\ y-3=2(x-2),\ y+1=\sqrt{3}(x+2),$$

$$y-2=\frac{1}{\sqrt{3}}(x-1),\ x\cos\frac{\pi}{3}+y\sin\frac{\pi}{3}=1,\ x\cos\frac{\pi}{3}-y\sin\frac{\pi}{3}=1,$$

$$x\sin\frac{\pi}{3}-y\cos\frac{\pi}{3}=1.$$

2. Determine the angle which each of the above lines makes with the axis of x, and the perpendicular from the origin.

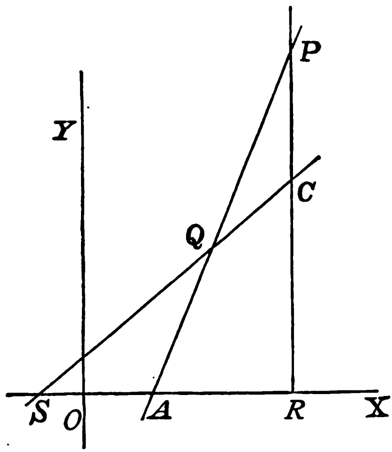
3. If the equations $\frac{x}{a}+\frac{y}{b}=1$, $Ax+By=C$, $y=nx+c$, $x\cos\alpha+y\sin\alpha=p$ represent the same line, then

$$Aa=Bb;\ p=\frac{C}{(A^2+B^2)^{\frac{1}{2}}}=\frac{ab}{(a^2+b^2)^{\frac{1}{2}}}=\frac{c}{(1+n^2)^{\frac{1}{2}}};$$

$$n\sin\alpha+\cos\alpha=0.$$

We can now find the equations to straight lines which are related in some way to others whose equations are known.

32. *To find the equations to the straight lines which pass through a given point, and make a given angle with a given straight line.*



Let C be the given point whose coordinates are x_1, y_1 and let the equation to the given line AP be reduced to the form

$$y = x \tan \alpha + c,$$

where $$PAX = \alpha.$$

Let $CPQ = CQP = \beta$; then $PRX = \alpha + \beta$, $QSX = \alpha - \beta$, and the equations to PR and QS are

$$y - y_1 = (x - x_1) \tan (\alpha \pm \beta).$$

If the axes are oblique, these expressions become very complicated, and are rarely useful.

33. Conversely, if we know the equations to two straight lines we can find the angle between them.

Let $$lx + my = d,$$
$$l'x + m'y = d',$$

be the equations to the known lines, and ϕ the angle between them; let θ, θ' be the angles they make with the axis of x respectively.

Then $$\tan\theta = -\frac{l}{m}, \quad \tan\theta' = -\frac{l'}{m'};$$

$$\therefore \tan\phi = \tan(\theta \sim \theta') = \frac{\tan\theta \sim \tan\theta'}{1 + \tan\theta\tan\theta'} = \frac{lm' \sim l'm}{ll' + mm'}.$$

34. If these straight lines are parallel, $\tan\phi = 0$,

$$\therefore \frac{l}{m} = \frac{l'}{m'}.$$

If they are at right angles, $\tan\phi = \infty$,

$$\therefore 1 + \frac{ll'}{mm'} = 0, \quad \text{or} \quad \frac{l'}{m'} = -\frac{m}{l}.$$

The equation $l'x + m'y = d'$ represents a straight line parallel to $lx + my = d$ if $\frac{l'}{m'} = \frac{l}{m}$, whatever be the inclination of the axes.

For the ratio $\frac{l}{m}$ is that of the intercepts on the axes of y and x respectively (Art. 29); and if this ratio be the same for two lines, these lines must be parallel.

Thus the equation to a straight line which is parallel to $lx + my = d$ may be written $lx + my = d'$; to a straight line perpendicular to the same line, $mx - ly = d'$, where d' is indeterminate in each case, as it manifestly ought to be, since there are an infinite number of straight lines parallel or perpendicular to a fixed line. In the latter case, however, the axes must be rectangular.

Thus a straight line parallel to $\frac{x}{a} + \frac{y}{b} = 1$ may be written $\frac{x}{a} + \frac{y}{b} = \lambda$, where λ is indeterminate; that perpendicular to the same line, $ax - by = \lambda$.

If the straight line parallel to $lx + my = d$ pass through the point (x_1, y_1) its equation must be $lx + my = lx_1 + my_1$, for this represents a straight line parallel to $lx+my=d$, and is true when $x=x_1$, $y=y_1$.

We may remark that the equation

$$y-y_1=m(x-x_1)$$

represents any straight line through the point (x_1, y_1), and the indeterminate constant m must be determined by some other condition about the line.

Ex. 8.

1. Find the equations to the straight lines which pass through the origin, and make angles of 15^0 with $x+y=2$.

2. Find the equations to the lines which pass through the point a, 0, and make angles $\frac{\pi}{4}$ with $\frac{x}{a}+\frac{y}{b}=1$.

3. Find the equations to the lines which cut off a length b from the axis of y, and make angles β with

$$x\cos\alpha+y\sin\alpha=p.$$

4. Find the angles between the following lines:

$$x+y=3, \text{ and } x=\sqrt{3}y; \quad \frac{x}{a}+\frac{y}{b}=1, \text{ and } \frac{x}{b}+\frac{y}{a}=1.$$

5. Find the equations to the straight lines which are parallel to $Ax+By=C$ and pass through the following points respectively (10, 0), (a, b), $(Ba, -aA)$.

6. Find the equations to the straight lines which are parallel to $y=mx+b$, and at a distance a from the origin.

7. Find the equation to the straight line which is perpendicular to $Ax+By+C=0$, and cuts off a length b from the axis of y.

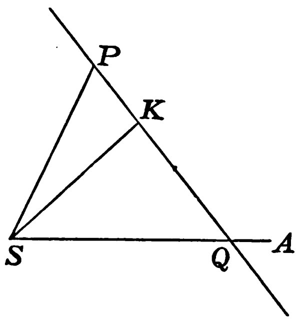
8. Find the equation to the straight line which is perpendicular to $\frac{x}{a}+\frac{y}{b}=1$, and passes through the point a, b.

9. If AB be parallel to $x\cos\alpha+y\sin\alpha=p$ and $OA \cdot OB=c^2$, find its equation.

10. If AB be parallel to $y=mx+b$ and $OA+OB=c$, find its equation.

35. *To find the polar equation to a straight line.*

Let PQ be the line required, P any point (r, θ) on it.



Let the perpendicular SK from the pole S make an angle $KSQ = a$ with the initial line.

Then $$SP \cos PSK = SK,$$

or, $$r \cos (\theta - a) = p,$$

the equation required.

This might also be obtained by writing $r \cos \theta$, $r \sin \theta$ for x and y respectively in the equation $x \cos \theta + y \sin \theta = p$.

The general polar equation to a straight line may be obtained from the rectangular equation

$$lx + my = d,$$

by writing $r \cos \theta$, $r \sin \theta$ for x and y respectively.

It becomes $$lr \cos \theta + mr \sin \theta = d,$$

or $$l \cos \theta + m \sin \theta = \frac{d}{r}.$$

36. *Polar equation to the straight line through two fixed points.*

Let (r_1, θ_1) (r_2, θ_2) be the coordinates of the fixed points, (r, θ) of any other point.

Then the area of the triangle of which these points are the vertices is

$$\frac{r_1 r_2 r}{2}\left(\frac{\sin(\theta_1-\theta_2)}{r}+\frac{\sin(\theta_2-\theta)}{r_1}+\frac{\sin(\theta-\theta_1)}{r_2}\right): \text{(Art. 13.)}$$

Now, if (r, θ) lie on the straight line joining the two points (r_1, θ_1), (r_2, θ_2) this area must vanish.

Hence $$\frac{\sin(\theta_1-\theta_2)}{r}+\frac{\sin(\theta_2-\theta)}{r_1}+\frac{\sin(\theta-\theta_1)}{r_2}=0$$

is the equation required.

This equation, if expanded, is easily shewn to be of the form

$$l\cos\theta+m\sin\theta=\frac{d}{r}.$$

Ex. 9.

1. Shew that the equations $r\cos\theta=a$, $r\sin\theta=a$ represent straight lines perpendicular and parallel to the initial line at a distance a from the pole.

2. Interpret the equations:

$$\theta=\frac{\pi}{3},\quad \theta+a=\pi,\quad r\sin\left(\theta-\frac{\pi}{3}\right)=p,\quad \frac{l}{r}=\cos\theta+\sin\theta.$$

3. Find the polar equations to the straight lines which pass through the following pairs of points:

$$a, 0;\ b, \frac{\pi}{2}:\ -a, 30^\circ;\ a, 60^\circ:\ a, 60^\circ;\ a, 120^\circ.$$

4. ABC is a triangle: if A be the pole, AB the initial line, find the equation to the straight lines through A, B, C respectively which are perpendicular to the opposite sides, and the coordinates of their point of intersection.

5. If $r\cos(\theta-a)=p$, $l\cos\theta+m\sin\theta=\frac{d}{r}$, represent the same line

$$\frac{m}{l}=\tan a,\ d=p\,(l^2+m^2)^{\frac{1}{2}}.$$

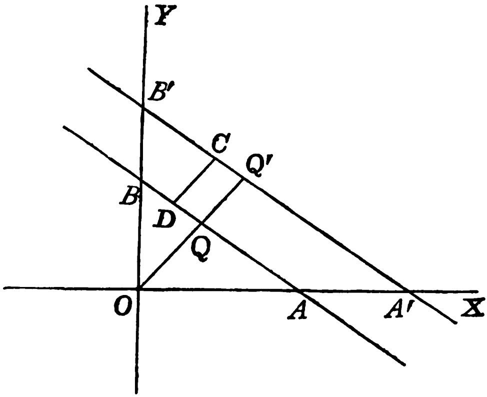
We may apply these equations to the solution of various problems.

37. *To find the distance of a fixed point from a given straight line.*

Let C be the given point (h, k) and AB the given straight line, whose equation is

$$lx+my=d.$$

Through C draw CD perpendicular to AB, and $A'CB'$ parallel to AB, cutting the axes in A', B' respectively. Draw OQQ' perpendicular to AB and therefore to $A'B'$.



Then the equation to $A'B'$ is

$$lx + my = lh + mk. \quad \text{(Art. 34.)}$$

Now $$CD = QQ' = OQ' - OQ.$$

But $$OQ' = \frac{lh + mk}{(l^2 + m^2)^{\frac{1}{2}}}, \quad OQ = \frac{d}{(l^2 + m^2)^{\frac{1}{2}}};$$

$$\therefore \; CD = \frac{lh + mk - d}{(l^2 + m^2)^{\frac{1}{2}}}.$$

Hence to find the perpendicular from a given point on a given straight line, take all the terms to the left hand in the equation to the given straight line, write the coordinates of the point instead of x and y in the expression which is now equal to zero, and divide by the square root of the sum of the squares of the coefficients of x and y; the expression thus obtained is the length of the perpendicular required.

If this expression be negative, this shews that $OQ' < OQ$, or that the point lies on the negative side of the line.

If the equation to the straight line be written in the form

$$x \cos a + y \sin a - p = 0,$$

the perpendicular from $(x_1 y_1)$ is $x_1 \cos a + y_1 \sin a - p$.

It is obvious that if the point (x, y) be not on the line whose equation is

$$x \cos a + y \sin a - p = 0,$$

$x \cos a + y \sin a - p$ cannot be zero; we now see what it does represent.

38. *To find the distance between two parallel straight lines.*

Let their equations be $lx + my = d_1$,

$$lx + my = d_2,$$

then the distance between them must be the difference of their distances from the origin, that is $\dfrac{d_1 \sim d_2}{(l^2 + m^2)^{\frac{1}{2}}}$.

39. *To find the point of intersection of two straight lines.*

Since both straight lines pass through this point, both equations must be true for that point, if therefore we treat the two equations as simultaneous, the resulting values of x and y will be the coordinates of the point of intersection required.

Hence, for example, to find the coordinates of the foot of the perpendicular from a given point on a given straight line, write down the equation to the straight line through that point perpendicular to the given straight line, and treat these equations as simultaneous.

Ex. Find the perpendicular from the point (hk) on the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, and the coordinates of the point where it meets that line.

The length is $\dfrac{\dfrac{h}{a} + \dfrac{k}{b} - 1}{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)^{\frac{1}{2}}}$ or $\dfrac{bh + ak - ab}{(a^2 + b^2)^{\frac{1}{2}}}$ (Art. 37).

The equation to the perpendicular is

$$ax - by = ah - bk \quad \text{(Art. 34).}$$

Eliminating y between this equation, and $\frac{x}{a}+\frac{y}{b}=1$, or

$$bx+ay=ab,$$

we obtain $$(a^2+b^2)\,x=a^2h-abk+ab^2,$$

$$x=\frac{ah-bk+ab}{a^2+b^2}\cdot a.$$

Similarly $$y=\frac{bk-ah+ab}{a^2+b^2}\cdot b.$$

Ex. 10.

1. Find the distance of the point (2, 3) from the line $$x+y=1.$$

2. Find the distance of the point (3, 0) from the line $$\frac{x}{2}+\frac{y}{3}=1.$$

3. Find the distance of the point (0, 1) from the line $$x-3y=1.$$

4. Find the distance of the point $(-1, 3)$ from the line $$3x+4y+2=0.$$

5. Find the distance of the point $(-a, -b)$ from the line $$\frac{x}{a}+\frac{y}{b}=1.$$

6. Find the distance of the point (a, b) from the line $$ax-by=0.$$

7. Find the distance of the point (h, k) from the line $$Ax+By+C=D.$$

8. Find the distance of the origin from the line $$hx+ky=c^2.$$

9. Find the distance of the point (h, k) from the line $$hx+ky=c^2.$$

10. Find the distance of the point $(a, 0)$ from the line $$y=mx+\frac{a}{m}.$$

11. Find the distance of the point h, k, from the line

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1.$$

12. Find the coordinates of the foot of the perpendicular in each of the preceding cases.

13. Find the distance between the lines

$$y=(x-a)\tan\alpha,$$

$$\text{and } y=(x-b)\tan\alpha;$$

also between $\qquad \dfrac{x}{a}+\dfrac{y}{b}=2,\quad \text{and } \dfrac{x}{a}+\dfrac{y}{b}=\dfrac{1}{2}.$

14. Find the coordinates of the points of intersection of the four lines in questions 1—4.

15. Find the coordinates of the intersection of

$$x\cos\alpha+y\sin\alpha=p,\ \text{and } x\cos\beta+y\sin\beta=p.$$

16. Find the coordinates of the angular points of the parallelogram whose sides have for their equations

$$x=a,\ x=b,\ x\cos\alpha+y\sin\alpha=p_1,\ x\cos\alpha+y\sin\alpha=p_2.$$

17. Find the equations to the diagonals of the parallelogram in question 16, the coordinates of their point of intersection, and the area of the parallelogram.

18. Three consecutive angular points of a parallelogram are a, 0; h, k; 0, b, respectively; find the coordinates of the other angular point, and the equations to the diagonals.

19. Determine the angles of the parallelograms in the preceding questions.

20. Determine the angle between the straight lines

$$\frac{x}{a}+\frac{y}{b}=1,\quad \frac{x}{b}+\frac{y}{a}=1,$$

and find the relation between a and b when this angle is 150°.

40. *To find the equation to a straight line which passes through the intersection of two given straight lines.*

Let the equation to the two given lines be

$$lx + my = d \text{ ....(1)},$$

$$l'x + m'y = d' \text{ ....(2)}.$$

Then $\quad lx + my - d = \lambda\,(l'x + m'y - d') \text{ ....(3)}$

is the equation required, where λ may have any value.

For this is the equation to some straight line, and it passes through the intersection of (1) and (2), since if either of these equations be satisfied, the other is satisfied also.

Since there are an infinite number of straight lines which pass through any point, λ is indeterminate, and must in any particular case be determined by the circumstances of the problem.

For instance, we will find the equation to the two straight lines which bisect the angles between (1) and (2).

Let (xy) be a point on one of these lines; then, if δ_1, δ_2 be the perpendiculars on the lines from (xy), $\delta_1 = \pm\delta_2$.

But

$$(l^2 + m^2)^{\frac{1}{2}}\,\delta_1 = lx + my - d, \quad (l'^2 + m'^2)^{\frac{1}{2}}\,\delta_2 = l'x + m'y - d',$$

therefore from (3)

$$(l^2 + m^2)^{\frac{1}{2}} = \pm\lambda\,(l'^2 + m'^2)^{\frac{1}{2}},$$

and the equations are

$$\frac{lx + my - d}{(l^2 + m^2)^{\frac{1}{2}}} = \pm\frac{l'x + m'y - d}{(l'^2 + m'^2)^{\frac{1}{2}}}.$$

41. *To find the condition that three points may be on the same straight line.*

Let $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$, be the coordinates of the points; then, since these points lie on a straight line, the area of the triangle of which these are the angular points must be equal to 0, but twice the area of this triangle is equal to

$$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3; \text{ (Art. 13.)}$$

$$\therefore\ x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$$

is the condition required.

Generally however the best way of proving that three points lie on a straight line, is to write down the equation to the line through two of the points and see whether the coordinates of the third satisfy it.

42. *To find the condition that three straight lines pass through the same point.*

This is the same as saying that the same values of x and y satisfy the equations to the three lines.

Let these three equations be

$$l_1x + m_1y = d_1 \text{.....................} (1),$$

$$l_2x + m_2y = d_2 \text{.....................} (2),$$

$$l_3x + m_3y = d_3 \text{.....................} (3).$$

Multiply (2) by λ, (3) by μ, and add;

$$\therefore\ (l_1 + \lambda l_2 + \mu l_3)\, x + (m_1 + \lambda m_2 + \mu m_3)\, y = d_1 + \lambda d_2 + \mu d_3$$

for all values of λ and μ, and $\therefore$ when $l_1 + \lambda l_2 + \mu l_3 = 0$, and $m_1 + \lambda m_2 + \mu m_3 = 0$.

But then $d_1 + \lambda d_2 + \mu d_3 = 0$ also.

Now if $l_1 + \lambda l_2 + \mu l_3 = 0$, and $m_1 + \lambda m_2 + \mu m_3 = 0$,

$$\lambda = \frac{l_3m_1 - l_1m_3}{l_2m_3 - l_3m_2},$$

$$\mu = \frac{l_1m_2 - l_2m_1}{l_2m_3 - l_3m_2};$$

$$\therefore\ d_1(l_2m_3 - l_3m_2) + d_2(l_3m_1 - l_1m_3) + d_3(l_1m_2 - l_2m_1) = 0$$

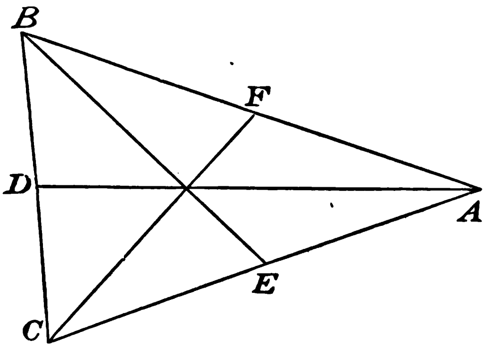
is the condition required.

Generally, however, the best way is to find the point of intersection of two straight lines, and to make its coordinates satisfy the third. Such straight lines are said to be concurrent.

Problems which prove that three straight lines pass through a point may generally be much simplified by a judicious choice of axes.

43. We will, by way of example, prove the well-known properties of a triangle, that the three sets of three straight lines passing through the angles, and (1) bisecting the sides, (2) perpendicular to the sides, (3) bisecting the angles, are concurrent.

In the first case, since the magnitudes of angles are not involved, we may take any axes: take *CB*, *CA* as the axes of x, y respectively.



Let *D*, *E*, *F* be the middle points of the sides, then the equation to *AD* is

$$\frac{x}{CD} + \frac{y}{AC} = 1, \text{ or } \frac{2x}{a} + \frac{y}{b} = 1.$$

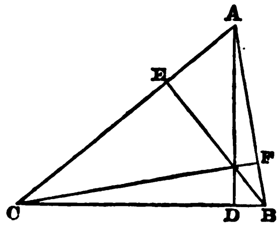
Similarly, the equation to *BE* is $\frac{x}{a} + \frac{2y}{b} = 1$, and that to *CF*, $\frac{x}{a} = \frac{y}{b}$, since the coordinates of *F* are $\frac{a}{2}, \frac{b}{2}$.

Subtract the equation to *BE* from that to *AD*, and we get $\frac{x}{a} = \frac{y}{b}$, the equation to *CF*;

$\therefore$ *AD*, *BE*, *CF* are concurrent.

(2) Next let *AD*, *BE*, *CF* be perpendicular to the sides.

Here, since right angles are involved, it is best to use rectangular coordinates.



Take BC, DA as axes, then

$$DB = c\cos B,\quad DC = -b\cos C,$$

$$DA = c\sin B = b\sin C;$$

therefore the equations to AB, AC are

$$\frac{x}{\cos B} + \frac{y}{\sin B} = c,\quad \frac{y}{\sin C} - \frac{x}{\cos C} = b \text{ respectively.}$$

Hence the equations to CF, BE, which are perpendicular to them, and pass respectively through the points $(-b\cos C, 0)$, $(c\cos B, 0)$, are

$$y\sin B - (x + b\cos C)\cos B = 0,$$

$$y\sin C + (x - c\cos B)\cos C = 0.$$

Multiply by $\sin C$ and $\sin B$ respectively and subtract, then we get $x = 0$; that is, AD passes through the intersection of BE, CF.

(3) Next, to prove that the bisectors of the angles are concurrent. We will use the property that if a straight line bisect an angle, the perpendiculars from any point on it on the arms of the angle must be equal.

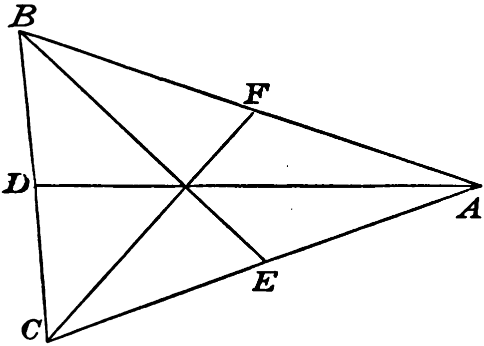
Take any rectangular axes, and let the equations to BC, CA, AB be

$$x\cos\alpha + y\sin\alpha = p \text{ ...... (1),}$$

$$x\cos\beta + y\sin\beta = q \text{ ...... (2),}$$

$$x\cos\gamma + y\sin\gamma = r \text{ ...... (3) respectively.}$$

Let AD, BE, CF be the bisectors.



Then the equation to AD is

$$x \cos\beta + y \sin\beta - q = x \cos\gamma + y \sin\gamma - r,$$

since these expressions are the lengths of the perpendiculars from (x, y) on (2), (3) respectively.

Similarly the equation to BE is

$$x \cos\alpha + y \sin\alpha - p = x \cos\gamma + y \sin\gamma - r;$$

therefore at the intersection of these two lines

$$x \cos\alpha + y \sin\alpha - p = x \cos\beta + y \sin\beta - q.$$

But this is the equation to CF, which therefore passes through the intersection of AD and BE.

Ex. 11.

1. Find the equation to the straight line which passes through the point (h, k) and through the intersection of

$$\frac{x}{a_1} + \frac{y}{b_1} = 1 \text{ and } \frac{x}{a_2} + \frac{y}{b_2} = 1.$$

2. Find the equation to the straight line which passes through the intersection of

$$A_1x + B_1y + C_1 = 0, \; A_2x + B_2y + C_2 = 0,$$

and also through the origin.

3. Find the equation to the straight line which passes through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $y = mx$, and is perpendicular to the former line.

4. Find the equation to the straight line which passes through the intersection of $l_1x + m_1y = d_1$, $l_2x + m_2y = d_2$, and is parallel to $l_3x + m_3y = d_3$.

5. Find the equation to the straight line which passes through the intersection of

$$x \cos \alpha + y \sin \alpha = p, \quad x \cos \beta + y \sin \beta = p,$$

and also through that of $y = mx + c$, $x = my + c$.

6. If $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $y = mx$ intersect in a point, find the value of m.

7. Find the equations to the straight lines which pass through the intersection of $y = 2x + 4$, $y = 3x + 6$, and bisect the supplementary angles which they include.

8. Find the condition that the intersection of

$$x \cos \alpha + y \sin \alpha = p, \quad x \sin \alpha + y \cos \alpha = q$$

should lie on the straight line which joins the points (2, 3), (3, 2).

44. Although the equation to a straight line is of one dimension only, yet it does not follow that every equation of more than one dimension does not represent straight lines.

Take for instance the equation

$$xy - a(x + y) + a^2 = 0.$$

This may be put in the form $(x - a)(y - a) = 0$,

$$\therefore \text{ either } x - a = 0, \text{ or } y - a = 0,$$

that is to say, the locus of the equation is two straight lines. Generally, whenever an equation can be resolved into simple factors, it is satisfied by putting each factor separately equal to zero, and is therefore the locus of the various lines whose equations are so obtained.

The locus of an equation may be a point, or may be impossible. For instance, let

$$(x - y)^2 + (x + y + a)^2 = 0.$$

Here $x-y=0$, and also $x+y+a=0$, since if the sum of two squares be zero, each of them must vanish;

$$\therefore\ x=y=-\frac{a}{2}$$

is the only point which satisfies the equation.

Similarly, if we have the equation

$$x^2+(x-y)^2+a^2=0,$$

the locus is impossible, since no real values of x and y can be found such that $x^2+(x-y)^2$ shall be negative.

The easiest way of testing an equation of the second degree which does not split up into factors by inspection is to treat it as a quadratic in either x or y.

Ex. (i) $x^2-4xy-5y^2+2x-16y=3$ may be written

$$x^2-2(2y-1)x+(2y-1)^2=(2y-1)^2+5y^2+16y+3=(3y+2)^2;$$

$$\therefore\ x-2y+1=\pm(3y+2);$$

$$\therefore\ x-5y=1 \text{ or } x+y+3=0, \text{ two straight lines.}$$

(ii) $x^2-6xy+10y^2+2x-6y+1=0,$

$$(x-3y+1)^2+y^2=0;$$

$$\therefore\ x-3y+1=0,\ y=0 \text{ simultaneously,}$$

$$x=-1,\ y=0, \text{ a point on the axis of } x.$$

(iii) The equation

$$Ax^2+2Bxy+Cy^2=0$$

represents two straight lines through the origin, or the origin itself.

For it is equivalent to

$$Ax=(-B\pm\sqrt{B^2-AC})\,y,$$

$$\text{or } Ax+By=\pm\sqrt{(B^2-AC)}\,y.$$

If B^2-AC is positive, this equation represents two real lines, but if B^2-AC is negative, the only real values of x and y are zero. In this case, however, it is better to say that the equation represents two imaginary lines.

For the condition that the general equation of the second degree should represent right lines, the student is referred to Chapter VII.

It is, however, easy to find the angle between them if it does.

Let the equation

$$Ax^2+2Bxy+Cy^2+2Dy+2Ey+F=0$$

represent the two lines

$$y=x\tan\theta+b,\quad y=x\tan\theta'+b';$$

then it must be equivalent to

$$\{y-(x\tan\theta+b)\}\,\{y-(x\tan\theta'+b')\}=0;$$

$$\text{or } y^2-(\tan\theta+\tan\theta')\,xy+x^2\tan\theta\tan\theta'+x(b\tan\theta'+b'\tan\theta)$$
$$-y(b+b')+bb'=0;$$

$$\therefore\ \tan\theta+\tan\theta'=-\frac{2B}{C},\quad \tan\theta\tan\theta'=\frac{A}{C}.$$

Now, if ϕ be the angle between these lines

$$\tan\phi=\tan(\theta\sim\theta')=\frac{\tan\theta\sim\tan\theta'}{1+\tan\theta\tan\theta'}$$

$$=\left\{\frac{4(B^2-AC)}{(A+C)^2}\right\}^{\frac{1}{2}}=\frac{2\sqrt{B^2-AC}}{A+C}.$$

We omit the ambiguity of sign, and consider ϕ to be the acute angle between the lines.

If $\phi=\frac{\pi}{2}$, $A+C=0$;

$$\therefore\ \text{the equation } y^2+\lambda xy-x^2=0$$

always represents two straight lines through the origin at right angles to each other.

EXAMPLES ON CHAPTER II.

1. If θ be the angle between the lines whose equations are $lx+my=d$, $l'x+m'y=d'$ respectively, then

$$\cos\theta=\frac{ll'+mm'}{(l^2+m^2)^{\frac{1}{2}}(l'^2+m'^2)^{\frac{1}{2}}}.$$

2. Find the area included between the lines

$$x=y, \quad x+y=0, \quad x=c.$$

3. Find that contained by the lines

$$x+y=a, \quad 2x=y+a, \quad 2y=x+a.$$

4. Find that contained by the lines

$$\frac{x}{a}+\frac{y}{b}=1, \quad y=2x+b, \quad x=2y+a.$$

5. Find that contained by the lines

$$x\cos\alpha+y\sin\alpha=p, \quad x\cos\beta+y\sin\beta=q, \quad x\cos\gamma+y\sin\gamma=r.$$

6. Find the equations to the straight lines which pass through the point (h, k), and form with the axes a triangle of given area.

7. If the straight lines

$$x\cos\alpha_1+y\cos\beta_1=p_1, \quad x\cos\alpha_2+y\cos\beta_2=p_2.$$

have equal portions intercepted between the axes,

$$p_1\cos\alpha_2\cos\beta_2=p_2\cos\alpha_1\cos\beta_1.$$

8. Interpret the equations

(i) $xy=0$, (ii) $x^2-y^2=0$, (iii) $x^2y=xy^2$,

(iv) $AB(x^2+y^2)+(A^2+B^2)xy+Bx+Ay=0$,

(v) $x^2\cos^2\theta-y^2\sin^2\theta=p^2-2py\sin\theta$,

(vi) $x^3+2x^2y-2xy^2-4y^3=0$,

(vii) $x^2-2xy-3y^2+2x-2y+1=0$,

(viii) $x^2-4xy+5y^2-6y+9=0$,

(ix) $x^2-4xy+3y^2+6y-9=0$.

9. Interpret the equations

(i) $(r^2-a^2)^2+b^2(\theta^2-\alpha^2)^2=0$, (ii) $(r-a)^2+b^2(\theta-\alpha)^2=0$,

(iii) $(r-a)^2(\theta-\alpha)^2+(r-b)^2(\theta-\beta)^2=0$.

10. The equation

$$\frac{x^2}{a^2}+\frac{b^2}{y^2}+\frac{a^2y^2}{b^2x^2}=\frac{y^2}{b^2}+\frac{a^2}{x^2}+\frac{b^2x^2}{a^2y^2}$$

represents the sides and diagonals of a parallelogram.

11. The straight lines bisecting the angles between those represented by $ax^2+2bxy+cy^2=0$ are represented by

$$b(x^2-y^2)=(a-c)\,xy.$$

12. The distance of (x_1y_1) from each of two straight lines through the origin is δ, prove that the straight lines are represented by the equation $(x_1y-xy_1)^2=(x^2+y^2)\,\delta^2$.

13. Shew that the straight lines bisecting the angles between the lines $y(h-a)=k(x-a)$ and $y(h+a)=k(x+a)$ are represented by

$$\{(x-h)^2+(y-k^2)\}\,hk=(x-h)\,(y-k)\,(h^2-k^2+a^2).$$

14. The equation

$$2y^2-3xy-2x^2-y+2x=0$$

represents two straight lines at right angles to each other.

15. The equation

$$y^2-2xy\sec\alpha+x^2=0$$

represents two straight lines including an angle α.

16. If the equation

$$ax^2+by^2+2cxy+2a'x+2b'y+c'=0$$

represent two parallel straight lines, $ab'=a'c$ and $a'b=b'c$.

17. The equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2$$

represents two straight lines through the point (α, β).

(*In Questions* 18—25 *the axes are inclined at an angle* ω.)

18. Find the equation to that straight line which passes through the point (a, b), and of which the portion intercepted between the axes is bisected at that point.

19. BOA is a given angle, AB is of fixed length and passes through the point (a, b): find its equation.

20. Find the equations to the straight lines which pass through the point $(a, 0)$, and are perpendicular to the axes.

21. Find the equations to the straight lines which pass through the point $(0, b)$, and (i) make angles a with the axis of x, (ii) make angles β with the axis of y.

22. The straight lines

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

will be perpendicular to each other if

$$AA' + BB' - (AB' + A'B) \cos \omega = 0.$$

23. The same straight lines will be equally inclined to the axis of x in opposite directions, if $\frac{B}{A} + \frac{B'}{A'} = 2 \cos \omega$.

24. The equation

$$x^2 + 2xy \cos \omega + y^2 \cos 2\omega = 0$$

represents two straight lines through the origin, which make equal angles with the axis of x, and are perpendicular to each other.

25. If the angle between $\frac{x}{l} = \frac{y}{m}$ and $\frac{x}{l'} = \frac{y}{m'}$, be $\frac{\pi}{2} - \omega$, then

$$(ll' + mm') \cos \omega + (lm' + l'm) \cos^2 \omega \pm (lm' - l'm) \sin^2 \omega = 0.$$

26. The equations

$$\frac{c}{r} = A \cos(\theta - a) + B \sin(\theta - \beta) \ldots\ldots\ldots\ldots (1),$$

$$\frac{c}{r} = A' \cos(\theta - a) + B' \cos(\theta - \beta) \ldots\ldots\ldots\ldots (2),$$

$$\frac{c}{r} = A \sin(\theta - a) - B \cos(\theta - \beta) \ldots\ldots\ldots\ldots (3),$$

$$\frac{c}{r} = A' \sin(\theta - a) + B' \cos(\theta - \beta) \ldots\ldots\ldots\ldots (4),$$

represent straight lines such that (3) and (4) are perpendicular to (1), (2).

27. Find the coordinates of the point of intersection of the lines $\frac{2a}{r} = \cos\left(\theta - \frac{\pi}{2}\right)$, $\frac{a}{r} = \cos\left(\theta - \frac{\pi}{6}\right)$, and the angle between them.

28. If the hour-hand and minute-hand of a watch be $\frac{5}{8}$ of an inch and an inch in length respectively, find the distance of the line joining their extremities from the centre of the watch at half-past one o'clock.

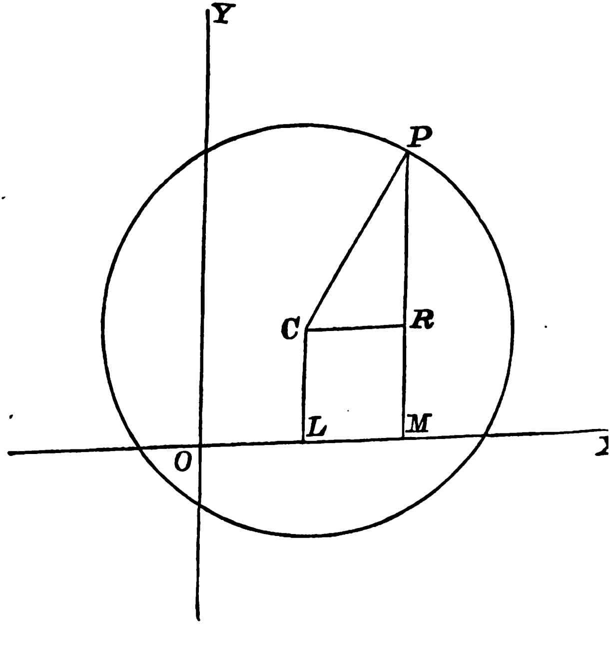
29. $ABCD$ is a rectangle and P any point: straight lines are drawn through A, B, C, D perpendicular to PA, PB, PC, PD respectively: prove that two of the diagonals of the quadrilateral so formed are parallel to the sides of the rectangle, and that the third diagonal is perpendicular to the line joining P with the intersection of the other two.

30. If on the sides of a triangle, taken in turn as diagonals, be constructed parallelograms the sides of which are parallel to two fixed lines, the other diagonals of these parallelograms will pass through a point.

CHAPTER III. *The Circle.*

45. Next in simplicity to the straight line comes the circle: we proceed to find its equation.

Let C be the centre, and let the radius be c.



Let OM, PM, the coordinates of P, any point on the circle, be x, y respectively.

Draw CR parallel to the axis of x meeting PM in R.

Let the coordinates of C be a and b.

Then we have to express the condition that CP is equal to c, in terms of x, y, a, b.

First, let the coordinates be rectangular.

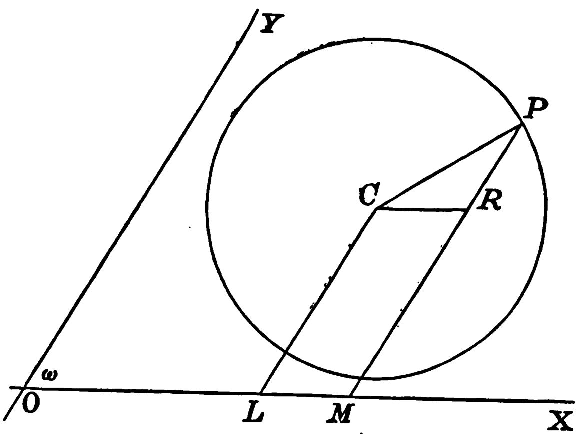
Then $CR^2 + PR^2 = CP^2$.

But $CR = x - a$, $PR = y - b$;

$$\therefore \ (x-a)^2 + (y-b)^2 = c^2$$

is the equation required.

Secondly, let the axes be inclined at an angle ω.



Then $CR^2 + RP^2 - 2CR \, . \, RP \cos CRP = CP^2$.

But $CRP = \pi - \omega$;

$$\therefore \ (x-a)^2 + (y-b)^2 + 2(x-a)(y-b) \cos \omega = c^2$$

is the equation referred to oblique axes.

46. *Reduction of the equation.*

Let O and C coincide; then $a=0$, $b=0$, and the equation becomes $x^2+y^2=c^2$.

This is the form most generally used.

Any results obtained for this form can be generalized by writing $x-a$, $y-b$, for x, y respectively.

In the equation $(x-a)^2+(y-b)^2=c^2$, if $a^2+b^2=c^2$, the origin is on the curve, the equation to which becomes

$$x^2+y^2=2(ax+by).$$

If in addition $a=0$, the centre is on the axis of y, which is therefore a diameter.

Similarly, if $b=0$, $a=c$, the axis of x is a diameter.

47. *To find the condition that the general equation of the second degree*

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \qquad \text{.........(1)}$$

shall represent a circle.

If possible, let it be the equation to that circle whose centre is the point a, b, and radius c.

First, let the coordinates be rectangular.

Then the equation must be identical with

$$x^2+y^2-2ax-2by+a^2+b^2-c^2=0.$$

Divide by A and equate coefficients of xy, y^2, x, y and the constant terms. Therefore

$$B=0, \quad \frac{C}{A}=1,$$

$$\frac{2D}{A}=-2a,$$

$$\frac{2E}{A}=-2b,$$

$$\frac{F}{A}=a^2+b^2-c^2.$$

The equation (1) will therefore represent a circle if $B=0$, $C=A$, or the general equation to a circle is

$$Ax^2+Ay^2+2Dx+2Ey+F=0.$$

Hence, if the equation

$$x^2+y^2+2Dx+2Ey+F=0$$

represents a circle, the coordinates of the centre are

$$-D,\ -E$$

and the radius is $(D^2+E^2-F)^{\frac{1}{2}}$.

Secondly, let the coordinates be oblique.

The equation $(x-a)^2+(y-b)^2+2(x-a)(y-b)\cos\omega=c^2$ when expanded and rearranged becomes

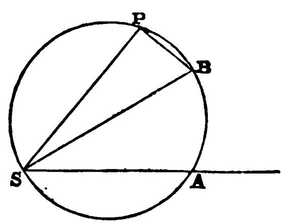
$$x^2+2xy\cos\omega+y^2-2(a+b\cos\omega)x-2(b+a\cos\omega)y$$
$$+a^2+2ab\cos\omega+b^2-c^2=0.$$

Hence we must have $A=C$, $\frac{B}{A}=\cos\omega$.

48. *To find the polar equation to a circle.*

(i) Let the centre be pole and the radius c, then the equation is evidently $r=c$.

(ii) Let the pole S be on the circumference, and let the diameter SB make an angle a with the initial line SA.



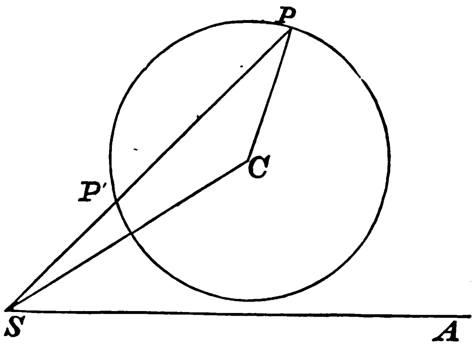
Let the coordinates of P, any point on the circle, be r, θ.

Join BP.

Then $$SP=SB\cos BSP,$$

or $$r = 2c\cos(\theta - a),$$
the equation required.

(iii) Let the coordinates of the centre C be l, a.



Then, in the triangle SCP,
$$SP^2 - 2SP \,.\, SC\cos PSC + SC^2 - CP^2 = 0,$$
or $$r^2 - 2lr\cos(\theta - a) + l^2 - c^2 = 0,$$
the equation required.

N.B. In this equation if r_1, r_2 be the two values SP, SP' of r corresponding to any value of θ, we know that
$$r_1 r_2 = l^2 - c^2.$$

This proves that, if from any point a straight line be drawn cutting a circle, the rectangle contained by its segments is constant.

If the pole be within the circle, $l^2 < r^2$ and $\therefore$ $r_1 r_2$ have opposite signs, that is, they are drawn in opposite directions.

Ex. 12.

1. Determine the radii of the circles denoted by the following equations:
$$x^2 + y^2 = 9a^2,$$
$$(x + y)^2 + (x - y)^2 = 8a^2,$$
$$\frac{x^2 + y^2}{a^2 + b^2} = 1.$$

2. Determine the coordinates of the centre, and the radius of each of the circles denoted by the following equations:

$$x^2+y^2-2a(x-y)=c^2,$$

$$x^2+y^2+ax+by=a^2+b^2,$$

$$x^2+y^2-3x-4y+4=0,$$

$$x^2+y^2=ax,$$

$$x^2+y^2=by,$$

$$x^2+y^2=a(x+a),$$

$$x^2+y^2=ax+by.$$

3. Find the equation to the circle whose radius is a, and coordinates of the centre a, $-a$.

4. To that whose radius is $5a$, and coordinates of centre $3a$, $4a$.

5. To that whose radius is c, and coordinates of centre $b+c$, $b-c$.

6. To that which passes through the origin and cuts off lengths a, b from the axes.

7. To that which passes through the origin, and two given points (x_1y_1), (x_2y_2).

8. To that which passes through the three points (x_1y_1), (x_2y_2), (x_3y_3).

9. To that which passes through the three points $(a, 0)$, $(0, b)$, $(2a, 2b)$.

10. In questions 6 and 9 find the coordinates of the centre and the radius of each circle.

11. Find the equation to the circle which has its centre in the line $lx=my$, and cuts off chords of length $2a$, $2b$ from the axes.

12. Find the equation to the four circles whose radius is $\sqrt{2}a$, and which cut off chords from each axis equal to $2a$.

13. Find the equation to the circle which passes through the origin and cuts off equal lengths a from the lines $y=x$, $x+y=0$.

14. The equation to the circle whose centre is the origin and radius a, the axes being inclined at an angle ω, is

$$x^2+2xy\cos\omega+y^2=a^2.$$

15. Find the angle between the axes when the equation

$$x^2+xy+y^2-a(x+y)-b^2=0$$

represents a circle.

Find also its radius and the coordinates of its centre.

16. The axes being inclined at an angle ω, find the equation to the circle which passes through the origin and the points $(h, 0)$, $(0, k)$.

17. Find the angle between the axes when the equation

$$x^2+y^2=a^2+\sqrt{3}xy+ax$$

represents a circle: find the radius and coordinates of the centre.

18. If the equation

$$a^2(x+y-a)^2=2b^2xy$$

represent a circle, determine the angle between the axes, the centre, and the radius.

19. Prove analytically, Euc. III. Props. 3, 9, 14, 15.

20. Find the polar equation to the circle which has its centre at the pole and cuts off a chord of given length from the line

$$r\cos(\theta-a)=p.$$

21. Prove, using polar coordinates, Euc. III. 7, 8.

22. If two chords of a circle intersect at right angles, the sum of the squares of the segments is together equal to the square of the diameter.

23. Find the length of the chord of the circle

$$x^2+y^2=c^2,$$

the equation to the chord being $\frac{x}{a}+\frac{y}{b}=1$.

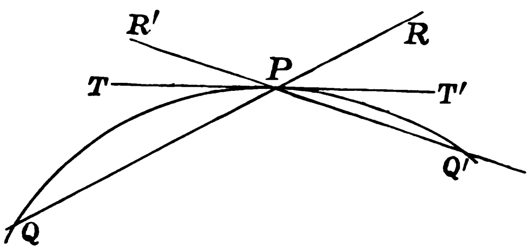
24. The equation to the circle whose diameter is the straight line joining the points (x_1y_1), (x_2y_2) is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Tangent and Normal.

49. We might proceed to find the equation to the tangent to a circle at any point from the consideration that it is perpendicular to the diameter through that point; we will, however, give a definition of a tangent, and a method of finding its equation, which will be applicable to all curves.

Def. Let QPQ' be a curve, P a point on it, Q any other point on it; draw the secant QPR; let Q move



along the curve to P; then the limiting position of the secant QPR, when Q *moves up to and ultimately coincides* with P, is called the tangent to the curve QPQ' at P.

It is evident that on whichever side of P we take Q, for every position of Q there is a definite position of PQ; there must therefore be some position when the point Q is neither on *one* side of P nor on the *other*, but *at* P: this position is called the tangent at P.

50. *To find the equation to the tangent at any point of a circle.*

Let the point be (x_1y_1), and the equation to the circle

$$x^2+y^2=c^2 \qquad (1),$$

and first let us find the secant passing through the points (x_1y_1), (x_2y_2).

The equation to the straight line through these points has been already found, it is (Art. 23)

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad (2).$$

Now if the points P, Q coincide, $x_2=x_1$; $y_2=y_1$, and the fraction $\frac{y_2-y_1}{x_2-x_1}$ assumes the indeterminate form $\frac{0}{0}$.

We have not, however, introduced the condition that these points should lie on the circle.

Since $$x_1^2+y_1^2=c^2=x_2^2+y_2^2,$$

$$\therefore y_2^2-y_1^2=x_1^2-x_2^2,$$

$$\therefore \frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1}.$$

Substituting in equation (2) we obtain

$$y-y_1+\frac{x_2+x_1}{y_2+y_1}(x-x_1)=0.$$

Now let Q coincide with P, or $x_2=x_1$, $y_2=y_1$, and the equation becomes

$$y-y_1+\frac{x_1}{y_1}(x-x_1)=0,$$

or $$yy_1+xx_1=y_1^2+x_1^2=c^2:$$

$$xx_1+yy_1=c^2$$

is therefore the equation required.

If the centre be not the origin but the point h, k, transfer the origin to that point and we obtain

$$(x-h)(x_1-h)+(y-k)(y_1-k)=c^2.$$

51. *To find the points where the tangent cuts the axes.*

Putting y and x successively equal to zero in the equation to the tangent, we obtain

$$x=\frac{c^2}{x_1},\ y=\frac{c^2}{y_1}.$$

52. Def. The normal to a curve at any point is the perpendicular to the tangent at that point.

To find the equation to the normal at the point (x_1y_1).

Since the normal is perpendicular to the tangent

$$xx_1+yy_1=c^2,$$

its equation must be

$$(x-x_1)\,y_1-(y-y_1)\,x_1=0\,;\ \text{(Art. 34.)}$$

or

$$xy_1-yx_1=0.$$

The normal therefore passes through the centre.

53. To find the condition that the line

$$lx+my=d$$

should touch the circle $x^2+y^2=c^2$.

The simplest method of finding the condition that a straight line should touch a circle is to make the perpendicular from the centre equal to the radius.

Now the perpendicular from the origin on $lx+my=d$ is

$$\frac{d}{\sqrt{l^2+m^2}}\,;\ \therefore \frac{d}{\sqrt{l^2+m^2}}=c \text{ or } d^2=(l^2+m^2)\,c^2$$

is the condition required.

The straight line $x\cos\theta+y\cos\theta=c$, of course touches the circle, since the perpendicular from the origin is equal to c, and it is the simplest form of the equation to take in all questions in which the point of contact is not involved.

The student should always consider carefully before beginning any problem whether the coordinates of the point of contact are (or are not) involved.

54. We will give another method of finding this condition which is applicable to any curve, and agrees with our definition of a tangent.

If we eliminate y between

$$lx + my = d,$$
$$x^2 + y^2 = c^2,$$

we shall obtain a quadratic in x, the roots of which will give us the abscissæ of the points where the line *cuts* the circle.

If the line *touch* the circle, the points of section must coincide, and the roots become equal.

The equation $lx + my = d$ may be written

$$my = d - lx.$$

Multiply both sides of the equation $x^2 + y^2 = c^2$ by m^2 and substitute for y;

$$\therefore\ m^2x^2 + (d - lx)^2 = m^2c^2,$$

$$(l^2 + m^2)\,x^2 - 2ldx + d^2 - m^2c^2 = 0.$$

If this equation has equal roots

$$l^2d^2 = (d^2 - m^2c^2)(l^2 + m^2),$$
$$d^2 = (l^2 + m^2)\,c^2,$$

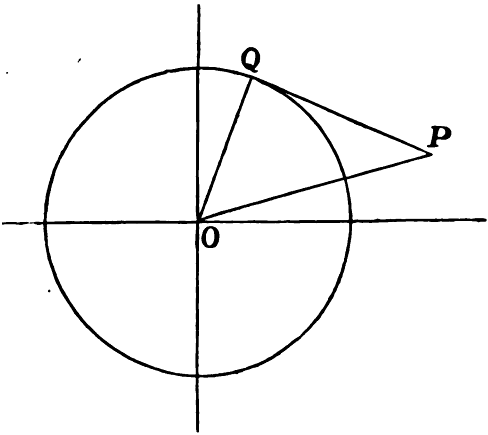
the condition we obtained before.

55. *To find the length of the tangent drawn from the point* xy *to the circle*

$$\mathrm{x}^2 + \mathrm{y}^2 = \mathrm{c}^2.$$

Let PQ be the tangent, then

$$PQ^2 = OP^2 - OQ^2 = x^2 + y^2 - c^2.$$



ence that expression which is equal to zero when the (xy), is on the circle, is equal to the square of the h of the tangent from (xy) when the point is without ircle.

' the point be within the circle,

$$x^2+y^2-c^2 \text{ is negative,}$$

uare root is therefore impossible; this shews that no etrical tangent can be drawn to the circle from a within it.

;. We saw (Art. 25) that if a straight line makes an θ with the axis of x and pass through a point (x_1y_1), ay write $x=x_1+r\cos\theta$, $y=y_1+r\sin\theta$, where r is the ıce between the points (xy) and (x_1y_1).

ɔw write these values in the equation to the circle and ge by powers of r; it becomes

$$r^2+2(x_1\cos\theta+y_1\sin\theta)r+x_1^2+y_1^2-c^2=0.$$

ence if r_1, r_2 be the roots of this equation

$$r_1r_2=x^2+y^2-c^2$$

l values of θ: this proves Euc. III. 35—37.

Ex. 13.

1. Write down the equations to the tangents to the circle

$$x^2+y^2=c^2$$

which pass through the points on the circle,

$$c,\ 0;\ \frac{3}{5}c,\ -\frac{4}{5}c;\ -h,\ k.$$

2. Find the equations to the tangents to the circle

$$x^2+y^2=c^2$$

which have the following properties respectively;

(i) Make a given angle with the axis of x:

(ii) Are parallel to $\frac{x}{a}+\frac{y}{b}=1$:

(iii) Are perpendicular to

$$Ax+By+C=0:$$

(iv) Pass through a given point on the axis of y:

(v) Are at a distance δ from the point $a\cos\alpha,\ a\sin\alpha$:

(vi) Cut off a triangle of area $\frac{c^2}{2}$ from the axes.

3. Find the condition that the lines

$$Ax+By+C=0,$$

$$y-y_1=(x-x_1)\tan\alpha,$$

$$y=nx+b,$$

should touch the circle.

4. Shew that the equation to the tangent to the circle

$$x^2+y^2=ax+by,$$

which passes through the origin, is

$$ax+by=0.$$

5. Prove that the circles and lines whose equations are here given touch each other respectively, and find the points of contact in each case:

$x^2+y^2+ax+by=0$, and $ax+by+a^2+b^2=0$;

$x^2+y^2-2ax-2by+b^2=0$, and $x=2a$;

$x^2+y^2-2c(x+y)+c^2=0$, the axes, and

$$(x-c)\cos\theta+(y-c)\sin\theta=c;$$

$x^2+y^2=ax+by$, and $ax-by+b^2=0$.

6. Find the condition that the straight line

$$x\cos\theta+y\sin\theta=p$$

should touch the circle

$$x^2+y^2=2(ax+by).$$

7. Prove that the tangent at the point (x_1y_1) to the circle

$$x^2+2xy\cos\omega+y^2=c^2$$

is $$(x_1+y_1\cos\omega)x+(y_1+x_1\cos\omega)y=c^2.$$

8. Prove that the straight line

$$x\cos\alpha+y\cos\beta=c$$

will touch the circle

$$x^2+2xy\cos(\alpha+\beta)+y^2=c^2.$$

9. Prove that the straight line

$$r\cos(\theta-\alpha)=a$$

touches the circle $r=a$ at the point a, α.

10. Prove that the tangent to the circle

$$r=l\cos(\theta-\alpha),$$

at the point for which $\theta=\beta$, is

$$r\cos(\theta+\alpha-2\beta)=l\cos^2(\beta-\alpha).$$

11. Find the tangents to the circle $r=l\cos(\theta-\alpha)$ at the pole, and at the extremity of the diameter through the pole.

12. Determine the point of contact of that tangent to the circle $r=l\cos(\theta-\alpha)$ which makes an angle γ with the initial line.

57. We are often required to find the locus of a point which moves subject to some given law: no general rule can be given for finding the equation to such a locus; it generally however results from elimination between two or more equations.

We will give a few examples.

The figures are simple, and it will be useful to the student to draw them for himself.

(i) To find the locus of a point the distances of which from two given points are in a constant ratio.

Let O, A be the two given points, P a point on the locus.

Take O as origin, OA as axis of x; let $OA=a$, and let the coordinates of P be (x, y): let $OP=mAP$.

$$\text{Now} \qquad OP=(x^2+y^2)^{\frac{1}{2}},\ AP=\{(x-a)^2+y^2\}^{\frac{1}{2}};$$

$$\therefore\ x^2+y^2=m^2\{(x-a)^2+y^2\},$$

$$(1-m^2)(x^2+y^2)+2am^2x-m^2a^2=0.$$

The locus is therefore (Art. 46) a circle, of which the centre is on the axis of x.

(ii) To find the locus of the intersection of two straight lines, which pass each through a given point and contain a given angle. (Of course we know from geometry that this locus is a circle, we will however obtain this result analytically.)

Let A, B be the given points, and let $AB=2a$.

Let P be a point on the locus, and $APB=\alpha$.

Take the middle point of AB as origin, AB as axis of x.

Let AP make an angle θ with the axis of x.

Then its equation is (Art. 24)

$$y=(x-a)\tan\theta \text{.......................}(1).$$

Let BP make an angle ϕ with the axis, then its equation is

$$y=(x+a)\tan\phi \text{.......................}(2).$$

But $\theta - \phi = \alpha$;

$$\therefore \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan\alpha.$$

Now, from (1) and (2),

$$\tan\theta = \frac{y}{x-a}, \quad \tan\phi = \frac{y}{x+a};$$

$$\therefore \frac{y(x+a) - y(x-a)}{(x^2 - a^2) + y^2} = \tan\alpha;$$

$$\therefore x^2 + y^2 - 2ay\cot\alpha = a^2,$$

the equation to the circle whose centre is on the axis of y at a distance $a\cot\alpha$ from the origin, and radius $a\operatorname{cosec}\alpha$.

(iii) ABC is a triangle, P a point such that the sum of its distances from the sides is constant; find the locus of P.

Let the sum of the distances be c, and the equations to the the sides

$$x\cos\alpha + y\sin\alpha = p,$$

$$x\cos\beta + y\sin\beta = q,$$

$$x\cos\gamma + y\sin\gamma = r.$$

Then if (x, y) be the coordinates of P, the distances of P from the sides are

$$x\cos\alpha + y\sin\alpha - p, \quad x\cos\beta + y\sin\beta - q, \quad x\cos\gamma + y\sin\gamma - r,$$

respectively.

$$\therefore x(\cos\alpha + \cos\beta + \cos\gamma) + y(\sin\alpha + \sin\beta + \sin\gamma) = p + q + r + c,$$

the equation to a straight line.

(iv) C is a fixed point, and through C a straight line is drawn to cut a fixed circle in P and Q; find the locus of a point R on this line such that CR is (α) an arithmetic, (β) a geometric, (γ) a harmonic mean between CP and CQ.

Take the centre of the given circle as the origin of rectangular axes, and let its equation be

$$x^2 + y^2 = c^2 \quad \text{.........................(1).}$$

Let CPQ make an angle θ with the axis of x, and let the coordinates of C be h, k: then the equation to CQ may be written

$$\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r,\ \text{(Art. 25)}\ \ldots\ldots\ldots\ldots(2),$$

where r is the distance between the points (xy), (hk).

Substituting the values of x and y obtained from equations (2) in (1), we obtain a quadratic in r, which gives the lengths of CP and CQ.

Rearranging this quadratic we get

$$r^2+2r(h\cos\theta+k\sin\theta)+h^2+k^2-c^2=0\ \ldots\ldots\ldots(3).$$

If r_1, r_2 be the roots of this quadratic, and (x, y) be now the coordinates of R the point in question, and if $CR=\rho$; then, since R is on (2),

$$x=h+\rho\cos\theta,\ y=k+\rho\sin\theta.$$

Now, from (3),

$$r_1+r_2=-2(h\cos\theta+k\sin\theta),\ r_1r_2=h^2+k^2-c^2.$$

Then, (α) if ρ be an arithmetic mean between r_1 and r_2,

$$\rho=\frac{r_1+r_2}{2}=-(h\cos\theta+k\sin\theta);$$

$$\therefore\ \rho^2+h\rho\cos\theta+k\rho\sin\theta=0,$$

or $$(x-h)^2+(y-k)^2+h(x-h)+k(y-k)=0,$$

the equation to the circle on OC as diameter.

(β) If ρ be a geometric mean between r_1 and r_2,

$$\rho^2=r_1r_2 \text{ or } (x-h)^2+(y-k)^2=h^2+k^2-c^2,$$

the equation to the circle whose centre is C and radius CO.

(γ) If ρ be a harmonic mean between r_1 and r_2,

$$\frac{2}{\rho}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{r_1+r_2}{r_1r_2};$$

$$\therefore\ 2r_1r_2=(r_1+r_2)\rho;$$

$$\therefore\ h^2+k^2-c^2+(h\cos\theta+k\sin\theta)\rho=0;$$

$$\therefore\ h^2+k^2-c^2+h(x-h)+k(y-k)=0,$$

or $$hx+ky=c^2, \text{ a straight line.}$$

All these loci must pass through the points where tangents from C meet the circle, since at these points the values of r become equal, and therefore their means coincide with those equal values.

Portions of these loci are, however, without the circle, can we interpret the equations in these cases?

If θ in (2) have such a value that the line does not meet the circle, the values of r_1, r_2 which are the roots of (3) are imaginary.

Their sum and product are, however, real, and so therefore their means, and so we arrive at the anomaly of impossible points lying on a real line, and being such that the point of bisection of the line joining them is real.

This anomaly arises from the fact that in symbolical algebra we can attach a meaning to the impossible roots of equations while we are unable to interpret these expressions geometrically.

Our algebra is therefore more general than our geometry.

Ex. 14.

1. The locus of a point, the algebraic sum of whose distances from the sides of a polygon is constant, is a straight line.

2. O is a fixed point, OPQ a straight line; if P move on a fixed circle through O, and $OP \cdot OQ$ is constant, the locus of O is a straight line, and if P be on a straight line, the locus of Q is a circle.

3. The locus of a point the distances of which from two fixed lines are in a given ratio is a straight line.

4. ABC is a given triangle, PP' a straight line parallel to BC, cutting AB, AC in P, P' respectively; through P, P' straight lines are drawn perpendicular to AB, AC and intersecting in D: find the locus of D.

5. If through P, P' in the preceding problem, straight lines are drawn perpendicular to AC, AB intersecting in E, find the locus of E.

6. ABC is a triangle, $AB : AC$ is a given ratio; if B moves along a straight line, so does C.

7. Find the locus of a point such that its distance from the line

$$x \cos a + y \sin a = p,$$

is a third proportional to a given line b, and its distance from a given point (h, k).

8. AB, AC are fixed straight lines, DE a line of fixed length terminated by them; from D, E are drawn perpendiculars to AB, AC intersecting in P: find the locus of P.

9. AB is a given diameter of a circle of which C is the centre; DE a chord such that $DCE = 2a$; AD, BE intersect in F: prove that the locus of F is a circle.

10. ABC is a triangle of which BC is fixed, find the locus of A if $\tan B = m \tan C$.

Poles and Polars.

58. *From a given point tangents are drawn to a circle, to find the equation to the straight line passing through the points of contact.*

Let h, k be the coordinates of the given point, let x_1, y_1 be the coordinates of one point of contact, then the equation to the tangent at $(x_1 y_1)$ is

$$xx_1 + yy_1 = c^2;$$

but, since this passes through (h, k),

$$hx_1 + ky_1 = c^2.$$

Similarly, if x_2, y_2 be the coordinates of the other point of contact,

$$hx_2 + ky_2 = c^2,$$

since (h, k) lies on the tangent through $(x_2 y_2)$.

Hence, since

$$hx_1 + ky_1 = c^2,$$
$$hx_2 + ky_2 = c^2,$$

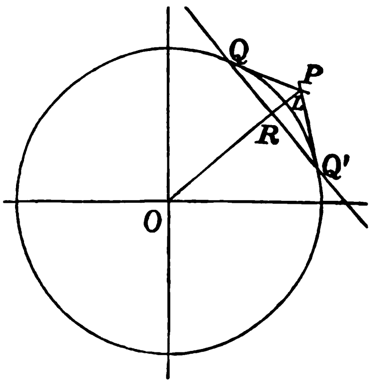
(x_1, y_1), (x_2, y_2) satisfy the equation

$$hx + ky = c^2;$$

this, however, is the equation to a straight line, and since it is satisfied by the coordinates of two points, $(x_1 y_1)$, $(x_2 y_2)$, it is the equation of the straight line which passes through those points: it is therefore the equation to the straight line required.

59. We have seen that when the point (h, k) is without the circle, the equation

$$hx + ky = c^2$$

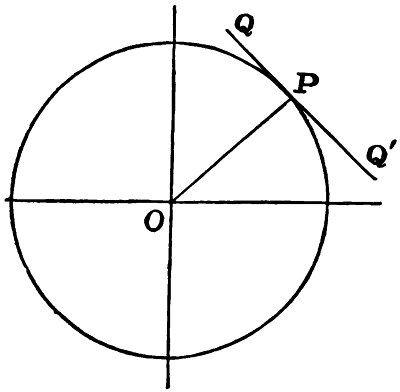


represents the chord of contact of tangents through (h, k); when (h, k) is on the circle, the same equation represents the tangent at (h, k); what will then this equation represent when the point (h, k) is within the circle, so that no real tangents can be drawn through it to the circle?

It is still the equation to a straight line, and since its form is unchanged whatever be the *position* of the point (h, k) the equation must represent some geometrical facts which are independent of that *position*.

Let P be the point (h, k) either without, on, or within the circle, then the equation to OP is

$$\frac{x}{h} = \frac{y}{k} \quad\text{.............................}\quad (1);$$



∴ (Art. 34), the line

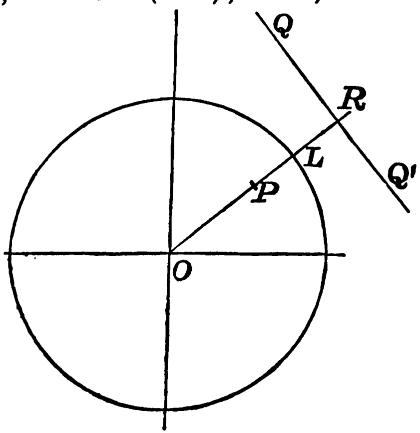
$$hx+ky=c^2 \quad \text{......................} \quad (2)$$

is perpendicular to OP.

Again, the distance of the origin from the line (2) is

$$\frac{c^2}{(h^2+k^2)^{\frac{1}{2}}}, \text{ (Art. 30).}$$

Now, $\qquad OP=(h^2+k^2)^{\frac{1}{2}}, \ OL=c;$



therefore if in OP or OP produced we take a point R such that

$$OR : OL :: OL : OP,$$

and through R draw QRQ' perpendicular to OP, the equation to QRQ' will be

$$hx + ky = c^2.$$

60. QRQ' is called the *polar* of P: conversely, P is called the *pole* of QRQ'.

The polar of a point may be defined either geometrically or algebraically.

Geometrically, thus: let O be the centre of a circle, P any point, join OP and divide it, produced if necessary, in R, so that OR is a third proportional to OP and the radius: through R draw a straight line QRQ' at right angles to OP: this straight line is called the polar of P.

Conversely, let QQ' be any straight line, draw OR perpendicular to QQ', and in OR, produced if necessary, take a point P such that OP is a third proportional to OR and the radius, then P is called the pole of QQ'.

Algebraically: let the coordinates of any point be h, k; then the straight line represented by the equation

$$hx + ky = c^2$$

is called the polar of (h, k), with respect to the circle

$$x^2 + y^2 = c^2.$$

Conversely, let the equation to any straight line be thrown into the form

$$hx + ky = c^2,$$

then (h, k) is the pole of the line.

Thus; required the pole of

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Multiply by c^2,

$$\therefore \frac{c^2}{a} x + \frac{c^2}{b} y = c^2,$$

$\therefore \frac{c^2}{a}, \frac{c^2}{b}$, are the coordinates of the pole.

The student is recommended to pay particular attention to the preceding articles; there is no part of the subject which it is more necessary to understand thoroughly.

61. *If Q lies on the polar of* P, *then the polar of* Q *passes through* P.

Let the coordinates of P be h, k, of Q, x', y', then, since Q is on the polar of P, Q lies on the line

$$hx + ky = c^2 \quad\text{.......................(1)},$$

$$\therefore hx' + ky' = c^2 \quad\text{.......................(2)}.$$

But the polar of (x', y') is

$$xx' + yy' = c^2 \quad\text{.......................(3)}.$$

In this equation if we put $x = h$, we get from (2), $y = k$.

Therefore the point (h, k) lies on the line (3), and conversely the line (3) passes through the point (h, k).

62. Similarly, if the polar of Q passes through P, then Q lies on the polar of P.

Let the coordinates of P be h, k, of Q, x', y', then the polar of Q is

$$xx' + yy' = c^2;$$

but, since this equation represents a straight line which passes through (h, k),

$$\therefore hx' + ky' = c^2;$$

$\therefore x', y'$, are the coordinates of a point which satisfies the condition

$$hx + ky = c^2;$$

that is, the point Q lies on the polar of P.

63. If the origin be not the centre, and the point (a, b) be the centre, we must in these equations write $x-a$, $y-b$, for x and y.

64. To find the polar of the origin with respect to the circle

$$(x-a)^2+(y-b)^2=c^2.$$

The polar of the point (h, k) is

$$(x-a)(h-a)+(y-b)(k-b)=c^2,$$

or $$(h-a)x+(k-b)y-ha-kb+a^2+b^2-c^2=0.$$

Put $h=0$, $k=0$, then

$$ax+by=a^2+b^2-c^2,$$

the equation required.

Two or more Circles.

65. *To find the equation to the straight line which passes through the intersection of two circles which cut each other.*

Let the equations to the circles be

$$x^2+y^2-2a_1x-2b_1y+a_1^2+b_1^2-c_1^2=0\ldots\ldots\ldots(1),$$

$$x^2+y^2-2a_2x-2b_2y+a_2^2+b_2^2-c_2^2=0\ldots\ldots\ldots(2).$$

Subtract one of these equations from the other: then

$$2(a_1-a_2)x+2(b_1-b_2)y=a_1^2-a_2^2+b_1^2-b_2^2-(c_1^2-c_2^2)\ldots(3).$$

This is the equation to the straight line required, for it is the equation to *some* straight line, and it passes through the intersection of (1) and (2), because whenever equations (1) and (3) are satisfied simultaneously, equation (2) is satisfied also.

66. *Let the expressions on the left-hand side of equations* (1), (2) *be denoted by* S_1, S_2, *respectively, then equation* (3) *may be written*

$$S_1 - S_2 = 0, \text{ or } S_1 = S_2 \text{..........................(4).}$$

Let us investigate the geometrical meaning of this expression.

We have seen that when the circles cut one another the chord of intersection is represented by equations (3) or (4); they may however not cut one another, in which case (3) or (4) will still represent a straight line.

Now we know (Art. 55) that $x^2 + y^2 - c^2$ is the square on the tangent drawn from (x, y) to the circle

$$x^2 + y^2 - c^2 = 0.$$

Similarly, $(x-a)^2 + (y-b)^2 - c^2$ is the square on the tangent drawn from (x, y) to the circle

$$(x-a)^2 + (y-b)^2 - c^2 = 0,$$

or S is the square of the tangent drawn from (x, y) to the circle $S = 0$.

Hence the meaning of equation (4) is, that for all points represented by it, the squares on the tangents, and therefore the tangents themselves, drawn from them to the circles, are equal to one another.

Hence, if we subtract the expression on the left-hand side of the equation to one circle from the similar expression in the equation to another circle we get the equation to a straight line, and, if from any point in this straight line we draw tangents to the circles, these tangents will be equal to one another.

This straight line is called the *radical axis* of the two circles.

67. Let the equations to three circles be

$$S_1 = (x-a_1)^2 + (y-b_1)^2 - c_1^2 = 0,$$
$$S_2 = (x-a_2)^2 + (y-b_2)^2 - c_2^2 = 0,$$
$$S_3 = (x-a_3)^2 + (y-b_3)^2 - c_3^2 = 0.$$

Then the equations to the radical axes of these circles taken two and two together are

$$S_1 - S_2 = 0,$$

$$S_2 - S_3 = 0,$$

$$S_3 - S_1 = 0.$$

At the point where the straight lines represented by the first two of these equations intersect, we have

$$S_1 - S_2 = S_3 - S_2 = 0,$$

or $$S_3 - S_1 = 0,$$

that is, the third straight line passes through the intersection of the first two.

Hence the three radical axes of any three circles meet in a point.

This point is called the *radical centre* of the three circles.

68. In the equation to the circle

$$(x-a)^2 + (y-b)^2 - c^2 = 0,$$

let $$c = 0, \quad \therefore\ x - a = 0,\ y - b = 0,$$

the equation therefore represents the point (a, b).

A point may therefore be considered as a circle of infinitely small radius. If in the equations to two circles we put the radius of one equal to zero, and subtract, we obtain the equation to the radical axis as before, which is now a straight line such that tangents from any point in it to a circle are equal in length to the distance of that point from a given point.

If both radii vanish, and both circles become points, the radical axis becomes the straight line, every point in which is equally distant from the two given points.

69. *To find the common tangents to two circles.*

Let the equations to the two circles be

$$x^2+y^2=c_1^2 \quad \text{.............................(1)},$$

$$(x-a)^2+y^2=c_2^2 \quad \text{.......................(2)}.$$

Let $\qquad x\cos\theta+y\sin\theta=p$

be the equation to the common tangent.

Then since this straight line touches (1),

$$p=\pm c_1.$$

Also, since it touches (2), the distance from the point a, 0, must be c_2.

But this distance is

$$\pm(a\cos\theta-p),$$

$$\therefore\ \pm(a\cos\theta\mp c_1)=c_2;$$

$$\therefore\ \cos\theta=\pm\frac{c_1\pm c_2}{a};$$

$$\sin\theta=\frac{\{a^2-(c_1\pm c_2)^2\}^{\frac{1}{2}}}{a}.$$

Hence the equation to the tangents may be written

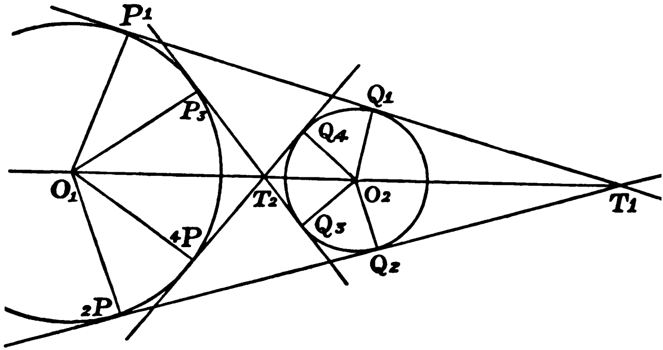
$$x(c_1\pm c_2)\pm y\{a^2-(c_1\pm c_2)\}^{\frac{1}{2}}=ac_1.$$

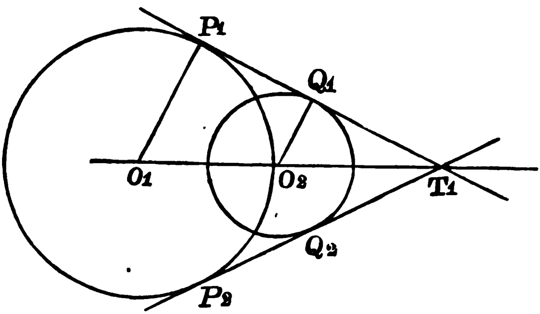
70. In the above equations; first, let $a>c_1+c_2$, or

$$O_1O_2>O_1P_1+O_2Q_2,$$

then, since the circles do not meet, there can be *four* common tangents drawn to them.

$$x(c_1+c_2)\pm y\{a^2-(c_1+c_2)^2\}^{\frac{1}{2}}=ac_1,$$





is the equation to the internal tangents which cut the axis of x in the point T_2 such that

$$O_1T_2 = \frac{O_1P_3}{O_1P_3 + O_2Q_3}\, O_1O_2,$$

$$\text{or } x = \frac{c_1}{c_1 + c_2}\, a\,;$$

and $$x(c_1 - c_2) \pm y\,\{a^2 - (c_1 - c_2)^2\}^{\frac{1}{2}} = ac_1,$$

the equations to the external tangents which cut O_1O_2 produced at a point T_1 such that

$$O_1T_1 = \frac{c_1}{c_1 - c_2} a.$$

T_1, T_2, are called the external and internal centres of similitude.

If $c_1 + c_2 > a$, the two circles cut one another, and only two tangents can be drawn to touch both circles.

This is shewn analytically by one of the values of $\cos\theta$ becoming greater than 1, which shews that no angle exists having the required cosine.

If $c_1 - c_2 > a$, both values of $\cos\theta$ are greater than 1; in this case all the equations become irrational, or no such tangents can be drawn.

In this case, since

$$O_1O_2 < O_1P_1 - O_2Q_1,$$
$$O_1O_2 + O_2Q_1 < O_1P_1,$$

therefore one circle lies entirely within the other.

EXAMPLES ON CHAPTER III.

1. Write down the polars with respect to the circle $x^2 + y^2 = c^2$ of the following points:

$$(c, c); \quad (2c, 3c); \quad (a + b, a - b).$$

2. Find the poles of the following lines:

$$Ax + By + C = 0; \quad \frac{x}{a} + \frac{y}{b} = 2; \quad y - y_1 = (x - x_1)\tan\theta; \quad x = a; \quad y = b.$$

3. If d be variable in the equation $lx + my = d$, the locus of the pole is a straight line.

4. If the pole always lie on the line $\frac{x}{a} + \frac{y}{b} = 1$, the equation to the polar is $(ax - c^2)\cos\theta + (by - c^2)\sin\theta = 0$, where θ is any angle.

5. If the pole lie on the circle $x^2+y^2=4c^2$, the polar will touch the circle $x^2+y^2=\frac{c^2}{4}$.

6. Prove that all circles represented by the equation

$$(x-a_1)^2+(y-b_1)^2-c_1^2=\lambda\{(x-a_2)^2+(y-b_2)^2-c_2^2\}$$

have a common radical axis.

7. Find the radical axes and radical centre of the circles

$$(x-1)^2+(y-2)^2=6,$$
$$(x-2)^2+(y-3)^2=8,$$
$$(x-3)^2+(y-1)^2=10.$$

8. The abscissæ of the centres of two circles are a_1, a_2, and the lengths of tangents from the origin l_1, l_2 respectively: shew that the radical axis cuts the axis of x at the point whose abscissa is $\frac{l_1^2-l_2^2}{2(a_1-a_2)}$.

9. A fixed circle is cut by a series of circles, all of which pass through two fixed points: shew that the radical centre is a fixed point.

10. Three circles have fixed centres, and their radii are $r_1+\rho$, $r_2+\rho$, $r_3+\rho$, where ρ is variable; shew that their radical centre lies on a fixed straight line.

11. If a series of circles be such that the polar of a fixed point with reference to any one of them is a fixed straight line, they will have a common radical axis.

12. The four tangents which are common to two circles which do not intersect, and which are terminated at the points of contact, have their middle points on the radical axis.

13. Find the equations to the four common tangents to the circles $x^2+y^2=4c^2$, $x^2+y^2-8cx+15c^2=0$.

14. Find the equations to the common tangents to

$$x^2+y^2=c^2,\quad x^2+y^2+a^2+2ac=2(a+c)y.$$

15. Prove that the circles

$$x^2+y^2=(c+a)^2, \quad (x-a)^2+y^2=c^2$$

have only one common tangent, and find its equation.

16. Shew that if α, β be the angular coordinates of the extremities of the chord of the circle $r=c\cos\theta$, the equation to the chord is $r\cos(\alpha+\beta-\theta)=\frac{c}{2}\cos(\alpha-\beta)$.

17. Find the length of that chord of the circle $x^2+y^2=c^2$, the equation to which is $\frac{x}{a}+\frac{y}{b}=1$.

18. Find the equation to the circle whose diameter is the common chord of the circles $x^2+y^2=c^2$ and $(x-a)^2+y^2=c^2$.

19. Find the equation to the straight line which joins the centres of $x^2+2x+y^2=0$ and $x^2+2y+y^2=0$.

20. Find also the tangents at the origin to these two circles.

21. Find the length of the common chord of

$$(x-a)^2+(y-b)^2=c^2,$$
$$(x-b)^2+(y-a)^2=c^2,$$

and hence prove that the condition that these two circles should touch each other is $2c^2=(a-b)^2$.

22. Find the length of the line drawn from (h, k) making an angle α with the axis of x to cut the circle

$$(x-a)^2+(y-b)^2=c^2,$$

and deduce the condition that it may touch the circle.

23. Two circles, whose radii are a and b, cut at an angle α, shew that the length of the common chord is

$$\frac{2ab\sin\alpha}{(a^2+2ab\cos\alpha+b^2)^{\frac{1}{2}}}.$$

24. If the centre of the first circle be the origin and that of the second on the axis of x, determine its equation.

25. If the chord of a circle pass through the middle point of a fixed chord, the tangents at its extremities cut off equal intercepts from the fixed chord, measured from that middle point.

26. The length of the tangent from (f, g) to the circle

$$x^2+y^2+2Ax+2By+C=0,$$

is $$(f^2+g^2+2Af+2Bg+C)^{\frac{1}{2}}.$$

27. The circles $$x^2+y^2+2ax+b^2=0,$$ $$x^2+y^2+2bx+a^2=0,$$

cut orthogonally.

28. Find the equations to the circles which pass through the point (h, k) and touch $x^2+y^2=c^2$.

29. Shew that the equation

$$(hx+ky-c^2)^2=(h^2+k^2-c^2)(x^2+y^2-c^2)$$

represents the tangents to the circle $x^2+y^2=c^2$ from the point (h, k).

30. If $\frac{x}{a}+\frac{y}{b}=1$ touch the circle

$$x^2+y^2+Ax+By+C=0,$$

then $$4\left\{\frac{A}{a}+\frac{B}{b}+1+C\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}=\left(\frac{A}{b}-\frac{B}{a}\right)^2.$$

EXAMPLES ON LOCI.

1. ABC is a triangle, the base of which BC is fixed, and of length $2a$, find the locus of A, when

(i) $AB^2-AC^2=d^2$,

(ii) $AB=mAC$,

(iii) $AB^2+AC^2=c^2$.

2. If B and C move on two parallel lines, and AB, AC be inclined to those lines at angles α, β respectively, and $AB=mAC$, find the locus of A.

3. ACB, DCE are two straight lines of given lengths which intersect in C, the middle point of AB; DE is fixed, and AB moves parallel to itself: if AD meet BE in P, find the locus of P.

4. ABC is a triangle, DE a straight line parallel to the base; BE, CD intersect in P: find the locus of P.

5. AB, AC are fixed straight lines, DE a straight line such that $AD+AE=l$; P is a point in DE such that $DP=mEP$: find the locus of P.

6. A is a given point, P a point on a fixed circle: AP is divided in Q so that $AQ=mPQ$: shew that the locus of Q is a circle.

7. A straight line through a point R cuts two fixed circles in P, P', Q, Q' respectively: if $RP.RP'=\mu RQ.RQ'$, find the locus of R.

8. A circle of given radius moves so that its radical axis with reference to a fixed circle always passes through a fixed point: shew that its centre is on a circle of which the fixed point is centre.

9. If a chord of $x^2+y^2=c^2$ touch the circle

$$(x-a)^2+(y-b)^2=d^2,$$

the locus of its middle point is

$$(x^2+y^2-ax-by)^2=d^2(x^2+y^2).$$

10. From a point P perpendiculars are drawn to the sides of a regular polygon of n sides: if the sum of the squares on these perpendiculars be constant, the locus of the point P is a circle.

CHAPTER IV. *Conic Sections. Focal Properties.*

71. A conic section is the locus of a point, the distance of which from a fixed point is in a constant ratio to its distance from a fixed straight line.

The fixed point is called the *focus*, the fixed straight line the *directrix*, the fixed ratio the *eccentricity*.

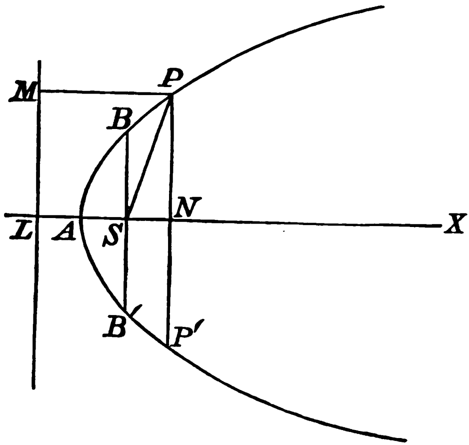
A conic section is generally called a conic, for shortness. There are three kinds of conics, according as the eccentricity is equal to, less than, or greater than, unity. When the eccentricity is unity, the conic is called a *parabola*, when less than unity, an *ellipse*, when greater, a *hyperbola*.

A conic section is so called, because if a right cone be cut by a plane which does not pass through the vertex, and is not perpendicular to its axis, the section is always one of these curves.

The perpendicular from the focus on the directrix is called the *axis*, the chord through the focus perpendicular to the axis the *latus rectum*, the straight line joining the focus with any point the *radius vector*, the point where the axis meets the curve the *vertex* of the conic.

Many properties are common to all conics, and are most easily deduced from the polar equation.

In drawing the figures, certain letters are always used in certain positions: thus the focus is always denoted by S, and the vertex, or point in which SL, the perpendicular on the directrix, cuts the curve, is denoted by A.



Any point on the conic is usually denoted by P.

The eccentricity is usually denoted by the letter e.

Thus, LM is the directrix, SP the radius vector of P, LX the axis, BSB' the latus rectum, and $SP = e \, . \, PM$.

72. *To find the polar equation to a conic.*

Let S (in the preceding figure) be the focus, SL the axis, LM the directrix, P any point on the conic.

Draw PM, PN perpendicular to the directrix and axis respectively, and let

$$SP = r, \ LSP = \theta, \ SL = c.$$

Then $\quad SP = e \, . \, PM = e \, . \, LN = e(SL + SN)$.

Now $\quad SN = SP \cos PSX = -r \cos \theta$;

$$\therefore \ r = e(c - r\cos\theta), \text{ or } r(1 + e\cos\theta) = ec;$$

$$\therefore \ r = \frac{ec}{1 + e\cos\theta}, \text{ the equation required.}$$

Let BSB', the latus rectum, be denoted by $2l$;

$$\therefore l = ec,$$

and
$$r = \frac{l}{1 + e\cos\theta}.$$

73. **Focal properties.** (i) The curve is symmetrical with respect to the axis, for if we change θ into $-\theta$, the value of r is unaltered.

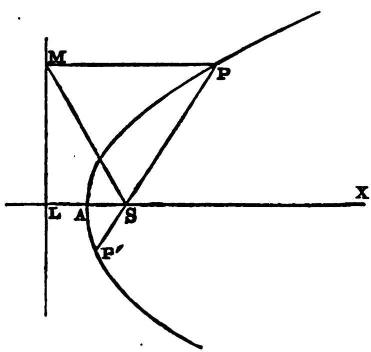
Hence if we draw PN perpendicular to the axis, and produce it to P', so that $P'N = PN$, P' is a point on the curve.

(ii) The least value of r is when $\theta = 0$, that is SA.

For $\frac{l}{1 + e\cos\theta}$ is least when $\cos\theta$ is greatest, that is when $\theta = 0$, $r = \frac{l}{1+e}$.

This result may be obtained geometrically thus.

Divide SL in A so that $SA : AL :: e : 1$, then A is a point on the curve.



Now
$$SA : SL :: e : 1 + e;$$

$$\therefore SA = \frac{ec}{1+e} = \frac{l}{1+e}.$$

(iii) **The latus rectum is double the harmonic mean between the segments of any focal chord.**

Let PSP' be any focal chord; (see fig. on preceding page).

Let $$LSP=\theta, \text{ and } \therefore LSP'=\pi-\theta.$$

Let $$SP=r,\ SP'=r';$$

$$\therefore\ r=\frac{l}{1+e\cos\theta},\ r'=\frac{l}{1+e\cos(\pi-\theta)}=\frac{l}{1-e\cos\theta};$$

$$\therefore\ \frac{1}{r}+\frac{1}{r'}=\frac{2}{l},$$

or l is the harmonic mean between r and r'.

74. *To find the equation to the directrix.*

Let M, in the preceding figure be any point (r, θ), on the directrix.

Then $$SM\cos MSL=SL,$$

or $$r\cos\theta=\frac{l}{e},$$

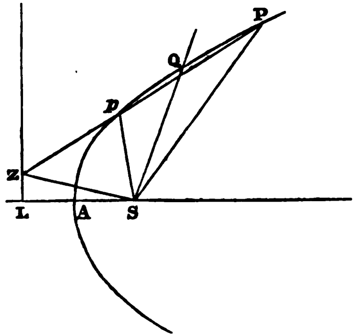
$$\frac{l}{r}=e\cos\theta, \text{ the equation required.}$$

75. *To find the equations to a chord and to a tangent.*

Let P, p be the two points on the curve $(r_1\theta_1)$ $(r_2\theta_2)$ through which the chord passes, (r, θ) the coordinates of any other point Q on the chord; then (Art. 35) the equation to Pp is

$$\frac{\sin(\theta_1-\theta_2)}{r}+\frac{\sin(\theta_2-\theta)}{r_1}+\frac{\sin(\theta-\theta_1)}{r_2}=0\ldots\ldots\ldots\ldots\text{(i)}.$$

Now multiply by l and write $1+e\cos\theta_1$, $1+e\cos\theta_2$, for $\frac{l}{r_1}$, $\frac{l}{r_2}$, respectively.



The equation becomes

$$\frac{l}{r}\sin(\theta_1-\theta_2)+\sin(\theta_2-\theta)(1+e\cos\theta_1)$$

$$+\sin(\theta-\theta_1)(1+e\cos\theta_2)=0\text{......(ii).}$$

Now $\sin(\theta_2-\theta)(1+e\cos\theta_1)+\sin(\theta-\theta_1)(1+e\cos\theta_2)$

$$=-2\sin\frac{\theta_1-\theta_2}{2}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)$$

$$+e\{\cos\theta_1\sin(\theta_2-\theta)+\cos\theta_2\sin(\theta-\theta_1)\}$$

$$=-2\sin\frac{\theta_1-\theta_2}{2}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)-e\cos\theta\sin(\theta_1-\theta_2)$$

$$=-2\sin\frac{\theta_1-\theta_2}{2}\left\{\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)+e\cos\theta\cos\frac{\theta_1-\theta_2}{2}\right\}.$$

Hence, substituting in (ii), and dividing by $\sin(\theta_1-\theta_2)$, the equation becomes

$$\frac{l}{r}=\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\sec\frac{\theta_1-\theta_2}{2}+e\cos\theta.$$

In this equation to the chord put $\theta_2=\theta_1$; then $r_2=r_1$, and the chord becomes the tangent; the equation therefore becomes

$$\frac{l}{r}=\cos(\theta-\theta_1)+e\cos\theta.$$

76. From these equations we can obtain various important properties of the curve.

For instance we can prove

(i) If a chord Pp cut the directrix in Z and QS bisect the angle PSp, QSZ is a right angle.

(ii) If the tangent at P cut the directrix in Z, ZSP is a right angle.

(iii) If TP, TQ be the tangents at P, Q respectively, $TSP=TSQ$.

(i) Let the equation to Pp be

$$\frac{l}{r}=\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\sec\frac{\theta_1-\theta_2}{2}+e\cos\theta;\ \therefore\ ASQ=\frac{\theta_1+\theta_2}{2}.$$

Now the equation to the directrix is $\frac{l}{r}=e\cos\theta$;

therefore at Z, $\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)=0$;

$$\therefore\ \theta\sim\frac{\theta_1+\theta_2}{2}=\frac{\pi}{2},$$ or ZSQ is a right angle.

(ii) Next let $\theta_2=\theta_1$, then the chord becomes the tangent at P, and we have at Z, $\cos(\theta-\theta_1)=0$, therefore $\theta\sim\theta_1=\frac{\pi}{2}$, or ZSP is a right angle.

(iii) Let the equations to the tangents TP, TQ be

$$\frac{l}{r}=\cos(\theta-\theta_1)+e\cos\theta,$$

$$\frac{l}{r}=\cos(\theta-\theta_2)+e\cos\theta.$$

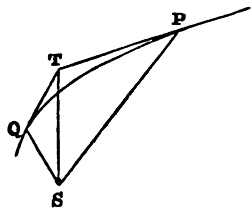
Subtract, and we have, at T,

$$\cos(\theta-\theta_1)-\cos(\theta-\theta_2)=0;$$

$$\therefore \sin\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\sin\frac{\theta_1-\theta_2}{2}=0;$$

now θ_1 is not equal to θ_2,

$$\therefore \theta=\frac{\theta_1+\theta_2}{2}, \text{ that is } TSP=TSQ.$$



77. *Equation referred to rectangular axes.*

In the equation

$$\frac{l}{r}=1+e\cos\theta,$$

write $\pi-\theta$ for θ, so that $PSX=\theta$, and multiply by r, the equation becomes

$$r=l+er\cos\theta.$$

Now square and write x^2+y^2 for r^2, x for $r\cos\theta$;

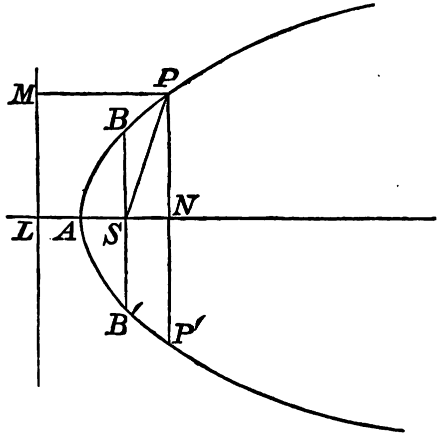
$$\therefore x^2+y^2=(l+ex)^2,$$

or $$(1-e^2)x^2-2lex+y^2=l^2,$$

the equation required.

78. This equation may easily be obtained from the definition of a conic, remembering that $l=ec$.

Let P be a point on the curve; $\therefore SP = e.PM$.



But $SP^2 = x^2 + y^2$, $PM = LN = LS + SN = c + x$;

$$\therefore x^2 + y^2 = e^2 (c + x)^2,$$

or $$(1 - e^2) x^2 - 2e^2 cx + y^2 = e^2 c^2, \text{ as before.}$$

79. *Equation referred to axis and tangent at vertex.*

Let us transform the origin to the point A, without altering the direction of the axes.

Now $$SA = \frac{ec}{1+e}.$$

We must therefore write $x - \dfrac{ec}{1+e}$ for x in the preceding equation which becomes

$$(1 - e^2) x^2 - 2ecx + y^2 = 0,$$

or $$y^2 = 2lx - (1 - e^2) x^2.$$

This equation may be written

$$y^2 = 2lx + nx^2,$$

where $n = e^2 - 1$, and is therefore negative for the ellipse, zero for the parabola, positive for the hyperbola.

In this equation if $x=0$, $y^2=0$.

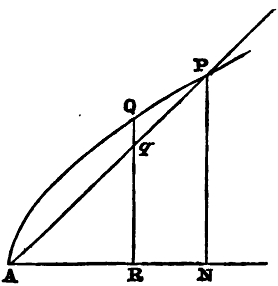
The axis of y therefore cuts the curve in two coincident points at the origin, that is, it is the tangent at the origin.

80. If P be a point on a conic, the straight line AP lies within the conic.

Let AN, PN, the coordinates of P, be h, k, and those of Q, any point on the conic, x', y'; let QR meet AP in q $(x'y'')$, and let the equations to the conic and to AP be

$$y^2=2lx+nx^2,$$

$$y=mx, \text{ respectively;}$$



then $$m^2h^2=2lh+nh^2, \therefore h=\frac{2l}{m^2-n}.$$

Now $$\frac{y'^2}{x'^2}=\frac{2l}{x'}+n,$$

$$\frac{y''^2}{x'^2}=m^2;$$

$$\therefore \frac{y'^2-y''^2}{x'^2}=\frac{2l-(m^2-n)x'}{x'}.$$

This is positive if $x'<\dfrac{2l}{m^2-n}<h$, negative if $x'>h$, that is, AP lies within the curve, which is therefore of the form represented in the figure.

81. If from a point O straight lines OPp, OQq be drawn in fixed directions, cutting a conic in P, p, Q, q, then $OP.Op : OQ.Oq$ is independent of the position of O.

Let OPp, OQq make angles α, β respectively with the axis of x, and let the coordinates of O be h, k.

Then along OP, $x=h+r\cos\alpha$, $y=k+r\sin\alpha$;

therefore substituting in the equation to the conic

$$y^2=2lx+nx^2,$$

and rearranging

$$r^2(\sin^2\alpha-n\cos^2\alpha)+2r\{k\sin\alpha-(l+nh)\cos\alpha\}$$
$$+k^2-2lh-nh^2=0;$$

therefore $OP.Op=$ the product of the roots of this equation,

$$=\frac{k^2-2lh-nh^2}{\sin^2\alpha-n\cos^2\alpha},$$

so $$OQ.Oq=\frac{k^2-2lh-nh^2}{\sin^2\beta-n\cos^2\beta};$$

$$\therefore\ \frac{OP.Op}{OQ.Oq}=\frac{\sin^2\beta-n\cos^2\beta}{\sin^2\alpha-n\cos^2\alpha},$$ which is independent of h and k.

Cor. If a circle cut a conic in P, Q, p, q, and these points be joined, the pairs of chords so formed are equally inclined to the axis.

For let any pair intersect in O, and make angles α, β with the axis, then

$$OP.Op=OQ.Oq;\ \therefore\ \tan^2\alpha=\tan^2\beta;\ \therefore\alpha=-\beta,$$

for it cannot be β, since the tangents intersect.

EXAMPLES ON CHAPTER IV.

1. If, in the conic $\frac{l}{r}=1+e\cos\theta$, r, r' be the lengths of two radii vectores at right angles to each other,

$$\left(\frac{1}{r}-\frac{1}{l}\right)^2+\left(\frac{1}{r'}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}.$$

2. If the angle between the focal distances of two points on the conic be α, the tangents at those points will intersect on the conic

$$\frac{l}{r} = \cos\frac{\alpha}{2} + e\cos\theta.$$

What is the eccentricity of this conic, and how is it situated?

3. P is a point on the conic

$$\frac{l}{r} = 1 + e\cos\theta,$$

PT the tangent at P, if $PST = \alpha$, the locus of T is a conic.

4. Shew that the equation to the normal at $(r_1\theta_1)$ is $r\sin(\theta - \theta_1) = e(r\sin\theta - r_1\sin\theta_1)$, and hence that if the normal at P cut the axis in G,

$$SP = e\,.\,SG.$$

5. The equation

$$\frac{l}{r} = \cos(\theta - \theta_1) + e\cos\theta$$

represents the polar of (r_1, θ_1) with respect to the curve

$$\frac{l}{r} = 1 + e\cos\theta.$$

6. The equation to the tangent to the curve $y^2 = 2lx + nx^2$, at the point x_1y_1, is

$$yy_1 = l(x + x_1) + nxx_1.$$

7. The equation to the normal to the same curve at the same point is

$$(l + nx_1)\,y + xy_1 = \{l + (n+1)x_1\}\,y_1.$$

8. If a circle touch a conic at P, and cut it in Q, q, prove that PQ, Pq make equal angles with the axis, as also do Qq and the tangent at P.

9. PQ is a chord which subtends a right angle at A: prove that it always passes through a fixed point on the axis.

10. K is a fixed point on a conic; PQ a chord; if PKQ be a right angle, PQ passes through a fixed point on the normal at K.

CHAPTER V. *The Parabola.*

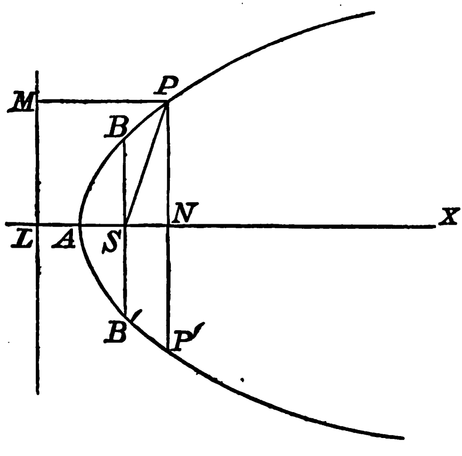
82. IN the equation in Art. 77, put $e=1$, the curve becomes a parabola, and its equation becomes

$$y^2=2lx,$$

since $e=1$, $l=c$, $SA=AL=\frac{1}{2}SL$.

Let $l=2a$, then the equation becomes

$$y^2=4ax,$$



where a is the distance between the focus and vertex, and the latus rectum $=4a$

83. *To trace the form of the parabola from its equation.*

Since $y^2 = 4ax$, or

$$x = \frac{y^2}{4a},$$

x cannot be negative, that is, the curve lies wholly on the positive side of the axis of y.

Since $y^2 = 4ax$, $y = \pm 2 (ax)^{\frac{1}{2}}$,

therefore, since this equation is unaltered if we write $-y$ for y, to every point P on the curve on the positive side of the axis there corresponds another point P', on the negative side, such that $P'N = PN$.

The curve is therefore symmetrical with respect to the axis of x.

Again, if $x = 0$; $y = 0$, and has no other value, therefore the curve does not cut either axis at any other point besides the origin.

Also, the greater value we give to x, the greater value we get for y, and when x is infinite y is infinite, hence the curve goes off to an infinite distance on each side of the axis of x.

84. *To find the distance of any point from the focus.*

Let P be any point on the curve,

then
$$\begin{aligned} SP &= PM \\ &= LN \\ &= LA + AN \\ &= a + x. \end{aligned}$$

Ex. 15.

1. If the distance of a point from the focus be equal to the latus rectum, then its abscissa is equal to $3a$.

2. Prove that the length of a side of an equilateral triangle inscribed in a parabola, so that one angle coincides with the vertex, is $8\sqrt{3}a$.

3. Find the length of the side of an equilateral triangle of which one angle coincides with the focus, and the others lie on the parabola $y^2=4ax$.

4. A double ordinate PNP' of a parabola $=8a$, prove that $AN=4a$, and that PAP' is a right angle.

5. If a circle pass through a fixed point and touch a fixed straight line, its centre lies on a parabola.

6. The diameter of the circle passing through the vertex and the extremities of the latus rectum is five-fourths of the latus rectum.

7. Two parabolas have the same axis and vertex, but the latus rectum of one is double that of the other; prove that any chord of the greater passing through the common vertex is bisected by the lesser parabola.

8. If $(x_1 y_1)$, $(x_2 y_2)$, be two points on the parabola $y^2=4ax$, and $x_2-x_1=a$, $y_1 y_2=a^2$, determine x_1, y_1, x_2, y_2.

9. Prove that the area of a triangle inscribed in the parabola $y^2=4ax$ is

$$\frac{1}{8a}(y_1-y_2)(y_2-y_3)(y_3-y_1),$$

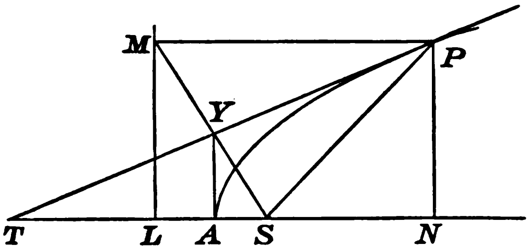
where y_1, y_2, y_3, are the ordinates of the angular points.

10. If a circle cut a parabola in four points, the algebraic sum of their ordinates is zero.

85. *To find the equation to the tangent at any point of a parabola.*

Since the equation to the parabola is essentially unsymmetrical, it is convenient in problems respecting it to use the unsymmetrical form of the equation to a straight line,

$$y=mx+c.$$



Let (x_1, y_1), (x_2, y_2), be two points P and Q on a parabola, then the equation to the straight line joining them is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1):$$

but since these points are on the parabola,

$$y_1^2 = 4ax_1, \quad y_2^2 = 4ax_2;$$

$$\therefore\ y_2^2 - y_1^2 = 4a(x_2 - x_1);$$

$$\therefore\ \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1};$$

the equation to the straight line joining (x_1, y_1), (x_2, y_2), becomes therefore

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1).$$

In this equation, multiply by $y_1 + y_1$,

then $$y(y_1 + y_2) - y_1^2 - y_1 y_2 = 4a(x - x_1),$$

or, remembering that $y_1^2 = 4ax_1$,

$$y(y_1 + y_2) - y_1 y_2 = 4ax.$$

Now let Q move up to P, and therefore $x_2 = x_1$, $y_2 = y_1$, then PQ becomes the tangent at P, and the equation becomes

$$2yy_1 - y_1^2 = 4ax,$$

$$\text{or, } 2yy_1 = 4a(x + x_1),$$

$$yy_1 = 2a(x + x_1),$$

the equation required.

86. In this equation put $y=0$,

$$\therefore x=-x_1,$$

therefore the tangent PT cuts the axis at a point T such that $AT=AN$.

Now $$ST=AS+AT=a+x_1=SP,$$

$$\therefore SPT=STP=TPM,$$

the tangent therefore bisects the angle MPS.

87. *To find the equation to the tangent to a parabola in terms of the tangent of the angle it makes with the axis.*

Let the line $$y=mx+c$$

cut the parabola $$y^2=4ax,$$

then we shall find the abscissæ of the points of intersection by substituting for y in this equation, and finding the roots of the quadratic

$$(mx+c)^2=4ax.$$

Now if the straight line *touch* the parabola, these roots must be equal;

$$\therefore m^2x^2+2(mc-2a)x+c^2=0,$$

must have equal roots;

$$\therefore m^2c^2=(mc-2a)^2;$$

$$\therefore c=\frac{a}{m};$$

therefore the straight line

$$y=mx+\frac{a}{m},$$

or $$y=x\tan\theta+a\cot\theta,$$

touches the parabola $y^2=4ax$.

This equation may be written

$$y = mx + \frac{a}{m}.$$

This form of the equation is often useful in problems which do not involve the coordinates of the point of contact.

Example:

Tangents at right angles intersect in the directrix.

$$\text{Let} \quad y = mx + \frac{a}{m},$$

$$y = nx + \frac{a}{n},$$

be the equations to the tangents; then since they are at right angles to each other,

$$n = -\frac{1}{m};$$

$$\therefore \; y = -\left(\frac{x}{m} + ma\right),$$

is the equation to the tangent at right angles to the first.

Subtract this equation from that to the first, then

$$(x+a)\left(m + \frac{1}{m}\right) = 0;$$

now $m + \frac{1}{m}$ cannot vanish,

$$\therefore x = -a,$$

the equation to the directrix.

88. *To find the coordinates of the point of contact.*

$$\text{Let} \qquad (x \tan\theta + a \cot\theta)^2 = y^2 = 4ax,$$

$$\therefore x \tan\theta - a \cot\theta = 0,$$

$$x = a \cot^2\theta.$$

$$\text{Similarly} \qquad y = 2a \cot\theta.$$

89. *To find the locus of the point in which the perpendicular from the focus on the tangent meets that tangent.*

Let the equation to the tangent be

$$y = mx + \frac{a}{m} \quad \text{.........................(1)},$$

then the equation to the straight line through $(a, 0)$ perpendicular to (1) is

$$my + x = a, \text{ or } y + \frac{x}{m} = \frac{a}{m}.$$

At the intersection of these two straight lines we must have

$$x\left(m + \frac{1}{m}\right) = 0, \therefore x = 0.$$

This is the equation to the tangent at the vertex (Art. 79): hence the locus of the foot of the perpendicular from the focus on a tangent is the tangent at the vertex.

90. *To find the equation to the normal at any point of a parabola.*

Let x_1, y_1 be the coordinates of the point, then

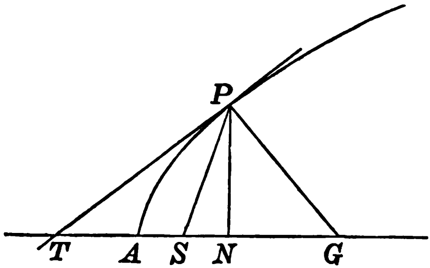
$$yy_1 = 2a(x + x_1),$$

is the equation to the tangent at (x_1, y_1), therefore the equation to the straight line through (x_1, y_1) perpendicular to this tangent must be

$$2a(y - y_1) + y_1(x - x_1) = 0,$$

the required equation to the normal.

91. Let PT, PG, be the tangent and normal at P, then TN is called the subtangent, NG the subnormal.



In the equation to the normal

$$2a(y-y_1)+y_1(x-x_1)=0,$$

put $y=0$;

$$\therefore\ x-x_1=2a, \quad \text{or } NG=2a.$$

Now $SG=SN+NG=x-a+2a=x+a=SP=ST$;

therefore the circle described on GT as diameter will pass through P and have S for its centre.

92. *To find the equation to the normal in terms of the tangent of the angle it makes with the axis of x.*

The equation to the normal may be written

$$y=-\frac{y_1}{2a}x+y_1+\frac{x_1y_1}{2a}.$$

Let $-\frac{y_1}{2a}=m$, then

$$y_1=-2am, \quad \frac{x_1y_1}{2a}=\frac{y_1^3}{8a^2}=-m^3a;$$

$$\therefore\ y=m\{x-a(2+m^2)\}$$

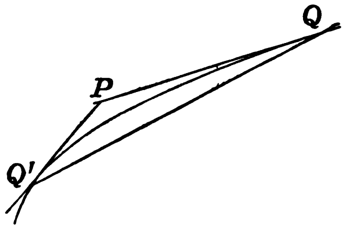
is the equation required. If however m be the angle which the tangent at P makes with the axis, we must write $-\frac{1}{m}$ for m in this equation.

93. *To find the equation of the chord of contact of tangents drawn through a given point.*

Let the coordinates of P be h, k, of Q and Q' x_1, y_1; x_2, y_2, respectively; then the equation to PQ is

$$yy_1 = 2a(x + x_1),$$

and to PQ', $$yy_2 = 2a(x + x_2).$$



Now since (h, k) lies on each of these lines, we must have

$$ky_1 = 2a(h + x_1),$$

$$ky_2 = 2a(h + x_2),$$

that is, (x_1, y_1), (x_2, y_2), satisfy the condition

$$ky = 2a(x + h).$$

Now this is the equation to some straight line, and the coordinates of Q and Q' satisfy it, it is therefore the equation to QQ'.

P is called the pole of QQ', QQ' the polar of P, and whether the point (h, k) be within, on, or without the parabola, the straight line represented by the equation

$$ky = 2a(x + h)$$

is said to be the polar of (h, k).

94. *To find the pole of a given line.*

Let the equation to the straight line be of the form

$$y = mx + c.$$

Divide by m and multiply by $2a$,

$$\therefore \frac{2a}{m} y = 2ax + \frac{2ac}{m};$$

comparing this with the equation to the polar, we get

$$\frac{c}{m}, \frac{2a}{m} \text{ as the pole of the given line.}$$

95. By reasoning precisely similar to that by which we proved the analogous proposition in the case of the circle, we may prove that if P be the pole of QQ', then the polar of every point on QQ' passes through P.

Example. The polar of the focus a, 0, is

$$x + a = 0,$$

the directrix.

Hence the pole of every chord through the focus lies on the directrix: hence also the directrix is the locus of the intersection of tangents at the extremities of focal chords: but we have proved it to be the locus of tangents which intersect at right angles: hence tangents at the extremities of focal chords intersect at right angles in the directrix.

Ex. 16.

1. The points of contact of two tangents being given, find their point of intersection.

2. Find the distances of the vertex and focus from the tangent $y = mx + \frac{a}{m}$.

3. Two tangents make angles $\tan^{-1}m$, $\tan^{-1}m'$ with the axis, find their point of intersection, its polar, and the point of intersection of the corresponding normals.

4. A common tangent is drawn to a parabola, and the circle described on its latus rectum as diameter; prove that the angle between the lines drawn from the focus to the points of contact is bisected by the latus rectum.

5. The portion of the tangent at the vertex intercepted between the vertex and any diameter is bisected by the tangent at the extremity of that diameter.

6. If a circle pass through the vertex and focus, cutting the tangent at the vertex in Q, the tangent to the circle at Q will touch the parabola.

7. If the tangent at P cut the axes in T, Y, respectively,

$$TP \, . \, TY = TN \, . \, TS.$$

8. The circle described on SP as diameter touches the tangent at the vertex.

9. Pp is a focal chord; the circle described on Pp as diameter touches the directrix.

10. A straight line touches the curves

$$y^2 = 4ax, \text{ and } x^2 + y^2 = c^2,$$

find its equation.

11. If P, Q, be two points (x_1, y_1), (x_2, y_2) on the parabola, find the coordinates of the point of intersection of the normals at P, Q.

12. If PG, the normal at P, cut the curve again at p, find the coordinates of p, and the length of Pp.

96. *To find the locus of the middle points of parallel chords of a parabola.*

Let $QQ'R$ be a straight line cutting the parabola in Q, Q' and making an angle θ with the axis of x.

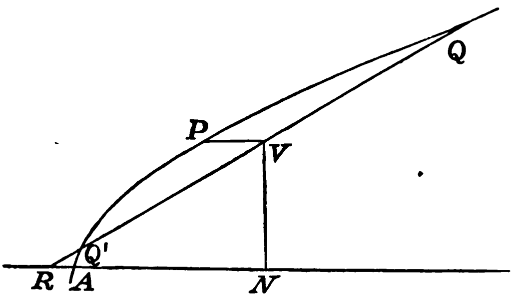
Let V be the middle point of QQ', and let its coordinates be x', y'.

Let r be the distance between V and any point (x, y) on QQ'.

Then the equation to QQ' may be written

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r, \quad \text{(Art. 25.)}$$

or $$x=x'+r\cos\theta, \quad y=y'+r\sin\theta.$$



If we substitute these values for x and y in the equation $y^2=4ax$, and rearrange,

$$r^2\sin^2\theta+2r(y'\sin\theta-2a\cos\theta)+y'^2-4ax'=0 \text{.........(1)}.$$

Now if V be the middle point of QQ', the values of r obtained from this equation must be equal and opposite;

$$\therefore\ y'\sin\theta-2a\cos\theta=0, \quad \text{or } y'=2a\cot\theta,$$

the equation to a straight line parallel to the axis.

Now let the chord QQ' move parallel to itself till Q, Q' coincide, then the chord becomes the tangent at P.

Hence the middle points of all chords parallel to a tangent lie on the straight line parallel to the axis through the point of contact.

Such a straight line is called a diameter, and the chords it bisects are called the ordinates of that diameter.

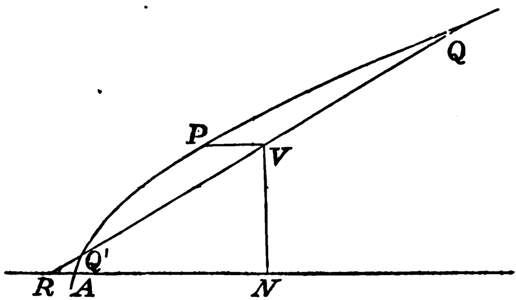
Since (see fig. of Art. 85), $SY \perp PT$,

$$\therefore SYA = YTA = \theta,$$

$$\therefore SY = a \operatorname{cosec} \theta, \ SP = SY \operatorname{cosec} \theta = a \operatorname{cosec}^2 \theta.$$

$4SP$ is called the parameter of the diameter PV.

97. *To find the equation to the parabola, referred to a diameter and the tangent at its extremity as axes.*



Let the coordinates of a point Q be (xy) when referred to the axes, and (XY) when referred to the diameter PV as axis of x, and to the tangent at P, (which is parallel to QQ') as axis of y; let the coordinates of V be x', y' referred to the old axes, then in equation (1) of the preceding article we may write Y for r.

Now $PV = X$, and $y'^2 = 4a(x' - X)$, since $x' - X$, y' are the coordinates of P referred to the old axes,

$$\therefore 4aX = 4ax' - y'^2.$$

Hence remembering that the coefficient of r in equation (1) of preceding article is zero, we have

$$Y^2 \sin^2 \theta = 4aX,$$

or $$Y^2 = 4a \operatorname{cosec}^2 \theta \,.\, X.$$

If $SP = a'$ this becomes $Y^2 = 4a'X$, the equation required.

The equation to the tangent at $x'y'$ is still

$$Yy' = 2a'(X + x'),$$

since the investigation of Art. 85 does not depend on the the value of a.

98. The polar equations to a conic section, and to the tangent, become, in the case of the parabola,

$$\frac{l}{r} = 1 + \cos\theta, \quad \text{or} \quad r = a\sec^2\frac{\theta}{2},$$

and, $$\frac{l}{r} = \cos(\theta - \theta_1) + \cos\theta, \text{ respectively.}$$

EXAMPLES ON CHAPTER V.

1. If P be any point on a parabola, and PK drawn perpendicular to AP cut the axis in K, then NK is equal to the latus rectum.

2. If SPG is an equilateral triangle, SP is equal to the latus rectum.

3. If $GL \perp SP$, then $PL = 2a$.

4. Find the equations to the common chords of the curves

$$y^2 = 4ax, \quad x^2 + y^2 = 2cx.$$

5. If PM be the perpendicular on the directrix from P,

$$SM^2 = 4a \cdot SP.$$

6. If Pp be a focal chord, the triangle $PAp \propto (Pp)^{\frac{1}{2}}$.

7. If Pp be a focal chord, and the tangent at Q be parallel to it, $SP \cdot Sp = 2l \cdot SQ$.

8. If Pp be a focal chord, and Q, q, its points of intersection with the circle whose centre is S and radius SA, then

$$PQ \cdot pq = a^2.$$

9. If Pp be a focal chord, QV the diameter bisecting it,

$$SQ = QV, \text{ and } PV = 2SQ.$$

10. A circle touches a parabola at A, cuts it at B, C, and cuts the axis in E: BC cuts the axis in D: if DE be bisected in K, BK is the normal at B.

11. BC, CD are two chords, such that the lengths of the diameters intercepted between their middle points E, F and the curve are equal: prove that EF is parallel to the tangent at C.

12. Two equal parabolas have the same axis, and a chord Qq of the one making a constant angle with the axis cuts the other in P; prove that $PQ \cdot Pq$ is constant.

13. QT, qT are two tangents, and Qq is bisected in V, prove that T lies on the diameter through V, and that if this diameter cut the curve in P, $PT = PV$.

14. If there are three tangents to a parabola, the triangle formed by their intersections is half that whose angular points are the points of contact.

15. Two parabolas have their axes perpendicular to each other; prove that if they cut each other in four points, these points will lie on a circle.

16. Three parabolas, whose axes are parallel to each other, intersect; prove that their common chords are concurrent.

17. PQ is a chord through a given point O; prove that $PO \cdot OQ$ is least when PQ is perpendicular to the axis.

18. On any chord of a parabola as diameter a circle is described, cutting the curve again in two points: if these two points be joined, the portion of the axis intercepted between these two chords is equal to the latus rectum.

19. PSQ is a focal chord, QNQ' a double ordinate of the axis; prove that PQ' passes through the foot of the directrix.

20. If a triangle be inscribed in a parabola, and a similar one described about it, the sides of the former are four times those of the latter.

21. Two normals meet in O at right angles; ON is drawn perpendicular to the axis, and NQ measured along the axis towards the vertex equal to a: QO is a normal.

22. Two tangents to the parabola $y^2 = 4ax$, are

$$y = x\tan\theta_1 + a\cot\theta_1,$$
$$y = x\tan\theta_2 + a\cot\theta_2,$$

find the locus of their intersection when $\cot\theta_1 + \cot\theta_2 = k$.

23. Find the locus, when $\cot\theta_1 - \cot\theta_2 = k$.

24. Find the locus, when $\tan\theta_1 . \tan\theta_2 = k$.

25. Find the locus, when $\sin\theta_1 . \sin\theta_2 = k$.

26. If a straight line be drawn from the focus cutting the tangent $y = x\tan\theta + a\cot\theta$ at an angle α, it will intersect it in the tangent

$$y = x\tan\alpha + a\cot\alpha.$$

27. Two normals to a parabola are always at right angles, find the locus of their intersection.

28. The point of intersection of the perpendicular from the focus on any normal to $y^2 = 4ax$, lies on the parabola $y^2 = a(x - a)$.

29. Find the locus of the vertex of a parabola which has a given focus and touches a given straight line.

30. In the radius vector SP, $SQ = AN$; find the locus of Q.

31. If Q be on the focal chord Pp, and $SQ = Pp$, find the locus of Q.

32. Given a diameter and its tangent, find the locus of the focus.

33. Find also the locus of the vertex.

34. The poles of all straight lines through the foot of the directrix lie on the latus rectum.

35. The pole of any tangent to the parabola $y^2 = 4ax$, with respect to the circle $x^2 + y^2 = c^2$, lies on the parabola

$$ay^2 + c^2x = 0.$$

36. The area and base of a triangle being given, find the locus of the intersection of perpendiculars from the extremities of the base on the opposite sides.

37. Find the locus of the centre of a circle inscribed in a sector of a given circle, one of the radii of the sector being fixed.

38. A is the origin, B a point on the axis of y, BQ a line parallel to the axis of x; in AQ, P is taken such that its ordinate is equal to BQ: find the locus of P.

39. Find the locus of the centre of a circle which touches a given circle and given line.

40. SY is the perpendicular from the focus on the tangent PY, find the locus of the centre of the circle circumscribing SYP.

41. Find the locus of the focus of a parabola, which has a given vertex and touches a given line.

42. A parabola touches two given straight lines, and has the direction of its axis fixed: the locus of the focus is a straight line.

43. YPR is a straight line bisected at P, of which Y lies on the tangent at the vertex, P on the parabola, and R on the axis; prove that YPR touches another parabola.

44. The locus of points from which pairs of tangents intercept constant lengths on the tangent at the vertex, is an equal parabola with the same axis.

45. Two equal parabolas have the same axis and vertex, but are turned in opposite directions: chords of one are tangents to the other: shew that the locus of the middle points of these chords is a parabola, whose latus rectum is one-third of that of either parabola.

Chapter VI. *Central Conics.*

99. We have seen (Art. 77) that the equation to a conic referred to the axis and latus rectum as axes is

$$(1-e^2)x^2 - 2e^2cx + y^2 - e^2c^2 = 0 \text{............} (1),$$

where e is the eccentricity and c the perpendicular from the focus on the directrix: this equation can be made more simple, and the properties of the curve more easily investigated, by transformation. Change the origin to the point on the axis of x at a distance h from the old origin, then we must write $x+h$ for x in equation (1), y remaining unaltered.

Expand, and rearrange by powers of x; then the equation becomes

$$(1-e^2)x^2 + y^2 + 2x\{(1-e^2)h - e^2c\} + (1-e^2)h^2 - 2e^2ch - e^2c^2 = 0.$$

If now we put $h = \dfrac{e^2c}{1-e^2}$, the coefficient of x vanishes, and the equation becomes

$$(1-e^2)x^2 + y^2 - \frac{e^2c^2}{1-e^2} = 0,$$

or

$$(1-e^2)^2x^2 + (1-e^2)y^2 = e^2c^2.$$

In this equation, if $e < 1$, let

$$\frac{ec}{1-e^2} = a, \quad \frac{ec}{(1-e^2)^{\frac{1}{2}}} = b,$$

and it becomes, dividing by e^2c^2,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If $e>1$ let

$$\frac{ec}{e^2-1}=a,\quad \frac{ec}{(e^2-1)^{\frac{1}{2}}}=b,$$

and the equation becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

These equations represent the ellipse and hyperbola respectively.

100. We have now to discuss the ellipse and hyperbola whose equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

respectively.

We will first determine the shape of these curves, then discuss the properties in which they agree, and afterwards those in which they differ.

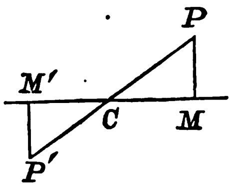
It is obvious that if in any result in the ellipse we obtain an expression involving b^2, we shall obtain the corresponding expression for the hyperbola by changing the sign of b^2, but if we obtain an expression involving b we shall not get any corresponding expression for the hyperbola, since we should have to substitute the impossible quantity $b\sqrt{-1}$ for b.

101. Definition. If there be a point such that all chords of a curve drawn through that point are bisected in it, that point is called the centre of the curve, and chords drawn through it are called diameters.

The origin is the centre, and chords through the origin diameters.

Let P be the point (xy) on either of the curves

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$



Then since this equation is not changed if we write $-x, -y$ for x, y, $(-x, -y)$ is also a point on the curve, that is, if P' be the point in which PC produced cuts the curve,

$$CM' = CM, \quad P'M' = PM;$$

and $$\therefore CP' = CP;$$

$$\therefore C \text{ is the centre.}$$

102. To examine the form of the curve represented by the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

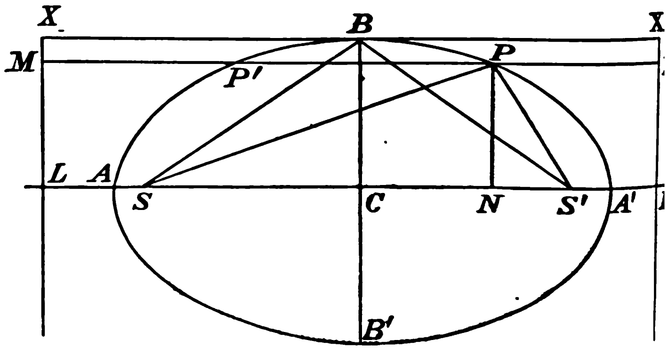
Since the equation only involves the squares of x and y, it will be unaltered if we substitute either $-x$, or $-y$, or both, for x and y.

It is therefore symmetrical with respect to the axes of x and y, that is, the axes divide it into four equal and similar portions.

Since the equation may be put into the form

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

if $x^2 > a^2$, y is impossible; similarly, if $y^2 > b^2$, x is impossible; the curve has therefore no infinite branches and is entirely



contained in the rectangle whose sides are the straight lines represented by the equations,

$$x = \pm a,$$
$$y = \pm b.$$

In the equation $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

as x increases from 0 to $\pm a$, y diminishes from $\pm b$ to 0, the curve must therefore be of the form represented in the figure, where

$$CA = CA' = a, \quad CB = CB' = b.$$

AA', BB' are called the major and minor axes, or sometimes the transverse and conjugate axes.

103. To examine the form of the hyperbola, whose equation is

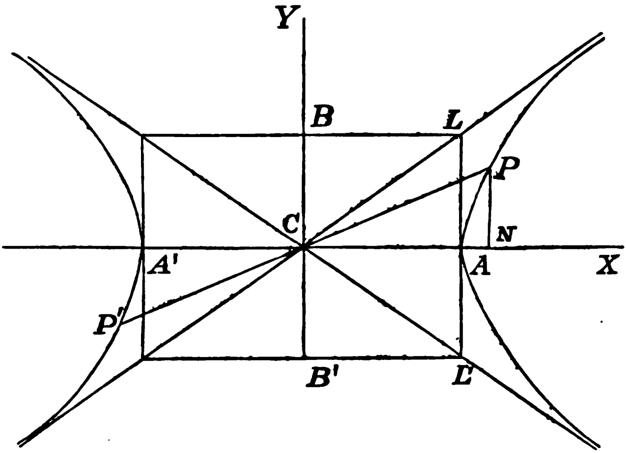
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This equation to the curve may be written

$$\frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2}.$$

Here for every value of y there correspond two values of x equal but of opposite signs, but if $x^2 < a^2$, y is impossible: x and y may both increase without limit.

Hence the curve extends indefinitely on both sides of the axes, and no part of it is included between the lines $x = a$, $x = -a$.



Take C the centre as origin, CX, CY the axes, make CA, CA' each equal to a, CB, CB' to b, then the curve passes through A, A'.

AA' is called the transverse axis, BB' the conjugate axis.

Let $y = x \tan\theta$ be the equation to any diameter, then, substituting in the equation to the curve,

$$x^2\left(\frac{1}{a^2} - \frac{\tan^2\theta}{b^2}\right) = 1;$$

$$\therefore \text{ if } \frac{1}{a^2} < \frac{\tan^2\theta}{b^2}, \text{ or } \tan\theta > \pm\frac{b}{a},$$

x is impossible.

Through A draw LAL' perpendicular to CA, and through B, B', BL, $B'L'$ parallel to CA; join CL, CL' and produce these lines both ways to any distance, then

$$\tan LCA = \frac{b}{a},$$

therefore the curve lies wholly between the lines LC, $L'C$. These straight lines, the equation to which may be written

$$\frac{x^2}{a^2} = \frac{y^2}{b^2};$$

are called the asymptotes to the hyperbola: their properties will be investigated hereafter.

Since for every point P there is another point P' such that $CP' = CP$, the curve must have two branches, passing through the points A, A' respectively, and be of the form represented in the figure.

104. Since CS or $h = \frac{e^2 c}{1 - e^2}$, which is positive if $e < 1$, negative if $e > 1$, S lies between A and C in the ellipse, but A is between S and C in the hyperbola.

Again, $SC = h = \frac{e^2 c}{1 - e^2} = ea$ in both curves: therefore, if L be the foot of the directrix,

$$\therefore SC : AC :: SA : AL;$$

$$\therefore SC \pm AS : AC \pm AL :: SA : AL,$$

$$\text{or } AC : CL :: SA : AL :: CS : CA,$$

$$CL \,.\, CS = CA^2;$$

$$\therefore CL = \frac{a}{e}.$$

Again, since in the ellipse

$$a = \frac{ec}{1 - e^2}, \quad b = \frac{ec}{(1 - e^2)^{\frac{1}{2}}};$$

$$\therefore 1 - e^2 = \frac{b^2}{a^2}, \quad e^2 = \frac{a^2 - b^2}{a^2}.$$

So in the hyperbola $e^2 = \dfrac{a^2+b^2}{a^2}$.

Again if l be the semi-latus rectum, $l = ec = \dfrac{b^2}{a}$.

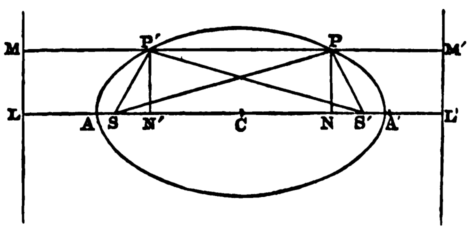
These formulæ, connecting a, b, e and l, are of constant use in the solution of problems.

105. Since to every point P on one side of the axis of y there corresponds a point P' on the other side at an equal distance, to the focus S and directrix LM there must correspond another focus S' and directrix $L'M'$, such that $CS' = CS$ and $CL' = CL$.

106. *To express the focal distances of any point in terms of the abscissa of the point.*

Let P be a point (xy) on either curve, $CN = x$, $PN = y$.

Draw MPM' parallel to the axis cutting the directrices in M, M' respectively. Join SP, $S'P$.



Then $SP = e \,.\, PM$, $S'P = e \,.\, PM'$;

$\therefore$ in the *ellipse* $SP + S'P = eMM' = eLL'$.

But $LL' = 2CL = 2\dfrac{a}{e}$, $\therefore SP + S'P = 2a$.

Again, $SP = e \,.\, PM = e \,.\, NL = e(CL + CN)$

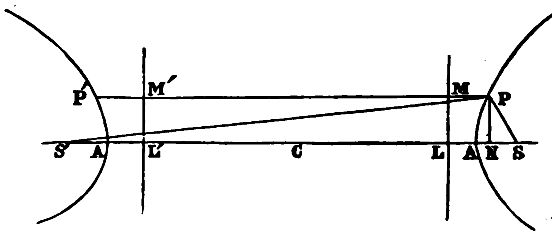
$$= e\left(\frac{a}{e} + x\right) = a + ex,$$

$$S'P = a - ex.$$

So, in the *hyperbola* $S'P - SP = e \cdot MM' = 2a$,

$$SP = e \cdot PM = ex - a,$$

$$S'P = ex + a.$$



COR. In the ellipse $SB = a$, $SB = S'B$.

107. *The auxiliary circle.*

The equation to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may be put in the form

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

Here for any value of x we have two equal and opposite values of y.

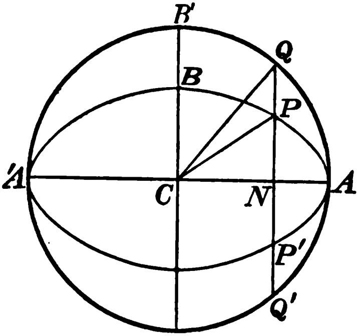
Let AQB' be the circle described on the axis major as diameter, then the equation to this circle is

$$x^2 + y^2 = a^2,$$

or $$y^2 = a^2 - x^2,$$

$$\therefore\ y = \pm (a^2 - x^2)^{\frac{1}{2}}$$

Let $CN = x$: divide QN in P so that $PN = \frac{b}{a} QN$. Then $PN = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$, that is, P is a point on the ellipse.



Hence, if on the axis major of an ellipse as diameter a circle be described, the ellipse will cut all its ordinates in the ratio $\frac{a}{b}$. This circle is called the auxiliary circle.

The circle described on the transverse axis of the hyperbola has the same name.

108. *Polar equation to a central conic. Centre pole.*

In the equation

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

write $r\cos\theta$ for x, $r\sin\theta$ for y, and divide by r^2;

$$\therefore \frac{1}{r^2} = \frac{\cos^2\theta}{a^2} \pm \frac{\sin^2\theta}{b^2}.$$

Now $b^2 = \pm a^2(1-e^2)$; therefore this equation may be written

$$\frac{1}{r^2} = \frac{(1-e^2)\cos^2\theta + \sin^2\theta}{a^2(1-e^2)},$$

or

$$\frac{1}{r^2} = \frac{1-e^2\cos^2\theta}{a^2(1-e^2)}.$$

In the ellipse $e < 1$, therefore r is possible for all values of θ; r is greatest when $1-e^2\cos^2\theta$ is least, that is when $\theta=0$ or π: the axis major is therefore the greatest diameter of the ellipse, and the axis minor is the least, and the diameter which makes a greater angle with the axis major is less than one which makes a lesser angle.

In the hyperbola $e > 1$, therefore we may write

$$\frac{1}{r^2} = \frac{e^2\cos^2\theta - 1}{a^2(e^2-1)}.$$

Hence if $\cos^2\theta < \frac{1}{e^2}$, r is impossible, and the least value of r is when $\cos\theta = 1$, the transverse axis is therefore the least diameter of a hyperbola, a diameter increases as the angle it makes with the axis increases, and if this angle is greater than $\cos^{-1}\frac{1}{e}$, the diameter does not meet the hyperbola in real points.

109. Let P be a point on the ellipse, PN perpendicular to AC: produce NP to cut the auxiliary circle in Q, join CQ, and draw PFE parallel to CQ to cut the axes in F, E, respectively.

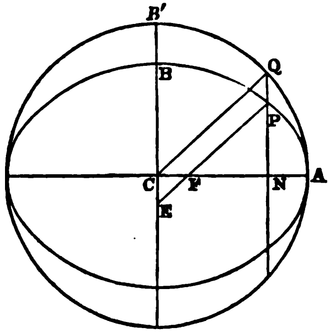
Then $$PF : CQ :: PN : QN$$
$$:: BC : AC,$$

but $$CQ = AC, \therefore PF = BC,$$

so $FE = AC - BC$, and is therefore constant.

Hence, if a straight line of fixed length move so that its extremity and a fixed point in it always lie on two straight lines at right angles to each other, the free extremity will trace out an ellipse.

Elliptic compasses are constructed from this property.



The simplest method of drawing an ellipse is, however, to fix two pins into the paper, with a string joining them, and to keep the string stretched by means of a pencil.

As the pencil moves the sum of the distances of its point from the two pins will be constant, it will therefore describe the ellipse of which the pins are the foci, and the length of the string the axis major.

110. *Conjugate hyperbola.*

We have seen that diameters which make an angle with the transverse axis greater than $\tan^{-1}\frac{b}{a}$ do not meet the hyperbola in real points; they however will meet the curve denoted by the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

It is evident that this curve is a hyperbola in which the axis of y meets the curve in real points.

Comparing this equation with

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we see that if we interchange x and y, a and b, properties of the ordinates of one curve become those of the abscissæ of the other.

Hence the equation

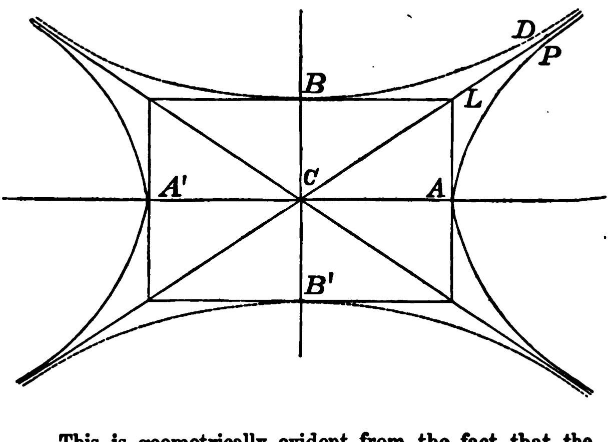
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

represents a hyperbola, having the same axes as the original hyperbola, the transverse axis of the original hyperbola being the conjugate of this, and vice versâ.

The asymptotes of this hyperbola will be

$$\frac{x}{a} = \pm\frac{y}{b}.$$

That is the two hyperbolas have the same asymptotes.



This is geometrically evident from the fact that the rectangle of which the asymptotes are diagonals, and the sides the perpendiculars to the axes at their extremities is the same for both.

The polar equation to this hyperbola is evidently

$$r^2=\frac{a^2b^2}{a^2\sin^2\theta-b^2\cos^2\theta}.$$

We may observe that properties of the conjugate hyperbola may be obtained from the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, by writing $-a^2$ for a^2. In the figure the dotted lines represent the conjugate hyperbola.

Ex. 17.

1. Determine the eccentricities and latera recta of the following central conics:

(i) $x^2+2y^2=2$; (ii) $3x^2+4y^2=12$;

(iii) $2x^2+y^2=1$; (iv) $9x^2-16y^2=25$.

2. Determine the distances between the foci in the curves in q : 1.

3. The latus rectum of an ellipse is l and the eccentricity $\frac{1}{2}$, determine the axes.

4. The distance between the foci of a hyperbola is $2c$ and the eccentricity $\sqrt{2}$, determine the axes, and find the equation to the curve referred to them.

5. If S, S' be the foci, B the extremity of the axis minor, and SBS' be a right angle, determine the eccentricity.

6. If KSK' be the latus rectum and KCK' an equilateral triangle, determine the eccentricity.

7. Can the curves in the two preceding questions be hyperbolas?

8. An ellipse and hyperbola have the same foci and conjugate axis; if a, b be the semi-axes of the ellipse, a', b of the hyperbola, prove that $a'^2=a^2-2b^2$, and if e_1, e_2 be the eccentricities,

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=2.$$

9. Determine the inclination of a diameter to the axis major, when its square is (i) an arithmetic, (ii) geometric, (iii) harmonic mean between the squares on the axes.

10. The length of a diameter, its inclination to the axis, and the eccentricity are known; determine the axes.

11. The length of a diameter is known and its inclination to the axis minor, the length of which is also known; determine the axis major.

12. If $(r_1\theta_1)$, $(r_2\theta_2)$ be two points on an ellipse referred to the centre and the axis major,

$$\frac{1}{r_1^2} - \frac{1}{r_2^2} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\sin^2\theta_1 - \sin^2\theta_2).$$

TANGENT AND NORMAL.

111. Let (x_1y_1), (x_2y_2), be two points on the curve

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

then

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

is the equation to the chord passing through these points.

But

$$\frac{x_1^2}{a^2} \pm \frac{y_1^2}{b^2} = 1 = \frac{x_2^2}{a^2} \pm \frac{y_2^2}{b^2};$$

$$\therefore \frac{y_2^2 - y_1^2}{b^2} = \pm\frac{x_1^2 - x_2^2}{a^2}, \text{ or } \frac{y_2 - y_1}{x_2 - x_1} = \mp\frac{b^2}{a^2}.\frac{x_2 + x_1}{y_2 + y_1}.$$

The equation to the chord becomes therefore

$$\frac{(x - x_1)(x_2 + x_1)}{a^2} \pm \frac{(y - y_1)(y_2 + y_1)}{b^2} = 0,$$

or

$$\frac{x(x_1 + x_2)}{a^2} \pm \frac{y(y_1 + y_2)}{b^2} = 1 \pm \frac{y_1y_2}{b^2} + \frac{x_1x_2}{a^2}.$$

In this equation let $x_2 = x_1$, $y_2 = y_1$, then the chord becomes the tangent and the equation becomes

$$\frac{xx_1}{a^2} \pm \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} \pm \frac{y_1^2}{b^2} = 1.$$

The equation to the tangent at the point $(x_1 y_1)$ is therefore

$$\frac{xx_1}{a^2} \pm \frac{yy_1}{b^2} = 1.$$

112. *Equation to the tangent in terms of the tangent of the angle it makes with the axis of x.*

Let the straight line whose equation is $y = mx + c$ meet the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Substitute for y and arrange by powers of x;

$$\therefore x^2\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) + \frac{2mcx}{b^2} + \frac{c^2 - b^2}{b^2} = 0.$$

Multiplying up by a^2b^2, we have, if the equation have equal roots,

$$(m^2a^2 + b^2)(c^2 - b^2) = m^2a^2c^2;$$

$$\therefore c^2 = m^2a^2 + b^2$$

is the condition that the straight line shall touch the curve.

The equation to any tangent may therefore be written

$$y = mx + (m^2a^2 + b^2)^{\frac{1}{2}}.$$

If we change the sign of b^2, the equation to the tangent to the hyperbola becomes

$$y = mx + (m^2a^2 - b^2)^{\frac{1}{2}}.$$

If we write $\tan\theta$ for m in the above equations, and multiply by $\cos\theta$ they become

$$y\cos\theta - x\sin\theta = (a^2\sin^2\theta \pm b^2\cos^2\theta)^{\frac{1}{2}}.$$

Hence we see that if $\tan\theta < \frac{b}{a}$, the tangent to the hyperbola is impossible; in this case, however, the straight line

$$y\cos\theta - x\sin\theta = (b^2\cos^2\theta - a^2\sin^2\theta)^{\frac{1}{2}},$$

touches the conjugate hyperbola, whose equation is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

113. In the above equations, write $\frac{\pi}{2} + \theta'$ for θ, then θ' is the angle which the perpendicular on the tangent makes with the axis, and the above equations become

$$x\cos\theta' + y\sin\theta' = (a^2\cos^2\theta' \pm b^2\sin^2\theta')^{\frac{1}{2}}.$$

114. Perpendicular on the tangent from the centre.

By Art. 36, this may be written down in either of the forms,

$$\frac{a^2b^2}{(a^4y_1^2 \pm b^4x_1^2)^{\frac{1}{2}}}, \quad \frac{(m^2a^2 \pm b^2)^{\frac{1}{2}}}{(1+m^2)^{\frac{1}{2}}}, \text{ or } (a^2\cos^2\theta' \pm b^2\sin^2\theta')^{\frac{1}{2}}.$$

Since, in the ellipse, $a^2y_1^2 = a^2b^2 - b^2x_1^2$,

$$a^4y_1^2 + b^4x_1^2 = b^2\{a^4 - (a^2 - b^2)\,x_1^2\} = a^2b^2(a^2 - e^2x_1^2).$$

The length of the perpendicular is therefore $\frac{ab}{(a^2 - e^2x_1^2)^{\frac{1}{2}}}$.

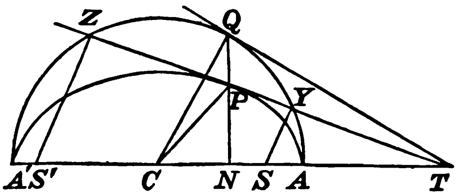
For the hyperbola this becomes $\frac{ab}{(e^2x_1^2 - a^2)^{\frac{1}{2}}}$.

It will be convenient in future to take S, A, on the positive side of C, and to accent the corresponding letters on the negative side.

115. Let P be any point (x_1, y_1) on the ellipse, let NP be produced to meet the auxiliary circle in Q, then the co-ordinates of Q are x_1, y_1', where $y_1' = \frac{a}{b}y_1$.

Now the equation to the tangent to the ellipse at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$



That to the tangent to the circle at Q is

$$xx_1+yy_1'=a^2.$$

In each of these equations put $y=0$.

Then the lines represented by them cut the axis of x in the same point T, such that $CT=\dfrac{a^2}{x_1}$.

Hence, if we wish to draw a tangent to an ellipse at any point P, draw the tangent QT to the auxiliary circle, cutting the axis in T, and join PT; PT is the tangent required.

Since x_1, or CN is necessarily greater than CA in the hyperbola, the construction above given fails in this case. Still, however, we have $xx_1=a^2$.

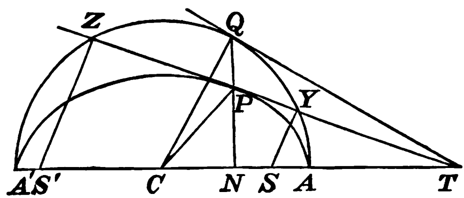
Let P be a point on the hyperbola, PN its ordinate, draw NQ to touch the auxiliary circle in Q, and QT perpendicular to AC: join TP; TP is the tangent to the hyperbola at P.

For $CT.CN=a^2$.

116. The perpendiculars from the foci on the tangent intersect them on the auxiliary circle, and the rectangle contained by them is equal to the square on half the conjugate axis.

Take the case of the ellipse.

Let $x \cos\theta + y \sin\theta = (a^2 \cos^2\theta + b^2 \sin^2\theta)^{\frac{1}{2}}$ be the equation to the tangent TP.



Let $SY, S'Z$ be the perpendiculars from S, S', respectively.

Then the equation to SY is (Art. 34),

$$x \sin\theta - y \cos\theta = (a^2 - b^2)^{\frac{1}{2}} \sin\theta,$$

since it is perpendicular to the tangent and passes through the point $\{(a^2-b^2)^{\frac{1}{2}}, 0\}$: square these equations to TY, SY, and add;

$$\text{then at } Y,\ x^2 + y^2 = a^2,$$

that is, Y lies on the auxiliary circle.

Similarly the equation to $S'Z$ is

$$x \sin\theta - y \cos\theta = -(a^2 - b^2)^{\frac{1}{2}} \sin\theta,$$

squaring and adding we get the same equation as before.

Again (Art. 37),

$$SY = (a^2 \cos^2\theta + b^2 \sin^2\theta)^{\frac{1}{2}} - (a^2 - b^2)^{\frac{1}{2}} \cos\theta,$$

$$S'Z = (a^2 \cos^2\theta + b^2 \sin^2\theta)^{\frac{1}{2}} + (a^2 - b^2)^{\frac{1}{2}} \cos\theta;$$

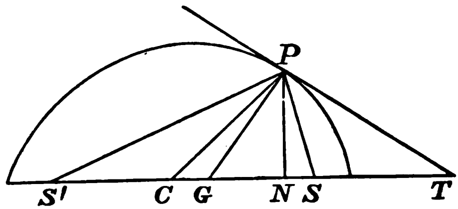
$$\therefore SY . S'Z = a^2 \cos^2\theta + b^2 \sin^2\theta - (a^2 - b^2) \cos^2\theta = b^2.$$

Changing the sign of b^2 in these expressions we still get, at Y or Z,

$$x^2 + y^2 = a^2, \text{ and } SY . S'Z = -b^2.$$

This shews that S, S' lie on opposite sides of the tangent in the hyperbola, as may be seen by drawing the figure.

117. *To find the equation to the normal at the point* $x_1\, y_1$.



Let PG be the normal; then since it passes through $(x_1\, y_1)$ its equation is of the form $l(x-x_1)+m(y-y_1)=0$; also, since it is perpendicular to the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

$$\frac{lx_1}{a^2}+\frac{my_1}{b^2}=0.$$

The equation therefore becomes

$$\frac{x_1y}{a^2}-\frac{x_1y_1}{a^2}=\frac{y_1x}{b^2}-\frac{x_1y_1}{b^2}, \text{ or } \frac{y-y_1}{a^2y_1}=\frac{x-x_1}{b^2x_1}.$$

In this equation put $y=0$;

$$\therefore x=\frac{a^2-b^2}{a^2}x_1=e^2x_1.$$

If therefore PG be the normal at P, cutting the axis at G, $CG=e^2CN$.

Similarly the equation to the normal to the hyperbola is,

$$\frac{y-y_1}{a^2y_1}+\frac{x-x_1}{b^2x_1}=0.$$

Here if $y=0$, $x=\dfrac{a^2+b^2}{a^2}x_1=e^2x_1$, as before.

118. ***The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal distances.***

We have in the ellipse $SP = a - ex_1$,

$$S'P = a + ex_1.$$

Now $$SG = SC - CG = ae - e^2x_1,$$

$$S'G = SC + CG = ae + e^2x_1;$$

$$\therefore SG : S'G :: SP : S'P;$$

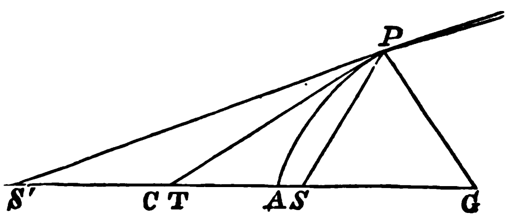
$\therefore$ PG bisects the angle $S'PS$.

If the normal bisects the internal angle, it is evident that the tangent must bisect the external angle between the same straight lines.

So in the hyperbola

$$SP = ex_1 - a,$$

$$S'P = ex_1 + a,$$



$$S'G = ae + e^2x_1, \quad SG = e^2x_1 - ae.$$

$$\therefore SG : S'G :: SP : S'P,$$

and PG bisects the angle between SP and $S'P$ produced.

Hence the tangent PT bisects the angle between the focal distances.

119. By a proof precisely similar to that of Arts. 58, 59, we can shew that, when (x_1y_1) is a point without the curve, the equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

represents the chord of contact of tangents through (x_1y_1).

This equation represents the polar of (x_1y_1) wherever (x_1y_1) may be; and it may be proved, as in the case of the circle and parabola, that the chords of contact of tangents drawn from any point on the line always pass through the point.

120. The equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2,$$

represents the two tangents through the point (hk). For it can be split up into two factors, is satisfied by (hk), and if $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, it becomes $\frac{hx}{a^2}+\frac{ky}{b^2}=1$, the equation to the polar of (hk).

Ex. 18.

1. Find the equations to tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

which fulfil the following conditions:

(i) Cut off a triangle of given area between the axes;

(ii) Are parallel to

$$\frac{x}{a}+\frac{y}{b}=1;$$

(iii) Make equal angles with the axes;

(iv) Are at a given distance from the point (h, k).

2. Find the equation to the tangent TPt, cutting the axes of the ellipse in T, t, respectively, when

$$CT + Ct = \sqrt{2}(a+b).$$

3. Find the equation to the tangent TPt, when the perimeter of the triangle TCt is equal to $(2+\sqrt{2})(a^2+b^2)^{\frac{1}{2}}$.

4. Find the equation to the tangent, when the perpendiculars from the foci are in a given ratio.

5. If $lx + my = d$ touch the ellipse, then

$$l^2a^2 + m^2b^2 = d^2(l^2+m^2).$$

6. If p be the perpendicular from the centre on the tangent which makes an angle θ with the axis,

$$p = a(1 - e^2\cos^2\theta)^{\frac{1}{2}}.$$

7. Find the equation to the tangent at the extremity of the latus rectum of the ellipse

$$x^2 + 2y^2 = 3.$$

8. Find the equation to the tangent which is parallel to SB.

9. The equation to the diameter drawn perpendicular to the tangent at (x_1y_1), is

$$\frac{xy_1}{b^2} = \frac{x_1y}{a^2}.$$

10. Prove that the locus of the point of intersection of this line with the line $x = x_1$ is the ellipse

$$\frac{x^2}{a^2} + \frac{b^2y^2}{a^4} = 1.$$

11. If h, k, be the intercepts on the axes of any tangent,

$$\frac{a^2}{h^2} + \frac{b^2}{k^2} = 1.$$

12. Two tangents are such that the product of the tangents of the angles they make with the axis of x is m, prove that they intersect on the curve

$$y^2 - b^2 = m(x^2 - a^2).$$

13. Two tangents intersect at right angles, prove that their point of intersection lies on the circle

$$x^2+y^2=a^2+b^2.$$

14. If PG be the normal at P, cutting the axis at G, and CK be perpendicular to it, determine PG, and CK in terms of PCA.

15. If the normal cut the curve again at Q, find PQ.

16. Find the equations to the normals which pass through the points c, 0; 0, c; h, k; respectively.

17. If PG cut the minor axis in g, find Gg.

18. Q is a point on the tangent at the extremity of the axis minor, QP the tangent at P; if $BQ=\sqrt{3}a$, then

$$PQ=BP.$$

19. PR, QR are two normals at P, Q, cutting at right angles in R; CRH is a semi-diameter; prove that the tangent at H is parallel to PQ.

20. If r be a radius CP of an ellipse and p the perpendicular on the tangent at P, then

$$p^2=\frac{a^2b^2}{a^2+b^2-r^2}.$$

DIAMETERS.

121. *To find the locus of the middle points of chords which make a fixed angle with the axis.*

Let β be the angle, (xy), (XY) two points on one of these chords, r the distance between these points.

Then (Art. 25) $\quad x=X+r\cos\beta,$

$$y=Y+r\sin\beta.$$

Now let (xy) be on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Substituting and rearranging by powers of r, we have

$$r^2\left(\frac{\cos^2\beta}{a^2}+\frac{\sin^2\beta}{b^2}\right)+2r\left(\frac{X\cos\beta}{a^2}+\frac{Y\sin\beta}{b^2}\right)$$
$$+\frac{X^2}{a^2}+\frac{Y^2}{b^2}-1=0 \;\ldots\ldots\ldots\ldots(1).$$

Now if (XY) be the middle point of the chord, the values of r deduced from this equation must be equal and opposite: the coefficient of r must therefore be zero;

$$\therefore \quad \frac{X\cos\beta}{a^2} + \frac{Y\sin\beta}{b^2} = 0.$$

That is, (XY) is a point on the straight line

$$\frac{x\cos\beta}{a^2} + \frac{y\sin\beta}{b^2} = 0.$$

The locus of the middle points of chords which make a fixed angle with the axis of x, is therefore a diameter.

Now let the chord move parallel to itself till its middle point coincides with the extremity of the diameter bisecting it; the two extremities coincide with that middle point, and the chord becomes the tangent at the extremity of the diameter. A diameter therefore bisects all chords parallel to the tangent at its extremity.

The diameter parallel to a tangent is said to be conjugate to the diameter at the extremity of which the tangent is drawn, or to the diameter which bisects chords parallel to the first.

122. *If any diameter be conjugate to another, the second is conjugate to the first.*

Let CD be conjugate to CP, that is, let CD be parallel to the tangent TP, then shall CP be conjugate to CD.

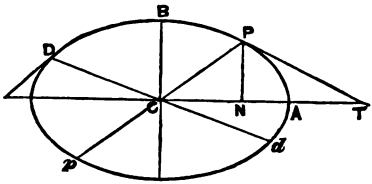
Let the coordinates of P be x_1y_1, of D x_2y_2.

Then since the equation to PT is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

that to CD must be $\quad \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 0;$

$\therefore \ \dfrac{x_1x_2}{a^2} + \dfrac{y_1y_2}{b^2} = 0$, is the condition that CD is conjugate to CP.



Since in this condition x_1, x_2, y_1, y_2 are interchangeable, the point (x_1y_1) is on the diameter conjugate to CD.

That is CP is conjugate to CD.

If $PCA=\alpha$, $DCA=\beta$, $CP=r$, $CD=r'$, we must have

$$x_1=r\cos\alpha,\quad y_1=r\sin\alpha,\quad x_2=r'\cos\beta,\quad y_2=r'\sin\beta,$$

and the condition becomes

$$\frac{\cos\alpha\cos\beta}{a^2}+\frac{\sin\alpha\sin\beta}{b^2}=0,$$

or
$$\tan\alpha\tan\beta=-\frac{b^2}{a^2}.$$

If then pCP be any diameter, PT the tangent at P, and if dCD be parallel to PT, pCP will bisect all chords parallel to CD, dCD will bisect all chords parallel to CP, and the tangent at D will be parallel to CP.

123. To find the coordinates of D in terms of those of P, and the lengths of CP and CD.

Let x_1, y_1 be the coordinates of P, x_2, y_2 of D.

Then $$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=0,\quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1=\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}.$$

Hence, $$x_2=-\frac{a}{b}y_1,\quad y_2=\frac{b}{a}x_1.$$

These are the coordinates of D taken as in the figure, those of d will of course be $\frac{a}{b}y_1,\ -\frac{b}{a}x_1$.

Again, $$CP^2 = x_1^2 + y_1^2 = x_1^2 + \frac{b^2}{a^2}(a^2 - x_1^2) = b^2 + e^2x_1^2,$$

$$CD^2 = x_2^2 + y_2^2 = (a^2 - x_1^2) + \frac{b^2}{a^2}x_1^2 = a^2 - e^2x_1^2;$$

therefore $CP^2 + CD^2 = a^2 + b^2$, or the sum of the squares on conjugate diameters is constant.

124. In the hyperbola the equation to CD becomes, by changing the sign of b^2,

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 0,$$

and we have

$$\tan\alpha \tan\beta = \frac{b^2}{a^2}.$$

If then $\tan\alpha < \frac{b}{a}$, $\tan\beta > \frac{b}{a}$, and therefore CD does not meet the hyperbola in real points.

It meets the conjugate hyperbola however in the points $\frac{a}{b}y_1$, $\frac{b}{a}x_1$, which satisfy the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

For the hyperbola we have,

$$CP^2 = x_1^2 + y_1^2 = x_1^2 + \frac{b^2}{a^2}(x_1^2 - a^2) = e^2x_1^2 - b^2,$$

$$CD^2 = x_2^2 + y_2^2 = e^2x_1^2 - a^2;$$

therefore $CP^2 - CD^2 = a^2 - b^2$, or the difference of the squares on conjugate diameters is constant.

125. Since in the ellipse, $SP = a - ex_1$, and in the hyperbola $SP = ex_1 + a$, and in both cases $S'P = ex_1 + a$, $\therefore SP \,.\, S'P = CD^2$.

This might also have been shown geometrically, since $CDB = PSY$, and therefore

$$SP \,.\, S'P : SY \,.\, S'Z :: CD^2 : BC^2,$$

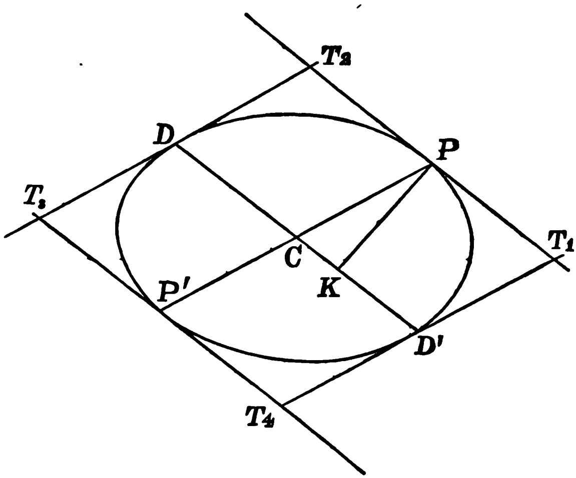
but $$SY \,.\, S'Z = BC^2; \quad \therefore SP \,.\, S'P = CD^2.$$

126. *The area of the parallelogram circumscribing the curve whose sides are parallel to conjugate diameters is constant.*

Let $T_1T_2T_3T_4$ be such a parallelogram, PCP', DCD' conjugate diameters; draw the normal PK.

Let x_1, y_1 be the coordinates of P. Then the parallelogram $T_1T_2T_3T_4 = 4PCDT_2 = 4CD \cdot PK$.

But
$$CD = (a^2 - e^2x_1^2)^{\frac{1}{2}},$$
$$PK = \frac{ab}{(a^2 - e^2x_1^2)^{\frac{1}{2}}} \text{ (Art. 114)};$$
$$\therefore CD \cdot PK = ab,$$
therefore the area of the parallelogram required $= 4ab$.



127. If α be the angle between two conjugate diameters whose lengths are a', b', respectively, since the

parallelogram contained by them is equal to $a'b'\sin\alpha$, we have

$$a'b'\sin\alpha = ab,$$

$$\text{or}\quad \sin\alpha = \frac{ab}{a'b'}.$$

128. *To find the equation to the ellipse referred to a pair of conjugate diameters as axes.*

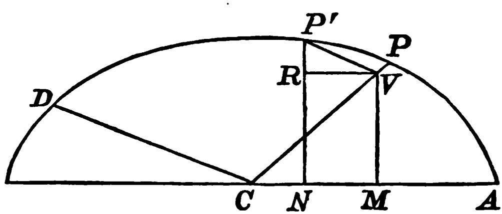
Let CP, CD be the new axes, α, β the angles they respectively make with the axis of x.

Let P' be any point on the ellipse; (x, y), (x', y') its coordinates referred to the old and new axes respectively.

Draw VR parallel to CA, then

$$x = CN = CM - RV = CV\cos PCA + P'V\cos DCA$$
$$= x'\cos\alpha + y'\cos\beta;$$

similarly, $$y = x'\sin\alpha + y'\sin\beta.$$



Substitute in the equation to the ellipse; thus

$$\frac{(x'\cos\alpha + y'\cos\beta)^2}{a^2} + \frac{(x'\sin\alpha + y'\sin\beta)^2}{b^2} = 1,$$

or $$x'^2\left(\frac{\cos^2\alpha}{a^2} + \frac{\sin^2\beta}{b^2}\right) + 2x'y'\left(\frac{\cos\alpha\cos\beta}{a^2} + \frac{\sin\alpha\sin\beta}{b^2}\right)$$
$$+ y'^2\left(\frac{\cos^2\alpha}{a^2} + \frac{\sin^2\beta}{b^2}\right) = 1.$$

Now the polar equation to the ellipse is

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}; \text{ Art. (108)}$$

$$\therefore\ \frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}=\frac{1}{CP^2}=\frac{1}{a'^2}:$$

similarly, $$\frac{\cos^2\beta}{a^2}+\frac{\sin^2\beta}{b^2}=\frac{1}{CD^2}=\frac{1}{b'^2}.$$

Also $\frac{\cos\alpha\cos\beta}{a^2}+\frac{\sin\alpha\sin\beta}{b^2}=0$, since the diameters are conjugate;

therefore the above equation becomes

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}=1.$$

Hence we see that, whatever be the angle between the axes, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

represents an ellipse referred to two conjugate diameters whose lengths are $2a$ and $2b$.

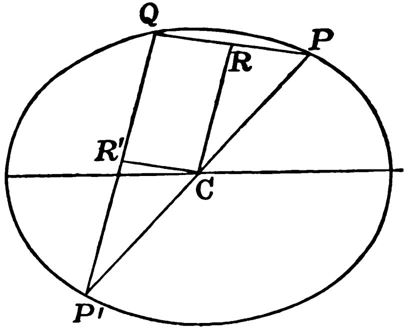
The proof is precisely the same for the hyperbola, changing the sign of b^2.

Since in Art. 111 no mention was made of the axes being rectangular, the equation

$$\frac{xx_1}{a^2}\pm\frac{yy_1}{b^2}=1,$$

to the tangent at the point (x_1, y_1), will hold good generally provided the curve be referred to two conjugate diameters.

129. *Supplemental chords.*



Let PCP' be any diameter of a central conic, Q any point on the curve: join QP, QP'; these are called supplemental chords.

The diameters parallel to a pair of supplemental chords are conjugate.

Bisect QP, QP', in R, R', respectively; join CR, CR'. Then we know that CR' bisects all chords parallel to $P'Q$, and CR all chords parallel to PQ.

But $$QR = \frac{1}{2} QP, \quad PC = \frac{1}{2} PP',$$

therefore CR is parallel to $P'Q$.

Similarly CR' is parallel to PQ,

therefore CR, CR', are conjugate diameters.

Hence if we wish to draw two conjugate diameters of an ellipse or hyperbola containing a given angle; take any diameter PP' of the curve and on it describe a segment of a circle containing the required angle; let this segment cut the curve in Q, then the diameters parallel to PQ, $P'Q$ are conjugate, and contain the required angle.

Ex. 19.

1. Write down the equations to the diameters respectively conjugate to the following lines,

$$y=x;\ x+y=0,\ ax=by;\ ay=bx;\ y\cos\theta=x\sin\theta;$$

$$\frac{x}{a}+\frac{y}{b}=0.$$

2. The length of each of the equal conjugate diameters is

$$\sqrt{2}(a^2+b^2)^{\frac{1}{2}}.$$

3. The length of a semidiameter is c, find the equation to its conjugate.

4. If the tangent at the vertex A cut any two conjugate diameters in T, t, then $AT\,.\,At=b^2$.

5. The length of a semidiameter is $\frac{a+b}{2}$, find the equation to its conjugate.

6. If α be the angle between two conjugate diameters of an ellipse, which make angles θ, θ', with the axis major,

$$\cos\alpha=e^2\cos\theta\cos\theta'.$$

7. Determine the corresponding equation in the case of the hyperbola.

8. The angle between the equal conjugate diameters being $\frac{\pi}{3}$, shew that the eccentricity is $\frac{\sqrt{6}}{3}$.

9. The equation to an ellipse being $2x^2+3y^2=4$, the diameters

$$y=2x,\ \ x+3y=0,$$

are conjugate.

10. The locus of the middle points of chords joining the extremities of conjugate diameters is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

11. If a', b', be the lengths of CP, CD, and the angles PCA, DCB be α, β, respectively, then

$$\frac{a'^2-b'^2}{a^2-b^2}=\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}.$$

12. S is a focus, CP, CD conjugate diameters; the distance of P from the diameter which is parallel to SD is equal to b.

13. If SP intersect CD in Q, then $PQ=a$.

14. If a', b' be the lengths of CP, CD, and α, β the angles which they make with the axis of x, then

$$a'^2 \sin 2\alpha + b'^2 \sin 2\beta = 0.$$

15. The normals at P and D meet the major axis in G, G', respectively, prove that $PG^2 + DG'^2 = \frac{b^2}{a^2}(a^2+b^2)$.

16. If T be a point (h, k) on the tangent at P, l the length of TP, and $2b'$ of the diameter conjugate to CP,

$$l^2 = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) b'^2.$$

17. Tangents to an ellipse are drawn of lengths equal to n times the conjugate semi-diameters at their extremities, prove that the locus of their other extremities is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + n^2.$$

18. If a', b', be the lengths of two semidiameters at right angles to each other,

$$\frac{1}{a'^2} + \frac{1}{b'^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

19. If θ be the angle which these diameters make respectively with the major and minor axes,

$$\cos 2\theta = \frac{\dfrac{1}{b'^2} - \dfrac{1}{a'^2}}{\dfrac{1}{b^2} - \dfrac{1}{a^2}}.$$

20. If the ellipse be referred to these diameters as axes of coordinates, its equation will become

$$\frac{x^2}{a'^2} + xy\left(\frac{1}{b'^2} - \frac{1}{a'^2}\right) \tan 2\theta + \frac{y^2}{b'^2} = 1.$$

21. If $2c$ be the length of the equal conjugate semidiameters, the equation

$$x^2 + y^2 = c^2$$

represents the ellipse referred to these as axes.

22. If α be the acute angle between the axes of coordinates, the semiaxes of the ellipse, $x^2+y^2=c^2$, are

$$\sqrt{2}c\cos\frac{\alpha}{2}, \text{ and } \sqrt{2}c\sin\frac{\alpha}{2}.$$

23. If e be the eccentricity of the same ellipse,

$$e=\left(\frac{2\cos\alpha}{1+\cos\alpha}\right)^{\frac{1}{2}}.$$

24. The equation to the auxiliary circle is

$$x^2+y^2+(2xy-c^2)\cos\alpha=c^2.$$

130. *The eccentric angle.*

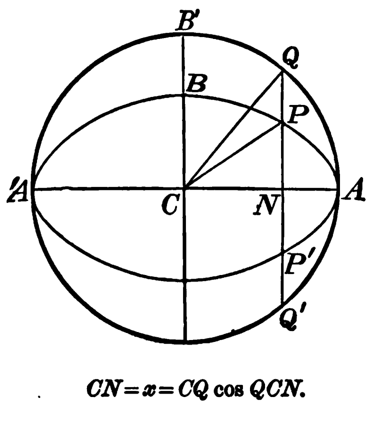
Since $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the equation to the ellipse, and

$$\cos^2\phi+\sin^2\phi=1.$$

If $$x=a\cos\phi,\ y=b\sin\phi.$$

Let P be the point (xy) on the ellipse, and let NP the ordinate at P when produced cut the auxiliary circle in Q. Join PC, QC.



Then $$CN=x=CQ\cos QCN.$$

If $\therefore\ QCN=\phi,\ x=a\cos\phi$, and $y=b\sin\phi$.

Since $PN=b\sin\phi$, and $QN=CQ\sin\phi=a\sin\phi$,

$$QN=\frac{a}{b}\,.\,PN.$$

QCA is called the *eccentric angle* of *P*.

Many properties of the ellipse analogous to those of the circle may be deduced by using coordinates in terms of the eccentric angle.

131. Equation to a chord in terms of the eccentric angles of its extremities.

Let $\phi_1,\ \phi_2$ be these angles, then (Art. 23) the equation to the chord is

$$y-b\sin\phi_1=\frac{b}{a}\cdot\frac{\sin\phi_2-\sin\phi_1}{\cos\phi_2-\cos\phi_1}(x-a\cos\phi_1);$$

$$\therefore\ \frac{y}{b}+\frac{x}{a}\cot\frac{\phi_1+\phi_2}{2}=\sin\phi_1+\cos\phi_1\cot\frac{\phi_1+\phi_2}{2};$$

$$\therefore\ \frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}.$$

Now let $\phi_2=\phi_1$, then the chord becomes the tangent, and its equation becomes

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

This equation might have been obtained by putting

$$x=a\cos\phi,\ \ y=b\sin\phi,$$

in the equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

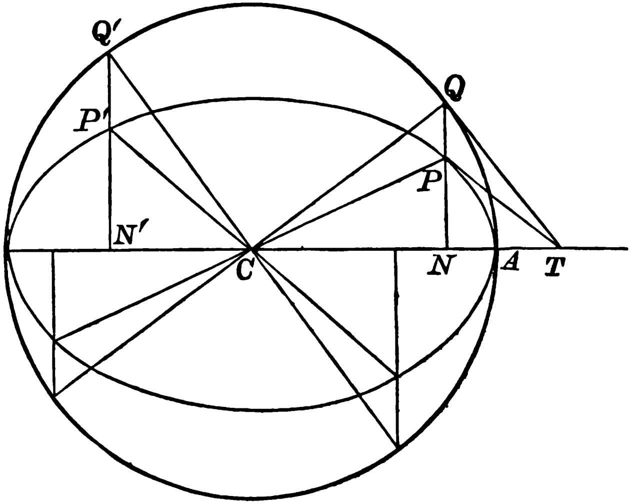
132. The equation to the normal in terms of ϕ is

$$a\frac{(x-a\cos\phi)}{\cos\phi}-b\frac{(y-b\sin\phi)}{\sin\phi}=0,$$

$$\text{or}\quad ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2.$$

133. **Many of the properties of conjugate diameters are easily obtained by means of the eccentric angle.**

In the figure let CP, CP' be conjugate, then QCA, $Q'CA$ are the eccentric angles of P, P'. Let $QCA = \phi$. Now the tangents at P, Q meet on the axis, since if we put $y=0$ in each of these equations we get $x = a \sec \phi$.



But CP' is parallel to PT, and

$$QN : PN :: Q'N' : P'N';$$

$\therefore$ CQ' is parallel to QT, and $\therefore$ $Q'CQ$ is a right angle.

The eccentric angles of two conjugate diameters differ therefore by a right angle.

This may be also proved thus.

The equation to CP is $\frac{y}{b} = \frac{x}{a} \tan \phi$.

That to CP' which is parallel to PT is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=0,$$

or
$$\frac{y}{b}=\frac{x}{a}\tan\left(\phi+\frac{\pi}{2}\right);$$

$\therefore$ the eccentric angle of P' is $\frac{\pi}{2}+\phi$.

Hence we may prove that if CP' is conjugate to CP, CP is to CP'. For the equation to the diameter conjugate to CP' or to

$$\frac{y}{b}=\frac{x}{a}\tan\left(\phi+\frac{\pi}{2}\right)$$

is
$$\frac{y}{b}=\frac{x}{a}\tan(\phi+\pi)$$

$$=\frac{x}{a}\tan\phi,$$

but this is the equation to CP.

The student will observe that all those properties of the ellipse which are proved in geometrical treatises by projecting the auxiliary circle into the ellipse, can be proved easily analytically by using the eccentric angle.

134. To obtain the corresponding properties of the hyperbola, we should have to write $-b^2$ for b^2, and therefore $b\sqrt{-1}$ for b.

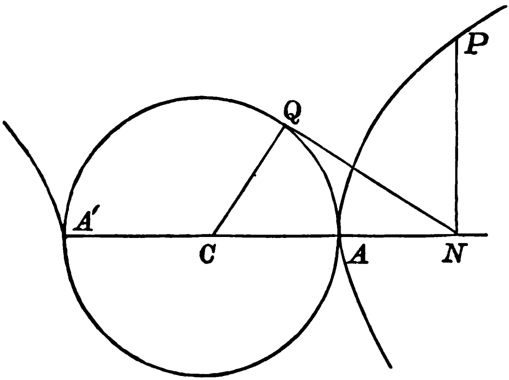
Thus $\tan\phi=\frac{a}{b}\frac{y}{x}\sqrt{-1}$, which is impossible, hence there is no angle in the hyperbola which corresponds to the eccentric angle in the ellipse.

However, since $\sec^2\phi-\tan^2\phi=1$, if in the hyperbola

$x=a\sec\phi$ we must have $y=b\tan\phi$.

The geometrical angle which corresponds to ϕ may be thus determined.

Let P be any point on the hyperbola, PN its ordinate, draw NQ touching the auxiliary circle.



Then $CN = CQ \sec QCA = a \sec QCA,$

therefore if we call QCA, ϕ, we must have $x = a \sec\phi$, $y = b\tan\phi$.

Since $QN = a\tan\phi,$

$$QN : PN :: a : b.$$

We can easily see that if PT the tangent at P cut CA in T, then QT is perpendicular to CA.

Ex. 20.

The Eccentric Angle.

1. If PT, PG be the tangent and normal at P, determine the lengths of PT, PG in terms of the eccentric angle.

2. Find the perpendiculars from the centre and foci on PT and PG.

3. Determine the eccentric angle at the extremity of the latus rectum.

4. If PG cut the curve again at Q, determine PQ, and the eccentric angle at Q.

5. If $PCA = \theta$, $PGA = \psi$, then $\tan\theta\tan\psi = \tan^2\phi$.

6. Write down the equation to the tangent at P', where CP' is conjugate to CP.

7. Find the lengths of CP, CP', and hence prove that

$$CP^2 + CP'^2 = a^2 + b^2.$$

8. If $PCA = \theta$, $P'CA = \theta'$, prove, by means of the eccentric angle, that $\tan\theta\tan\theta' = -\dfrac{b^2}{a^2}$.

9. The length of a diameter is $2c$, find its eccentric angle, and the equation to its conjugate.

10. If CP, CD be conjugate, and the ordinates of P, D meet another ellipse described on the same axis major in Q, E, respectively, then QC, CE are conjugate diameters of the second ellipse.

11. An ellipse and hyperbola have the same foci and conjugate axis, their semi-axes are a, b; a', b, respectively; shew that if ϕ_1, ϕ_2 be the eccentric angles with reference to the ellipse and hyperbola of a point of their intersection,

$$\sin^2\phi_1 = \tan^2\phi_2 = \frac{a^2 - a'^2}{a^2 + a'^2}.$$

12. PG meets the axes in G, g and DCD' the diameter conjugate to CP in F, prove that

$$PF.PG = BC^2, \quad PF.Pg = AC^2.$$

135. *Asymptotes.*

DEF. Lines are said to be asymptotes to a curve, when the curve continually approaches them, but never actually reaches them, though its distance from them may be made less than any assignable distance.

The equations to the lines CL, CL', the diagonals of the rectangle whose sides are lines drawn through the extremities of the axes perpendicular to them, are

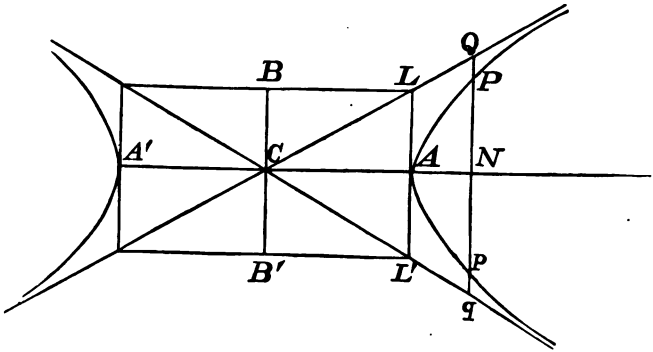
$$y = \pm\frac{b}{a}x,$$

$$\text{or} \quad \frac{y^2}{b^2} = \frac{x^2}{a^2}.$$

The equation to the hyperbola is

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1.$$

Now let P be a point on the hyperbola: let $CN=x_1$; let NP produced both ways cut CL, CL' produced in Q, q; let $PN=y_1$, $QN=y_2$.



Then $$\frac{y_2^2}{b^2}=\frac{x_1^2}{a^2}, \quad \frac{y_1^2}{b^2}=\frac{x_1^2}{a^2}-1,$$

$$\therefore \frac{y_2^2-y_1^2}{b^2}=1, \text{ or } y_2^2-y_1^2=b^2;$$

therefore y_1^2 is always less than y_2^2, that is to say, the hyperbola never cuts the lines CL, CL'.

Now as x_1 increases without limit, y_2 and y_1 also increase without limit, and therefore y_2+y_1 increases without limit.

But $$(y_2+y_1)(y_2-y_1)=b^2,$$

therefore y_2-y_1 diminishes without limit.

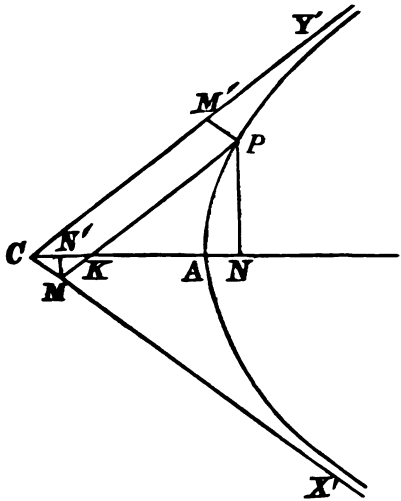
Hence CLQ is an asymptote.

Similarly $CL'q$ is an asymptote.

It is easy to prove that $pq=PQ$.

For if $-y_1$, $-y_2$ be written for y_1, y_2 respectively, y_1-y_2 is changed in sign but not in value.

136. *Equation referred to the asymptotes as axes.*



Let P be any point on the hyperbola, x, y its coordinates referred to the old, x', y' to the new axes.

Let $$Y'CA = X'CA = \alpha,$$

then $$\tan\alpha = \frac{b}{a}.$$

Now since $x' = CM = KM$, $y' = PM$,

$$\therefore\ x = CN = CN' + N'N = (CM + PM)\cos\alpha = (x' + y')\cos\alpha,$$

$$y = PN = PK\sin\alpha = (y' - x')\sin\alpha.$$

But $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

$$\therefore\ \frac{(x' + y')^2}{a^2}\cos^2\alpha - \frac{(x' - y')^2}{b^2}\sin^2\alpha = 1.$$

Also $$\cos^2\alpha = \frac{a^2}{a^2 + b^2},\ \sin^2\alpha = \frac{b^2}{a^2 + b^2},$$

$$\therefore\ (x' + y')^2 - (x' - y')^2 = a^2 + b^2,$$

or $$4x'y' = a^2 + b^2,$$

the equation required.

137. *To interpret this equation geometrically.*

Draw PM' parallel to CX', then the area of the parallelogram $CMPM'$

$$=xy \sin\omega = \frac{a^2+b^2}{4} \sin\omega = \frac{ab}{2},$$

since $$\sin\omega = 2\sin\alpha\cos\alpha = \frac{2ab}{a^2+b^2}.$$

Hence if from any point on a hyperbola, straight lines be drawn parallel to the asymptotes, the area of the parallelogram thus formed is invariable.

138. The equation to the conjugate hyperbola is evidently

$$xy = -\frac{a^2+b^2}{4},$$

since either x or y is necessarily negative, and the axes are the same as those of the original hyperbola.

139. *Equation to the tangent, the asymptotes being axes.*

Let $(x_1\, y_1)$, $(x_2\, y_2)$ be two points on the hyperbola,

then $$y_1 = \frac{a^2+b^2}{4x_1};\quad y_2 = \frac{a^2+b^2}{4x_2};$$

$$\therefore\ y_2 - y_1 = \frac{a^2+b^2}{4}\left(\frac{1}{x_2} - \frac{1}{x_1}\right) = \frac{a^2+b^2}{4}\,\frac{x_1-x_2}{x_1x_2};$$

$$\therefore\ \frac{y_2-y_1}{x_2-x_1} = -\frac{a^2+b^2}{4x_1x_2};$$

$\therefore$ the equation to the chord through $(x_1\, y_1)$, $(x_2 y_2)$

$$\text{is } y - y_1 + \frac{a^2+b^2}{4x_1\,x_2}(x - x_1) = 0.$$

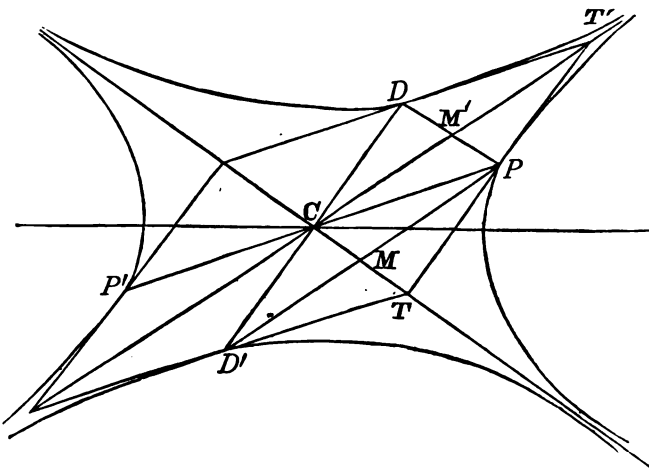
Now let $x_1 = x_2$ and write $\frac{a^2+b^2}{4y_1}$ for x_2.

Then the equation to the tangent becomes

$$y - y_1 + \frac{y_1}{x_1}(x - x_1) = 0,$$

or
$$\frac{x}{x_1} + \frac{y}{y_1} = 2.$$

140. Draw PMD', $PM'D$ parallel to the asymptotes cutting the conjugate hyperbola in D' D, respectively.



Then the equation to PD is $y = y_1$, $\therefore$ the coordinates of D are $-x_1$, y_1, or $M'D = M'P$: similarly $MD' = MP$.

141. Let TPT' be the tangent at P; then, putting y_1, x_1 successively equal to zero in the equation to the tangent,

$$CT = 2x_1 = 2CM,$$

$$CT' = 2y_1 = 2CM';$$

therefore $T'P = TP$, or the part of the tangent intercepted between the asymptotes is bisected at the point of contact.

142. The equation to the diameter conjugate to CP is

$$\frac{x}{x_1} + \frac{y}{y_1} = 0,$$

since it is parallel to the tangent at P.

Now this cuts the conjugate hyperbola

$$4xy + a^2 + b^2 = 0,$$

in points $\pm x_1 \mp y_1$.

Hence, if DCD' be this conjugate diameter, PD, PD' are bisected by the asymptotes, and the straight lines joining the extremities of conjugate diameters are parallel to the asymptotes.

143. The tangent to

$$4xy + a^2 + b^2 = 0,$$

at the point $(-x_1 y_1)$ is

$$\frac{x}{-x_1} + \frac{y}{y_1} = 2.$$

Tangents at the extremities of conjugate diameters therefore meet on the asymptotes, which are therefore the diagonals of the parallelograms so formed.

Hence the asymptotes may be considered themselves as conjugate diameters, since each bisects chords parallel to the other.

Ex. 21.

Asymptotes.

1. The equation to a hyperbola which has the axes as asymptotes and passes through the point (h, k) is $xy = hk$.

2. The straight line

$$Ax + By + C = 0$$

will touch the hyperbola $xy = c^2$, if $C^2 = 4ABc^2$.

3. If two tangents be drawn to a hyperbola, and the points in which they intersect the asymptotes be joined, the joining lines will be parallel to one another.

4. The equation to the diameter conjugate to

$$\frac{x}{h}+\frac{y}{k}=0 \text{ is } \frac{x}{h}=\frac{y}{k},$$

the hyperbola being referred to its asymptotes.

5. If the abscissæ of any number of points in a hyperbola, referred to its asymptotes, be in A. P., the ordinates will be in H. P.

6. If on any chord as diagonal be constructed a parallelogram, the sides of which are parallel to the asymptotes, the other diagonal will pass through the centre.

7. If two hyperbolas have common asymptotes, any chord of the one touching the other will be bisected at the point of contact.

8. Tangents are drawn to a hyperbola, and the portions intercepted by the asymptotes are divided in a constant ratio; prove that the locus of the points of section is a hyperbola.

9. The coordinates of the vertex of the hyperbola $xy=c^2$, are c, c, and of the focus $c \sec \alpha$, $c \sec \alpha$, where 2α is the angle between the asymptotes.

10. If 2α be the angle between the asymptotes of the hyperbola $xy=c^2$, the eccentricity is $\sec \alpha$.

11. The equation to the directrix of the same hyperbola is

$$x+y=2c \cos \alpha.$$

12. The equation to the normal at the point x_1, y_1, is

$$x_1(x+y \cos \omega - x_1)=y_1(y+x \cos \omega - y_1).$$

13. If A, S, be the vertex and focus of a hyperbola, and the tangent at A, and the directrix intersect an asymptote in E, R, respectively, then SE is parallel to AR.

14. P is the point (x_1, y_1) on the hyperbola $xy=c^2$, if CP cut the ellipse

$$\frac{x^2}{x_1^2}+\frac{y^2}{y_1^2}=1$$

in Q, then the tangent to the ellipse at Q is parallel to that to the hyperbola at P.

144. *Rectangular Hyperbola.*

In the equation to the hyperbola let $b=a$.

Then the equation becomes

$$x^2-y^2=a^2.$$

In this hyperbola the asymptotes are evidently at right angles, hence it is called the rectangular hyperbola.

Ex. 22.

Rectangular Hyperbola.

1. A diameter is equal to its conjugate.

2. The eccentricity is $\sqrt{2}$.

3. If the perpendicular from the focus meet an asymptote in R, $SR=AC$.

4. The distance of any point from the centre is a geometric mean between its focal distances.

5. If PN be the ordinate at P, and NQ touch the auxiliary circle at Q, then $PN=QN$.

6. If PG, the normal at P, cut the axis in G, then $PG=CP$.

7. If the tangent at P cut the asymptotes in T, t, respectively, then $Tt=2CP$.

8. If TG, tG, be joined, the angle TGt is a right angle.

9. The equation referred to the asymptotes is $2xy=a^2$.

10. If from any point on the curve straight lines be drawn to the extremities of any diameter, these make equal angles with the asymptotes.

11. The equation referred to polar coordinates, the centre being pole, is

$$r^2=a^2 \sec 2\theta.$$

12. The equation to the normal at (x_1, y_1) when the asymptotes are axes is the same as that to the tangent at $(x_1 y_1)$ when the axes of the curve are axes of coordinates.

13. If the product of the tangents of the inclination to the axis of x of a pair of tangents to the rectangular hyperbola $xy=c^2$ be k^2, they will intersect on the diameter $y=kx$.

14. If a right-angled triangle be inscribed in a rectangular hyperbola, the perpendicular from the right angle on the hypotenuse is a tangent to the curve.

15. If the axes be inclined at an angle θ to the axes of the curve the equation to the rectangular hyperbola is

$$(x^2-y^2)\cos 2\theta - 2xy \sin 2\theta = a^2.$$

EXAMPLES ON CHAPTER VI.

1. An ellipse and hyperbola are confocal; and, at the common points, the tangents to the ellipse are parallel to the asymptotes of the hyperbola: prove that the axes of the ellipse are in the duplicate ratio of those of the hyperbola.

2. Two straight lines are drawn parallel to the axis major of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at a distance $\frac{ab}{\sqrt{a^2-b^2}}$ from it; prove that the part of any tangent intercepted between them is divided by the point of contact into parts which subtend equal angles at the centre.

3. A circle is described passing through the foci, and common tangents are drawn to the conic and circle; if the points of contact with the circle on the same side of the axis major are joined, the joining line will pass through the extremity of the axis minor.

4. TL, TM are two tangents from a point T; λ, μ are the eccentric angles of L, M; prove that

$$TL = (a^2 \sin^2\lambda + b^2 \cos^2\lambda)^{\frac{1}{2}} . \tan\frac{\lambda - \mu}{2}.$$

5. If θ, ϕ be the eccentric angles of the extremities of any focal chord of an ellipse,

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{e-1}{e+1}.$$

6. If PQ be a focal chord, DE the diameter parallel to it, and AA' the transverse axis, $AA' . PQ = DE^2$.

7. The curves

$$(x^2+y^2)^{\frac{1}{2}} = x\cos\theta + y\sin\theta, \text{ and } \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2},$$

intersect on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

8. If to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, there be drawn the four tangents

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = \pm 1,$$

$$x\sin\phi - y\cos\phi = \pm(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}},$$

they will intersect on the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = a + b$, and the perimeter of the parallelogram will be $4(a+b)$.

9. The length of the perpendicular from the centre on a chord joining the extremities of any two diameters at right angles is $\frac{ab}{(a^2+b^2)^{\frac{1}{2}}}$.

10. TP, TQ are two tangents at right angles, prove that $\sin^2 SPT + \sin^2 SQT$ is constant.

11. If perpendiculars be drawn from the centre on two tangents at right angles, the semi-diameters equal in length to these perpendiculars are conjugate, and the ordinates at their extremities intersect the perpendiculars in the auxiliary circle.

12. If CP meet the directrix in Q, then SQ is perpendicular to the tangent at P.

13. Find the equation to that normal to an ellipse which makes an angle $\tan^{-1} m$ with the axis of x.

14. If from a point O there be drawn four normals OP, OQ, OR, OS, and p, q, r, s be taken such that their coordinates are respectively equal to the intercepts on the axes of the tangents at P, Q, R, S, then p, q, r, s lie on a straight line.

15. Prove that the sum of the eccentric angles of P, Q, R, S in the preceding question is an odd multiple of π, and that the tangents parallel to PQ and RS meet on the equal conjugate diameters.

16. If O be any point, and from O two straight lines be drawn cutting a conic in P, p, Q, q respectively, and if CD, CE be parallel to OP, OQ, then

$$OP \, . \, Op : OQ \, . \, Oq :: CD^2 : CE^2.$$

17. If a parallelogram touch an ellipse at the extremities of a pair of conjugate diameters, one of which is $ly = mx$, shew that its diagonals are conjugate and are determined by the equation

$$\pm \frac{ay}{bx} = \left(\frac{ma - lb}{ma + lb}\right)^{\pm 1}.$$

18. If PCP', DCD', two conjugate diameters, subtend angles α, β at either extremity of the axis minor,

$$CP^2 = \frac{a^2 \tan^2 \beta + b^2 \tan^2 \alpha}{\tan^2 \alpha + \tan^2 \beta}.$$

19. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equation to the ellipse referred to a pair of conjugate diameters, then the circle

$$x^2 + 2xy \cos \omega + y^2 = r^2$$

will touch the ellipse if

$$\left(\frac{1}{r^2} - \frac{1}{a^2}\right)\left(\frac{1}{r^2} - \frac{1}{b^2}\right) = \frac{\cos^2 \omega}{r^4}.$$

20. Two conjugate diameters of an ellipse are drawn, and their extremities joined to a point on a concentric circle of given radius; shew that the sum of the squares on these four lines is constant.

21. Prove that the line joining the centre with the intersection of normals at the extremities of two conjugate diameters is perpendicular to the line joining those extremities.

22. If an ellipse and hyperbola have the same centre and foci, they will cut each other orthogonally, and if from any point in the circle through the points of intersection tangents be drawn to the two curves, they will be at right angles to one another.

23. From P, a point in an ellipse, straight lines are drawn to A, A', the extremities of the axis major, and from A, A', AQ, $A'Q'$ are drawn perpendicular to these lines: if AQ, $A'Q'$ intersect in R, the locus of R is the ellipse

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=\frac{a^2}{b^2}.$$

24. The locus of the extremity of the straight line formed by adding the abscissa to the ordinate of any point on the circle $x^2+y^2=c^2$, is an ellipse.

25. SQ, $S'Q$ are drawn parallel to a pair of conjugate diameters and intersect in Q: the locus of Q is a concentric ellipse.

26. If SQ, $S'Q$ are perpendicular to a pair of conjugate diameters, the locus is also a concentric ellipse.

27. A series of ellipses have their equal conjugate diameters of the same magnitude, one being common while the other varies in position: prove that tangents drawn from any point in the fixed diameter will touch the ellipses in points on the circumference of a circle.

28. The locus of the vertices of an equilateral triangle about the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is

$$4\,(b^2x^2+a^2y^2-a^2b^2)=3\,(x^2+y^2-a^2-b^2)^2.$$

29. Pairs of tangents at right angles are drawn to the ellipse: prove that the locus of the middle points of their chords of contact is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}.$$

30. On any straight line through C, three points P, Q, R are taken such that $CP \, . \, CR = CQ^2$: if the loci of Q and R be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{b^2}+\frac{y^2}{a^2}=\frac{a^2}{b^2}, \text{ respectively,}$$

that of Q is $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{a^2}{b^2}\left(\frac{x^2}{b^2}+\frac{y^2}{a^2}\right).$$

31. A diameter CP meets the auxiliary circle in Q, shew that the locus of the intersection of the tangent to the ellipse at P with that to the auxiliary circle at Q is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\frac{x^2y^2}{a^2b^2}\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

32. If from C the centre of the rectangular hyperbola $x^2-y^2=a^2$, a perpendicular CQ be drawn to any tangent QT, the equation to the locus of Q is

$$(x^2+y^2)^2=a^2(x^2-y^2).$$

33. From each point of the circle $x^2+y^2=c^2$, a straight line is drawn, making an angle α with the radius at that point: shew that the middle points of the parts of these lines intercepted by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, lie on the curve

$$\frac{a^2(c^2\sin^2\alpha-y^2)}{b^2x^2}+\frac{b^2(c^2\sin^2\alpha-x^2)}{a^2y^2}=2.$$

Examine the cases when the ellipse becomes the auxiliary circle, or the rectangular hyperbola.

34. If TP, TQ be two tangents to the ellipse and the eccentric angles of P and Q be ϕ, $\phi-\beta$, the locus of T is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\frac{\beta}{2}.$$

35. A series of hyperbolas having the same asymptotes is cut by a line parallel to an asymptote, and through the points of section straight lines are drawn parallel to the other asymptote, and equal to a semi-axis; the locus of their extremities is a parabola.

36. If a perpendicular be drawn from (h, k) on any tangent to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the locus of the point of intersection with the tangent is the curve

$$\{x(x-h) + y(y-k)\}^2 = a^2(x-h)^2 - b^2(y-k)^2.$$

37. If SP meet the perpendicular from the foot of the directrix on the tangent at P in R, the locus of R is a circle.

38. If CP meet the same perpendicular in Q, the locus of Q is a straight line perpendicular to the axis.

39. NPQ is an ordinate of the axis cutting the ellipse in P and the auxiliary circle in Q: if PT, the tangent at P, cut CQ in T, prove that the locus of T is

$$\left(\frac{x^2}{a} + \frac{y^2}{b}\right)^2 = x^2 + y^2.$$

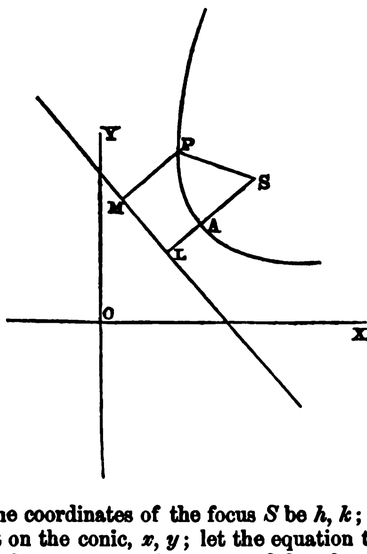
40. CQ is a radius of the auxiliary circle: prove that the polar of Q with respect to the ellipse intersects CQ in a point whose locus is

$$a^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = x^2 + y^2.$$

CHAPTER VII.

General Equation of the Second Degree.

145. *General equation to a conic referred to any axes.*



Let the coordinates of the focus S be h, k; those of P, any point on the conic, x, y; let the equation to the directrix LM be $x\cos\alpha + y\sin\alpha = p$, and let the eccentricity be e.

Then $$SP = e \cdot PM.$$

But $SP = \{(x-h)^2 + (y-k)^2\}^{\frac{1}{2}}$, (Art. 11):

$$PM = x \cos a + y \sin a - p; \quad \text{(Art. 37)}$$

$$\therefore\ (x-h)^2 + (y-k)^2 = e^2 (x \cos a + y \sin a - p)^2.$$

Expanding and rearranging we obtain for the equation to the conic,

$$x^2 (1 - e^2 \cos^2 a) - 2e^2 xy \cos a \sin a + y^2 (1 - e^2 \sin^2 a)$$
$$+ 2(e^2 p \cos a - h)x + 2(e^2 p \sin a - k)y + h^2 + k^2 - e^2 p^2 = 0 \ldots (1).$$

146. *The general equation of the second degree*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \ldots\ldots\ldots\ldots (2)$$

always represents a conic section.

Since the coefficients of the various terms of the equation (1) in the preceding Article can be made to vary as much as we please by the alteration of the constants a, h, k, e, p, and since any equation is unaltered if we multiply all the terms by any constant, the two equations (1) and (2) represent the same locus.

Since the degree of an equation cannot be altered by transformation of coordinates (Art. 20), the general equation of the second degree in oblique coordinates will still be of the second degree when transformed to rectangular axes, and will therefore represent a conic.

We have seen, however, (Arts. 44, 47), that this equation may represent two straight lines, a point, or a circle.

In these cases, we must consider the two straight lines as a particular case of the hyperbola, the point of an ellipse when the axes vanish, and the circle of an ellipse whose axes are equal.

147. To examine the nature of the conic represented by the general equation.

Since the locus of an equation is not altered if we divide by any constant, we may write the general equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + 1 = 0 \ldots\ldots (1).$$

Let this equation represent the same locus as

$$x^2(1-e^2\cos^2 a)-2e^2xy\cos a\sin a+y^2(1-e^2\sin^2 a)$$
$$+2(e^2p\cos a-h)\,x+2\,(e^2p\sin a-k)\,y+h^2+k^2-e^2p^2=0\ldots(2).$$

Writing λ for $\frac{1}{h^2+k^2-e^2p^2}$, and equating coefficients, we have

$$A=\lambda\,(1-e^2\cos^2 a),\ B=-\lambda e^2\cos a\sin a,\ C=\lambda(1-e^2\cos^2 a),$$
$$D=\lambda\,(e^2p\cos a-h),\ E=\lambda\,(e^2p\sin a-k).$$

Hence

$$AC-B^2=\lambda^2\{(1-e^2\cos^2 a)(1-e^2\sin^2 a)-e^4\cos^2 a\sin^2 a\}$$
$$=\lambda^2(1-e^2).$$

The conic is an ellipse, parabola, or hyperbola, according as $1-e^2$ is $>,=,$ or <0, that is, the general equation of the second degree represents an ellipse, parabola, or hyperbola, if $AC>,\ =,$ or $<B^2$.

For the complete discussion of the general equation of the second degree, and for many other theorems and methods which do not fall within the design of this work, the student is referred to Salmon's *Conic Sections.*

We will however discuss some of the more important properties which may be deduced from the general equation, and shew how the curve represented may be transformed to its axes.

148. We will first see what the equation

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0\ldots\ldots\ldots(1)$$

represents, when one or more of the constants vanish.

Since the nature of the conic depends upon the value of B^2-AC,

If $A=0$, or $C=0$ (or both), and B be real, the conic is a hyperbola; but if B also vanish, a parabola.

If $B=0$, the conic is an ellipse if A and C are of like sign, a hyperbola if of unlike.

If $D=0$, $E=0$, the origin is the centre, since we may write $-x$, $-y$, instead of x, y without altering the equation.

If in addition $B=0$, the axes of coordinates are those of the curve, or if the coordinates be oblique, they are conjugate diameters.

If $B=0$, $D=0$, the axis of y is a diameter, since $-x$ may be written instead of x without altering the equation. Similarly, if $B=0$, $E=0$, the axis of x is a diameter.

If $F=0$, the origin is a point on the curve.

If in addition $D=0$, the axis of y is a tangent.

We may notice that if the equation represent a parabola, the three first terms form a perfect square.

149. In the general equation write $x'+r\cos\theta$, $y'+r\sin\theta$, for x and y respectively, where r is the distance between (xy) and $(x'y')$. Rearrange: the equation becomes

$$r^2(A\cos^2\theta+2B\cos\theta\sin\theta+C\sin^2\theta)$$
$$+2r\{(Ax'+By'+D)\cos\theta+(Bx'+Cy'+E)\sin\theta\}$$
$$+Ax'^2+2Bx'y'+Cy'^2+2Dx'+2Ey'+F=0.$$

We may consider this equation as a quadratic in r, or (if we divide by r^2) in $\frac{1}{r}$.

If one of the values of $\frac{1}{r}$ be 0, then r is infinite.

The condition for this is that

$$A\cos^2\theta+2B\cos\theta\sin\theta+C\sin^2\theta=0,$$

$$\text{or }\tan\theta=\frac{-B\pm(B^2-AC)^{\frac{1}{2}}}{C}.$$

Here, if $B^2-AC>0$, or the curve be a hyperbola, there are two directions in which straight lines through $x'y'$ meet the curve in one point at infinity, and since $\tan\theta$ is independent of x' and y', these directions are the same for every point in the plane.

If $B^2-AC=0$ there is one such direction, which is that of the diameters of the parabola.

If $B^2-AC<0$, $\tan\theta$ is impossible, and there is no such direction, that is, the curve has no infinite branches.

Again, if the coefficient of r be zero, the chord through $x'y'$ is bisected at that point, but this coefficient is of one dimension in $x'y'$; this proves that the locus of the middle points of parallel chords is a straight line.

150. If in the equation of the preceding Article we write x for $r\cos\theta$, y for $r\sin\theta$, we really transform the origin to the point $x'y'$.

The equation becomes

$$\begin{aligned}&Ax^2+2Bxy+Cy^2\\&\quad+2(Ax'+By'+D)x+2(Bx'+Cy'+E)y\\&\quad+Ax'^2+2Bx'y'+Cy'^2+2Dx'+2Ey'+F=0\ldots\ldots(1).\end{aligned}$$

We observe that the quadratic terms in x, y are not altered by this transformation.

We have seen (Art. 148) that if the coefficients of x, y are zero, the origin is the centre.

The coordinates of the centre are therefore the roots of the equations,

$$Ax'+By'+D=0,$$

$$Bx'+Cy'+E=0;$$

$$\therefore\ x'=\frac{DC-EB}{B^2-AC},\ y'=\frac{EA-DB}{B^2-AC}.$$

Hence we obtain another proof that the parabola has no centre, since if $B^2-AC=0$, x' and y' become infinite.

Let $Ax'^2+2Bx'y'+Cy'^2+2Dx'+2Ey'+F=H$.

Then the equation (1) becomes

$$Ax^2+2Bxy+Cy^2+H=0,$$

when the centre is origin and two diameters at right angles to each other the axes of coordinates.

Now turn the axes through an angle θ, that is, write $X\cos\theta-Y\sin\theta$, $X\sin\theta+Y\cos\theta$ for x, y respectively (Art. 16).

Then if the axes of coordinates are those of the curve, the coefficient of XY must vanish (Art. 148).

The transformed equation becomes

$$(A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta)X^2$$
$$-2\{(A-C)\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta)\}XY$$
$$+(A\sin^2\theta - 2B\sin\theta\cos\theta + C\cos^2\theta)Y^2 + H = 0.$$

If, then, the curve be referred to its axes,

$$(A-C)\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) = 0, \text{ or}$$

$$\tan 2\theta = \frac{2B}{C-A}.$$

Since the different values of 2θ deduced from this equation differ by multiples of π, and therefore those of θ by multiples of $\frac{\pi}{2}$, they will all give the same positions of the axes.

In the transformed equation, let the coefficients of X^2, XY, Y^2 be called A', B', C' respectively, then we easily see that

$$A' + C' = A + C,\ B'^2 - A'C' = B^2 - AC.$$

These quantities are called the invariants of the conic.

To find the axes of the conic denoted by the general equation we may therefore transform to the centre, and turn the axes through an angle, such that

$$\tan 2\theta = \frac{2B}{C-A}.$$

The curve will then be referred to its axes.

Or, assuming α, β to be the axes, and the equation to the curve $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, we must have

$$A + C = \frac{1}{\alpha^2} + \frac{1}{\beta^2};$$

$$B^2 - AC = -\frac{1}{\alpha^2\beta^2},$$

whence α and β may be determined.

151. To find the equation to a conic referred to two tangents as axes.

Let a, b, be the distances of the points of contact from the origin, respectively.

In the general equation put $y = 0$; then, since the axis of x touches the curve, the equation must reduce to a quadratic, both the roots of which are equal to a; therefore the equation

$$Ax^2 + 2Dx + F = 0$$

must have two roots each equal to a;

$$\therefore \frac{D}{A} = -a, \quad \frac{F}{A} = a^2, \quad \text{and therefore } \frac{F}{D} = -a.$$

Similarly,

$$\frac{E}{C} = \frac{F}{E} = -b, \quad \frac{F}{C} = b^2;$$

therefore dividing by F, and substituting the values of the coefficients thus determined,

$$\frac{x^2}{a^2} + \frac{2B}{F}xy + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0,$$

$$\text{or } \left(\frac{x-a}{a}\right)^2 + \left(\frac{y-b}{b}\right)^2 = 1 - \frac{2B}{F}xy = 1 + 2\lambda\frac{xy}{ab},$$

where
$$\frac{\lambda}{ab} = \frac{-B}{F}.$$

Here the curve is an ellipse, parabola or hyperbola, as $1-\lambda^2 >, =,$ or < 0.

The above equation may also be written

$$\frac{x^2}{a^2} + \frac{2xy}{ab} + \frac{y^2}{b^2} - 2\left(\frac{x}{a} + \frac{y}{b}\right) + 1 = 2(1+\lambda)\frac{xy}{ab};$$

$$\text{or } \left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = \frac{2(1+\lambda)xy}{ab}.$$

Putting $1 + \lambda = \frac{\mu^2}{2}$, we obtain the equation in the form

$$\frac{x}{a} + \frac{y}{b} - 1 = \mu\left(\frac{xy}{ab}\right)^{\frac{1}{2}};$$

Here, since $\lambda^2 =$ or > 1 in the parabola or hyperbola, and $2(1+\lambda) = \mu^2$, λ must be positive, or the equation will either represent a straight line or become impossible.

152. In the case of the parabola this equation admits of a still further simplification.

Here $\lambda = 1$; hence the equation becomes

$$\frac{x}{a} + 2\left(\frac{xy}{ab}\right)^{\frac{1}{2}} + \frac{y}{b} = 1,$$

where the root may have either sign.

Extracting the square roots of both sides, we obtain

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{1} = 1,$$

where the roots may have either sign.

There is no necessity for expressing the ambiguities in this equation, since they vanish when the equation becomes rational.

153. *Any straight line through a pole of a conic section is harmonically divided by the curve and the polar.*

Take the pole as origin, and let the axes be the tangents. Then the equation to the curve may be written

$$\frac{x^2}{a^2} + 2\lambda\frac{xy}{ab} + \frac{y^2}{b^2} - 2\left(\frac{x}{a} + \frac{y}{b}\right) + 1 = 0 \quad \ldots\ldots \text{(1)},$$

and the equation to the polar is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)},$$

since it passes through the points of contact of the tangents.

Let the equation to any straight line through the pole be

$$\frac{x}{m} = \frac{y}{n} = r,$$

where r is the length from the origin.

Substitute mr, nr, for x and y respectively in equation (1), and divide by r^2;

$$\therefore\ \frac{m^2}{a^2} + 2\lambda\frac{mn}{ab} + \frac{n^2}{b^2} - 2\left(\frac{m}{a} + \frac{n}{b}\right)\frac{1}{r} + \frac{1}{r^2} = 0.$$

Now, if $\frac{1}{r_1}$, $\frac{1}{r_2}$ be the roots of this quadratic in $\frac{1}{r}$, that is, r_1, r_2 the values of r at the points where the line $\frac{x}{m}=\frac{y}{n}=r$ cuts the conic,

$$\frac{1}{r_1}+\frac{1}{r_2}=2\left(\frac{m}{a}+\frac{n}{b}\right):$$

but if r_3 be the value of r where this line cuts the polar (2),

$$\frac{mr_3}{a}+\frac{nr_3}{b}=1, \text{ or } \frac{m}{a}+\frac{n}{b}=\frac{1}{r_3};$$

$$\therefore\ \frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{r_3},$$

that is, r_3 is a harmonic mean between r_1 and r_2.

154. *To find the conditions that the general equation of the second degree shall represent two straight lines, or a point.*

The equation must be the difference or sum of two squares.

Multiply every term by A, then it may be written

$$(Ax+By+D)^2=(B^2-AC)\,y^2+2\,(BD-AE)\,y+D^2-AF.$$

Now, if the right-hand side of this equation is of the form $(Gy+H)^2$, the equation will represent two straight lines, if of the form $-(Gy+H)^2$, a point, the intersection of

$$Ax+By+D=0, \text{ and } Gy+H=0.$$

The condition is therefore

$$(B^2-AC)\,(D^2-AF)=(BD-AE)^2,$$

which may be written

$$ACF+2BDE-AE^2-CD^2-B^2F=0.$$

If this condition be satisfied, the sign of the expression $(B^2-AC)\,y^2+2\,(BD-AE)\,y+D^2-AF$ is that of B^2-AC.

If, then, $B^2 > AC$, the equation represents two straight lines, or a hyperbola which coincides with its asymptotes.

If $B^2 = AC$, the condition becomes $BD = AE$, and the equation represents two parallel straight lines, two coincident straight lines, or is impossible, as

$$D^2 >, =, \text{ or } < AF.$$

If $B^2 < AC$, the equation represents a point.

We have already seen (Art. 47) that if the equation represents a circle,

$$A = C,\ B = 0.$$

155. *To find the equation to the tangent at the point* x_1, y_1.

Using the same method as in Articles 50, 85, &c.

Let $(x_1\, y_1)$, $(x_2\, y_2)$ be two points on the curve, then

$$A(x_2^2 - x_1^2) + 2B(x_2y_2 - x_1y_1) + C(y_2^2 - y_1^2)$$
$$+ 2D(x_2 - x) + 2E(y_2 - y_1) = 0;$$

$$\therefore \{A(x_2 + x_1) + 2By_2 + 2D\}(x_2 - x_1)$$
$$+ \{2Bx_2 + C(y_2 + y_1) + 2E\}(y_2 - y_1) = 0;$$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = -\frac{A(x_2 + x_1) + 2By_2 + 2D}{2Bx_1 + C(y_2 + y_1) + 2E}.$$

Substituting this expression in the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

multiplying up, re-arranging, and then putting $x_2 = x_1$, $y_2 = y_1$, the chord becomes the tangent and its equation becomes

$$(Ax_1 + By_1 + D)x + (Bx_1 + Cy_1 + E)y$$
$$= Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 = -(Dx_1 + Ey_1 + F);$$

$$\therefore Axx_1 + B(xy_1 + x_1y) + Cy_1^2 + D(x + x_1) + E(y + y_1) + F = 0$$

is the equation required.

This equation is easily remembered if we notice that in the equation to the curve, x^2 is changed into xx_1, $2xy$ into xy_1+x_1y, y^2 into yy_1, $2x$ into $x+x_1$, $2y$ into $y+y_1$.

As before, if (x_1y_1) represent a point not on the curve, this equation represents the polar of that point.

If $F=0$, the tangent through the origin is $Dx+Ey=0$. The polar of the origin is $Dx+Ey+F=0$.

EXAMPLES ON CONICS.

1. Find the coordinates of the vertex, and centre of the conic represented by the equation of Art. 145, and prove by means of the equation that

$$CL=e^2CS, \text{ and that } CS\,.\,CL=CA^2.$$

2. Discuss the curves denoted by the following equations, transforming them, when possible, to their principal axes:

$$y^2-4xy+4x^2+6x-3y=0:$$
$$(x-y)^2=2(x+y):$$
$$3(x^2+y^2)+2xy=4:$$
$$2xy=x+y:$$
$$(3x+y)\sqrt{5}y+3x-10y=3\sqrt{5}:$$
$$2x^2+5xy+2y^2-3(x+y)+1=0:$$
$$5x^2+8xy+5y^2-6(x+y)+2=0.$$

3. If two conic sections touch one another at two points, they cannot intersect at any other point.

4. The coordinates of the centre of the conic

$$\left(\frac{x-a}{a}\right)^2+\left(\frac{y-b}{b}\right)^2=1+2\lambda\frac{xy}{ab},$$

are

$$\frac{a}{1-\lambda},\ \frac{b}{1-\lambda}.$$

5. The length of the diameter through the origin is

$$\frac{\{2(1+\lambda)(a^2+2ab\cos\omega+b^2)\}^{\frac{1}{2}}}{1-\lambda}.$$

6. If the origin be transferred to the centre and the axes unaltered in direction, the equation becomes

$$\frac{x^2}{a^2} - \frac{2\lambda xy}{ab} + \frac{y^2}{b^2} = \frac{1+\lambda}{1-\lambda}.$$

7. The diameters of the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1,$$

are parallel to the line $\qquad \frac{x}{a} = \frac{y}{b}.$

8. The straight line $\frac{x}{a} + \frac{y}{b} = \frac{1}{2}$ is a tangent to this parabola, at the point $\left(\frac{a}{4}, \frac{b}{4}\right)$.

9. The coordinates of the focus are

$$\frac{ab^2}{a^2 + 2ab \cos \omega + b^2}, \quad \frac{a^2 b}{a^2 + 2ab \cos \omega + b^2};$$

those of the vertex

$$\frac{ab^2 (a \cos \omega + b)^2}{(a^2 + 2ab \cos \omega + b^2)^2}, \quad \frac{a^2 b (a + b \cos \omega)^2}{(a^2 + 2ab \cos \omega + b^2)^2};$$

and the length of the latus rectum is

$$\frac{4a^2 b^2 \sin^2 \omega}{(a^2 + 2ab \cos \omega + b^2)^{\frac{3}{2}}}.$$

10. The directrix of the parabola $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$, is

$$(a + b \cos \omega) x + (b + a \cos \omega) y = ab \cos \omega.$$

11. The equation to the parabola whose focus is the origin, and directrix $\frac{x}{a} + \frac{y}{b} = 1$, is

$$\frac{x^2}{b^2} - \frac{2xy}{ab} + \frac{y^2}{a^2} + \frac{2x}{a} + \frac{2y}{b} = 1.$$

12. OA, OB are two tangents to a parabola; any other tangent cuts them in P, Q respectively; prove that

$$\frac{OP}{OA} + \frac{OQ}{OB} = 1.$$

13. If a tangent to $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$ meet the axes in P and Q, and perpendiculars be drawn from P, Q to the opposite axes, they will intersect in the line

$$\frac{x+y\cos\omega}{a}+\frac{y+x\cos\omega}{b}=\cos\omega.$$

14. The equation

$$2\lambda y=ax^2+2bxy+cy^2$$

represents a conic section which touches the axis of x at the origin.

15. Find the coordinates of the centre of the curve in *q* : 14, and the diameters through the origin, and parallel to the axis of x.

16. Determine the condition that the line

$$lx+my=d$$

should touch the conic denoted by the general equation.

17. Find the equation to the normal to this conic at the point x_1y_1.

18. Find the equation to the diameter which bisects the chord

$$\frac{x}{a}+\frac{y}{b}=1.$$

19. Find the condition that $x=y$, $lx+my=d$, should be parallel to a pair of conjugate diameters.

20. O is any point, OPp, OQq any straight lines cutting a conic in P, p, Q, q; prove that Pq, Qp intersect on the polar of O.

21. The equation

$$(x+a)^{\frac{1}{2}}+(y+a)^{\frac{1}{2}}=(2a)^{\frac{1}{2}}$$

represents a parabola of which the vertex is origin, the axis inclined at an angle of 45° to the angle of x, and the latus rectum $4\sqrt{2}a$.

22. A triangle is inscribed in a conic so that the centre of the inscribed circle is a focus, shew that the radius is

$$\frac{l}{1+(1+e^2)^{\frac{1}{2}}}.$$

23. Two triangles ABC, $A'B'C'$ are described about an ellipse; the sides with the same letters are parallel. If any tangent cut the sides $B'C'$, $C'A'$, $A'B'$ in P, Q, R, then AP, BQ, CR will be parallel.

24. If a circle be described through the foci of a central conic, the angle between the tangents at the point of intersection is equal to that between the normal and conjugate axis.

25. Chords are drawn at right angles to each other through the vertex of a conic; the pole of the line through their other points of section lies on a straight line.

26. From any point on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, tangents are drawn to $\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1$: if a diameter parallel to the tangent to the outer ellipse at the point cut these tangents, the portion of each cut off will be $\frac{ab}{\lambda}$.

27. Two tangents to one hyperbola are asymptotes to another: if the second touch one asymptote of the first, it will also touch the other.

28. A hyperbola of given eccentricity has its centre on a given circle. One asymptote passes through a fixed point in the circumference: prove that the other asymptote and the transverse axis will also pass through fixed points on the circumference of the circle.

29. ABC is a triangle, AB is fixed, and C moves on a rectangular hyperbola through A, B; if P, Q be the points in which AB, BC intersect the circle on AB as diameter, AQ, BP intersect on the hyperbola.

30. S, H are the foci of an ellipse, and the extremities of a diameter of a rectangular hyperbola: shew that the tangent and normal of the ellipse, where it meets the hyperbola, are parallel to the asymptotes.

31. The length of the diameter through the origin of the conic

$$\frac{x}{a} + \frac{y}{b} - 1 = \frac{2(xy)^{\frac{1}{2}}}{c} \text{ is } \frac{c\{ab(a^2+b^2)\}^{\frac{1}{2}}}{c^2 - ab}.$$

32. If $Ax^2+2Bxy+Cy^2=1$ represent a rectangular hyperbola, $A+C=2B\cos\omega$.

33. A circle is drawn touching a parabola at P, and the axis at the focus. Shew that the abscissa of P is $\frac{1}{2}$ of the latus rectum.

34. Normals are drawn to a parabola at the extremities of any focal chord; shew that the chord joining their further extremities is parallel to the focal chord, and three times as long.

35. Rectangles circumscribe an ellipse; shew that the parallelograms formed by joining the points of contact touch a confocal ellipse.

36. PQ is a chord of a parabola, normal at P; if

$$PSQ=\theta,\ SP=SQ\cos^2\frac{\theta}{2}.$$

37. If parallel tangents be drawn to two confocal ellipses, the difference of the squares of their distances from the centre is constant.

38. PP', DD' are conjugate diameters of the curve

$$y^2=\frac{b^2}{a^2}(2ax-x^2).$$

If the coordinates of P be (h, k), find those of P', D, D'. If these four points be the extremities of the latera recta, what is the eccentricity?

39. Two tangents are drawn to a parabola from a point on the directrix: shew that they cut off from the tangent at the vertex a segment equal to the distance of the given point from the focus.

40. ABC, $A'B'C'$ are six points on a parabola: AA' is parallel to BC, BB' to CA, CC' to AB, prove that the area of ABC is one-eighth of that of $A'B'C'$.

41. Three tangents to the parabola $y^2=4ax$ are inclined to the axis at angles θ, ϕ, ψ, and form a triangle; prove that the radius of the circumscribing circle is $\frac{a}{2\sin\theta\sin\phi\sin\psi}$.

42. The segment of the normal to a rectangular hyperbola which is intercepted between the axes is equal to the parallel diameter.

43. On a chord of a rectangular hyperbola as diameter a circle is described, prove that the tangent to the hyperbola at a point where the circle cuts it is perpendicular to the chord.

44. The circles described on parallel chords of a rectangular hyperbola as diameters pass through two fixed points.

45. Two parabolas are described touching one another, and having their axes at right angles; prove that the straight lines joining the foci to the points of contact are at right angles.

46. S, S' are the foci of an ellipse: a circle touches the ellipse at P and cuts $S'P$ in Q; prove that $SP = SQ$.

47. The two conics

$$ax^2 + 2bxy + cy^2 = 1,$$
$$a'x^2 + 2b'xy + c'y^2 = 1,$$

will be confocal if

$$\frac{a-c}{b} = \frac{a'-c'}{b'} \text{ and } \frac{ac-b^2}{b} = \frac{a'c'-b'^2}{b'}.$$

48. The normal at P, a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets the curve again at Q; PP' is any chord, PR perpendicular to PP': if $P'Q$ meet PR in R, the locus of R is

$$\frac{x\cos\phi}{a} - \frac{y\sin\phi}{b} = \frac{a^2+b^2}{a^2-b^2},$$

where ϕ is the eccentric angle of P.

49. On a chord of this ellipse a circle is described; shew that the normals at the two other points of intersection intersect the chord in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

50. Normals at the extremities of a focal chord bisect the base angles of the triangle which has the chord for base, and the other focus for vertex.

51. Normals at the extremities of a focal chord intersect on the line which is parallel to the axis major and bisects the chord.

52. If the tangents through the pole include with the polar a constant area, the pole lies on a concentric similar conic.

53. If through two given points lines be drawn parallel to a pair of conjugate diameters of a conic, the locus of their intersection will be a similar conic through the points.

54. Two straight lines are drawn, and two circles touch these lines and each other: prove that the locus of their point of contact is an ellipse.

55. A chord of an ellipse is drawn through a fixed point, and a conic touches the ellipse at the extremities of this chord, and also passes through the centre: the locus of the centre of this conic is a circle.

56. Two tangents are drawn to the parabola $y^2=4ax$; if the product of the cosines of their inclinations to the axis be k, the locus of their point of intersection will be the ellipse

$$k^2x^2=(x-a)^2+y^2.$$

57. The locus of the point of intersection of two tangents to an ellipse, which intercept on a diameter distances from the centre, the product of which is constant, is a concentric hyperbola, and two arcs of a concentric ellipse.

58. If the bisectors of angles between pairs of tangents to an ellipse be parallel to a fixed line, the locus of the point of intersection will be a rectangular hyperbola.

59. If a tangent be drawn to one of two confocal conics perpendicular to a tangent to the other, the locus of their point of intersection is a concentric circle.

60. If the extremities of the base of a triangle move along fixed lines, the vertex moves on a conic section.

If the vertical angle be the supplement of that between the lines, this conic becomes a straight line.

61. Given a focus, tangent, and latus rectum $2l$, of a conic, the locus of the other focus is

$$l^2\{x^2+(c-y)^2\}=c^2y^2,$$

where the tangent is axis of x, and the given focus the point $(c, 0)$.

62. If the axes of a system of coaxial conics be represented by $2a$, $2b$; then if $a \pm b = c$, the locus of the pole of a given straight line is a parabola touching the axes in two points, which lie on a circle with the two points where the given straight line cuts the axes.

63. Shew that if a circle be described about the triangle formed by a tangent and asymptotes of $xy = c^2$, the locus of its centre is

$$(x + y \cos \omega)(y + x \cos \omega) = c^2,$$

where ω is the angle between the asymptotes.

64. P is a point on a parabola, Q a point in the diameter through P such that $PQ = mSP$; shew that the locus of Q is a parabola.

65. Confocal ellipses and hyperbolas intersect; if the transverse axis ($2a$) of the hyperbola is equal to the conjugate axis of the ellipse, shew that the locus of the point of intersection is the curve

$$x^4 + y^4 = a^2(x^2 + y^2).$$

66. A conic is drawn, touching the axes at A, B, and passing through a point C such that AC, BC are parallel to the axes: if AOB be of constant area, the locus of the centre is a hyperbola.

67. ABC is a triangle; BC is fixed: if $\tan B \tan \frac{C}{2} = k$, the locus of A is a conic.

68. The straight line AB is bisected at O, and through O a fixed straight line is drawn: on this two points PQ are taken such that $PQ = a$: prove that the locus of the intersection of AB, BQ is a hyperbola.

69. O is a fixed point, Q any point on a fixed line. From Q, QP is drawn perpendicular to OQ, and subtends an angle β at O. Prove that the locus of P is a straight line.

70. If a conic be drawn touching the asymptotes of a given conic in P, Q, the locus of intersection of the chords of intersection is PQ.

71. A straight line AB is terminated by the axes and passes through a given point: prove that the locus of its middle point is a conic.

72. AOB, COD are two straight lines which bisect one another at right angles: if P be a point such that

$$PA \cdot PB = PC \cdot PD,$$

the locus of P is a rectangular hyperbola.

73. A parabola touches the axes at A, B, which are fixed: a variable tangent cuts the axes in C, D. Prove that the locus of the centre of the circle described about OCD is a straight line.

74. Two of the normals drawn from a point P to a parabola make equal angles with a fixed straight line: shew that the locus of P is a parabola.

75. Find the locus of the intersection of normals at the extremities of a chord of

$$Ax^2 + By^2 + 2Cx + 2Dy + 1 = 0,$$

which passes through a fixed point (hk).

Shew that the locus is a cubic, which becomes a conic if (hk) be on the axis.

76. P is a point on a fixed diameter of an ellipse, PQ the perpendicular, to the polar of P. Prove that the locus of Q is a rectangular hyperbola.

77. Given a focus, a tangent, and the eccentricity of a conic, prove that the locus of the centre is a circle.

78. SY, HZ are drawn from the foci of a conic perpendicular to a tangent: SZ, HY intersect in P: prove that the locus of P is a conic.

79. An ellipse is described having for axes the tangent and normal at P, a point on a fixed ellipse, and touching one of the axes of the fixed ellipse at the centre. Prove that the locus of the focus of the moving ellipse is two circles of radii $a \pm b$.

80. If A be any point on a conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and AB, AC chords equally inclined to the tangent at A, prove that BC produced meets the tangent in a point M which is independent of the inclination of AB, AC to the tangent, and which lies on the curve $\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2} = (a^2 - b^2)^2$.

CHAPTER VIII.

Abridged Notation. Trilinear Coordinates.

156. Let u represent any expression in x and y, then $u = 0$ will be the equation to some line, which will be straight or curved as u is of one or more dimensions.

Similarly we may denote any other line by $v = 0$.

We shall use small letters u, v, to denote expressions of one dimension, capital letters such as S, U, to denote expressions of two or more dimensions.

The equation $U + \lambda S = 0$, where λ is constant, denotes any curve of the degree which is generally that of the highest of the two expressions U and S, passing through all the points of intersection of $U = 0$, $S = 0$, for then $U + \lambda S = 0$ also.

Thus $u + \lambda v = 0$ represents any straight line through the intersection of $u = 0$ and $v = 0$.

If $u = 0$ and $v = 0$ are parallel, $u + \lambda v = 0$ must be parallel to either of them, for if not, at the intersection of $u = 0$ and $u + \lambda v = 0$ we should have $v = 0$, which is impossible if $u = 0$.

Since λ is indeterminate, $u + \lambda v = 0$ may be made to pass through any point.

Now let $w = 0$ be another straight line, then $u + \lambda v = 0$ and $w = 0$ intersect in some point which may be anywhere.

Hence $u + \lambda v + \mu w = 0$ represents any straight line through this point, that is, any straight line whatever. This equation is generally written

$$lu + mv + nw = 0.$$

157. *Straight line at infinity.*

Let $lx + my = d$ be any straight line. Then, if we increase x and y without limit, we must diminish l and m without limit.

This may be otherwise expressed by saying that if $l = 0$, $m = 0$, x and y must be infinite, since no finite values of x and y can satisfy $0 . x + 0 . y = d$.

Hence we may consider $d = 0$, the equation to a straight line altogether at an infinite distance.

Thus $u = d$ may be considered to cut the line $u = 0$, at the point for which $d = 0$, that is, at infinity; hence $u = d$ is parallel to $u = 0$.

Similarly the conics $S = 0$, $S = d$ intersect only on the line at infinity, that is, they do not intersect at any finite point.

158. Let us consider the meaning of the equations in which each term, or one term, is the product of two linear expressions such as u or v.

Let the equations $u = 0$, $v = 0$, $w = 0$, $z = 0$, represent four straight lines, then the equation

$$uv = lwz$$

represents a conic, for it is of the second degree.

If either $u = 0$ or $v = 0$, then must either $w = 0$ or $z = 0$; the conic therefore passes through the intersection of $u = 0$ with $w = 0$ or $z = 0$, and also of $v = 0$, with either of these straight lines.

It is therefore a conic circumscribed about the quadrilateral whose opposite sides are the pairs $u = 0$, $v = 0$, and $w = 0$, $z = 0$.

Let us take special cases.

Let $w=0$ and $z=0$ coincide, then their points of intersection with $u=0$, $v=0$, also coincide, and these lines cut the conic in two coincident points, that is, are tangents.

Hence $uv=lw^2$ represents a conic which touches $u=0$, $v=0$, at the points where they cut the line $w=0$.

$w=0$ is, therefore, the polar of the point of intersection of $u=0$, $v=0$ with respect to this conic, and conversely, if we have three straight lines $u=0$, $v=0$, $w=0$, $uv=lw^2$ represents any conic which touches the first two where they intersect the third.

u and v may be imaginary, while uv is real; in this case $uv=0$ represents a point within the conic, while $w=0$ is still the polar.

Again, in the equation $uv=lwz$, let $z=d$, a constant, then the line represented by $z=0$ is at infinity, hence the conic is a hyperbola, whose asymptotes are parallel to $u=0$, $v=0$.

If also $w=d$, the polar of the intersection of $u=0$, $v=0$ is at infinity, the equation therefore represents a hyperbola whose asymptotes are $u=0$, $v=0$.

It may happen that $u=0$, $v=0$ are parallel, in this case $uv=ld^2$ will represent a parabola, since v must be equal to $u+k$, and the quadratic terms must form a perfect square.

159. Now let $S=0$ represent a conic, therefore $S=luv$ will also represent a conic which intersects $S=0$ at the points where $u=0$, $v=0$ meet it.

These lines will therefore be the chords of intersection of the two conics.

Now let $u=v$, then the two lines coincide, and their points of intersection with $S=0$ coincide, and therefore $S=lu^2$ represents a conic having double contact with $S=0$ at the points where $u=0$ meets it.

For further discussion of methods of abridged notation the student is referred to Salmon's *Conic Sections*, or to Whitworth's *Modern Geometry*, to the latter of which books I am indebted for much of the matter of this chapter.

TRILINEAR COORDINATES.

160. In the method of Trilinear coordinates the perpendicular distances of a point from the sides of a given triangle are used to determine its position, and are called the Trilinear coordinates of the point.

The triangle is called the triangle of reference.

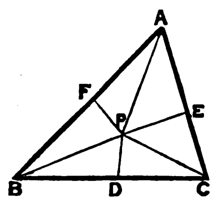
The distances from the sides a, b, c of the triangle ABC are denoted by α, β, γ respectively.

It does not lie within the compass of this work fully to discuss the methods and applications of trilinear coordinates. It will suffice to explain the elementary principles, and to obtain some of the more remarkable results of those principles.

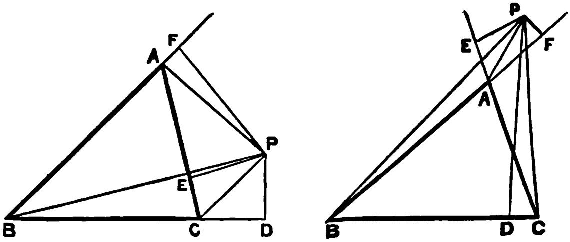
For a complete discussion the student is referred to Ferrers' *Trilinear Coordinates*, or to Whitworth's *Modern Geometry*. It will be seen that many of the results are obtained from the elimination of constants or variables between linear equations. It will be a great advantage to the student to have read the theory of determinants in either of the works referred to, or in Todhunter's *Theory of Equations*.

161. If two of the quantities α, β, γ are known, the point is fixed, and the other is therefore determinate: there must therefore be some equation connecting α, β, γ.

Let P be the point $(\alpha\beta\gamma)$, join PA, PB, PC.



Let us make the convention that α, β, γ are to be considered positive when drawn in the direction from an angle to the opposite side, negative when towards the angle, then if the point lie within the triangle, as in Fig. 1, α, β, γ will



be positive; if in the space between two sides produced and the third side, as in Fig. (2), one coordinate is negative, and the others positive; if in the space included between two sides produced, two are negative, and one positive.

Thus in Fig. (2) $PE=-\beta$, in Fig. (3) $PE=-\beta$, $PF=-\gamma$.

Let the area of the triangle $ABC=2S$.

Then, in Fig. (1)

$$\Delta PBC+\Delta PCA+\Delta PAB=\Delta ABC$$

$$\therefore\ a\alpha+b\beta+c\gamma=2S.$$

In Fig. (2) $PBC+PAB-PCA=ABC$,

$$\text{and } PCA=\frac{1}{2}AC\,.\,PE=-\frac{1}{2}b\beta.$$

In Fig. (3) $PBC-PAB-PCA=ABC$,

$$\text{while here } PCA=-\frac{1}{2}b\beta,\ PAB=-\frac{1}{2}c\gamma,$$

$$\therefore \text{ in all cases } a\alpha+b\beta+c\gamma=2S.$$

162. Every equation in trilinear coordinates can be made homogeneous in α, β, γ.

For since $\dfrac{a\alpha+b\beta+c\gamma}{2S}=1$, we can multiply every term of a degree below the highest by this expression, or any of its powers.

163. We may now write α, β, γ instead of u, v, w respectively in Arts. (156—159), and we see that

$$l\alpha + m\beta + n\gamma = 0$$

is the most general form of the equation to a straight line. This is also evident by transforming to any rectangular system, for then we may write the equation to each side in the form

$$x \cos\theta + y \sin\theta = p.$$

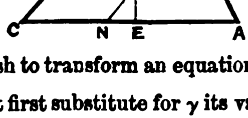
Hence the general equations of the first and second degrees in trilinears represent the same loci as in Cartesian coordinates.

164. We can easily reduce an equation in trilinear coordinates to the corresponding equation in oblique coordinates, taking CA, CB as axes.

Let P be a point: (x, y), (α, β, γ) its coordinates referred to the two systems respectively; then in the figure

$$CN = x, PN = y, PF = \alpha, PE = \beta;$$

$$\therefore \alpha = x \sin C, \beta = y \sin C.$$



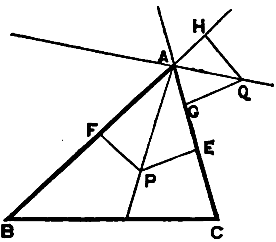
Hence, if we wish to transform an equation in trilinear coordinates, we must first substitute for γ its value $\dfrac{2S - a\alpha - b\beta}{c}$, and then write $x \sin C$, $y \sin C$ for α, β respectively.

The converse process will, of course, be to transform to two sides of the triangle of reference as axes, substitute $\alpha \operatorname{cosec} C$, $\beta \operatorname{cosec} C$ for x and y, and make the equation homogeneous.

These processes are, however, rarely used.

The expressions for the distance between two points, the distance of a point from a line, the angles lines make with each other, become extremely complicated in trilinear coordinates: problems in which such questions are involved are best solved by Cartesian coordinates.

165. We have seen (Art. 157) that $u = lv$ represents a locus through the intersections of $u = 0$ and $v = 0$: $\beta = l\gamma$ will



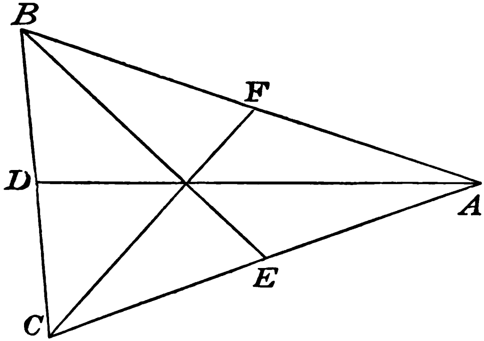
therefore represent a straight line through A. Let AP be a straight line through A, P the point α, β, γ, then $\frac{PE}{PF} = \frac{\sin PAE}{\sin PAF}$, which is the same for all points along AP, therefore $\frac{\beta}{\gamma} = \frac{\sin PAE}{\sin PAF}$.

If, however, a line QA be drawn, cutting the exterior angle at A, we have $\frac{QG}{QH} = \frac{\sin QAG}{\sin QAH}$; but QG, QH are of opposite signs, therefore $\frac{\beta}{\gamma} = -\frac{\sin QAG}{\sin QAH}$.

We see then that $\beta = l\gamma$ represents a straight line through A, and that l is the ratio of the sines of the angles into which A is divided by the line, and is negative, if the line cuts the exterior angle.

As a specimen of the advantages of occasionally using trilinear coordinates, we will prove the three properties of a triangle we proved in Art. 43, that the bisectors of the angles, of the sides, and the perpendiculars from the angles on the sides meet in a point.

(i) Let AD bisect the angle A, then its equation is $\beta=\gamma$.

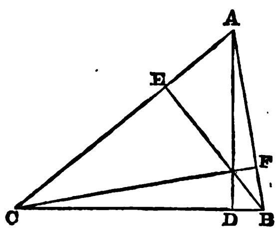


Similarly the equations to the other bisectors are $\gamma=\alpha$, $\alpha=\beta$.

They are therefore all satisfied by $\alpha=\beta=\gamma=\dfrac{2S}{a+b+c}$.

(ii) Next let AD bisect BC, then $\dfrac{\sin CAD}{\sin BAD}=\dfrac{c}{b}$; the equation to AP is therefore $\beta b=\gamma c$; that to BE must be $\gamma c=\alpha a$, and at their intersection $\alpha a=\beta b$, which represents CF.

(iii) Next let AD be perpendicular to BC. Then $DAC=\dfrac{\pi}{2}-C$, $DAB=\dfrac{\pi}{2}-B$:



The equation to AD is therefore $\beta \cos B = \gamma \cos C$.

Similarly the equation to BE is $\gamma \cos C = a \cos A$, and therefore, at their intersection, $a \cos A = \beta \cos B$, the equation to CF.

166. To find the coordinates of the points where the line

$$la + m\beta + n\gamma = 0$$

meets the sides of the triangle of reference.

Let D, E, F be the points where it meets the sides, then at E we have $\beta = 0$,

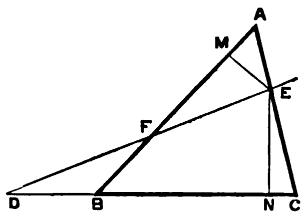
$$\text{and } \therefore\ aa + c\gamma = 2S,$$

$$la + n\gamma = 0\ ;$$

$$\therefore a = \frac{2nS}{na - lc},\ \gamma = \frac{2lS}{lc - na}.$$

Similarly the coordinates of D are

$$0,\ \frac{2nS}{nb - mc},\ \frac{2mS}{mc - nb},$$



and of F,

$$\frac{2mS}{ma - lb},\ \frac{2lS}{lb - ma},\ 0.$$

Again, $AE = EM \operatorname{cosec} A = \dfrac{2lS \operatorname{cosec} A}{lc - na} = \dfrac{lbc}{lc - na}$,

so $AF = \dfrac{lbc}{lb - ma}$.

Now the more nearly l, m, n are proportional to a, b, c, the smaller do the denominators of these expressions become, and therefore the further is the line

$$l\alpha + m\beta + n\gamma = 0$$

from the triangle of reference.

167. **Straight line at infinity.**

We have seen (Art. 157) that $k=0$ represents a straight line at infinity, as it is an inconsistent equation.

Instead of k we may write $k\dfrac{a\alpha + b\beta + c\gamma}{2S}$

Hence $a\alpha + b\beta + c\gamma = 0$, represents a straight line at infinity.

This is a very convenient notation, and must be considered an abbreviation for the statement in the preceding article, that the more nearly l, m, n are proportional to a, b, c, the further is the line $l\alpha + m\beta + n\gamma = 0$ from the triangle of reference.

168. **To find the equation to the straight line which passes through two given points.**

Let the two points be $(\alpha_1 \beta_1 \gamma_1)$, $(\alpha_2 \beta_2 \gamma_2)$, and

$l\alpha + m\beta + n\gamma = 0$, the equation required.

Then
$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0,$$
$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0.$$

Eliminating l, m, n by cross-multiplication, we have

$$(\beta_1\gamma_2 - \beta_2\gamma_1)\,\alpha + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\,\beta + (\alpha_1\beta_2 - \alpha_2\beta_1)\,\gamma = 0.$$

This is the equation to some straight line, and therefore to the line required.

If we have to find the condition that three given points should be collinear, we have only to write $\alpha_3, \beta_3, \gamma_3$ for α, β, γ respectively in the above result.

169. To find the condition that three straight lines should be concurrent.

$$\text{Let } l_1\alpha + m_1\beta + n_1\gamma = 0,$$
$$l_2\alpha + m_2\beta + n_2\gamma = 0,$$
$$l_3\alpha + m_3\beta + n_3\gamma = 0$$

be the equations to the three lines.

If they are concurrent, the same values of α, β, γ must satisfy them all; eliminating by cross-multiplication we have

$$l_1 m_2 n_3 - l_1 m_3 n_2 + l_2 m_3 n_1 - l_2 m_1 n_3 + l_3 m_1 n_2 - l_3 m_2 n_1 = 0.$$

170. Hence we can find the condition that two straight lines may be parallel.

If they are parallel they intersect at an infinite distance, that is, *on the straight line at infinity*,

$$a\alpha + b\beta + c\gamma = 0.$$

Hence we must write a, b, c for l_3, m_3, n_3 in the preceding condition, which therefore becomes

$$(m_1 n_2 - m_2 n_1)\,a + (n_1 l_2 - n_2 l_1)\,b + (l_1 m_2 - l_2 m_1)\,c = 0.$$

171. To find the equation to a straight line parallel to a given straight line.

We have seen that $u = k$ is parallel to $u = 0$,

$$l\alpha + m\beta + n\gamma = k = \frac{k}{2S}(a\alpha + b\beta + c\gamma)$$

is therefore parallel to $l\alpha + m\beta + n\gamma = 0$.

This is the same thing as saying that the straight line parallel to $l\alpha + m\beta + n\gamma = 0$ passes through the intersection of this straight line with the line at infinity, and that therefore its equation is

$$l\alpha + m\beta + n\gamma = \lambda\,(a\alpha + b\beta + c\gamma).$$

If, in addition, it passes through the point $\alpha_1\beta_1\gamma_1$, we have

$$l\alpha_1 + m\beta_1 + n\gamma_1 = \lambda(a\alpha_1 + b\beta_1 + c\gamma_1),$$

and therefore $\dfrac{l\alpha + m\beta + n\gamma}{l\alpha_1 + m\beta_1 + n\gamma_1} = \dfrac{a\alpha + b\beta + c\gamma}{a\alpha_1 + b\beta_1 + c\gamma_1}$

is the equation to the straight line through $(\alpha_1\beta_1\gamma_1)$ parallel to $l\alpha + m\beta + n\gamma = 0$.

This may also be written

$$l\alpha + m\beta + n\gamma = \frac{l\alpha_1 + m\beta_1 + n\gamma_1}{2S}(a\alpha + b\beta + c\gamma).$$

EQUATION OF THE SECOND DEGREE.

172. The general equation may be written

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0,$$

and represents the same locus as the general equation of the second degree in Cartesian coordinates, that is a conic.

We will discuss particular cases, when the conic is related in some special way to the triangle of reference.

(i) Let the equation reduce to $l\alpha^2 + 2l'\beta\gamma = 0$, the other constants being zero.

This may be written $\alpha^2 = \lambda\beta\gamma$, and by Art. 158 represents a conic which touches $\beta = 0$, $\gamma = 0$, at the points where they intersect $\alpha = 0$. That is to say, $\alpha^2 = \lambda\beta\gamma$ represents a conic which touches AB, AC at B and C.

173. To find the condition that

$$l\alpha + m\beta + n\gamma = 0$$

shall touch this conic.

This equation may be written in the form

$$\alpha = \mu\beta + \nu\gamma$$

by putting $\dfrac{m}{l}$, $\dfrac{n}{l}$ equal to $-\mu$, $-\nu$ respectively.

Where this meets $\alpha^2 = \lambda\beta\gamma$,

$$(\mu\beta + \nu\gamma)^2 = \lambda\beta\gamma.$$

If the roots of this equation be equal,

$$4\mu\nu=\lambda.$$

The equation to any tangent may therefore be written in the form

$$a=\mu\beta+\frac{\lambda}{4\mu}\gamma;$$

or putting $2\mu=m$

$$2a=m\beta+\frac{\lambda}{m}\gamma.$$

This conic cannot be a circle unless the triangle be isosceles.

174. Conic described about the triangle of reference.

If the general equation represent a conic described about the triangle of reference, the line $a=0$ must meet the curve in the points for which either $\beta=0$, or $\gamma=0$.

In the equation

$$la^2+m\beta^2+n\gamma^2+2l'\beta\gamma+2m'\gamma a+2n'a\beta=0,$$

let $a=0$, $\therefore\ m\beta^2+n\gamma^2+2l\beta\gamma=0$.

This must vanish when $\beta=0$, or when $\gamma=0$;

$$\therefore\ m=0,\ n=0, \text{ and by symmetry } l=0.$$

The equation is therefore reduced to

$$l'\beta\gamma+m'\gamma a+n'a\beta=0,$$

which may be written (suppressing the accents)

$$\frac{l}{a}+\frac{m}{\beta}+\frac{n}{\gamma}=0.$$

175. The conic $l\beta\gamma+m\gamma a+na\beta=0$, which may be written

$$l\beta\gamma+(m\gamma+n\beta)a=0,$$

is cut by the straight line $m\gamma+n\beta=0$ in the two points where this straight line cuts $\beta=0$, and $\gamma=0$, that is, in two coincident points;

$$m\gamma+n\beta=0, \quad \text{or } \frac{\beta}{m}+\frac{\gamma}{n}=0,$$

is therefore the tangent at A.

Similarly the tangents at B and C are

$$\frac{a}{l}+\frac{\gamma}{n}=0, \quad \frac{a}{l}+\frac{\beta}{m}=0, \text{ respectively.}$$

In the equation $\frac{a}{l}+\frac{\beta}{m}+\frac{\gamma}{n}=0$ put a, β, γ successively equal to zero, then we get the tangents at A, B, C.

Hence we obtain this theorem.

If a conic circumscribe a triangle, the three points where the tangents at the angles cut the opposite sides are collinear.

176. To find the condition that the line

$$\lambda a+\mu\beta+\nu\gamma=0$$

shall touch the conic $\frac{l}{a}+\frac{m}{\beta}+\frac{n}{\gamma}=0.$

Eliminating a, we obtain

$$\lambda l\beta\gamma=(m\gamma+n\beta)(\mu\beta+\nu\gamma).$$

If the roots of this equation are equal

$$(m\mu+n\nu-l\lambda)^2=4mn\mu\nu.$$

Taking the square root

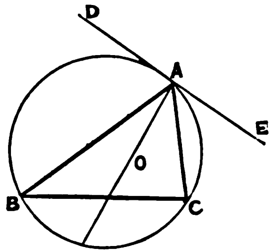
$$m\mu+2(mn\mu\nu)^{\frac{1}{2}}+n\nu=l\lambda,$$

or $$(l\lambda)^{\frac{1}{2}}+(m\mu)^{\frac{1}{2}}+(n\nu)^{\frac{1}{2}}=0,$$

where the roots are of ambiguous sign.

177. **To find the condition that the circumscribed conic shall be a circle.**

Let DE be the tangent to the circle at A.



Then $DAB = C$, $EAC = B$. (Euc. III. 32).

Therefore the equation to DE is

$$\beta \sin C + \gamma \sin B = 0, \text{ or } \frac{\beta}{b} + \frac{\gamma}{c} = 0.$$

Now, if the equation to the conic be

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

the equation to the tangent at A must be

$$\frac{m}{\beta} + \frac{n}{\gamma} = 0, \quad \text{or} \quad \frac{\beta}{m} + \frac{\gamma}{n} = 0;$$

this must be coincident with $\frac{\beta}{b} + \frac{\gamma}{c} = 0$, and similarly for the equations to the tangents at B and C.

l, m, n must therefore be proportional to a, b, c respectively, and the equation to the circumscribing circle is therefore

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0.$$

178. To find the equation to a conic inscribed in the triangle of reference.

We have seen that $u^2 = kvw$ is the equation to the conic which touches $v=0$, $w=0$ at the points where these lines meet $u=0$, or with respect to which $u=0$ is the polar of the intersection of $v=0$, $w=0$.

Let A be the pole of

$$l\alpha + m\beta + n\gamma = 0,$$

then the equation must be of the form

$$(l\alpha + m\beta + n\gamma)^2 = k\beta\gamma.$$

Now if $k = 4mn$, this equation may be put in either of the forms

$$(l\alpha + m\beta - n\gamma)^2 = -4nl\gamma\alpha,$$

$$\text{or} \quad (l\alpha - m\beta + n\gamma)^2 = -4lm\alpha\beta\,;$$

this shews that B and C are the poles of

$l\alpha + m\beta - n\gamma = 0$, and $l\alpha - m\beta + n\gamma = 0$, respectively.

The conic therefore touches the three sides of the triangle ABC.

We may write the equation in the form

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0,$$

since this equation when cleared of radicals becomes identical with

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 + 2lm\alpha\beta + 2mn\beta\gamma + 2nl\gamma\alpha = 0,$$

where, however, either *three* or *one* of the coefficients of the products must be negative, or the equation will be a perfect square and therefore represent a straight line.

179. To find the condition that the straight line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$

shall touch the conic

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (\nu\gamma)^{\frac{1}{2}} = 0.$$

We may write this latter equation in the form

$$m\beta + n\gamma + 2(mn\beta\gamma)^{\frac{1}{2}} = l\alpha.$$

Multiplying by λ and substituting for $\lambda\alpha$, we obtain

$$(m\lambda + l\mu)\beta + (n\lambda + l\nu)\gamma + 2\lambda(mn\beta\gamma)^{\frac{1}{2}} = 0.$$

If the roots of this equation be equal, we must have

$$(m\lambda + l\mu)(n\lambda + l\nu) = mn\lambda^2,$$

$$l\lambda\mu n + l^2\mu\nu + ml\lambda\nu = 0,$$

or, dividing by $l\lambda\mu\nu$,

$$\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\gamma} = 0.$$

We may observe that the condition that

$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$

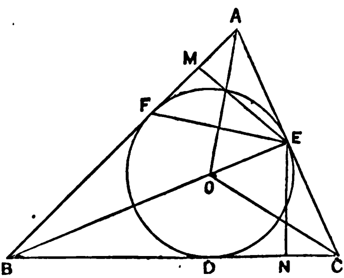
should touch the circumscribed conic

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0$$

is $$(l\lambda)^{\frac{1}{2}} + (m\mu)^{\frac{1}{2}} + (n\nu)^{\frac{1}{2}} = 0.$$

If we write α, β, γ for λ, μ, ν in this equation we get the equation to the inscribed conic, and similarly if we make the same substitution in the condition for the line touching this conic we get the circumscribed conic.

180. To find the equation to the circle inscribed in the triangle of reference.

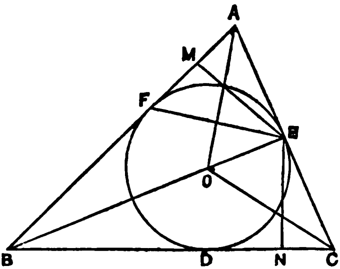


Let O be the centre, D, E, F the points of contact with the sides BC, CA, AB respectively.

Let the radius be r; join OA, OC, OE, EF, and draw EM, $EN \perp AB$, AC respectively.

Then $EM = AE \sin A = OE \cot \frac{A}{2} \sin A = 2r \cos^2 \frac{A}{2}$.

Similarly $EN = 2r \cos^2 \frac{C}{2}$. Hence the coordinates of E are, $2r \cos^2 \frac{C}{2}$, 0, $2r \cos^2 \frac{A}{2}$. Similarly the coordinates of F are $2r \cos^2 \frac{B}{2}$, $2r \cos^2 \frac{A}{2}$, 0.



Now the equation to the straight line through $(\alpha_1 \beta_1 \gamma_1)$, $(\alpha_2 \beta_2 \gamma_2)$ is $(\beta_1\gamma_2 - \beta_2\gamma_1)\alpha + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\beta + (\alpha_1\beta_2 - \alpha_2\beta_1)\gamma = 0$:

Therefore the equation to EF is (dividing by $2r$)

$$-\alpha \cos^4 \frac{A}{2} + \beta \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} + \gamma \cos^2 \frac{A}{2} \cos^2 \frac{C}{2} = 0,$$

$$\text{or} \qquad \alpha \cos^2 \frac{A}{2} - \beta \cos^2 \frac{B}{2} - \gamma \cos^2 \frac{C}{2} = 0.$$

The equations to FD and DE may be written down by symmetry.

Now FE is the polar of A.

The equation to the circle must ∴ be of the form

$$\left(\alpha \cos^2 \frac{A}{2} - \beta \cos^2 \frac{B}{2} - \gamma \cos^2 \frac{C}{2}\right)^2 = k\alpha\beta \text{ (Art. 159).}$$

Now if $k = 4 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2}$, this equation becomes

$$\alpha^2 \cos^4 \frac{A}{2} + \beta^2 \cos^4 \frac{B}{2} + \gamma^2 \cos^4 \frac{C}{2} =$$

$$2\left(\beta\gamma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} + \gamma\alpha \cos^2 \frac{C}{2} \cos^2 \frac{A}{2} + \alpha\beta \cos^2 \frac{A}{2} \cos^2 \frac{B}{2}\right),$$

which is symmetrical with respect to $\alpha, \beta, \gamma, A, B, C$, and therefore is the equation required.

This equation may be written

$$\alpha^{\frac{1}{2}} \cos \frac{A}{2} + \beta^{\frac{1}{2}} \cos \frac{B}{2} + \gamma^{\frac{1}{2}} \cos \frac{C}{2} = 0.$$

By a similar process we may find the equations to the escribed circle which touches BC and the other sides produced to be $(-\alpha)^{\frac{1}{2}} \cos \frac{A}{2} + \beta^{\frac{1}{2}} \cos \frac{B}{2} + \gamma^{\frac{1}{2}} \cos \frac{C}{2} = 0$, and so for the other escribed circles.

181. To interpret the equation $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$.

If l, m, n are all of the same sign, this equation is impossible; one of these quantities must therefore be of different sign to the others.

We may therefore write the equation

$$l\alpha^2 - m\beta^2 - n\gamma^2 = 0.$$

Since this may be written in the form

$$(l^{\frac{1}{2}}\alpha + m^{\frac{1}{2}}\beta)(l^{\frac{1}{2}}\alpha - m^{\frac{1}{2}}\beta) = n\gamma^2,$$

we see that $\gamma = 0$ is the polar of the intersection of

$$l^{\frac{1}{2}}\alpha + m^{\frac{1}{2}}\beta = 0, \text{ and } l^{\frac{1}{2}}\alpha - m^{\frac{1}{2}}\beta = 0,$$

that is, AB is the polar of C.

Similarly, since the equation may be written

$$(l^{\frac{1}{2}}\alpha + n^{\frac{1}{2}}\gamma)(l^{\frac{1}{2}}\alpha - n^{\frac{1}{2}}\gamma) = m\beta^2,$$

CA is the polar of B.

Again, since the equation may be written

$$(m^{\frac{1}{2}}\beta + \sqrt{-1}\, n^{\frac{1}{2}}\gamma)(m^{\frac{1}{2}}\beta - \sqrt{-1}\, n^{\frac{1}{2}}\gamma) = l\alpha^2,$$

and the imaginary lines $m^{\frac{1}{2}}\beta \pm \sqrt{-1}\, n^{\frac{1}{2}}\gamma = 0$ intersect in the real point A, we see that A is within the conic, that these lines are the imaginary tangents from A, and that BC is the polar of A.

The equation therefore represents a conic so related to the triangle ABC, that each vertex is the pole of the opposite side.

If two triangles ABC, $A'B'C'$ are so related to a conic that A, B, C are the poles of $B'C'$, $C'A'$, $A'B'$, these triangles are said to be conjugate.

If the two triangles coincide, ABC is said to be self-conjugate.

We see then that the triangle of reference is self-conjugate with respect to the conic

$$l\alpha^2+m\beta^2+n\gamma^2=0.$$

EXAMPLES ON CHAPTER VIII.

1. If $u+v+w=0$, $-u+v+w=0$, $u-v+w=0$, $u+v-w=0$, form a quadrilateral, the diagonals will be represented by $u=0$, $v=0$, $w=0$.

2. If $u=0$, $v=0$, $w=0$ represent 3 sides of a quadrilateral and $u+v+w=0$ the fourth, then $u+v=0$, $v+w=0$, $w+u=0$ represent the three diagonals, and $u-v=0$, $v-w=0$, $w-u=0$ the lines joining the vertices with the intersections of those diagonals which do not pass through them.

3. If $s=0$ be the equation to the straight line at infinity,

$$u+v+s=0,\ u+v-s=0,\ u-v+s=0,\ u-v-s=0$$

represent the sides of a parallelogram whose diagonals are

$$u=0,\ v=0.$$

4. Interpret the equation

$$(x\cos\alpha+y\sin\alpha-p)\ (x\cos\beta+y\sin\beta-q)$$
$$=l\ (x\cos\gamma+y\sin\gamma-r)\ (x\cos\delta+y\sin\delta-s),$$

and find the condition that it shall represent a circle.

5. O is the orthocentre of the triangle of reference: prove that the equation to OC is $\alpha\sec A=\beta\sec B$.

6. Find the equation to the straight lines which join the middle points of the sides of the triangle of reference.

7. Find the equations to the straight lines which join the feet of the perpendiculars from the angles on the sides.

8. Find the equation to the straight line which passes through A, and is parallel to $l\alpha+m\beta+n\gamma=0$.

9. Transform the equation

$$l\alpha+m\beta+n\gamma=0$$

to oblique axes AC, AB.

10. By means of quest. 9 find the length of the perpendicular from $(\alpha_1\beta_1\gamma_1)$ on $l\alpha+m\beta+n\gamma=0$.

11. Find the equation to the straight line joining the centres of the circumscribed and inscribed circles of the triangle.

12. Interpret the equations

$$a\alpha=\beta b+\gamma c:$$

$$\alpha+\beta+\gamma=0:$$

$$\alpha\cos A+\beta\cos B+\gamma\cos C=0.$$

13. $\beta\gamma=\lambda^2\alpha^2$ is the equation to a conic: prove that

$$\frac{\beta}{\mu}+\mu\gamma=2\lambda\alpha,\ \frac{\beta}{\mu'}+\mu'\gamma=2\lambda\alpha$$

touch the conic, and that the polar of their intersection is

$$\beta+\mu\mu'\gamma=(\mu+\mu')\lambda\alpha.$$

14. Prove that the equation

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+(a\alpha+b\beta+c\gamma)\ (l\alpha+m\beta+n\gamma)=0$$

always represents a circle, and may represent any circle.

15. If $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ be two points on the conic

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0,$$

the equation to the straight line joining them is

$$\frac{l\alpha}{\alpha'\alpha''}+\frac{m\beta}{\beta'\beta''}+\frac{n\gamma}{\gamma'\gamma''}=0,$$

and that to the tangent at $(\alpha'\beta'\gamma')$

$$\frac{l\alpha}{\alpha'^2}+\frac{m\beta}{\beta'^2}+\frac{n\gamma}{\gamma'^2}=0.$$

16. Find the coordinates of the middle point of the chord in the preceding question.

17. Find the equation to the straight line joining the middle point of AB, and the intersection of tangents to the conic about ABC at A and B, and hence find the coordinates of the centre.

18. Find the condition that the conic may be a parabola.

19. Find the equation to the straight line joining A and the middle point of its polar with respect to

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0,$$

and hence find the coordinates of the centre of the conic.

20. Find the condition that the conic may be a parabola.

LIST OF FORMULÆ.

The coordinates are rectangular unless otherwise specified.

The Point. The distance between (x_1y_1) and (x_2y_2) is

$$\{(x_1-x_2)^2+(y_1-y_2)^2\}\ .$$

The area of a triangle is

$$\pm\frac{1}{2}\{x_1y_2-x_2y_1+x_2y_3-x_3y_1+x_3y_1-x_1y_3\}.$$

If the coordinates be polar, these expressions become

$$\{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\}^{\frac{1}{2}},$$

$$\pm\frac{r_1r_2r_3}{2}\left\{\frac{\sin(\theta_1-\theta_2)}{r_3}+\frac{\sin(\theta_2-\theta_3)}{r_1}+\frac{\sin(\theta_3-\theta_1)}{r_2}\right\}.$$

Transformations. Origin changed but not axes: $x=x'+h$, $y=y'+k$. Axes changed but not origin: $x=x'\cos\theta-y'\sin\theta$, $y=x'\sin\theta+y'\cos\theta$.

Also $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$, $\tan\theta=\frac{y}{x}$.

The straight line.

$lx+my=d$: general equation, any axes.

$\frac{x}{a}+\frac{y}{b}=1$: in terms of intercepts on any axes.

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$: through the points (x_1y_1), (x_2y_2).

If the line makes an angle θ with the axis of x, and r is the distance between two points (xy), (x_1y_1),

$$x=x_1+r\cos\theta,\ y=y_1+r\sin\theta.$$

$x\cos a+y\sin a=p$: where p is the perpendicular from the origin which makes $\angle\, a$ with axis of x.

$r \cos(\theta - a) = p$, general polar equation.

$$\frac{\sin(\theta_1 - \theta_2)}{r} + \frac{\sin(\theta_2 - \theta)}{r_1} + \frac{\sin(\theta - \theta_1)}{r_2} = 0,$$

polar equation to the straight line through $(r_1\theta_1)$, $(r_2\theta_2)$.

If $lx + my = d$, $l'x + m'y = d'$ are parallel, $\frac{l}{l'} = \frac{m}{m'}$; if at right angles $ll' + mm' = 0$.

The distance of (xy) from $lx + my = d$ is $\frac{lx + my - d}{(l^2 + m^2)^{\frac{1}{2}}}$.

Circle.

Equations: $x^2 + y^2 = c^2$, centre origin;

$(x - a)^2 + (y - b)^2 = c^2$, centre (a, b);

$x^2 + y^2 + 2Ax + 2By + C = 0$, general equation.

$r = l \cos(\theta - a)$, polar equation, pole on the curve.

Equations to tangent; at the point (x_1, y_1), $xx_1 + yy_1 = c^2$;

to any tangent, $x \cos\theta + y \sin\theta = c$.

$hx + ky = c^2$ is the polar of (h, k).

$S_1 - S_2 = 0$, radical axis of $S_1 = 0$, $S_2 = 0$.

Parabola.

$y^2 = 4ax$, equation to curve.

$yy_1 = 2a\,(x + x_1)$ to the tangent at (x_1, y_1).

$y = x \tan\theta + a \cot\theta$ to any tangent making an angle θ with the axis: $a \cot^2\theta$, $2a \cot\theta$ the point of contact.

$y = 2a \cot\theta$, diameter bisecting chords parallel to this tangent.

$x + a$, focal distance of (x, y).

$y^2 = 4a'x$, equation to curve referred to a diameter and its tangent.

LIST OF FORMULÆ.

The coordinates are rectangular unless otherwise specified.

The Point. The distance between (x_1y_1) and (x_2y_2) is

$$\{(x_1-x_2)^2+(y_1-y_2)^2\}^{\frac{1}{2}}.$$

The area of a triangle is

$$\pm\frac{1}{2}\{x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3\}.$$

If the coordinates be polar, these expressions become

$$\{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\}^{\frac{1}{2}},$$

$$\pm\frac{r_1r_2r_3}{2}\left\{\frac{\sin(\theta_1-\theta_2)}{r_3}+\frac{\sin(\theta_2-\theta_3)}{r_1}+\frac{\sin(\theta_3-\theta_1)}{r_2}\right\}.$$

Transformations. Origin changed but not axes: $x=x'+h,\ y=y'+k$. Axes changed but not origin: $x=x'\cos\theta-y'\sin\theta,\ y=x'\sin\theta+y'\cos\theta$.

Also $x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2,\ \tan\theta=\frac{y}{x}$.

The straight line.

$lx+my=d$: general equation, any axes.

$\frac{x}{a}+\frac{y}{b}=1$: in terms of intercepts on any axes.

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$: through the points (x_1y_1), (x_2y_2).

If the line makes an angle θ with the axis of x, and r is the distance between two points (xy), (x_1y_1),

$$x=x_1+r\cos\theta,\ y=y_1+r\sin\theta.$$

$x\cos\alpha+y\sin\alpha=p$: where p is the perpendicular from the origin which makes $\angle\,\alpha$ with axis of x.

Any Conics.

$$\frac{l}{r}=1+e\cos\theta,$$

polar equation, focus pole, axis initial line.

$$\frac{l}{r}=\cos(\theta-\theta_1)+e\cos\theta, \text{ tangent at } (r_1\theta_1).$$

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0,$$

represents a conic: the sum of the three first terms equated to zero give the directions of the asymptotes.

The conic is an ellipse, parabola, or hyperbola as

$$B^2-AC<, =, 0 \text{ or } >0.$$

Abridged Notation.

$u=\lambda v$, passes through the intersection of $u=0$, $v=0$.

$uv=\lambda wz$, circumscribes quadrilateral, of which $u=0$, $v=0$, and $w=0$, $z=0$, are pairs of opposite sides.

If $uv=\lambda w^2$, $w=0$ is the polar of the intersection of $u=0$, $v=0$.

Trilinear Coordinates.

$$a\alpha+b\beta+c\gamma=2S.$$

$l\alpha+m\beta+n\gamma=0$, represents any straight line.

$l^2\alpha^2=\beta\gamma$, a conic touching AB, AC, at B and C.

$2l\alpha=m\beta+\dfrac{\gamma}{m}$, any tangent to this conic.

$\dfrac{l}{\alpha}+\dfrac{m}{\beta}+\dfrac{n}{\gamma}=0$, a conic circumscribing the triangle, to which $\dfrac{l}{\alpha}+\dfrac{m}{\beta}=0$ is the tangent at C.

$\dfrac{a}{\alpha}+\dfrac{b}{\beta}+\dfrac{c}{\gamma}=0$, the circumscribing circle.

$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0$, the inscribed conic.

$l\alpha^2+m\beta^2+n\gamma^2=0$, a conic to which the triangle is self-conjugate.

ANSWERS.

Ex. 4. 1. i. (2, 1). ii. (10, 7). iii. $\left(\frac{a}{a+b}, \frac{b}{a+b}\right)$.
iv. $(\pm 8, \pm 6)$. v. $\left(6 \text{ or } 3, \frac{2}{3} \text{ or } \frac{1}{3}\right)$. vi. (0, 0, or $4a$, $4a$).
vii. $(\pm 2a, a)$. (viii) $(\pm a, 0)$. 2. $5a$; $\sqrt{5}b$; a; $\sqrt{10}a$;
$\sqrt{37}a$; $\sqrt{3}b$; $\left(1+\frac{\sin 2a}{2}\right)^{\frac{1}{2}} a$; $\sqrt{\iota}a$.

3. $\sqrt{2}$; $7\sqrt{2}$; $2\sqrt{2}$; $\sqrt{2}a$; $(h^2+16k^2)^{\frac{1}{2}}$; $\sqrt{2}(a-b)$; $10a$.
4. $2a\sin\frac{\omega}{2}$; $a\left(1+\frac{\sin^2\omega}{2}\right)^{\frac{1}{2}}$; $4\cos\frac{\omega}{2}$; $\sqrt{13+12\cos\omega}$;
$\sqrt{13-12\cos\omega}$. 5. $\{a^2+b^2+2ab\cos(\theta-\phi)\}^{\frac{1}{2}}$; $2a\sin\theta$; $2a\cos\theta$;
$a\sqrt{5+2\sqrt{3}}$. 6. (2, 2); (1, −2); (0, 0); $\left(\frac{3h}{2}, k\right)$. 7. (i) $\frac{3}{2}$;
(ii) 12. (iii) $\frac{1}{2}(x_1y_2-x_2y_1)$. (iv) ab. (v) $\frac{r^2}{2}$. (vi) $\frac{5\sqrt{3}}{4}a^2$.

Ex. 6. 1. $x'+y'=0$, $x'+y'+4=0$, $x'=y'$,
$x'^2+y'^2+2(x'+y')+1=0$, $x'^2-y'^2+2(x'-y')=1$, $y'^2+2y'=4x'+3$.
2. $\sqrt{2}y'=c$, $2x'y'+a^2=0$, $y'^2=2a(\sqrt{2}x'-a)$. 3. $r^2\cos 2\theta=a^2$,
$r=a$, $r=4a\cot\theta\operatorname{cosec}\theta$, $r\cos(\theta-a)=a$. 4. $\frac{2a}{r}=1-\cos\theta$.

5. $y=x\tan\frac{\pi}{3}$, $x^2+y^2=c^2$, $x+y=l$, $x^2-y^2=a^2$,
$(x^2+y^2)^2=a^2(x^2-y^2)$.

EXAMPLES ON CHAPTER I.

1. $a\cos\frac{n\pi}{4}$, $a\sin\frac{n\pi}{4}$, where n is integral.
2. $a\cos\frac{2n+1}{8}\pi$, $a\sin\frac{2n+1}{8}\pi$, where n is integral.
3. $(a, 0)$, $(2a, a)$, $(2a, 2a)$, $(2a, a)$, $(0, a)$, $(0, 0)$.
5. (1) $\frac{a}{2}$, $\frac{a}{2}\cot A$; $r\cot\frac{B}{2}$, r. (2) $\frac{a}{2}\cot A\operatorname{cosec} C$,
$\frac{a}{2}\cot A\operatorname{cosec} B$; $r\operatorname{cosec} A$, $r\operatorname{cosec} A$. (3) R, $\frac{\pi}{2}-C$;

$r \operatorname{cosec}\frac{A}{2}, \frac{A}{2}$. 6. $\frac{a}{3}, \frac{b}{3}$. 7. (i) 0, 0; $-3a$, $3a$; $3a$, $3a$; $9a^2$. (ii) 0, 2, 2, 0, 2. 9. $x=0$. 13. $\tan\theta = \frac{y \sin\omega}{x + y\cos\omega}$; $r = (x^2+y^2+2xy\cos\omega)^{\frac{1}{2}}$.

Ex. 6. 2. $x+y=a+b$; $kx=hy$; $x=h$; $y+k=0$; $x-y+1=0$; $x-y+1=0$; $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$.
3. $y-2=\frac{x+1}{\sqrt{3}}$. 4. $x=y$. 5. $x\cos a - y\sin a = b$.

Ex. 7. The angles are $\frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{4}, -\tan^{-1}\frac{2}{3}, -\tan^{-1}\frac{3}{2}, \tan^{-1}\frac{4}{3}, -\tan^{-1}\frac{4}{3}, \tan^{-1}2, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}, -\frac{\pi}{3}$ respectively, and the perpendiculars 2, -3, 1, 0, 0, $\frac{6}{\sqrt{13}}, \frac{1}{5}, \frac{1}{5}, \frac{1}{\sqrt{5}}, \frac{2\sqrt{3}-1}{2}, \frac{2\sqrt{3}-1}{2}$, 1, 1, 1.

Ex. 8. 1. $x+\sqrt{3}y=0, \sqrt{3}x+y=0$. 2. $(a+b)y=(a-b)(x-a)$, $(a-b)y+(a+b)(x-a)=0$. 3. $(y-b)\cot(a\pm\beta)+x=0$.
4. $\frac{5\pi}{12}, \tan^{-1}\frac{b^2\sim a^2}{2ab}$. 5. $Ax+By=10A$, or $Aa+Bb$, or 0.
6. $y=mx\pm a(1+m^2)^{\frac{1}{2}}$. 7. $Bx=A(y-b)$. 8. $ax-by=a^2-b^2$.
9. $x\cos a+y\sin a=\pm c(\sin a\cos a)^{\frac{1}{2}}$. 10. $y=mx+\frac{mc}{1+m}$.

Ex. 9. 3. $\frac{1}{r}=\frac{\cos\theta}{a}+\frac{\sin\theta}{b}$; $\frac{1}{r}=\frac{\sqrt{3}+1}{a}(\sin\theta-\cos\theta)$; $2r\sin\theta=\sqrt{3}a$. 4. $\theta=\frac{\pi}{2}-B$; $r\cos\theta=b\cos A$, $r\cos(\theta-A)=c\cos A$, $\theta=\frac{\pi}{2}-B$, $r=a\cot A$.

Ex. 10. 1. $2\sqrt{2}$. 2. $\frac{3}{\sqrt{13}}$. 3. $\frac{2\sqrt{10}}{5}$. 4. $\frac{11}{5}$.
5. $\frac{3ab}{(a^2+b^2)^{\frac{1}{2}}}$. 6. $\frac{a^2-b^2}{(a^2+b^2)^{\frac{1}{2}}}$. 7. $\frac{Ah+Bk+C-D}{(A^2+B^2)^{\frac{1}{2}}}$. 8. $\frac{c^2}{(h^2+k^2)^{\frac{1}{2}}}$.

9. $\dfrac{h^2+k^2-c^2}{(h^2+k^2)^{\frac{1}{2}}}$.   10. $\dfrac{a}{m}(1+m^2)^{\frac{1}{2}}$.   11. $\dfrac{b^2h^2+a^2k^2-a^2b^2}{(a^4k^2+b^4h^2)^{\frac{1}{2}}}$.

12. 0, 1; $\dfrac{30}{13}$, $-\dfrac{6}{13}$; $\dfrac{2}{5}$, $-\dfrac{1}{5}$; $-\dfrac{58}{25}$, $\dfrac{31}{25}$; $\dfrac{2b^2-a^2}{a^2+b^2}a$, $\dfrac{2a^2-b^2}{a^2+b^2}b$; $\dfrac{2ab^2}{a^2+b^2}$, $\dfrac{2a^2b}{a^2+b^2}$; $\dfrac{(Bh-Ak)B-A(C-D)}{A^2+B^2}$, $\dfrac{(Bh-Ak)A-B(C-D)}{A^2+B^2}$; $\dfrac{hc^2}{h^2+k^2}$, $\dfrac{kc^2}{h^2+k^2}$; 0, $\dfrac{a}{m}$; $\dfrac{(a^2-b^2)k^2+b^4}{a^4k^2+b^4h^2}a^2h$, $\dfrac{(b^2-a^2)h^2+a^4}{a^4k^2+b^4h^2}b^2k$.   13. $(a\sim b)\sin\alpha$; $\dfrac{3ab}{2(a^2+b^2)^{\frac{1}{2}}}$.   14. (1) and (2) intersect in $(4, -3)$, (1) and (3) in $(1, 0)$, (1) and (4) in $(6, -5)$, (2) and (3) in $\left(\dfrac{20}{11}, \dfrac{3}{11}\right)$, (2) and (4) in $\left(\dfrac{14}{3}, -4\right)$, (3) and (4) in $\left(-\dfrac{2}{13}, -\dfrac{5}{13}\right)$.

15. $p\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$, $p\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$.   16. a, $\dfrac{p_1-a\cos\alpha}{\sin\alpha}$; a, $\dfrac{p_2-a\cos\alpha}{\sin\alpha}$; b, $\dfrac{p_1-b\cos\alpha}{\sin\alpha}$; b, $\dfrac{p_2-b\cos\alpha}{\sin\alpha}$.

17. $x\cos\alpha+y\sin\alpha\pm\dfrac{2x-(a+b)}{2(a-b)}(p_2-p_1)=\dfrac{p_1+p_2}{2}$; $x=\dfrac{a+b}{2}$; $y=\dfrac{p_1+p_2-(a+b)\cos\alpha}{2\sin\alpha}$; $(a-b)(p_1-p_2)\operatorname{cosec}\alpha$.

18. $a-h$, $b-k$; $\dfrac{x}{a}+\dfrac{y}{b}=1$, $(y-k)(a-2h)=(x-h)(b-2k)$.

19. a, $\tan^{-1}\dfrac{ab-(bh+ak)}{ah+bk-(h^2+k^2)}$.   20. $\tan^{-1}\dfrac{a^2-b^2}{2ab}$, $\sqrt{3}(a^2-b^2)=2ab$.

Ex. 11. 1. $(x-h)\left(\dfrac{1}{a_1}-\dfrac{1}{a_2}\right)+(y-k)\left(\dfrac{1}{b_1}-\dfrac{1}{b_2}\right)=0$.

2. $(A_1C_2-A_2C_1)x+(B_1C_2-B_2C_1)y=0$.   3. $\dfrac{y}{b}-\dfrac{x}{a}=\dfrac{mb-a}{ma+b}$.

4. $l_3x+m_3y=\dfrac{d_1(l_3m_2-l_2m_3)+d_2(l_1m_3-l_3m_1)}{l_1m_2-l_2m_1}$.

$\frac{(a^2+b^2)^{\frac{1}{2}}}{2}$. 3. $x^2+y^2-2a(x-y)+a^2=0$.
4. $x^2+y^2=2a(3x+4y)$. 5. $x^2+y^2+2b^2+c^2=2\{(b+c)x+(b-c)y\}$.
6. $x^2+y^2=ax+by$. 7. This may be obtained thus: let $x^2+y^2+2Ax+2By=0$ be the equation; then $x_1^2+y_1^2+2Ax_1+2By_1=0$, and $x_2^2+y_2^2+2Ax_2+2By_2=0$, whence A and B may be obtained. 8. Write x_1-x_3, y_1-y_3, x_2-x_3, y_2-y_3, for x_1, y_1, x_2, y_2 in the result of 7.
9. $3(x^2+y^2)+2(a^2+b^2)=(5a^2+2b^2)\frac{x}{a}+(5b^2+2a^2)\frac{y}{b}$.

10. $\left(\frac{a}{2}, \frac{b}{2}\right)$ and $\frac{(a^2+b^2)^{\frac{1}{2}}}{2}$; $\left(\frac{5a^2+2b^2}{6a}, \frac{5b^2+2a^2}{6b}\right)$

and $\frac{(4a^6+53a^4b^2+53a^2b^4+4b^6)^{\frac{1}{2}}}{6a^2b^2}$.

11. $(m^2-l^2)(x^2+y^2)-2(mx+ly)(m^2-l^2)^{\frac{1}{2}}(a^2-b^2)^{\frac{1}{2}}$
$+l^2a^2-m^2b^2=0$.
12. $x^2+y^2=\pm 2a(x\pm y)$. 13. $x^2+y^2=\sqrt{2}ay$.

15. $\frac{\pi}{3}$, $\left(\frac{a}{3}, \frac{a}{3}\right)$ and $\left(\frac{a^2+3b^2}{3}\right)^{\frac{1}{2}}$.

16. $x^2+2xy\cos\omega+y^2=hx+ky$. 17. $\frac{5\pi}{6}$, $2a$, $(\sqrt{3}a, \sqrt{2}a)$.

18. $\cos\omega=\frac{a^2-b^2}{a^2}$. 20. If the given length be $2l$, $r=(p^2+l^2)^{\frac{1}{2}}$. 23. $2\left(c^2-\frac{a^2b^2}{a^2+b^2}\right)^{\frac{1}{2}}$.

Ex. 13. 1. $x=c$; $3x-4y=5c$; $hx-ky+c^2=0$.
2. (i) $x\sin\alpha-y\cos\alpha=\pm c$; (ii) $(bx+ay)^2=c^2(a^2+b^2)$; (iii) $(Bx-Ay)^2=(A^2+B^2)c^2$; (iv) If the given point be $(0, b)$, the equation to the two tangents is $(b^2-c^2)x^2=c^2(y-b)^2$; (v) In the equation, $x\cos\theta+y\sin\theta=c$, $\cos(\theta-\alpha)=\frac{c\pm\delta}{a}$; (vi) $x+y=\sqrt{2}c$. 3. $(A^2+B^2)c^2=C^2$, $(x_1\sin\alpha-y_1\cos\alpha)^2=c^2$, $b^2=(1+n^2)c^2$. 5. The points of contact are $(-a, -b)$; $(2a, b)$; $(c, 0)$, $(0, c)$; $\left(2c\cos^2\frac{\theta}{2}, 2c\sin^2\frac{\theta}{2}\right)$; $(0, b)$.

6. $(a\cos\theta+b\sin\theta-p)^2=a^2+b^2$. 11. $\theta=\frac{\pi}{2}+\alpha$, $r\cos(\theta-\alpha)=l$. 12. $\theta=\frac{\gamma+\alpha}{2}\pm\frac{\pi}{4}$.

5. $(1-m)p\left(y\cos\frac{\alpha+\beta}{2}-x\sin\frac{\alpha+\beta}{2}\right)$

$$=c\left\{(y-x)\cos\frac{\alpha-\beta}{2}+p\left(\cos\frac{\alpha+\beta}{2}-\sin\frac{\alpha+\beta}{2}\right)\right\}.$$

6. $m=1$. 7. $y+(\sqrt{2}-1)(x+2)=0$, $(\sqrt{2}-1)y=x+2$.

8. $p+q=5(\cos\alpha+\sin\alpha)$.

EXAMPLES ON CHAPTER II.

2. c^2. 3. 0. 4. $\dfrac{a^2+5ab+b^2}{3}$.

5. $\dfrac{\{p\sin(\beta-\gamma)+q\sin(\gamma-\alpha)+r\sin(\alpha-\beta)\}^2}{2\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$.

6. If $\frac{c^2}{2}$ be the given area, $y-k=m(k-h)$ will represent either line if m be one of the roots of $(mh-k)^2+mc^2=0$.

8. (i) the axes; (ii) the lines $x=\pm y$; (iii) the axes, and $x=y$; (iv) $Ax+By+1=0$, $Bx+Ay=0$; (v) $x\cos\theta\pm y\sin\theta=\pm p$; (vi) $x+2y=0$, $x=\pm\sqrt{2}y$; (vii) $x+1=3y$, or $x+y+1=0$; (viii) the point 6, 3; (ix) $x+3=3y$, or $x=y+3$.

9. (i) represents the 4 points $r=\pm a$, $\theta=\pm\alpha$; (ii) the point (a, α); (iii) the two points (a, β), (b, α). 18. $\frac{x}{a}+\frac{y}{b}=2$.

19. $\frac{x}{\alpha}+\frac{y}{\beta}=1$, where α, β are determined from the equations $\frac{a}{\alpha}+\frac{b}{\beta}=1$, $\alpha^2-2\alpha\beta\cos\omega+\beta^2=c^2$. 20. $x\cos\omega+y=a\cos\omega$, $x+y\cos\omega=a$. 21. (i) $y-b=\dfrac{x\sin\alpha}{\sin(\omega-\alpha)}$, or $y+\dfrac{x\sin\alpha}{\sin(\omega+\alpha)}=b$; (ii) $b-y=x\dfrac{\sin(\beta\pm\omega)}{\sin\beta}$. 27. $\left(2a, \frac{\pi}{2}\right)$, $\frac{\pi}{3}$.

28. $\dfrac{5}{\sqrt{178-80\sqrt{2}}}$ inches.

Ex. 12. 1. $3a$, $4a$, $(a^2+b^2)^{\frac{1}{2}}$. 2. $(a, -a)$ and $(2a^2+c^2)^{\frac{1}{2}}$, $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and $\dfrac{\sqrt{5}(a^2+b^2)^{\frac{1}{2}}}{2}$, $\left(\frac{3}{2}, 2\right)$ and $\frac{3}{2}$, $\left(-\frac{a}{2}, 0\right)$ and $\frac{a}{2}$, $\left(0, -\frac{b}{2}\right)$ and $\frac{b}{2}$, $\left(-\frac{a}{2}, 0\right)$ and $\frac{\sqrt{5}}{2}a$, $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and

$\frac{(a^2+b^2)^{\frac{1}{2}}}{2}$. 3. $x^2+y^2-2a(x-y)+a^2=0$.

4. $x^2+y^2=2a(3x+4y)$. 5. $x^2+y^2+2b^2+c^2=2\{(b+c)x+(b-c)y\}$.
6. $x^2+y^2=ax+by$. 7. This may be obtained thus: let $x^2+y^2+2Ax+2By=0$ be the equation; then $x_1^2+y_1^2+2Ax_1+2By_1=0$, and $x_2^2+y_2^2+2Ax_2+2By_2=0$, whence A and B may be obtained. 8. Write x_1-x_3, y_1-y_3, x_2-x_3, y_2-y_3, for x_1, y_1, x_2, y_2 in the result of 7.

9. $3(x^2+y^2)+2(a^2+b^2)=(5a^2+2b^2)\frac{x}{a}+(5b^2+2a^2)\frac{y}{b}$.

10. $\left(\frac{a}{2}, \frac{b}{2}\right)$ and $\frac{(a^2+b^2)^{\frac{1}{2}}}{2}$; $\left(\frac{5a^2+2b^2}{6a}, \frac{5b^2+2a^2}{6b}\right)$

and $\frac{(4a^6+53a^4b^2+53a^2b^4+4b^6)^{\frac{1}{2}}}{6a^2b^2}$.

11. $(m^2-l^2)(x^2+y^2)-2(mx+ly)(m^2-l^2)^{\frac{1}{2}}(a^2-b^2)^{\frac{1}{2}}$
$+l^2a^2-m^2b^2=0$.

12. $x^2+y^2=\pm 2a(x\pm y)$. 13. $x^2+y^2=\sqrt{2}ay$.

15. $\frac{\pi}{3}$, $\left(\frac{a}{3}, \frac{a}{3}\right)$ and $\left(\frac{a^2+3b^2}{3}\right)^{\frac{1}{2}}$.

16. $x^2+2xy\cos\omega+y^2=hx+ky$. 17. $\frac{5\pi}{6}$, $2a$, $(\sqrt{3}a, \sqrt{2}a)$.

18. $\cos\omega=\frac{a^2-b^2}{a^2}$. 20. If the given length be $2l$,

$r=(p^2+l^2)^{\frac{1}{2}}$. 23. $2\left(c^2-\frac{a^2b^2}{a^2+b^2}\right)^{\frac{1}{2}}$.

Ex. 13. 1. $x=c$; $3x-4y=5c$; $hx-ky+c^2=0$.
2. (i) $x\sin a-y\cos a=\pm c$; (ii) $(bx+ay)^2=c^2(a^2+b^2)$; (iii) $(Bx-Ay)^2=(A^2+B^2)c^2$; (iv) If the given point be $(0, b)$, the equation to the two tangents is $(b^2-c^2)x^2=c^2(y-b)^2$; (v) In the equation, $x\cos\theta+y\sin\theta=c$, $\cos(\theta-a)=\frac{c\pm\delta}{a}$; (vi) $x+y=\sqrt{2}c$. 3. $(A^2+B^2)c^2=C^2$, $(x_1\sin a-y_1\cos a)^2=c^2$, $b^2=(1+n^2)c^2$. 5. The points of contact are $(-a, -b)$; $(2a, b)$; $(c, 0)$, $(0, c)$; $\left(2c\cos^2\frac{\theta}{2}, 2c\sin^2\frac{\theta}{2}\right)$; $(0, b)$.

6. $(a\cos\theta+b\sin\theta-p)^2=a^2+b^2$. 11. $\theta=\frac{\pi}{2}+a$, $r\cos(\theta-a)=l$. 12. $\theta=\frac{\gamma+a}{2}\pm\frac{\pi}{4}$.

Ex. 14. 4, 5. Straight lines.
7. $(x-h)^2+(y-k)^2=b(x\cos\alpha+y\sin\alpha-p)$. 8. If $BAC=\omega$, and $CD=c$, the locus is $x^2+2xy\cos\omega+y^2=\dfrac{c^2}{1+\cos^2\omega}$.
10. Take BC for the axis of x, and the straight line bisecting it at right angles for the axis of y, and let $BC=2a$, the locus is $x=\dfrac{m-1}{m+1}a$.

EXAMPLES ON CHAPTER III.

1. $x+y=c$; $2x+3y=c$; $(a+b)x+(a-b)y=c^2$.
2. $\left(-\dfrac{Ac^2}{C}, -\dfrac{Bc^2}{C}\right)$; $\left(\dfrac{c^2}{2a}, \dfrac{c^2}{2b}\right)$; $\left(\dfrac{c^2\sin\theta}{x_1\sin\theta_1-y_1\cos\theta}, \dfrac{-c^2\cos\theta}{x_1\sin\theta-y_1\cos\theta}\right)$; $\left(\dfrac{c^2}{a}, 0\right)$, $\left(0, \dfrac{c^2}{b}\right)$. 6. They all pass through the points of intersection of $(x-a_1)^2+(y-b_1)^2=c_1^2$, $(x-a_2)^2+(y-b_2)^2=c_2^2$. 7. $x+y=3$ of (1) and (2); $2x-4y+5=0$ of (2) and (3); $4x-2y=1$ of (3) and (1); $\left(\dfrac{7}{6}, \dfrac{11}{6}\right)$ the radical centre. 13. $(x-8c)^2=15y^2$ and $(3x-8c)^2=7y^2$. 14. $x^2=c^2$, and, if $a>c$, $\{2y-(a+1)\}^2c^2=(a^2+2ac-3c^2)x^2$. 15. $x=a+c$.
17. $2\left\{\dfrac{c^2(a^2+b^2)-a^2b^2}{a^2+b^2}\right\}^{\frac{1}{2}}$. 18. $x^2-ax+y^2=\dfrac{2c^2-a^2}{2}$.
19. $x+y+1=0$. 20. The axes. 21. $\{4c^2-2(a-b)^2\}^{\frac{1}{2}}$.
22. $(a-h)\cos\alpha+(b-k)\sin\alpha\pm[c^2-\{(a-h)\sin\alpha-(b-k)\cos\alpha\}^2]^{\frac{1}{2}}$ and $\therefore$ the condition is $\{(a-h)\sin\alpha-(b-k)\cos\alpha\}^2=c^2$.
24. $\{x-(a^2+2ab\cos\alpha+b^2)^{\frac{1}{2}}\}^2+y^2=b^2$.
28. $(x-a)^2+(y-b)^2=(h-a)^2+(k-b)^2$, where $(a^2+b^2)^{\frac{1}{2}}-\{(a-h)^2+(b-k)^2\}^{\frac{1}{2}}=c^2$.

EXAMPLES ON LOCI. P. 93.

1. Take the middle point of BC as origin, BC as axis of x. (i) $4ax=d^2$, (ii) $(1-m^2)(x^2+y^2)+2(1+m^2)ax+(1-m^2)a^2=0$, (iii) $2(x^2+y^2+a^2)=c^2$. 2. A straight line parallel to the loci of B and C. 3. Take D as origin, DE as axis of x, and AB parallel to that of y; let $AB=2b$, $DE=a$; the locus is

$2bx = a(y+b)$. 4. The straight line which bisects the angle BAC. 5. Take ADB, AEC as axes, the locus is $mx+y=\frac{ml}{m+1}$. 7. If the fixed circles be $x^2+y^2=c^2$, $(x-a)^2+y^2=c'^2$, the locus is the circle $x^2+y^2-c^2=\mu\{(x-a)^2+y^2-c'^2\}$.

Ex. 15. 3. $4(2\pm\sqrt{3})a$. 8. $x_1=\frac{\sqrt{5}-2}{4}a$, $x_2=\frac{\sqrt{5}+2}{4}a$, $y_1^2=(\sqrt{5}-2)a^2$, $y_2^2=(\sqrt{5}+2)a^2$.

Ex. 16. 1. $x=(x_1x_2)^{\frac{1}{2}}$, $y=\frac{y_1+y_2}{2}$. 2. $\frac{a}{m(1+m^2)^{\frac{1}{2}}}$, $\frac{a(1+m^2)^{\frac{1}{2}}}{m}$. 8. The tangents intersect in the point $\left\{\frac{a}{mm'}, a\left(\frac{1}{m}+\frac{1}{m'}\right)\right\}$; the polar of this point is $(m+m')y=2(mm'x+a)$; the normals intersect in the point for which $x=a\left(2+\frac{1}{m^2}+\frac{1}{mm'}+\frac{1}{m'^2}\right)$, and $y=\frac{m+m'}{m^2m'^2}a$.
10. The equation to the two tangents is

$$[\{(c^2+4a^2)^{\frac{1}{2}}-c\}x-2ac]^2=2\{(c^2+4a^2)^{\frac{1}{2}}-c\}cy^2.$$

11. $x=2a+x_1+(x_1x_2)^{\frac{1}{2}}+x_2$, $y=\frac{y_1y_2(y_1+y_2)}{8a^2}$.

12. The coordinates of p are $\frac{(x_1+2a)^2}{x_1}$, $-\left(y_1+\frac{8a^2}{y_1}\right)$, and the length of Pp is $\frac{2\{a(a+x_1)^3\}^{\frac{1}{2}}}{x_1}$.

EXAMPLES ON CHAPTER V.

4. $x=2(c-2a)$, and $2ax^2=(c-2a)y^2$. 22. $y=ak$. 23. $y^2-4ax=k^2a^2$. 24. $kx=a$. 25. $(x-a)^2+y^2=\frac{a^2}{k^2}$. 27. $y^2=a(x-3a)$. 29. Let S be the given focus, SY the perpendicular from it on the given line; the locus is the circle described on SY as diameter. 30. Let $SQ=r$, $QSX=\theta$; the locus is $r=2a\cot\frac{\theta}{2}$. 31. With the same notation as in qu. 30, $r=4a\operatorname{cosec}^2\theta$. 32, 33. Straight lines through the extremity of the given diameter. 36—41. Parabolas.

Ex. 17. 1. (i) $\frac{1}{\sqrt{2}}$, $\sqrt{2}$; (ii) $\frac{1}{2}$, 3; (iii) $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$; (iv) $\frac{25}{12}$, $\frac{15}{8}$. 2. 2; 2; $\sqrt{2}$; $\frac{125}{18}$. 3. $\frac{8}{3}l$, $\frac{4}{\sqrt{3}}l$.
4. $\sqrt{2}c$, $\sqrt{2}c$, $x^2-y^2=\frac{c^2}{2}$. 5. $\frac{1}{\sqrt{2}}$. 6. $\frac{\sqrt{13}\pm 1}{2\sqrt{3}}$.
7. In 6 the curve may be a hyperbola. 9. If θ be the inclination required, (i) $\tan^2\theta=\frac{b^2}{a^2}$; (ii) $\tan^2\theta=\frac{b}{a}$; (iii) $\tan^2\theta=1$. 10. If $2b$ be the given length, a the given inclination, $a^2=\frac{1-e^2\cos^2 a}{1-e^2}l^2$, $b^2=(1-e^2\cos^2 a)\,l^2$.
11. If β be the given inclination, $a^2=\frac{b^2-l^2\cos^2\beta}{l^2\sin^2\beta}\,.\,b^2$.

Ex. 18. 1. (i) If θ be the angle of inclination to the axis of x, $\frac{c^2}{2}$ the area of the given triangle, $a^2\tan\theta\pm b^2\cot\theta+c^2=0$; (ii) In the ellipse $\frac{x}{a}+\frac{y}{b}=\pm\sqrt{2}$; in the hyperbola, the asymptote $\frac{x}{a}+\frac{y}{b}=0$; (iii) $(x\pm y)^2=a^2\pm b^2$; (iv) $(x\cos\theta+y\sin\theta)^2=a^2\cos^2\theta\pm b^2\sin^2\theta$, where $h\cos\theta+k\sin\theta=\delta$.
2. In the equation $x\cos\theta+y\sin\theta=(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}$, $(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}(\sec\theta+\operatorname{cosec}\theta)=l$. 3. In the same equation $(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}(\sec\theta+\operatorname{cosec}\theta+\sec\theta\operatorname{cosec}\theta)=2s$.
4. In the same equation $e^2\cos^2\theta=2e^2(1+m^2)-(1-m)^2$.
7. $x+\sqrt{2}y=\sqrt{6}$. 8. $(1-e^2)^{\frac{1}{2}}x+ey=(1-e^4)^{\frac{1}{2}}a$.

$$14.\quad PG=b\left\{\frac{1-e^2(2-e^2)\cos^2\theta}{1-e^2\cos^2\theta}\right\}^{\frac{1}{2}},$$

$$CK=\frac{(1-e^2)\,b\cos\theta\sin\theta}{\{(1-e^2\cos^2\theta)\,(1-e^2(2-e^2)\cos^2\theta)\}^{\frac{1}{2}}}.$$

15. $\frac{2\,(a^4y_1^2+b^4x_1^2)^{\frac{3}{2}}}{a^6y_1^2+b^6x_1^2}$. 16. $bcy\pm\{(a^2-b^2)^2-a^2c^2\}^{\frac{1}{2}}(x-c)=0$; $acx\pm\{(a^2-b^2)^2-b^2c^2\}^{\frac{1}{2}}y=0$. 17. $\frac{e^2}{(1-e^2)^{\frac{1}{2}}}(a^2-e^2x_1^2)^{\frac{1}{2}}$.

Ex. 19. 1. $b^2x+a^2y=0$; $b^2x=a^2y$; $a^3y+b^3x=0$; $ay+bx=0$; $b^2x\cos\theta+a^2y\sin\theta=0$; $ay=bx$. 3. $(a^2-c^2)^{\frac{1}{2}}y=\pm(c^2-b^2)^{\frac{1}{2}}x$.
5. $(3a+b)^{\frac{1}{2}}y=\pm(3b+a)^{\frac{1}{2}}x$. 7. The same as in qu. 6.

Ex. 20. 1. $(b^2\cos^2\phi+a^2\sin^2\phi)^{\frac{1}{2}}\tan\phi$, $\frac{b}{a}(b^2\cos^2\phi+a^2\sin^2\phi)^{\frac{1}{2}}$.

2. $\frac{a(1-e^2)^{\frac{1}{2}}}{(1-e^2\cos^2\phi)^{\frac{1}{2}}}$, $\frac{b(1\mp e\cos\phi)^{\frac{1}{2}}}{(1\pm e\cos\phi)^{\frac{1}{2}}}$, $\frac{ae^2\sin 2\phi}{2(1-e^2\cos^2\phi)^{\frac{1}{2}}}$,

$$\frac{ae\sin\phi(1\mp e\cos\phi)^{\frac{1}{2}}}{(1\pm e\cos\phi)^{\frac{1}{2}}}.$$

3. $\tan\phi=\frac{(1-e)^{\frac{1}{2}}}{e}$. 4. $PQ=\frac{2ab(b^2\cos^2\phi+a^2\sin^2\phi)^{\frac{3}{2}}}{b^4\cos^2\phi+a^4\sin^2\phi}$,

$\tan\phi'=\frac{1-e^4\cos^2\phi}{1-2e^2+e^4\cos^2\phi}\tan\phi$. 6. $\frac{y\cos\phi}{b}=1+\frac{x\sin\phi}{a}$.

7. $(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}$, $(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{2}}$.

9. $\cos^2\phi=\frac{c^2-b^2}{a^2-b^2}$, $(a^2-c^2)^{\frac{1}{2}}y=\pm(c^2-b^2)x$.

EXAMPLES ON CHAPTER VI.

13. $(mx-y)(m^2b^2+a^2)^{\frac{1}{2}}=m(a^2-b^2)^{\frac{1}{2}}$

EXAMPLES ON CONICS.

2. Two straight lines; a parabola; the transformed equation is $2x^2+y^2=2$; a rectangular hyperbola; a hyperbola; the two straight lines $2x+y=1$, $2y+x=1$; the point of intersection of these two lines.

15. $\frac{b\lambda}{b^2-ac}$, $\frac{-a\lambda}{b^2-ac}$; $ax+by=0$, $(b^2-ac)y+a\lambda=0$.

16. $(AC-B)d^2+2\{(CD-EB)l+(EA-BD)m\}d$
$+(m^2A-2mlB+Cl^2)F=(mD-lE)^2$.

17. $(x-x_1)(Bx_1+Cy_1+E)=(y-y_1)(Ax_1+By_1+D)$.

18. $(Ax+By+D)b=(Bx+Cy+E)a$. 19. $(A+B)m=(B+C)l$.

88. $2a-h$, $-k$; $a\left(1-\frac{k}{b}\right)$, $\frac{b}{a}(h-a)$; $a\left(1+\frac{k}{b}\right)$,
$\frac{-b}{a}(h-a)$; $\frac{\sqrt{5}-1}{2}$.

EXAMPLES ON CHAPTER VIII.

4. A conic; (see Art. 158). The conic becomes a circle, if $l=-1$, $\alpha+\beta=\pi+\gamma+\delta$. 6. $a\alpha=b\beta+c\gamma$, &c.
7. $\alpha\cos A=\beta\cos B+\gamma\cos C$, &c. 8. $(ma-lb)\beta+(na-lc)\gamma=0$.
9. $(lb-ma)y+(lc-na)x=lbc$.

10. $$\frac{l\alpha_1+m\beta_1+n\gamma_1}{\{l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C\}^{\frac{1}{2}}}.$$

11. $\alpha\cos B+\beta\cos C+\gamma\cos A=\alpha\cos C+\beta\cos A+\gamma\cos B$.
12. $a\alpha=b\beta+c\gamma$ represents the straight line joining the middle points of AB, AC; $\alpha+\beta+\gamma=0$ the straight line through the three points where the bisectors of the external angles meet the opposite sides; this proves that these three points are collinear; let $AD\perp BC$, in BC produced take D' such that $BD':DC::BD:DC$, or such that BD, BC, BD' are in A. P.; in CA and AB take similar points E', F'; then $D'E'F'$ is a straight line, whose equation is $\alpha\cos A+\beta\cos B+\gamma\cos C=0$.

16. $\frac{\alpha'+\alpha''}{2}$, $\frac{\beta'+\beta''}{2}$, $\frac{\gamma'+\gamma''}{2}$. 17. $a\alpha-b\beta=\frac{mb-la}{n}\gamma$;

$$\frac{\alpha}{l(la-mb-nc)}=\frac{\beta}{m(mb-nc-la)}=\frac{\gamma}{n(nc-la-mb)}.$$

18. $l^2a^2+m^2b^2+n^2c^2-2mnbc-2nlca-2lmab=0$.

19. $\frac{\beta}{lc+na}=\frac{\gamma}{la+mb}$; the coordinates of the centre satisfy the equations $a\alpha+b\beta+c\gamma=2S$,

$$\frac{\alpha}{nb+mc}=\frac{\beta}{lc+na}=\frac{\gamma}{la+mb}.$$

20. $\frac{l}{a}+\frac{m}{b}+\frac{n}{c}=0$.

THE END.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

A LIST

OF

EDUCATIONAL BOOKS

FOR PRIVATE AND

UNIVERSITY STUDENTS,

ALSO

FOR USE IN PUBLIC SCHOOLS,

GRAMMAR SCHOOLS,

AND

MIDDLE CLASS, ELEMENTARY, AND

NATIONAL SCHOOLS.

PUBLISHED BY

GEORGE BELL AND SONS,

YORK STREET, COVENT GARDEN,

LONDON, W.C.

DEIGHTON, BELL AND CO., CAMBRIDGE.

PLATO. By W. H. Thompson, D.D., Master of Trinity College, Cambridge.
Vol. I., "Phaedrus," 7*s*. 6*d*. Vol. II., "Gorgias," 7*s*. 6*d*.

SOPHOCLES. By Rev. F. H. Blaydes, M.A. Vol. I. 18*s*.
CONTENTS:—Œdipus Tyrannus. Œdipus Coloneus. Antigone.

TACITUS: The Annals. By the Rev. P. Frost. Price 15*s*.

TERENCE. By E. St. J. Parry, M.A. 18*s*.

VERGIL. By the late J. Conington, M.A. Corpus, Professor of Latin at Oxford. Vol. I., The Bucolics and Georgics. *Third edition*, 12*s*. Vol. II., The Æneid, Books I. to VI. *Second edition*, 14*s*. Vol. III., *Second edition*, 14*s*.

Uniform with above.

A COMPLETE LATIN GRAMMAR. By J. W. Donaldson, D. D. *Third edition*. 14*s*.

A COMPLETE GREEK GRAMMAR. By J. W. Donaldson, D. D. *Third edition*. 16*s*.

For other Critical and Annotated Editions see pages 71-73.

Grammar-School Classics.

A Series of Greek and Latin authors, with English notes, edited by eminent scholars; especially adapted for use in public and grammar schools. Foolscap 8vo.

CÆSAR DE BELLO GALLICO. Edited by George Long, M. A. *New edition*. 5*s*. 6*d*.

CÆSAR DE BELLO GALLICO. Books I.—III. With English notes for junior classes. By George Long, M. A. *New edition*. 2*s*. 6*d*.

CATULLUS, TIBULLUS, AND PROPERTIUS. Selected Poems. Edited by the Rev. A. H. Wratislaw, of Bury St. Edmunds School, and F. N. Sutton, B. A. With short biographical notices of the Poets. 3*s*. 6*d*.

CICERO: De Senectute, De Amicitia, and Select Epistles. Edited by George Long, M. A. *New edition*. 4*s*. 6*d*.

CORNELIUS NEPOS. Edited by the late Rev. J. F. Macmichael, Head Master of the Grammar School, Ripon. 2*s*. 6*d*.

HOMER: Iliad. Books I.—XII. Edited by F. A. Paley, M.A. 6*s*. 6*d*.

HORACE. Edited by A. J. Macleane, M. A,, late Head Master of King Edward's School, Bath. With a short Life. *New edition, revised.* 6*s*. 6*d*.

JUVENAL: Sixteen Satires (expurgated). By Herman Prior, M.A., late Scholar of Trinity College, Oxford. 4*s*. 6*d*.

MARTIAL: Select Epigrams. Edited by F. A. Paley, M. A., and the late W. H. Stone, Scholar of Trinity College, Cambridge. With a Life of the Poet. 6*s*. 6*d*.

OVID: The Six Books of the Fasti. Edited by F. A. Paley, M. A. *New edition.* 5*s*.

SALLUST: Catilina and Jugurtha. With a Life. Edited by George Long, M.A. 5*s*.

TACITUS: Germania and Agricola. Edited by the Rev. P. Frost, late Fellow of St. John's College, Cambridge. 3*s*. 6*d*.

VIRGIL: Bucolics, Georgics, and Æneid, Books I.—IV. By the Rev. J. G. Sheppard, D.C.L. Abridged from Professor Conington's edition. 5*s*. 6*d*.

—— Æneid, Books V.—XII. Abridged from Professor Connington's edition, by H. Nettleship, Fellow of Lincoln College, Oxford, and W. Wagner, Ph.D. 5*s*. 6*d*.

XENOPHON: The Anabasis. With Life, Itinerary, Index, and Three Maps. Edited by the late J. F. Macmichael, late Head Master of the Grammar School, Ripon. *New edition,* revised, corrected and enlarged. 5*s*.

—— The Cyropædia. Edited by G. M. Gorham, M. A., late Fellow of Trinity College, Cambridge. *New edition.* 6*s*.

—— Memorabilia. Edited by Percival Frost, M. A., late Fellow of St. John's College, Cambridge. 4*s*. 6*d*.

Uniform with the Series.

THE NEW TESTAMENT, in Greek. With English Notes and Preface, Synopsis, and Chronological Tables. By J. F. Macmichael, B. A. Fcap. 8vo. (730 pp.). 7s. 6d.

Cambridge Greek and Latin Texts.

AESCHYLUS. Ex novissima recensione F. A. Paley, A. M. 3*s*.

CÆSAR DE BELLO GALLICO. Recognovit G. Long, A. M. 2*s*.

CICERO de Senectute et de Amicitia, et Epistolæ Selectæ. Recensuit G. Long, A. M. 1*s*. 6*d*.

CICERONIS Orationes, Vol. I. (in Verrem). Ex recensione G. Long, A. M. 3*s*. 6*d*.

EURIPIDES. Ex recensione F. A. Paley, A.M. 3 vols. 3*s*. 6*d*. each.

CONTENTS OF VOL. I.—Rhesus. Medea. Hippolytus. Alcestis. Heraclidae. Supplices. Troades. Index.

CONTENTS OF VOL. II.—Ion. Helena. Andromache. Electra. Bacchae. Hecuba. Index.

CONTENTS OF VOL. III.—Hercules Furens. Phoenissae. Orestes. Iphigenia in Tauris. Iphigenia in Aulide. Cyclops. Index.

HERODOTUS. Recensuit J. G. Blakesley, S. T. B. 2 vols. 7*s*.

HOMERI ILIAS I.—XII. Ex novissima recensione F. A. Paley, A. M. 2*s*. 6*d*.

HORATIUS. Ex recensione A. J. Macleane, A.M. 2*s*. 6*d*.

JUVENAL ET PERSIUS. Ex recensione A. J. Macleane, A. M. 1*s*. 6*d*.

LUCRETIUS. Recognovit H. A. J. Munro, A.M. 2*s*. 6*d*.

SALLUSTI CRISPI CATILINA ET JUGURTHA, Recognovit G. Long, A. M. 1*s*. 6*d*.

TERENTI COMOEDIAE. Gul. Wagner relegit et emendavit. 3*s*.

THUCYDIDES. Recensuit J. G. Donaldson, S. T. P. 2 vols. 7*s*.

VERGILIUS. Ex recensione J. Conington, A.M. 3*s*. 6*d*.

XENOPHONTIS EXPEDITIO CYRI. Recensuit J. F. Macmichael, A. B. 2*s*. 6*d*.

NOVUM TESTAMENTUM GRAECUM, Textus Stephanici, 1550. Accedunt variae lectiones editionum Bezae, Elzeviri, Lachmanni, Tischendorfii, Tregellesii, curante F. H. Scrivener, A.M. *New edition.* 4*s.* 6*d.* An Edition with wide margin for notes, 7*s.* 6*d.*

EURIPIDES. Alcestis. *School edition*, with Notes and Introduction. By F. A. Paley, M.A., abridged from the larger edition. Fcap. 8vo.

AESCHYLUS. Prometheus Vinctus. *School edition*, with Notes and Introduction. By F. A. Paley, M.A., abridged from the larger edition. Fcap. 8vo. [*Preparing.*

Public School Series.

A Series of Classical Texts, annotated by well-known scholars, with a special view to the requirements of upper forms in Public Schools, or of University Students.

CICERO. The Letters of Cicero to Atticus, Bk. I. With Notes, and an Essay on the Character of the Writer. Edited by A. Pretor, M.A., late of Trinity College, Fellow of St. Catharine's College, Cambridge. Crown 8vo. 4*s.* 6*d.*

DEMOSTHENES de Falsa Legatione. *Fourth edition, carefully revised.* By R. Shilleto, M.A., Fellow of St. Peter's College, Cambridge. Crown 8vo. 6*s.*

—— The Oration Against the Law of Leptines. With English Notes and a Translation of Wolf's Prolegomena. By W. B. Beatson, M. A., Fellow of Pembroke College. Small 8vo. 6*s.*

PLATO. The Apology of Socrates and Crito, with Notes. critical and exegetical, by Wilhelm Wagner, Ph. D. Small 8vo. 4*s.* 6*d.*

—— The Phaedo. With Notes, critical and exegetical, and an Analysis, by Wilhelm Wagner, Ph. D. Small 8vo. 5*s.* 6*d.*

—— The Protagoras. The Greek Text revised, with an Analysis and English Notes. By W. Wayte, M.A., Fellow of King's College, Cambridge, and Assistant Master at Eton. 8vo. *Second edition.* 4*s.* 6*d.*

PLAUTUS. Trinummus. With Notes, critical and exegetical. By Wilhelm Wagner, Ph. D. Small 8vo. 4*s.* 6*d.*

SOPHOCLIS TRACHINIÆ. With Notes and Prolegomena. By Alfred Pretor, M.A., Fellow of St. Catherine's College, Cambridge. [*Preparing.*

TERENCE. With Notes, critical and explanatory. By Wilhelm Wagner, Ph. D. Post 8vo. 10*s.* 6*d.*

THEOCRITUS. With Short Critical and Explanatory Latin Notes. By F. A. Paley, M. A. *Second edition*, corrected and enlarged, and containing the newly discovered Idyll. Crown 8vo. 4*s.* 6*d.*

Others in preparation.

Critical and Annotated Editions.

AETNA. Revised, emended and explained by H. A. J. Munro, M.A., Fellow of Trinity College, Cambridge. 8vo. 3*s.* 6*d.*

ARISTOPHANIS Comoediae quae supersunt cum perditarum fragmentis tertiis curis recognovit additis adnotatione critica summariis descriptione metrica onomastico lexico Hubertus A. Holden, LL. D. [Head-Master of Ipswich School; Classical Examiner, University of London.] 8vo.

Vol. I. containing the Text expurgated with Summaries and critical notes, 18*s.*

The Plays sold separately; Acharnenses, 2*s.* Equites, 1*s.* 6*d.* Nubes, 1*s.* 6*d.* Vespae, 2*s.* Pax, 1*s.* 6*d.* Aves, 2*s.* Lysistrata, et Thesmophoriazusae, 3*s.* Ranae, 2*s.* Ecclesiazusae et Plutus, 3*s.*

Vol. II. Onomasticon Aristophaneum continens indicem geographicum et historicum. 5*s.* 6*d.*

—— Pax, with an Introduction and English Notes. By F. A. Paley, M. A. Fcap. 8vo. 4*s.* 6*d.*

EURIPIDES. Fabulæ Quatuor. Scilicet, Hippolytus Coronifer, Alcestis, Iphigenia in Aulide, Iphigenia in Tauris. Ad fidem Manuscriptorum ac veterum Editionum emendavit et Annotationibus instruxit J. H. Monk, S.T.P. *Editio nova.* 8vo. Crown 8vo. 12*s.*

Separately—Hippolytus Cloth, 5*s.* Alcestis. Sewed, 4*s.* 6*d.*

HORACE. Quinti Horatii Flacci Opera. The text revised, with an Introduction, by H. A. J. Munro, M.A., Fellow of Trinity College, Cambridge, Editor of "Lucretius." Illustrated from antique gems, by C. W. King, M. A., Fellow of Trinity College, Cambridge. Large 8vo. 1*l.* 1*s.*

LIVY. Titi Livii Historiæ. The first five Books, with English Notes. By J. Prendeville. *Eighth edition.* 12mo. roan, 5*s.* Or Books I. to III. 3*s.* 6*d.* IV. and V. 3*s.* 6*d.*

LUCRETIUS. Titi Lucreti Cari de Rerum Natura Libri Sex. With a Translation and Notes. By H. A. J. Munro, M.A., Fellow of Trinity College, Cambridge. *Third edition revised throughout.* 2 vols. 8vo. Vol. I. Text, 16*s.* Vol. II. Translation, 6*s.* Sold separately.

OVID. P. Ovidii Nasonis Heroides XIV. Edited, with Introductory Preface and English Notes, by Arthur Palmer, M.A., Fellow of Trinity College, Dublin. Demy 8vo. 6*s.*

PLAUTUS. Aulularia. With Notes, critical and exegetical, and an Introduction on Plautian Prosody. By Wilhelm Wagner, Ph.D. 8vo. 9*s.*

PROPERTIUS. Sex. Aurelii Propertii Carmina. The Elegies of Propertius, with English Notes. By F. A. Paley, M.A., Editor of "Ovid's Fasti," "Select Epigrams of Martial," &c. *Second edition.* 8vo. cloth. 9*s.*

THUCYDIDES. The History of the Peloponnesian War. With Notes and a careful Collation of the two Cambridge Manuscripts, and of the Aldine and Juntine Editions. By Richard Shilleto, M.A., Fellow of S. Peter's College, Cambridge. Book I. 8vo. 6*s.* 6*d.* Book II. *in the press.*

GREEK TESTAMENT. With a Critically revised Text; a digest of Various Readings; Marginal references to verbal and idiomatic usage; Prolegomena; and a critical and exegetical commentary. For the use of theological students and ministers. By Henry Alford, D.D., late Dean of Canterbury. 4 vols. 8vo. Sold separately.

Vol. I., *Seventh Edition,* the Four Gospels. 1*l.* 8*s.* Vol. II., *Sixth Edition,* the Acts of the Apostles, Epistles to the Romans and Corinthians. 1*l.* 4*s.*—Vol. III., *Fifth Edition,* the Epistles to the Galatians, Ephesians, Philippians, Colossians, Thessalonians,—to Timotheus, Titus, and Philemon. 18*s.*—Vol. IV. Part I., *Fourth edition.* The Epistle to the Hebrews: The Catholic Epistles of St. James and St. Peter. 18*s.*—Vol. IV. Part II., *Fourth edition.* The Epistles of St. John and St. Jude, and the Revelation. 14*s.* Or Vol. IV. in one Vol. 32*s.*

A LATIN GRAMMAR. By T. Hewitt Key, M.A., F. R. S., Professor of Comparative Grammar, and Head Master of the Junior School, in University College. *Sixth Thousand, with new corrections and additions.* Post 8vo. 8*s.*

Latin and Greek Class Books.

CHURCH (A. J.) Latin Prose Lessons. By Alfred J. Church, M. A., Head Master of the Grammar School, Retford. *A new edition.* Fcap. 8vo. 2*s.* 6*d.*

DAVIS and BADDELEY. Scala Græca: a Series of Elementary Greek Exercises. By the Rev. J. W. Davis, M. A., Head Master of Tamworth Grammar School, and R. W. Baddeley, M. A. *Second edition.* Fcap. 8vo. cloth. 2*s.* 6*d.*

FROST (P.) Eclogæ Latinæ: or First Latin Reading Book, with English Notes and a Dictionary. By the Rev. P. Frost, M. A., St. John's College, Cambridge. *New edition.* Fcap. 8vo. 2*s.* 6*d.*

—— Materials for Latin Prose Composition. *New edition.* Fcap. 8vo. 2*s.* 6*d.*

——— Key. 4*s.*

—— A Latin Verse Book. An Introductory Work on Hexameters and Pentameters. Fcap. 8vo. 3*s.* *New edition, revised and enlarged.*

——— Key. 5*s.*

—— Analecta Græca Minora, with Introductory Sentences, English Notes, and a Dictionary. *New edition, revised and enlarged.* Fcap. 8vo. 3*s.* 6*d.*

—— Materials for Greek Prose Composition. Constructed on the same plan as the "Materials for Latin Prose Composition." Fcap. 8vo. 3*s.* 6*d.*

——— Key. 5*s.*

GRETTON (F. E.) A First Cheque Book for Latin Verse Makers. By the Rev. F. E. Gretton, Head Master of Stamford Free Grammar School, sometime Fellow of St. John's College, Cambridge. 1*s.* 6*d.*

——— A Latin Version for Masters. 2*s.* 6*d.*

—— Reddenda; or Passages with Parallel Hints for Translation into Latin Prose and Verse. Crown 8vo. 4*s.* 6*d.*

GRETTON (F. E.) Reddenda Reddita; or Passages from English Poetry with a Latin Verse Translation. Crown 8vo. 6*s.*

HOLDEN (H. A.) Foliorum Silvula. Part I. Being Passages for Translation into Latin Elegiac and Heroic Verse, edited by Hubert A. Holden, LL.D., late Fellow of Trinity College, Head Master of Queen Elizabeth's School, Ipswich. *Sixth edition.* Post 8vo. 7*s.* 6*d.*

—— Foliorum Silvula. Part II. Being Select Passages for Translation into Latin Lyric and Comic Iambic Verse. By Hubert A. Holden, LL.D. *Third edition.* Post 8vo. 5*s.*

—— Foliorum Silvula. Part III. Being Select Passages for Translation into Greek Verse, edited with Notes by Hubert A. Holden, LL. D. *Third edition.* Post 8vo. 8*s.*

—— Folia Silvulæ, sive Eclogæ Poetarum Anglicorum in Latinum et Græcum conversæ quas disposuit Hubertus A. Holden, LL.D. Volumen Prius continens Fasciculos I. II. 8vo. 10*s.* 6*d.* Volumen Alterum continens Fasciculos III. IV. 8vo. 12*s.*

—— Foliorum Centuriæ. Selections for Translation into Latin and Greek Prose, chiefly from the University and College Examination Papers. By Hubert A. Holden, LL. D. *Fifth edition.* Post 8vo. 8s.

KEY (T. H.) A Short Latin Grammar, for Schools. By T. H. Key, M. A., F. S. A., Professor of Comparative Grammar in University College, London, and Head Master of the School. *Eighth edition.* Post 8vo. 3*s.* 6*d.*

MACLEANE (A. J.) Selections from Ovid: Amores, Tristia, Heroides, Metamorphoses. With English Notes, by the Rev. A. J. Macleane, M. A. *New edition.* Fcap. 1*s.* 6*d.*

MASON (C. P.) Analytical Latin Exercises; Accidence and Simple Sentences, Composition and Derivation of Words and Compound Sentences. By C. P. Mason, B. A., Fellow of University College, London. Post 8vo. 3*s.* 6*d.*

PRESTON (G.) Greek Verse Composition, for the use of Public Schools and Private Students. Being a revised edition of the "Greek Verses of Shrewsbury School." By George Preston, M. A., Fellow of Magdalene College. Crown 8vo. 4*s.* 6*d.*

Classical Tables.

GREEK VERBS. A Catalogue of Verbs, Irregular and Defective; their leading formations, tenses in use, and dialectic inflexions, with a copious Appendix, containing Paradigms for conjugation, Rules for formation of tenses, &c. &c. By J. S. Baird, T. C. D. *New edition, revised.* 2*s*. 6*d*.

GREEK ACCENTS (Notes on). On Card, 6*d*.

HOMERIC DIALECT. Its Leading Forms and Peculiarities. By J. S. Baird, T. C. D. 1*s*. 6*d*.

GREEK ACCIDENCE. By the Rev. P. Frost, M.A. 1*s*.

LATIN ACCIDENCE. By the Rev. P. Frost, M.A. 1*s*.

LATIN VERSIFICATION. 1*s*.

NOTABILIA QUÆDAM: or the Principal Tenses of most of the Irregular Greek Verbs and Elementary Greek, Latin, and French Constructions. *New edition.* 1*s*. 6*d*.

RICHMOND RULES TO FORM THE OVIDIAN *DISTICH*, &c. By J. Tate, M. A. *New edition.* 1*s*. 6*d*.

THE PRINCIPLES OF LATIN SYNTAX. 1*s*.

Translations, Selections, and Illustrated Editions.

*** Many of the following books are well adapted for school prizes. See also pages 87-90.

AESCHYLUS. Translated into English Prose, by F.A. Paley, M. A., Editor of the Greek Text. *Second edition, revised.* 8vo. 7*s*. 6*d*.

—— Translated by Anna Swanwick. With Introductions and Notes. *New edition.* Crown 8vo. 2 vols. 12*s*.

—————— Folio Edition with Thirty-three Illustrations from Flaxman's designs. Price £2 2*s*.

ANTHOLOGIA GRÆCA. A Selection of Choice Greek Poetry, with Notes. By Rev. F. St. John Thackeray, Assistant Master, Eton College. *New edition, corrected.* Fcap. 8vo. 7*s*. 6*d*.

ANTHOLOGIA LATINA. A Selection of Choice Latin Poetry, from Nævius to Boëthius, with Notes. By Rev. F. St. John Thackeray, Assistant Master, Eton College. *New edition, enlarged.* Fcap. 8vo. 6*s*. 6*d*.

ARISTOPHANES, The Peace. The Greek text, and a metrical translation on opposite pages, with notes and introduction, &c. By Benj. Bickley Rogers, M.A., late fellow of Wadham College, Oxford. Fcap. 4to. 7*s*. 6*d*.

—— The Wasps. Text and metrical translation, with notes and introduction. By Benj. B. Rogers, M.A. Fcap. 8vo. 7*s*. 6*d*. [*In the Press.*

CORPUS POETARUM LATINORUM. Edited by Walker. 1 thick vol. 8vo. Cloth, 18*s*.

Containing:—CATULLUS, LUCRETIUS, VIRGILIUS, TIBULLUS, PROPERTIUS, OVIDIUS, HORATIUS, PHAEDRUS, LUCANUS, PERSIUS, JUVENALIS, MARTIALIS, SULPICIA, STATIUS, SILIUS ITALICUS, VALERIUS FLACCUS, CALPURNIUS SICULUS, AUSONIUS and CLAUDIANUS.

HORACE. The Odes and Carmen Sæculare. Translated into English Verse by the late John Conington, M. A., Corpus Professor of Latin in the University of Oxford. *Fifth edition.* Fcap. 8vo. 5*s*. 6*d*.

—— The Satires and Epistles. Translated in English Verse by John Conington, M. A. *Third edition.* 6*s*. 6*d*.

—— Illustrated from Antique Gems by C. W. King, M. A., Fellow of Trinity College, Cambridge. The text revised with an Introduction by H. A. J. Munro, M. A., Fellow of Trinity College, Cambridge, Editor of Lucretius. Large 8vo. £1 1*s*.

MVSÆ ETONENSES sive Carminvm Etonæ Conditorvm Delectvs. Series Nova, Tomos Dvos complectens. Edidit Ricardvs Okes, S. T. P. Coll. Regal. apvd Cantabrigienses Præpositvs. 8vo. 15*s*.

Vol. II., to complete Sets, may be had separately, price 5*s*.

PROPERTIUS. Verse translations from Book V. with a revised Latin Text and brief English notes. By F. A. Paley, M. A. Fcap. 8vo. 3*s*.

PLATO. Gorgias, literally translated, with an Introductory Essay containing a summary of the argument. By the late E. M. Cope, M. A., Fellow of Trinity College. 8vo. 7*s*.

—— Philebus. Translated with short Explanatory Notes by F. A. Paley, M. A. Small 8vo. 4*s*.

—— Theaetetus. Translated with an Introduction on the subject-matter, and short explanatory notes. By F. A. Paley, M.A. Small 8vo. 4*s*.

PLATO. An Analysis and Index of the Dialogues. With References to the Translation in Bohn's Classical Library. By Dr. Day. Post 8vo. 5*s.*

REDDENDA REDDITA; or, Passages from English Poetry with a Latin Verse Translation. By F. E. Gretton, Head Master of Stamford Free Grammar School. Crown 8vo. 6*s.*

SABRINÆ COROLLA in hortulis Regiae Scholae Salopiensis contexuerunt tres viri floribus legendis. *Editio tertia.* 8vo. 8*s.* 6*d.*

SERTUM CARTHUSIANUM Floribus trium Seculorum Contextum. Cura Gulielmi Haig Brown, Scholæ Carthusianæ Archididascali. 8vo. 14*s.*

THEOCRITUS. Translated into English Verse by C. S. Calverley, M.A., late Fellow of Christ's College, Cambridge. Crown 8vo. 7*s.* 6*d.*

TRANSLATIONS into English and Latin. By C. S. Calverley, M.A., late Fellow of Christ's College, Cambridge. Post 8vo. 7*s.* 6*d.*

—— Into Greek and Latin Verse. By R. C. Jebb, Fellow of Trinity College and Public Orator in the University of Cambridge. 4to. cloth gilt. 10*s.* 6*d.*

VIRGIL in English Rhythm. With Illustrations from the British Poets, from Chaucer to Cowper. By the Rev. Robert Corbet Singleton, first Warden of S. Peter's College, Radley. A manual for master and scholar. *Second edition, re-written and enlarged.* Large crown 8vo. 7*s.* 6*d.*

A HISTORY OF ROMAN LITERATURE. By W. S. Teuffel, Professor at the University of Tübingen. Translated, with the Author's sanction, by Wilhelm Wagner, Ph. D., of the Johanneum, Hamburg. Two vols. Demy 8vo. 21*s.*

"Professor Teuffel skilfully groups the various departments of Roman literature according to periods and according to subjects, and he well brings out the leading characteristics of each." —*Saturday Review.*

THE THEATRE OF THE GREEKS. A Treatise on The History and Exhibition of the Greek Drama, with a Supplementary Treatise on the Language, Metres, and Prosody of the Greek Dramatists, by John William Donaldson, D.D., formerly Fellow of Trinity College, Cambridge. With numerous Illustrations from the best ancient authorities. *Eighth edition. Post 8vo.* 5*s.*

MATHEMATICAL AND OTHER CLASS BOOKS.

Cambridge School and College Text Books.

A Series of Elementary Treatises adapted for the use of students in the Universities, Schools, and candidates for the Public Examinations. Uniformly printed in Foolscap 8vo.

ARITHMETIC. By Rev. C. Elsee, M.A., late Fellow of St. John's College, Cambridge; Assistant Master at Rugby. Intended for the use of Rugby School. Fcap. 8vo.. *Fifth edition.* 3*s.* 6*d.*

ELEMENTS OF ALGEBRA. By the Rev. C. Elsee, M. A. *Second edition, enlarged.* 4*s.*

ARITHMETIC. For the Use of Colleges and Schools. By A. Wrigley, M. A. 3*s.* 6*d.*

AN INTRODUCTION TO PLANE ASTRONOMY. For the Use of Colleges and Schools. By P. T. Main, M. A., Fellow of St. John's College. *Second edition.* 4*s.*

ELEMENTARY CONIC SECTIONS treated Geometrically. By W. H. Besant, M. A., Lecturer and late Fellow of St. John's College. 4*s.* 6*d.*

ELEMENTARY STATICS. By the Rev. Harvey Goodwin, D. D., Bishop of Carlisle. *New edition, revised.* 3*s.*

ELEMENTARY DYNAMICS. By the Rev. Harvey Goodwin, D. D., Bishop of Carlisle. *Second edition.* 3*s.*

ELEMENTARY HYDROSTATICS. By W. H. Besant, M. A., late Fellow of St. John's College. *Sixth edition.* 4*s.*

AN ELEMENTARY TREATISE ON MENSURATION. By B. T. Moore, M. A., Fellow of Pembroke College. With numerous Examples. 5*s.*

THE FIRST THREE SECTIONS OF NEWTON'S PRINCIPIA, with an Appendix; and the Ninth and Eleventh Sections. By John H. Evans, M.A. The Fifth Edition, edited by P. T. Main, M. A. 4*s.*

ELEMENTARY TRIGONOMETRY. With a Collection of Examples. By T. P. Hudson, M. A., Fellow of Trinity College. 3*s*. 6*d*.

GEOMETRICAL OPTICS. By W. S. Aldis, M. A., Trinity College. 3*s*. 6*d*.

COMPANION TO THE GREEK TESTAMENT. Designed for the Use of Theological Students and the Upper Forms in Schools. By A. C. Barrett, A. M., Caius College. *Third edition, revised and enlarged.* Fcap. 8vo. 5*s*.

AN HISTORICAL AND EXPLANATORY TREATISE ON THE BOOK OF COMMON PRAYER. By W. G. Humphry, B. D. *Fifth edition revised.* Fcap. 8vo. 4*s*. 6*d*.

MUSIC. By H. C. Banister, Professor of Harmony and Composition at the Royal Academy of Music. *Third edition, revised.* 5*s*.

Others in Preparation.

Arithmetic and Algebra.

HIND (J.) Principles and Practice of Arithmetic. Comprising the Nature and Use of Logarithms, with the Computations employed by Artificers, Gaugers, and Land Surveyors. Designed for the Use of Students, by J. Hind, M. A., formerly Fellow and Tutor of Sidney Sussex College. *Ninth edition*, with Questions. 4*s*. 6*d*.

A Second Appendix of Miscellaneous Questions (many of which have been taken from the Examination Papers given in the University during the last few years) has been added to the present edition of this work, which the Author considers will conduce greatly to its practical utility, especially for those who are intended for mercantile pursuits.

*** Key, with Questions for Examination. *Second edition.* 5*s*.

—— Elements of Algebra. Designed for the Use of Students in Schools and in the Universities. By J. Hind, M.A. *Sixth edition, revised.* 540 pp. 8vo. 10*s*. 6*d*.

WATSON (J.) A Progressive Course of Examples in Arithmetic. With Answers. By James Watson, M. A., of Corpus Christi College, Cambridge, and formerly Senior Mathematical Master of the Ordnance School, Carshalton. *Second edition, revised and corrected.* Fcap. 8vo. 2*s*. 6*d*.

Geometry and Euclid.

ALDIS (T. S.) Text Book of Geometry. By T. S. Aldis, M. A., Trinity College, Cambridge. Small 8vo. 4*s*. 6*d*. Part I.—Angles—Parallels—Triangles—Equivalent Figures—Circles. 2*s*. 6*d*. Part II. Proportion. *Just published*. 2*s*. Sold separately.

The object of the work is to present the subject simply and concisely, leaving illustration and explanation to the teacher, whose freedom text-books too often hamper. Without a teacher, however, this work will possibly be found no harder to master than others.

As far as practicable, exercises, largely numerical, are given on the different Theorems, that the pupil may learn at once the value and use of what he studies.

Hypothetical constructions are throughout employed. Important Theorems are proved in more than one way, lest the pupil rest in words rather than things. Problems are regarded chiefly as exercises on the theorems.

Short Appendices are added on the Analysis of Reasoning and the Application of Arithmetic and Algebra to Geometry.

EUCLID. The Elements of Euclid. A new Text based on that of Simson, with Exercises. Edited by H. J. Hose, formerly Mathematical Master of Westminster School. Fcap. 8vo. 4*s*. 6*d*. Exercises separately, 1*s*.

CONTENTS:—Books I.—VI.; XI. 1—21; XII. 1, 2.

—— The Elements of Euclid. The First Six Books, with a Commentary by Dr. Lardner. *Tenth edition*. 8vo. 6*s*.

—— The First Two Books Explained to Beginners; by C. P. Mason, B. A. *Second edition*. Fcap. 8vo. 2*s*. 6*d*.

—— The Enunciations and Figures belonging to the Propositions in the First Six and part of the Eleventh Books of Euclid's Elements (usually read in the Universities), prepared for Students in Geometry. By the Rev. J. Brasse, D. D. *New edition*. Fcap. 8vo. 1*s*. On cards, in case, 5*s*. 6*d*.

Without the Figures, 6*d*.

McDOWELL (J.) Exercises on Euclid and in Modern Geometry, containing Applications of the Principles and Processes of Modern Pure Geometry. By J. McDowell, B. A., Pembroke College. Crown 8vo. 8*s*. 6*d*.

BESANT (W. H.) Elementary Geometrical Conic Sections. By W. H. Besant, M. A. 4*s*. 6*d*.

TAYLOR (C.) The Geometry of Conics. *Second edition* revised and enlarged. By C. Taylor, M.A., Fellow of St. John's College. 8vo. 4*s.* 6*d.*

Pages 80 to 112 to complete the First *edition* may be had separately, price 1*s.*

GASKIN (T.) Solutions of the Geometrical Problems, consisting chiefly of Examples, proposed at St. John's College. from 1830 to 1846. With an Appendix on the General Equation of the Second Degree. By T. Gaskin, M. A. 8vo. 12*s.*

Trigonometry.

ALDOUS (J. C. P.) The Shrewsbury Trigonometry. A Step to the Study of a more complete treatise. By J. C. P. Aldous, Jesus College, Cambridge, Senior Mathematical Master of Shrewsbury School. 2*s.*

HUDSON (T. P.) Elementary Trigonometry. With a Collection of Examples. By T. P. Hudson, M. A., Fellow and Assistant Tutor of Trinity College. 3*s.* 6*d.*

HIND (J.) Elements of Plane and Spherical Trigonometry. With the Nature and Properties of Logarithms and Construction and Use of Mathematical Tables. Designed for the use of Students in the University. By J. Hind, M.A. *Fifth edition.* 12mo. 6*s.*

MOORE (B. T.) An Elementary Treatise on Mensuration. By B. T. Moore, M. A., Fellow of Pembroke College. With numerous Examples. 5*s.*

Analytical Geometry and Differential Calculus.

TURNBULL (W. P.) An Introduction to Analytical Plane Geometry. By W. P. Turnbull, M.A. Fellow of Trinity College. 8vo. 12*s.*

O'BRIEN (M.) Treatise on Plane Co-ordinate Geometry. Or the Application of the method of Co-ordinates to the solutions of problems in Plane Geometry. By M. O'Brien, M.A. 8vo. 9*s.*

VYVYAN (T. G.) Elementary Analytical Geometry for Schools and Beginners. By T. G. Vyvyan, Fellow of Gonville and Caius College, and Mathematical Master of Charterhouse. *Second edition, revised.* Crown 8vo. 7*s.* 6*d.*

WALTON (W.) Problems in illustration of the Principles of Plane Co-ordinate Geometry. By W. Walton, M.A. 8vo. 16*s*.

WHITWORTH (W. A.) Trilinear Co-ordinates, and other methods of Modern Analytical Geometry of Two Dimensions. An Elementary Treatise. By W. Allen Whitworth, M. A., Professor of Mathematics in Queen's College, Liverpool, and late Scholar of St. John's College, Cambridge. 8vo. 16*s*.

ALDIS (W. S.) An Elementary Treatise on Solid Geometry. *Revised.* By W. S. Aldis, M.A. *Second edition, revised.* 8vo. 8*s*.

GREGORY (D. F.) A Treatise on the Application of Analysis to Solid Geometry. By D. F. Gregory, M. A., and W. Walton, M. A. *Second edition.* 8vo. 12*s*.

PELL (M. B.) Geometrical Illustrations of the Differential Calculus. By M. B. Pell. 8vo. 2*s*. 6*d*.

O'BRIEN (M.) Elementary Treatise on the Differential Calculus, in which the method of Limits is exclusively made use of. By M. O'Brien, M. A. 8vo. 10*s*. 6*d*.

BESANT (W. H.) Notes on Roulettes and Glissettes. By W. H. Besant, M. A. 8vo. 3*s*. 6*d*.

Mechanics and Natural Philosophy.

EARNSHAW (S.) Treatise on Statics: Containing the Theory of the Equilibrium of Forces, and numerous Examples illustrative of the General Principles of the Science. By S. Earnshaw, M.A. *Fourth edition.* 8vo. 10*s*. 6*d*.

WHEWELL (Dr.) Mechanical Euclid. Containing the Elements of Mechanics and Hydrostatics. By the late W. Whewell, D. D. *Fifth edition.* 5*s*.

FENWICK (S.) The Mechanics of Construction; including the Theories of the Strength of Materials, Roofs, Arches, and Suspension Bridges. With numerous Examples. By Stephen Fenwick, F. R. A. S., of the Royal Military Academy, Woolwich. 8vo. 12*s*.

GARNETT (W.) A Treatise on Elementary Dynamics for the use of Colleges and Schools. By William Garnett, B. A. (late Whitworth Scholar), Fellow of St. John's College, and Demonstrator of Physics in the University of Cambridge. Crown 8vo. 6*s*.

GOODWIN (Bp.) Elementary Statics. By H. Goodwin, D. D., Bp. of Carlisle. Fcap. 8vo. *New edition.* 3*s*. cloth.

GOODWIN (BP.) Elementary Dynamics. By H. Goodwin, DD., Bishop of Carlisle. Fcap. 8vo. *second edition.* 3*s.* cloth.

WALTON (W.) Elementary Problems in Statics and Dynamics. Designed for Candidates for Honours, first three days. By W. Walton, M. A. 8vo. 10*s.* 6*d.*

POTTER (R.) An Elementary Treatise on Mechanics. For the use of Junior University Students. By Richard Potter, A. M., F. C. P. S., late Fellow of Queens' College, Cambridge. Professor of Natural Philosophy and Astronomy in University College, London. *Fourth edition, revised.* 8*s.* 6*d.*

—— Elementary Hydrostatics Adapted for both Junior University readers and Students of Practical Engineering. By R. Potter, M. A. 7*s.* 6*d.*

BESANT (W. H.) Elementary Hydrostatics. By W. H. Besant, M. A., late Fellow of St. John's College. Fcap. 8vo. *Sixth edition.* 4*s.*

—— A Treatise on Hydromechanics. By W. H. Besant, M. A. 8vo. *New edition in the press.*

GRIFFIN (W. N.) Solutions of Examples on the Dynamics of a Rigid Body. By W. N. Griffin, M. A. 8vo. 6*s.* 6*d.*

LUNN (J. R.) Of Motion. An Elementary Treatise. By J. R. Lunn, M.A., late Fellow of St. John's, Camb. 8vo. 7*s.* 6*d.*

BESANT (W. H.) A Treatise on the Dynamics of a Particle. *Preparing.*

ALDIS (W. S.) Geometrical Optics. By W. S. Aldis, M.A., Trinity College, Cambridge. Fcap. 8vo. 3*s.* 6*d.*

—— A Chapter on Fresnel's Theory of Double Refraction. By W. S. Aldis, M.A. 8vo. 2*s.*

POTTER (R.) An Elementary Treatise on Optics. Part I. Containing all the requisite Propositions carried to first Approximations; with the construction of optical instruments. For the use of Junior University Students. By Richard Potter, A. M., F. C. P. S., late Fellow of Queens' College, Cambridge. *Third edition, revised.* 9*s.* 6*d.*

—— An Elementary Treatise on Optics. Part II. Containing the higher Propositions with their application to the more perfect forms of Instruments. By Richard Potter, A. M., F. C. P. S. 12*s.* 6*d.*

—— Physical Optics; or, the Nature and Properties of Light. A Descriptive and Experimental Treatise. By Richard Potter, A. M., F. C. P. S. 6*s.* 6*d.*

POTTER (R.) Physical Optics. Part II. The Corpuscular Theory of Light discussed Mathematically. By Richard Potter, M. A. 7*s.* 6*d.*

HOPKINS (W. B.) Figures Illustrative of Geometrical Optics. From Schelbach. By W. B. Hopkins, B. D. Folio. Plates. 10*s.* 6*d.*

MAIN (P. T.) The First Three Sections of Newton's Principia, with an Appendix; and the Ninth and Eleventh Sections. By John H. Evans, M. A. The *Fifth edition*, edited by P. T. Main, M. A. 4*s.*

—— An Introduction to Plane Astronomy. For the use of Colleges and Schools. By P. T. Main, M. A., Fellow of St. John's College. Fcap. 8vo. cloth, 4*s.*

MAIN (R.) Practical and Spherical Astronomy. For the Use chiefly of Students in the Universities. By R. Main, M.A., F.R.S., Radcliffe Observer at Oxford. 8vo. 14*s.*

—— Brünnow's Spherical Astronomy. Part I. Including the Chapters on Parallax, Refraction, Aberration, Precession, and Nutation. Translated by R. Main, M. A., F. R. S., Radcliffe Observer at Oxford. 8vo. 8*s.* 6*d.*

GOODWIN (BP.) Elementary Chapters on Astronomy, from the "Astronomie Physique" of Biot. By Harvey Goodwin, D. D., Bishop of Carlisle. 8vo. 3*s.* 6*d.*

—— Elementary Course of Mathematics. Designed principally for Students of the University of Cambridge. By Harvey Goodwin, D.D., Lord Bishop of Carlisle. *Sixth edition*, revised and enlarged by P. T. Main, M. A., Fellow of St. John's College, Cambridge. 8vo. 16*s.*

—— Problems and Examples, adapted to the "Elementary Course of Mathematics." By Harvey Goodwin, D.D. With an Appendix, containing the Questions proposed during the first three days of the Senate-House Examination, by T. G. Vyvyan, M. A. *Third edition.* 8vo. 5*s.*

—— Solutions of Goodwin's Collection of Problems and Examples. By W. W. Hutt, M. A., late Fellow of Gonville and Caius College. *Third edition, revised and enlarged.* By T. G. Vyvyan, M. A. 8vo. 9*s.*

SMALLEY (G. R.) A Compendium of Facts and Formulæ in Pure Mathematics and Natural Philosophy. By G. R. Smalley, F.R.A. S., of St. John's Coll., Cam. Fcap. 8vo. 3*s.* 6*d.*

TAYLOR (JOHN). A Collection of Elementary Examples in Pure Mathematics, arranged in Examination Papers; for the use of Students for the Military and Civil Service Examinations. By John Taylor, late Military Tutor, Woolwich Common. 8vo. 7*s.* 6*d.*

FILIPOWSKI (H. E.) A Table of Anti-Logarithms. Containing, to seven places of decimals, natural numbers answering to all logarithms from ·00001 to ·99999, and an improved table of Gauss' Logarithms, by which may be found the Logarithm of the sum or difference of two quantities. With Appendix, containing a Table of Annuities for 3 joint lives at 3 per cent. By H. E. Filipowski. *Third edition.* 8vo. 15*s.*

BYRNE (O.) A system of Arithmetical and Mathematical Calculations, in which a new basis of notation is employed, and many processes, such as involution and evolution, become much simplified. Invented by Oliver Byrne.

—— Dual Arithmetic; or, the Calculus of Concrete Quantities, Known and Unknown, Exponential and Transcendental, including Angular Magnitudes. With Analysis. Part I. 8vo. 14*s.*

In it will be found a method of obtaining the logarithm of any number in a few minutes by direct calculation; a method of solving equations, which involve exponential, logarithmic and circular functions, &c. &c.

—— Dual Arithmetic. Part II. The Descending Branch, completing the Science, and containing the Theory of the Application of both Branches. 8vo. 10*s.* 6*d.*

—— Dual Tables (Ascending and Descending Branches). Comprising Dual Numbers, Dual Logarithms, and Common Numbers; Tables of Trigonometrical Values, Angular Magnitudes, and Functions, with their Dual Logarithms. 4to. 21*s.*

ELLIS (LESLIE). The Mathematical and other Writings of Robert Leslie Ellis, M. A., late Fellow of Trinity College, Cambridge. Edited by William Walton, M. A., Trinity College, with a Biographical Memoir by Harvey Goodwin, D. D., Bishop of Carlisle. 8vo. 16*s.*

CHALLIS (PROF.) Notes on the Principles of Pure and Applied Calculation, and Applications to the Theories of Physical Forces. By Rev. J. Challis, M.A., F.R.S., &c., Plumian Professor of Astronomy, Cambridge. Demy 8vo. 15*s.*

—— The Mathematical Principle of Physics. An *Essay.* By the Rev. James Challis, M.A., F.R.S. Demy *8vo. 5s.*

MISCELLANEOUS EDUCATIONAL WORKS.

History, Topography, &c.

BURN (R.) Rome and the Campagna. An Historical and Topographical Description of the Site, Buildings, and Neighbourhood of Ancient Rome. By Robert Burn, M. A., late Fellow and Tutor of Trinity College, Cambridge. With Eighty-five fine Engravings by Jewitt, and Twenty-five Maps and Plans. Handsomely bound in cloth. 4to. 3*l*. 3*s*.

DYER (T. H.) The History of the Kings of Rome. By Dr. T. H. Dyer, Author of the "History of the City of Rome;" "Pompeii: its History, Antiquities," &c., with a Prefatory Dissertation on the Sources and Evidence of Early Roman History. 8vo. 16*s*.

"It will mark, or help to mark, an era in the history of the subject to which it is devoted. It is one of the most decided as well as one of the ablest results of the reaction which is now in progress against the influence of Niebuhr."—*Pall Mall Gazette.*

—— A Plea for Livy, throwing a new light on some passages of the first Book, and rectifying the German doctrine of the imperative mood. 8vo. 1*s*.

—— Roma Regalis, or the Newest Phase of an Old Story. 8vo. 2*s*. 6*d*.

An examination of the views and arguments respecting Regal Rome, put forth by Professor Seeley in a recent edition of "Livy," Book I.

—— The History of Pompeii; its Buildings and Antiquities. An account of the city, with a full description of the remains and the recent excavations, and also an Itinerary for visitors. Edited by T. H. Dyer, LL. D. Illustrated with nearly Three Hundred Wood Engravings, a large Map, and a Plan of the Forum. *Third edition*, brought down to 1873. 2 vols. post 8vo. [*In the press.*

—— Ancient Athens: Its History, Topography, and Remains. By Thomas Henry Dyer, LL. D., Author of "The History of the Kings of Rome." Super-royal 8vo. cloth, 1*l*. 5*s*.

This work gives the result of the excavations to the present time, and of a recent careful examination of the localities by the Author. It is illustrated with plans, and wood engravings taken from photographs.

LONG (G.) The Decline of the Roman Republic. 8vo.

Vol. I. From the Destruction of Carthage to the End of the Jugurthine War. 14*s.*

Vol. II. To the Death of Sertorius. 14*s.*

Vol. III. Including the third Mithridatic War, the Catiline Conspiracy, and the Consulship of C. Julius Cæsar. 14*s.*

Vol. IV. History ot Cæsar's Gallic Campaigns and of the contemporaneous events in Rome. 14*s.*

Vol. V. Concluding the work. 14*s.*

"If any one can guide us through the almost inextricable mazes of this labyrinth, it is he. As a chronicler, he possesses all the requisite knowledge, and what is nearly, if not quite as important, the necessary caution. He never attempts to explain that which is hopelessly corrupt or obscure: he does not confound twilight with daylight; he warns the reader repeatedly that he is standing on shaking ground; he has no framework of theory into which he presses his facts."—*Saturday Review.*

PEARSON (C. H.) A History of England during the Early and Middle Ages. By C. H. Pearson, M.A., Fellow of Oriel College, Oxford, and Lecturer in History at Trinity College, Cambridge. *Second edition, revised and enlarged.* 8vo. Vol. I. to the Death of Cœur de Lion. 16*s.* Vol. II. to the Death of Edward I. 14*s.*

—— Historical Maps of England. By C. H. Pearson. Folio. *Second edition, revised.* 31*s.* 6*d.*

An Atlas containing Five Maps of England at different periods during the Early and Middle Ages.

BOWES (A.) A Practical Synopsis of English History; or, A General Summary of Dates and Events for the use of Schools, Families, and Candidates for Public Examinations. By Arthur Bowes. *Fourth edition.* 8vo. 2*s.*

BEALE (D.) Student's Text-Book of English and General History, from B. C. 100 to the Present Time, with Genealogical and Literary Tables, and Sketch of the English Constitution. By D. Beale. Crown 8vo. 2*s.* 6*d.*

STRICKLAND (AGNES). The Lives of the Queens of England; from the Norman Conquest to the Reign of Queen Anne. By Agnes Strickland. Abridged by the Author for the use of Schools and Families. Post 8vo. Cloth. 6*s.* 6*d.*

HUGHES (A. W.) Outlines of Indian History: comprising the Hindu, Mahomedan, and Christian Periods (down to the Resignation of Sir J. Lawrence). With Maps, Statistical Appendices, and numerous Examination Questions. Adapted specially for Schools and Students. By A. W. Hughes, Bom. Uncov. Civil Service, and Gray's Inn. Small post 8vo. 3*s*. 6*d*.

HELPS (SIR A.) The Life of Hernando Cortez, and the Conquest of Mexico. Dedicated to Thomas Carlyle. 2 vols. crown 8vo. 15*s*.

—— The Life of Christopher Columbus, the Discoverer of America. *Fourth edition.* Crown 8vo. 6*s*.

—— The Life of Pizarro. With Some Account of his Associates in the Conquest of Peru. *Second edition.* Cr. 8vo. 6*s*.

—— The Life of Las Casas, the Apostle of the Indies. *Third edition.* Crown 8vo. 6*s*.

TYTLER (PROF.) The Elements of General History. *New edition.* Revised and brought down to the present time. Small post 8vo. 3*s*. 6*d*.

ATLASES. An Atlas of Classical Geography; Twenty-four Maps by W. Hughes and George Long, M.A. *New edition,* with coloured outlines. Imperial 8vo. 12*s*. 6*d*.

This Atlas has been constructed from the best authorities by Mr. W. Hughes, under the careful supervision of Mr. Long. The publishers believe that by this combination they have secured the most correct Atlas of Ancient Geography that has yet been produced.

—— A Grammar School Atlas of Classical Geography. Containing Ten Maps selected from the above. Imperial 8vo. *New edition.* 5*s*.

CONTENTS:—The Provinces of the Roman Empire. Gallia. Italia. Graecia (including Epirus and Thessalia, with part of Macedonia). The Coasts and Islands of the Aegaean Sea. Asia Minor, and the Northern part of Syria. Palaestina, with part of Syria, Assyria, and the Adjacent Countries. Sicilia; and a Plan of Rome.

—— First Classical Maps. By the Rev. J. Tate, M.A. *Third edition.* Imperial 8vo. 7*s*. 6*d*.

—— Standard Library Atlas of Classical Geography. *Twenty-two large Coloured Maps according to the latest authorities.* With a complete Index (accentuated), giving the latitude and longitude of every place named in the Maps. Imperial 8vo. 7*s*. 6*d*.

Philology.

RICHARDSON (Dr.) New Dictionary of the English Language. Combining Explanation with Etymology, and copiously illustrated by Quotations from the best Authorities. *New edition*, with a Supplement containing additional words and further Illustrations. The Words, with those of the same family, are traced to their origin. The Explanations are deduced from the primitive meaning through the various usages. The Quotations are arranged chronologically, from the earliest period to the present time. In 2 vols. 4to. £4 14*s*. 6*d*.; half-bound in russia, £5 15*s*. 6*d*.; in russia, £6 12*s*. The Supplement separately, 4to. 12*s*.

An 8vo. Edition, without the Quotations, 15*s*.; half-russia, 20*s*.; russia, 24*s*.

ADAMS (Dr.) The Elements of the English Language. By Ernest Adams, Ph. D. *Thirteenth edition*. Post 8vo. 4*s*. 6*d*.

KEY (Prof.) Philological Essays. By T. Hewitt Key, Professor of Comparative Grammar in University College, London. 8vo. 10*s*. 6*d*.

—— Language, its Origin and Development. By T. Hewitt Key, Professor of Comparative Grammar in University College. 8vo. 14*s*.

This work is founded on the Course of Lectures on Comparative Grammar delivered during the last twenty years in University College. The evidence being drawn chiefly from two of the most familiar members of the Indo-European family, Latin and Greek, especially the former, as that to which the writer's hours of study, for half a century, have been almost wholly devoted.

DONALDSON (J. W.) Varronianus. A Critical and Historical Introduction to the Ethnography of Ancient Italy and to the Philological Study of the Latin Language. *Third edition, revised and considerably enlarged*. By J. W. Donaldson, D. D. 8vo. 16*s*.

SMITH (Archdn.) Synonyms and Antonyms of the English Language. Collected and Contrasted by the Ven. C. J. Smith, M. A. *Second edition*. Post 8vo. 5*s*.

—— Synonyms Discriminated. Showing the accurate signification of words of similar meaning. By the Ven. C. J. Smith. Demy 8vo. 16*s*.

PHILLIPS (Dr.) A Syriac Grammar. By G. Phillips, D. D., President of Queens' College. *Third edition, revised and enlarged*. 8vo. 7*s*. 6*d*.

BEAMONT (W. J.) A Concise Grammar of the Arabic Language. By the Rev. W. J. Beamont, M. A. Revised by Sheikh Ali Nady el Barrany, one of the Sheikhs of the El Azhar Mosque in Cairo. 12mo. 7*s.*

WEBSTER. A Dictionary of the English Language. By Dr. Webster. Re-edited by N. Porter and C. A. Goodrich. The Etymological portion being by Dr. C. F. Mahn, of Berlin. With Appendix and Illustrations complete in one volume. £1 11*s.* 6*d.*

Besides the meaning and derivation of all words in ordinary use, this volume will be found to contain in greater fulness than any other Dictionary of the English Language hitherto published, scientific and technical terms, accompanied in many instances by explanatory woodcuts and an appendix giving supplementary lists, explanations, and 70 pages of elaborate diagrams and illustrations. In its unabridged form as above, it supplies at a moderate price as complete a literary and scientific reference book as could be obtained in the compass of a single volume.

"For the student of English etymologically Wedgwood, Ed. Muller, and Mahn's Webster are the best dictionaries. While to the general student Mahn's Webster and Craig's 'Universal Dictionary' are most useful."—*Athenæum.*

"THE BEST PRACTICAL ENGLISH DICTIONARY EXTANT." —*Quarterly Review.*

Divinity, Moral Philosophy, &c.

SCRIVENER (DR.) Novum Testamentum Græcum, Textus Stephanici, 1550. Accedunt variæ lectiones editionum Bezæ, Elzeviri, Lachmanni, Tischendorfii, et Tregellesii. Curante F. H. Scrivener, A. M., LL. D. 16mo. 4*s.* 6*d.*

An Edition with wide Margin for Notes. 7*s.* 6*d.*

This Edition embodies all the readings of Tregelles and of Tischendorf's Eighth or Latest Edition.

—— Codex Bezæ Cantabrigiensis. Edited, with Prolegomena, Notes, and Facsimiles, by F. H. Scrivener, M. A. 4to. 26*s.*

—— A Full Collation of the Codex Sinaiticus, with the Received Text of the New Testament; to which is prefixed a Critical Introduction. By F. H. Scrivener, M. A. *Second edition, revised.* Fcap. 8vo. 5*s.*

"Mr. Scrivener has now placed the results of Tischendorf's discovery within reach of all in a charming little volume, which ought to form a companion to the Greek Testament in the library of every Biblical student."—*Reader.*

SCRIVENER (DR.) An Exact Transcript of the Codex Augiensis, Græco-Latina Manuscript in Uncial Letters of St. Paul's Epistles, preserved in the Library of Trinity College, Cambridge. To which is added a Full Collation of Fifty Manuscripts containing various portions of the Greek New Testament deposited in English Libraries: with a full Critical Introduction. By F. H. Scrivener, M.A. Royal 8vo. 26*s.*

The CRITICAL INTRODUCTION *is issued separately, price* 5*s.*

—— A Plain Introduction to the Criticism of the New Testament. With Forty Facsimiles from Ancient Manuscripts. Containing also an account of the Egyptian versions, contributed by Canon Lightfoot, D.D. For the use of Biblical Students. By F. H. Scrivener, M.A., LL.D. Prebendary of Exeter. 8vo. *New edition.* 16*s.*

—— Six Lectures on the Text of the New Testament, and the MSS. which contain it, chiefly addressed to those who do not read Greek. By Rev. F. H. Scrivener. With facsimiles from MSS. Crown 8vo. 6*s.*

ALFORD (DEAN). Greek Testament. *See* p. 9.

BARRETT (A. C.) Companion to the Greek Testament. For the use of Theological Students and the Upper Forms in Schools. By A. C. Barrett, M.A., Caius College; Author of "A Treatise on Mechanics and Hydrostatics." *Third edition, enlarged and improved.* Fcap. 8vo. 5*s.*

This volume will be found useful for all classes of Students who require a clear epitome of Biblical knowledge. It gives in a condensed form a large amount of information on the Text, Language, Geography, and Archæology; it discusses the alleged contradictions of the New Testament and the disputed quotations from the Old, and contains introductions to the separate books. It may be used by all intelligent students of the sacred volume; and has been found of great value to the students of Training Colleges in preparing for their examinations.

SCHOLEFIELD (J.) Hints for Improvement in the Authorized Version of the New Testament. By the late J. Scholefield, M.A. *Fourth edition.* Fcap. 8vo. 4*s.*

TERTULLIAN. Liber Apologeticus. The Apology of Tertullian. With English Notes and a Preface, intended as an introduction to the Study of Patristical and Ecclesiastical Latinity. By H. A. Woodham, LL. D. *Second edition.* 8vo. 8*s.* 6*d.*

PEROWNE (CANON). The Book of Psalms; a New Translation, with Introductions and Notes, Critical and Explanatory. By the Rev. J. J. Stewart Perowne, B. D., Fellow of Trinity College, Cambridge; Canon Residentiary of Llandaff. 8vo. Vol. I. *Third edition.* 18*s.* Vol. II. *Third edition,* 16*s.*

—— The Book of Psalms. Abridged Edition for Schools. Crown 8vo. 10*s.* 6*d.*

WELCHMAN (ARCHDN.) The Thirty-Nine Articles of the Church of England. Illustrated with Notes, and confirmed by Texts of the Holy Scripture, and Testimonies of the Primitive Fathers, together with References to the Passages in several Authors, which more largely explain the Doctrine contained in the said Articles. By the Ven. Archdeacon Welchman. *New edition.* Fcap. 8vo. 2*s.* Interleaved for Students. 3*s.*

PEARSON (BISHOP). On the Creed. Carefully printed from an Early Edition. With Analysis and Index. Edited by E. Walford, M. A. Post 8vo. 5*s.*

HUMPHRY (W. G.) An Historical and Explanatory Treatise on the Book of Common Prayer. By W. G. Humphry, B. D., late Fellow of Trinity College, Cambridge, Prebendary of St. Paul's, and Vicar of St. Martin's-in-the-Fields, Westminster. *Fifth edition, revised and enlarged.* Small post 8vo. 4*s.* 6*d.*

—— The New Table of Lessons Explained, with the Table of Lessons and a Tabular Comparison of the Old and New Proper Lessons for Sundays and Holy-days. By W. G. Humphry, B.D., Fcap. 1*s.* 6*d.*

DENTON (W.) A Commentary on the Gospels for the Sundays and other Holy Days of the Christian Year. By the Rev. W. Denton, A. M., Worcester College, Oxford, and Incumbent of St. Bartholomew's, Cripplegate. *New edition.* 3 vols. 8vo. 54*s.*

Vol. I.—Advent to Easter. 18*s.*

Vol. II.—Easter to the Sixteenth Sunday after Trinity. 18*s.*

Vol. III.—Seventeenth Sunday after Trinity to Advent; and Holy Days. 18*s.*

DENTON (W.) Commentary on the Epistles for the Sundays and other Holy Days of the Christian Year. By the Rev. W. Denton, Author of "A Commentary on the Gospels," &c.

Vol. I.—Advent to Trinity. 8vo. 18*s.*
Vol. II.—Trinity and Holy Days. 18*s.*

—— Commentary on the Acts. By the Rev. W. Denton. Vol. I. 8vo. 18*s.* Vol. II. in preparation.

JEWEL (BP.) Apology for the Church of England, with his famous Epistle on the Council of Trent, and a Memoir. 32mo. 2*s.*

BARRY (DR.) Notes on the Catechism. For the use of Schools. By the Rev. A. Barry, D. D., Principal of King's College, London. *Second edition, revised.* Fcap. 2*s.*

BOYCE (E. J.) Catechetical Hints and Helps. A Manual for Parents and Teachers on giving instruction to Young Children in the Catechism of the Church of England. By Rev. E. J. Boyce, M. A. *Second edition.* Fcap. 2*s.*

—— Examination Papers on Religious Instruction. Sewed. 1*s.* 6*d.*

MONSELL (DR.) The Winton Church Catechist. Questions and Answers on the Teaching of the Church Catechism. By Rev. J. S. B. Monsell, LL.D., Author of "Our New Vicar." *Third Edition.* Cloth, 3*s.*; or in Four Parts, sewed, price 9*d.* each.

SADLER (M. F.) The Church Teacher's Manual of Christian Instruction. Being the Church Catechism Expanded and Explained in Question and Answer, for the use of the Clergyman, Parent, and Teacher. By the Rev. M. F. Sadler, Rector of Honiton. *Third edition.* 2*s.* 6*d.*

KEMPTHORNE (J.) Brief Words on School Life. A Selection from short addresses based on a course of Scripture reading in school. By the Rev. J. Kempthorne, late Fellow of Trinity College, Cambridge, and Head Master of Blackheath Proprietary School. Fcap. 3*s.* 6*d.*

SHORT EXPLANATION of the Epistles and Gospels of the Christian Year, with Questions for Schools. Royal 32mo. 2*s.* 6*d.*; calf, 4*s.* 6*d.*

BUTLER (BP.) Analogy of Religion; with Analytical Introduction and copious Index, by the Rev. Dr. Steere, Bishop in Central Africa. Fcap. *New edition.* 3*s.* 6*d.*

BUTLER (Bp.) Three Sermons on Human Nature, and Dissertation on Virtue. Edited by W. Whewell, D. D. With a Preface and a Syllabus of the Work. *Fourth and cheaper edition.* Fcap. 8vo. 2*s.* 6*d.*

WHEWELL (Dr.) Lectures on the History of Moral Philosophy in England. By W. Whewell, D. D. *New and improved edition*, with Additional Lectures. Crown 8vo. 8*s.*

*** The Additional Lectures sold separately. Crown 8vo. Price 3*s.* 6*d.*

—— Elements of Morality, including Polity. By W. Whewell, D. D. *New edition*, in 8vo. 15*s.*

—— Astronomy and General Physics considered with reference to Natural Theology (Bridgewater Treatise). *New edition, with new preface*, uniform with the Aldine Editions. 5*s.*

DONALDSON (Dr.) Classical Scholarship and Classical Learning considered with especial reference to Competitive Tests and University Teaching. A Practical Essay on Liberal Education. By J. W. Donaldson, D. D. Crown 8vo. 5*s.*

—— The Theatre of the Greeks. *New and cheaper edition.* Post 8vo. 5*s.*

STUDENT'S GUIDE TO THE UNIVERSITY OF CAMBRIDGE. *Revised and corrected in accordance with the latest regulations. Third edition.* Fcap. 8vo. 6*s.* 6*d.*

This volume is intended to give useful information to parents desirous of sending their sons to the University, and to indicate the points on which to seek further information from the tutor.

Suggestions are also given to the younger members of the University on expenses and course of reading.

"Partly with the view of assisting parents, guardians, schoolmasters, and students intending to enter their names at the University—partly also for the benefit of undergraduates themselves—a very complete, though concise, volume has just been issued, which leaves little or nothing to be desired. For lucid arrangement, and a rigid adherence to what is positively useful, we know of few manuals that could compete with this Student's Guide. It reflects no little credit on the University to which it supplies an unpretending, but complete, introduction."—*Saturday Review.*

KENT'S Commentary on International Law, revised with Notes and Cases brought down to the present time. Edited by J. T. Abdy, LL. D., Barrister-at-Law, Regius Professor of Laws in the University of Cambridge. 8vo. 16*s.*

LEAPINGWELL (G.) A Manual of the Roman Civil Law, arranged according to the Syllabus of Dr. Hallifax, Designed for the use of Students in the Universities and Inns of Court. By G. Leapingwell, LL.D. 8vo. 12*s.*

MAYOR (REV. J. B.) A Guide to the Choice of Classical Books. By J. B. Mayor, M.A., Professor of Classical Literature at King's College, late Fellow and Tutor of St. John's College, Cambridge. Crown 8vo. 2*s.*

FRENCH, GERMAN, AND ENGLISH CLASS BOOKS.

Foreign Classics.

A carefully edited series for use in schools, with English notes, grammatical and explanatory, and renderings of difficult idiomatic expressions. Fcap. 8vo.

CHARLES XII. par Voltaire. Edited by L. Direy. *Third edition, revised.* 3*s.* 6*d.*

GERMAN BALLADS from Uhland, Goethe, and Schiller; with Introductions, copious and biographical notices. Edited by C. L. Bielefeld. 3*s.* 6*d.*

AVENTURES DE TELEMAQUE, par Fenelon. Edited by C. J. Delille. *Second edition, revised.* 4*s.* 6*d.*

SELECT FABLES of La Fontaine. Edited by F. Gasc, *New edition, revised.* 3*s.*

PICCIOLA, by X. B. Saintine. Edited by Dr. Dubuc. *Fourth edition, revised.* 3*s.* 6*d.*

SCHILLER'S Wallenstein. Complete Text, comprising the Weimar Prologue, Lager, Piccolomini, and Wallenstein's Tod. Edited by Dr. A. Buchheim, Professor of German in King's College, London. *Revised edition.* 6*s.* 6*d.* *Or* the Lager and Piccolomini, 3*s.* 6*d.* Wallenstein's Tod, 3*s.* 6*d.*

—— Maid of Orleans; with English Notes by Dr. Wilhelm Wagner, Editor of Plato, Plautus, &c., and Translator of Teuffel's "History of Roman Literature." 3*s.* 6*d.*

GOETHE'S HERMANN AND DOROTHEA. With Introduction, Notes, and Arguments. By E. Bell, B.A., Trinity College, Cambridge, and E. Wölfel. 2*s.* 6*d.*

French Class Books.

BREBNER (W.) Twenty Lessons in French. With double vocabulary giving the pronunciation of French words, notes, and appendices. By W. Brebner. Post 8vo. 4*s.*

CLAPIN (A. C.) French Grammar for Public Schools. By the Rev. A. C. Clapin, M. A., St. John's College, Cambridge, and Bachelier-ès-lettres of the University of France. Fcap. 8vo. *Second edition greatly enlarged.* 2*s.* 6*d.* Or in two parts separately. Part I. Accidence, 2*s.* Part II. Syntax, 1*s.* 6*d.*

GASC (F. E. A.) First French Book; being a New, Practical, and Easy Method of Learning the Elements of the French Language. Fcap. 8vo. *New edition.* 1*s.* 6*d.*

—— Second French Book; being a Grammar and Exercise Book, on a new and practical plan, and intended as a sequel to the "First French Book." *New edition.* Fcap. 8vo. 2*s.* 6*d.*

—— Key to First and Second French Books. Fcap. 8vo. 3*s.* 6*d.*

—— French Fables, for Beginners, in Prose, with an Index of all the Words at the end of the work. *New edition.* 12mo. 2*s.*

—— Select Fables of La Fontaine. *New edition, revised.* Fcap. 8vo. 3*s.*

—— Histoires Amusantes et Instructives; or, Selections of Complete Stories from the best French modern authors, who have written for the young. With English notes. *New edition.* Fcap. 8vo. 2*s.* 6*d.*

—— Practical Guide to Modern French Conversation: containing:—I. The most current and useful Phrases in Every-day Talk; II. Everybody's Necessary Questions and Answers in Travel-Talk. Fcap. 8vo. 2*s.* 6*d.*

—— French Poetry for the Young. With English Notes, and preceded by a few plain Rules of French Prosody. Fcap. 8vo. 2*s.*

—— Materials for French Prose Composition; or, Selections from the best English Prose Writers. With copious foot notes, and hints for idiomatic renderings. *New edition.* Fcap. 8vo. 4*s.* 6*d.* KEY, 6*s.*

—— Prosateurs Contemporains; or, Selections in Prose chiefly from contemporary French literature. With English notes. 12mo. 5*s.*

GASC (F. E. A.) Le Petit Compagnon; a French Talk-Book for Little Children. 16mo. 2*s.* 6*d.*

—— An Improved Modern Pocket Dictionary of the French and English Languages; for the every-day purposes of Travellers and Students. Containing more than Five Thousand modern and current words, senses, and idiomatic phrases and renderings, not found in any other dictionary of the two languages. *A new edition with additions and corrections.* 16mo. Cloth, 4*s.*

—— Modern French and English Dictionary, with upwards of Fifteen Thousand new words, senses, &c., hitherto unpublished. Demy 8vo. 15*s.*

GOMBERT (A.) The French Drama; being a Selection of the best Tragedies and Comedies of Molière, Racine, P. Corneille, T. Corneille, and Voltaire. With Arguments in English at the head of each scene, and notes, critical and Explanatory, by A. Gombert. Sold separately at 1*s.* each.

CONTENTS.

MOLIERE:

*Le Misanthrope.	Les Précieuses Ridicules.
*L'Avare.	L'Ecole des Femmes.
*Le Bourgeois Gentilhomme.	L'Ecole des Maris.
*Le Tartuffe.	Le Médecin malgré Lui.
Le Malade Imaginaire.	M. de Pourceaugnac.
Les Femmes Savantes.	Amphitryon.
Les Fourberies de Scapin.	

RACINE:

La Thébaide, ou les Frères Ennemis.	Bajazet.
Alexandre le Grand.	Mithridate.
Andromaque.	Iphigénie.
Les Plaideurs.	Phédre.
Britannicus.	Esther.
Bérénice.	*Athalie. (*In the press.*)

P. CORNEILLE:

Le Cid.	Horace.	Polyeucte.
Cinna.		Pompée.

T. CORNEILLE:

Ariane.

VOLTAIRE:

Brutus.	Zaire.	Mérope.
Alzire.	Orestes.	La Mort de César.
Le Fanatisme.		Semiramis.

* *New Editions of those marked with an asterisk have lately been issued, carefully revised by the Rev. W. Holmes and M. Gasc. Fcap. 8vo. Neatly bound in cloth.* 1*s.* *each. Others will follow.*

LE NOUVEAU TRÉSOR: or, French Student's Companion; designed to facilitate the Translation of English into French at Sight. By M. E. S. *Sixteenth edition.* Fcap. 8vo. 3*s*. 6*d*.

CONTENTS:—Grammatical Introduction, 100 Lessons, Vocabulary. Conversational Sentences, Alphabetical Arrangement of the Verbs. General Table of Reference.

See also "Foreign Classics," p. 96.

German Class Books.

BUCHHEIM (DR. A.) Materials for German Prose Composition; consisting of Selections from Modern English writers, with grammatical notes, idiomatic renderings of difficult passages, and a general introduction. By Dr. Buchheim, Professor of German Language and Literature in King's College, and Examiner in German to the London University. *Third edition, revised.* Fcap. 4*s*. 6*d*.

In this edition the notes in Part I. have been entirely revised and increased in accordance with the suggestions of experienced teachers.

CLAPIN (A. C.) A German Grammar for Public Schools. By the Rev. A. C. Clapin, Compiler of a French Grammar for Public Schools, assisted by F. Holl-Müller, Assistant Master at the Bruton Grammar School. Fcap. 2*s*. 6*d*.

KOTZEBUE. Der Gefangene (the Prisoner). Edited, with English Notes Explanatory and Grammatical, by Dr. W. Stromberg. The first of a selection of German Plays, suitable for reading or acting. 1*s*.

See also "Foreign Classics," p. 96.

English Class Books.

ADAMS (Dr. E.) The Elements of the English Language. By Ernest Adams, Ph. D. *Thirteenth edition.* Post 8vo. 4*s.* 6*d.*

—— The Rudiments of English Grammar and Analysis. *New edition, enlarged.* Fcap. 8vo. 2*s.*

MASON (C. P.) First Notions of Grammar for Young Learners. By C. P. Mason, B. A., Fellow of University College, London. Fcap. 8vo. Cloth, 8*d.*

—— First Steps in English Grammar, for Junior Classes. Demy 18mo. *New edition, enlarged.* 1*s.*

—— Outlines of English Grammar for the use of Junior Classes. Cloth, 1*s.* 6*d.*

—— English Grammar: including the Principles of Grammatical Analysis. *Nineteenth edition,* with a new Etymological Appendix. Post 8vo. 3*s.* 6*d.*

—— The Analysis of Sentences applied to Latin. Post 8vo. 2*s.* 6*d.*

—— Analytical Latin Exercises: Accidence and Simple Sentences, Composition and Derivation of Words, and Compound Sentences. Post 8vo. 3*s.* 6*d.*

—— The First Two Books of Euclid explained to Beginners. *Second edition.* Fcap 8vo. 2*s.* 6*d.*

Edited for Middle-Class Examinations.

With notes on the Analysis and Parsing, and explanatory remarks.

—— Milton's Paradise Lost, Book I. With a Life of Milton. *Third edition.* Post 8vo. 2*s.*

—— Milton's Paradise Lost. Book II. With a Life of the Poet. *Second edition.* Post 8vo. 2*s.*

—— Milton's Paradise Lost. Book III. With a Life of Milton. Post 8vo. 2*s.*

—— Goldsmith's Deserted Village. With a Short Life of the Poet. Post 8vo. 1*s.* 6*d.*

MASON (C. P.) Cowper's Task. Book II. With an Outline of the Poet's Life. Post 8vo. 2*s.*

—— Thomson's "Spring." With a short Life. Post 8vo. 2*s.*

—— Thomson's "Winter." With short Life. Post 8vo. 2*s.*

MENET (J.) Practical Hints on Teaching. Containing Advice as to Organization, Discipline, Instruction, and Practical Management. By the Rev. John Menet, M. A. Perpetual Curate of Hockerill, and late Chaplain of the Hockerill Training Institution. *Fourth edition.* Containing in addition Plans of Schools which have been thoroughly tested, and are now being adopted in various localities. Crown 8vo. Cloth, 2*s.* 6*d.*; paper cover, 2*s.*

TEST LESSONS IN DICTATION, for the First Class of Elementary Schools. This work consists of a series of extracts, carefully selected with reference to the wants of the more advanced pupils; they have been used successfully in many Elementary Schools. The book is supplementary to the exercises given in the "Practical Hints on Teaching." Paper cover, 1*s.* 6*d.*

SKEAT (W. W.) Questions for Examinations in English Literature; with a Preface containing brief hints on the study of English. Arranged by the Rev. W. W. Skeat, late Fellow of Christ's College. 2*s.* 6*d.*

This volume will be found useful in preparing for the various public examinations, in the universities, or for government appointments.

DELAMOTTE (P. H.) Drawing Copies. By P. H. Delamotte, Professor of Drawing in King's College and School, London. Containing 48 outline and 48 shaded plates. Oblong 8vo. 12*s.*; sold also in parts at 1*s.* each.

This volume contains forty-eight outline and forty-eight shaded plates of architecture, trees, figures, fragments, landscapes, boats, and sea-pieces. Drawn on stone by Professor Delamotte.

POETRY for the School Room. *New edition.* Fcap. 8vo. 1*s.* 6*d.*

GATTY (MRS.) Select Parables from Nature, for Use in Schools. By Mrs. Alfred Gatty. Fcap. 8vo. Cloth, 1*s.*

SCHOOL RECORD for Young Ladies' Schools; a useful form of Register of Studies and Conduct. *6d.*

GEOGRAPHICAL TEXT-BOOK; a Practical Geography, calculated to facilitate the study of that useful science, by a constant reference to the Blank Maps. By M. E . . . S . . . . . 12mo. *2s.*

II. The Blank Maps done up separately. 4to. *2s.* coloured.

JOHNS (C. A.) A First Book of Geography. By the Rev. C. A. Johns, B.A., F.L.S., Author of "Botanical Rambles," "Flowers of the Field," &c. Illustrated. 12mo. *2s. 6d.*

LOUDON (Mrs.) Illustrated Natural History. *New edition.* Revised by W. S. Dallas, F. L. S. *With nearly* 500 *Engravings.* *7s.*

—— Handbook of Botany. Newly Edited and greatly enlarged by David Wooster. Fcap. *2s. 6d.*

HAYWARD. The Botanist's Pocket-Book, containing in a tabulated form, the Chief Characteristics of British Plants, with the botanical names, soil or situation, colour, growth, and time of flowering of every plant, arranged under its own order; with a Copious Index. By W. R. Hayward. Crown 8vo. Cloth limp, *4s. 6d.*

STÖCKHARDT. Experimental Chemistry, founded on the Work of Dr. Julius Adolph Stöckhardt. A hand-book for the study of the science by simple experiments. By C. W. Heaton, Professor of Chemistry in the Medical School of Charing Cross Hospital. Post 8vo. *5s.*

BONNEY (T. G.) Cambridgeshire Geology. A Sketch for the use of Students. By T. G. Bonney, F.G.S., &c., Tutor and Lecturer in Natural Science, St. John's Coll. Cambridge. 8vo. *3s.*

FOSTER (B. W.) Double Entry Elucidated. By B. W. Foster. *Seventh edition.* 4to. *8s. 6d.*

CRELLIN (P.) A New Manual of Book-keeping, combining the Theory and Practice, with Specimens of a set of Books. By Phillip Crellin, Accountant. Crown 8vo. *3s. 6d.*

This volume will be found suitable for merchants and all classes of traders: besides giving the method of double entry, *it* exhibits a system which combines the results of double entry *without* the labour which it involves.

PICTURE SCHOOL BOOKS. Written in simple language, and with numerous illustrations. Royal 16mo.

SCHOOL PRIMER. *6d.*

SCHOOL READER. By J. Tilleard, Hon. Member of and Examiner to the College of Preceptors. Numerous Illustrations. *1s.*

POETRY BOOK FOR SCHOOLS. *1s.*

THE LIFE OF JOSEPH. *1s.*

THE SCRIPTURE PARABLES. By the Rev. J. E. Clarke. *1s.*

THE SCRIPTURE MIRACLES. By the Rev. J. E. Clarke. *1s.*

THE NEW TESTAMENT HISTORY. By the Rev. J. G. Wood, M.A. *1s.*

THE OLD TESTAMENT HISTORY. By the Rev. J. G. Wood, M.A. *1s.*

THE STORY OF BUNYAN'S PILGRIM'S PROGRESS. *1s.*

THE LIFE OF CHRISTOPHER COLUMBUS. By Sarah Crompton. *1s.*

THE LIFE OF MARTIN LUTHER. By Sarah Crompton. *1s.*

GRANT. Course of Instruction for the Young, by the late Horace Grant.

Arithmetic for Young Children. A Series of Exercises exemplifying the manner in which Arithmetic should be taught to Young Children. *1s. 6d.*

Arithmetic. Second Stage. For Schools and Families, exemplifying the mode in which Children may be led to discover the main principles of Figurative and Mental Arithmetic. 18mo. *3s.*

Exercises for the Improvement of the Senses, and providing instruction and amusement for Children who are too young to learn to read and write. 18mo. *1s.*

Geography for Young Children. With Illustrations for Elementary Plan Drawing. 18mo. *2s.*

These are not class-books, but are especially adapted for use by teachers who wish to create habits of observation in their pupils and to teach them to think.

BOOKS FOR YOUNG READERS. In Eight Parts. Limp Cloth. *8d.* each ; or extra binding, *1s.* each.

Part I. contains simple stories told in monosyllables of not more than four letters, which are at the same time sufficiently interesting to preserve the attention of a child. Part II. exercises the pupil by a similar method in slightly longer easy words; and the remaining parts consist of stories graduated in difficulty, until the learner is taught to read with ordinary facility.

Bell's Reading Books

For Schools and Parochial Libraries.

The popularity which the Series of Reading-Books, known as "Books for Young Readers," has attained is a sufficient proof that teachers and pupils alike approve of the use of interesting stories with a simple plot in place of the dry combinations of letters and syllables, making no impression on the mind, of which elementary reading-books generally consist. There is also practical testimony to the fact that children acquire the power of reading much more rapidly when the process involves something more than the mere mechanical exercise of the faculties of sight and memory.

The publishers have therefore thought it advisable to extend the application of this principle to books adapted for more advanced readers; and to issue for general use in schools a series of popular works which they venture to think will in practice be found more adapted for the end in view than the collections of miscellaneous and often uninteresting extracts which are generally made to serve the purpose.

These volumes will be printed in legible type, and strongly bound in cloth, and will be sold at 1*s.* or 1*s.* 6*d.* each, post 8vo.

The first of the series, viz. CAPTAIN MARRYAT'S MASTERMAN READY, which was written expressly for young people, is now ready, condensed, price 1*s.* 6*d.*; also,

MRS. GATTY'S PARABLES FROM NATURE (selected), foolscap, 8vo., price 1*s.*

The following are in preparation :—

ROBINSON CRUSOE.

OUR VILLAGE. By MISS MITFORD (selections).

GRIMM'S GERMAN TALES. ,,

ANDERSEN'S DANISH TALES. ,,

FRIENDS IN FUR AND FEATHERS.

CHISWICK PRESS :—PRINTED BY WHITTINGHAM AND WILKINS, TOOKS COURT, CHANCERY LANE.