ELEMENTARY

APPLIED MECHANICS.

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PART II.

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WITH NUMEROUS DIAGRAMS
AND A SERIES OF GRADUATED EXAMPLES CAREFULLY WORKED OUT.

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WILLIAM JOHN MACQUORN RANKINE, LL.D.

To the Memory of

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PREFACE.

In this the second volume of our work on Applied Mechanics, the subject of Transverse Stress and Strain is treated in a systematic manner. Equations to, and Diagrams of, Bending Moments and Shearing Forces on Beams and Cantilevers are given for the various manners of loading to be met with in practice; the loads being fixed, or moving, or both combined; most of these are here given for the first time, and the mathematics employed is of an elementary character.

In treating of the Resistance to Bending and Shearing, the Cross Sections ordinarily met with in practice, together with some which are not much employed are given, so that their relative resistances may be compared.

Students not familiar with the Integral Calculus will, nevertheless, be able to apply the results given in connection with Curvature, Slope, and Deflection of Beams; the steps in the integration have been filled in, which may be an assistance to some.

Chapters on Twisting, Bending of Struts, and some other important points have been added to make the work more complete.

We have to thank Mr. Peter Alexander, M.A., for having kindly examined the proofs, and for many valuable suggestions.

T. A.
A. W. T.

May, 1883.

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ERRATA.

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3	5th line; for "X_1" read "x_1."
13	4th line from bottom; for "y_c" read "y."
64	7th ,, ,, for $\left(\frac{W^2}{U}\right)$ read $\left(\frac{W}{U}\right)^2$.
125	Eqn. 12 may also be written $\frac{R - \sqrt{KW_2}}{W_1} \cdot 2r$.
160	Delete the 7th, 8th, and 9th lines from foot of page.
,,	Delete, "it fails to give all the maxima," on the 3rd and 4th lines from foot of page.

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APPLIED MECHANICS.

PART SECOND.

TRANSVERSE STRESS.

In Part First we have considered the internal stress at any point within a solid, and have shown that it can be expressed by means of three principal stresses. We began with one principal stress, the other two being zero; this was illustrated by pieces strained under one direct simple stress, such as tie rods and struts; and at each point in these pieces the strain was similar in every respect. We next considered two principal stresses, the third being zero or identical with one of those two; this was illustrated by small rectangular prisms of earth under foundations, or loaded with the weight of superincumbent earth, the prism being strained by two (or three) direct simple stresses upon its pairs of opposite faces. There we saw that the strain at all points, in certain parallel planes, was similar in every respect; varying, however, as we passed from points in one to points in another of those parallel planes. It was pointed out that earth might have the stress in one horizontal direction artificially increased by a direct external stress, in which case there would be three principal stresses at each point, the intensities of which might be different at different points.

In all such examples, the internal stresses were due to strain produced in the simplest manner possible, viz., by *direct* external stresses; and in many the stresses at internal points were given, without specifying what the solid was, or in what manner it was strained. These exercises served to illustrate methods, but it will afterwards appear that the data specifying the stress at such points were obtained by supposing that the body was strained by external stresses, definite though by no means either simple or direct.

We now come to consider the stresses at points within solids, due to strains produced in the next simplest manner, viz., by external stresses which are all *parallel*. Pieces under such stresses are called *beams*, and the stress is called *transverse stress*. The case in which both ends of the beam are supported will be primarily considered. For simplicity, the external stresses, as shown on the diagrams, are all vertical; they consist of the two upward thrusts concentrated at the extremities, and the loads concentrated on intermediate portions and acting downwards. These external stresses are uniform in the direction normal to the paper; and whatever be the breadth of the beam, they may be replaced by forces all in one plane, the plane of the paper.

On fig. 1, $A\,A'\,B'\,B$ is the longitudinal section of a beam of length $2c$, depth h, and breadth b, and OX is any line chosen as axis. W_1 is a force in the plane of the paper,

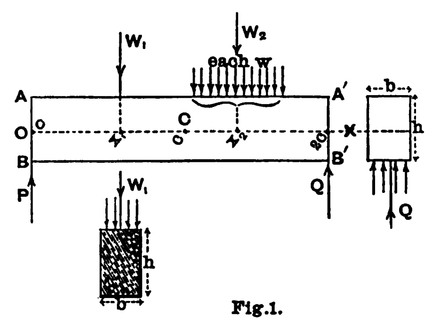


Fig.1.

replacing a stress spread uniformly over the breadth of the beam, as shown on the cross section. Similarly P and Q are forces at the extremities and in the plane of the paper. In order to have these forces specified, it is necessary to know their amounts, and the distances measured from some origin O, say at one end of the beam, to the points where their lines of action cross OX. Such distances are called the abscissæ of the places of application of the loads. Thus P acts at O, W_1 at X_1, and Q at $2c$. C is the centre of span, its abscissa is c.

The varieties of load to be considered are:—

1°. Loads concentrated at one or more points of the span as W_1.

2°. Loads *uniformly* spread over the whole or parts of the span, as w lbs. per running foot. Such a load is represented on fig. 1 by a set of arrows, each equal to w, and consequently they are one foot apart; to save trouble, it is more convenient, as on fig. 2, to represent such a load by means of a parallelogram surrounding all the arrows.

3°. Combinations of such loads.

The loads concentrated at points might be the ends of cross beams resting on such points, or the wheels of carriages, &c. The weight of the beam itself is often to be considered as a load spread uniformly over the span.

To find the relations among the external forces, we consider the equilibrium of the beam as a whole. The beam is to be considered as perfectly rigid and indefinitely strong. In order to find the supporting forces P and Q, we require to know the amounts and positions of the loads, and the length of the beam.

Since the forces are all parallel and in one plane, there are two conditions of equilibrium:—

I. The algebraic sum of the forces is zero.

II. The algebraic sum of the moments of the forces about any point is zero.

From the first condition we have

$$P + Q = W_1 + W_2 + W_3 + \&c.$$
$$= \Sigma(W) \qquad (1.)$$

where $\Sigma(W)$ represents the sum of all the quantities

W_1, W_2, W_3, &c.

If we take moments about O, then P has no moment, and Q tends to turn the beam in one direction about O, while the loads all tend to turn it in the other direction. By the second condition the sum of these moments is zero, and we may, if we choose, put the moment of Q equal to the sum of the moments of the loads, thus—

$Q \times$ leverage = sum of the products got by multiplying each load by its leverage;

$$\text{or} \qquad Q \,.\, 2c = W_1 x_1 + W_2 x_2 + W_3 x_3 + \text{\&c.}$$
$$= \Sigma(Wx);$$
$$\text{hence} \qquad Q = \frac{\Sigma(Wx)}{2c}. \qquad (2.)$$

where $\Sigma\,(Wx)$ represents the sum of all the quantities $W_1 x_1$, $W_2 x_2$, &c.

P may be found in a similar manner by taking moments about the other end; or it may be found at once since we know $P + Q$ by equation (1).

An uniform load, such as w lbs. per running foot spread over a portion of span, is to be treated as one force equal to the amount, and concentrated at the middle of that portion.

Examples.

1. The span of a beam is 20 feet, and there is a load of 80 tons at five feet from the left end. Find the supporting forces.

$$2c = 20; \; W_1 = 80, \text{ and } x_1 = 5.$$
$$Q \,.\, 2c = W_1 x_1.$$
$$\therefore \quad Q = \frac{W_1 x_1}{2c} = \frac{400}{20} = 20 \text{ tons.}$$
$$P + Q = 80 \text{ tons,}$$
$$P = 60 \text{ tons.}$$

Otherwise,

$$Q = \frac{\text{load}}{\text{span}} \times \text{segment remote from } Q$$

$$= \frac{80}{20} \times 5 = 20 \text{ tons};$$

and $$P = \frac{\text{load}}{\text{span}} \times \text{segment remote from } P$$

$$= \frac{80}{20} \times 15 = 60 \text{ tons.}$$

2. A beam of span 24 feet supports loads of 20, 30, and 40 tons concentrated, in order, at points which divide its length into four equal parts. Find the supporting forces.

$$W_1 = 20;\ W_2 = 30;\ W_3 = 40.$$

$$x_1 = 6;\ x_2 = 12;\ x_3 = 18;\ 2c = 24.$$

$$Q = \frac{\Sigma(Wx)}{2c} = \frac{20 \times 6 + 30 \times 12 + 40 \times 18}{24} = 50 \text{ tons.}$$

$$P = \Sigma(W) - Q = (20 + 30 + 40) - 50 = 40 \text{ tons.}$$

3. A beam 30 feet span supports three wheels of a locomotive which transmit each 6, 14, and 8 tons; the distances measured from the left end of the beam to the wheels are 8, 18, and 24 feet respectively. Find the supporting forces.

$$W_1 = 6;\ W_2 = 14;\ W_3 = 8;\ \Sigma(W) = 28 \text{ tons.}$$

$$x_1 = 8;\ x_2 = 18;\ x_3 = 24;\quad 2c = 30 \text{ feet.}$$

Ans. $P = 11{\cdot}6$ tons; $Q = 16{\cdot}4$ tons.

4. The span of a beam is 60 feet; an uniform load of 2000 lbs. per running foot is spread over the portion of the span beginning at 40 and ending at 50 feet from the left end. Find the supporting forces.

See fig. 1, and suppose the spread load alone on the beam.

Replacing $w\,w\ldots$ by $W_2 = w\,(50-40)$

$= 2000 \times 10 = 20{,}000$ lbs.

concentrated at $x_2 = \frac{1}{2}(40 + 50) = 45$ ft.

$$P = 5000, \text{ and } Q = 15{,}000 \text{ lbs.}$$

5. A beam 60 feet span, and weighing 100 tons, supports an uniform load of 2 tons per running foot, which extends from the left end of the span to a point 20 feet therefrom. Find the supporting forces.

$w = 2$; $W_1 = w \times 20 = 40$ tons;

$x_1 = 10$, the middle point of uniform load.

Weight of beam, $W_2 = 100$ tons;

$x_2 = c = 30$.

Ans. $P = 83\frac{1}{3}$ tons; $Q = 56\frac{2}{3}$ tons.

6. A beam 60 feet span, and weighing 100 tons, supports a locomotive as in exercise 3, and an uniform load of 2 tons

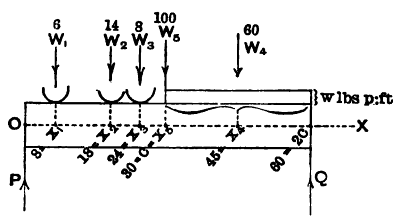


Fig. 2.

per running foot which extends from the middle to the right end of the span. Find the supporting forces.

$$Q \times 60 = 6 \times 8 + 14 \times 18 + 8 \times 24 + 60 \times 45 + 100 \times 30.$$

Ans. $Q = 103{\cdot}2$, and $P = 84{\cdot}8$ tons.

7. A beam 36 feet span weighs one ton per lineal foot.

The first half is loaded uniformly with 2 tons, and the second half with 3 tons per running foot. Find P and Q.

Ans. $P = 58\cdot5$, and $Q = 67\cdot5$ tons.

8. A beam 42 feet span supports five wheels of a heavy locomotive. The fore wheel is one foot from the left end, and the distances between the wheels, in order, are 5, 8, 10, and 7 feet, and the loads transmitted, in order, are 5, 5, 11, 12, and 9 tons. Find the supporting forces. See fig. 3.

$$W_1 = 5;\ W_2 = 5;\ W_3 = 11;\ W_4 = 12;\ W_5 = 9;\ \Sigma(W) = 42.$$

$$x_1 = 1;\ x_2 = 6;\ x_3 = 14;\ x_4 = 24;\ x_5 = 31;\ 2c = 42.$$

$$\therefore Q \times 42 = 5 \times 1 + 5 \times 6 + 11 \times 14 + 12 \times 24 + 9 \times 31.$$

Ans. $P = 24$, and $Q = 18$ tons.

For a system of loads such as W_1, W_2, &c., there is a point at which, if they were all concentrated, the supports would share the load as they do for the actual distribution at different points. This point is called the *centre of gravity* of the load system; its position will be marked G, and its abscissa OG will be denoted by $\bar{x}$. Hence, supposing the total force $\Sigma(W)$ concentrated at G, we have

$$Q\,.\,2c = \Sigma(W)\,.\,\bar{x} \text{ for the single force } \Sigma(W).$$

$$Q\,.\,2c = \Sigma(Wx) \text{ for the actual distribution; see equation 2 on page 4.}$$

Hence, $\Sigma(W)\,.\,\bar{x} = \Sigma(Wx)$,

$$\text{or} \qquad \bar{x} = \frac{\Sigma(Wx)}{\Sigma(W)},$$

which gives the position of G.

Having calculated $\bar{x}$, we can now find P and Q as for the single load $\Sigma(W)$ at $\bar{x}$ from the left end; see example 1 second method.

$$\left.\begin{array}{c}\text{The supporting force}\\ \text{at either end}\end{array}\right\} = \frac{\text{total load}}{\text{span}} \times \text{remote segment},$$

$$\text{or} \qquad P = \frac{\Sigma(W)}{2c}(2c - \bar{x});$$

$$Q = \frac{\Sigma(W)}{2c} \cdot \bar{x}.$$

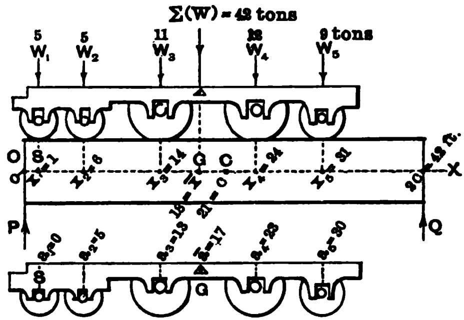


Fig. 8.

9. Solve exercise 8 by the method just described.

We have

$$\bar{x} = \frac{\Sigma(Wx)}{\Sigma(W)} = \frac{5 \times 1 + 5 \times 6 + 11 \times 14 + 12 \times 24 + 9 \times 31}{5 + 5 + 11 + 12 + 9}$$

$$\doteq 18 \text{ feet},$$

and the other segment $(2c - \bar{x}) = 24$ feet.

$$\therefore \; P = \frac{\text{total load}}{\text{span}} \times \text{remote segment}$$

$$= \frac{42}{42} \times 24 = 24 \text{ tons},$$

$$\text{and} \qquad Q = \frac{\text{total load}}{\text{span}} \times \text{remote segment}$$

$$= \frac{42}{42} \times 18 = 18 \text{ tons}.$$

The position of G relative to the loads W_1, W_2... can be found, although the span of the beam and the position of the loads upon it be unknown, provided the amounts of the loads and their distances apart be given. On fig 3, let S be the point where the first load W_1 is situated, and let $a_1 (=0)$, a_2, a_3, $\bar{a}$, a_4, a_5 be the distances from S to W_1, W_2, W_3, G, W_4, W_5 respectively. Taking moments about S, the moment of $\Sigma(W)$ acting at $\bar{a}$ will equal the sum of the moments of W_1, W_2... acting at a_1, a_2....

That is $\qquad \Sigma(W) \times \bar{a} = \Sigma(Wa)$;

$$\therefore \qquad \bar{a} = \frac{\Sigma(Wa)}{\Sigma(W)} \quad \text{gives position of } G$$

measured from S the left end of the load.

10. In example 9, find the position of the centre of gravity of the load measured from the fore wheel.

$$W_1 = 5,\ W_2 = 5,\ W_3 = 11,\ W_4 = 12,\ W_5 = 9,\ \Sigma(W) = 42.$$

$$a_1 = 0,\ \ a_2 = 5,\ \ a_3 = 13,\ \ a_4 = 23,\ \ a_5 = 30.$$

Then $\bar{a} = \dfrac{\Sigma(Wa)}{\Sigma(W)}$

$$= \frac{0 + 5 \times 5 + 11 \times 13 + 12 \times 23 + 9 \times 31}{5 + 5 + 11 + 12 + 9} = 17 \text{ feet,}$$

being less than $\bar{x}$ by one, as it evidently should be.

11. In example 3, find $\bar{a}$ the distance of G from the fore wheel.

$$W_1 = 6,\ W_2 = 14,\ W_3 = 8.$$

$$a_1 = 0,\ \ a_2 = 10,\ \ a_3 = 16 \quad \therefore \quad \bar{a} = 9{\cdot}57 \text{ feet.}$$

It is convenient to calculate $\bar{a}$ if it be required to find values of P and Q, as in examples 3 and 8, corresponding to the given load system shifted into some new position upon the beam; thus—

12. In example 8, find P and Q when the locomotive shifts till its fore wheel is 6 feet from the left end.

$\bar{a} = 17$, hence adding six feet we have $\bar{x} = 23$, so that P and Q will now be the same as for a single load $\Sigma(W) = 42$

tons concentrated at G, a point dividing the span into the segments 23 and 19.

$$\therefore \quad Q = \frac{\text{load}}{\text{span}} \times \text{remote segment}$$

$$= \frac{42}{42} \times 23 = 23 \text{ tons.}$$

$$\text{Similarly} \quad P = \frac{42}{42} \times 19 = 19 \text{ tons.}$$

In like manner P and Q may be found with great convenience for other positions of the locomotive, all of whose wheels *must*, however, be on the beam; because, if one wheel goes off, the beam is under a different load system altogether.

13. Find P and Q in example 3 when the locomotive is shifted so that its fore wheel is 10 ft. from the left end of the beam.

$$\text{Here} \quad \bar{a} = 9{\cdot}57 \text{ feet, and } \Sigma(W) = 28.$$

$$\textit{Ans.} \quad P = 11{\cdot}73,\ Q = 18{\cdot}27 \text{ tons.}$$

Neutral Plane and Neutral Axis.

The phenomena which accompany transverse stress are:—

Every horizontal straight line parallel to the axis of the beam becomes a curve, one line on the diagram showing the curved condition of all lines lying in the same horizontal layer.

All points in the beam, except those over the supports, arrive at a lower level.

The consequence is, that some horizontal layers are shorter and others are longer than they were before the stress was applied. The $\left\{\begin{matrix}\text{top}\\\text{bottom}\end{matrix}\right\}$ layer is that which is most $\left\{\begin{matrix}\text{compressed}\\\text{extended}\end{matrix}\right\}$, and one nearer the $\left\{\begin{matrix}\text{top}\\\text{bottom}\end{matrix}\right\}$ is more $\left\{\begin{matrix}\text{compressed}\\\text{extended}\end{matrix}\right\}$ than one not so near. Since this condition of being extended diminishes gradually as you pass upwards from layer to layer, and passes into a condition of being compressed, there must be one intermediate layer which is neither extended nor compressed. This layer is indefinitely thin, is in fact a plane, and is called the *neutral plane* of the beam, and the line which is its trace upon the diagram is called the *neutral axis of* the beam.

On fig 4, the straight line OX, the neutral axis, while the beam is unstrained, is chosen as an axis of reference. Let S be any point on the neutral axis after the beam has been strained, s its distance from O measured along the curve; let S' be the point on OX directly above S, and x its distance from O measured along OX. It is to be observed that the curvature, although exaggerated on the diagrams, is really in practice so slight that x and s will be sensibly equal to each other, and x may be put for the amount of either unless where it is absolutely necessary to distinguish between them. $S'S$ is called the *deflection* of the point S; the greatest value of this is called *the deflection* of the beam, and when the beam is symmetrically loaded it occurs at the centre of span.

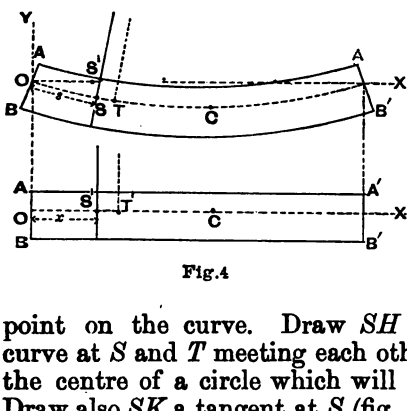


Fig.4

Let T be another point on the curve. Draw SH and TH normals to the curve at S and T meeting each other at H; then H will be the centre of a circle which will coincide with the arc ST. Draw also SK a tangent at S (fig. 5), meeting any horizontal line KL at K. Then

$ds =$ arc ST is the small difference between the values of s for the two points T and S.

$dx = S'T'$ is the small difference of the values of x, the abscissæ of T and S.

$ds = dx$ so far as value is concerned.

$\rho = SH$ is called the *radius of curvature* at S,

$\frac{1}{\rho}$, its reciprocal, is called the *curvature* at S, and

H is called the *centre of curvature* at S.

$i =$ the angle SKL is called the *slope* at S; its greatest value is at one end, and is called *the slope* of the beam.

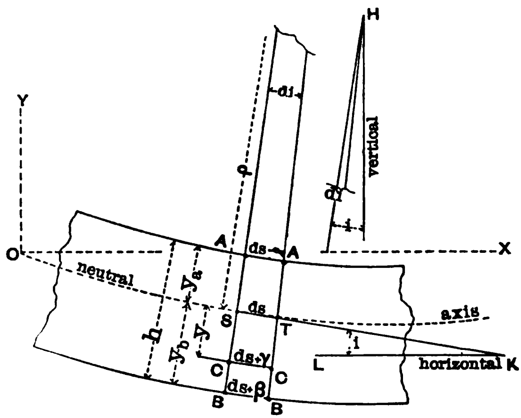


Fig. 5.

$di =$ the difference of the slopes at T and S,

$$= \frac{\text{arc } ST}{SH} = \frac{ds}{\rho} = \frac{dx}{\rho} \text{ sensibly.}$$

These angles are in circular measure.

Let y_a be the height of the top layer $A\,A$ above, y_b the depth of the bottom layer $B\,B$ below the neutral axis, and let $\{\mp\}\, y$ be the distance of any layer CC $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right\}$ the neutral axis.

The portions of these layers intercepted between the two radii $H\,S$ and $H\,T$ before being strained were all equal to

ds in length. Let $(ds-\alpha)$, $(ds+\beta)$ and $(ds\pm\gamma)$ be their lengths respectively when strained; then by similar triangles

$$\frac{\text{arc } AA}{AH}=\frac{\text{arc } ST}{SH}, \text{ or } \frac{ds-\alpha}{\rho-y_a}=\frac{ds}{\rho};$$

$$\therefore \quad ds-\alpha=ds-y_a\cdot\frac{ds}{\rho}, \text{ or } \alpha=y_a\frac{ds}{\rho}; \quad \therefore \quad \frac{\alpha}{ds}=\frac{y_a}{\rho}.$$

$$\text{Again } \frac{\text{arc } BB}{BH}=\frac{\text{arc } ST}{SH}, \text{ or } \frac{ds+\beta}{\rho+y_b}=\frac{ds}{\rho};$$

$$\therefore \quad ds+\beta=ds+y_b\cdot\frac{ds}{\rho}, \text{ or } \beta=y_b\frac{ds}{\rho}; \quad \therefore \quad \frac{\beta}{ds}=\frac{y_b}{\rho}.$$

$$\text{Again } \frac{\text{arc } CC}{CH}=\frac{\text{arc } ST}{SH}, \text{ or } \frac{ds\pm\gamma}{\rho\pm y}=\frac{ds}{\rho};$$

$$\therefore \quad ds\pm\gamma=ds\pm y\cdot\frac{ds}{\rho}, \text{ or } \gamma=y\frac{ds}{\rho}; \quad \therefore \quad \frac{\gamma}{ds}=\frac{y}{\rho}.$$

Now the intensity of the longitudinal strain on the layer AA at the point A is

$$\frac{\text{augmentation of arc } AA}{\text{original length of arc } AA}=\frac{-\alpha}{ds}. \qquad \text{(Part 1, page 5.)}$$

Similarly the intensity of the longitudinal strain on the layer BB at the point B is $\frac{\beta}{ds}$, and that on CC at any point C is $\frac{\pm\gamma}{ds}$. Hence from the above equations we have

$$\frac{\alpha}{ds} : \frac{\beta}{ds} : \frac{\gamma}{ds} :: y_a : y_b : y_c, \quad \text{or in words—}$$

The intensity of the longitudinal strain on each layer at the point where it crosses a section AB, is proportional to the distance of the layer from the neutral axis.

Elements of the Stress at an Internal Point of a Beam.

To specify the stress at a point within a beam, it is necessary and sufficient to find the intensity and obliquity of the stress at that point upon any two planes through it; for convenience we take two planes at right angles to the plane of the paper; see Part 1, page 51. In fig 6, OX and OY are rectangular axes; OX coincides with the neutral axis of the beam, and the origin O is at the middle of its length. Let distances measured along OX to the $\left\{\begin{matrix}\text{left}\\\text{right}\end{matrix}\right\}$, along $OY \left\{\begin{matrix}\text{downwards}\\\text{upwards}\end{matrix}\right\}$, be $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$. Then c and $-c$ will be the abscissæ of the two ends of the beam; x_1, x_2, &c., the abscissæ of the weights. Let H be any point in the beam, its co-ordinates being x and y; that is, on the diagram, x is the distance of H to the right or left of O, and y its distance above or below the neutral axis. Of the planes at right angles to the paper and passing through H, choose two, viz., AB and CD, vertical and horizontal. According to custom, CD may be called the *plan* through H, and AB the *cross-section*, or shortly the *section;* further, it is called the section at x, meaning that the abscissa of every point in the section is x. H may be *any* of the points on the cross section at the distance y from the neutral plane; and as all these points are exactly under the same stress, it is unnecessary to say which of them H is; or, in other words, it is not necessary to give the third or Z co-ordinate of H required to specify its distance from the plane of the paper. The trace of the neutral plane upon the cross section is a horizontal straight line, dividing the cross section into an upper and under portion, and this line is called the *neutral axis* of the cross section. In order to specify the stress at H, we find the intensities and obliquities of the stresses at that point upon the two rectangular planes through it. The stresses upon these two planes are due to the strain upon the beam; that is, to the fact of its being bent at the section

AB, leaving out of account the particular forces which actually cause the beam to be bent. We may suppose that the beam is bent by these particular forces, and surrounded by an envelope of some rigid material, and then that these particular forces are removed.

Upon this consideration it is evident that the stress on *CD* will have no *normal component.* Certainly, if one of the weights of the actual load happened to be at *A*, then a

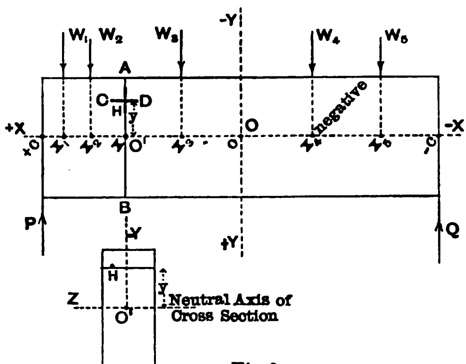


Fig.6.

normal stress on *CD* would be directly transmitted to it; such a stress, however, being accidental is left out of the present investigation. Having remarked this about the plane *CD*, we leave its further consideration for some time, and give our attention to the section *AB*. At the point *H* on the section *AB*, we see there will be a normal component stress; and we know, further, that it is a *thrust* since the horizontal fibre through *H* is compressed. If, however, *H* be on the neutral axis of the cross section, there is no normal component stress, since a horizontal fibre through such a point is unstrained. On the other hand, if *H* be below the neutral axis, that is, if its ordinate *y* be positive, there is a normal component stress; and we know, further, that it would be a *tension*, since the horizontal fibre through such a point is stretched. Generally then, at a point *H* on a section *AB*, there will be a normal component stress of opposite signs for points situated on opposite sides of the neutral axis of the cross section; and generally, also, there will be a tangential component stress acting in the same direction for all points on the section, as will afterwards be seen.

To ascertain the stresses at points on the section AB; suppose the beam cut into two portions at that section, and consider the equilibrium of one, say the left portion, as a rigid body, fig. 7. The forces acting on it are P, W_1, W_2, shown on the figure by strong arrows, together with the stress upon the cut surface, shown by fine arrows. Let P be greater than the sum of W_1 and W_2; then the vertical components of the fine arrows must act downwards to conspire with W_1 and W_2 in balancing P. For some positions of the section, $W_1 + W_2 +$ &c., the sum of the loads on the portion of the beam to the left of the section AB may

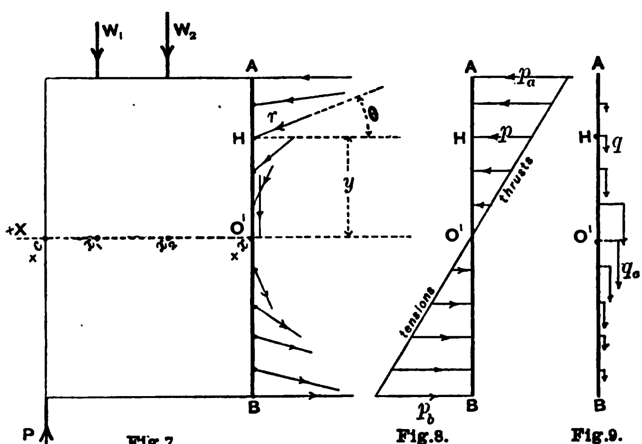


Fig.7. Fig.8. Fig.9.

exceed P, and on such sections all the vertical components of the fine arrows must act upwards; while for other positions of the section, the sum of W_1, W_2, &c., may equal P, and on these sections the fine arrows will be normal to AB. Let r be the intensity of the stress on the section at the point H, and θ its obliquity. Resolve the fine arrows into vertical and horizontal components,—that is, resolve the stress at each point into a tangential and normal component stress. On figs. 8 and 9, p and q are the components of r shown separately, one set on each diagram; p_a and p_b are the values of p at the highest and lowest points of the cross section, and the value of p at the neutral axis is zero; q_o is the value of q at the neutral axis, and, as will afterwards appear, the value of q at the highest and lowest points is zero.

The equilibrium of these forces gives the three following conditions:—

I. The algebraic sum of the arrows p is zero.

II. The algebraic sum of the external forces (strong arrows) together with the arrows q is zero.

III. The algebraic sum of the moments of all the forces about O' is zero.

Condition I. is equivalent to:—The sum of the arrows p which are thrusts acting on the portion of the section above the neutral axis, equals the sum of the arrows p which are tensions acting on the portion of the section below the neutral axis. Hence the resultant of all the arrows p is a pair of equal and opposite forces not in one straight line; or, in other words, a couple in the plane of the paper. Since this couple, by condition III., balances the sum of the moments of the external forces about O', therefore the moment of the couple is equal to that sum, and acts in the opposite direction.

DEFINITIONS. F_x, the algebraic sum of the external forces acting on a portion of a beam included between one end and the cross section at x, and comprising the supporting force (if any) at that end and the loads on that portion, is called the *shearing force* at that cross section.

M_x, the moment of these external forces about any point on the cross section at x, is called the *bending moment* at that cross section.

F_x, the amount of the tangential component stress on the cross section at x, is called the *resistance to shearing* of that cross section.

M$_x$, the couple which is the moment of the total stress on the cross section at x about any point of the cross section, or what is the same thing, the moment of the normal component stress on the cross section about any point in the plane of the paper, is called the *moment of resistance to bending* of that cross section.

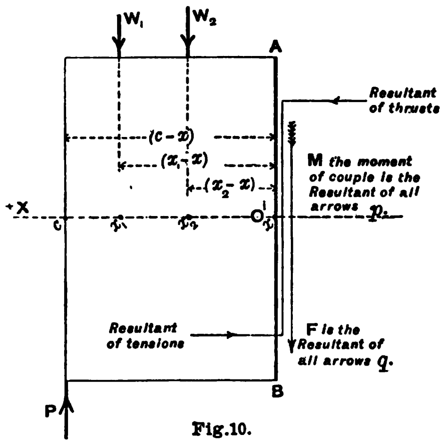


Fig.10.

On fig. 10, these four quantities are

$$F_x = P - W_1 - W_2.$$

$$M_x = P(c-x) - W_1(x_1 - x) - W_2(x_2 - x).$$

$\mathbf{F}_x$ = Resultant of arrows q.

$\mathbf{M}_x$ = Resultant of the pair of equal and opposite forces, one of which is the resultant of all the thrusts p, the other of all the tensions p.

The conditions of equilibrium for any portion of the beam comprised between one end and a cross section at x are

$$F_x = \mathbf{F}_x. \qquad (1.)$$

$$M_x = \mathbf{M}_x. \qquad (2.)$$

Since F_x and M_x are calculated from external forces alone, they are independent of the size or form of the cross section, and depend only upon the amount and distribution of the load; and, when calculated at a sufficient number of cross sections, they form the data from which to design the beam. On the other hand, F_x and M_x depend only upon the size and form of the cross section at x, and upon the material of which the beam is to be made. Having fixed upon a material for which we know the working strengths to resist tension, thrust, and shearing, the two equations above enable us to design the different dimensions of the cross section at x in any required form. The form to be adopted in any particular case depends in some degree upon the shapes in which the material is naturally obtained, or in which it can be manufactured cheaply; and when it can with equal facility be manufactured in several forms, that one is to be preferred which takes greatest advantage of any difference in the resistance of the material to the various *kinds* of stress; since by doing so we require the least quantity of material. The form of cross section chosen must be suitable for the particular nature of the load, and locality of the beam. Having thus designed the cross sections at a sufficient number of places, we are said to have designed the beam so as to have sufficient *strength* to resist the given loads.

Besides the above qualities of suitability, cheapness and economy of material, a beam must also have sufficient *stiffness*. We will, further on, derive equations to enable us to find the form of economy mentioned above, and also to select the ratio of the dimensions of the cross section to fulfil the condition of stiffness.

The cross section at x either being given or having been designed, equations 1 and 2 enable us to calculate the elements of the stress at any internal point H.

THE CANTILEVER.

A beam may be supported by *one* prop placed exactly below the centre of gravity of the load, as shown on figure 11. It is evident from the definitions, that in this case th shearing force and bending moment are maxima at the point of support; because the external forces upon one of the portions into which the section through this point divides the beam, say upon the left portion, are the loads on that portion and they all act in one direction. The bending moment at that section is the *arithmetical* sum of each of these loads into its leverage; and for any other section to the left, the number of loads to be reckoned in calculating the bending

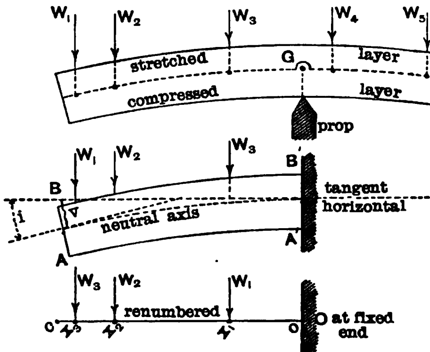


Fig.11.

moment may be fewer, and at the same time all the leverages are shorter. In considering the left portion alone, it is unnecessary to draw the other; and instead, the left portion may be supposed, as on the figure, to terminate at its right extremity in a vertical plane; this plane is supposed to give the necessary resistance to balance the shearing force and bending moment at the point of support. Such a piece is called a *Cantilever;* as shown on diagram, the right is its *fixed* and the left its *free* end; its length is c. When the cantilever is strained, that is bent, the tangent to the axis at the fixed end remains horizontal; the upper layer is stretched while the lower is compressed, and the lettering will be accordingly. The deflection v of the cantilever is the difference of level of the two ends, and its slope i is the inclination of the tangent to the axis at its free end with the horizontal.

We shall now proceed to calculate the bending moments on, and construct bending moment diagrams for, beams and cantilevers under various loadings; in these diagrams, the horizontal and vertical dimensions are respectively the spans, and bending moments at the different points of the span, drawn to scale. Each of these diagrams requires two scales,—a scale say of feet for horizontal, and a scale such as foot-tons for vertical, measurements; there may be required, also, a scale say of tons for loads, if the loads are drawn to scale. Such diagrams must be drawn upon a large scale if intended to be used as graphical solutions, in which case only the construction requires to be known. The diagrams of this treatise are too small for such purposes, but they serve to show clearly the constructions, and are principally useful as maps upon which to note the analytical results. In every case, these diagrams are constructed so that one of their boundaries is straight, is in fact the span. This is a matter of importance, as the diagram so constructed assumes a suitable form for a practical purpose which will be afterwards pointed out.

One boundary of such diagrams is generally a parabola in certain simple positions, and we will now give a short chapter on the equations to, and the construction of, this curve.

THE PARABOLA.

On fig. 12, X and Y are two rectangular axes passing through A any point of reference. Let distances measured from A to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$, and from the axis of X $\left\{\begin{matrix}\text{downwards}\\ \text{upwards}\end{matrix}\right\}$, be $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$.

The points P_1, P_2, &c., have their ordinates $1P_1$, $2P_2$, &c., proportional to their abscissæ $A1$, $A2$, &c.; that is, if X and Y be the co-ordinates of any point P, then

$$Y = mX$$

where m is any number, whole or fractional, and the locus of P is a straight line passing through A. In the figure $m = \frac{1}{4}$, and for any point P, X and Y are both positive or both negative. The co-ordinates of points P_1, P_2, &c., are marked; *e.g.* for P_3, $X = 3$, and $Y = \frac{3}{4}$.

On fig. 13, P_1, P_2, &c., are points corresponding to the points marked similarly on fig. 12, but in this case the points P_1, P_2, &c., have their ordinates $1P_1$, $2P_2$, &c., proportional to the squares of their abscissæ; that is

$$Y = mX^2.$$

This is the *principal equation* to the parabola, and the quantity m is the *modulus*. The locus of this equation, P,

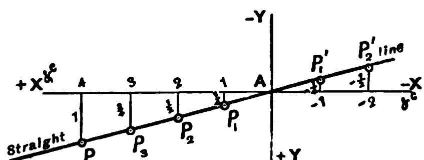


Fig.12.

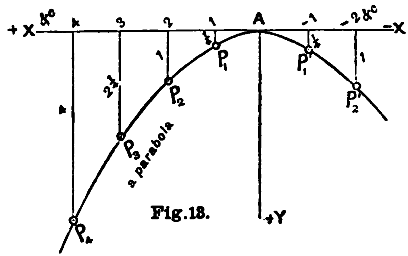


Fig.13.

is the parabola; and it is altogether on one side of the axis of X, since, although X the abscissa of any point P is positive or negative, the quantity X^2 is always positive.

In the figure $m = \frac{1}{4}$, and for any point P, while X may be positive or negative, Y is always positive. The co-ordinates of the points P_1, P_2, &c., are marked; *e.g.* for P_3, $X = 3$, and $Y = 2\frac{1}{4}$. The point A is called the vertex, and the line AY the axis of the parabola; the curve is symmetrical about this axis. The points P_1, P_2, &c., thus found are points on the curve; and if a sufficient number of such points be found and a fair curve drawn through them, the curve will be sensibly a parabola.

Whenever we have an equation of the above form, we conclude that the locus is a parabola whose vertex is at the origin, and whose axis lies along the axis of Y.

Parabolic Segment.

Any line as BC, fig. 14, meeting the curve in two points, and drawn parallel to the axis of X, may be called a *right chord;* and the figure enclosed between this chord and the curve may be called a *parabolic right segment.* This chord is the *base* of the segment, and AO the distance of the vertex A from the base is the *height* of the segment.

The following is a convenient construction for drawing a parabolic right segment of a given height and on a given base BC, fig. 14. Plot A at the required height above O the middle of the base, and complete the rectangle $BOAH$. Let it be required to construct accurately twelve points at equal horizontal intervals. Divide AH and HB each into six equal intervals, and number them as on the figure. From each number on AH draw vertical lines, and from the point A draw a ray to each number on HB. Then P_1, the point of intersection of the vertical through 1 and the ray $A1$, is a point on the parabola; so also is P_2, the point of intersection of the vertical through 2 and the ray $A2$; &c.

It is evident that, for instance, DP_4 is proportional to the square of AD; for, calling $1P_1$ unity, then $DJ=4$, and $DP_4 = 4DJ = 16 = 4^2$, while the abscissæ of the points P_1 and P_4 are proportional to 1 and 4; similarly with the other points. Hence *one fourth* is the common ratio of the depth of each point below AH to the square of its distance laterally from AO. We will return to the subject of drawing a parabolic segment, and will now show how to draw a

TANGENT AT ANY POINT OF A PARABOLA,

say P_4; on P_4A and P_4D construct a parallelogram and draw its diagonal P_4E, this line will be the tangent required. P_4K is the same fraction of P_4D that P_4L is of P_4A, and LP_3 is parallel to P_4K. If KP_3 were parallel to P_4L, then would KL be a parallelogram similar to DA, and so P_3 would be on the diagonal P_4E. But since KP_3 is not parallel to P_4L, but converges towards it,

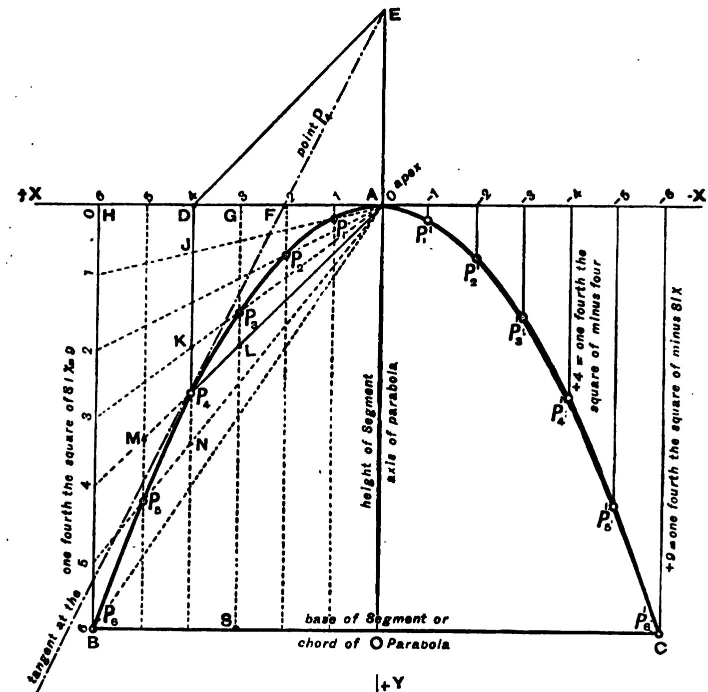


Fig.14.

the point P_3 where KP_3 meets LG lies to the right of the line P_4E. Produce EP_4 through P_4. P_4N is the same fraction of P_4D that P_4M is of P_4A; if NP_5 were parallel to P_4M, then P_5 would be on the diagonal EP_4 produced; but, since NP_5 diverges from P_4M, the point P_5 where NP_5 meets MP_5 lies to the right of the line P_4E

produced through P_4. In the same manner every point on the curve, either above or below P_4, lies to the right of P_4E; that is, P_4E is a tangent at the point P_4. Now the diagonal P_4E bisects the other diagonal AD in F, and the most convenient way of drawing a tangent at any point P_4 is to project P_4 on the horizontal through the apex A, bisect AD in F, and draw P_4F; this is the tangent required.

TO PLOT THE LOCUS OF AN EQUATION OF THE FORM $Y=mX^2$;

in other words, to draw the parabola whose modulus is m. For instance, let $m=\frac{1}{4}$. Draw any line BC parallel to the axis of X, fig. 14, as base; lay off OA equal to $\frac{1}{4}\,OB^2$, then draw the segment by plotting accurately a number of points P_1, P_2, &c., by the construction already given, and draw a fair curve through them. Usually we fix only a few points on the curve accurately, and from these the rest of the curve is sketched in. By making the number of these points sufficiently great, we can draw the curve as accurately as we please.

A parabolic segment might be constructed on, and cut out of, a piece of cardboard, and used exactly like a set square. The parabola could then be quickly drawn on our diagrams thus:—If the apex A be given, place the parallel rollers to the axis of X, shift the rollers, slide the cardboard segment along them till the apex is at A, and draw the curve. Again, suppose we are given (fig. 14) the axis OE and a point B on the curve, and that we wish to draw the curve; place the rollers at right angles to the given axis OE, slide the cardboard segment with its apex on this axis till the curved edge passes through B, and then draw the curve. Instead of cardboard we may use a parabolic segment cut out of a slip of pear-tree or brass, and with one such segment we can, if we choose, draw all parabolas. Thus, suppose our pear-tree segment to have the modulus $\frac{1}{4}$, and that we require to draw a segment whose modulus is 1. We may choose as suitable for horizontals, a scale of 10 parts to an inch; lay off the base with this scale and draw the curve with the pear-tree segment; if we now measure the verticals on the scale, we find for every point $Y=\frac{1}{4}X^2$. If we draw a *new scale* of 40 parts to an inch for vertical measurements, then for every point on the curve we will have, as required,

$$Y = 1 \times X^2 = X^2.$$

In like manner the curve drawn by this pear-tree segment may represent *any* parabola, if the verticals be measured on a suitable scale; this is similar to the common practice of exaggerating the vertical scale for sections. All the diagrams which immediately follow may be very conveniently drawn with *one such segment,* since it is much easier to draw an additional scale than to construct a new parabola; for example, suppose the segment, fig. 14, so drawn, and that we wish it to represent the parabola

$$Y = \tfrac{1}{3}X^2.$$

Using a horizontal scale of four parts to an inch, we have by measurement $OB = 6$, and OA or $HB = 9$; that is, $HB = \frac{1}{4}OB^2$; but we wish HB to measure $\frac{1}{3}OB^2$, that is 12. It is only necessary then to divide HB into 12 equal parts, and lay off a scale of such parts to be used for verticals; this scale is evidently that of 3 inches divided into 16 equal parts.

Equations to the Parabola.

In figs. 15 and 16, let O the middle of the span be the origin of rectangular co-ordinates, the span BC being taken as the axis of X; let distances measured from O to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$, and from the axis of $X \left\{\begin{matrix}\text{upwards}\\ \text{downwards}\end{matrix}\right\}$, be $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$. In fig. 15, the axis of the curve passes through O, whereas in fig. 16 it passes to one side. Let x, y, be the co-ordinates of any point P; and for the apex let $x = K$, and $y = H$.

In fig. 15, $K = 0$, since K is on the axis of Y; and in order to find the equation to the curve with the origin at O, we have

$$Y = -mX^2 \qquad \text{(origin at } A.)$$

Instead of X and Y, we substitute their values, thus

$$y - H = -mx^2;$$
$$y = H - mx^2.$$

In what immediately follows, let C_0, C_1, and C_2 be *constant* quantities; then, if we have an equation of the form

$$y = C_0 + C_2 x^2,$$

consisting of a term not containing x, and a term in x^2, we conclude that the locus is a parabola, with its axis vertical,

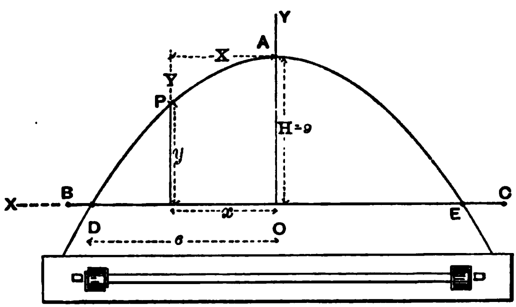


Fig. 15.

its apex on the axis of Y and at the distance C_0 from the origin; and that the modulus of the parabola is C_2, the co-efficient of x^2, so that the principal equation is

$$Y = C_2 X^2.$$

For example, suppose we wish to plot a number of points above the span BC, such that the co-ordinates of each point may fulfil the equation

$$y = \tfrac{1}{4}(36 - x^2) = 9 - \tfrac{1}{4}x^2;$$

we conclude that all the points are on a parabola whose axis is the vertical through O; that the apex A is at the height $H = 9$, the term not containing x; that the modulus is $-\frac{1}{4}$, the co-efficient of x^2; and, therefore, that the principal equation is

$$Y = -\tfrac{1}{4}X^2,$$

$$\therefore \quad H = 9 = \tfrac{1}{4}OD^2;$$

$$\text{or} \quad OD = 6.$$

To draw the locus;—lay off $OA = 9, OD = OE = 6$, and construct points on the curve, as already shown on fig. 14; or more quickly, by means of the pear-tree segment we may draw a curve through the points D and E just found, and construct a scale for verticals upon which OA measures 9.

Again, on fig. 16, the equation to the curve with the origin

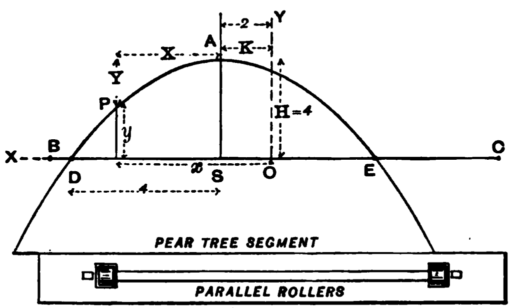


Fig. 16.

at O is derived from the principal equation by substituting $x-K$ and $y-H$ for X and Y, thus:—

$$y - H = -m(x-K)^2,$$

$$\text{or} \quad y = H - mx^2 + 2mKx - mK^2;$$

$$\text{or} \quad y = (H - mK^2) + 2mKx - mx^2,$$

which is an equation of the form

$$y = C_0 + C_1 x + C_2 x^2,$$

consisting of a term not containing x, a term in x, and a term in x^2; and we conclude that the locus is a parabola, with its axis vertical. In this equation, if C_1C_2 is $\left\{\begin{array}{c}\text{positive}\\\text{negative}\end{array}\right\}$, the apex is to the $\left\{\begin{array}{c}\text{right}\\\text{left}\end{array}\right\}$ of the axis of Y; if $C_0 - \frac{C_1^2}{4C_2}$ is $\left\{\begin{array}{c}\text{positive}\\\text{negative}\end{array}\right\}$, the apex is $\left\{\begin{array}{c}\text{above}\\\text{under}\end{array}\right\}$ the axis of X.

To find the value of K and of H, arrange the equation thus:—

$$y = m(2K - x)x + H - mK^2.$$

Now, when $y = H$, its value is a maximum, and the corresponding value of x is K; so that to ascertain K, we have only to find that value of x which makes

$$y = m(2K - x)x + H - mK^2 = \text{maximum}.$$

This is a maximum when $(2K - x)x$ is a maximum, since the rest of the expression is constant for all values of x.

Suppose that $2K$ is the length of a line, which is divided at a point into two segments; let x be the length of one of them, then $2K - x$ is the length of the other, and $(2K - x)x$ is the rectangle contained by the two. We know by Euclid that this rectangle is greatest when the segments are equal, each being half of the line $2K$. So that, when $x = K$, $(2K - x)x$ is a maximum, $m(2K - x)x + H - mK^2$ is also a maximum, and $y = H$.

For example, suppose we wish to plot a number of points whose co-ordinates x, y, fulfil the equation

$$y = \tfrac{1}{4}(4 - x)x + 3 = 3 + x - \tfrac{1}{4}x^2.$$

From the form of this equation, we conclude that all the points are on a parabola, whose axis is vertical and to the left of O; that the apex A is above BC; that the modulus is $-\frac{1}{4}$, the co-efficient of x^2, and therefore that the principal equation is

$$Y = -\tfrac{1}{4}X^2.$$

The distance K at which the axis lies to the left of O is found thus:—the value of x which makes

$$y = \tfrac{1}{4}(4 - x)\,x + 3 = \text{a maximum},$$

is that in which $4 - x = x$, or $x = 2$; that is $K = 2$.

Again, to find the height of the segment; when

$$x = K,\ y = H, \text{ and } H = \tfrac{1}{4}(4 - 2)\,2 + 3 = 4.$$

Substituting in the principal equation, we have

$$4 = H = \tfrac{1}{4}\,SD^2;$$

$$\therefore \qquad \text{half-base} = SD = \pm 4.$$

To draw the locus:—lay off $OS = 2$, and draw a vertical through S; from S lay off $SD = SE = 4$, and $SA = 4$; and construct points on the curve, as in fig. 14: or more quickly, by means of the pear-tree segment, we may draw a curve through the points D and E just found, and construct a scale for verticals upon which SA measures 4.

Bending Moments and Bending Moment Diagrams for Fixed Loads.

Definition.—*The Bending Moment at any cross section of a beam, or,* as we may more conveniently say, *at any point of the span, is,—The sum of the moments about that point of all the external forces, acting upon the portion of the span on either side of the point.*

For convenience in the case of beams supported at both ends, we calculate this bending moment from the forces acting upon the portion to the left of the point. These forces comprise (figs. 6 and 10) the supporting force P acting upwards at the left end, and the loads acting downwards between that end and the point. Taking the centre of span as origin, the abscissa of P is c; and, if x be the abscissa of the point about which moments are taken, then $(c - x)$ is the leverage of P, and P tends to break the beam at the point by bending the left portion *upwards* with a moment $P(c - x)$; the abscissa of W_1 is x_1, its leverage is

$(x_1 - x)$, and it tends to break the beam at the point by bending it *downwards* with a moment $W_1(x_1 - x)$; all the other loads to the left of the point have an effect on the beam similar to that of W_1; and since, in this case, the left portion of the beam is bent upwards at the point, the moment $P(c - x)$ exceeds the sum of all these moments $W_1(x_1 - x) + W_2(x_2 - x)$, &c.; and if M_x represent the bending moment at any point x, then

$$M_x = P(c - x) - W_1(x_1 - x) - W_2(x_2 - x), \text{ \&c.},$$

all the loads on the portion of the beam to the left of the point being taken into account.

A BENDING MOMENT DIAGRAM is a figure having a horizontal straight line for its base, equal in length to the span on a scale for horizontals which should accompany the diagram. Above this base is an outline or *locus* consisting of a curve, a polygon with straight sides, or a polygon with curved sides, and such that the height of any point on the outline gives the bending moment at the point of the span over which it stands, measured on a scale for verticals which also should accompany the drawing. It is evident that this outline always meets the horizontal base at both ends, since the bending moment at each end is zero. It will be seen that x and M_x are respectively the abscissa and ordinate of a point on this locus or outline; an equation between those two quantities, such that, when you substitute into it any value for x, it gives you the corresponding value of M_x, is called the *equation to the bending moment.*

An approximate method of drawing a bending moment diagram is, to calculate the bending moments at a number of points of the span, say at equal short intervals, plot these to scale, and then join the tops of the verticals with straight lines, or draw through these points a fair curve. Such a diagram will give the bending moments accurately at the points which were plotted, and approximately at intermediate points; and its principal use is to mark the calculated results thereon. On the other hand, if upon investigation we find the locus to be of a form which we can draw readily, then, drawing the diagram first, we may afterwards by measurement from it find the value of the bending moment at any point of the span. Such a method of proceeding is called a *graphical solution.*

The MAXIMUM BENDING MOMENT is that value of the bending moment, than which no other value is greater; if this value be at one particular point of the span, that point is called the *point of maximum bending moment;* sometimes this value extends between two points of the span, in which case any point intermediate may be so called. The determination of the maximum bending moment, and of the point at which it occurs, is of great importance. A graphical method is peculiarly successful in giving these, as we know or readily find the *highest* point on the diagram; the height or ordinate of this point is, of course, the maximum bending moment, and its abscissa is the point at which it occurs. In all the cases which follow, that point on the diagram is either the angular point or side of a polygon, or the apex of a parabolic right segment.

Span loaded with unequal weights fixed at irregular intervals.—Let a beam 42-feet span (fig. 17) support weights, viz., $W_1 = 5$, $W_2 = 5$, $W_3 = 11$, $W_4 = 12$, and $W_5 = 9$ tons at points whose abscissæ, reckoned from O the centre, are $x_1 = 20$, $x_2 = 15$, $x_3 = 7$, $x_4 = -3$, and $x_5 = -10$ feet. This is the problem of example 9, fig. 3, and we have $P = 24$ tons, at $c = 21$ ft., and $Q = 18$ tons, at $-c = -21$ ft.

To calculate the bending moment at any point x, $x = -3$ for instance, we may take the forces upon the right portion, and

$$M_{-3} = Q \times 18 - W_5 \times 7 = 18 \times 18 - 9 \times 7 = 261 \text{ ft. tons.}$$

We will now calculate the bending moments at the points where the weights stand, in each case systematically from the forces upon the left portion.

	Ft.-Tons.
$M_{x_1} = P(c - x_1)$;	
$M_{20} = 24 \times 1 =$	24
$M_{x_2} = P(c - x_2) - W_1(x_1 - x_2)$;	
$M_{15} = 24 \times 6 - 5 \times 5 =$	119

$$M_{x_3} = P(c-x_3) - W_1(x_1-x_3) - W_2(x_2-x_3); \qquad \text{Ft.-Tons.}$$
$$M_7 = 24\times 14 - 5\times 13 - 5\times 8 = \ldots\ldots\ldots\ldots \; 231$$
$$M_{x_4} = P(c-x_4) - W_1(x_1-x_4) - W_2(x_2-x_4) - W_3(x_3-x_4);$$
$$M_{-3} = 24\times 24 - 5\times 23 - 5\times 18 - 11\times 10 = \; 261$$
$$M_{x_5} = P(c-x_5) - W_1(x_1-x_5) - W_2(x_2-x_5) - W_3(x_3-x_5) - W_4(x_4-x_5);$$
$$M_{-10} = 24\times 31 - 5\times 30 - 5\times 25 - 11\times 17 - 12\times 7 = \; 198$$

In the interval between W_3 and W_4, consider any two sections A and B, the latter being situated further to the right; let the distance between them be d. The leverage of

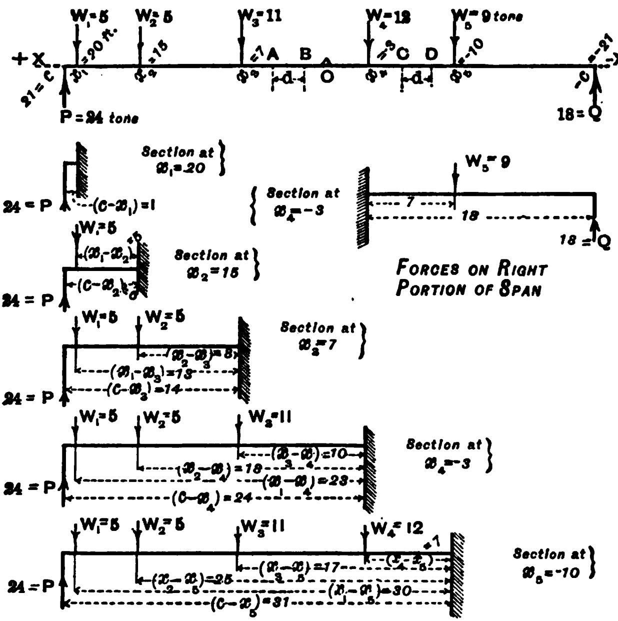


Fig. 17.

P is greater at B than at A by the quantity d, and the *upward* bending moment of P is greater at B than at A by the quantity $P.d$; again, the leverages of W_1, W_2, and W_3, respectively, are greater at B than at A by the quantity d;

and the *downward* bending moment due to W_1, W_2, and W_3 is greater at B than at A by the quantity $(W_1 + W_2 + W_3)\,d$. Hence M_B is to be derived from M_A by adding $P.d$ and subtracting $(W_1 + W_2 + W_3)\,d$; in our example $P > (W_1 + W_2 + W_3)$, so that the quantity to be added *exceeds* the quantity to be subtracted; that is

$$M_B > M_A,\ \text{by}\ (P - W_1 - W_2 - W_3)\,d$$

a quantity proportional to d.

Again, for any two points C and D in the interval between W_4 and W_5, M_D is to be derived from M_C by adding $P.d$ and subtracting $(W_1 + W_2 + W_3 + W_4)d$; in this case, however, $P < (W_1 + W_2 + W_3 + W_4)$, so that the quantity to be added is *smaller* than the quantity to be subtracted; that is

$$M_D < M_C,\ \text{by}\ (W_1 + W_2 + W_3 + W_4 - P)\,d$$

a quantity proportional to d.

In both cases, as we pass towards the right from one point to another in the interval between two weights, the change in the bending moment is uniform, and is proportional to the horizontal distance passed over,—uniformly increasing or uniformly decreasing according as the supporting force P is greater or less than the sum of all the weights to the left of the points being considered; and if for some interval, P is equal to the sum of the weights to the left, the bending moment at points in that interval is constant.

Bending Moment Diagram.—With a scale of feet for horizontals, lay off the span, fig. 18, and plot the positions of the loads; at each of these points erect a vertical equal to the bending moment thereat upon a suitable scale of, say, ft.-tons for verticals; join the tops of these ordinates by straight lines, and join each end of span to the top of the nearest ordinate. The lines just drawn give ordinates which vary uniformly in each interval, so that at any point whatever, the ordinate gives the bending moment at that point in ft.-tons when measured on the vertical scale. The bending moment at each load having been calculated analytically, a diagram thus constructed is a *graphical solution* for every other point.

Maximum Bending Moment.—In our example, since $P > W_1$, the line AB slopes *up* towards the right, that is, the bending moment increases in this interval; BC also slopes *up*

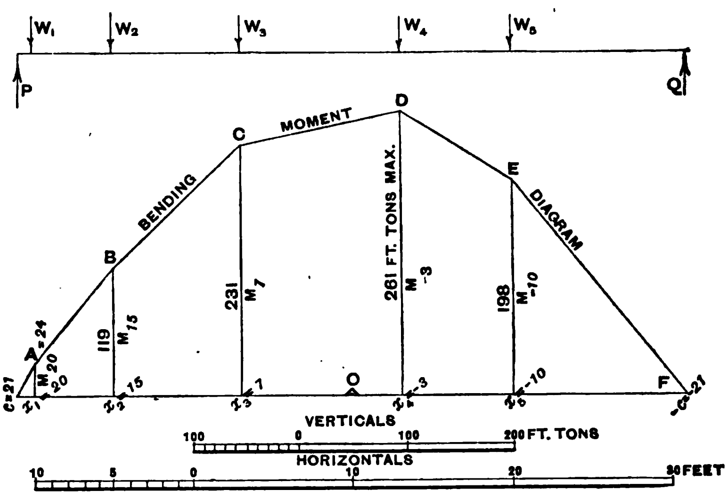


Fig. 18.

towards the right since $P > (W_1 + W_2)$, and CD slopes *up* towards the right since $P > (W_1 + W_2 + W_3)$; DE slopes *down* towards the right since $P < (W_1 + W_2 + W_3 + W_4)$, and, lastly, EF slopes *down* towards the right since $P < (W_1 + W_2 + W_3 + W_4 + W_5)$. That is, beginning at the left end and passing towards the right, we find that the sides of the polygon slope $\left\{ \begin{matrix} \text{up} \\ \text{down} \end{matrix} \right\}$ towards the right while P is $\left\{ \begin{matrix} \text{greater} \\ \text{less} \end{matrix} \right\}$ than the sum of the weights we have passed. For some loads there is one interval where P is *equal* to the sum of the weights passed, and in such cases the side of the polygon above that interval is horizontal. Evidently the highest ordinate is that of the angle of the polygon made by the last side which slopes up to the right, either with the first side which slopes down, or with the horizontal side if there is one. On fig. 18, that angle is D, and its ordinate is the maximum bending moment; so that, in our example, the maximum bending moment occurs at W_4, that is at the point $x = -3$, and its value is 261 ft.-tons. If one side of the polygon be horizontal, the ordinate to any point of this line is constant, and is a maximum.

The *Point of Maximum Bending Moment* is found thus—From P subtract the quantities W_1, W_2, W_3, &c., in succession until the remainder becomes zero, or first negative; when the remainder becomes zero, the max. bending moment occurs at the weight last subtracted, at the weight next in order, and at every point between them; when the remainder becomes negative for the first time, the maximum occurs at the weight last subtracted, and at that point only. In our example, from $P = 24$, subtract $W_1 = 5$, $W_2 = 5$, $W_3 = 11$, and the remainder is $+3$; subtract $W_4 = 12$, and the remainder is negative for the first time; at this weight, that is, at $x_4 = -3$, the maximum bending moment 261 ft.-tons occurs.

Graphical Solution.—The following purely graphical solution for the same problem requires no analysis, but must be drawn with accurate instruments and upon a large scale. Draw the vertical lines P, W_1, W_2, W_3, W_4, W_5, and Q, fig. 19, at the given horizontal distances apart upon a scale for dimensions; draw ab, bc, cd, de, ef equal respectively to the forces W_1, W_2, W_3, W_4, W_5 upon a scale for forces, that is equal to 5, 5, 11, 12, and 9 tons in our example. Choose any pole O', and join it to a, b, c, d, e, and f. From H' any point on P draw $H'A'$ parallel to $O'a$, and meeting W_1 at A'; draw $A'B'$, $B'C'$, $C'D'$, $D'E'$, and $E'K'$ parallel respectively to $O'b$, $O'c$, $O'd$, $O'e$, and $O'f$. Join $H'K'$; and draw $O'g$ parallel to $H'K'$, meeting af in g. Then, upon the scale for forces, $fg = Q$, and $ga = P$. In the example, $fg = 18$ tons, and $ga = 24$ tons; so far this is a graphical solution for finding the supporting forces. The polygon $H'A'B'C'D'E'K'$ is a bending moment diagram; but as it is inconvenient to have $H'K'$ sloping, we draw another polygon thus:—Choose a new pole O on the horizontal line passing through g, and such that the distance Og is some convenient integral number upon the scale for dimensions, on the diagram $Og = 10$;

draw *Oa, Ob, Oc, Od, Oe, Of,* and *Og*. From *H* any point on *P*, draw *HA* parallel to *Oa*, and meeting W_1 at *A*; similarly,

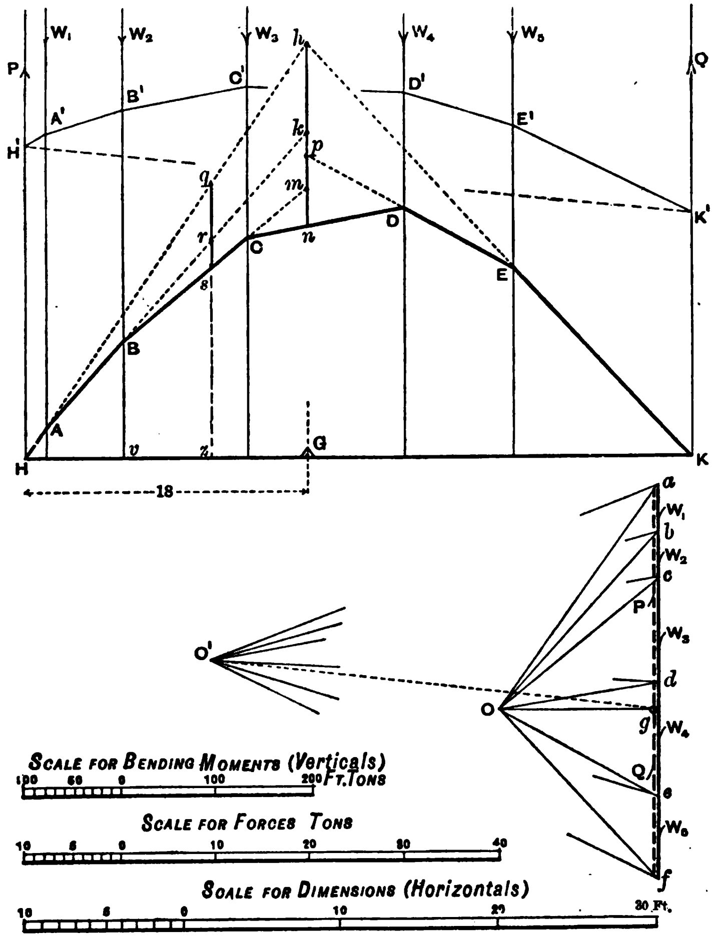


Fig.19.

draw *AB, BC, CD, DE,* and *EK* parallel respectively to *Ob, Oc, Od, Oe,* and *Of*.

We have a new polygon, *HABCDEK*, which is the bending moment diagram, and *HK* is horizontal as desired. The vertical ordinates give the bending moments at each point of the span, upon a scale for verticals obtained by subdividing the divisions on the scale for forces by the number which *Og* measures on the scale for dimensions; thus on the figure this new scale is made by subdividing each division on the scale for forces into 10 equal parts, because we made $Og = 10$ on the scale for dimensions. We can now find the bending moment at *z*, any point of the span, by measuring the ordinate *zs* upon the proper scale; for instance, the ordinate at *D* measures 261, the bending moment in foot tons at that point.

Produce the two sides *HA* and *KE* to meet at *h*, then *G* the point below *h* is the centre of gravity of the loads W_1, W_2, W_3, W_4, and W_5; on fig. 19, $HG = 18$ on the scale for dimensions; see example 9, fig. 3.

Proof.—Draw the vertical *hn*, and produce all the sides of the polygon to meet it; at *z*, any point of span, draw the vertical *zq*, and let the sides of the polygon that lie to the left, or those sides produced, meet it in the points *r* and *s*; then *sr* is called the *intercept* on the vertical through *z*, made by the two sides of the polygon which meet at the angle *B*. Other intercepts on this vertical are *zq* and *rq*, made by the pairs of sides from the angles *H* and *A* respectively. Again, on the vertical through *h* we have the intercepts *Gh*, *kh*, *mk*, and *nm* made respectively by the pairs of sides from the angles *H*, *A*, *B*, and *C*; and also the intercepts *Gh*, *ph*, and *np* made respectively by the pairs of sides from the angles *K*, *E*, and *D*.

The triangle *Bsr* on the base *sr*, and of height *zv*, is similar to the triangle *Ocb* on the base *cb*, and of height *gO*, because their sides are respectively parallel; the bases of these triangles measured on one scale will be in the same proportion to each other as the heights measured on any other scale; hence

on scale of forces		on scale of dimensions
$\overline{sr} : \overline{cb}$	: :	$\overline{zv} : \overline{gO},$

$$\text{or} \qquad \overline{sr} \begin{smallmatrix}\text{on scale}\\ \text{of forces}\end{smallmatrix} = \frac{\overline{cb} \begin{smallmatrix}\text{on scale}\\ \text{of forces}\end{smallmatrix} \times \overline{zv} \begin{smallmatrix}\text{on scale of}\\ \text{dimensions}\end{smallmatrix}}{gO \begin{smallmatrix}\text{on scale of}\\ \text{dimensions}\end{smallmatrix}}$$

$$= \frac{W_2 \text{ tons} \times \overline{zv} \text{ feet}}{10}$$

$$= \frac{\text{Moment of } W_2 \text{ about } z}{10}.$$

But $\overline{sr}$ measures 10 times as much upon the scale for verticals as it does upon the scale for forces, since we made the scale for verticals by subdividing each division on the scale for forces into ten parts; therefore

$$\overline{sr} \text{ on vertical scale} = \text{Moment of } W_2 \text{ about } z.$$

That is, measuring on the vertical scale, the intercept on the vertical through any point z, made by the pair of sides from any angle of the polygon, equals the moment about z of W the force at that angle.

To show that if $P = \overline{ga}$, and $Q = \overline{fg}$, they will balance $W_1 + W_2 + W_3 + W_4 + W_5$. It is evident that $P + Q = \Sigma W$; and if you produce Hh to meet the vertical through K, the intercept by the pair of sides from H equals the moment of ga, that is of P, about K; the five intercepts made on the vertical through K by the pairs of sides from A, B, C, D, and E, are the moments about K of the weights W_1, W_2, W_3, W_4, and W_5 respectively, and it is evident that the first of these intercepts is identically equal to the sum of the other five. Hence the moment of P equals the sum of the moments of W_1, W_2, W_3, W_4, and W_5 about K; that is, ga exactly represents the value of P.

To show that G is the centre of gravity of the weights W_1, W_2, W_3, W_4, and W_5. The sum of the moments about G of all the weights to the left of G, that is of W_1, W_2, and W_3, is $hk + km + mn = hn$; again, the sum of the moments about G of all the weights to the right of G, that is of W_4 and W_5, is $np + ph = hn$; hence the sum of the moments about G of all the weights to the left equals the sum of the moments about G of all the weights to the right; and since these moments tend to turn the beam in opposite directions, the one sum destroys the other, or the sum of the moments of all the weights W about G is zero; that is, G is the centre of gravity of the weights.

To show that $\overline{zs}$, the ordinate of the polygon, measured on the vertical scale equals the bending moment at z,

$$\begin{aligned}\overline{zs} &= \overline{zq} - \overline{qr} - \overline{rs}, \text{ all measured on the scale for verticals} \\ &= \text{Mom. of } P - \text{Mom. of } W_1 - \text{Mom. of } W_2, \text{ all about } z \\ &= \text{Bending Moment at } z.\end{aligned}$$

Corollaries.—If there is only one concentrated load, the maximum bending moment occurs at the load, and the bending moment diagram is a triangle. As this is a very important case, we will give it special consideration afterwards. If the load is at the centre, the maximum bending moment is at the centre, and the bending moment diagram is an isosceles triangle.

If the loads on the two halves of the span are similar and symmetrically placed about the centre, and if there be no weight at the centre, then P equals the sum of the loads on the left half; and when you subtract the weights from P according to the rule, there will be no remainder when you have subtracted that weight on the left half which is nearest the centre; in this case the maximum bending moment occurs equally at either of the weights nearest the centre, and at any point between them, so that the bending moment diagram is a polygon symmetrical about the vertical through the centre, and has the side above the centre horizontal; the centre, then, is *one* of the points at which the maximum occurs. If, however, a weight be at the centre, then P equals the sum of the weights on the left half of span, and half the weight at the centre; so that in subtracting the weights from P, according to the rule, the remainder is positive till we subtract the one at the centre, when the remainder is negative for the first time; in this case the maximum bending moment occurs at the centre, and the bending moment diagram is a polygon symmetrical about the vertical line through the centre, and having its highest apex on that line. In these two, which are the only possible cases of loading with similar loads concentrated at points, and symmetrically arranged about the centre, the bending moment at the centre is the value of the maximum bending moment. Taking the origin at the centre, as Rankine does, M_0 equals the bending moment at the centre; for loading symmetrical about the centre

$$M_0 = \text{Maximum Bending Moment.}$$

Examples.

14. A beam 24 feet span is loaded with 20, 30, and 40 tons at points dividing it into equal intervals. Find the maximum bending moment and the point where it occurs.

As in example No. 2, we have $P = 40$ tons, deduct 20 and it leaves 20, deduct 30 and the remainder is negative for the first time; hence the maximum occurs under the load 30, that is at 12 feet from the left end, and

$$\text{max. } M_{12} = P \times 12 - W_1 \times 6;$$
$$= 40 \times 12 - 20 \times 6 = \ldots\ldots\ldots\ 360 \text{ ft.-tons.}$$

15. In example No. 14, find the bending moments at the other two weights.

$$M_6 = P \times 6 = 40 \times 6 = \ldots\ldots\ldots\ 240 \text{ ft.-tons.}$$
$$M_{18} = P \times 18 - W_1 \times 12 - W_2 \times 6;$$
$$= 40 \times 18 - 20 \times 12 - 30 \times 6 = 300 \quad ,,$$
$$\text{or} \quad M_{18} = Q \times 6 = 50 \times 6 = \ldots\ldots\ldots\ 300 \quad ,,$$

16. In example No. 14, find the bending moments at points midway between the weights.

$$M_9 = \text{an average of } M_6 \text{ and } M_{12} = 300 \text{ ft.-tons.}$$
$$M_{15} = \text{an average of } M_{12} \text{ and } M_{18} = 330 \quad ,,$$
$$\text{or} \quad M_{15} = P \times 15 - W_1 \times 9 - W_2 \times 3;$$
$$= 40 \times 15 - 20 \times 9 - 30 \times 3 = 330 \quad ,,$$

17. In example No. 3, find the bending moments at intervals of five feet.

$M_5 = P \times 5 = 11{\cdot}6 \times 5 =$ .... 58 ft.-tons.

$M_{10} = P \times 10 - W_1 \times 2 = 11{\cdot}6 \times 10 - 6 \times 2 =$ 104 „

$M_{15} = P \times 15 - W_1 \times 7 = 11{\cdot}6 \times 15 - 6 \times 7 =$ 132 „

$M_{20} = P \times 20 - W_1 \times 12 - W_2 \times 2;$

$= 11{\cdot}6 \times 20 - 6 \times 12 - 14 \times 2 =$ .... 132 „

$M_{25} = P \times 25 - W_1 \times 17 - W_2 \times 7 - W_3 \times 1;$

$= 11{\cdot}6 \times 25 - 6 \times 17 - 14 \times 7 - 8 \times 1 =$ 82 „

18. Find the maximum bending moment in example No. 3.

Ans. $M_{18} = 148{\cdot}8$ ft.-tons, maximum.

19. Find the maximum bending moment in example No. 12.

Ans. $M_{19} = 256$ ft.-tons, maximum.

Or, taking the centre as origin, $M_2 = 256$ ft.-tons, maximum, and is at the wheel transmitting $W_3 = 11$ tons.

Thus for the position of the locomotive given in No. 12, that is with its fore wheel 6 feet from the left end, the maximum bending moment 256 ft.-tons occurs at the wheel transmitting $W_3 = 11$ tons; whereas for the position of the locomotive given in figs. 17 and 18, that is with its fore wheel one foot from the left end, the maximum bending moment 261 ft.-tons occurred at the wheel transmitting $W_4 = 12$ tons. In like manner, we may find the maximum bending moment for the locomotive in a variety of positions, in each case observing the wheel under which it occurs. For the two positions which we have investigated, observe that the maximum in the one case is greater than the maximum in the other, and occurs at a different place. It may be that the maximum for some third position is greater than either, and occurs at some other place; so that we do not know that 261 ft.-tons is the greatest possible bending moment that can be produced upon the beam by the locomotive, nor are we even sure that 261 ft.-tons is the greatest bending moment that can possibly occur at the point two feet to the left of the centre of span, for it is possible that some new position of the locomotive may produce a greater bending moment there.

20. A beam 40 feet span supports four weights $W_1 = 50$, $W_2 = 10$, $W_3 = 20$, and $W_4 = 30$ cwts. at points whose abscissæ, measuring from the centre to left and right, are $x_1 = 10$, $x_2 = 2$, $x_3 = -12$, and $x_4 = -16$ feet. Find the supporting force at the left end, the maximum bending moment, and the place where that maximum occurs.

Ans. $P = 50$ cwts.; and since $P - W_1 = 0$, the maximum bending moment occurs at x_1, at x_2, and at every intermediate point, and

$$M_{10 \text{ to } 2} = 500 \text{ ft.-cwts. maximum.}$$

21. A beam 50 feet span has weights of 5, 8, 9, 12, 9, 8, and 5 cwts. placed at equal intervals of 5 feet, in order, upon the span, and with the load 12 cwts. at the centre. Find the maximum bending moment.

Ans. $M_0 = 500$ ft.-cwts. maximum.

Cantilever loaded with unequal weights fixed at irregular intervals.—A cantilever (fig. 20) 12 feet long supports three weights, $W_1 = 8$, $W_2 = 6$, and $W_3 = 4$ tons at points whose abscissæ reckoned from the fixed end are $x_1 = 12$, $x_2 = 10$, and $x_3 = 4$ feet. This loading is shown also on fig. 11, but on that figure the cantilever projects beyond W_1, and so c is greater than x_1; it is evident, however, that the part which so projects is not strained, so that although a cantilever does so project, yet, for purposes of calculation, its length may be considered as the distance from the fixed end to the most remote load; this is shown on fig. 20, where $c = x_1$.

We will now calculate the bending moments at the fixed end, and at points where the weights stand, systematically from the forces upon the left hand portion.

$M_{x_1 \text{ or } c} =$ .... 0 ft.-tons.

$M_{x_2} = W_1(x_1 - x_2)$;

$M_{10} = 8 \times 2 =$ .... 16 „

$$M_{x_3} = W_1(x_1 - x_3) + W_2(x_2 - x_3)\,;$$

$$M_4 = 8 \times 8 + 6 \times 6 = \ldots\ldots\ldots 100 \text{ ft.-tons.}$$

$$M_0 = W_1x_1 + W_2x_2 + W_3x_3\,;$$

$$= 8 \times 12 + 6 \times 10 + 4 \times 4 = \ldots\ldots\ldots 172 \quad ,,$$

The Bending Moment at any point is the sum of the products got by multiplying each weight to the left of that point by its distance therefrom; the above is an arithmetical sum, since all the weights tend to bend the left portion downwards. Having found the bending moment at one point, we may derive the bending moment at another point nearer the fixed end and such that no weight intervenes, by adding the product of the sum of all the weights to the left into the distance between the two points, since no *new* weights have to be considered, and all the leverages have increased by the distance between the two points. Hence the bending moments increase uniformly in each interval as you move towards the fixed end.

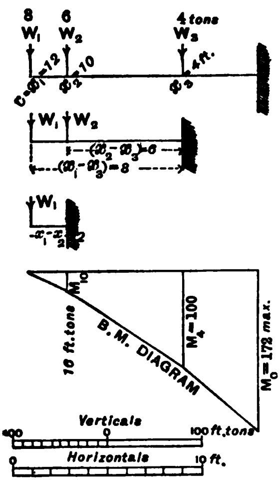


Fig.20.

Bending Moment Diagram.—With a scale of feet for horizontals, lay off the length (fig. 20) and plot the positions of the loads; at each of these points, and at the fixed end, draw a vertical ordinate *downwards*, equal to the bending moment thereat, upon a suitable scale of ft.-tons for verticals; join the ends of these ordinates by straight lines. The ordinates are drawn *downwards* to signify that the moments on a cantilever are of a different sign, as compared to those on a beam. In either case the moments are all of one sign, which will always be reckoned as positive.

Maximum Bending Moment.—It is evident that the maximum bending moment is at the fixed end, and is

$$M_0 = \Sigma(Wx).$$

Graphical Solution.—Draw a vertical line through K, the fixed end, and draw the vertical lines W_1, W_2, W_3 at the given horizontal distances apart upon a scale for dimensions (fig. 21). Draw ab, bc, cd equal respectively to the forces W_1, W_2, W_3 upon a scale for forces; that is, equal to 8, 6, and 4 tons in our example. Choose a pole O on the horizontal line passing through a, and such that the distance Oa is some convenient integral number upon the scale for dimensions; on the diagram $Oa = 10$; and draw Ob, Oc, and Od. From A any point on W_1 draw AB parallel to Ob, and meeting W_2 at B; similarly, draw BC, CD, and AK parallel, respectively, to Oc, Od, and Oa. Then $ABCDK$ is the bending moment diagram, its vertical ordinates give the bending moment at each point of the span upon a scale for verticals, obtained by subdividing

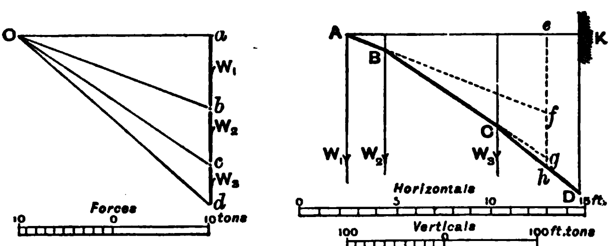


Fig. 21.

the divisions on the scale for forces by the number which Oa measures on the scale for dimensions; thus, on the figure this new scale is made by subdividing each division on the scale for forces into 10 equal parts, since $Oa = 10$ on the scale for dimensions.

Proof.—At any point e draw the ordinate eh, and produce those sides of the polygon that lie to the left till they meet it. On the figure, they meet it at e, f, g, h. As in the proof to fig. 19, the intercept on the vertical through e made by the pair of sides from any angle of the polygon, will, when measured upon the scale for verticals, give the moment about e of the force at that angle. Now the ordinate at e is the sum of the intercepts made by the pairs of sides from each angle to the left, and so will give on the vertical scale the sum of the moments of the forces to the left of e; that is, the bending moment at e. On fig. 21, ef the intercept by the pair of sides from A, gives the moment of W_1 about e; fg, the intercept by the pair of sides from B, gives the moment of W_2 about e, and gh gives the moment of W_3 about e. Hence eh gives the sum of these three moments, that is the bending moment at e.

Beam loaded at the centre.—Fig. 22. Let W be the load at the centre. By symmetry $P = Q = \frac{1}{2}W$. For a section distant from the centre x towards the left, consider the forces on the left hand portion. The only force is P, and its leverage is $(c - x)$; hence

$$M_x = P(c-x) = \frac{W}{2}(c-x),$$

the equation to the bending moment.

The value of M_x is zero at the end, that is where $x = c$; it increases uniformly as x decreases, that is, as you approach the centre, and it is greatest where $x = 0$, that is at the centre; by symmetry for the other half of the span, the value will decrease uniformly till it is again zero at the right hand end; hence the maximum bending moment

$$M_0 = \frac{W}{2}c = \frac{1}{4}.Wl.$$

In Rankine's "Applied Mechanics," the maximum bending moment in each case is given in the above form, viz.,

maximum bending moment = constant × total load × span.

$$\text{or} \qquad M_0 = m.W.l.$$

Throughout a great portion of that work, m stands for this constant, which he calls the numerical co-efficient of the maximum bending moment expressed in terms of the load and span. The value of m depends upon the *manner* of loading and of support. In the case we have just solved, we specify the mode of support by calling the piece a beam, and the manner of loading when we say that W is at the centre, and we find $m = \frac{1}{4}$.

Bending Moment Diagram.—Upon a convenient scale of feet, lay off BC, fig. 22, equal to the span; construct a triangle with its apex above the centre O. Draw a scale of foot-lbs. to measure verticals upon, such that OA shall measure upon it one-fourth of the product of the load in lbs. into the span in ft.

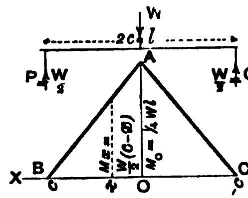


Fig 22.

Graphical Solution.—The simplest graphical solution is to draw the bending moment diagram as above; one which is purely graphical may be made as in fig. 19, and it will not be necessary to use the first pole O', &c., as we know that $P = Q = \frac{W}{2}$. Draw ab vertical and equal to the load W; from its middle point draw a horizontal line, choose O at a distance from ab equal to some convenient integral number on the scale for dimensions, and draw Oa and Ob. Then fig. 22 is constructed by drawing BA parallel to Oa, and AC parallel to Ob; and a scale for verticals is obtained by subdividing the scale for forces by the number chosen for the distance of O from ab.

Cantilever loaded at the end.—Fig. 23. To find the bending moment at a section distant x from the fixed end K, consider the loads to the left of that section. The only force is W, and its leverage about the section at x is $(c-x)$, and we have

$$M_x = W(c-x),$$

the equation to the bending moment.

The value of M_x is zero when x equals c, that is at the free end; it uniformly increases as x decreases, and is a maximum when $x = 0$, that is at the fixed end; the maximum bending moment is

$$M_0 = Wc = W.l,$$

and the value of the constant is $m = 1$.

Bending Moment Diagram.—Upon a convenient scale of feet, lay off KA, fig. 23, equal to the length; draw below KA the right angled triangle FAK, with the right angle at the fixed end, and construct a scale of ft.-lbs. for verticals, such that KF may measure upon it the product of the load in lbs. into the length in feet.

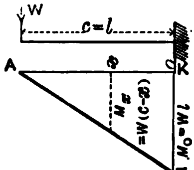


Fig. 23.

Graphical Solution. — The simplest graphical solution is to draw the bending moment diagram as above; one which is purely graphical may be made as on fig. 21. Make ab equal to W, and from a draw a horizontal line; choose a point O such that Oa is a convenient integer on the scale for dimensions, and join Ob. Draw AF, fig. 23, parallel to Ob, and a scale for verticals is obtained by subdividing the scale for forces by the value of Oa.

Beam uniformly loaded.—Fig. 24. Let w be the intensity of the uniform load in lbs. per foot of span. This is represented by a load area, consisting of a rectangle of height w feet standing on the span, and weighing one lb. per square foot. The total load W is the area of this rectangle, so that $W = 2wc$, or $w = \frac{W}{2c}$.

To find the bending moment at a section distant x from the centre O.—Consider the load area standing upon the portion of the span to the left of that section; it consists of a rectangle of length $(c-x)$ ft., its area is $w(c-x)$ sq. ft., and its weight is $w(c-x)$ lbs.

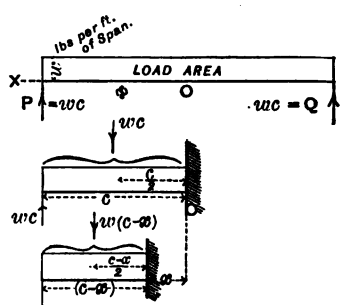


Fig. 24.

This weight may be considered to be concentrated at the centre of gravity of the area,—that is at its middle point; this gives a bending moment about the section equal to that for the actual distribution. We have two forces to the left of the section, $P = wc$ lbs., half the total load acting upwards with a leverage of $(c-x)$ ft., and $w(c-x)$ lbs. acting downwards with a leverage of $\frac{1}{2}(c-x)$ ft.; hence

$$M_x = P(c-x) - w(c-x) \times \frac{c-x}{2}$$

$$= wc(c-x) - \frac{w}{2}(c-x)^2$$

$$= \frac{w}{2}(c-x)(2c-c+x) = \frac{w}{2}(c-x)(c+x)$$

$$= \frac{w}{2}(c^2-x^2);$$

$$= \frac{W}{4c}(c^2-x^2), \text{ the equation to the bending moment.}$$

The value of M_x is zero where $x = \pm c$, that is at the two ends; it increases as x decreases numerically from c and from $-c$, that is from both ends towards the centre; and it is greatest where $x = 0$, that is at the centre. By making $x = 0$, we have $M_0 = \frac{1}{4}Wc$; or by considering the section at O, the maximum bending moment is

$$M_0 = P.c - wc.\frac{c}{2}$$

$$= wc.c - \frac{w}{2}c^2 = \frac{wc^2}{2};$$

$$= \tfrac{1}{4}Wc, \quad \text{putting } w = \frac{W}{2c};$$

$$= \tfrac{1}{8}Wl, \quad \text{putting } c = \tfrac{1}{2}l.$$

In this case, the value of the constant is $m = \frac{1}{8}$.

Bending Moment Diagram.—Examining the equation to the bending moment, we see that the ordinate M_x equals a constant term, minus a term in x^2; hence we know that the locus is a parabola, with its axis vertical and with its apex A (fig. 25) exactly above the origin O at the height $\frac{Wc}{4}$; and that the modulus of the parabola is $\frac{W}{4c}$, the co-efficient of x^2, so that the principal equation to the parabola, that is taking A as origin, is

$$Y = \frac{W}{4c} X^2.$$

Graphical Solution.—With a scale of feet for horizontals, lay off the span BC, fig. 25, and draw a vertical OA upwards through O; apply the parallel rollers to BC; place *any* parabolic segment cut upon pear-tree or card-board against the rollers (see fig. 15), with its apex on the vertical through O; shift the rollers till the curved edge passes through B and C, which it will do simultaneously, and draw the curve BAC; construct a scale of ft.-lbs. for verticals, such that

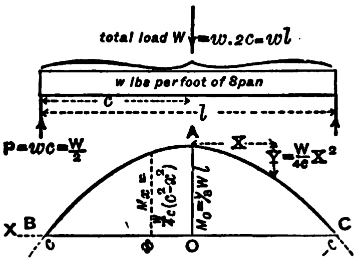


Fig. 25.

$$OA = \tfrac{1}{8} W.l.$$

Cantilever uniformly loaded.—Fig. 26. Consider a section at the distance x from the fixed end. The load area standing on the portion to the left of that section is a rectangle of length $(c-x)$ ft., and height w ft.; its area represents a weight of $w(c-x)$ lbs., which may be considered to be concentrated at the centre of gravity of the rectangle; and so, to the left of the section, there is to be taken into account only one force $w(c-x)$ lbs., having a leverage about the section of $\frac{1}{2}(c-x)$ ft.; hence

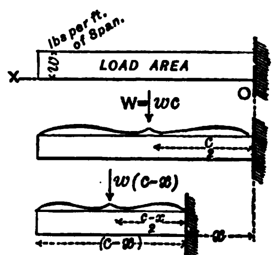


Fig. 26.

$$M_x = w(c-x) \cdot \frac{c-x}{2} = \frac{w}{2}(c-x)^2$$

$$= \frac{W}{2c}(c-x)^2,$$

the equation to the bending moment.

The value of M_x is zero where $x = c$, that is at the free end; it increases as x decreases; and it is greatest where $x = 0$, that is at the fixed end. Putting $x = 0$ we have the greatest value, or directly from the figure we find the maximum bending moment

$$M_0 = W \cdot \frac{c}{2},$$
$$= \tfrac{1}{2} Wl.$$

In this case, the value of the constant is $m = \frac{1}{2}$.

Bending Moment Diagram.—For the sake of comparison with the diagrams for beams, we may consider the bending moments on a cantilever to be negative, when the equation becomes

$$M_x = -\frac{W}{2c}(c-x)^2$$
$$= \frac{W}{2c}(2c-x)x - \frac{Wc}{2};$$

the locus is a parabola with its axis vertical and to the left of O, and with its apex on the axis of X. The value $x = c$ makes $(2c-x)x$ greatest, and therefore makes $M_c = 0$ a *positive* maximum; so that the apex lies to the left of O at a distance c,—that is, the apex is at the free end. Since the coefficient of x^2 is $\frac{W}{2c}$, the principal equation to the parabola is

$$Y = \frac{W}{2c} X^2.$$

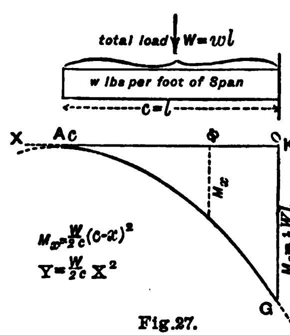


Fig. 27.

Graphical Solution.—With a convenient scale of feet, lay off KA, fig. 27, equal to the length; place the parallel rollers to KA; set *any* parabolic segment against the rollers, with its apex at A; draw the curve AG, till it meets the vertical through K at G. Construct a scale of ft.-lbs. for verticals, such that KG may measure upon it one-half the product of the total load in lbs. into the span in feet.

Examples.

22. A cantilever 12 feet long bears four loads $W_1 = 8$, $W_2 = 6$, $W_3 = 9$, and $W_4 = 12$ tons at distances from the fixed end of $x_1 = 12$, $x_2 = 8$, $x_3 = 6$, and $x_4 = 2$ feet.

Find the bending moment at each weight, and also the maximum bending moment.

Ans. $M_{12} = 0$, $M_8 = 32$, $M_6 = 60$, $M_2 = 152$, and $M_0 = 212$ ft.-tons.

23. In the previous example, find the bending moments at points midway between each pair of weights.

Ans. They are averages of those at the weights on each side of such points, or they may be calculated independently.

$$M_{10} = 16,\ M_7 = 46,\ M_4 = 106 \text{ ft.-tons.}$$

24. A cantilever is loaded with weights of 8, 6, and 4 tons at distances of 12, 10, and 4 feet from the fixed end. Find the bending moment at each weight, and draw a bending moment diagram upon a large scale; see fig. 20.

25. Draw also a bending moment diagram by the graphical construction, fig. 21, to large scales. From either diagram, by measurement, find the bending moments at intervals of two feet.

Ans. $M_0 = 172$, $M_2 = 136$, $M_4 = 100$, $M_6 = 72$, $M_8 = 44$, $M_{10} = 16$, $M_{12} = 0$ ft.-tons.

26. A cantilever 12 feet long is uniformly loaded with 3 cwts. per foot run. Find the equation to the bending moment.

Ans. $$M_x = \frac{3}{2}(12-x)^2 \text{ ft.-cwts.}$$

27. In the previous example, find the bending moments

at intervals of two feet by substituting for x into the equation.

Ans. $M_0 = \frac{3}{2}(12-0)^2 = 216$ ft.-cwts.

$M_2 = \frac{3}{2}(12-2)^2 = 150$ ft.-cwts.

$M_4 = 96$, $M_6 = 54$, $M_8 = 24$, $M_{10} = 6$, $M_{12} = 0$.

28. In example No. 26, find M_4, directly, by taking a section at the point $x = 4$.

Ans. $M_4 = 3(12-4) \times \frac{12-4}{2} = 96$ ft.-cwts.

29. A cantilever 20 ft. long is uniformly loaded with 2 tons per foot run. Find the maximum bending moment.

Ans. $M_0 = m.W.l = \frac{1}{2} \times 40 \times 20 = 400$ ft.-tons.

30. For the previous example, find the bending moment at the centre.

The bending moment diagram is a parabola, with its apex at the free end; for the middle point, the horizontal ordinate is half that for the fixed end, measuring from the apex; and since the verticals vary as the squares of the horizontals, the bending moments at the centre and at the fixed end are in the proportion of one and four.

Ans. $M_{10} = 100$ ft.-tons.

31. Find the principal equation to the parabola which, in each case, forms the bending moment diagram in examples Nos. 26 and 29.

Ans. $Y = \frac{3}{2}X^2$, and $Y = X^2$.

32. A beam 20 ft. span supports a load of 20 tons at its centre. Find the maximum bending moment, and the bending moments at points midway between the centre and each end.

Ans. $M_0 = m.W.l = \frac{1}{4} \times 20 \times 20 = 100$ ft.-tons,

and $M_5 = M_{-5} = \frac{1}{2}M_0 = 50$ ft.-tons.

33. A beam 20 ft. span supports a load of 20 tons uniformly distributed. Find M_0 and M_5.

$$Ans.\ M_0 = m.W.l = \tfrac{1}{8} \times 20 \times 20 = 50 \text{ ft.-tons};$$

and $$M_5 = \tfrac{3}{4}M_0,$$

as may readily be seen from a bending moment diagram.

34. Find the equation to the bending moment in the previous example, and calculate M_6 from that equation. Also find the principal equation to the parabola which forms the bending moment diagram.

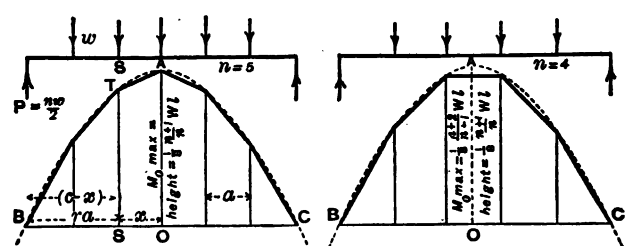
$$Ans.\ M_x = \frac{W}{4c}(c^2 - x^2) = \tfrac{1}{2}(100 - x^2);\ M_6 = 32 \text{ ft.-tons}.$$

$$Y = \tfrac{1}{2}X^2.$$

35. In example 33, find M_6, directly, by taking a section at the point $x = 6$.

$$M_6 = P(10-6) - 1 \times (10-6) \times \frac{10-6}{2}$$
$$= 10 \times 4 - 4 \times 2 = 32 \text{ ft.-tons}.$$

Beam divided at a number of points into equal intervals, and loaded with equal weights at these points.—Fig. 28. Let W,



n equal loads at equal intervals. W = total load. l = span.
If W be the same in both so also is M_0 the max.
Fig. 28.

the total load, be distributed over the span l, in n parts each equal to w, and at equal intervals a apart; then

$nw = W$, $(n+1)a = 2c = l$, and $P = \frac{1}{2}nw$. Take S, a point directly under one of the weights, and let $BS = ra$, then r will be a whole number; and if x be the distance of S from O the centre, then $ra = (c-x)$. On the portion of the beam to the left of S, there are in all r forces, viz., $P = \frac{1}{2}nw$ acting upwards with a leverage about S of ra, and $(r-1)$ forces each equal to w and acting downwards; the nearest to S has a leverage a, the next a leverage $2a$, the next a leverage $3a$, &c., and the last a leverage $(r-1)a$; hence

$$M_x = P.ra - w.a - w.2a - w.3a \ldots - w.(r-1)a$$

$$= \frac{nw}{2}.ra - wa(1+2+3\ldots+\overline{r-1})$$

$$= \frac{nw}{2}.ra - wa\frac{(r-1)r}{2}$$

$$= \frac{w}{2}.ra(n+1-r).$$

But $w = \frac{W}{n}$, $ra = c - x$, $a = \frac{2c}{n+1}$, $\therefore\ r = \frac{(n+1)(c-x)}{2c}$;

substituting, we have

$$M_x = \frac{W}{2n}(c-x)\left\{n+1-\frac{(n+1)(c-x)}{2c}\right\}$$

$$= \frac{W}{2n}(c-x)(n+1)\frac{c+x}{2c}$$

$$= \frac{W}{4c}\ \frac{n+1}{n}(c^2-x^2),$$

which is the equation to the locus of T.

This equation gives the bending moment only at points where weights are, that is for values of x which are multiples of a, but not at intermediate points; it only differs from the equation we had for an uniform load, by the constant factor $\frac{n+1}{n}$. The locus of T, the tops of the ordinates at the points where the weights are situated, is parabola with its axis vertical, and its apex above O. Putting y instead of M_x for the ordinate to the curve at any point, we have

$$y = \frac{W}{4c}\frac{n+1}{n}(c^2 - x^2),$$

so that y is a maximum where $x = 0$, and

$$y_0 = \frac{W}{4c}\cdot\frac{n+1}{n}c^2 = \tfrac{1}{8}\frac{n+1}{n}W.l$$

a maximum, and the height of the apex A in every case.

On fig. 28, it will be seen that if n be odd, a weight comes exactly at the centre; so that y_0, the ordinate of the parabola, is the maximum bending moment; hence the maximum bending moment

$$M_0 = \tfrac{1}{8}\frac{n+1}{n}W.l, \qquad (n \text{ odd}),$$

and the value of the constant is $m = \tfrac{1}{8}\,\dfrac{n+1}{n}$.

When n is even, the maximum bending moment M_0 is less than y_0, and equals the ordinate of the parabola at the weight on either side of the centre; so that to find the value of M_0, it is only necessary to substitute for x half of an interval, that is $\tfrac{1}{2}a$, or $\dfrac{c}{n+1}$; hence the maximum bending moment

$$M_0 = y_{\left(\frac{c}{n+1}\right)} = \frac{W}{4c}\frac{n+1}{n}\left\{c^2 - \left(\frac{c}{n+1}\right)^2\right\}$$

$$= \tfrac{1}{8}\,\frac{n+2}{n+1}\,W.l \qquad (n \text{ even.})$$

and the value of the constant is $m = \tfrac{1}{8}\,\dfrac{n+2}{n+1}$.

The chords of the parabola give the bending moments at points intermediate between the weights, since the bending moment varies uniformly in these intervals.

COROLLARY. If the load be distributed in equal portions at equal intervals, the maximum bending moment exceeds that for the uniform distribution in the ratio $\frac{n+1}{n}$ or $\frac{n+2}{n+1}$, according as n, the number of parts into which the load is divided, is odd or even.

Thus suppose M_U is the maximum for uniform distribution; then if the load be concentrated at the centre, that is, if $n = 1$, —

$$\text{max. bending moment} = \frac{1+1}{1} M_U = 2M_U;$$ see figs. 22 & 25;

if concentrated equally at two points dividing the span into three equal intervals, that is if $n = 2$, —

$$\text{maximum bending moment} = \frac{2+2}{2+1} M_U = \frac{4}{3} M_U;$$

if concentrated equally at three points dividing the span into four equal intervals, that is if $n = 3$,—

$$\text{maximum bending moment} = \frac{3+1}{3} M_U = \frac{4}{3} M_U; \text{ \&c.}$$

The last two are each $\frac{4}{3} M_U$, and are therefore equal to each other; and for the same total load W placed on the span at equal intervals, the maximum bending moment M_0 will be the same whether the load be divided into an *even* number of equal parts, or the *next consecutive odd* number of equal parts.

The Bending Moment Diagram is the polygon formed by the chords of the parabola.

COROLLARY. If n be great, the polygon nearly coincides with the parabola, $\frac{n+1}{n}$ approaches unity, and the parabola is nearly the *same* as that for the load uniformly distributed; that is, if the load be concentrated equally at a great number of points equally apart, as, for instance, when a girder supports, at equal intervals, the ends of cross girders which carry equal loads, then the bending moments will be nearly the same as for the total load uniformly distributed.

Graphical Solution.—With a scale of feet for horizontals, lay off the span BC, fig. 28, and draw a vertical OA upwards through O. Apply the parallel rollers to BC; place *any* parabolic segment cut on pear-tree against the rollers (see fig. 15) with its apex on the vertical through O; shift the rollers till the curved edge passes through B and C, which it will do simultaneously, and draw the dotted curve BAC. Draw up verticals to meet the parabola from the points at which the weights are, and draw the chords of the parabola. Construct a scale of ft.-lbs. for verticals such that $OA = \frac{1}{8}\frac{n+1}{n} W.l$; where W = total load in lbs., l = span in feet, and n = the number of equal parts into which the load is divided.

Examples.

36. A beam 40 ft. span supports seven loads, each two tons, and placed symmetrically on the span at intervals of five feet. Calculate the maximum bending moment by substituting in the proper equation, and calculate at each load the height of the parabola which gives the bending moments.

Here $W = 14$ tons, $c = 20$ ft., $l = 40$ ft., and $n = 7$.

The maximum bending moment is

$$M_0 = y_0 = \frac{1}{8}\frac{n+1}{n} W.l = \frac{1}{8} \times \frac{8}{7} \times 14 \times 40 = 80 \text{ ft.-tons.}$$

The equation to the parabola is

$$y = \frac{W}{4c} \cdot \frac{n+1}{n}(c^2 - x^2) = \frac{14}{80} \times \frac{8}{7}(400 - x^2) = \frac{1}{5}(400 - x^2).$$

$\therefore$ at the weights, $M_{5 \text{ or } -5} = \frac{1}{5}(400 - 25) = 75$ ft.-tons.

$$M_{10 \text{ or } -10} = 60,\ M_{15 \text{ or } -15} = 35 \text{ ft.-tons.}$$

37. In example No. 36, calculate M_0 independently, by taking a section at the centre.

$$M_0 = P.c - 2\times 5 - 2\times 10 - 2\times 15$$
$$= 7\times 20 - 10 - 20 - 30 = 80 \text{ ft.-tons.}$$

38. A beam 39 ft. span supports twelve loads, each 10 cwts., and placed symmetrically on the span at intervals of 3 ft. Find the maximum bending moment, and the height of the parabola which gives the bending moments; also, from the equation to the parabola find the bending moments $M_{1\cdot 5}$, $M_{4\cdot 5}$, and M_6, all measured from the centre.

Here $W = 120$ cwts.; $c = 19\cdot 5$ ft.; $l = 39$ ft., and $n = 12$.

Max. bending momt., $M_0 = \frac{1}{8}\cdot\frac{n+2}{n+1}W.l = 630$ ft.-cwts.

Height of parabola, $y_0 = \frac{1}{8}\frac{n+1}{n}W.l = 633\frac{3}{4}$.

Equation to parabola, $y = \frac{W}{4c}\cdot\frac{n+1}{n}(c^2 - x^2) = \frac{5}{3}(380\frac{1}{4} - x^2)$;

hence $M_{1\cdot 5} = \frac{5}{3}\{380\frac{1}{4} - (\frac{3}{2})^2\} = 630 = M_0$;

$M_{4\cdot 5} = 600$; $M_{7\cdot 5} = 540$;

and $M_6 = 570$, an average of $M_{4\cdot 5}$ and $M_{7\cdot 5}$, as it is the ordinate of the middle point of the chord joining the tops of $y_{4\cdot 5}$ and $y_{7\cdot 5}$.

39. A beam 80 ft. span supports a load of 100 tons. Find the bending moments at the centre, and at twenty feet from the end of span; first, if the load be uniformly distributed; and, second, if it be distributed in equal amounts at intervals of two feet.

For uniform load, $M_0 = \frac{1}{8}W.l = \frac{1}{8}\times 100\times 80 = 1000$ ft.-tons.

and $M_{20} = \frac{3}{4}M_0 = \ldots\ldots\ldots\ldots$ 750 „

For distributed load, $n = 39, W = 100, l = 80$, and $c = 40$;

$$M_0 = \tfrac{1}{8}\frac{n+1}{n}W.l = \tfrac{1}{8} \times \tfrac{40}{39} \times 100 \times 80 = 1026 \text{ ft.-tons.}$$

$$M_{20} = \frac{W}{4c}\frac{n+1}{n}(c^2 - 20^2) = \ldots\ldots\ldots\ldots\ldots\ldots \quad 769 \quad ,,$$

In this case the bending moments exceed by $\frac{1}{39}$th, that is by $\frac{1}{n}$th, of themselves, the bending moments for the uniform load.

Beam uniformly loaded and with a load at its centre.—Fig. 29. Let U be the amount of the uniform load, then the bending moment at x due to it alone is $\frac{U}{4c}(c^2 - x^2)$; let W be the load at the centre, then the bending moment at x due to it alone is $\frac{W}{2}(c-x)$; summing these, we have

$$M_x = \frac{U}{4c}(c^2 - x^2) + \frac{W}{2}(c-x)$$

$$= \frac{U}{4c}(c-x)\left(c + x + \frac{2Wc}{U}\right),$$

the equation to the bending moment for positive values of x, that is, for the left half of the span. Putting y instead of M_x, we have

$$y = \frac{U}{4c}(c-x)\left(c + x + \frac{2Wc}{U}\right)$$

a curve, the ordinates of which are the bending moments for the left half of span; this curve is a parabola with its axis vertical, and its apex above BC the span. To find the position of the apex A_1, it is only necessary to find that value of x which makes y greatest; now y is greatest when

$$(c-x)\left(c + x + \frac{2Wc}{U}\right)$$

is a maximum; and since the sum of these two factors is constant, their product is greatest when they are equal; putting then

$$c+x+\frac{2Wc}{U}=c-x,$$

we have

$$x=-\frac{W}{U}c$$

as the value of x which makes y greatest; the negative sign denotes that A_1 lies to the *right* of O, so that OS_1 is to be laid off towards the right and equal to $\frac{W}{U}c$. The height of A_1 is the value of y when we substitute this value for x; that is,

$$S_1A_1=\frac{U}{4c}\left(c+\frac{W}{U}c\right)\left(c-\frac{W}{U}c+\frac{2Wc}{U}\right)$$

$$=\left(\frac{W+U}{U}\right)^2\times(\tfrac{1}{8}Ul)$$

$$=\left(\frac{W+U}{U}\right)^2 \text{ times the height of } A_0, \text{ the}$$

apex of the parabola for the uniform load alone.

A_2DC is the same parabola with its apex at the symmetrical point A_2, and the portion DC gives the bending moments for the right half of the span. Since the co-efficient of x^2 is $4c$, the principal equation to the parabolas A_1DB and A_2DC, each referred to its own apex as origin, is

$$Y=\frac{U}{4c}X^2;$$

this is also the principal equation to the parabola BA_0C for the uniform load alone, so that all three parabolas are identical.

We might suppose the diagram for the uniform load alone to consist of two parabolas lying on the top of each other; and that upon the addition of the load W at the centre, they both move upwards, while the one moves towards the right and the other towards the left.

The *Bending Moment Diagram* is *BDC*. It can be shown that the tangent at *D* to the parabola A_2DC cuts off *CE* equal to the height of A_0. *ODEC* is an *approximate* bending moment diagram made with *straight* lines; and it is *safe*, since the ordinate of any point on *DE* is greater than the ordinate for the corresponding point on *DC*.

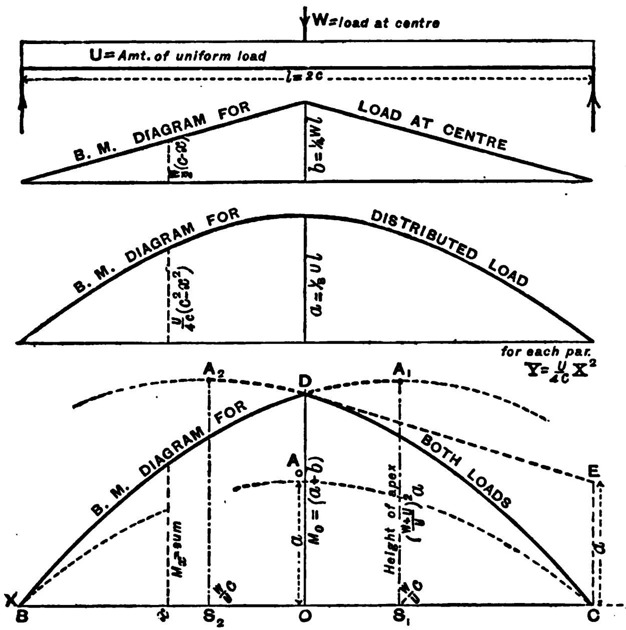


Fig. 29.

Graphical Solution.—With a scale of feet for horizontals, lay off *BC* equal to the span, fig. 29. From the centre *O* lay off OS_1 and OS_2 to the right and left, each equal to $\frac{W}{U}c$, and draw verticals upwards from S_1, *O*, and S_2. Apply the parallel rollers to the span *BC*; place *any* parabolic segment against the rollers, as in fig. 16, with its apex on the vertical through S_1; shift the rollers till the curved edge passes through *B*, the end of the span on the *opposite* side of the centre from S_1, and draw the curve BDA_1. Again, place the segment with its apex on the vertical through S_2; shift the rollers till the curved edge passes through the end C, and draw the curve CDA_2; then BDC is the bending moment diagram. The scale, of say ft.-lbs., for verticals is to be constructed such that OD measures $\frac{1}{8}(U+2W)l$, where U and W are in lbs. and l is in feet. The same scale may be constructed as follows:—Place the parabolic segment with its apex on the vertical through O; shift the rollers till the curved edge passes through B and C; draw the curve BA_0C, and make a scale of ft.-lbs. for verticals, such that OA_0 measures upon it $\frac{1}{8}Ul$, where U is in lbs. and l is in feet.

COROLLARY.—For the same uniform load, although different loads be put at the centre, A_1DB is always the *same* parabola; as the load at the centre increases, the apex A_1 moves from the centre, and the arc DB is a part of the wing of that parabola further from the apex. Now the wing of a parabola gets flatter as its distance from the apex increases; hence, if U be constant and W be increased, BD becomes flatter and flatter; and if W be very great compared to U, BD is sensibly a straight line.

Cantilever uniformly loaded and with a load at its free end.—Fig. 30. As in the previous case, add the bending moments at the section distant x from O the fixed end, due to the loads separately; thus

$$-M_x = \frac{U}{2c}(c-x)^2 + W(c-x).$$

We consider the bending moments on a cantilever negative as compared with those on a beam, and so they will be represented by ordinates drawn *down* from the span instead of *up* as in the case of beams. If we further put y instead of M_x for the ordinate at the point on the curve corresponding to *any* value of x, then y will be M_x only for values of x from 0 to c; and

$$-y = \frac{U}{2c}(c-x)^2 + W(c-x) = \frac{U}{2c}(c-x)\left(c-x+\frac{2Wc}{U}\right)$$

$$\text{or} \quad y = \frac{U}{2c}(c-x)\left(x-c-\frac{2Wc}{U}\right).$$

This curve is a parabola with its axis vertical and its apex above the span. To find the position of the apex A, it is only necessary to find that value of x which makes y greatest; now y is greatest when $(c-x)\left(x-c-\frac{2Wc}{U}\right)$ is a maximum; and since the sum of these two factors is constant, their product is greatest when they are equal; putting then

$$x-c-\frac{2Wc}{U}=c-x,$$

we have $$x=\left(1+\frac{W}{U}\right)c$$

as the distance of A to the *left* of O, or $\frac{W}{U}c$ as the distance of A to the left of E; that is, the apex of the parabola is beyond the free end by the distance $\frac{W}{U}c$, or the same fraction of the span that W the concentrated load is of U the distributed load. The height of A above OE is the value of y when we substitute the above value for x; or—

$$\text{Height of } A=\frac{U}{2c}\left(c-\frac{U+W}{U}c\right)\left(\frac{U+W}{U}c-c-\frac{2Wc}{U}\right)$$

$$=\left(\frac{W^2}{U^2}\right)(\tfrac{1}{2}Ul)$$

$$=\left(\frac{W}{U}\right)^2$$ times the maximum bending moment for the uniform load alone. Since the co-efficient of x^2 is $\frac{U}{2c}$, the principal equation to the parabola AEF is

$$Y=\frac{U}{2c}X^2;$$

and it is therefore the *same* parabola as that for the uniform load alone.

The *Bending Moment Diagram* is *EFO*, formed by the parabola *EG*, whose apex is at *E* and which makes the diagram for the uniform load alone, shifted without turning till its apex is at *A*.

Graphical Solution.—With a scale of feet for horizontals, lay off *OE* (fig. 30) equal to the length, and produce it to *S*, so that $ES = \frac{W}{U}c$, or is the same fraction of the length as the load at the end is of the uniform load, and draw a vertical upwards through *S*; apply the parallel rollers to the line *OE*; place any parabolic segment against the rollers with its apex on the vertical through *S*; shift

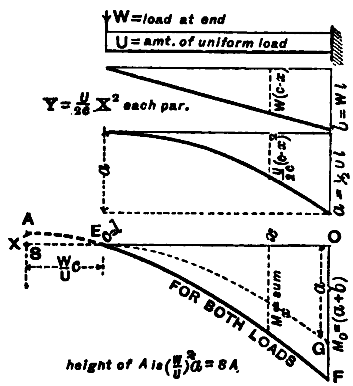


Fig. 30.

the rollers till the curved edge passes through the free end *E*, and draw the curve *AEF*; *OEF* is the bending moment diagram. The scale of say ft.-lbs. for verticals is to be constructed such that *OF* measures $\left(\frac{U}{2} + W\right)l$, where *U* and *W* are in lbs., and *l* is in feet. The same scale may be constructed as follows:—Place the parabolic segment with its apex on the vertical through the free end *E*, and move the rollers till the apex comes to *E*; draw the dotted curve *EG*, and make a scale of ft.-lbs. for verticals such that *OG* measures upon it $\frac{1}{2}Ul$, where *U* is in lbs., and *l* is in feet.

COROLLARY. If W be great compared to U, then EF is sensibly a straight line; because if U be constant AEF is the *same* parabola, no matter what W may be; as W increases, A moves from E, and the arc EF, part of the wing of that parabola further from the apex, becomes flatter and flatter.

Beam uniformly loaded on a portion of the span.—Fig. 31. Let $2c$ be the length of the span, and O its centre; let G be the centre of the load area, and $2k$ the extent of the load. The intensity of the uniform load is w lbs. per foot run; and, as in the previous case, we take w as the height of the load area in feet, so that every square foot of load area represents one lb. The total load area is a rectangle of height w ft., and length $2k$ ft.; its area is $2wk$ square feet, so that the total load on the span is $2wk$ lbs., which may be supposed to be concentrated at G the centre of gravity of the load area; this gives the supporting forces P and Q as for the actual distribution. Let $\bar{x} = OG$, the distance from the centre of span to the centre of gravity of the load; then

$$P = \frac{2wk}{2c}(c + \bar{x}) = \frac{wk}{c}(c + \bar{x}).$$

For any section between the left end of the span and of the load area, the only force to the left of the section is P, and the bending moment may be calculated just as if the whole load were at G; for such sections the bending moment increases uniformly from zero at the left end till we come to the section through the left end of the load area, and the bending moment diagram for that part of the span is a straight line sloping up from the left end of span till it meets the vertical through the left end of the load area. Similarly for the other end, the diagram is a straight line sloping up from the right end till it meets the vertical through the right end of the load area; the moments at the two ends of the load area are easily calculated from P and Q.

For sections within the load area, it will be convenient to choose G as origin; and let x, which must be not greater than k, be the distance to the section which may be either to the right or left of G; in our diagram it is to the right. We will consider x positive in the direction from the origin G *towards* the centre; in the diagram x is positive to the right. The load area standing upon the part of span to the left of our section consists of a rectangle of height w ft., and length $(k+x)$ ft.; its area is $w(k+x)$ square feet, representing a weight of $w(k+x)$ lbs. which may be supposed to

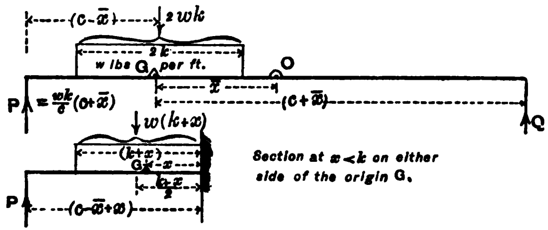


Fig. 31.

be concentrated at the middle of the area. To the left of the section there are two forces, P acting upwards with a leverage of $(c-\bar{x}+x)$ feet, and $w(k+x)$ lbs. acting downwards with a leverage of $\frac{1}{2}(k+x)$ feet; hence at this section

$$M_x = P(c-x+x) - w(k+x)\,.\,\frac{k+x}{2}$$

$$= \frac{wk}{c}(c+\bar{x})\,(c-\bar{x}+x) - \frac{w}{2}(k+x)^2;$$

arranging in powers of x,

$$M_x = \left(wkc - \frac{wk\bar{x}^2}{c} - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{2k\bar{x}}{c} - x\right)x.$$

This is the equation to a parabola with its axis vertical and its apex above the span; it is the equation to the bending moment, for values of x from k to $-k$; for any other value of x we may put y for the corresponding ordinate, but y will no longer be M_x. To find the distance of the apex from G, it is only necessary to find that value of x which makes M_x greatest; now M_x is greatest when the product $\left(\frac{2k\bar{x}}{c} - x\right)x$

is greatest; the sum of the two factors of this product is constant, so that the product is greatest when the factors are equal: putting $x = \frac{2k\bar{x}}{c} - x$

we have $$x = \frac{k}{c}\bar{x} \quad \text{or} \quad \frac{2k}{2c}\bar{x},$$

that is, $GS = \frac{\text{length of load}}{\text{span}} GO.$

It is evident that always

$$\bar{x} < c \quad \therefore \quad \frac{k}{c}\bar{x} < k;$$

or $\frac{k}{c}\bar{x}$, the horizontal distance that the apex is from G is less than k the half extent of load; that is, the apex is always above a point which is *within* the load area.

To find the height of the apex above the span; substitute the above value of x into the equation to the parabola, and

$$M_{\max} = \left(wkc - \frac{wk\bar{x}^2}{c} - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{2k\bar{x}}{c} - \frac{k\bar{x}}{c}\right)\frac{k\bar{x}}{c}$$

$$= \frac{wk}{2c^2}(c^2 - \bar{x}^2)(2c - k)$$

$$= \frac{W}{4c}(c^2 - \bar{x}^2)\left(2 - \frac{k}{c}\right).$$

The *position* of this maximum is easily remembered, but as the above expression for its value is complicated, it may be easier to calculate the bending moment at that point directly.

COROLLARY 1. If the centre of load be over the centre of span, the maximum bending moment is at their common centre.

COROLLARY 2. If k increases, $\bar{x}$ decreases ultimately, so that $(c^2 - \bar{x}^2)$ is increasing; at the same time, the product $k(2c - k)$ increases and becomes greatest when $k = c$; hence $M_{\max.}$ is greatest for the whole span loaded.

The *Bending Moment Diagram BHEC* (fig. 32), consists of the above parabola with its axis vertical and its apex A situated as described above, and drawn both ways till it meets the verticals through the ends of the load at H and E, and of the straight lines HB and EC. In the above equation to the parabola, the coefficient of x^2 is $\frac{w}{2}$, and the principal equation to the curve is

$$Y = \frac{w}{2} X^2;$$

this is the principal equation to the curve on fig. 25, so that in the present case the parabola is the *same* as that for an uniform load of intensity w over the whole span. BH and CE are the same as if the whole load were at G, therefore they meet at R, a point on the vertical through G; if the load were concentrated at G, BRC would be the bending moment diagram. At any point as L, the ordinate to HR is greater than that to HA, because the former is the product of P into BL, while the latter is that same product minus the

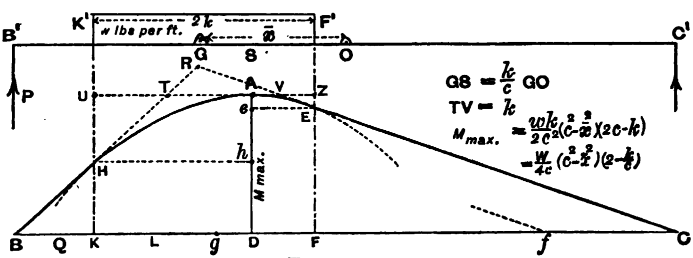


Fig. 32.

moment of the load area to the left of L; that is every point in HR is outside the curve. For such a point as Q, by substituting gQ for x in the equation to the parabola, its ordinate is again less than that of BH, so that BH is a tangent at H to the parabola, and EC is a tangent at E; T and V are the middle points of AU and AZ, therefore

$$TV = \tfrac{1}{2} UZ = k$$

the half extent of load.

COROLLARY. Since HAE is always the same parabola for the same value of w, it is evident that A will be higher when the parabola passes through B and C; that is, when the whole span is loaded. This is the same result as corollary 2, page 68, derived in this case by geometry.

Graphical Solution.—With a scale of feet for horizontals, lay off $B'C'$ (fig. 32) equal to the span, and $K'F'$ equal to the extent of load in its proper position. From G the centre of the load, lay off GS towards O the centre of the span, and equal to the same fraction of GO, as the extent of load is of the span. Apply the parallel rollers to $B'C'$; place any parabolic segment against the rollers with its apex at any point A on the vertical through S, and draw the curve HAE between the verticals through K' and F'; draw UAZ with the rollers; bisect AU in T, and AZ in V; produce TH to meet the vertical through B' in B, and VE to meet the vertical through C' in C, and join BC. The accuracy of the drawing will be tested by observing that BC should be horizontal, and that the slopes should meet the vertical through G in *one* point R. A scale of say ft.-lbs. for verticals is constructed as follows:—Place the segment with its apex on the vertical through the centre O, shift the rollers till the curved edge passes through B and C, and draw the curve BA_0C (not shown on fig. 32); then the height of the apex A_0 should measure $\frac{1}{8}wl^2$, where w is the intensity in lbs. per foot, and l is the span in feet.

Another method.—If the extent of the load be small, it will generally happen that BC will not be quite horizontal on account of the shortness of VE the line to be produced, and the following construction may be more satisfactory. Lay off the span BC, construct any triangle BRC with its apex R on the vertical through G; lay off Cf equal to half the extent of load, draw fT parallel to CR, and TV horizontal; TV is then equal to fC half the extent of load; make TA equal to TU, and DA is the maximum bending moment. This may be sufficient, but as many points on the parabolic arcs HA and EA as may be required can be plotted as on fig. 14. The scale for verticals is to be such that the height of R may measure upon it what the bending moment at G would be if the load were concentrated there.

Graphical Solution for Particular Case.—Fig. 33. When the load extends to one end, say B, of the span, the first graphical solution given above may be made shorter and more direct thus:—Lay off BC equal to the span; from G lay off GS towards O, so that $GS = \frac{k}{c} GO$; apply the rollers to BC; place any parabolic segment against the rollers with

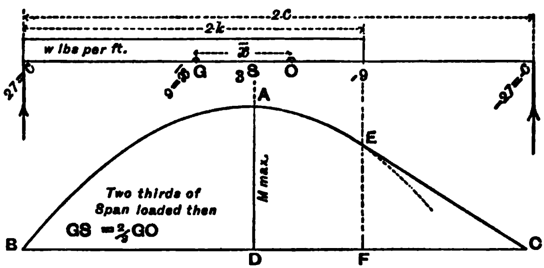


Fig. 33.

its apex on the vertical through S, and shift the rollers till the curved edge passes through B; draw the curve BAE, and join EC. Construct a scale as described above.

Cantilever uniformly loaded on a portion of its length.—Fig. 34. Let w lbs. per ft. run be the intensity of the load, and let k ft. be its extent. The loaded part AD may be considered to be a cantilever of length k uniformly loaded, so that the bending moment diagram is the parabola AE, as in fig. 27, for the whole length loaded. Suppose now the whole load concentrated at B the centre of gravity of the load, then BEF would be the bending moment diagram, as in fig. 23; and for points between D and K, the moment is the same as for the actual distribution of the load, because for sections at such points, the whole load is to the left whether we consider it concentrated at B or spread over AD. It is evident that for sections between B and D, the concentrated load would be all to the left, while only part of the actual distributed load is so situated. *EF* is a tangent at *E* to *AE*, since *B* is the middle point of *AD*.

The *Bending Moment Diagram AEFK* consists of the above parabola, with its axis vertical and its apex at the free end, drawn till it meets the vertical through the end of the load at *E*, and of *EF* the tangent at *E* to the curve.

Graphical Solution.—With a scale of feet for horizontals, lay off *AK* (fig. 34) equal to the length, and from the free end *A* lay off *AD* the extent of the load. Apply the parallel rollers to *AK*; place any parabolic segment against the rollers with its apex at the free end *A*, and draw the curve *AEG* to meet the verticals through the right end of the load and the fixed end, at *E* and *G* respectively; *EG* may only be dotted. Bisect *AD* in *B*; join *BE* with a dotted line, and produce it with a full line to meet the vertical through the fixed end at *F*, then *KAEF* is the bending moment diagram. Construct a scale of say foot

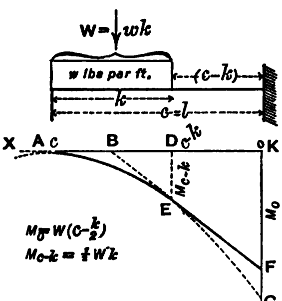


Fig. 34.

lbs. for verticals such that *KF* may measure $W\left(c - \frac{k}{2}\right)$, where $W =$ total load in lbs.; $c =$ length, and $k =$ extent of load, both in feet.

Examples.

40. A beam 54 feet span is loaded uniformly for two thirds of its length from the left end with 10 cwt. per foot run. Find the position and magnitude of the maximum bending moment. See fig. 33.

In this case $c = 27$ ft., and $OG = 9$ ft., measured to the left of O.

From G lay off $GS = 6$ ft. $= \frac{2}{3}GO$, since the load extends over two thirds of the span, and the maximum moment occurs at S, that is at 3 ft. to the left of the centre. Suppose the whole load $W = 360$ cwt. is concentrated at G; then

$$P = \frac{360}{54} \times 36 = 240 \text{ cwt.}$$

Taking a section at S, the portion of the span to the left is 24 ft., so that the load upon it is 240 cwt. acting downwards, and if supposed to be concentrated at its centre, its leverage about the section is 12 ft.; at the same time P acts upwards with a leverage of 24 ft., and

$$M_8 = 240 \times 24 - 240 \times 12 = 2880 \text{ ft.-cwts. max.}$$

41. The left half of a beam 32 feet span is uniformly loaded with one ton per foot run. Find the position and magnitude of the maximum bending moment.

Ans. The maximum occurs at the section four feet to the left of the centre, and its value is $M_4 = 72$ ft.-tons.

42. A beam 50 ft. span is uniformly loaded from the right end for an extent of 10 feet, with two tons per foot run. Find the position and magnitude of the maximum bending moment.

Ans. The maximum occurs at the section 16 feet to the right of the centre, and its value is $M_{-16} = 81$ ft.-tons.

43. A beam 36 feet span is loaded uniformly from the middle point towards the left to an extent of 12 feet, with 2 tons per foot run. Find the position and magnitude of the maximum bending moment. See fig. 32.

In this case $OG = 6$ ft., and $GS = \frac{1}{3}OG = 2$ feet, since the extent of load is one third of span; the maximum bending moment is at S, four feet to the left of the centre. Suppose the whole load W, 24 tons, concentrated at G, we have

$$P = \frac{24}{36} \times 24 = 16 \text{ tons.}$$

Taking a section at S, the extent of the load to the left is 8 ft., and is equivalent to 16 tons acting downwards with a leverage of 4 feet, while P acts upwards with a leverage of 14 feet; hence

$$M_4 = 16 \times 14 - 16 \times 4 = 160 \text{ ft.-tons.}$$

44. A beam 40 feet span is loaded uniformly with 2 tons per foot run, beginning at one foot from the left end, and ending at nine feet from the right end. Find the position and magnitude of the maximum bending moment.

Ans. At one foot to the left of centre, $M_1 = 360$ ft.-tons.

45. In example No. 40, find the bending moments at intervals of six feet.

Substitute into the formula, which when reduced becomes $M_x = 2700 + 5(12-x)x$; remembering that G is the origin, and that measurements *towards* C are positive, the values of x are -12, -6, 0, $+6$, $+12$, and $+18$; for the other two points required, calculate Q and find the moments from the left end, or by proportion (see fig. 33) one of them is one third, and the other is two thirds, of FE the value calculated from the formula when we substitute $+18$.

A graphical solution is obtained by drawing fig. 33 upon a large scale, and measuring the ordinates at the above points.

Ans. 1260, 2160, 2700, $2880_{max.}$, 2700, 2160; 1440, 720 ft.-cwts.

46. A cantilever 18 feet long is loaded uniformly for two-thirds of its length from the free end, with 10 cwt. per foot run. Find the bending moments at intervals of two feet. See fig. 34.

Look upon the loaded part as a cantilever uniformly loaded, then

$$M_x = \frac{w}{2}(c-x)^2$$

$$= 5(18-x)^2;$$

$$\therefore \quad M_{18} = 0, \text{ and } M_6 = 720 = DE.$$

Now consider BK a cantilever loaded at B with 120 cwts.;

then
$$M_x = W\left(c - \frac{k}{2} - x\right)$$

$$= 120(12-x);$$

so that
$$M_6 = 720 = DE, \text{ and } M_0 = 1440 \text{ ft.-cwts.}$$

Ans. 0, 20, 80, 180, 320, 500, 720; 960, 1200, 1440 ft.-cwts.

A graphical solution is obtained by drawing fig. 34 upon a large scale, and measuring the ordinates at intervals of two feet.

THEOREM.

Fig. 35. The quadrant of a parabola and a right-angled triangle stand on a common horizontal base, with the right angle of each at one end (the right end in the figure); let a be the height of the apex of the parabola, and b the height of the vertex of the triangle, above that end. If at each point of the base, the ordinate of the parabola be added to that of the hypotenuse of the triangle, and a new curve be plotted; it will be the *same* parabola with its axis still vertical, and having its apex shifted beyond that end (the right in the figure) of the base above which the apex was, and the curve will still pass through the other end of the base.

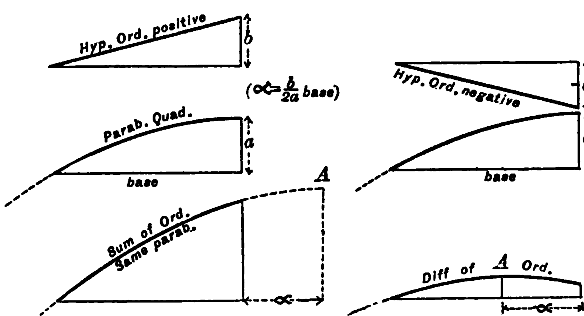


Fig. 35.

In like manner, if the ordinate of the parabola be deducted from that of the hypotenuse, a similar result is obtained; the apex of the new figure, however, is shifted to the other side. The horizontal distance through which the apex shifts is the same fraction of the base, that the height of the vertex of the triangle is of twice the height of the apex of the parabola; or if α be the lateral distance through which the apex shifts, then

$$\alpha = \frac{b}{2a} \times \text{ base of the quadrant.}$$

The proof for the case of addition is already given in the text to fig. 29, and that for subtraction readily follows.

Beam uniformly loaded on its two segments with loads of different intensity.—Fig. 36. Let w_1 and w_2 be the intensities of the loads (say in lbs. per foot run) on the left and right segments respectively. Considering the bending moments on the left segment alone, we have the diagram $B'ae$ due to the load on that segment; and it is the same parabola as if the whole span were loaded with the intensity w_1 (see fig. 33), with the apex a shifted so that G_1S_1 is the same fraction of G_1O that the left segment is of the span: next we have the straight line $B'K$ for the bending moments on the left segment due to the load on the right. $B'KC'$ is drawn as if W_2 the total load on the right segment were concentrated at G_2, the diagram being the same as far as the left segment is concerned. The total bending moment at any point on the left segment is the sum of the ordinates of $B'ae$ and $B'K$ at that point; this gives BA_1E the same parabola as before, with its apex shifted horizontally through S_1T_1 towards the centre, the direction in which $B'K$ slopes upwards, the curve still passing through B. The distance through which the apex a shifts laterally to A_1 is

$$S_1T_1 = \frac{\overline{D_1b}}{2\overline{D_1a}} \times B'D_1.$$

Now if the load on the left segment be constant while that on the right segment varies, then $\overline{D_1b}$ is the only quantity in the above expression which varies; hence

$$\begin{aligned} S_1T_1 \text{ is proportional to } & \overline{D_1b}; \\ \text{,, \quad ,, \quad} & KH; \\ \text{,, \quad ,, \quad} & W_2; \\ \text{,, \quad ,, \quad} & w_2. \end{aligned}$$

But when $w_2 = w_1$, the apex must shift through S_1O so as to be at the centre, for the beam would then be uniformly loaded throughout; therefore

$$\frac{S_1T_1}{S_1O} = \frac{w_2}{w_1}.$$

In like manner for the right segment, the bending moment diagram is the same parabola as if the whole span were loaded with the intensity w_2, but with its apex A_2 shifted so that

$$\frac{S_2T_2}{S_2O} = \frac{w_1}{w_2}.$$

Let $2c =$ the span, $2k_1 =$ the left and longer segment, and $2k_2 =$ the right segment; then

$$OZ = 2k_1 - c, \text{ and } k_1 + k_2 = c.$$

$$G_1O = (c-k_1), \qquad G_2O = (c-k_2).$$

$$S_1O = G_1O - G_1S_1, \qquad S_2O = G_2O - G_2S_2$$

$$= G_1O - \frac{2k_1}{2c}G_1O, \qquad = G_2O - \frac{2k_2}{2c}G_2O$$

$$= \frac{c-k_1}{c}G_1O, \qquad = \frac{c-k_2}{c}G_2O$$

$$= \frac{(c-k_1)^2}{c} = \frac{k_2^2}{c}. \qquad = \frac{(c-k_2)^2}{c} = \frac{k_1^2}{c}.$$

Now $\quad S_1Z = S_1O + OZ = \dfrac{(c-k_1)^2}{c} + 2k_1 - c = \dfrac{k_1^2}{c}$

$$= S_2O;$$

hence also $S_2Z = S_1O$.

The quantity $\dfrac{w_2}{w_1}$ is either equal to, greater than, or less than $\dfrac{k_1^2}{k_2^2}$. If

$$\frac{w_2}{w_1} = \frac{k_1^2}{k_2^2} = \frac{S_2O}{S_1O},$$

then $\quad S_1T_1 = \dfrac{w_2}{w_1}S_1O = \dfrac{S_2O}{S_1O} \cdot S_1O = S_2O = S_1Z,$

and T_1 would coincide with Z; similarly

$$S_2T_2 = \frac{w_1}{w_2}S_2O = \frac{S_1O}{S_2O} \cdot S_2O = S_1O = S_2Z,$$

and T_2 would also coincide with Z; both the apexes A_1 and A_2 would be on the vertical through Z the junction of the two segments. If

$$\frac{w_2}{w_1} > \frac{k_1^2}{k_2^2},$$

then $$S_1T_1 > S_1Z, \text{ and } S_2T_2 < S_2Z,$$

that is, T_1 and T_2 lie to the right of Z. If

$$\frac{w_2}{w_1} < \frac{k_1^2}{k_2^2},$$

T_1 and T_2 lie to the left of Z. When the two apexes are over Z, their common height is the maximum bending moment for the whole span; when both are over segment

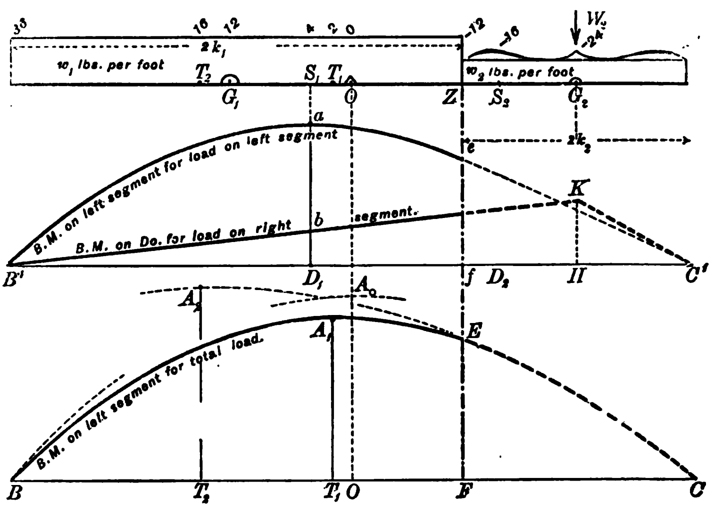


Fig. 36.

number one, as in fig. 36, the maximum is at T_1, since A_1 is the highest point on BA_1E, and E is the highest point on EC; when both are over segment number two, the maximum is at T_2.

Maximum Bending Moment.—Find the products $w_1k_1^2$

and $w_2k_2^2$; if they are equal, the maximum bending moment is at the junction of the segments; if they are unequal, choose the segment for which the product is greater; from G its middle point lay off towards the centre of span $\overline{GS} = \frac{k}{c}\overline{GO}$; from S lay off ST towards the centre, the same fraction or number of times SO that the intensity of the load on the other segment is of the intensity upon this. At the point T the maximum bending moment occurs, the amount of which may be readily calculated by taking a section there.

Graphical Solution.—With a scale of feet for horizontals, lay off the span BC, fig. 36, and mark F, G_1, and G_2, the junction and middle points of the segments. Make $G_1S_1 = \frac{k_1}{c} G_1O$, and $S_1T_1 = \frac{w_2}{w_1} S_1O$; similarly make $G_2S_2 = \frac{k_2}{c} G_2O$, and $S_2T_2 = \frac{w_1}{w_2} S_2O$. Draw verticals through T_1, T_2, O, and F; apply the parallel rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through T_1; shift the rollers till the curved edge passes through B, and draw BA_1E, stopping at E where the curve meets the vertical through F; then BA_1E is the bending moment diagram for the segment BF. Construct a scale for verticals thus:—Shift the parabolic segment till its apex is on the vertical through O; move the rollers till the curved edge passes through B and C, and draw BA_0C; make a scale of ft.-lbs. such that $OA_0 = \frac{1}{8} w_1 l^2$, where w_1 is the intensity in lbs. of the load on segment number one, and l is the span in feet. For the other segment the figure may be drawn with the same parabolic segment, but it will be to a different vertical scale, viz., that one upon which $OA_0 = \frac{1}{8}w_2l^2$. If necessary, it may readily be reduced to the same vertical scale as the other, by means of a pair of proportional compasses, reducing the point A_2 and a sufficient number of points between E and C; or having reduced the point A_2, the quadrant A_2EC may be constructed as on figure 14. In drawing the curves for the two segments with the same parabolic segment, E will be defined at *different* heights by the two curves; and the fact that they ought to coincide furnishes a ready means of setting the proportional compasses, so as to reduce one half of the figure to the vertical scale of the other.

Beam uniformly loaded and supported on three props.—Fig. 37. Let BC be a beam with a load U uniformly distributed on it and supported on three props, one at each end and one at the centre; let P, W, and Q be the forces with which they press upwards, so that $P + W + Q = U$ always, and $P = Q$ by symmetry. If $W = 0$, the central prop bears no share, $P = Q = \frac{U}{2}$, and the bending moment diagram is the parabola HAK, as on fig. 25. If the central prop bears a share, then the beam is loaded with a uniform load U, and a *negative* load W at the centre; and the bending moment diagram is two parabolas, each the same as HAK, but with its apex away from the centre, and in a direction opposite to that on fig. 29. The horizontal distance through which each apex moves is given by the equation $a = \frac{W}{U}c$. Suppose HAK to be two parabolas lying one on the top of the other,

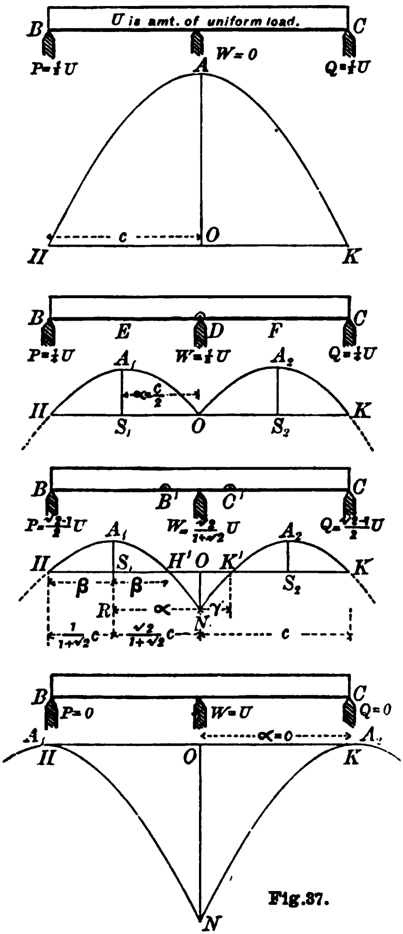


Fig. 37.

and let the prop press up with a greater and greater force, then the two parabolas shift away from each other. When $W = \frac{U}{2}$, A_1 is over S_1, and A_2 is over S_2, the middle points of OH and OK respectively, and the bending moment at O is zero; this is the best value of W, if the beam may only be bent so that its convex side shall be down. It is evident that $S_1A_1 = \frac{1}{4}\, OA$; and that the beam might be sawn through at O, when it would be two beams of span c, each uniformly loaded and supported at the two ends

If the beam may be bent both upwards and downwards, let the central prop press upwards till $W > \frac{U}{2}$, then OS_1 is greater than $\frac{OH}{2}$, or in symbols $a > \frac{c}{2}$; and it can be seen from the figure that the ordinates from H' to K' are *negative*, from H' to H positive, and from K' to K positive, while at H, H', K, and K' the bending moments are zero. Hence the beam will be bent with the convex side downwards from H to H', and from K' to K, and upwards from H' to K'. A hinge might be put on the beam at H' and K', since there is no tendency to bend at these points, which are called *points of contrary flexure.* There are three maxima bending moments, two equal positive ones at S_1 and S_2, and a negative one at O. If the material of the beam may be bent upwards and downwards *equally well*, then the best value of W is that which makes the positive and negative maxima equal, as their common value in this case is less than the greatest value would be in any other; for, suppose them equal, then if W be increased the parabolas move outwards, and the ordinate at O will increase; while, if W be made smaller, the parabolas will approach and the ordinates at S_1 and S_2 will increase. Let the three maxima be equal to each other, then $ON = S_1A_1$, or

$$A_1S_1 : A_1R :: 1 : 2;$$

hence $$S_1H' : RN :: 1 : \sqrt{2}$$

since the curve is a parabola; or in symbols

$$\beta : a :: 1 : \sqrt{2}.$$

Practically this could be accomplished by causing the prop to press upwards till

$$W = \frac{OS_1}{OH} U = \frac{a}{a+\beta} U$$

$$= \frac{\sqrt{2}}{\sqrt{2}+1} U = \cdot 586 U;$$

or by fixing hinges at the two points H' and K' at the proper distances from O, the central prop may be made to bear the above share of U.

If the prop at the centre press upwards so that $W = U$, then $P = Q = 0$, and $OS_1 = OH$; A_1 coincides with H, and A_2 with K; the bending moment is everywhere negative, and its maximum value ON equals OA. OH and OK are cantilevers, as on fig. 27.

Beam uniformly loaded and supported on many props.—Fig. 38. Let BE be a beam bearing an uniform load and supported on many (5 in the figure) props; this is only an extension of the previous case. The end C, fig. 37, instead of resting on

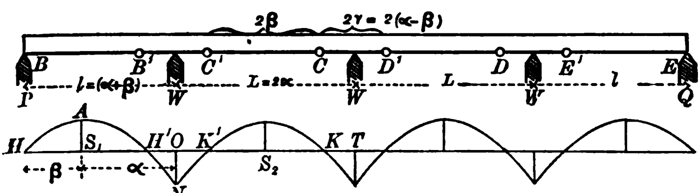


Fig. 38.

a prop, might be hinged to the end of a cantilever, which in turn might have its other end hinged to a beam, &c.; see fig. 38. Here a double cantilever, as $B'C'$, over each intermediate prop is hinged on each side to the end of a beam; each cantilever bears an uniform load over itself, as well as half the load on the beam concentrated at its end, and is therefore in the condition of the cantilever shown on fig. 30. Each intermediate beam as $C'C$ is uniformly loaded, and is supported by a hinge at each end; the two end beams $B'B$ and $E'E$ are supported at one end by a hinge, and at the other by one of the extreme props; each central span L consists of a beam and two cantilevers; the end spans l of a beam and a cantilever.

If the hinges be put in the positions indicated by H' and K' on fig. 37, then the negative maxima bending moments over the props at O, T, &c., are equal to the positive maxima bending moments at S_1, S_2, &c., the centres of the beams; the common value of all these maxima will be *less* than the greatest for the hinges in any other position, and

$$l : L :: \alpha + \beta : 2\alpha$$
$$:: 1 + \sqrt{2} : 2\sqrt{2}.$$
$$:: \cdot 854 : 1.$$

Beam loaded both uniformly and with unequal weights fixed at irregular intervals.—Fig. 39. Let U be the amount of the uniform load; the parabola BA_0C is the bending moment diagram due to it, and a is the height of its apex A_0. Let $Oefgh...K$ be the bending moment diagram due to the loads W_1, W_2, W_3, W_4, &c.; from one end O draw lines parallel to the sides, and meeting the vertical through b the centre of the span at the points b_1, b_2, &c.; let b_1, b_2, &c., stand for the distances of these points from b, those which are below being considered negative. If to the ordinates of the parabola BA_0 we add those of the straight line Ob_1, the locus is the *same* parabola with its apex to the right of b at a distance given by the equation

$$\overline{bd}_1 = \frac{b_1}{2a}c,$$

and the curve still passes through B; let A_1EB be the parabola so placed, then BE, the portion of this parabola

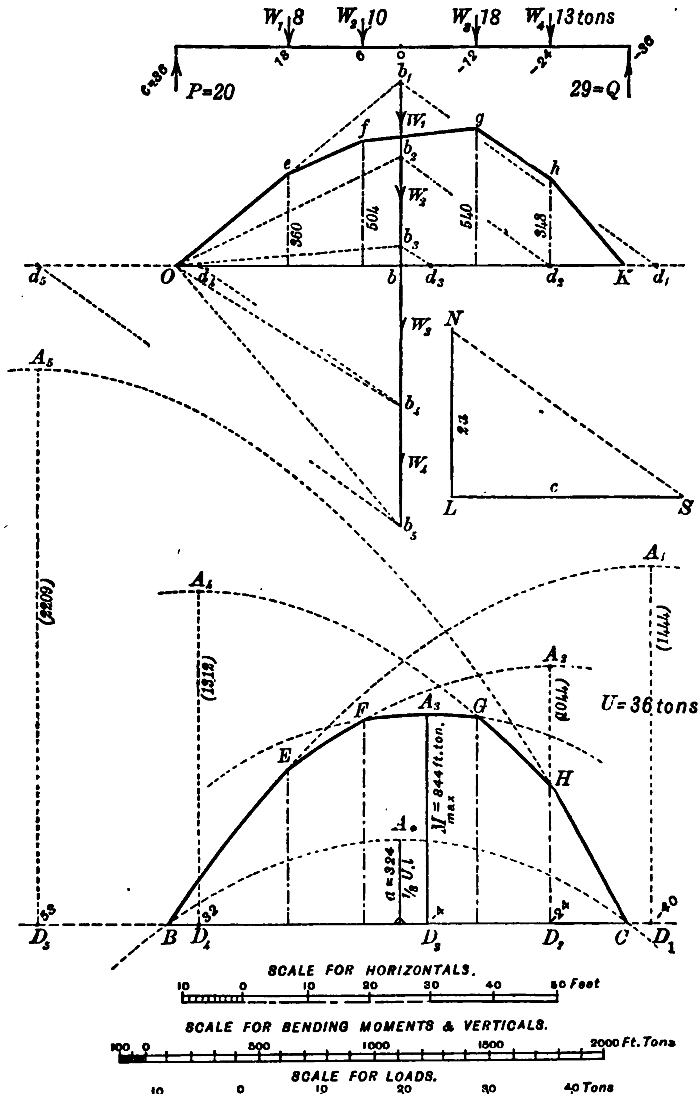

W1 8
W2 10
W3 18
W4 13 tons
P=20
29=Q
b1
b2
b3
b4
b5
W1
W2
W3
W4
e
f
g
h
360
504
540
348
d5
O
d4
b
d3
d2
K
d1
N
L
S
A5
A4
A3
A2
A1
(1444)
U=36 tons
F
G
E
H
M max = 844 ft. ton.
a=324
1/8 U.l
D5
B
D4
D3
D2
C
D1
SCALE FOR HORIZONTALS.
10 0 10 20 30 40 50 Feet
SCALE FOR BENDING MOMENTS & VERTICALS.
100 0 500 1000 1500 2000 Ft. Tons
SCALE FOR LOADS.
10 0 10 20 30 40 Tons

Fig. 39.

from the end of span to the vertical through the first weight, is the bending moment diagram for that portion of the span. If to the ordinates of the parabola BA_0 we add those of the straight line Ob_2, the locus is the *same* parabola with its apex to the right of b at a distance given by the equation

$$\overline{bd_2} = \frac{b_2}{2a}c,$$

and the curve still passes through the point B; if we now raise the parabola through the vertical distance between the two parallels Ob_2 and ef, which is accomplished by drawing the curve through E instead of B, we have the curve in the position A_2FE; and EF the portion between the verticals through the first and second weights is the bending moment diagram for that portion of the span. Similarly FA_3G is the parabola with A_3 to the right of b at a distance $\overline{bd_3} = \frac{b_3}{2a}c$, and drawn through F; A_4GH is the parabola with A_4 to the *left* of b (since b_4 is negative) at a distance $\overline{bd_4} = \frac{b_4}{2a}c$, and drawn through G; lastly, A_5HC is the parabola with A_5 to the *left* of b at a distance $\overline{bd_5} = \frac{b_5}{2a}c$, the curve being drawn through H and passing through C the other end of the span. The bending moment diagram thus completed is $BEFGC$.

It is convenient to call the portions of the span "fields"; thus, from the left end to the first weight is the first field, from the first weight to the second the second field, &c.; we may also say that the parabola A_1EB *bounds* the 1st field, A_2FE the 2nd field, &c.

The equations given above for the horizontal distances of the apexes from the centre of span may be simplified as follows:—

Let d_n = the horizontal distance from O to A_n,
b_n = the height of the point b corresponding to that apex,

$$s_n = \text{slope of boundary of } n^{\text{th}} \text{ field} = \frac{\text{vert.}}{\text{hor.}},$$

$$u = \text{intensity of uniform load};$$

the general expression then becomes

$$d_n = \frac{b_n}{2a}c;$$

now b_n = slope of boundary of n^{th} field multiplied by c

$$= s_n\, c,$$

$$a = \tfrac{1}{2}uc^2, \text{ and therefore}$$

$$d_n = \frac{s_n\, c}{2.\tfrac{1}{2}uc^2}c = \frac{s_n}{u}.$$

Maximum Bending Moment.—Of the ordinates to the five apexes on the figure, it will be seen that only that of A_3 is really a bending moment; and the apex A_3 is said to lie in its own field, that is the third. In the figure *OefghK*, such a side as *ef*, which slopes *up* to the right, shifts A_2 the apex for that field to the right of the centre; the field itself, however, lies wholly to the left of the centre, so that the apex *cannot* lie in its own field; hence, for any side sloping up to the right and wholly to the left of the centre, the apex cannot lie in its own field. For such a side as *fg* sloping up to the right the apex shifts to the right; but since part or whole of *fg* also lies to the right of the centre, the apex *may* lie in its own field. If there be two sides lying to the right of the centre and sloping up to the right, then the apex of one, but not of both, may lie in its own field. Suppose that *gh* also sloped up towards the right, then *fg* is necessarily steeper than *gh*, and the apex for *gh* would be closer to the centre than that for *fg*; if that for *fg* lies in its own field, that for *gh* cannot lie in its own. Further, if a number of the sides of the polygon lie to one side of the centre and slope down towards it, the apex for any one, but for one only, may lie in its own field. Suppose the apex for one such to lie in its own field, then, since each of the sides further from the centre is necessarily less steep, its apex is closer to the centre, and so cannot lie in its own field; on the other hand, each of the sides closer to the centre is steeper, its apex is further from the centre, and so cannot lie in its own field.

If one apex lies in its own field, the height of that apex is the maximum bending moment; if no apex lies in its own field, the maximum is at the weight at which is the maximum for the weights alone. On fig. 39, the maximum bending moment is the height of A_3; but if A_3 had *not* been between F and G, the maximum would have been the height of G.

Graphical Solution.—With a scale of feet for horizontals, lay off BC equal to the span, fig. 39, and draw BA_0C with any parabolic segment. Construct a scale for verticals and bending moments such that a the height of A_0 shall measure upon it $\frac{1}{8}U.l$, where $U =$ amount of uniform load, and l = span; draw the bending moment diagram $OefghK$, as on fig. 18, by calculating the bending moments at the weights and plotting e, f, g, h to the same vertical scale as BA_0C; from one end O draw parallels to each side, and so determine the points b_1, b_2, &c.; or, as on fig. 19, lay off on the vertical through the centre $\overline{b_1b_2} = W_1$, $\overline{b_2b_3} = W_2$, &c., on a scale for loads c times larger than the scale for verticals already constructed; determine the point b and through it draw a horizontal. Make the horizontal LS equal to c, and the vertical LN equal to $2a$; draw parallels to NS as $\overline{b_1d_1}$ from b_1, $\overline{b_2d_2}$ from b_2, &c., and so determine the points d_1, d_2, &c. Apply the parallel rollers to BC; place the parabolic segment against them with its apex on the vertical through D_1, move the rollers till the curved edge passes through B, and draw BE meeting the vertical through W_1 in E. Shift the segment till the apex is on the vertical through D_2; move the rollers till the curved edge passes through E, and draw EF. Shift the segment till the apex is on the vertical through D_3; move the rollers till the curved edge passes through F, and draw FG meeting the vertical through W_3 in G, &c. The accuracy of the drawing is checked by observing whether the last curve passes through C the other end of the span. Fig. 29 is a particular case of this. The construction is simplified when the Shearing Force Diagram is drawn first, as on fig. 86; and to this diagram the student is referred.

Examples.

47. A beam 72 feet span is loaded in two segments; the left segment is two thirds of the span, and is loaded uniformly at the rate of two tons per foot; the right segment is uniformly loaded at the rate of one ton per foot. Find the position and amount of the maximum bending moment.

Fig. 36 is this example drawn to scale.

The data in symbols are

$$2c = 72,\ 2k_1 = 48,\ 2k_2 = 24,\ w_1 = 2, \text{ and } w_2 = 1.$$

Now $2 \times 48^2 > 1 \times 24^2$, hence the maximum is in the left segment.

$$G_1O = c - k_1 = 36 - 24 = 12 \text{ ft.}$$

Since this segment is $\frac{2}{3}$ of span,

$$G_1S_1 = \tfrac{2}{3}G_1O = \tfrac{2}{3} \times 12 = 8 \text{ ft.};$$

hence $OS_1 = 4$ ft.

Since w_2 is half of w_1,

$$S_1T_1 = \tfrac{1}{2}S_1O = \tfrac{1}{2} \times 4 = 2 \text{ ft.};$$

hence $OT_1 = 2$ feet; that is, the maximum occurs at 2 feet to the left of the centre.

To find P; suppose $W_1 = w_1 \times 2k_1 = 96$ tons to be concentrated at G_1 the middle of the left segment, and $W_2 = w_2 \times 2k_2 = 24$ tons to be concentrated at G_2 the middle of the right segment, and take moments about the right end, thus;—

$$P \times 2c = W_1(2k_2 + k_1) + W_2k_2;$$

$$P \times 72 = 96 \times 48 + 24 \times 12.$$

$$\therefore \quad P = 68 \text{ tons.}$$

And $M_2 = P(c-2) - w_1(c-2) \times \left(\frac{c-2}{2}\right) = 1156$ ft.-tons max.

48. A beam 50 ft. span is loaded with 4 cwts. per foot for a distance of 30 feet beginning at the left end, and the remainder is loaded with 9 cwts. per foot. Find the maximum bending moment.

Intensity on left seg. × sq. of its length $= 4 \times 30^2 = 3600$;

Intensity on right seg. × sq. of its length $= 9 \times 20^2 = 3600$;

since these are equal, the maximum bending moment is at the junction of the segments, that is 5 ft. to the right of the centre.

To find P, we have

$$P \times 50 = 120 \times 35 + 180 \times 10. \quad \therefore \quad P = 120.$$

Taking a section at the junction of the segments

$$M_{-5} = P \times 30 - 4 \times 30 \times \tfrac{30}{2}$$
$$= 1800 \text{ ft.-cwts. maximum.}$$

49. A beam is 49 feet span; its left segment, 28 feet long, is uniformly loaded with one ton per foot; and its right segment, 21 feet long, is loaded with two tons per foot. Find the maximum bending moment.

Intensity on left seg. × sq. of its length $= 1 \times 28^2 = 784$;

Intensity on right seg. × sq. of its length $= 2 \times 21^2 = 882$;

since $882 > 784$, the maximum occurs in the right segment. In this case

$$G_2O = 14; \text{ and } G_2S_2 = \tfrac{3}{7} \times 14 = 6 \text{ ft.},$$

because the right segment is $\frac{3}{7}$ of the span; hence

$$S_2O = 8 \text{ ft.};$$
$$S_2T_2 = \tfrac{1}{2} \times 8 = 4 \text{ ft.},$$

since the intensity of the load on the other segment is half of the intensity on this; and

$$OT_2 = 4 \text{ ft.};$$

that is, the maximum occurs at 4 feet to right of centre.

It may be convenient to calculate the bending moment from the right end, thus;—

$$Q \times 2c = W_1 \times 14 + W_2 \times 38{\cdot}5;$$

$$Q \times 49 = 28 \times 14 + 42 \times 38{\cdot}5. \quad \therefore Q = 41 \text{ tons};$$

and $$M_{-4} = Q \times 20{\cdot}5 - w_2 \times 20{\cdot}5 \times \frac{20{\cdot}5}{2} = 420\tfrac{1}{4} \text{ ft.-tons. max.}$$

50. A beam 30 feet span has two thirds of its length from its left end loaded with 5 cwts. per foot, and the remainder with 20 cwts. per foot. Find the maximum bending moment.

Ans. At the junction $M_{-5} = 1000$ ft.-cwts. max.

51. A beam is 64 feet span; its left segment, 40 feet long, is loaded with 3 cwts. per foot, the right segment with 5 cwts. per foot. Find the maximum bending moment.

Ans. At 3 feet to right of centre, $M_{-3} = 1837{\cdot}5$ ft.-cwts. max.

52. The beam of example No. 51 has its left segment loaded with 1 ton per foot, and its right with 5 tons per foot. Find the maximum bending moment.

Ans. It occurs in the right segment at 10 ft. to the right of the centre, and

$$M_{-10} = 1210 \text{ ft.-tons max.}$$

53. In example No. 47, find the position and height of A_2, fig. 36.

$$G_2O = (c - k_2) = 24;$$

$$G_2S_2 = \tfrac{1}{3} \times G_2O = 8. \quad \therefore \quad OS_2 = 16 \text{ ft.}$$

Then $$S_2T_2 = \tfrac{2}{1} S_2O = 32. \quad \therefore \quad OT_2 = 16 \text{ ft.}$$

To find the height T_2A_2; substitute into the equation which gives the ordinates of CE, thus;—

$$T_2A_2 = Q \times CT_2 - w_2 \times CT_2 \times \frac{CT_2}{2}$$
$$= 1352 \text{ on vertical scale.}$$

Note that the ordinate of the other parabola BA_1 is the bending moment at T_2.

54. A beam is uniformly loaded, and a prop in the centre bears one-third of the load. Find the maximum bending moment. Fig. 37.

Let U = the amount of the uniform load,

W = the upward thrust of central prop $= \frac{1}{3}U$.

Here $a = \frac{1}{3}c$; and since $a < \frac{1}{2}c$, the two parabolas intersect above the span, and there are no negative bending moments; there is a positive minimum at the centre, and a positive maximum at a on each side of the centre.

The base of the segment $HA_1H' = 2\beta = \frac{4}{3}c = \frac{2}{3}l$.

$$\text{Height } S_1A_1 : \text{Height } OA :: (\tfrac{2}{3})^2 : 1^2.$$
$$\therefore \quad S_1A_1 = \tfrac{4}{9}OA,$$
$$\text{or} \quad M_a = \tfrac{4}{9} \times \tfrac{1}{8}U.l = \tfrac{1}{18}U.l, \text{ maximum.}$$

55. In the previous example, if the central prop bear two thirds of the load, find the position and magnitude of the maxima bending moments positive and negative, and the position of the points of contrary flexure.

$$a = \tfrac{2}{3}c; \; 2\beta = \tfrac{2}{3}c = \tfrac{1}{3}l.$$
$$\therefore \quad M_{\pm\frac{2}{3}c} = \tfrac{1}{9} \times \tfrac{1}{8}U.l, \text{ max. positive.}$$
$$M_0 = -\tfrac{1}{24}U.l, \text{ max. negative.}$$

The points of contrary flexure are at $\pm \frac{1}{3}c$.

56. A beam 72 feet span is loaded with 8 and 10 tons at points 18 and 6 feet to the left of the centre, and with 18 and 13 tons at points 12 and 24 feet to the right of the centre; there is also an uniform load of half a ton per foot of span. Find the position and value of the maximum bending moment. Fig. 39.

Data:—

$W_1=8$, $W_2=10$, $W_3=18$, $W_4=13$, $u=\frac{1}{2}$, and $U=36$ tons;

$x_1=18$, $x_2=6$, $x_3=-12$, $x_4=-24$, and $c=36$ feet.

For the loads W_1, W_2, &c., alone, we find, as in examples 14, 19, 20, &c., that the maximum is under the load W_3; hence only the boundary of the polygon from W_2 to W_3, that is, the boundary of field 3, lies (in part) to one side of the centre and slopes down towards the centre. Taking the total load then, it is possible only for A_3 to lie in its own field. To find the position of A_3, we have

$$\text{Slope of } fg = P - W_1 - W_2 = 20 - 8 - 10$$
$$= 2 \text{ vertical to } 1 \text{ horizontal.}$$

This also is the slope of $\overline{Ob_3}$, and

$$\overline{bb_3} = 2 \times \overline{Ob} = 2c = 72 \text{ on vertical scale.}$$

Again, the height of A_0 for the uniform load alone is

$$a = \tfrac{1}{8}U.l = 324 \text{ on vertical scale.}$$

Now A_3 lies to the right of the centre at a distance given by the equation

$$\overline{bd_3} = \frac{\overline{bb_3}}{2a}.c = \frac{72}{648} \times 36 = 4 \text{ ft.};$$

$$\text{or otherwise, } \overline{bd_3} = \frac{s_3}{u} = \frac{2}{\frac{1}{2}} = 4 \text{ ft.};$$

and since field 3 extends 12 ft. to the right of the centre, it is evident that A_3 lies in field 3. The *position* of the maximum bending moment is at 4 ft. to the right of the centre.

To find the value of the max. bending moment, we have

for loads W alone, $M_{-4} = P \times 40 - W_1 \times 22 - W_2 \times 10 = 524$;

for uniform load, $M_{-4} = \dfrac{U}{4c}(c^2 - 4^2) = 320$;

and for the total load, $M_{-4} = 844$ ft.-tons maximum.

57. In the previous example, suppose the uniform load to be $\frac{1}{8}$th of a ton per foot of span, and find the maximum bending moment.

Here $u = \frac{1}{8}$, $U = 9$; $a = 81$,

$$\text{and} \qquad \overline{bd}_3 = \frac{72}{2 \times 81} \times 36 = 16 \text{ ft.},$$

or otherwise, $\overline{bd}_3 = \frac{s_3}{u} = \frac{2}{\frac{1}{8}} = 16$ ft. to the right of the centre; hence A_3 is not in field 3, and the maximum is at W_3 as for the loads W_1, W_2, &c., alone.

For loads W alone, $M_{-12} = 540$;

for uniform load, $M_{-12} = \frac{U}{4c}(c^2 - 12^2) = 72$;

for total load, $M_{-12} = 612$ ft.-tons max.

58. Suppose the beam shown on figs. 17 and 18 to bear an additional uniform load of one ton per ft. of span, and find the maximum bending moment.

On fig 18, we see that only the side CD of the polygon lies (in part) to one side of, and slopes down towards, the centre; it is therefore possible only for A_4, the apex of the parabola which bounds field 4, to lie in its own field.

To find the position of A_4, we have

$$\text{Slope of } CD = P - W_1 - W_2 - W_3 = 24 - 5 - 5 - 11$$
$$= 3 \text{ vertical to 1 horizontal};$$

hence, if a line is drawn from the left extremity at this slope, that is parallel to CD, it cuts off an intercept on the vertical through the centre of height

$$b_4 = 3c = 63 \text{ on vertical scale};$$

again, for the uniform load alone the height of A_0 is

$$a = \tfrac{1}{8}U.l = 220{\cdot}5;$$

the distance of A_4 to the *right* of the centre is

$$d_4 = \frac{b_4}{2a}.c = \frac{63}{441} \times 21 = 3 \text{ feet};$$

or otherwise, $d_4 = \frac{s_4}{u} = \frac{3}{1} = 3$ feet,

which is exactly at W_4 the right extremity of field 4. Hence the maximum is at W_4, and

For loads W alone, $M_{-3} = 261$, (see fig. 18.)

for uniform load, $M_{-3} = \frac{U}{4c}(c^2 - 3^2) = 216$,

for total load, $M_{-3} = 477$ ft.-tons max.

59. Suppose the beam shown on figs. 17 and 18 to bear an additional uniform load of 3 tons per foot of span, and find the maximum bending moment.

Here a is three times as great as in the previous example; d_4 is only one third of its former value, that is, $d_4 = 1$ ft.; and the maximum bending moment occurs at one foot to the right of the centre.

For loads W alone, $M_{-1} = 255$,

for uniform load, $M_{-1} = \frac{U}{4c}(c^2 - 1^2) = 660$,

for total load, $M_{-1} = 915$ ft.-tons max.

60. Find by analysis the positions of the apexes in the example solved graphically on fig. 39.

		Vert.		Horiz.
Slope upwards of	$Oe = s_1 = P$, ...	$= 20$	to	1
„ „	$ef = s_2 = P - W_1$	$= 12$	„	1
„ „	$fg = s_3 = P - W_1 - W_2$, ...	$= 2$	„	1
„ „	$gh = s_4 = P - W_1 - W_2 - W_3$, ...	$= -16$	„	1
„ „	$hK = s_5 = P - W_1 - W_2 - W_3 - W_4$,	$= -29$	„	1

Intercepts on the vertical through the centre by the parallels to the sides of the polygon drawn from the left end O; heights positive;

$$b_1 = 20c = 720. \qquad b_3 = 2c = 72.$$

$$b_2 = 12c = 432. \qquad b_4 = -16c = -576.$$

$$b_5 = -29c = -1044.$$

Considering distances to left of centre as positive, and reckoning the sign from the figure, the abscissæ of the apexes are—

$$d_1 = \frac{s_1}{u} = -40; \quad d_2 = \frac{s_2}{u} = -24; \quad d_3 = \frac{s_3}{u} = -4;$$

$$d_4 = \frac{s_4}{u} = 32; \qquad d_5 = \frac{s_5}{u} = 58.$$

$$\text{Height of } A_1 : \text{ht. of } A_0 :: (c + d_1)^2 : c^2.$$

$$\therefore \text{ Ht. of } A_1 = \left(\frac{c+d_1}{c}\right)^2 a = \left(\frac{76}{36}\right)^2 \times 324 = 1444 \text{ vert. scale.}$$

Had the other parabolas been drawn through the end B or C, we would have had the heights of their apexes as follows:—

$$\text{Height of } A_2 = \left(\frac{c+d_2}{c}\right)^2 a = \left(\frac{60}{36}\right)^2 \times 324 = 900,$$

$$\text{,,} \quad A_3 = \left(\frac{c+d_3}{c}\right)^2 a = \left(\frac{40}{36}\right)^2 \times 324 = 400,$$

$$\text{,,} \quad A_4 = \left(\frac{c+d_4}{c}\right)^2 a = \left(\frac{68}{36}\right)^2 \times 324 = 1156,$$

$$\text{,,} \quad A_5 = \left(\frac{c+d_5}{c}\right)^2 a = \left(\frac{94}{36}\right)^2 \times 324 = 2209.$$

For the second parabola, however, we add the vertical distance between ef and the parallel Ob_2, that is, the difference of the slopes of Oe and ef multiplied by the distance of W_1 from the left end; similarly for the 3rd and 4th parabolas; thus—

$$\text{Additional for } A_2 = (20 - 12) \times 18 = 144,$$
$$\text{,, ,, } A_3 = (20 - 2)\ 30 - (20 - 12) \times 12 = 444,$$
$$\text{,, ,, } A_4 = (29 - 16) \times 12 = 156;$$

and the height of apex

$A_1 =$	....	1444	on vert.	scale.
$A_2 =$	$900 + 144 =$	1044	,,	,,
$A_3 =$	$400 + 444 =$	844	,,	,,
$A_4 =$	$1156 + 156 =$	1312	,,	,,
$A_5 =$	....	2209	,,	,,

61. A beam has a span of 64 feet; $W_1 = 9$, $W_2 = 10$, $W_3 = 50$, $W_4 = 42$ tons; and $x_1 = 8$, $x_2 = -8$, $x_3 = -14$, $x_4 = -22$ ft. Find P and Q, and the slopes of the boundaries of the bending moment diagram.

Ans. $P = 30$, and $Q = 81$ tons.

				Vert.		Horiz.	
Slope,	1st	field	$=s_1=P=$	30	to	1	upwards to right.
,,	2nd	,,	$=s_2=P-W_1=$	21	,,	1	,,
,,	3rd	,,	$=s_3=P-W_1-W_2=$	11	,,	1	,,
,,	4th	,,	$=s_4=P-W_1-W_2-W_3=$	-39	,,	1	downw'ds to right.
,,	5th	,,	$=s_5=P-W_1-W_2-W_3-W_4=$	-81	,,	1	,,

The boundary of field 2 lies in part, and the boundary of field 3 lies altogether to the right of the centre, and both slope down towards the centre. Hence, if an uniform load be combined with these loads, the maximum bending moment may be in field 2, or in field 3, or at W_3, depending on the intensity of the uniform load; since it is possible that A_2 or A_3, but not both, may lie in its own field, or that neither may so lie.

62. An uniform load of 3 tons per foot of span is combined with the loads on the beam in No. 61. Find the maximum bending moment.

$$\text{Here} \qquad U = 192, \text{ and } a = \tfrac{1}{8}U.l = 1536.$$

Find first the horizontal position of A_2; a line drawn from the left end, parallel to the boundary of field 2, intercepts *above* the span, on the vertical through the centre an amount

$$b_2 = \text{slope of boundary} \times c = 21 \times 32 = 672\,;$$

hence the horizontal distance of A_2 to the *right* of the centre is

$$d_2 = \frac{b_2}{2a}c = \frac{672}{2 \times 1536} \times 32 = 7 \text{ feet,}$$

$$\text{or, otherwise,} \qquad d_2 = \frac{s_2}{u} = \frac{21}{3} = 7 \text{ ,, };$$

field 2 extends 8 ft. to the right, so that the apex A_2 is, and no other apex can be, in its own field; the height of A_2 is, therefore, the maximum bending moment.

For the loads W alone, $M_{-7} = 30 \times 39 - 9 \times 15 = 1035$,

for the uniform load, $M_{-7} = \dfrac{U}{4c}(c^2 - 7^2) = 1462{\cdot}5$,

for the total load, $M_{-7} = 2497{\cdot}5$ ft.-tons max.

63. With the data in No. 61 combine an uniform load of 1 ton per foot of span, and find the max. bending moment.

$$\text{Here} \qquad U = 64, \text{ and } a = 512.$$

To find first the horizontal position of A_2; a line drawn from the left end, parallel to the boundary of field 2, intercepts *above* the span upon the vertical through the centre

$$b_2 = \text{slope of boundary} \times c = 21 \times 32 = 672\,;$$

the distance of A_2 to the *right* of the centre is

$$d_2 = \frac{b_2}{2a}c = \frac{672}{2 \times 512} \times 32 = 21 \text{ feet},$$

or, otherwise, $d_2 = \frac{s_2}{u} = \frac{21}{1} = 21$ „ ;

but since field 2 only extends 8 feet to right of centre, A_2 is not in its own field.

To find next the horizontal position of A_3; a line drawn from the left end, parallel to the boundary of field 3, intercepts *above* the span upon the vertical through the centre

$$b_3 = \text{slope of boundary} \times c = 11 \times 32 = 352;$$

the distance of A_3 to the right of the centre is

$$d_3 = \frac{b_3}{2a}c = \frac{352}{2 \times 512} \times 32 = 11 \text{ feet},$$

or, otherwise, $d_3 = \frac{s_3}{u} = \frac{11}{1} = 11$ „ ;

field 3 extends from 8 feet to the right, to 14 feet to the right, of the centre; hence A_3 lies in its own field, and its height is found as follows;—

For loads W alone, $M_{-11} = 30 \times 43 - 9 \times 19 - 10 \times 3 = 1089$,

for the uniform load, $M_{-11} = \frac{U}{4c}(c^2 - 11^2) = 451 \cdot 5$,

for total load, $M_{-11} = 1540 \cdot 5$ ft.-tons max.

64. With the data in No. 61 combine an uniform load of half a ton per foot of span, and find the maximum bending moment.

From No. 61 it is possible only for A_2 or A_3 to be in its own field. For A_2 we have

$$d_2 = \frac{s_2}{u} = \frac{21}{\cdot 5} = 42 \text{ feet},$$

which is far beyond field 2; for A_3 we have

$$d_3 = \frac{s_3}{u} = \frac{11}{\cdot 5} = 22 \text{ feet},$$

which is beyond field 3. Hence no apex lies in its own field, and the maximum bending moment is at the weight at which it occurs for the loads W alone, in this example at W_3; for, in subtracting W_3, the remainder changed sign, that is, beginning at the left hand, the boundary of field 4 is the first which slopes down towards the right.

For loads W, $M_{-14} = 30 \times 46 - 9 \times 22 - 10 \times 6 = 1122$,

for uniform load, $M_{-14} = \frac{U}{4c}(c^2 - 14^2) = 207$,

for total load, $M_{-14} = 1329$ ft.-tons max.

65. In example No. 61, find when the max. bending moment lies in field 2, when in field 3, and when at W_3, for different intensities of the uniform load.

Let u be the intensity of the uniform load, then

$$d_2 = \frac{21}{u}, \text{ and } d_3 = \frac{11}{u}.$$

Suppose that A_2 is at the right extremity of field 2, then

$$d_2 = \frac{21}{u} = 8,$$

or $$u = \frac{21}{8} = 2\tfrac{5}{8} \text{ tons};$$

that is, A_2 is within, at the right extremity of, or beyond its own field, according as u is greater than, equal to, or less than $2\frac{5}{8}$ tons per foot.

Suppose that A_3 is at the right extremity of field 3, then

$$d_3 = \frac{11}{u} = 14,$$

or $$u = \frac{11}{14} \text{ tons};$$

again, suppose that A_3 is at the left extremity of field 3 then

$$d_3 = \frac{11}{u} = 8,$$

or
$$u = \frac{11}{8};$$

that is, A_3 is within its own field if u has any value between $\frac{11}{8}$ and $\frac{11}{14}$, and is nearer the centre the larger u becomes.

Hence the maximum bending moment is at W_3 if u is less than or equal to $\frac{11}{14}$, between W_3 and W_2 if u is between $\frac{11}{14}$ and $\frac{11}{8}$, at W_2 if u is between $\frac{11}{8}$ and $2\frac{5}{8}$, and between W_2 and the centre of the span if u is greater than $2\frac{5}{8}$ tons per foot.

66. The span of a beam is 40 feet, and at a distance of 12 feet to the right of the centre there is a load of 20 tons; there is also an uniform load of half a ton per foot of span. Find the maximum bending moment.

For the concentrated load, the bending moment diagram is a triangle with its apex over the weight; part of the left side of this triangle lies to the right of, and slopes down towards, the centre; so that the apex for field No. 1 *may* lie in its own field, and the maximum bending moment is either between the centre and the weight or at the weight.

The apex A_1 is situated to the right of the centre at a distance

$$d_1 = \frac{s_1}{u}.$$

For W alone, $P = \frac{20}{40} \times 8 = 4$ tons; therefore s_1, the slope of the left side of the triangle, is 4 vert. to 1 horiz.;

hence
$$d_1 = \frac{4}{\frac{1}{2}} = 8 \text{ feet};$$

and this being within field No. 1, is the point where the maximum occurs.

$$\text{For } W \text{ alone,} \qquad M_{-8} = P \times 28 = 112,$$

$$\text{for uniform load,} \quad M_{-8} = \frac{U}{4c}(c^2 - 8^2) = 84,$$

$$\text{for both loads,} \qquad M_{8} = 196 \text{ ft.-tons max.}$$

67. Find the maximum bending moment in No. 66 if the uniform load is a quarter of a ton per foot of span.

$$d_1 = \frac{4}{\frac{1}{4}} = 16 \text{ feet;}$$

this is beyond field No. 1; hence the maximum is at the weight, that is, at 12 feet to right of centre.

Ans. $M_{-12} = 160$ ft.-tons maximum.

Bending Moments and Bending Moment Diagrams for Moving Loads and for Travelling Load Systems.

In Part First, page 15, the action of a live load when applied to a tie or strut is described; the action is somewhat similar when a live load is applied to a beam. Thus for a beam loaded at the centre, the load W may at one instant be in contact with the central point of the beam, and yet not be resting any of its weight on the beam; the next instant its whole weight may be resting on the beam. It does not follow directly from Hooke's Law but is a matter for demonstration, that for an instant, the strain thus produced is *double* that which the dead load produces, provided the greatest strain does not exceed the proof strain.

One way of applying the actual weight W to the centre as a dead load is, as in the case of a tie, to put it on bit by bit; another way is to put the whole weight W on the end of the beam, when the strain is zero, and then push it very slowly towards the centre, when the strain gradually increases to the full intensity due to W as a dead load. If W, on the other hand, be pushed from the end to the centre in an indefinitely short time, it will be the same as if it had been applied suddenly at the centre; in this case, then, W is applied as a live load.

Definition. A load which passes along a beam, and which thus occupies at different instants every possible position upon the span, is called a *moving* or *travelling load.*

A moving load may be dead or live or of intermediate importance, but not of greater importance than a live load. A travelling crane, which moves very slowly, and so as not to set the suspended weight swinging, is practically a dead moving load. The action of a moving load on a railway bridge is of intermediate importance; when the span of the bridge is short, say less than 20 feet, this importance is about equal to that of a live load; and when the span is long, say more than 40 or 50 feet, it is only a little greater than that of a dead load.

The Commissioners on the Application of Iron to Railway Structures at p. xviii. of their report say,—"That as it has appeared that the effect of velocity communicated to a load is to increase the deflection that it would produce if set at rest upon the bridge; also that the dynamical increase in bridges of less than 40 feet in length is of sufficient importance to demand attention, and may even for lengths of 20 feet become more than one-half of the statical deflection at high velocities, but can be diminished by increasing the stiffness of the bridge; it is advisable that for short bridges especially, the increased deflection should be calculated from the greatest load and highest velocity to which the bridge may be liable; and that a weight which would statically produce the same deflection should, in estimating the strength of the structure, be considered as the greatest load to which the bridge is subject."

Classes of moving loads. An uniform load coming on at one end of the span, covering an increasing segment till it is *all* on, then moving to a central position on the span, and passing off at the other end, is called an *advancing load.* A train of trucks, shorter than the span of a bridge, coming on at one end, travelling across and going off at the other end of the bridge, is an approximate example of, and is generally to be reckoned as, an advancing load. The reason that it is called approximate, is that although the weight of the trucks may be uniform per foot of length, yet they are not continuously in contact with the bridge but transmit the load thereto by means of wheels at a number of points. An advancing load may be equal in length to the span; in which case, in passing across, it covers the whole span for an instant. If the load be longer than the span, it will continue to cover it for a definite time while passing, but as time does not come into our consideration, it will be included in the advancing load equal in length to the span.

A load concentrated at a point, and which moves backwards and forwards on the span, is called a *rolling load;* a wheel which rolls along a beam is a practical example of this. In reality the load is distributed over a small area, and if now the load be taken to be uniformly distributed over this small area, it may be considered as an advancing load of small extent; on the diagrams it is represented by a wheel or circle.

A Travelling Load System is a load transmitted to the beam in definite amounts at points fixed relatively to each other, the whole load moving into all possible positions on the span; a locomotive engine is a practical example of such a system, and a rolling load is its simplest form. On the diagrams, the load is represented by a number of circles or wheels with their centres fixed on a frame (see fig. 3), or for ease in drawing by a number of vertical arrows connected by a thick horizontal line (see fig. 43).

It will not be necessary to consider moving loads upon cantilevers as in practice there is seldom such a thing. It is only necessary to suppose the load fixed in the position most remote from the fixed end; this, it is evident, gives the greatest bending moment at each point, the maximum being at the fixed end.

Beam under an advancing load equal in length to the span.— Suppose the load to come on from the left end and cover a segment of the span, the bending moment diagram is shown on fig. 33; when the whole span is covered, on fig. 25; and when the load is passing off, by fig. 33 reversed. Since the parabolas in these two figures are the *same,* it is evident that the apex *A* is higher on fig. 25 than upon fig. 33, because on the former the base of the parabolic segment is th whole span; the ordinate, not only for A but for every point on fig. 25, is greater than the corresponding ordinate upon fig. 33. Hence the maximum bending moment at each point of the span occurs when the whole span is loaded; of these maxima, the maximum is at the centre, and this case resolves into that of a beam uniformly loaded.

Beam under an advancing load less in length than the span.—Fig. 40. Let $2c$ = span; $2k$ = extent of load; w = intensity of load; C the origin, and centre of span; G the centre of load; and let the load be upon the span in any position.

Here $k < c$, and $W = 2wk$ = total load.

To find P, we may suppose the whole load concentrated at G, and we have

$$P = \frac{2wk}{2c}(c + x - y),$$

where x is the abscissa of any point of the span reckoned positive to the left of C, and y is the distance of the same point reckoned positive to the left of G. Taking a section at the point x, we have two forces acting on the portion of the span to the left of the section, viz., P acting upwards with a leverage $(c - x)$, and a load area equivalent to a force $w(k - y)$ acting downwards with a leverage $\frac{1}{2}(k - y)$; hence the bending moment at this section is

$$M_x = P(c - x) - w(k - y) \times \frac{k - y}{2}$$

$$= \frac{wk}{c}(c + x - y)(c - x) - \frac{w}{2}(k - y)^2$$

$$= \left(\frac{wk}{c}(c^2 - x^2) - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{2kx}{c} - y\right)y \text{..............(1.)}$$

As the load moves about, y varies and the bending moment M_x at the section x depends upon the position of the load, that is, upon the value of y. To find the position of the load which gives the greatest bending moment at the point x, it is only necessary to find the value of y which makes M_x a maximum; now, M_x is greatest when the product $\left(\frac{2kx}{c} - y\right)y$ is greatest; and as the sum of the two factors of this product is constant, the product is greatest when the factors are equal to each other, that is when

$$\frac{2kx}{c} - y = y;$$

so that we have $$y = \frac{kx}{c} \quad \text{.............................(2.)}$$

or $$y \; : \; k \; :: \; x \; : \; c \quad \text{....................(3.)}$$

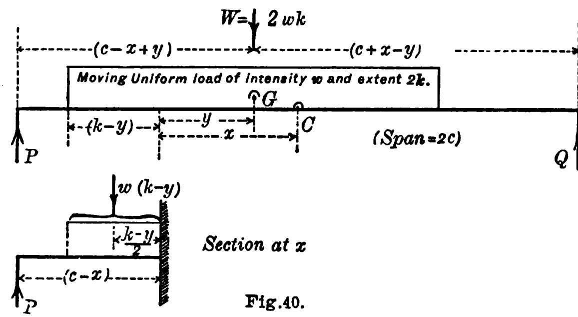


Fig.40.

This proportion expressed in words gives the following

RULE. The greatest bending moment at any point of the span occurs when there is directly over it, that point in the load which is situated in the extent of the load in a position similar to that in which the point is situated in the extent of the span.

Substituting in (1) the value of y in (2), we have

$$\text{max.} M_x = \left(\frac{wk}{c}(c^2 - x^2) - \frac{wk^2}{2}\right) + \frac{w}{2}\left(\frac{kx}{c}\right)^2$$

$$= \frac{wk}{2c^2}(c^2 - x^2)(2c - k) = \frac{2wk}{4c}(c^2 - x^2)\left(2 - \frac{k}{c}\right)$$

$$= \frac{W}{4c}(c^2 - x^2)\left(2 - \frac{k}{c}\right) \quad \text{.......................... (4.)}$$

This is the equation to the maxima bending moments, and may be written thus—

$$_{\text{max.}}M_x = C_o\ (c^2 - x^2)$$

where C_o is a constant quantity; the locus is therefore a parabola with its apex above the centre of span, and the maximum of these maxima—that is, the maximum for the whole span—is at the centre, or where $x = 0$;

$$_{\text{max.}}M_0 = \frac{Wc}{4}\left(2 - \frac{k}{c}\right) = \tfrac{1}{8}\left(2 - \frac{k}{c}\right)Wl,$$

$$= \tfrac{1}{4}W(l - k) \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

For the maximum bending moment, the coefficient

$$m = \tfrac{1}{8}\left(2 - \frac{k}{c}\right) \text{ (see page 46)};$$

and the principal equation to the parabola is

$$Y = \frac{W}{4c}\left(2 - \frac{k}{c}\right)X^2.$$

Compare the above with the result of investigation to fig. 31, from which this might have been deduced.

By the preceding rule the maximum bending moment at the centre of the span occurs when the centre of the load is over the centre of the span.

Graphical Solution.—Fig. 41. With a scale of feet lay off the span and draw a vertical upwards through O the centre; apply the rollers to the span; place any parabolic segment against the rollers with its apex on the vertical through O, shift the rollers till the curved edge passes through the ends of the span, which it will do simultaneously, and draw the curve. Construct a scale of ft.-lbs. for verticals such that $OA = \frac{1}{4}W(l - k)$, where W is in lbs., and l and k are in feet.

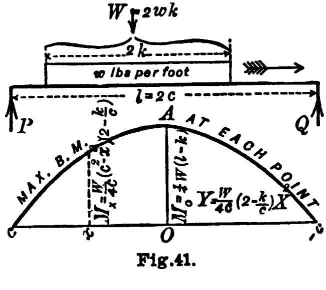


Fig. 41.

Note that to give the maximum bending moment at any point, the load assumes a different position for each point according to the above rule, and that it is possible to fulfil the condition of the rule for every point without any of the load going off the span.

COROLLARY I. Suppose the extent of load equal to the span; then $k = c$, $\left(2 - \frac{k}{c}\right) = 1$, and we have

$$_{\max.}M_x = \frac{W}{4c}(c^2 - x^2), \text{ and } _{\max.}M_o = \tfrac{1}{8}W.l,$$

the same as for the span uniformly loaded (fig. 25), and as shown in the preceding case. Note further, that the rule for fixing the position of the load so as to give the maximum bending moment at any point is fulfilled simultaneously for *all* points of the span; as it is evident that when the centre of load is over the centre of span, every point of the load is over the corresponding point of the span.

COROLLARY II. Suppose the extent of load to be zero; then $k = 0$, $\left(2 - \frac{k}{c}\right) = 2$, and the load is a rolling load for which

$$_{\max.}M_x = \frac{W}{2c}(c^2 - x^2), \text{ and } _{\max.}M_o = \tfrac{1}{4}W.l.$$

When the rule for finding the position of the load which gives the maximum bending moment at any point is applied to this case, it is found that the maximum occurs at any point of span when the rolling load is at that point.

As this is an important case, and leads to cases still more important, we will give a separate investigation.

Beam under a rolling load.—Fig. 42. Consider any point of the span at the distance x from the centre, distances to the left being reckoned positive. Let R be the amount of the rolling load, and suppose it over the point in consideration.

We may calculate the bending moment M_x from either of the two equations

$$M_x = P(c - x),$$

or

$$M_x = Q(c + x).$$

If the load moves to the right, then the upward supporting force P' is less than P, and

$$M'_x = P'(c-x)$$
$$< M_x\ ;$$

if now the load moves to the left, the supporting force Q' is less than Q, and

$$M''_x = Q'(c-x)$$
$$< M_x\ ;$$

thus M_x decreases whether R moves to the right or left, that is, M_x, the bending moment at any point x, is greatest when R the rolling load, is over the point.

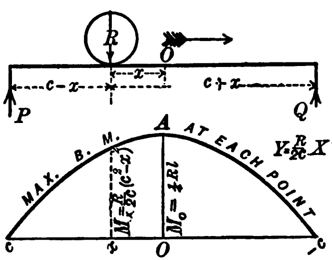


Fig. 42.

Let R be over the point x, then

$$_{\text{max.}}M_x = P(c-x) = \frac{R}{2c}(c+x)(c-x),$$
$$= \frac{R}{2c}(c^2-x^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

This is the equation to the maxima bending moments; the bending moment diagram is a parabola, with its axis vertical and its apex above the centre of span; and the maximum of these maxima, that is the maximum bending moment for the whole span occurs at the centre when the load is over the centre; putting $x = 0$, we have

$$_{\max.}M_0 = \frac{R}{2}c = \tfrac{1}{4}R.l \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.)$$

the value of the constant $m = \frac{1}{4}$, and the principal equation to the parabola is

$$Y = \frac{R}{2c}X^2.$$

Graphical Solution.—Fig. 42. With a scale of feet for horizontals lay off the span, and draw a vertical upwards through O the centre; apply the rollers to the span, place any parabolic segment against the rollers with its apex on the vertical through O, shift the rollers till the curved edge passes through the ends of the span, which it will do simultaneously, and draw the curve. Construct a scale of ft.-lbs. for verticals, such that $OA = \frac{1}{4}Rl$.

Examples.

68. An advancing load as long as or longer than the span, and of intensity 2 tons per foot, comes upon a beam 32 feet long.

Find the maximum bending moment for the whole span for all positions of the load.

Ans. $_{\max.}M_0 = \frac{1}{8}W.l = \frac{1}{8} \times 64 \times 32 = 256$ ft.-tons.

69. For the previous example find the maxima bending moments at intervals of four feet.

$$_{\max.}M_x = \frac{W}{4c}(c^2 - x^2) = \frac{64}{4 \times 16}(16^2 - x^2).$$

$$= 256 - x^2.$$

$\therefore\ M_{\pm 16} = 256 - 16^2 = 0,\ M_{\pm 12} = 256 - 12^2 = 112,\ M_{\pm 8} = 192,$

$M_{\pm 4} = 240$, and $M_0 = 256$ ft.-tons, all maxima.

70. In example 68, if the advancing load be 20 feet long, find the maximum bending moment for the whole span.

Ans. $_{\text{max.}}M_0 = \frac{1}{4}W(l-k) = \frac{1}{4} \times 40 \times (32-10) = 220$ ft.-tons.

71. In the previous example, find the maxima at intervals of 4 feet.

Substitute into equation (4) page 105, thus—

$$_{\text{max.}}M_x = \frac{W}{4c}(c^2 - x^2)\left(2 - \frac{k}{c}\right),$$

$$= \frac{55}{64}(256 - x^2)\text{; and}$$

$M_{16}=0$, $M_{12}=96\cdot25$, $M_8=165$, $M_4=206\cdot25$, and $M_0=220$ ft.-tons, all maxima.

72. In example 70, find $_{\text{max.}}M_8$ by placing the load in its proper position upon the span, and taking a section at $x = 8$.

Eight feet to the left of the centre is a *fourth* of the span to the left of the centre; to give the maximum bending moment, the load is to be placed so that the point which is a *fourth* of the extent of the load to the left of its centre shall be over the point. That is, the point of the load 5 ft. to the left of G is to be over the point of the span 8 feet to the left of C; hence G is 3 feet to the left of C, and

$$P = \frac{40}{32} \times 19 = \frac{95}{4}.$$

Taking a section at $x = 8$, we have upon its left a load area extending for 5 feet and equivalent to 10 tons; hence

$$_{\text{max.}}M_8 = \frac{95}{4} \times 8 - 10 \times \frac{5}{2},$$

$$= 165 \text{ ft.-tons.}$$

73. A beam 42 feet span is subject to an advancing load of 3 tons per foot and 12 feet long. Find the maximum bending moment at 7 feet on either side of the centre.

Where is the centre of the load situated when this maximum is produced?

Ans. ${}_{\text{max.}}M_{\underline{+}7}=288$ ft.-tons. Five feet from centre of span.

74. In the previous example, find the maximum bending moment for the whole span, for all positions of the above load.

$$Ans. \quad {}_{\text{max.}}M_0 = 324 \text{ ft.-tons.}$$

75. Find the principal equation to the parabola which is the curve of maxima bending moments in example 71.

$$Ans. \quad Y = \frac{W}{4c}\left(2-\frac{k}{c}\right)X^2 = \frac{55}{64}X^2.$$

76. A beam 30 feet span is subject to a rolling load of 40 tons; find the maximum bending moment for whole span. At what point does it occur, and how is the load then situated?

Ans. ${}_{\text{max.}}M_0 = \frac{1}{4}R.l = 300$ ft.-tons. R is at centre.

77. In the previous example, find the maxima bending moments at intervals of 5 feet. How must the load be situated in each case?

$$ {}_{\text{max.}}M_x = \frac{R}{2c}(c^2-x^2) = \tfrac{4}{3}(225-x^2);$$

$\therefore \quad M_{\underline{+}15}=0,\ M_{\underline{+}10}=166\tfrac{2}{3},\ M_{\underline{+}5}=266\tfrac{2}{3}$, and $M_0=300$ ft.-tons, all maxima. The load R in each case is over the point.

78. If the load in the examples 76 and 77 be spread uniformly over 3 inches, instead of being concentrated at a point, how much are the above results in error?

$$ {}_{\text{max.}}M_x = \frac{R}{4c}(c^2-x^2)\left(2-\frac{\text{extent of load}}{\text{extent of span}}\right);$$

this differs from the above expression by the factor

$$\tfrac{1}{2}\left(2-\frac{2k}{l}\right) = \tfrac{1}{2}\left(2-\frac{\cdot 25}{30}\right) = \frac{239}{240},$$

hence the results above would be in excess by a $\frac{1}{240}$th part, or by $\frac{5}{12}$ths per cent.

Beam under a travelling load system of two equal weights at a fixed interval apart.—Fig. 43. Let R be the total load, and $W_1 = W_2$ the weights numbered from the left end; let $4s$ be their distance apart, so that if G be the origin for loads, the abscissæ of W_1 and W_2 are $2s$ and $-2s$ respectively; the origin for the span is O the centre, and $\bar{x}$ is the distance from O to G.

To find the maximum bending moment at any point x.

First let W_1 be over the point x, the whole load being on the span; then P may be calculated as if the whole load R were at G, that is,

$$P = \frac{R}{2c}(c - \bar{x}), \text{ and}$$

$$_1M_x = P(c-x) = \frac{R}{2c}(c-\bar{x})(c-x) \quad \text{..........(1.)}$$

This is the equation to the bending moment at any point x, when W_1 is over the point, the whole load being on the span.

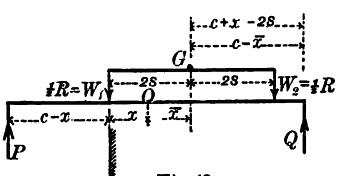


Fig. 43.

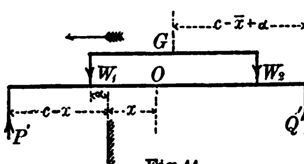


Fig. 44.

If the load travels a little to the right, P diminishes, and therefore the bending moment at x diminishes; if the load travels until W_1 is at a small distance a to the left of the section, then fig. 44

$$P' = \frac{R}{2c}(c - \bar{x} + a),$$

and

$$M'_x = P'(c-x) - W_1 a$$

$$= \frac{R}{2c}(c - \bar{x} + a)(c-x) - \frac{R}{2}a$$

$$= \frac{R}{2c}(c-\bar{x})(c-x) - \frac{R}{2c}ax$$

$$= {}_1M_x - \frac{Ra}{2c}.x$$

$$< {}_1M_x, \text{ if } x \text{ is positive.}$$

Hence ${}_1M_x$ is the maximum bending moment for values of x from 0 to c; that is, for any point in the left half of the span the maximum occurs when W_1 is over it, provided the whole load be then on the span. By symmetry the maximum for any point in the right half of the span occurs when W_2, the right weight, is over it.

Bending Moment Diagram.—Fig. 45. Substituting for $\bar{x}$ its value $(2s-x)$, we have

$${}_1M_x = \frac{R}{2c}(c - 2s + x)(c - x), \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

the equation to the maxima bending moments for first half of span. The locus is a parabola with its axis vertical, and

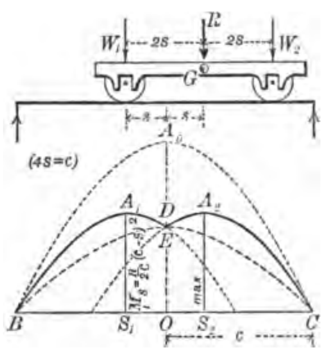


Fig.45.

the principal equation is $Y = \frac{R}{2c}X^2$; it is therefore the *same* parabola as for a rolling load R. To find the horizontal distance to the apex A_1, find that value of x which makes the product $(c - 2s + x)(c - x)$ greatest; since the sum of the factors is constant, this occurs when they are equal; thus—

$$c-2s+x = c-x$$

or $$x = s \qquad (3.)$$

That is, the apex A_1 lies to the left of O at a distance s, one quarter of the distance between the two weights. To find the height of A_1, put $x=s$, and

$$_1M_s = \frac{R}{2c}(c-s)^2, \qquad (4.)$$

the maximum of maxima for first half of span.

It is evident that A_2 will lie to the right at a distance s, and that the two parabolas will intersect at D on the vertical through the centre. It is convenient to call the first half of the span *field* 1, and to say that this field is *governed* by W_1; and we observe that the maximum in field 1 occurs when W_1, being in its own field, lies as far to one side of O the centre, as G lies to the other.

If it be possible for W_1 to occupy every point in its field without W_2 going off the span, we say that W_1 can *overtake* its field. In the present problem it is necessary that $4s$, the distance between the weights, be not greater than c the half-span, in order that each weight may be able to overtake its field. The problem divides into two cases.

Case I.—$4s =$ or $< c$.

On fig. 45, $4s = c$; and it is evident that one weight $\frac{1}{2}R$ may be at any point of the half-span, while the other weight is not on the span. Hence the locus of the maximum bending moment at each point when only one weight is on the span is the parabola BEC, due to a rolling load $\frac{1}{2}R$ as in fig. 42.

The apex E of this parabola coincides with D, the intersection of the pair of parabolas. This may be seen thus:— Shift the load until W_2 is over the centre; then, since W_1 is over B the extremity of the span, we may either consider it not yet on the span when OE is the bending moment, or we may consider that W_1 is just on the span when OD is the bending moment. Also note that $OE=\frac{1}{2}OA_0$, as BEC is the parabola due to the rolling load $\frac{1}{2}R$, while BA_0C is the parabola which would be due to a rolling load R.

The apex A_1 is higher than the apex E; the two parabolic arcs BA_1 and BE intersect at B, and every point on the arc BA_1D is outside of BEC. Hence the locus BA_1DA_2C is everywhere outside of BEC, and gives the maximum bending moment for each point of span.

Again, on fig. 46, $4s < c$; that is, A_1 and A_2 are closer together than on fig. 45, hence D is higher than E and BA_1DA_2C is again outside of BEC. BF and HC, parts of the parabola BEC, are shown by heavy dotted lines, and they indicate the bending moments, when one weight only is on the span.

Graphical Solution; $4s \gtreqless c$.—Figs. 45 and 46. With a scale of feet lay off BC equal to the span, and upon each

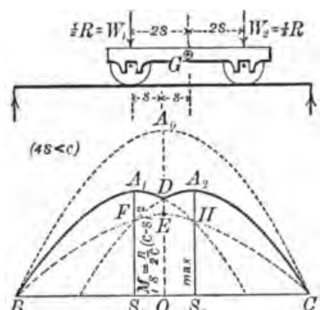


Fig. 46.

side of the centre lay off $OS_1 = OS_2 = s$; apply the rollers to BC, place any parabolic segment against the rollers with its apex on the vertical through S_1, shift the rollers till the curved edge passes through B and draw BA_1D; similarly draw DA_2C. Then BA_1DA_2C is the diagram of maximum bending moment at each point, the common height of A_1 and A_2 being maximum for the whole span.

Place the segment with its apex on the vertical through the centre, and move the rollers till the curved edge passes through B and C; mark A_0, and construct a scale of ft.-lbs. for verticals such that $OA_0 = \frac{1}{4}Rl$.

Case II.—$4s > c$; fig. 47.

In this case the apexes A_1 and A_2 are farther apart than on fig. 45, hence D is below E; hEf, the central portion of the parabola BEC due to only one weight on the span, lies outside of BA_1DA_2C and gives the maximum at each point for that portion. It will be seen that

$$OH = OF = 4s - c;$$

for, suppose W_2 at F, then W_1 is at B, and Ff representing the bending moment at F is the ordinate of BEC or of DA_2C, according as we consider that W_1 is not yet on, or is just on, the span. In this case, W_1 cannot overtake the portion HO, neither can W_2 overtake the portion OF, of their respective fields. There are two equal maxima at A_1 and A_2, and a third maximum at E; and the greatest of these is the maximum for the whole span. The point E may thus be the same height as, or higher or lower than, A_1.

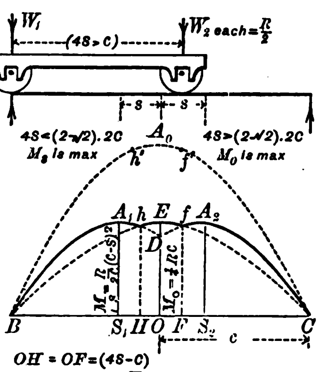


Fig. 47.

Suppose that E is of the same height as A_1, then A_0 will be twice as high as A_1; and we will have

$$OB : S_1B :: \sqrt{2} : 1;$$

that is $c : (c - s) :: \sqrt{2} : 1$

or $4s : 2c :: (2 - \sqrt{2}) : 1$;

hence $4s = (2 - \sqrt{2}).2c$,

or the distance between the weights $= (2 - \sqrt{2})$ times the span.

Thus if the distance between the weights equals or exceeds the quantity $(2 - \sqrt{2}).2c$, or about $\frac{7}{12}$ths of the span, the maximum bending moment for the whole span is at the centre.

Graphical Solution; $4s > c$.—Fig. 47. Lay off BC equal to the span; on each side of the centre lay off $OS_1 = OS_2 = s$, and $OH = OF = 4s - c$; draw verticals through S_1, H, O, F, S_2; apply the rollers to BC, place any parabolic segment against the rollers with its apex on the vertical through S_1, shift the rollers till the curved edge passes through B, and draw BA_1h, stopping at the vertical through H; similarly draw CA_2f. Shift the segment till the apex is on the vertical through the centre, move the rollers till the curved edge passes through B and C, and draw BA_0C. Bisect OA_0 in E; and plot as many points in hEf as may be necessary, by bisecting the ordinates of $h'A_0f'$, or in any manner construct the parabola BEC. Then BA_1hEfA_2C is the diagram of maximum bending moment at each point. Construct a scale for verticals such that $\overline{OA}_0 = \frac{1}{4}R.l.$

The complete interpretation of the locus BA_1DA_2C is shown in fig. 48. Suppose the beam to extend beyond the

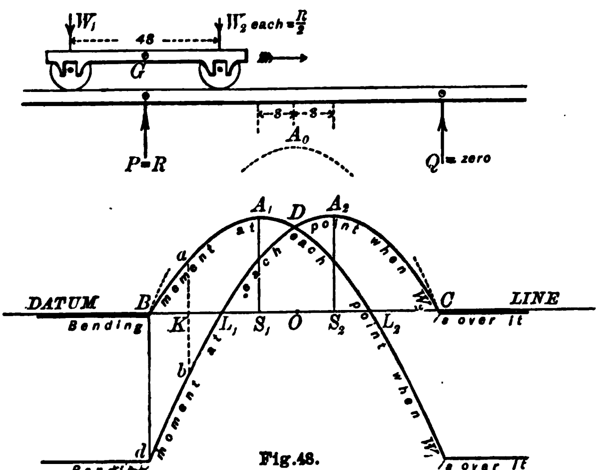


Fig. 48.

supports at B and C, and to be fixed at these supports so that P and Q may act upwards or downwards. In the figure the travelling load is standing with G over B, so that $P = R$, and Q is zero; hence at L_1, the point under W_2, the bending moment is zero, and this is the point at which CA_2D meets BC. Let the load move towards the left until W_2 is over any point as K; Q now acts downwards; at the point K, the beam is bent upwards and the bending moment is negative; the value of this *negative* moment is $Q.\overline{KC}$, and it is given by the *downward* ordinate $\overline{Kb}$. When W_2 arrives over B, the bending moment at B is negative; and its value, $Q.\overline{BC} = W_1.4s$, is given by the ordinate $\overline{Bd}$.

As the load moves farther to the left, the bending moment at each point, as W_2 comes over it, is of the constant value Bd; BC is now a cantilever under the downward load Q, and the bending moment at each point of BC is now negative, increasing indefinitely as the load moves towards the left. For all positions of the load, with no restriction on the value of $4s$, BA_1DA_2C gives the maximum positive bending moment at each point; and the height of A_1 is the greatest positive bending moment that can possibly be produced by the load system.

Beam under a travelling load system of two unequal weights at a fixed interval apart.—Fig. 49. Let R be the total load; W_1 and W_2 the weights numbered from the left end; let G their centre of gravity be the origin for loads, $2h_1$ and $-2h_2$ being the abscissæ of W_1 and W_2, so that the distance between the weights is $2h_1 + 2h_2$. Let W_1 be over any point of the span whose abscissa is x measured (positive to left) from the

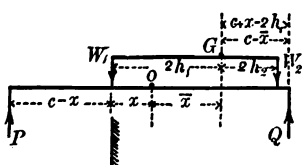


Fig. 49.

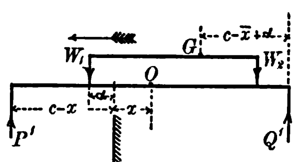


Fig. 50.

centre as origin, and let it be understood all through that the whole load is on the span.

As in the previous case,

$$_1M_x = \frac{R}{2c}(c - \bar{x})(c - x) \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

is the equation to the bending moment at any point x when W_1 is over it.

If the load travels a little to the right, P diminishes and therefore M_x diminishes; if the load travels until W_1 is at a small distance a to the left of the section, then (fig. 50),

$$M'_x = \frac{R}{2c}(c - \bar{x} + a)(c - x) - W_1 a$$

$$= \frac{R}{2c}(c - \bar{x})(c - x) + \left\{ \frac{R}{2c}(c - x) - W_1 \right\} a$$

$$= {}_1M_x - \left\{ x - (c - \frac{W_1}{R}.2c) \right\} \frac{Ra}{2c}$$

$$< {}_1M_x, \text{ if } x > (c - \frac{W_1}{R}.2c).$$

That is, the bending moment at x any point of the span, when W_1 is over it, is greater than when the load is in any other position, provided that the point itself is situated between B the left end, and F a point whose distance from the centre is

$$\overline{OF} = c - \frac{W_1}{R}.2c,$$

or from the left end is

$$\overline{BF} = \frac{W_1}{R}.2c.$$

Further, $\overline{CF} = \frac{W_2}{R}.2c$, and the bending moment is greatest at any point of $\overline{CF}$ when W_2 is over it. $\overline{BF}$ and $\overline{CF}$ are fields 1 and 2, and they are commanded by W_1 and W_2 respectively.

Bending Moment Diagram.—Fig. 51. For $\bar{x}$ substitute $2h_1 - x$ in equation 1, and we have

$$_1M_x = \frac{R}{2c}(c - 2h_1 + x)(c - x) \quad \ldots\ldots\ldots\ldots\ldots \quad (2.)$$

the equation to the maxima bending moments for field 1.

The locus is a parabola whose axis is vertical, and principal equation is $Y = \frac{R}{2c}X^2$; it is therefore the same parabola as for a rolling load R.

The abscissa of the apex A_1, that is OS_1, is found as before by equating the factors of equation 2, thus:—

$$c-2h_1+x = c-x,$$

or

$$x = h_1.$$

That is, the apex A_1 lies to the left of O at a distance $OS_1 = h_1$; similarly A_2 lies to the right of O at a distance $OS_2 = h_2$, each being half the distance between W and G. Putting $x = h_1$, we have

$$S_1A_1 = \frac{R}{2c}(c-h_1)^2 \dots\dots (3.)$$

Similarly

$$S_2A_2 = \frac{R}{2c}(c-h_2)^2 \dots\dots (4.)$$

If the point S_1 does not lie in field 1, the ordinates which are the bending moments for field 1 continually increase from zero at B the left end to their greatest value at F the other end of the field; if S_1 lies in field 1, then

$$_1M_{h_1} = S_1A_1 = \frac{R}{2c}(c-h_1)^2 \dots\dots (5.)$$

the maximum of maxima for field 1.
Similarly, if S_2 be situated in field 2, the height of A_2 will be the maximum bending moment for field 2.

Suppose $W_2 > W_1$; then since both parabolas are the *same* as that for the rolling load R, and are therefore the *same* as each other, A_2 is higher than A_1 because the quadrant CA_2S_2 stands on a longer base than BA_1S_1; and

$$_2M_{-h_2} = S_2A_2 = \frac{R}{2c}(c-h_2)^2 \dots\dots (6.)$$

the max. of maxima for field 2, and max. for whole span.

Now F is both in field 1 and field 2; and when W_1 arrives at F, the ordinate of the first parabola gives the maximum bending moment at F; again, when W_2 arrives at F, the ordinate of the second parabola also gives the maximum bending moment at F; that is, the ordinates at F are equal, or the two parabolas intersect at D a point on the vertical through F.

It is well to observe that the maximum in either field occurs when the weight commanding the field, while lying in its own field, is as far from the centre of the span upon one side as G is upon the other. If it be impossible in one of the fields for the weight so to lie, then for that field the bending moment continuously increases towards the end of the field not coinciding with the end of the span.

Case I. $2h_1 + 2h_2 \overline{\overline{<}} \frac{\text{smaller weight}}{\text{total weight}} \times \text{span}$, or distance between the weights $\overline{\overline{<}}$ shorter field. In this case it is evident that each weight can overtake its field.

Suppose now that the distance between the weights is *equal to* the smaller field, and that the load stands with the

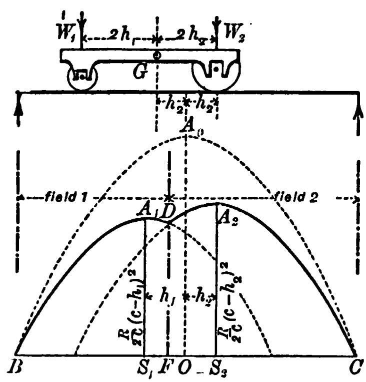


Fig.51.

greater weight just over F the junction of the fields, then the smaller weight is directly over the support. We may consider that the whole load is on the span, so that FD, the ordinate of the intersection of the parabolas, gives the bending moment at F; or we may consider that the greater load alone is on the span, so that the ordinate at F to the parabola for the greater load alone as a rolling load also gives the bending moment at F; that is, the last-named parabola passes through D. The parabola BDC for the greater load alone can be derived from BA_0C by taking the same fraction of its ordinates as the greater load is of the total load; it is therefore everywhere flatter than BA_0C; and since BA_1D and DA_2C are the *same* parabola as BA_0C, it follows that the locus BA_1DA_2C is everywhere outside of the parabola for either load alone.

Again, suppose the distance between the weights to be decreased, then the apexes A_1 and A_2 approach, and D their point of intersection rises; here again the locus BA_1DA_2C is outside of the parabola for either load alone. Therefore, for Case I., the locus BA_1DA_2C gives the maximum bending moment at each point of span due to the load coming on at either end, passing across, and going off at the other end.

Graphical Solution.—Fig. 51. Find G the centre of gravity of the loads, by dividing the total distance between the loads inversely as the loads by arithmetic or by the construction on fig. 19. Divide the span BC in F, so that $\overline{BF} = \frac{W_1}{R} 2c$; or draw parallels in opposite directions from B and C, and of lengths equal to the loads respectively, when the line joining their extremities will cut $\overline{BC}$ in F; and observe that the distance between the weights is not greater than BF. From the centre of the span lay off $OS_1 = h_1$ towards the left, and $OS_2 = h_2$ towards the right; draw verticals through O, F, S_1, and S_2; on the diagram, the vertical through F is drawn with dash and dot, to indicate that it separates two fields; those through S_1 and S_2 shown by fine continuous lines give maxima bending moments, and that through O shown by a dotted line is only constructive. Apply the rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, and draw BA_1D; similarly, draw CA_2D, and the segment should now pass through C and D simultaneously if the drawing has been accurately constructed. The figure BA_1DA_2C is the diagram of maximum bending moment at each point of the span. Shift the segment till its apex is on the vertical through the centre; move the rollers till the curved edge passes through B and C, and mark A_0; construct a scale for verticals such that $OA_0 = \frac{1}{4}R.l.$

Case II. $2h_1 + 2h_2 > \frac{\text{smaller weight}}{\text{total weight}} \times$ span, or distance between the weights > shorter field. Fig. 52. The two apexes A_1 and A_2 are further apart than in the previous case; D occupies a lower position and is no longer on BEC, of which a portion kEh is above BA_1DA_2C; and the diagram showing the maximum bending moment at each point is now $BkEhA_2C$.

To find the points h and k; let W_1 be over B, then W_2 is at H, where BH is the distance between the weights. We may either consider that W_1 is not yet on the span, when the ordinate at H to BEC gives the bending moment there; or that W_1 is just on the span, when the ordinate at H to DA_2C gives the same moment; that is, the parabolas intersect at h. Again, equating the value of the ordinate of the parabola BA_1D to that of BEC, we find the abscissæ of the points of intersection; thus

$$\frac{R}{2c}(c + x - 2h_1)(c - x) = \frac{W_2}{2c}(c^2 - x^2);$$

$$(c - x)\{R(c + x - 2h_1) - W_2(c + x)\} = 0 \text{ .... (7.)}$$

putting $c - x = 0$, then $x = c$, or the parabolas intersect at B, which we already know; putting

$$R(c + x - 2h_1) - W_2(c + x) = 0,$$

then $$W_1(c + x) = R.2h_1,$$

and $$x = \frac{R}{W_1}.2h_1 - c = OK \text{ .... (8.)}$$

which gives K the point on $\overline{OB}$ under the other point of intersection of the parabolas.

Now $$CK = CO + OK = \frac{R}{W_1}.2h_1,$$

but $$R : W_1 : W_2 :: h_1 + h_2 : h_2 : h_1;$$

$$\therefore \quad CK = \frac{h_1 + h_2}{h_2} . 2h_1 = \frac{h_1}{h_2}(2h_1 + 2h_2)$$

$$= (2h_1 + 2h_2)\frac{W_2}{W_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.)$$

= The distance between the weights increased in the ratio of the weights.

It *may* happen that E and A_2 are of the same height, in which case there are two equal maxima for the whole span, one at the centre, the other at S_2; again, E may be higher

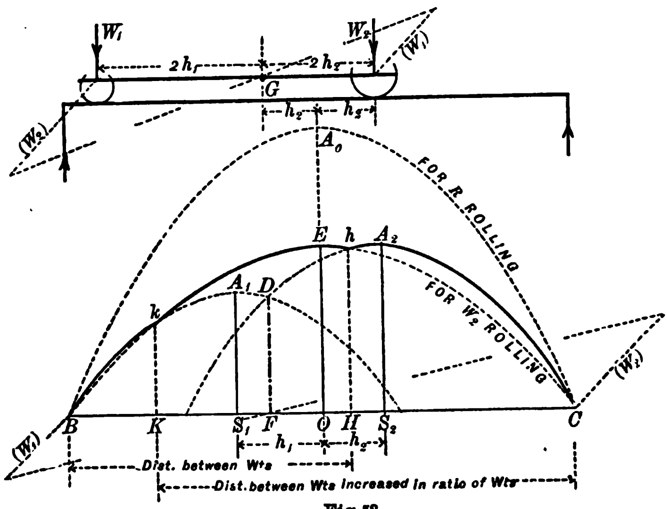


Fig. 52.

than A_2 when the maximum for whole span is at the centre. To investigate this, we have (equation 4)—

$$S_2A_2 = \frac{R}{2c}(c - h_2)^2,$$

and $$OE = \tfrac{1}{4} W_2 l = \tfrac{1}{2} W_2 c;$$

equating these quantities, we have

$$\frac{R}{2c}(c - h_2)^2 = \tfrac{1}{2} W_2 c;$$

$$\text{or} \qquad (c-h_2)^2 = \frac{W_2}{R}c^2; \text{ or } c-h_2 = \sqrt{\frac{W_2}{R}}.c;$$

$$\therefore \qquad h_2 = \left(1 - \sqrt{\frac{W_2}{R}}\right)c \ \ldots\ldots\ldots\ldots\ldots (10.)$$

$$\text{Also} \qquad h_1 = \frac{W_2}{W_1}h_2 = \frac{W_2}{W_1}\left(1 - \sqrt{\frac{W_2}{R}}\right)c \ldots (11.)$$

$$\therefore \qquad h_1 + h_2 = \left(1 + \frac{W_2}{W_1}\right)\left(1 - \sqrt{\frac{W_2}{R}}\right)c;$$

$$\text{or} \qquad \frac{h_1 + h_2}{c} = \frac{W_1 + W_2}{W_1} \cdot \frac{\sqrt{R} - \sqrt{W_2}}{\sqrt{R}};$$

$$\text{or} \qquad \frac{2h_1 + 2h_2}{2c} = \frac{R}{R - W_2} \cdot \frac{\sqrt{R} - \sqrt{W_2}}{\sqrt{R}};$$

$$\text{or, distance between weights} = \frac{\sqrt{R}}{\sqrt{R} + \sqrt{W_2}} . 2c \ \ldots\ldots (12.)$$

That is, if the height of A_2 be equal to the height of E the apex of the parabola due to the rolling load W_2, the distance between the weights W_1 and W_2 is equal to the fraction $\frac{\sqrt{R}}{\sqrt{R} + \sqrt{W_2}}$ multiplied by the span; W_2 being the greater weight, and R the total load. This fraction is always greater than $\frac{1}{2}$, and therefore it is possible for W_2 to be at the centre without W_1 being on the span. If the weights are fixed at this fraction of the span apart, then during their transit there are two equal maxima for the whole span, one at O when W_2 is over the centre, and another at S_2 when W_2 is over S_2.

If the distance between the weights is increased, the apexes A_1 and A_2 are lowered, and A_2 becomes lower than E the apex of the parabola for W_2 alone; so that during the transit of the load, the maximum for the whole span would occur at the centre while W_2 is over it.

It is thus shown that, when the distance between the weights equals or exceeds the above fraction of the span, the maximum bending moment for the whole span occurs at the centre when the greater weight is over it, and equals the maximum due to the greater load alone as a rolling load, viz.,

$$_{\max.}M_0 = \tfrac{1}{4}W_2.l \text{ .... } (13.)$$

By taking W_1 small enough, the fraction $\dfrac{\sqrt{R}}{\sqrt{R}+\sqrt{W_2}}$ may be made to differ from one-half by a quantity as small as we please; on the other hand, this fraction is greatest when W_2 takes its smallest possible value. Since W_2 is the greater weight, it is always greater than $\frac{1}{2}R$; putting W_2 equal to $\frac{1}{2}R$, we find that the greatest value of the fraction is equal to $(2-\sqrt{2})$ or ·586, and we have the following:—

If the distance between the unequal weights exceeds the half span by $\frac{86}{1000}$ths of the span, or about $\frac{1}{12}$th part, then during the transit of the load, the maximum bending moment for the whole span occurs at the centre when the greater weight is over it; should the weights be very unequal, the maximum will occur at the centre if the distance between the weights be only a little (less than a twelfth of the span) greater than the half span.

Graphical Solution.—Fig. 52. Construct BA_1DA_2C as in Case I., and observe that the distance between the weights is greater than BF; this determines that it belongs to Case II. Lay off $BH = 2h_1 + 2h_2$, and $CK = (2h_1 + 2h_2)\dfrac{W_2}{W_1}$. Draw verticals through H and K; ink in the portions Bk and Ch of the locus BA_1DA_2C, and construct a number of points on kEh by cutting the ordinates of BA_0C in the ratio $\dfrac{W_2}{R}$. If K does not lie on CB, the arc Bk is inadmissible, and the locus is $BEhA_2C$. Construct the scale for verticals such that $OA_0 = \frac{1}{4}R.l$.

Note.—During the transit W_2 really commands from C to K.

Corollary 1.—If CK equals CB, the points k and B coincide, and the parabolas touch at B; if CK exceeds CB

then k is beyond B. In either case BEC is outside of BA_1D, and W_2 commands the whole span.

Corollary 2.—If the beam extends past the supports as in fig. 48, and if P and Q are able to act upwards and downwards, then the locus BA_1DA_2C gives the maximum positive bending moment at each point for all cases; each curve drawn as on that figure gives the bending moment at each point when the corresponding weight comes over it.

Corollary 3.—Let both weights (fig. 52) be confined to the span, then

1°. If $FC \overline{>} (2h_1 + 2h_2)$, the locus for maximum bending moment at each point is the arc BD, chord Dh, and arc hA_2C;

2°. If $FC < (2h_1 + 2h_2)$, the locus is arc Bh', chord $h'D$, chord Dh, and arc hA_2C. The point h' is above H' and on the arc BA_1D, and $CH' = (2h_1 + 2h_2)$. H' and h' are not shown on the figure.

The maximum will be at S_2 or H.

Examples.

79. A beam, 20 feet span, bears a travelling load of 10 tons concentrated in equal portions on two wheels 8 feet apart. Find the maximum bending moment.

In this case, $W_1 = W_2$, and $4s < c$, so that (fig 46), $OS_1 = OS_2 = s$. The maximum occurs at 2 feet on either side of the centre, whether the load is confined to the span like a travelling crane or makes a transit like a truck; its amount is found by assuming the load to be standing with the left wheel two feet to left of centre.

Ans. $P = 4$ tons; ${}_1M_2 = {}_2M_{-2} = 32$ ft.-tons, max. for whole span.

80. In the preceding example, find the greatest bending moment that occurs at the point midway between the left end and the centre of the span.

Since the point is in the left half of span, the maximum at the point occurs when the left wheel is over it.

Ans. $P = 5{\cdot}5$ tons; ${}_1M_5 = 27{\cdot}5$ ft.-tons max. at that point.

81. Find, for example 79, the equation to the maximum bending moment at each point of left half of span, and give its value at intervals of two feet.

$$ {}_1M_x = \frac{R}{2c}(c+x-2s)(c-x) $$

$$ = \tfrac{1}{2}(6+x)(10-x); \text{ max. for values of } x \text{ from } 0 \text{ to } 10. $$

Hence ${}_1M_0 = 30$ ft.-tons; ${}_1M_2 = 32$, max.; ${}_1M_4 = 30$; ${}_1M_6 = 24$; ${}_1M_8 = 14$; ${}_1M_{10} = 0$.

82. In example 79, if the wheels are 4 feet apart, find the maximum bending moment for the whole span.

$$ {}_1M_s = \frac{R}{2c}(c-s)^2; $$

$$ {}_1M_1 = \frac{10}{20}(10-1)^2 = 40{\cdot}5 \text{ ft.-tons.} $$

83. In example 79 find the bending moment at the centre, when the load is in any position such that one wheel is on each side of the centre.

Ans. $M_0 = 30$ ft.-tons.

Note.—The bending moment at the centre is the same for either wheel over it, and for the wheels in any position one on each side of the centre; this will be proved in the general case.

84. A beam 20 feet span bears a travelling load of 10 tons concentrated in equal portions on two wheels which are 12 feet apart and confined to the span. Find the maximum bending moment for whole span.

In this case $OS_1 = 3$, and W_1 may occupy the position S_1. Calculate M_3 with the load in its proper position, or find M_s the maximum from the formula, and

$$ {}_1M_3 = {}_2M_{-3} = 24{\cdot}5 \text{ ft.-tons max.} $$

Note.—The maximum may readily be derived from the diagram thus:—The apex is above the point $x = 3$, so that, base of quadrant $= (10-3) = 7$; the modulus $=$ load $\div$ span $= \frac{1}{2}$; hence, height of apex, or modulus into base squared, is 24·5.

85. Find the maximum if the load in the previous example makes a transit.

In this case $4s > \cdot 586l$, so that the max. for whole span occurs at the centre when either wheel is there situated, the other not being then on the span; its value is $M'_0 = 25$ ft.-tons maximum for whole span during transit, as we see by comparing with ${}_1M_3$.

86. If the wheels in example 84 are 16 feet apart, and the load is confined to the span, find the maximum for the whole span.

The left wheel cannot come to the point $x=4$; bring it as near as possible to that point, viz., to the point $x = 6$, and the maximum occurs here. Its value is to be found by calculating M_6 when the load is in the position $x = 6$, or by substituting $x = 6$ into the equation to the left parabola.

$$Ans. \quad {}_1M_6 = {}_2M_{-6} = 16 \text{ ft.-tons.}$$

87. Find the equations to the maximum bending moment at each point in the preceding example, and the amounts at intervals of 2 feet.

$${}_1M_x = \tfrac{1}{2}(2+x)(10-x), \text{ for values of } x \text{ from 6 to 10;}$$
$$\text{and } {}_2M_x = \tfrac{1}{2}(2-x)(10+x), \text{ for values of } x \text{ from } -6 \text{ to } -10.$$

For the central portion, the chords Dh and Df (fig. 47) give the maxima bending moments. The height of D is ${}_1M_0 = 10$, and of h is ${}_1M_6 = 16$; the difference of these heights is 6, and the upward slope of Dh is one vertical to one horizontal.

Ans. ${}_1M_0 = 10$, $M'_2 = 12$, $M'_4 = 14$, ${}_1M_6 = 16$, ${}_1M_8 = 10$, ${}_1M_{10} = 0$, all maxima; the right half is symmetrical.

88. Find, for example 86, the equations to the maximum bending moment at each point of span when the load makes a transit; and give the values at intervals of two feet.

The equations are the same as in the previous example for the values of x from 6 to 10 and from -6 to -10; for the central portion, the locus is now the parabola for one wheel alone rolling, viz.:—

$$M'_x = \tfrac{1}{4}(100-x^2), \text{ for values of } x \text{ from 6 to } -6.$$

$M'_0 = 25$, $M'_2 = 24$, $M'_4 = 21$, $M'_6 = {}_1M_6 = 16$, ${}_1M_8 = 10$, ${}_1M_{10} = 0$ ft.-tons, all maxima for the transit.

89. A beam 40 feet span, bears a travelling load of 5 tons concentrated on two wheels 10 feet apart, there being 2 tons on the left and 3 tons on the right wheel. Find the maximum bending moment; the equations to, and amounts at intervals of 2 feet of, the maxima bending moments.

Data:—$W_1 = 2$, $W_2 = 3$, $R = 5$ tons; $c = 20$, $2h_1 = 6$, $2h_2 = -4$ feet.

Dividing 40 directly as 2 and 3, we have BF and FC, fields 1 and 2, equal respectively to 16 and 24. Since the distance between the wheels is less than the shorter field, the example comes under Case I. (see fig. 51), and the maximum bending moment at each point is the locus BDC, whether the load is confined to the span or makes a transit.

The maximum bending moment for whole span occurs at $x = h_2 = -2$; its value is to be found by supposing the load standing with the right wheel two feet to the right of the centre, and then calculating the moment at that point.

$$P = \tfrac{11}{4} \text{ tons, and } {}_2M_{-2} = 40{\cdot}5 \text{ ft.-tons max.}$$

The equations to the maximum bending moment at each point are,

For field 1, ${}_1M_x = \frac{1}{8}(14+x)(20-x)$, for values of x from 20 to 4.

For field 2, ${}_2M_x = \frac{1}{8}(16-x)(20+x)$, for values of x from 4 to -20.

And evaluating at intervals of two feet, we have

$$M_{20} = 0,\ M_{18} = 8, \text{ \&c., to } M_4 = 36;\ M_2 = 38\tfrac{1}{2},\ M_0 = 40,\ M_{-2} = 40{\cdot}5 \text{ max.},\ M_{-4} = 40 \text{ ft.-tons, \&c.}$$

Note.—The height of A_2 may be calculated from fig. 51 thus:—$OS_2 = 2$, the base of quadrant then is 18; the modulus is the load divided by span, that is $\frac{1}{8}$; hence the height of apex, or modulus into base squared, is $40{\cdot}5$.

To make the graphical solution, lay off $OS_1 = 3$, $OS_2 = 2$, and draw BDC as previously described; make a vertical scale upon which $OA_0 = \frac{1}{4} \times 5 \times 40 = 50$.

90. Find the maximum bending moment in the previous example if the wheels are 15 ft. apart.

The distance apart is still less than the shorter field, and $2h_2$ now equals 6 feet.

Ans. ${}_2M_{-3} = 36\frac{1}{8}$ ft.-tons, max. for transit.

91. Give the equations to the maximum bending moment at each point of span, and the amounts of *the* maxima in example 89, if the wheels are 20 ft. apart.

The distance apart is greater than the shorter field; the equations to the maxima bending moments for the transit are (fig. 52)—

$$ {}_1M_x = \tfrac{1}{8}(8 + x)(20 - x) $$

for values of x between B and K, that is from 20 to 10, since $CK = 20 \times \frac{3}{2} = 30$ feet.

$$ {}_2M'_x = \tfrac{3}{40}(400 - x^2) $$

for values of x between K and H, that is from 10 to 0; since BH equals the distance between the weights, H coincides with O.

$$ {}_2M_x = \tfrac{1}{8}(12 - x)(20 + x) $$

for values of x between H and C, that is from 0 to -20.

The two maxima are ${}_2M'_0 = 30$, and ${}_2M_{-4} = 32$ ft.-tons, the heights of E and of A_2 respectively. We could have anticipated that the height of A_2 would exceed that of E, since the distance between the loads does not exceed $\frac{7}{12}$ths of the span.

92. In the previous example, suppose the load confined to the span, and show how the locus of maxima bending moments is affected.

For the central portion from F to H, the maxima are the ordinates of the chord DE or Dh since the points E and h coincide, and the height of A_2 is still the maximum for the whole span. The ordinates to the chord Dh may be found by supposing the right wheel fixed at H, the other wheel being then over the point of support may be neglected. Otherwise, the height of D is found by substituting $x = 4$, into either ${}_1M_x$ or ${}_2M_x$; and the height of h by substituting $x = 0$, into ${}_2M_x$; thus, height of D or $M_4 = 24$, height of h or $M_0 = 30$, and by proportion the intermediate ordinates of the chord Dh are easily found.

93. A beam 56 feet span bears a travelling load of 16 tons concentrated on two wheels 32 feet apart, 7 tons being on the left wheel and 9 tons on the right. Find the maximum bending moment during the transit.

The right hand side of equation 12 on page 125 may be written thus,—

$$\frac{R - \sqrt{R.W_2}}{W_1} 2c;$$

in our example, this quantity equals 32 feet, which happens to be the distance between the weights; hence there will be two equal maxima, one at 7 feet to the right of the centre when the greater load is over it; the other, at the centre also when the greater lond is over it, the smaller load not being then on the span. Place the load in those positions respectively, and calculate the moments.

$$Ans. \quad {}_2M_{-7} = {}_2M'_0 = 126 \text{ ft.-tons.}$$

94. A travelling load of 5 tons concentrated on two wheels 10 feet apart, 1 ton being on the left wheel and 4 tons on the right, passes over a beam of 40 feet span. Find the maxima bending moments at intervals of 4 feet, and the maximum for the whole span.

Distance between weights × ratio of weights = 40 feet = span, so that k coincides with B (fig. 52), and therefore arc BhE lies everywhere above arc BD; that is, the maximum at each point of span occurs when the greater weight is over it. Further, the height of A_2 is greater than that of E, since the distance between the weights does not exceed $\frac{7}{12}$ths of the span. Placing the greater load over points at intervals of 4 feet and calculating the bending moments, or substituting into the equations to the loci Bh and hA_2C, we have

${}_2M'_{20}=0, {}_2M'_{16}=14{\cdot}4, {}_2M'_{12}=25{\cdot}6$; ${}_2M_8=35, {}_2M_4=42, {}_2M_0=45,$
${}_2M_{-4}=44, {}_2M_{-8}=39, {}_2M_{-12}=30, {}_2M_{-16}=17, {}_2M_{-20}=0$ ft.-tons.
Since the greater weight lies 2 feet to the right of the

centre of gravity of the load, the maximum lies at 1 foot to the right of the centre of span, and

$_2M_{-1} = 45\frac{1}{8}$ ft.-tons max. for whole span during transit.

The equations to the loci from which these may be calculated are

for Bh, $\quad {}_2M'_x = \frac{4}{40}(c^2 - x^2) = \frac{1}{10}(400 - x^2)$
for values of x from 20 to 10;

and for hA_2C, $\quad {}_2M_x = \frac{5}{40}(c + 2h_2 - x)(c + x)$

$$= \tfrac{1}{8}(18 - x)(20 + x)$$

for values of x from 10 to -20.

95. A beam 30 feet span supports a load of 5 tons concentrated on two wheels 20 feet apart, 1 ton being on the left wheel and 4 tons on the right. The load being confined to the span, find the maximum bending moment.

In this example, S_2 lies 2 feet to the right of the centre, or 17 feet from the left end, while H lies 20 feet from the left end; hence H lies to the right of A_2 (fig. 52.) The highest ordinate of the locus, arc BD, chord Dh, arc hC, is the ordinate of h.

Ans. $\quad {}_2M_5 = 26\frac{2}{3}$ ft.-tons.

The Parabola.

Theorem. If to the ordinates of a parabola, axis vertical and apex to left or right of origin, the ordinates of another parabola, with axis vertical and passing through the origin, be added; then the new locus is a parabola, its modulus is the sum of their moduli, and its axis is vertical; its apex lies on the same side of the origin as the apex of the first parabola; and the abscissa of its apex is the same fraction of the abscissa of the first parabola's apex, that the modulus of the first parabola is of the sum of the moduli. Fig. 53. See also figs. 15 and 16.

For the parabolas whose apexes are A_1 and A_0 respectively,—

$$y_1 = (H_1 - mK_1^2) + 2mK_1x - mx^2.$$
$$y_0 = H_0 \qquad\qquad - m_0x^2.$$

Adding, $y'_1 = (H_0 + H_1 - mK_1^2) + 2mK_1x - (m_0 + m)x^2,$

or $$y'_1 = (H_0 + H_1 - mK_1^2) + (m_0 + m)\left(\frac{2mK_1}{m_0 + m} - x\right)x \ldots.(1.)$$

which is a parabola of modulus $(m_0 + m)$, its axis is vertical, its apex lies on the same side of origin as the apex of the

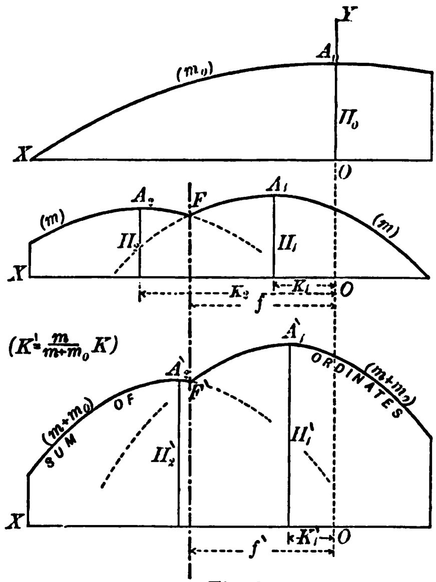


Fig.53.

first parabola; and if K'_1 is the value of x which makes $\left(\frac{2mK_1}{m_0 + m} - x\right)x$ greatest, then

$$K'_1 = \frac{m}{m_0 + m}K_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Q. E. D.

Theorem. If to the ordinates of a locus consisting of two parabolas, with axes vertical, with a common modulus m, and intersecting at F, there be added the ordinates of a parabola, axis vertical, apex above origin, and modulus m_0; then the new locus consists of a pair of parabolas with axes vertical, having the common modulus (m_0+m), and intersecting at F' on the same vertical as F. Fig. 53.

For the parabolas whose apexes are A_1 and A_2 respectively,—

$$y_1 = (H_1 - mK_1^2) + 2mK_1x - mx^2.$$

$$y_2 = (H_2 - mK_2^2) + 2mK_2x - mx^2.$$

If f be the abscissa of the point of intersection, then where $y_1 = y_2$, $x = f$; subtracting, we have

$$0 = (H_1 - H_2) - m(K_1^2 - K_2^2) + 2m(K_1 - K_2)f \ldots (3.)$$

which gives the value of f.

Again, by the previous theorem, equation (1)—

$$y'_1 = (H_0 + H_1 - mK_1^2) + 2mK_1x - (m_0 + m)x^2.$$

similarly, $$y'_2 = (H_0 + H_2 - mK_2^2) + 2mK_2x - (m_0 + m)x^2.$$

Let f' be the abscissa of the point of intersection, then where $y'_1 = y'_2$, $x = f'$; subtracting, we have

$$0 = (H_1 - H_2) - m(K_1^2 - K_2^2) + 2m(K_1 - K_2)f' \ldots\ldots (4.)$$

hence $$f' = f \ldots\ldots (5.)$$

Q. E. D.

Theorem. If two parabolas with axes vertical and a common modulus m intersect at a point, then the horizontal projection of any double chord drawn through that point is constant and is equal to the double horizontal chord drawn through it. And conversely, if a line whose horizontal projection is equal to the double horizontal chord through the point of intersection be placed with one extremity on each parabola, either the line or the line produced passes through the point of intersection. Fig. 54.

The equation to the parabola (fig. 16) is

$$y = (H - mK^2) + 2mKx - mx^2;$$

choosing the origin at O the point of intersection, then $H_1 = mK_1^2$, and $H_2 = mK_2^2$; hence $(H_1 - mK_1^2) = (H_2 - mK_2^2) = 0$; and considering K_1 and K_2 positive, we have

$$y_1 = 2mK_1x - mx^2 \quad \text{.......................... (6.)}$$

$$y_2 = -2mK_2x - mx^2 \quad \text{........................ (7.)}$$

The equation to any straight line through O (see page 21) may be written

$$y = \mu x \quad \text{.. (8.)}$$

equating (6) and (8), we find for V the point of intersection $x = 2K_1 - \frac{\mu}{m}$; similarly for N, $x = -2K_2 - \frac{\mu}{m}$.

Hence $\quad \overline{vn} = 2(K_1 + K_2) = ST$ .... (9.)

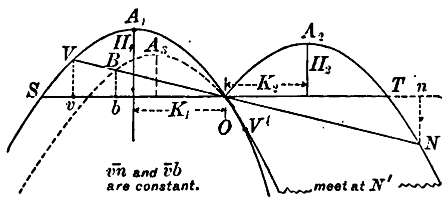


Fig.54.

Further, if VN be such that $\overline{vn} = 2(K_1 + K_2)$, and V moves on parabola No. 1 while N moves on parabola No. 2, it is evident that $\overline{VN}$ always passes through O. When V arrives at O, then $\overline{VN}$ is a tangent at O to parabola No. 1; and if V moves to V', then $N'V'$ produced passes through O. *Q. E. D.*

Theorem. While the double chord VON turns, if for each position there be taken on it a point B whose horizontal projection b divides $\overline{vn}$ in a constant ratio; then this point traces out a parabola whose axis is vertical, which has the same modulus m as the parabolas Nos. 1 and 2, and which passes through the point of intersection O. Fig. 54.

x, y being the coordinates of V; let X, Y, be the coordinates of B; and $\overline{vb} = C$ a constant; then

$$x = X + C, \text{ and } Y = \frac{y}{x} X.$$

Now $$y = 2mK_1x - mx^2, \text{ or } \frac{y}{x} = m(2K_1 - x);$$

$$\therefore \quad Y = m(2K_1 - C - X)X \quad \text{.................... (10.)}$$

the locus of B; and it is a parabola with axis vertical, of modulus m, and passing through O. The abscissa of the apex A_3 is equal to $K_1 - \frac{1}{2}C$. *Q. E. D.*

Travelling load concentrated on two wheels together with fixed loads.—Let the travelling load be such as is described in Case I., page 121. With any parabolic segment draw the diagram as on fig. 51; on this, to the same scales, draw the diagram for fixed loads as on fig. 39. Through S_1 draw a vertical, and find the points d_1, d_2, &c., as described for and shown on fig. 39; in this case $2S_1A_1 = LN$, and $BS_1 = LS$. With the same parabolic segment, placing its apex successively on the verticals through d_1, d_2, &c., draw the arcs BE, EF, &c., as on fig. 39, until the vertical dividing the two fields is reached. Again, draw a vertical through S_2 and find the points d_5, d_4, &c.; in this case $2S_2A_2 = LN$, and $CS_2 = LS$; with the same parabolic segment draw the arcs CH, HG, &c., until the point formerly fixed is reached.

For Case II., page 123, the method to be employed is somewhat similar:—Draw the diagrams as on figs. 52 and 39. Draw the arcs BE, EF, &c., of fig. 39 until the vertical through K is reached, and the arcs CH, HG, &c., also of fig. 39, until the vertical through H is reached, all with the parabolic segment used for drawing BA_1DA_2C of fig. 52, as for Case I. Complete the figure by drawing the arcs between the verticals through K and H with the same parabolic segment as BEC (fig. 52) is drawn with; here, $2OE = LN$, and $c = LS$, where c equals the half-span.

One case is of special interest, viz., that of case fig. 51, together with a fixed load W at F the junction of the fields. The effect of adding this weight is to make the apexes A_1 and A_2 move horizontally toward each other. Some particular value of W will make them come on the same vertical when it is evident that they must coincide; any value of W greater still will cause them to pass each other as on fig. 29; and it can easily be shown that the value of W which makes them coincide is

$$W = \frac{h_1 + h_2}{c} R.$$

Hence, if on fig. 51, besides the load shown on the beam, there be added at F the junction of the fields, a weight which is the same fraction of R that the distance between the wheels is of the span, then BA_0C is the bending moment diagram, just as if R were concentrated on one wheel as a rolling load. And conversely, if R be concentrated as a rolling load, and at the same time the weight above-mentioned act *upwards* at F, then BA_1DA_2C is the bending moment diagram for the combined load; that is, we may construct a bending moment diagram for case fig. 51 by drawing the parabola BA_0C, and above the same base a triangle corresponding to the *upward* weight at F; the distances between the triangle and parabola give the bending moments, but there is the disadvantage of having a figure which consists of *two* irregular outlines, and which fails to show the positions of the maxima A_1 and A_2.

Beam under a travelling load system of unequal weights fixed at irregular intervals, the load being confined to the span so that no weight passes off.—Fig. 55. Let R be the total load; W_1, W_2 ... W_r ... W_n, the weights numbered in order from the left end; G the centre of gravity and origin for the weights; $2h_1$, $2h_2$ ... $2h_r$ ... $2h_n$, the abscissæ of the weights, those to the right of G including their negative sign; O the origin for, and $2c$ the length of, the span, distances towards the left being positive; and let x be the abscissa of any section.

First. Let the load be in a position such that the r^{th} weight is over the section; x_1, x_2 ... x_{r-1}, the abscissæ of the $(r-1)$ weights, measured from O; $\bar{x}$ the abscissa of G, and c that of the left end of span; then

$$P = \frac{R}{2c}(c + \bar{x});$$

and $\quad {}_rM_x = \dfrac{R}{2c}(c+\bar{x})(c-x) - W_1(x_1-x) - W_2(x_2-x) - \ldots\ldots$

$$\ldots - W_{r-1}(x_{r-1}-x) - W_r(x_r-x) \ldots (1.)$$

the last term is zero, since $x_r = x$, and it may either be expressed as above or omitted.

Second. Let the load be situated at a short distance a to the right of its former position; then

$$P' = \frac{R}{2c}(c+\bar{x}-a);$$

and $\quad M'_x = \dfrac{R}{2c}(c+\bar{x}-a)(c-x) - W_1(x_1-a-x) - W_2(x_2-a-x) - \ldots$

$$\ldots - W_{r-1}(x_{r-1}-a-x)$$

$$= {}_rM_x - \frac{R}{2c}(c-x)a + (W_1 + W_2 + \ldots + W_{r-1})\,a \ldots (2.)$$

$$< {}_rM_x, \text{ if } \frac{R}{2c}(c-x) > \Sigma_1^{r-1}(W),$$

$$\text{or if} \quad c - \frac{\Sigma_1^{r-1}(W)}{R}.2c > x;$$

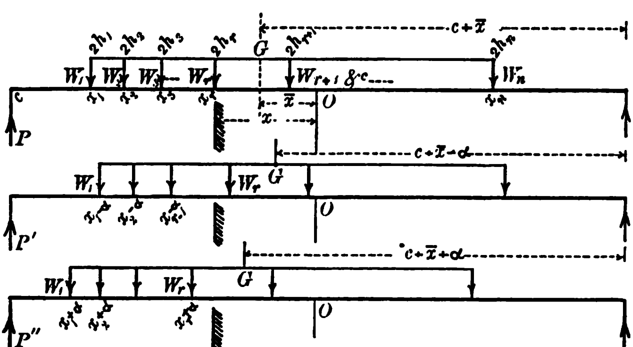


Fig. 55.

that is, the bending moment for the first position is the greater, if the distance of the section to the left of the centre is less than $c - \dfrac{\Sigma_1^{r-1}(W)}{R}.2c$; or what is the same thing, if the distance of the section from the left end is greater than $\frac{\Sigma_1^{r-1}(W)}{R} \cdot 2c$.

Third. Let the load be situated at a short distance a to the left of the first position; then

$$P'' = \frac{R}{2c}(c+\bar{x}+a); \text{ and}$$

$$M''_x = \frac{R}{2c}(c+\bar{x}+a)(c-x) - W_1(x_1+a-x) - W_2(x_2+a-x) - \ldots\ldots$$
$$\ldots\ldots - W_r(x\ +a-x),$$
$$= {}_rM_x + \frac{R}{2c}(c-x)a - (W_1 + W_2\ + \ldots +\ W\ )a \ldots\ldots\ldots (3.)$$
$$< {}_rM_x, \text{ if } \frac{R}{2c}(c-x) < \Sigma_1^r(W);$$
$$\text{or, if } c - \frac{\Sigma_1^r(W)}{R} \cdot 2c < x;$$

that is, the bending moment for the first position is greater than for the third, if the distance of the section to the left of the centre is greater than $c - \frac{\Sigma_1^r(W)}{R} \cdot 2c$; or, what is the same thing, if the distance of the section from the left end is less than $\frac{\Sigma_1^r(W)}{R} \cdot 2c$.

Hence there is a portion of the span, lying between the point whose distance from the left end is $\frac{\Sigma_1^r(W)}{R} . 2c$, and the point whose distance from the left end is $\frac{\Sigma_1^{r-1}(W)}{R} \cdot 2c$, such that the bending moment at any section in that portion is greater when the r^{th} weight is over it than for any other position of the load, if for each position all the weights are on the span as premised. This portion of the span we call the r^{th} *field*, and we say it is *commanded* by the r^{th} weight. The extent of this r^{th} field is

$$\frac{\Sigma_1^r(W)}{R} . 2c - \frac{\Sigma_1^{r-1}(W)}{R} . 2c = \frac{W_r}{R} . 2c \ldots\ldots\ldots\ldots\ (4.)$$

the same fraction of the span as the r^{th} weight is of the total load; hence, in order to mark the fields, the span is to be divided into as many portions as there are weights, these portions being proportional to the weights and in the same order. The maximum bending moment at any point occurs when the weight, which commands the field in which the point lies, comes over that point. Since no weight is to go off the span, sometimes there is a part of a field which the commanding weight cannot occupy, and then the weight is said not to be able to *overtake* that part of its field; as will be proved when we come to the graphical solution, the maximum bending moment for such points occurs when the commanding weight is as close thereto as it can be brought.

Into the expression for ${}_rM_x$ the maximum at any point of the r^{th} field, substitute as follows:—

$\bar{x} = (x - 2h_r)$, $(x_1 - x) = (2h_1 - 2h_r)$, $(x_2 - x) = (2h_2 - 2h_r)$, &c. ... $(x_{r-1} - x) = (2h_{r-1} - 2h_r)$; and we have

$$
\begin{aligned}
{}_rM_x &= \frac{R}{2c}(c + x - 2h_r)(c - x) - W_1(2h_1 - 2h_r) - W_2(2h_2 - 2h_r) - \dots \\
&\qquad\qquad \dots - W_{r-1}(2h_{r-1} - 2h_r) \\
&= \frac{R}{2c}(c + x - 2h_r)(c - x) + 2h_r\, \Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W.2h) \quad \dots(5.)
\end{aligned}
$$

this is the equation to the maxima bending moments for the r^{th} field. The locus is a parabola, its axis is vertical, its apex is above the span and may lie to either side of O the centre of span; its modulus is $\frac{R}{2c}$, a quantity which is the same for all fields and is the same as the modulus of the parabola due to R as a rolling load. The abscissa of A_r, the apex of the r^{th} parabola, is that value of x which makes ${}_rM_x$ or $(c + x - 2h_r)(c - x)$ greatest, that is where

$$x = h_r \quad \dots\dots\dots\dots (6.)$$

hence the apex of each parabola lies on the same side of, and horizontally half as far from, the centre of the span as the commanding weight is from G the centre of gravity of the load. The apexes for some of the fields may lie in their own fields, and in such fields the max. of the maxima is given by the ordinate of the apex; for other fields, the apexes may lie outside of their own fields, and in these there is no max. of maxima, but the maxima increase continuously from one end of the field to the other.

Bending Moment Diagram.—Fig. 56. The locus is the polygon BD_1D_2, &c., formed with parabolic arcs BD_1, D_1D_2, &c.; each parabola being the same as BA_0C that for the rolling load R, but lying with their apexes at the distances $OS_1 = h_1$, $OS_2 = h_2$, &c., where h_1, h_2, &c., are half the respective distances of W_1, W_2, &c., from G the centre of gravity of the load. The parabolas intersect in pairs on the verticals through the junctions of the fields; that is, through F_1, F_2, &c., points such that

$$BF_1 : F_1F_2, \&c. : BC :: W_1 : W_2, \&c. : R;$$

for, if F_r be the junction between the r^{th} field and the $(r+1)^{th}$ field, then F_r is the last point in the r^{th} field; the maximum at F_r occurs when W_r is over it and is given by the ordinate at F_r to the r^{th} parabola. Again, F_r is the first point in the $(r+1)^{th}$ field; the maximum at F_r occurs when W_{r+1} is over it and is given by the ordinate at F_r to the $(r+1)^{th}$ parabola. But the maximum at F_r is some one definite quantity, hence the two parabolas have a common ordinate at F_r.

Maximum Bending Moment for whole span.—In fig. 56, A_3 and A_4 lie respectively in their own fields; and the one which has the greater ordinate gives the maximum for whole span. In the general case one or more apexes will lie in their own fields; one at least, as we cannot conceive of such a series of curves lying so that every apex is inside of another of the curves. The ordinate of the apex that lies in its own field, if only one is so situated, is maximum for the whole span; the ordinate of the highest apex, if more than one be so situated.

Suppose the first curve continued to the right, past D_1; the second past D_2, &c.; then each parabola is the locus of the bending moment at each point as the corresponding weight comes over it, the whole load being on the span. If the load stands still in any position, as for instance that in which the load is drawn in fig. 56, the bending moment at the point where W_1 stands is the ordinate there of the 1st parabola, at the point where W_2 stands the ordinate of the 2nd parabola, &c. If then the ordinates at these points be drawn each to the proper parabola, and the tops of the ordinates be joined, we will have the bending moment diagram for that set of fixed loads; we will have in fact the diagram shown in fig. 18, because the load is now the fixed load shown in fig. 3.

Further, for the position of the load shown in fig. 56, the straight line joining the tops of the two ordinates, one drawn to parabola 2 from the point where W_2 stands, and the other to parabola 3 from the point where W_3 stands, will pass through D_2 the intersection of these parabolas, because the horizontal projection of that joining line is constant, being equal to the distance between W_2 and W_3; and we know that one end of this joining line coincides with D_2 when either of the weights W_2 or W_3 is over F_2; hence by the preceding theorem (fig. 54) it will always pass through D_2. Now, for the position of the load shown, the joining line gives the bending moments at all intermediate points, so that the ordinate of D_2 is the bending moment at F_2 for that position of the load; and similarly for any other position for which F_2 lies between the weights W_2 and W_3. In other words, the bending moment at F the junction of two fields is the same, whether the weight commanding the field on either side is over it, or whether the load stands in any position for which F lies between those weights. When the load stands in a position where those two weights are both to one side of F, then the joining line *produced* still passes through D, but the ordinates of the produced line are not bending moments. For instance, the line joining the tops of the ordinates on fig. 56, one drawn to parabola 4 from the point where W_4 stands, and the other to parabola 5 from the point where W_5 stands, will when produced pass through D_4 the junction of those parabolas; however, not the ordinate at F_4 to that produc line, but the ordinate to the line which terminates at C, gives the bending moment there.

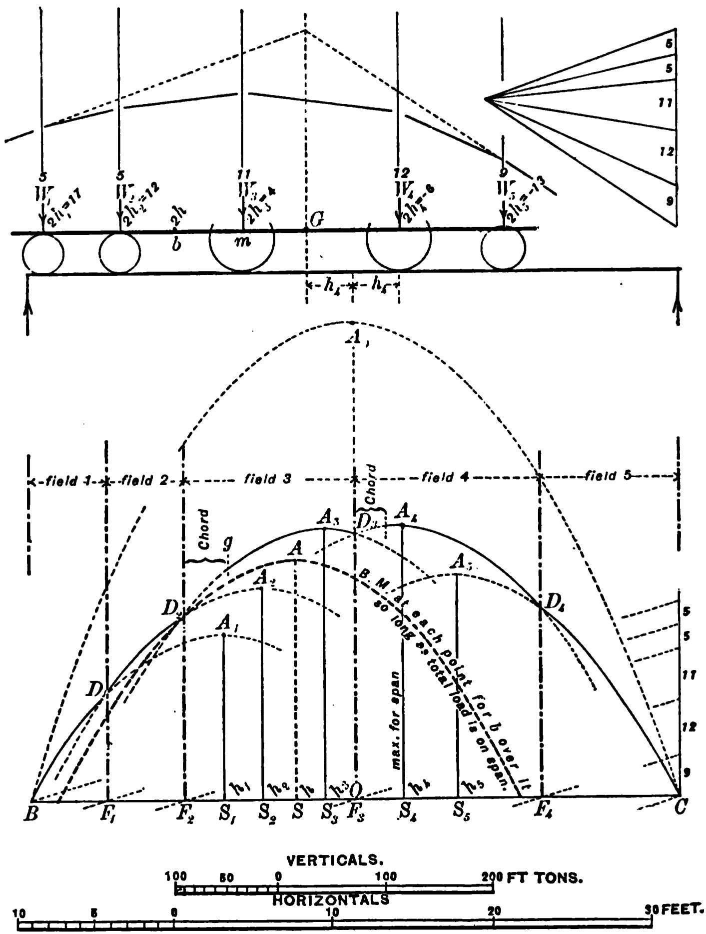


Fig. 56.

On fig. 56, observe that if the load moves more than one foot to the left, W_1 goes off the span; and there is a portion of field 3 at its left end, which W_3 cannot overtake, and for which the corresponding portion of parabola 3 is dotted, being inadmissible. For the position of the load shown, we saw that the bending moment at each point of that portion of field 3 is given by the ordinate to a straight line from D_2 to the top of the ordinate of parabola 3, at the point where W_3 is standing; that is, by the chord of parabola 3, from D_2 to the top of that ordinate. Now, the closer W_3 comes to F_2 the steeper will that chord be, and consequently the greater the bending moments at all these points. Hence the chord of the dotted or inadmissible part of parabola 3 gives the maximum bending moment at each point of the portion of field 3 which W_3 cannot overtake. Similarly for any portion of any field which the commanding weight cannot overtake, the chord of the parabola, instead of the arc, gives the maxima bending moments for the whole load on the span. Fig. 52 has already been quoted as an example of this, when we assumed the arc Bk, chord kD, chord Dh, and arc hC to be the diagram of maxima bending moments for load confined to the span.

Graphical Solution.—Fig. 56. Lay off the wheel base and find G the centre of gravity of the load either by analysis as at fig. 3, or graphically as in fig. 19 and indicated in fig. 56. Lay off BC equal to the span; divide it at the points F into fields proportional to the weights, either by arithmetic or as indicated on the diagram, and draw vertical lines through them to separate the fields. Lay off OS_1 equal to half the distance of W_1 from G, OS_2 equal to half the distance of W_2 from G, &c.,—each point S being on that side of the centre O on which the corresponding weight lies with respect to G; draw verticals through the points S; apply the parallel rollers to BC; place any parabolic segment against the rollers with its apex on the vertical through S_1; shift the rollers till the curved edge passes through B, and draw the arc BD_1, stopping at the vertical through F_1; shift the segment till the apex is on the vertical through S_2, move the rollers till the curved edge passes through D_1, and draw the arc D_1D_2, stopping at the vertical through F_2. Similarly draw arc after arc in succession freach field, and if the arc for the last field passes through C the extremity of the span, it checks the accuracy of the drawing. Lastly, shift the segment till the apex is on the vertical through the centre; move the rollers till the curved edge passes through the two extremities, mark A_0 and construct a scale for verticals and bending moments such that $\overline{OA_0} = \frac{1}{4}R.l.$ Find by inspection the portions of the fields which the commanding weights cannot overtake, and over such portions replace the arcs by chords. Then the locus gives the maximum bending moment at each point for all possible positions of the load, the load being confined to the span.

To find the locus of the bending moment at each point of span for a particular point of the wheel base over it.—Let b be any point on the wheel base, or that base produced, and suppose the load standing as on fig. 56; let ordinates be drawn from the points where W_2 and W_3 (the weights on each side of b) stand to the corresponding parabolas 2 and 3, and let V and N the tops of those ordinates be joined by a straight line, which, as we have already seen, passes through D_2 the intersection of those parabolas. Let B be the point on that joining line on the same vertical with b, then the ordinate of B is the bending moment at the point over which b stands. Now by theorem, fig. 54, when the load moves, the joining line, or double chord of the two parabolas, turns about D_2; and since $\overline{vb}$ the horizontal projection of VB is constant, the point B traces out a parabola. This parabola passes through D_2, and is the same parabola as the others of the series; also by comparing figures 54 and 56, it can be shown that its apex A lies to the left or right of O the centre, according as b lies to the left or right of G the centre of gravity, and at a distance given by the equation

$$\overline{OS} = \tfrac{1}{2}\overline{Gb}.$$

Graphically.—Construct as much of the preceding solution as determines D_2, the junction of the parabolas corresponding to W_2 and W_3 the weights on the two sides of b. Lay off OS equal to half of $\overline{Gb}$, and draw a vertical through

S; shift the parabolic segment till its apex is on that vertical, move the rollers till the curved edge passes through D_2, and draw the parabola whose apex is A. If b lies between the two end weights W_1 and W_2, the curved edge passes through D_1; if b is on the wheel base produced to the left, the curved edge passes through the extremity B. If the beam extends beyond the supports, as on fig. 48, the props being fixed and able to act upwards and downwards; then for the load in any position on the beam whatsoever, the locus BD_1D_2, &c., of fig. 56 gives the positive maxima bending moments, and the negative ordinates have the interpretation given at fig. 46.

Examples.

96. A beam 42 feet span supports the five wheels of the locomotive shown on fig. 3; find the locus of the maximum bending moment at each point for all positions of the load, it being understood that no wheel moves off the span. Fig. 56.

Loads..............	5,	5,	11,	12,	9 tons	= 42 tons.
Intervals...........		5,	8,	10,	7	feet = 30 feet.
Distances from G,	17,	12,	4,	-6,	-13 feet.	

Dividing the span in the ratio of the weights, we have the extent of the fields as follows:—

1st field from 21 to 16; 2nd field from 16 to 11;
3rd field from 11 to 0; 4th field from 0 to -12;
5th field from -12 to -21.

Substituting in the general equation

$$_rM_x = \frac{R}{2c}(c+x-2h_r)(c-x) + 2h_r\Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W.2h),$$

we have

$$_1M_x = \frac{R}{2c}(c+x-2h_1)(c-x) = (4+x)(21-x),$$

for values of x from 21 to 16;

$$
{}_2M_x = \frac{R}{2c}(c+x-2h_2)(c-x)+2h_2.W_1 - W_1.2h_1
$$
$$
= (9+x)(21-x)-25,
$$

for values of x from 16 to 11;

$$
{}_3M_x = \frac{R}{2c}(c+x-2h_3)(c-x)+2h_3(W_1+W_2)-(W_1.2h_1+W_2.2h_2)
$$
$$
= (17+x)(21-x)-105,
$$

admissible for values of x from 8 to 0.

$$
{}_4M_x = \frac{R}{2c}(c+x-2h_4)(c-x)+2h_4(W_1+W_2+W_3) - (W_1.2h_1+W_2.2h_2+W_3.2h_3)
$$
$$
= (27+x)(21-x)-315,
$$

admissible for values of x from -2 to -12;

$$
{}_5M_x = \frac{R}{2c}(c+x-2h_5)(c-x)+2h_5(W_1+W_2+W_3+W_4) - (W_1.2h_1+W_2.2h_2+W_3.2h_3+W_4.2h_4)
$$
$$
= (34+x)(21-x)-546,
$$

for values of x from -12 to -21.

To find the ordinates of the apexes; substitute $x = h_1 = 8\cdot5$ in the first equation, $x = h_2 = 6$ in the second, and so on, and we have—

$$
{}_1y_{8\cdot5} = 156\cdot25, \quad {}_2y_6 = 200, \; {}_3M_2 = 256,
$$
$$
{}_4M_{-3} = 261, \quad {}_5y_{-6\cdot5} = 210\cdot25.
$$

Only two are expressed by the letter M, as they alone of the five are bending moments; the others do not lie in their own fields. Hence the maximum bending moment for the whole span is

$$
{}_4M_{-3} = 261 \text{ ft.-tons.}
$$

97. Find the equation to the maxima bending moments in the 3rd field directly, and without using the general formula.

Choose any point whose abscissa is x, such that when the load is placed with the 3rd weight over it, the whole load will be on the span. The distance of G the centre of gravity from the right end will be $(c+x-2h_3)$, or $(17+x)$, and

$$P = \frac{42}{42} \times (17+x) = (17+x).$$

Taking a section at x and considering the forces on the left side of it, we have

$$\begin{aligned} {}_3M_x &= P(21-x) - W_1(5+8) - W_2 \times 8 \\ &= (17+x)(21-x) - 105. \end{aligned}$$

98. Find the maximum bending moment at any given points, say $x = 8$, and $x = 10$, in No. 96; and find the maximum for the whole span without using the general equations. Fig. 56.

1°. To find the maximum bending moment at any given point, divide the span into fields proportional to the weights; consider which field the point lies in, and place the load so that the corresponding weight is over the point. Then if the whole load be on the span, the bending moment calculated at the point for the load fixed in this position is a maximum. If when the load is so placed, some weights are off the span, move the load the least distance which will bring them all on the span, and calculate the bending moment at the point for the load fixed in that position.

2°. To find the maximum bending moment for the whole span.—By inspection, place the load so that any particular weight and the centre of gravity may be upon different sides of the centre of span and at equal distances therefrom; then if the whole load be on the span and the weight be in its own field, calculate the bending moment at the point where the weight stands for the load fixed in this position, and it will be the maximum at the point. For each weight which can be placed so as to fulfil these conditions, there is a maximum, and having calculated these maxima, the greatest is the maximum for the whole span.

In No. 96, the point $x = 8$ lies in the 3rd field; place the load so that W_3 is over it, and it will be found that all the wheels are on the span; calculating the bending moment at $x = 8$ with the load in this position, we have

$${}_3M_8 = 220 \text{ ft.-tons maximum at point.}$$

The same result may be obtained by substituting $x = 8$ into ${}_3M_x$.

Again, the point 10 lies in field 3, but when W_3 is placed over it, W_1 is not on the span; moving the load two feet to the right brings W_1 just on the span, while W_3 is at 8; that is, the load is in the very same position as previously. Calculating the bending moment at $x = 10$, with the load in this position, we have—

$$M'_{10} = 190 \text{ ft.-tons max. at point.}$$

The same result may be obtained by substituting $x = 8$, and $x = 11$, into ${}_3M_x$, and then adding one-third of their difference to the smaller.

In No. 96, we find that by placing W_3 at the point $x = 2$, it is in its own field while G is at the point $x = -2$, and the whole load is on the span; calculating the bending moment at 2 for the load in that position we have ${}_3M_2 = 256$. Again, by placing W_4 at the point $x = -3$, it is in its own field while G is at $x = 3$, and the whole load is on the span; calculating the bending moment at -3 for the load in that position, we have (see example at fig. 17) ${}_4M_{-3} = 261$. Now since no other weight can be placed according to the rule, it follows that ${}_4M_{-3} = 261$ ft.-tons. is the maximum for span not only for the position shown on fig. 17, but for all possible positions of the load on the span.

99. A beam 76 feet span is subjected to a moving load $R = 19$ tons confined to the span, and concentrated on three wheels as under:—

Loads.................	7	4	8	tons = 19 tons.
Intervals..............		30	12	feet = 42 feet.
Distances from G...	24	-6	-18	feet.

The 1st field extends from 38 to 10; 2nd field from 10 to -6; 3rd field from -6 to -38.

Substituting in the general equation, we have

$$ {}_rM_x = \tfrac{1}{4}(38 + x - 2h_r)(c - x) + 2h_r\Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W.2h). $$

$$ {}_1M_x = \tfrac{1}{4}(14 + x)(38 - x), \text{ for values of } x \text{ from 38 to 10.} $$

$$ {}_1M_{h_1} = {}_1M_{12} = 169, \text{ max. for field 1.} $$

$$ {}_2M_x = \tfrac{1}{4}(44 + x)(38 - x) - 210, \text{ for values of } x \text{ from 8 to} $$

-6; the chord to be taken from 8 to 10.

$$ {}_2M_{h_2} = {}_2M_{-3} = 210\tfrac{1}{4}, \text{ max. for field 2.} $$

$$ {}_3M_x = \tfrac{1}{4}(56 + x)(38 - x) - 342, \text{for values of } x \text{ from } -6 \text{ to } -38. $$

$$ {}_3M_{h_3} = {}_3M_{-9} = 210\tfrac{1}{4}, \text{ max. for field 3.} $$

The maximum for whole span is equally ${}_2M_{-3}$ or ${}_3M_{-9}$.

100. For example No. 96, find the equation to the bending moment at each point of span as b, a point of the load 8 feet to the left of G, comes over it. Fig. 56.

To find D_2 the point of intersection on the junction of the fields commanded by W_2 and W_3 the weights on either side of b, suppose the load fixed with either of these weights over the point F_2, where $OF_2 = 11$, and calculate the bending moment there; or, substitute $x = 11$ into ${}_2M_x$ or ${}_3M_x$ of No. 96, and we find the height of D_2 to be 175; hence the co-ordinates of D_2 are ;—$x = 11$, $y = 175$. The locus for the point b is the parabola whose apex A is on the vertical through S, where $OS = h = \frac{1}{2}Gb = 4$; this parabola passes through D_2, has its axis vertical, and its modulus is the load divided by the span; hence

$$ {}_bM_x = \frac{R}{2c}(c + x - 2h)(c - x) + C = (13 + x)(21 - x) + C; $$

since it passes through D_2, we have ${}_bM_{11} = 175$.

$$ \therefore \qquad 175 = (13 + 11)(21 - 11) + C, \text{ or } C = -65; $$

and $\qquad {}_bM_x = (13 + x)(21 - x) - 65,$

for values of x from 12 to 0, since part of the load goes off the span when b occupies any position not comprised between these two limits.

101. In example 89, find the locus of the bending moment at each point, as G the centre of gravity of the load comes over it. Fig. 51.

Put $x = 4$ in either equation, and we find the height of D to be 36; so that the co-ordinates of D are $x = 4$, $y = 36$. The parabola for G has its apex on the vertical through the centre O; its axis is vertical and it passes through D; the modulus is the load divided by the span; hence

$$_gM_x = \frac{R}{2c}(c+x-0)(c-x)+C = \tfrac{1}{8}(400-x^2)+C;$$

but $\quad _gM_4 = 36$, therefore

$$36 = \tfrac{1}{8}(400-16)+C, \text{ or } C = -12;$$

$\therefore \quad _gM_x = \tfrac{1}{8}(400-x^2)-12,$

for values of x from 14 to -16, that is for the whole load confined to the span.

If the load makes a transit, the above is still the locus for the central portion. For the left end portion, W_2 alone is on the span as G comes over each point; and for that portion the locus is the parabola for W_2 rolling; the apex is on the vertical through the point $x = 2$, since G is no longer the centre of gravity, but only a point lying 4 feet to left of W_2 which is its own centre of gravity; the parabola passes through the left end, and its equation is

$_gM'_x = \frac{3}{40}(20+x-4)(20-x)$, for values of x from 20 to 14.

Similarly $_gM''_x = \frac{2}{40}(20-x-6)(20+x)$, for values of x from -16 to -20.

Beam under both a rolling load and an uniform dead load.—Let R be the dead load equivalent to the actual rolling load (see page 102), and U the uniform load. For each load separately the bending moment diagram is a parabola with its apex over the centre. For the combined load, the diagram is a parabola whose apex is also over the centre, and whose modulus is the sum of their moduli; hence

$$M_x = \left(\frac{U}{4c} + \frac{R}{2c}\right)(c^2-x^2) = \frac{U+2R}{4c}(c^2-x^2), \text{ maxima};\ldots(1)$$

and $M_0 = \frac{c}{4}(U+2R) = \frac{1}{8}(U+2R).l$, max. of maxima,...(2)

the bending moments being in terms of a *dead* load throughout.

Graphical Solution.—With any parabolic segment draw a parabola with its apex over the centre and passing through the two ends, and make a scale for verticals upon which the middle ordinate $OA = \frac{1}{8}(U+2R).l$.

Beam under a travelling load concentrated on two wheels, together with an uniform dead load.—This divides into the same cases as for the travelling load alone, figs. 51 and 52.

The locus BDC still has D on the vertical through F, but the modulus of the curves is the sum of the moduli for the loads separately, that is, for the uniform load and the rolling load; the abscissa of each apex is the same fraction of the abscissa of the corresponding apex for the rolling load alone, OS_1 and OS_2, figs. 51 and 52, that the modulus is of the sum of the moduli; see theorem, fig. 53.

$$\text{Sum of moduli} = \frac{U+2R}{4c},$$

and the fraction mentioned above is

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U+2R};$$

hence the equations to the maxima bending moments are

$${}_1M_x = \frac{U+2R}{4c}\left(c+x-\frac{2R}{U+2R}.2h_1\right)(c-x) \text{ for first field;...(1)}$$

$${}_2M_x = \frac{U+2R}{4c}\left(c-x+\frac{2R}{U+2R}.2h_2\right)(c+x) \text{ for second field..(2)}$$

Such portions of the fields as were bounded by the chord instead of the arc are still so bounded. In the case where for the transit kEh superseded BDC, fig. 52, the same central portion is now bounded by the locus

$$M'_x = \frac{U+2W_2}{4c}(c^2-x^2)\text{......................(3)}$$

Graphical Solution.—For an uniform load U combined with any of the cases, figs. 45, 46, 47, 51, and 52, the graphical solution is the same in all particulars, case for case, except that OS_1 and OS_2 are now to be laid off a $\frac{2R}{U+2R}$th part of what they were laid off for the travelling load alone; the vertical scale is to be constructed so that

$$OA_0 = \tfrac{1}{8}(U+2R).l,$$

where R, as previously stated, is the dead load equivalent to the actual rolling load.

Beam under a travelling load system confined to the span, together with an uniform dead load.—In the bending moment diagram, the locus consists of the same number of parabolas as for the travelling load alone, and the fields are the same; any portion of a field bounded by the chord instead of the arc is still bounded by the chord of the new parabola. The modulus of each parabola is the sum of the moduli for R the dead load equivalent to the actual travelling load, and for the uniform load, namely,

$$\frac{R}{2c} + \frac{U}{4c} = \frac{U+2R}{4c}.$$

The distance OS of any apex from the centre of the span is now only a fraction of what it was for the travelling load alone; the fraction being the modulus for R rolling divided by the sum of the moduli, that is

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U+2R}. \quad \text{(Theorems at fig. 53.)}$$

For the travelling load alone, we had as the equation to the maxima bending moments for the r^{th} field,—

$${}_rM_x = \frac{R}{2c}(c+x-2h_r)(c-x) + 2h_r\Sigma_1^{r-1}(W) - \Sigma_1^{r-1}(W.2h);$$

altering the modulus and putting in the fraction, we have, for the combined load for the r^{th} field,

$$ {}_rM_x = \frac{U+2R}{4c}\left(c+x - \frac{2R}{U+2R}\cdot 2h_r\right)(c-x) + 2h_r\Sigma_1^{r-1}(W) - \Sigma^{r-1}(W.2h)\text{.......}(1) $$

Graphical Solution.—Exactly the same as that for the travelling load alone, fig. 56, excepting that

$$ OS_1 = \frac{2R}{U+2R}h_1, \qquad OS_2 = \frac{2R}{U+2R}h_2, \text{ \&c.,} $$

and the scale for verticals is such that $OA_0 = \frac{1}{8}(U+2R).l.$

Examples.

102. Suppose the weights in example 96 and shown at fig. 56, to be the equivalent dead loads for the actual weights; combine with these an uniform load of $\frac{2}{3}^{rds}$ of a ton per foot of span, that is, a total load $U = 28$ tons, and find the equations to the maxima bending moments for the various fields.

Summing the moduli for R rolling and for U spread uniformly, we have

$$ \frac{R}{2c} + \frac{U}{4c} = \frac{U+2R}{4c} = \frac{28+84}{84} = \frac{4}{3}, $$

the modulus of the parabolas for combined load.

The modulus for R rolling, divided by the sum of the moduli, is—

$$ \left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U+2R} = \frac{84}{28+84} = \frac{3}{4}. $$

Substituting into the general equation,

$$ {}_1M_x = \tfrac{4}{3}(21 + x - \tfrac{3}{4} \times 17)(21-x) = \tfrac{4}{3}(8{\cdot}25 + x)(21-x), \text{ for} $$

values of x from 21 to 16.

$$_2M_x = \tfrac{4}{3}(21 + x - \tfrac{3}{4} \times 12)(21 - x) + 12 \times 5 - 5 \times 17$$

$$= \tfrac{4}{3}(12 + x)(21 - x) - 25, \text{ for values of } x \text{ from 16 to 11.}$$

$$_3M_x = \tfrac{4}{3}(21 + x - \tfrac{3}{4} \times 4)(21 - x) + 4(5 + 5) - (5 \times 17 + 5 \times 12)$$

$$= \tfrac{4}{3}(18 + x)(21 - x) - 105, \text{ for values of } x \text{ from 8 to 0,}$$

the chord being taken for values from 11 to 8.

$$_4M_x = \tfrac{4}{3}(21 + x + \tfrac{3}{4} \times 6)(21 - x) - 6(5 + 5 + 11) - (5 \times 17 + 5 \times 12 + 11 \times 4)$$

$$= \tfrac{4}{3}(25{\cdot}5 + x)(21 - x) - 315, \text{ for values of } x \text{ from } -2$$

to -12, the chord being taken for values from 0 to -2.

$$_5M_x = \tfrac{4}{3}(21 + x + \tfrac{3}{4} \times 13)(21 - x) - 13(5 + 5 + 11 + 12) - (5 \times 17 + 5 \times 12 + 11 \times 4 - 12 \times 6)$$

$$= \tfrac{4}{3}(30{\cdot}75 + x)(21 - x) - 546, \text{ for values of } x \text{ from } -12$$

to -21.

The abscissæ of the apexes are—

$\tfrac{3}{4} \times 8{\cdot}5 = 6{\cdot}375$; $\tfrac{3}{4} \times 6 = 4{\cdot}5$; $\tfrac{3}{4} \times 2 = 1{\cdot}5$, lies in field 3;

$\tfrac{3}{4} \times (-3) = -2{\cdot}25$, lies in field 4; $\tfrac{3}{4} \times (-6{\cdot}5) = -4{\cdot}875$.

Here only two apexes lie in the corresponding fields; substituting for these, we have

$_3M_{1\cdot5} = 402$ max. in field 3; and

$_4M_{-2\cdot25} = 405{\cdot}75$ max. in field 4; and max. for span;

that is, for the load confined to the span, the greatest bending moment occurs at $2\tfrac{1}{4}$ ft. to the right of the centre, when the fourth weight is over it.

103. Suppose the weights in example 99 to be the equivalent dead loads for the actual weights, and combine an uniform load of $\tfrac{1}{2}$ a ton per foot of span with the travelling load. Find the maximum bending moment at each point of span, for the load confined to the span.

The sum of the moduli of the parabolas for U and for R rolling is

$$\frac{U+2R}{4c} = \frac{38+38}{152} = \tfrac{1}{2}, \text{ the modulus for the combined load.}$$

The modulus of the parabola for R, divided by the sum of the moduli, is

$$\left(\frac{R}{2c}\right) \div \left(\frac{R}{2c} + \frac{U}{4c}\right) = \frac{2R}{U+2R} = \frac{38}{38+38} = \tfrac{1}{2}.$$

In the general equation given for No. 99 replace $\frac{1}{4}$ by $\frac{1}{2}$, and in the trinomial factor replace $2h_r$ by h_r; otherwise the expression remains the same; thus

$${}_1M_x = \tfrac{1}{2}(38+x-12)(38-x) = \tfrac{1}{2}(26+x)(38-x),$$

for values of x from 38 to 10.

${}_1y_6 = \frac{1}{2}(32)^2 = 512$, height of apex of parabola 1, not in field 1.

${}_2M_x = \frac{1}{2}(41+x)(38-x)-210$, for values of x from 8 to -6; the chord to be taken from 8 to 10.

$${}_2M_{-1\cdot5} = \tfrac{1}{2}(39\cdot5)^2-210 = 570\tfrac{1}{8}, \text{ max. in field 2.}$$

${}_3M_x = \frac{1}{2}(47+x)(38-x)-342$, for values of x from -6 to -38.

${}_3y_{-4\cdot5} = \frac{1}{2}(42\cdot5)^2-342 = 561\frac{1}{8}$, height of apex of parabola 3, not in field 3.

Hence the max. for whole span is $570\frac{1}{8}$ ft.-tons, and it occurs at $1\frac{1}{2}$ feet to the right of the centre when the second weight is over it.

104. Supposing the weights in example 81 to be the equivalent dead loads for the actual weights, and taking into account the weight of the beam which is 5 tons uniformly distributed; find the general equation to the bending moments, and find the maximum bending moment for whole span.

Ans. ${}_1M_x = \frac{5}{8}(10+x-\frac{4}{5}\times 4)(10-x) = \frac{5}{8}(6\cdot8+x)(10-x).$

The maximum for whole span occurs at $\frac{4}{5}$ths of 2 feet on either side of the centre, and is

$$M_{1\cdot 6} = \tfrac{5}{8}(8\cdot 4)^2 = 44\cdot 1 \text{ ft.-tons.}$$

105. In example 91, we had a beam of 40 ft. span bearing a travelling load $W_1 = 2$, and $W_2 = 3$ tons fixed 20 feet apart. For the transit of the load, we found a maximum of 30 ft.-tons at the centre, and a maximum of 32 ft.-tons, four feet to the right of the centre. Supposing these weights to be the equivalent dead loads for the actual weights, and taking the weight of the beam, six tons uniformly distributed, into account, make the appropriate alteration upon the equations given at example 91.

For the first and third equations, $R = 5$, and $U = 6$ tons; hence the new modulus is

$$\frac{U+2R}{4c} = \frac{6+10}{80} = \frac{1}{5};$$

and the fraction $\quad \dfrac{2R}{U+2R} = \dfrac{10}{6+10} = \dfrac{5}{8}.$

For centre equation $W_2 = 3$, and $U = 6$ tons; hence the new modulus is

$$\frac{U+2W_2}{4c} = \frac{6+6}{80} = \frac{3}{20}.$$

Ans. $_1M_x = \frac{1}{5}(20+x-7\cdot 5)(20-x)$, for values of x from 20 to 10.

$_2M'_x = \frac{3}{20}(400-x^2)$, for values of x from 10 to 0.

$_2M_x = \frac{1}{5}(20-x-5)(20+x)$, for values of x from 0 to -20.

$_1y_{3\cdot 75} = \frac{1}{5}(16\cdot 25)^2 = 52\cdot 81$, height of apex, not in field.

$_2M'_0 = 60$, a maximum, at centre.

$_2M_{-2\cdot 5} = \frac{1}{5}(17\cdot 5)^2 = 61\cdot 25$, maximum in 2nd field, and for span.

In this example, where the distance between the wheels exactly equals the half-span, E coincides with h, fig. 52; hence A_2 will always be higher than E though an uniform load indefinitely great be compounded with it. When, however, the distance between the wheels is greater than the half-span, the diagram is as shown upon fig. 52; and if E be higher than A_2, much more will it be higher when we add the ordinates of the parabola due to any uniform load; if E be lower than A_2, there is some uniform load which being added will make them equal in height; an uniform load greater than this being added will cause E to be higher than A_2.

Beam under a travelling load system confined to the span, together with fixed loads at points.

Graphical Solution.—Construct the diagram for moving load, as on fig. 56, and above it construct the polygon for the fixed loads, as on fig. 39, to the same scales. Draw a vertical through the first apex A_1 up through the polygon, and produce those sides of the polygon that lie wholly or in part over field 1, to make intercepts on that vertical as on fig. 39; make LS equal to the semi-base, and LN equal to twice the height of the apex of the parabolic segment whose apex is A_1; and otherwise proceed exactly as on fig. 39, till the locus for the combined load over field 1 is completed; next draw a vertical through A_2 up through the polygon, and produce such sides of the polygon as lie wholly or in part over field 2 to make intercepts on that vertical; complete the construction of the locus for the combined load over field 2 exactly as on fig. 39, except that LS is now to be the semi-base, and LN the height of the apex of the parabolic segment whose apex is A_2. In like manner draw for each field till the locus for the combined load is completed for the whole span.

One particular case is of interest.—BA_0C, fig. 56, is the diagram for a rolling load R. Suppose combined with this load a system of fixed loads, one at each junction between two fields—on fig. 56, four loads, one at each of the points F_1, F_2, F_3, F_4;—then to obtain the diagram for the combined load, we add to the ordinates of the parabola BA_0C those of the polygon for the fixed loads; and, as on fig. 39, the result will be five parabolas like BA_0C, and the apex of each will move from the centre *away from* the corresponding point F.

Suppose, on the other hand, that the fixed weights at F_1, F_2, &c., act upwards; then subtracting the ordinates of the polygon from those of BA_0C, the result will be five parabolas like BA_0C, but each apex will now have moved from the centre *towards* the corresponding point F.

For a certain set of values of these negative forces at F_1, F_2, &c., the five parabolas will assume the exact position of the parabolas on fig. 56, and it can easily be shown, that the set of weights which will produce this effect is such that the weight at each junction is the same fraction of R, that the distance between the weights commanding the fields of which the point is the junction is of the span. Hence we have the following

ALTERNATIVE METHOD OF DRAWING A BENDING MOMENT DIAGRAM for a travelling load system confined to the span:— On a base equal to the span, construct a parabolic segment for the bending moment diagram due to the total load R as a *rolling load;* upon the same base, and on the same side of it, construct the polygon which is the bending moment diagram due to a set of fixed loads $W_{1\cdot 2}$, $W_{2\cdot 3}$, &c., one at the junction of each pair of fields (the suffixes denoting the pair of fields between which the load is applied), and of magnitudes given by the following equations:—

$$W_{1\cdot 2} = \frac{2h_1 - 2h_2}{l} R; \quad W_{2\cdot 3} = \frac{2h_2 - 2h_3}{l} R; \quad \&c.$$

Then the ordinate between the parabola and the polygon is the maximum bending moment at any point; the maximum for the whole span is found by drawing a tangent to the parabola parallel to that side of the polygon which slopes least, and taking the ordinate at the point of contact.

The following are objections to this alternative solution:— There are *two* irregular boundaries, and therefore it fails to suggest the proper form for practical designs; it fails to give all the maxima; it fails to give the maxima at the parts of the fields which the commanding weights cannot overtake; and it fails to show what portion of the span is commanded by any weight,—that is what position the load should occupy in order to produce the maximum at any point.

Beam under a travelling load system confined to the span, together with an uniform load, together with loads at fixed points.

Graphical Solution.—Draw the diagram for the first two combined, and construct the vertical scale as already described; above it draw the polygon which is the diagram for the fixed loads to the same scales. From apex A_1 draw a vertical up through the polygon, and produce such sides as lie over field 1 to make intercepts on it; proceed as on fig. 39, but having LS equal to the semi-base, and LN equal to twice the height of the apex of the parabolic segment whose apex is A_1, and complete the locus for field 1; proceed similarly for apex A_2 and field 2, for apex A_3 and field 3, &c., and so draw the locus field after field till the whole span is completed. The height of the highest apex gives the maximum.

Only one parabolic segment, cut on wood or cardboard, is required for the complete construction.

Beam subject to the transit of a travelling load system.

Graphical Solution.—We already solved this problem for the transit of two weights, and saw that it then had two cases. The general problem will have many cases, and we shall illustrate the solution by solving a particular problem, viz., for the data shown on fig. 56, where the solution for the load confined to the span is already given. Suppose the load to come on at the left end and pass off at the right. First, the weight W_5 alone is on the span; then W_5 and W_4; &c.; then the whole load; then W_5 passes off; and so on, till W_1 alone is on; this finally passes off and a transit has been made. Figures 58, 59, 60, 61 give the maxima at each point, while one only, two only, three only, four weights only are on the span; fig. 56 gives the maximum at each point, while all five are on; and figures 62, 63, 64, 65 give the maximum after one, two, three, and four weights respectively have passed off.

The manner of constructing those figures is as follows:—Prepare a number of parabolic segments (fig. 57), one corresponding to each of the manners of loading, the highest one being drawn with the parabolic segment that was used in the construction of fig. 56; divide its height so that the heights of the various apexes will be in the ratio of the loads, viz.,—

9; 21; 32; 37; 42; 33; 21; 10; 5.

Draw a number of ordinates to the curve; divide these similarly with proportional compasses, and draw curves

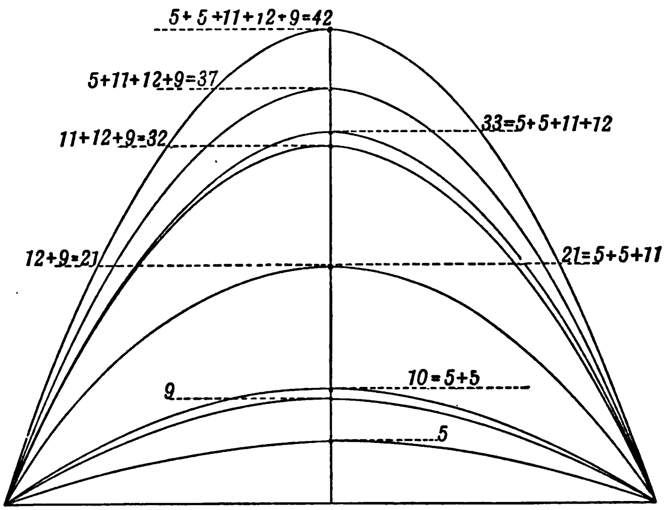


Fig. 57.

through the points thus found. Separate pieces of cardboard should then be cut out so as to correspond with each of these curves.

A description of the method for constructing fig. 60 will suffice for all. This figure is for the weights 11, 12, 9 tons on the span; and since the total load is 32 tons, select the cardboard segment whose height is proportional to 32; this ensures that the verticals are drawn to the same scale as those on fig. 56. Lay off BC equal to the span on the same horizontal scale as that used in fig. 56. Draw with a dotted line the locus BD_1D_2C, as for the load confined to the span and capable of moving into all positions on the span. This, however, is not the locus required, since it is evident that the range of motion of the load is limited by the next weight, 5 tons, not being allowed to come on to the span. Hence, place the load as far on the span as possible, but so that the next weight, 5 tons, may not be on the span; that is, place it so that the next weight is over B the extremity;

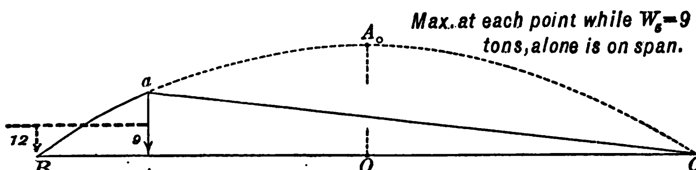


Fig. 58.

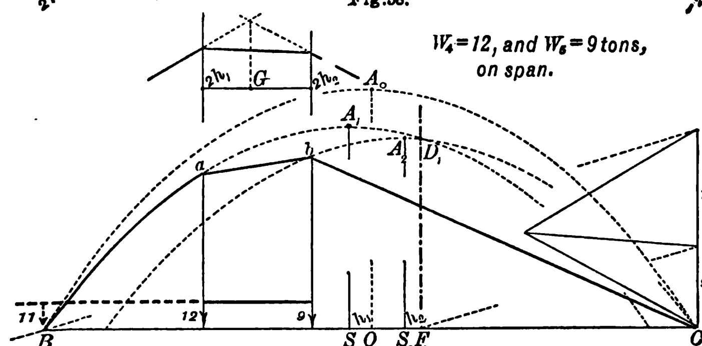


Fig. 59.

from each weight draw an ordinate to the corresponding parabola, for instance, from the weight 11 tons to the parabola whose apex is A_1. Ink in the curves for the portion of each field which has been travelled over by the weight commanding; the remainder of the locus is made by drawing the polygon whose angles are the tops of the ordinates where the weights stand. Thus, on fig. 60, the locus is the arc *Ba* of the first parabola since the weight 11 tons has travelled over that part of its field; and the arc D_1b of the second parabola since the weight 12 tons has travelled over that part of its field; the remainder of the locus consists of parts of the polygon *BabdC*, that is the polygon for the load fixed in the position shown on figure; it may be observed

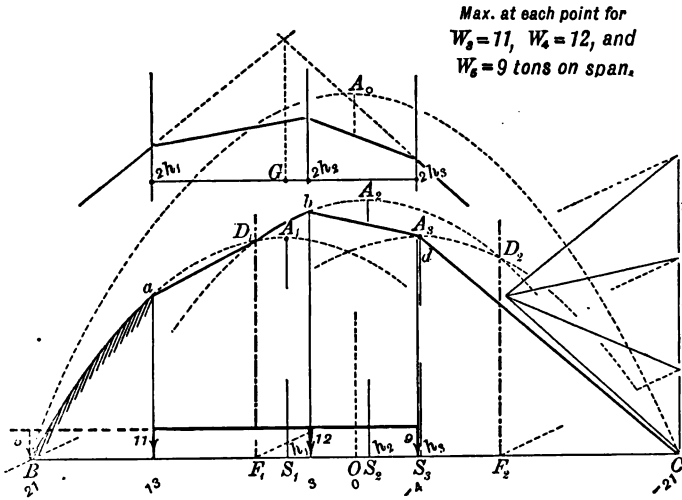


Fig. 60.

that aD_1 is a part of *ab* (theorem, fig. 54). Hence the locus, fig. 60, is arc *Ba*, chord aD_1, arc D_1b, and chords *bd*, *dC*.

It is evident that the arcs named give the maxima for the portions over which they stand for all the possible positions of the three weights on the span; and, that the chord aD_1 gives maxima for the portion of span over which it stands will be seen as follows:—The load cannot shift to the right; suppose it shifts to the left, then the new position of *ab* will still pass through D_1 (theorem, fig. 54), but the point *a* will be lower down on the first parabola; that is, as the load moves to the left, the chord D_1a turns about D_1 in the opposite direction from the hands of a watch, so that the ordinate of every point in it is decreasing. Again, *bd* produced passes through the point D_2; and if the load moves, the new position of bd still passes through D_2; that is, as the load moves to the left, the chord bd turns about D_2 in the opposite direction from the hands of a watch, so that the ordinate of every point in it is decreasing. Similarly dC turns about C in the same direction as the others, and the ordinate of every point in it is also decreasing.

If the seven figures, for the complete transit of the load, be superimposed upon figure 56 for the load confined to the

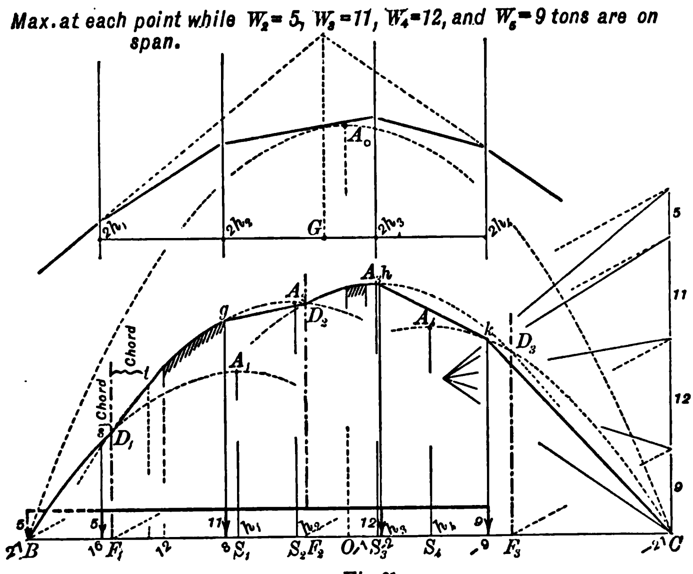


Fig. 61.

span, the portions which are shaded on figs. 60 and 61 will lie *outside* of the locus on figure 56; so that, replacing the corresponding portions of fig. 56 by those shaded parts, that is, everywhere following the highest locus, we have a graphical solution for the transit of the load. The labour of this graphical solution is great if we have to prepare the parabolic segments, but the solution can be drawn by means of *one* parabolic segment from the following considerations.

Suppose in the example that all the eight diagrams were drawn with the *same* parabolic segment, then it is evident that each would be to a *different* vertical scale, and that nothing could be determined by superposition. However, it can be seen by inspection that fig. 62 lies entirely within fig. 56: for, since the weight 9 tons on fig. 62 is over the point of support B, we may consider that it is not on the span as we did when constructing fig. 62; on the other hand, if we consider it to be on the span, then the total load is on the span; and for that position of the total load

Max. at each point for $W_1 = 5$, $W_2 = 5$, $W_3 = 11$, and $W_4 = 12$ tons on span.

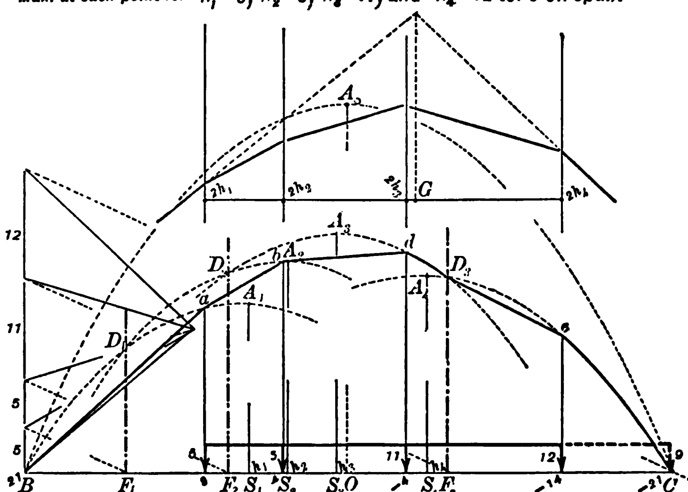


Fig.62.

we may plot the point a upon fig. 56 by drawing an ordinate to the first parabola from the point whose abscissa is 9, that is, from the point where the weight 5 tons stands on fig. 62. Similarly the point b is plotted on fig. 56 by drawing an ordinate to the second parabola from the point whose abscissa is 4, that is, from the point where the second weight, 5 tons, stands on fig. 62; likewise the points d and e are plotted. If the one figure be drawn exactly above the other, it will only be necessary to draw verticals from the points a, b, d, e, fig. 62, to meet the 1st, 2nd, 3rd, 4th parabolas respectively on fig. 56, and we have these points plotted on fig. 56. It will then be seen that these points are inside the locus on fig. 56; so that the whole locus, fig. 62, is inside that of fig. 56. Similarly, by drawing verticals

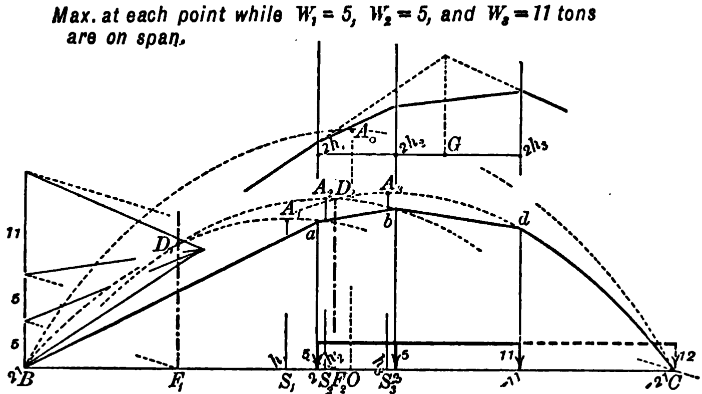


Fig. 63.

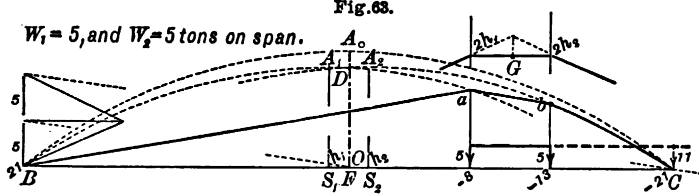


Fig. 64.

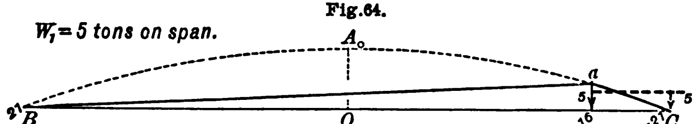


Fig. 65.

from the points a, b, d, fig. 63, to meet respectively the 1st, 2nd, 3rd parabolas on fig. 62, these points are plotted on fig. 62 and will be found to lie inside of it. Hence the locus fig. 63 lies inside the locus fig. 62, and much more will it lie inside of the locus, fig. 56. In this way it will be seen that the loci figs. 65, 64, 63, 62, 56 lie each within the other in order, and therefore that all lie within the locus fig 56

In the same way it can be shown that the loci only on figs. 61 and 60 lie partly outside of the locus fig. 56. On fig. 61 are shown the points *s*, *g*, *h*, *k* transferred from fig. 56 in the manner described. It is evident that *g* and *h* are the extremities of the chords that were parts of the locus for the load confined to the span; because, the reason we followed the chord at D_2, fig. 56, was that the commanding weight W_3 could not overtake that part; in other words, when W_3 came to *g*, then W_1 was over *B* and about to go off the span; that is, the load was in the position shown on fig. 61.

It will therefore always be the case that when, for the load confined to the span, we have chords for portions of the locus, then the diagram for one load off the span lies in part outside the locus; and the ends of such chords which are not junctions of fields are points of intersection.

Having ascertained by inspection the shaded parts of figs. 60 and 61 which alone lie outside of fig. 56, it will be necessary to plot points at intervals upon fig. 56 with the proportional compasses; the points thus got from fig. 61 have their ordinates reduced in the ratio of 42 and 37, those from fig. 60 in the ratio of 42 and 32, since the diagrams here spoken of are supposed to be drawn with the same parabolic segment, and therefore to different vertical scales.

We will complete the analytical solution for the transit of the load in this example. To find the equation to *Ba*, fig. 60; suppose the weight 11 tons to be under this arc and at any point whose abscissa is x, then it will be found by taking moments round *Q*, that

$$P = \frac{399 + 32x}{42}.$$

$$M_x = P(c - x) = \tfrac{1}{42}(399 + 32x)(21 - x),$$

for values of x from 21 to 13.

For the arc *tg*, fig. 61; let the weight 11 tons be under this arc at any point whose abscissa is x, then

$$P = \frac{544 + 37x}{42}.$$

$$M_x = P(c-x) - 5 \times 8 = \tfrac{1}{42}(544 + 37x)(21 - x) - 40,$$

for values of x from 12 to 8.

For the arc D_2h; let the weight 12 tons be under this arc, and at any point whose abscissa is x, then

$$P = \frac{914 + 37x}{42}.$$

$$M_x = P(c-x) - 5 \times 18 - 11 \times 10$$
$$= \tfrac{1}{42}(914 + 37x)(21 - x) - 200,$$

for values of x from 0 to -1.

The limits in each case are found by making the equation to the arc simultaneous with that to the arc on fig. 56, which is now superseded; the limits as given above are to the nearest whole number.

During the transit, it is to be observed that in this case the weight $W_3 = 11$ tons commands exactly the left half of the span, while the right half is commanded by $W_4 = 12$, and $W_5 = 9$ tons, just as on fig. 56 for the load confined to the span.

Calculating the ordinate at each foot of span from the equation just given for the values of x there specified, and from the equations already given at fig. 56 for other values of x, we find the maximum bending moment at each foot of span during the transit to be:—

abs.	B.M.	abs.	B.M.	abs.	B.M.	abs.	B.M.	abs.	B.M.	abs.	B.M.
20 ...	24·7	13 ...	155·3	6 ...	240	−1 ...	259·4	− 8 ...	236	−15 ...	138
19 ...	48·0	12 ...	171·7	5 ...	247	−2 ...	260	− 9 ...	225	−16 ...	120
18 ...	69·7	11 ...	186·5	4 ...	252	−3 ...	261	−10 ...	212	−17 ...	100
17 ...	89·8	10 ...	199·4	3 ...	255	−4 ...	260	−11 ...	197	−18 ...	78
16 ...	108·5	9 ...	210·6	2 ...	256	−5 ...	257	−12 ...	180	−19 ...	54
15 ...	125·6	8 ...	220·0	1 ...	255	−6 ...	252	−13 ...	168	−20 ...	28
14 ...	141·2	7 ...	231·0	0 ...	257	−7 ...	245	−14 ...	154		

The max. for the whole span is still $M_{-3} = 261$ ft.-tons.

Beam under a travelling load system of equal weights fixed at equal intervals and confined to the span.—Figs. 66, 67. The locus $BD_1D_2D_3C$, drawn as in the general case, will be symmetrical about the centre. If the number of weights be n, and their distances apart be $\frac{1}{n}$th of the span, then it is evident that the span will be divided into n equal fields whose common extent is the same as the distance between two weights, and that each weight will just be able to overtake its field; if the distance between two weights be less than $\frac{1}{n}$th of the span, each weight is still able to overtake its own field. Therefore, for the common distance between the n weights equal to or less than $\frac{1}{n}$th of the span, the bending moment diagram is the locus $BD_1D_2D_3 \ldots C$ everywhere following the curves; the maximum bending moment is at the centre, or at one quarter of the common interval on either side of the centre, according as n is odd or even. On the other hand, when the common interval between the n weights is greater than $\frac{1}{n}$th of the span, each weight will always be in its own field, and

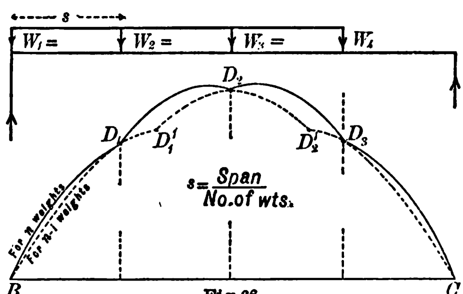


Fig. 66.

will only be able to overtake a portion of its field; the bending moment diagram is the locus $BD_1D_2D_3 \ldots C$ following the arcs for portions of fields overtaken by the weights commanding, and the chords for the remainder. If n be odd, the middle weight can always be placed at the centre of the span, and at that point the maximum bending moment will occur; if n be even, and it be possible for a weight to come as close to the centre as or closer than a quarter of the common interval, then the maximum bending moment will be on both sides of the centre, and at one quarter of the common interval therefrom; if a weight cannot come so close, the maximum will still be on both sides of the centre, and at points as near thereto as the weight on either side of it may approach.

For n even, the proof that the maximum is at the point, a quarter of the common interval on either side of the centre may be shown thus:—Let G be over the centre, then the weight nearest to the centre will be distant one-half of an interval; if the end weight be distant from the end at least a quarter of an interval, it will be possible for a weight to approach the centre, a quarter interval; hence, the span must equal the $\overline{n-1}$ equal intervals, and at least two quarter intervals more, that is

$$2c \overline{\gtrless} (n-1)s + \tfrac{1}{2}s;$$

$$\overline{\gtrless} \frac{2n-1}{2}s.$$

Theorem.—If a system of n equal weights at a common interval not greater than $\frac{1}{n}$th of the span be confined to the span, then the locus of the maximum bending moment at each point will entirely include the loci due to any smaller number of the same equal weights at the same common interval.

First. Let the common interval equal $\frac{1}{n}$th of the span. Place the load, fig. 66, with one weight over the support, then the other weights are exactly over the $n-1$ junctions of the fields. We may either consider that the n weights are on the span, or that $n-1$ weights only are on the span; hence the two loci, the one for n weights and the other for $n-1$, have common ordinates at the points where the weights are; that is, the locus for $n-1$ weights passes through the points D_1, D_2, D_3, &c., of, and lies entirely within, the locus for n weights.

On the figure, the locus drawn with full lines is that due to n weights, while the locus drawn with dotted lines is that due to $n-1$ weights.

Second. Let the common interval decrease by 2δ. In the new locus for the n weights, each pair of consecutive apexes will be closer by the amount δ, so that every point in the locus will be higher than before; in the new locus for the $n-1$ weights, any two consecutive apexes will be closer by δ, so that each point in the locus will be higher than before. In the locus for the n weights, the parabolas have a greater modulus than those in the locus for the $n-1$ weights, so that the increase of height due to the approach of the apexes by δ in the one case, will be greater than that due to the approach of the apexes by the amount δ in the other; that is, if the distance between the weights be decreased, every point in the locus drawn with full lines, fig. 66, will rise more than the corresponding point in the dotted locus, so that much more will the new dotted locus be entirely within the other.

It follows, then, that if the common interval between n equal weights be equal to or less than $\frac{1}{n}$th of the span, or, in other words, if each of the n weights can overtake its field, then the loci due to the n weights, to $n-1$ of the same weights at the same common interval, to $n-2$, &c.,

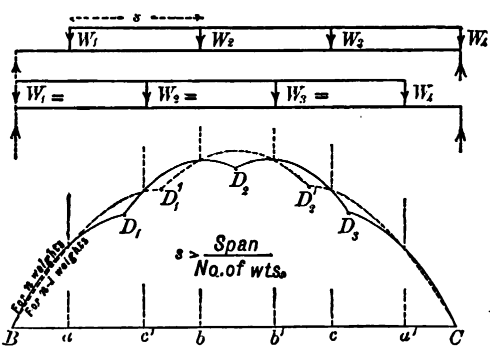


Fig. 67.

are, in order, each entirely within the one preceding it, and therefore all the others entirely within the first.

Corollary. If the interval between the n weights be greater than $\frac{1}{n}$th of the span, the locus due to these weights confined to the span will intersect that due to $n-1$ of the same equal weights, also confined to the span; for, reasoning as before, when the common interval is increased, the locus drawn with full lines on fig. 66 is lowered more than the other. From fig. 67, it will be evident that the points of intersection a, b, c, a', b', c' are found by laying off the common intervals from each end of the span, and that these points are the junctions between the alternate arcs and chords on the diagram for the n loads confined to span. In order that the n weights may be all on the span at one time, the common interval must not be greater than $\frac{1}{n-1}$th of the span; so that, for the case of $n-1$ of these weights, each could overtake its field; and the loci for $n-2$ weights, $n-3$, &c., are all within the locus for $n-1$ weights shown by dotted lines on fig. 67.

Beam subject to the transit of a system of equal weights at equal intervals.

I. *Load shorter than span.*

Let n be the number of weights, and s their common distance apart.

(a) Let the distance between the weights be less than or equal to $\frac{1}{n}$th part of the span, that is let $s \leqq \frac{2c}{n}$.

With any parabolic segment draw the series of n parabolas, as shown with the full lines fig. 66, and the maximum bending moment will be at the centre, or on both sides of the centre and at a quarter interval therefrom, according as n is odd or even. In fact the solution is the same as for the load confined to the span.

(b) Let the distance between the weights be greater than $\frac{1}{n}$th of the span, that is let $s > \frac{2c}{n}$.

In this case the double locus, as shown on fig. 67, is to be drawn; one for n weights with any parabolic segment, the other for $\overline{n-1}$ weights with another parabolic segment prepared as on fig. 57, and so that its modulus and the modulus of the first are in the ratio $(n-1):n$. Since however the points of intersection a, b, c ... a', b', c' may be defined as shown on fig. 67, both loci may be drawn withe *same* parabolic segment, inking in the arcs over Ba, ac', &c., alternately on the two loci; a separate vertical scale will be required for each; or if considered necessary, the one locus may be reduced to the same vertical scale as the other at a number of points and then drawn.

The maximum bending moment will be the highest apex in the one locus or in the other; it must therefore be at the centre, at a quarter interval from the centre, or equally at these three points. The maximum is readily found as follows:—if n be *odd*, place the central weight over the centre of span, and calculate the bending moment at that point; this will be the maximum required: if n be *even*, place either of the two central weights, first at the centre of span, and afterwards at a point in its own field a quarter interval distant from the centre; calculate the bending moment for these two points, and the greater will be the maximum for transit required.

II. *Load longer than span.*

Let n be the greatest number of weights that can be on the span at once.

(a) Let the common interval be an aliquot part of the span.

The series of n parabolas as in I. (a) is the locus for transit, fig. 66.

(b) Let the interval be not an aliquot part of the span.

The number of weights on the span will be alternately n and $n-1$; while n are on, *no* weight can overtake its field before a weight goes off; while $(n-1)$ are on, *no* weight can overtake its field before another weight comes on; it is evident therefore that the locus for n, and the locus for $(n-1)$ weights are to be drawn as in I. (b) and as shown on fig. 67.

Note.—In all the above cases of transit, during the passing on of the front of the load and the passing off of its rear, fewer weights than n or $(n-1)$ are for some time on the span; by the theorem immediately preceding, the loci for such smaller numbers of weights are all wholly within the dotted loci on figures 66 and 67.

Theorem. The locus of the points of intersection D_1, D_2, &c., of the regular locus $BD_1D_2 \dots C$, figs. 66 and 67, for equal weights at equal intervals is a parabola.

Let x be the abscissa of D_r the junction between the r^{th} and $(r+1)^{\text{th}}$ fields, and let the load stand with the r^{th} weight over that point, then

$$\text{Remote segment} = c + x - \text{distance of } r^{\text{th}} \text{ weight from } G,$$

$$= c + x - \left(\frac{n+1}{2} - r\right)s;$$

$$\therefore \quad P = \frac{nw}{2c}\left(c + x + rs - \frac{n+1}{2}s\right); \text{ and}$$

$$M_x = \frac{nw}{2c}\left(c + x + rs - \frac{n+1}{2}s\right)(c - x) - ws(1 + 2 + 3 + \dots$$
$$\dots + \overline{r-1}) \dots\dots\dots\dots\dots\dots (1.)$$

for such values of x as are abscissae of junctions of fields. Putting y for the ordinate of this locus at *any* point x,

$$y = \frac{nw}{2c}\left(c + x + rs - \frac{n+1}{2}s\right)(c - x) - \frac{ws}{2}r(r-1) \dots\dots\dots (2.)$$

but $$c - x = r \cdot \frac{2c}{n},$$

so that $$r = \frac{n(c-x)}{2c}.$$

Substituting this value for r in equation 2, arranging the terms and putting R for wn, we have

$$y = \frac{R}{4c}(c^2 - x^2)\left(2 - \frac{ns}{2c}\right) \dots\dots\dots\dots\dots\dots (3.)$$

Hence the locus of B, D_1, D_2, &c., C is a parabola, axis vertical, and apex over the centre; comparing it with the equation at fig. 41, we see that it is the same as the locus of bending moments for a travelling uniform load of extent ns; that is, for the total load R spread uniformly over the actual extent of the load and one interval s more.

It is evident that this parabola gives too short an ordinate at all points except the junctions of fields. Still we have—

COROLLARY. Let R remain constant, and let n increase indefinitely while s decreases indefinitely; then the locus of the intersections D nearly coincides with $BD_1D_2 \ldots C$, and the equation to the locus is the same as for R spread uniformly over an extent only greater than the actual extent of the load by s an indefinitely small amount. In the limit, when n is infinite, the load is uniform and the result, fig. 41, follows from the present investigation.

Note.—In the general case, the locus of D is not a parabola nor any conic section. The general equation for any conic section (see Todhunter's Conic Sections, chap. XIII.) may be written thus:—

$$x^2 + bxy + cy^2 + dx + ey + f = 0;$$

and if we suppose the locus to pass through the five points B, D_1, D_3, D_4, and C, example No. 96 and fig. 56, we find values for the five co-efficients; and the equation becomes

$$x^2 + \tfrac{7}{930}\,xy + \tfrac{1}{7440}\,y^2 + \tfrac{532}{310}\,y - 441 = 0;$$

by trial we see that this locus does not pass through D_2; that is, the locus is not a conic section.

Examples.

106. A beam 20 feet span is subject to the transit of 5 weights, each 2 tons and fixed at intervals of 3 feet. Find the maximum bending moment at each point during the transit.

All the loads may be on the span at once, therefore $n = 5$; since the common interval is less than a fifth of the span, the locus is that for the whole load on the span, fig. 66. For left half of span

$${}_1M_x = \frac{5w}{2c}(c + x - 2s)(c - x) = \tfrac{1}{2}(4 + x)(10 - x),$$

for values of x from 10 to 6.

$$ {}_2M_x = \tfrac{1}{2}(10+x-3)(10-x)-2\times 3 = \tfrac{1}{2}(7+x)(10-x)-6, $$

for values of x from 6 to 2.

$$ {}_3M_x = \tfrac{1}{2}(10+x-0)(10-x)-2\times 6-2\times 3 $$

$$ = \tfrac{1}{2}(10+x)(10-x)-18, $$

for values of x from 2 to 0.

By symmetry the values for the right half of the span may be obtained, and the maximum for span during transit is

$$ {}_3M_0 = 32 \text{ ft.-tons.} $$

107. A beam 30 feet span is subject to the transit of 5 weights each 3 tons, and fixed at intervals of 7 feet. Find the maximum bending moment at each point.

The interval is not an aliquot part of the span, and when 5 weights (the whole load), *or* when only 4 weights, are on the span, no weight can overtake its field; the parts of the fields overtaken in each case correspond exactly with the parts not overtaken in the other.

Hence we have to find the locus $BD_1D_2 \ldots C$, when 5 weights are on the span, and also the locus $BD'_1D'_2 \ldots C$, when 4 weights are on, and take parts alternately of the two loci; the limits being found by laying off a, b, c, &c., a', b', c', &c., at intervals of 7 feet from each end; see fig. 67. For left half of span—

$$ {}_1M_x = \frac{5w}{2c}(c+x-2s)(c-x) = \tfrac{1}{2}(1+x)(15-x), $$

for values of x from 15 to 13.

$$ {}_1M'_x = \frac{4w}{2c}(c+x-\tfrac{3}{2}s)(c-x) = \tfrac{2}{5}(4{\cdot}5+x)(15-x), $$

for values of x from 13 to 8.

$$ {}_2M_x = \frac{5w}{2c}(c+x-s)(c-x)-ws = \tfrac{1}{2}(8+x)(15-x)-21, $$

for values of x from 8 to 6.

$$ {}_2M'_x = \frac{4w}{2c}(c + x - \tfrac{1}{2}s)(c - x) - ws = \tfrac{2}{5}(11{\cdot}5 + x)(15 - x) - 21, $$

for values of x from 6 to 1.

$$ {}_3M_x = \frac{5w}{2c}(c + x - 0)(c - x) - w.2s - ws = \tfrac{1}{2}(15 + x)(15 - x) - 63, $$

for values of x from 1 to 0.

By symmetry the values for the right half of the span may be obtained. The two maxima are—

$$ {}_3M_0 = 49{\cdot}5 \text{ ft.-tons at the centre, and} $$

${}_2M'_{1\cdot75} = 49{\cdot}225$ ft.-tons at a quarter interval from the centre; the former is the maximum for the transit.

Note.—The solution would be exactly the same although there were *many* weights, each 3 tons, and fixed at the same intervals, viz., 7 feet.

108. A beam 42 feet span is subject to the transit of a row of trucks 50 feet or more in length; the common interval between the wheels is 4 feet, and the weight on each wheel is 2 tons. Find the maximum bending moment for transit.

The greatest number of wheels that can be on the span at once is $n = 11$; and since $s = 4$, the distance between the wheels is not an aliquot part of the span, and the two loci for 11 and for 10 weights respectively intersect; the one has a maximum at the centre, and the other at 1 foot, a quarter interval, from the centre; so that it is only necessary to take the bending moments, one at the centre when a weight is over it, and another at 1 foot on either side of the centre when a weight is there; in each case the load being supposed to extend beyond the span on both sides. A weight at the centre gives

$$ P = \frac{nw}{2};\ M_0 = \frac{nw}{2}.c - ws\left(1 + 2 + 3 + \dots + \frac{n-1}{2}\right) $$

$$ = \frac{nw}{2}c - ws.\tfrac{1}{2}\frac{n-1}{2}.\frac{n+1}{2} $$

$$ = \frac{w}{8}\{4nc - s(n^2 - 1)\} = 111 \text{ ft.-tons.} $$

When a weight is over the point 1 foot from the centre, another is just over the left point of support, and we may say that either 11 or 10 weights are on the span; that is, the two loci intersect at $x = 1$. The maximum for the 10 weights is the same as the ordinate at $x = 1$ to the locus for 11 weights, and its height is less than the above calculated value.

109. A beam 39 feet span is subject to the transit of the same load as in No. 108. Find the maximum bending moment during transit.

$${}_wM_0 = 9 \times 19{\cdot}5 - 2(4+8+12+16) = 95{\cdot}50 \text{ ft.-tons.}$$

Now $\left(c - \frac{8}{4}\right) = 18{\cdot}5.$

$$\therefore {}_wM_1 = \tfrac{20}{39} \times 18{\cdot}5 \times 18{\cdot}5 - 2(4+8+12+16) = 95{\cdot}51 \text{ ft.-tons;}$$

and this being the greater is the maximum for transit.

Only 10 weights can be on the span at one time; and if we suppose these spread uniformly over their actual length, that is over 9 intervals, we have the approximate locus fig. 41,

$$M_x = \tfrac{20}{78}(19{\cdot}5^2 - x^2)\left(2 - \tfrac{36}{39}\right);$$

and its maximum value is $M_0 = 105$ ft.-tons.

SHEARING FORCES AND SHEARING FORCE DIAGRAMS FOR FIXED LOADS.

DEFINITION.—*The Shearing Force at any cross section of a beam*, or, as we may more conveniently say, *at any point of the span is,—The algebraic sum of the external forces acting upon the portion of the beam to either side of the point.*

We will denote this shearing force by F_x, and calculate its amount systematically from the forces acting on the portion of the beam to the left of the point; thus for a beam

$$F_x = P - W_1 - W_2 - \&c., \qquad (1)$$

the external forces on the left portion.

Similarly for a cantilever—

$$F_x = -W_1 - W_2 - \&c. \qquad (2)$$

On the beam then, F_x is $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ according as P is $\left\{\begin{matrix}\text{greater}\\\text{less}\end{matrix}\right\}$ than the sum of the weights to the left of the section at x. The beam may be divided at a certain point into two segments, such that for every point in the $\left\{\begin{matrix}\text{left}\\\text{right}\end{matrix}\right\}$ segment, the shearing force is $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$; at this point the shearing force changes sign. The shearing force on either segment may be considered as positive; for convenience we have chosen as just stated. Having fixed the sign thus, on a cantilever as shown on the diagrams, that is with its fixed end to the right, the shearing force is everywhere negative; and in considerations regarding beams *and* cantilevers, it is necessary to take this into account; when, however, the cantilever alone is considered, it will be better to reckon the shearing forces as positive.

It may be observed that the symbolical expression for F_x does not depend for its form upon the position of the origin; in this respect it is unlike M_x.

A Shearing Force Diagram is a figure having a horizontal base representing the span on a convenient scale, and an outline or locus. For a cantilever this locus lies wholly under the base; for a beam, it lies above the base for the segment to the left, under the base for the segment to the right, and crosses the base at an intermediate point. The *height* of any point on the locus, measured on a scale for forces, say tons or lbs., gives the shearing force at the point of the span over which it stands. Both these scales should accompany the drawing.

Maximum Shearing Force.—On a cantilever it is evident that the greatest value is at the fixed end. On a beam there are two values each greater than the others near it, one at the left end and positive, one at the right and negative; the value of the greater of these is the greatest for the whole span.

This locus, in the diagrams for all the cases of fixed loads which we consider, consists of straight lines. For a portion of the span between any two adjacent loads, the straight line is evidently horizontal; for portions uniformly loaded it slopes at a rate given by the number which indicates the intensity of the load; thus if w lbs. per foot be the intensity of the uniform load, then w vertical to one horizontal is the slope. Where a weight is concentrated at a point, the line is vertical; that is, at such a point the locus makes a sudden change of level, the change being equal to the weight.

On a cantilever, the shearing force at the fixed end is equal to the load, and at the free end it is zero; we can readily draw the straight lines as above described to suit the nature of the load, and so complete the diagram.

On a beam the shearing force at the left end is P, at the right end it is $-Q$, and at some intermediate point the locus intersects the base and changes sign; the manner of fixing the position of this point will be explained immediately. The whole locus is then readily completed by drawing the lines as above to suit the nature of the load.

Theorem. The Shearing Force at any point of a beam or cantilever is the rate of variation of the bending moment at that point; and on the beam the shearing force changes sign at the point of maximum bending moment.

As before,

$$F_x = P - \Sigma(W),$$

where $\Sigma(W)$ means the sum of the loads to the left of x; and

$$M_x = P(c-x) - \Sigma(W.x),$$

where $\Sigma(W.x)$ means the sum of the products got by multiplying each load to the left of x by its leverage about x. Hence if we suppose F_x to be positive, and take the bending moment at any interval d further to the right, the second bending moment will exceed the first by $F_x . d$, if there be no load on the portion d; and by $F_x . d$, minus the load over the portion d into its leverage about x, if there be a load on the portion d.

Since the leverage of the load which is on the portion d is less than d, we can by taking d small enough make th product as small as we please; that is, $F_x.d$ is the change of the bending moment in passing from x through a small interval d. Now the rate of change of M_x means the change in passing from x through an unit interval, say one foot, if the change continued uniform throughout that interval, and at the same rate as at x; in other words, the change in M_x for d reckoned equal to unity, without taking into consideration any additional loads which may be over that interval; hence the

$$\text{Rate of change of } M_x = F_x; \qquad\qquad (1)$$

also, as we pass from the left to the right end of the span, if F_x be $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$, M_x is $\left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\}$, and at the point where F_x changes sign M_x is a maximum.

COROLLARY. For any system of fixed loads the shearing force only once changes sign, since there is only one maximum bending moment.

These general remarks will suffice for the analysis of shearing forces due to fixed loads, and we will now give the graphical solutions.

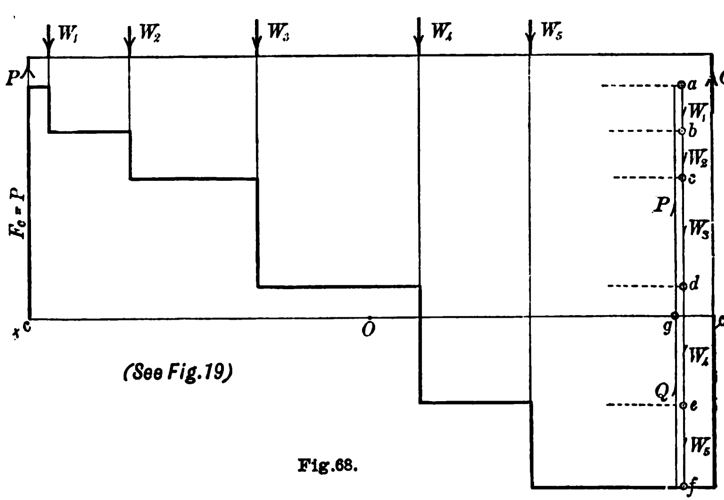


Fig. 68.

Beam under unequal weights at irregular intervals.—Fig. 68. P and Q are determined graphically by finding the point g, fig. 19. Through g draw a horizontal line and produce the lines of action of P, W_1, W_2, ... Q to cut it; join these lines of action in pairs by horizontals through a, b, c, d, e, f respectively. The scale used for forces is also the scale for shearing forces.

The construction is cyclical; through each of the points a, b, c, d, e, f, g a horizontal line being drawn to join the lines of action of the forces in pairs, viz:—From a joining P and W_1, from b joining W_1 and W_2, from c joining W_2 and W_3, from d joining W_3 and W_4, from e joining W_4 and W_5, from f joining W_5 and Q, and from g joining Q and P.

Cantilever under unequal weights at irregular intervals.—Fig. 69. From a, b, c, d respectively, fig. 21, draw horizontals joining the lines of action of the forces in pairs, the vertical through the fixed end being reckoned as a line of action. Thus draw horizontals, from a joining the vertical through the fixed end and W_1, from b joining W_1 and W_2, from c joining W_2 and W_3, and from d joining W_3 and the vertical through the fixed end. The scale for forces is also the scale for shearing forces.

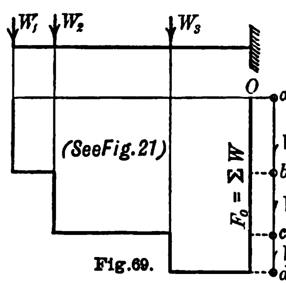


Fig. 69.

Corollaries. For a beam with W at the centre, the shearing force diagram consists of two rectangles of height $\frac{1}{2}W$, one standing above the left half and the other below the right half of span.

For a beam loaded with W at a point dividing the span into any two segments, the shearing force diagram consists of two rectangles, one standing above the left segment, the other below the right segment. The height of each is inversely proportional to the length of the segment on which it stands, and the sum of their heights is W.

In these two cases it is evident, and it can easily be proved for a general case fig. 68, that the area of the part of the shearing force diagram above the base is equal to the area of the part below the base.

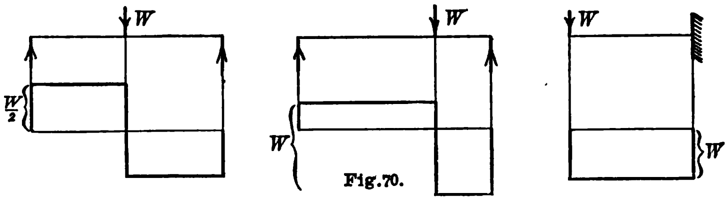


Fig.70.

For a cantilever with W at its free end, the shearing force diagram is evidently a rectangle of height W, standing below the span.

Beam uniformly loaded.—Fig. 71. Draw verticals, one upwards from the left end, another downwards from the right end of span, each equal to $\frac{1}{2}W$; join their extremities with a straight line, which will cut the base at O the centre.

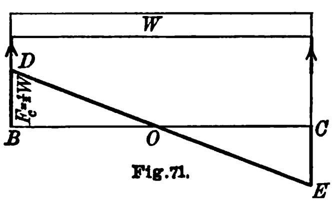


Fig.71.

The shearing force diagram consists of two right-angled triangles, whose common height is $\frac{1}{2}W$; one is above the left half, the other is below the right half, of span, and the right angle of each is at the end of span.

Cantilever uniformly loaded.—Fig. 72. Draw a vertical downwards from the fixed end of the base equal to W, and join its extremity to the free end of the base.

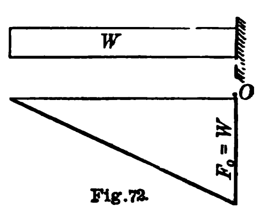


Fig.72.

The shearing force diagram is a right-angled triangle of height W, standing below the span, and with the right angle at the fixed end.

Beam under an uniform load together with weights fixed at intervals.—Fig. 73. The two diagrams figs. 68 and 71 are to be directly superimposed by shifting the portions of the triangle till they stand upon the steps of the other figure.

Another method is,—Draw *DE* to represent the shearing force for the uniform load alone, and on this line as a sloping base make the construction for fig. 68.

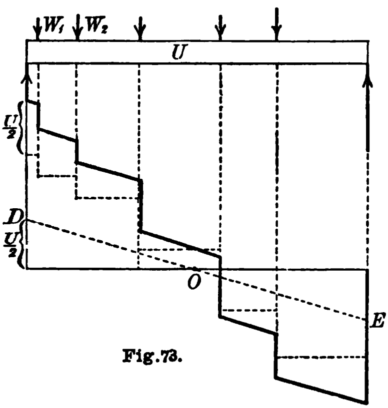


Fig. 73.

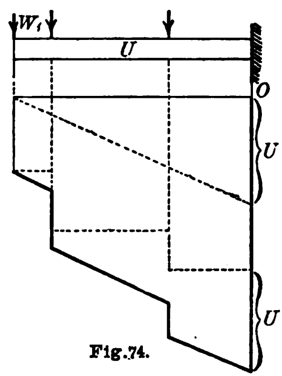


Fig. 74.

Cantilever under an uniform load together with weights fixed at intervals.—Fig. 74. The diagram fig. 72 is to be superimposed upon the diagram fig. 69, by shifting the portions of the triangles till they stand upon the steps of the other figure.

Cantilever uniformly loaded on a portion of its span.—Fig. 75. The left end of the load is the free end of the cantilever. Draw downwards from the fixed end a vertical equal to the total load, and through its extremity draw a horizontal to meet the vertical through the nearest end of the load; join the point of intersection to the free end with a straight line.

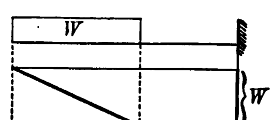


Fig. 75.

Beam uniformly loaded on a portion of its span.—Figs. 76 and 77. Fix the point S at which the maximum bending moment occurs as on fig. 32; draw verticals through S and through the two extremities of the load; lay off $MN = W$, the total load, upon the vertical through one of the extremities of the load; from N draw a horizontal meeting

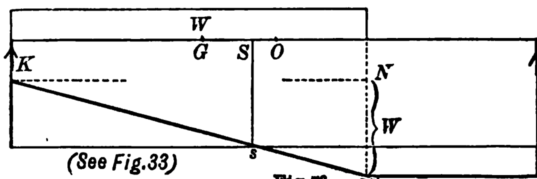


Fig. 76.

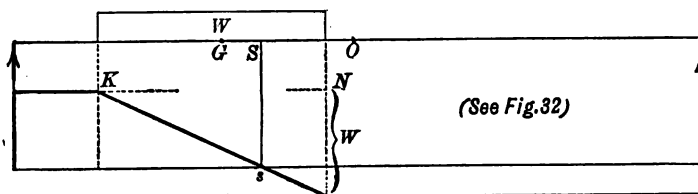
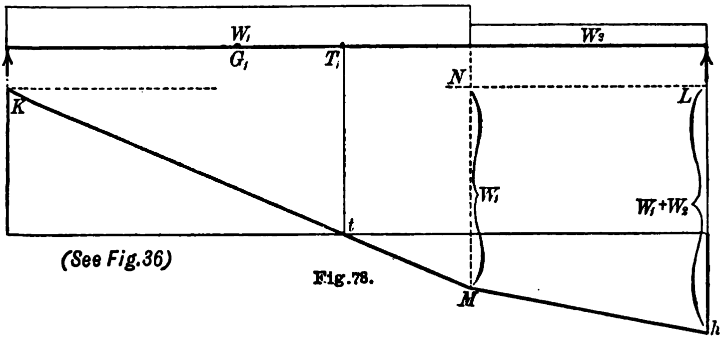


Fig. 77.

the vertical through the other end in K, and draw MK cutting the vertical through S in s. Draw for base a horizontal through s and extending the whole length of span, a horizontal through M and extending to the vertical through the right end of span, and a horizontal from N to the vertical through the left end of span.

Note that this construction determines P and Q the supporting forces.

Beam loaded uniformly on two segments with different intensities of load.— Fig. 78. Fix the point T_1 at which the maximum bending moment occurs, as on fig. 36; draw verticals through T_1, the junction of the loads, and the extremities of the span; lay off MN upon the vertical through the junction, equal to W_1 the load on that segment in which T_1 lies; project N at L and K on the verticals through the extremities, join MK cutting the vertical from



(See Fig.36)

Fig.78.

T_1 at t. Through t draw the base; make Lh equal to the total load, and draw Mh.

This construction also determines P and Q.

Shearing Forces due to Moving Loads.

In the same way as was explained (see page 102) when speaking of bending moments, the Shearing Strain produced by a moving load is greater than that produced by the same load when fixed. In the cases which follow, it is to be understood that the loads as given are dead loads, or the equivalent reduced dead loads. When the load is partly fixed and partly moving, the equivalent dead load is the sum of the actual dead load and the dead load equivalent to the actual moving load.

Definition. For any point x, the *Range of Shearing Force* due to a moving load is its extent; and the limits of this extent are the maximum positive and maximum negative values which F_x assumes during the transit of the moving load.

Beam under a rolling load.—Fig. 79. At any point of the span, F_x the shearing force is positive and equal to P, so long as R is to the right of the point; since P increases as R moves towards the left support, F_x is evidently a positive maximum when R is indefinitely close to, and on the right side of, the point. When R passes to the left of the point, $F_x = P - R = -Q$; since Q increases as R comes closer to the right support, it is again evident that F_x is a negative maximum when R is indefinitely close to, and on the left side of, the point. When R is indefinitely close to the point, P and Q have sensibly the same values as for R exactly at the point, and we have the following,—

To find the maximum shearing force at any point x:— Place R over the point, and calculate P and Q for that position of the load; these are respectively the max. positive and max. negative values of F_x.

During the passage of the load, the shearing force assumes all values lying between these maxima; and it is important to observe that the shearing force not only changes sign at any point as R passes over the point, but that it changes from its greatest positive to its greatest negative value, or *vice versa*, according as R is moving to

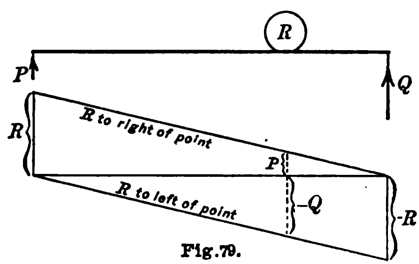


Fig. 79.

left or right, and does so even although the load be moving slowly.

The positive maximum at each point is the value of P as R comes to the point; and since P is proportional to the remote segment, it follows that the positive maximum at each point is proportional to the distance of the point from the right end of span; it is zero for the right end, and increases uniformly till it is R for the left end; for, when R is just to the right of the right end of span, P is zero, no load being on the span; and again, when R is just to the right of the left end of span, P is sensibly equal to R.

For a rolling load the range at each point is constant and is equal to R.

Graphical Solution.—From the left end of the base draw upwards a vertical equal to R, and join its extremity to the right end of the base; similarly from the right end, draw downwards a vertical equal to R, and join its extremity to the left end of the base; at each point the ordinate upwards gives the maximum positive, and the ordinate downwards the maximum negative, shearing force; while the double ordinate gives the range.

Beam under a travelling load system.—Figs. 80 and 81. At any point the shearing force increases as each weight in succession approaches from the right; and when a weight passes the point, it suddenly diminishes by an amount equal to that weight. At each point there is a maximum when a weight is just to the right of the point, and a minimum when it is just to the left.

In order to find for any point the maximum and minimum corresponding to a particular weight;—Place the load system so that this weight is over the point; from P subtract the weights to the left of that weight for the maximum, and further subtract that weight for the minimum. In figs. 80 and 81, a locus giving the maximum at each point due to a particular weight approaching, is shown by a full line; the locus giving the minimum due to the same weight is evidently parallel to the first, and below it at a constant distance equal to that weight; this parallel locus is shown by a dotted line. In each of these diagrams there are five loci drawn in full lines, and giving the maximum at each point due to the approach from the right of each of the five weights respectively; of the five dotted loci giving the minimum at each point due to the receding of each weight respectively, only one is shown, and it is drawn parallel to the locus shown by the lowest full line, and at a constant depth below it equal to the weight at the right end of the load. Having determined the full lines or positive loci, it is easy to draw the others.

The first locus Aa is the value of P when W_1 is over any point; so that as the load comes on from the right end, and so long as W_1 alone is on the span, Aa is a portion of the diagram due to W_1 as a rolling load; that is, it slopes at an angle whose tangent is $W_1 \div 2c$, or, in other words, at a rate proportional to W_1; further, Aa extends over a horizontal distance equal to that between W_1 and W_2. When W_2 comes over any point, W_1 and W_2 alone being on the span, then F_x is calculated by finding P and subtracting the constant quantity W_1; P is now increasing as for a rolling load $(W_1 + W_2)$ concentrated at its centre of gravity, so that the second link aa slopes at an angle whose tangent is $(W_1 + W_2) \div 2c$, or, in other words, at a rate proportional to $W_1 + W_2$; further, aa extends from the point where the preceding link ended, and continues through a horizontal distance equal to that between W_2 and W_3. Thus, the locus

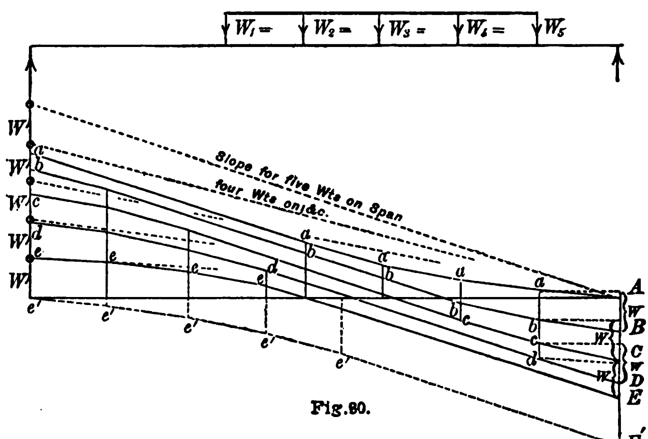


Fig. 80.

$Aaaa\ldots$ begins at the right end of the span, each link sloping more and more at rates in direct proportion to the sum of the weights on the span, and extending respectively over horizontal distances equal to those between the weights;

the *last* link extends over a distance which is the excess of the span over the extent of the load. On fig. 80, the first four links are short and equal, the last one is long. The other loci *Bbb*..., *Ccc*..., &c., consist of links sloping more and more as weights come on at the right end, and less and less as they go off at the left; for instance, *Ccc*..., fig. 80, consists of two equal short links and a long one increasing in slope, and two equal short ones decreasing in slope; the rate of slope of each link is directly proportional to the sum of the weights on the span at the time corresponding.

After having drawn the first locus *Aaa*..., the initial points *B*, *C*, *D*, &c., of the other loci are found as follows:—The ordinate of *B* represents the value of the shearing force at the right end of span when W_2 is just to the right of that point; if the load be placed in this position, the shearing force at the right end of span, and in the interval between that end and the point where W_1 stands, is constant; the value of the shearing force just to the left of W_1 is given by the ordinate of *a* the left extremity of the first link *Aa*, and this quantity diminished by W_1 is the value required. Hence *B* is on the vertical through the right end of span, and at a depth W_1 below the level of the first joint *a* on the locus *Aaa*...; similarly *C* is on the same vertical and at a depth W_2 below the level of the first joint *b* on the locus *Bbb*....

Graphical Solution.—Case I. Moving load of equal weights at equal intervals. Fig. 80. Lay the weights up, in order, on the vertical through the left end of span, and draw a ray from the right end of span to each junction; each slope is evidently the shearing force diagram for a rolling load equal to one of the weights, the sum of two, the sum of three, &c., respectively. Draw verticals at intervals from the right end equal to the common interval between the weights, and the same in number; draw the first link *Aa* parallel to the slope for one weight, the next link *aa* parallel to the slope for two weights, &c., similarly draw the locus *e′e′*... *E′* below the span. The ordinates to *Aaa*... and to *e′e′*... *E′* give respectively the maximum shearing force positive and negative at each point, during the transit of the load system; and in this case it is unnecessary to draw the other loci.

II.—General Case. Fig. 81. The example shown in the figure is the same as that for which the bending moment diagram is given in fig. 56. Lay the weights up, in order, on the vertical through the left end of span, and then down again in order; draw a ray from the right end of span to each junction; rays to joints in ascending order are drawn in full lines, and to joints in descending order are drawn in dotted lines. Draw another line Ae' equal to the span, so as

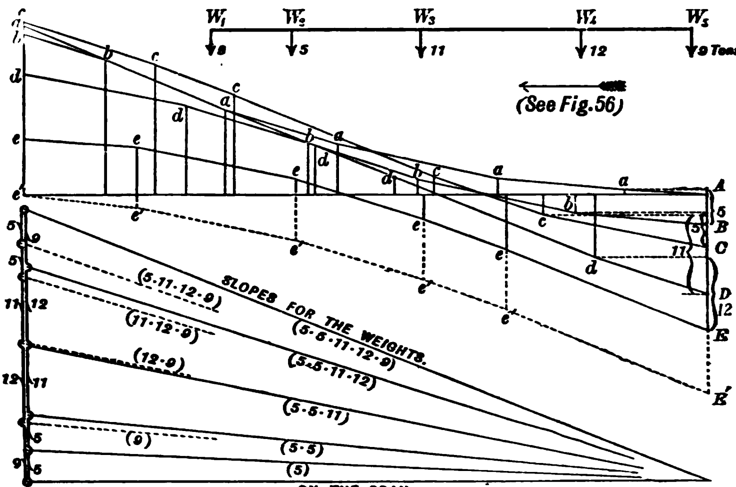


Fig. 81.

not to complicate the figure. Draw the locus Aaa ... link after link parallel to the slopes in ascending order, each link extending in order for a horizontal distance equal to that between the weights in pairs, the last link completing the locus; in the figure, the first link Aa extends for a distance equal to that between W_1 and W_2; the second link aa to that between W_2 and W_3, and so on; the last link extends for a distance equal to the difference between the span and the length of the load. Plot B on the vertical through the right end, at a depth equal to W_1 below the level of the first joint a in the locus already drawn; draw the locus *Bbb* ... parallel to the respective slopes in ascending order; the first link *Bb* is parallel to the slope for $(W_1 + W_2)$, each link extends till a new weight comes on, the second last link extends till W_1 goes off, and the last link is drawn parallel to the slope for $(W_2 + W_3 + W_4 + W_5)$. Plot C at a depth equal to W_2 below the level of the first joint b, and draw the locus *Ccc*...; the first link *Cc* is parallel to the slope for $(W_1 + W_2 + W_3)$; the other links are drawn parallel respectively to the slopes in ascending order, and when these are exhausted the remaining links are drawn parallel to the slopes in descending order; the extent of each link is determined as each weight after W_3 comes on, and then as weight after weight goes off. In the same way, for each weight, a locus is drawn in full lines, consisting of as many links as there are weights; the highest ordinate at any point gives the maximum positive shearing force thereat, for transit of load. Dotted loci are drawn, one parallel to each locus shown by a full line, and below it at a distance equal to the weight to which it corresponds; the deepest ordinate at each point gives the negative maximum. On the diagram, $e'e' \ldots E'$ is drawn parallel to $ee \ldots E$; and since, in this case, the locus $e'e' \ldots E'$ gives the maximum shearing force for every point of span, the other four dotted loci are not shown.

Beam under an advancing load of uniform intensity.—Fig. 82. At any point x, the positive maximum shearing force occurs when the front of the load is at the point. Suppose the load to be in the position shown at the top of the diagram, then the shearing force is positive and equal to P. If the load move towards the right, P will decrease, and F_x will equal that decreased value. If the load move towards the left, and if the total load be not yet on the span, then the new load is exactly the same as the first with an additional load to the left of x; this additional load is shared in some manner between the two supports, so that the increase of P is only a *fraction* of that added load; in reckoning F_x, however, we subtract from this increased value of P, the whole of the added load, that is we subtract more than we add. Again, if the load be shorter than the span and be wholly on the span; then, after the advance, the new load is the same as the first with a portion added to the left of x and an equal portion taken off at the tail of the load; the portion added increases by a fraction of itself the value of P, while the portion taken off decreases by a smaller fraction of itself the value of P; and in reckoning F_x, we subtract the whole of the added portion from this increased value of P. Hence, whatever be the length of the load, F_x is a positive maximum when the front of the load is at x. Similarly, by calculating the negative shearing forces from the supporting force Q, it can be shown that the shearing force is a negative maximum when the tail of the load is at the point. At each point, during the transit of the load, the shearing force assumes all values between the two maxima, and passes gradually through the whole range in the same time that the load takes to make a transit.

Shearing Force Diagram.—Length of load equal to, or greater than, span. Fig. 82. When the front of the load is at the left end of span, the whole span is covered, and

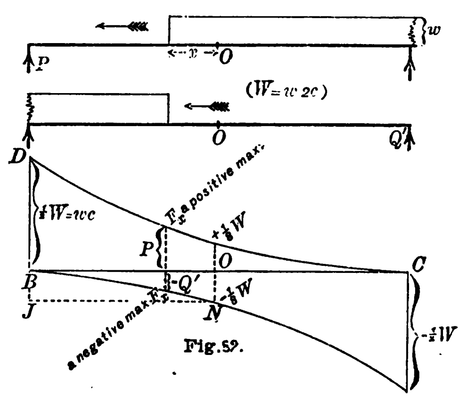


Fig. 82.

$P = \frac{1}{2}W$, where W is the load which covers span; hence the positive maximum at the left end is $\frac{1}{2}W$. When the front of the load is at the right end of span, P is zero as no load is on the span; hence the positive maximum at the right end is zero. When the front of load is at any intermediate point, the right segment is loaded, and $F_x = P$ a positive maximum; the value of this maximum increases as the point approaches the left end of span, because the length of the loaded segment is increasing, and because its centre of gravity is nearer the left end; or in symbols, this positive maximum

$$F_x = \frac{\text{load on span}}{\text{span}} \times \frac{\text{length of loaded segment}}{2} = \frac{w}{4c}(c+x)^2.$$

Hence, the locus giving F_x the positive maximum shearing force at each point is a parabola with its axis vertical and its apex at the right end of the span, and whose ordinate at the left end is $\frac{1}{2}W$; the locus giving the negative maximum is a similar parabola below the base and with its apex at the left end of span; the *range* at each point is given by the double ordinate.

Graphical Solution.—Length of load equal to, or greater than, span. With any parabolic segment draw a curve with its apex at the right end meeting the vertical through the left end at D, fig. 82; draw another below the base, and construct a scale for verticals upon which $\overline{BD} = \frac{1}{2}W$, where W is the load which covers the span.

Shearing Force Diagram.—Length of load less than span. Fig. 83. Let $2k$ be the length of the load. While the

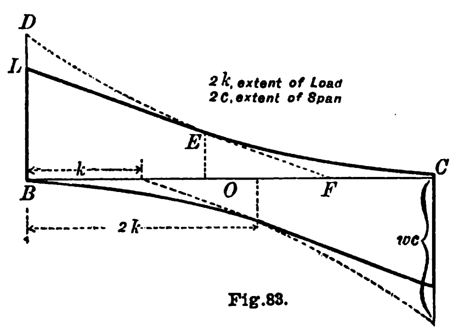


Fig. 83.

load advances from the right end, and so long as it completely covers the right segment, that is up to a distance $2k$ from C, the diagram is a portion of the parabola, fig. 82; for the further advance of the load, the maximum shearing force is increasing, because the centre of gravity of the load is approaching the left end, and the remainder of the locus is therefore a straight line. The straight portion of this locus and the locus for an equal rolling load (see fig 79) are parallel, and are separated from each other by a distance k measured horizontally; this straight portion, when produced, cuts the base at a point F such that $CF = k$, and it is therefore a tangent to the parabola.

Graphical Solution.—Length of load less than span. Fig. 83. Draw the parabola CD as in the previous case; construct a vertical scale such that $BD = wc$, that is equal to half the load which would be on the span supposing the whole span covered; ink in CE the portion of this parabola extending from the apex C through a horizontal distance equal to $2k$; and draw for the remainder of the span, a tangent to the parabola at the point E; this tangent is drawn by laying off $CF = k$, drawing FE, and producing it to L.

Beam under an uniform dead load together with a rolling load. —Fig. 84.

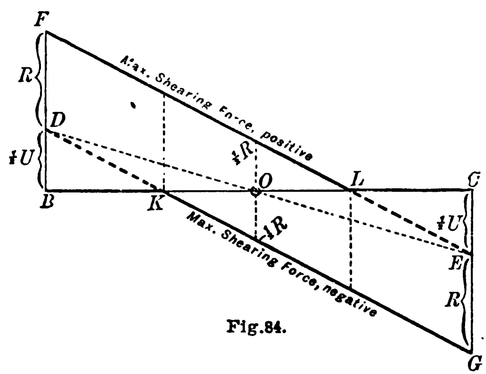


Fig. 84.

Graphical Solution.—Reduce the rolling load to its equivalent dead load, and call this quantity R. Lay off the base BC equal to the span, and construct the locus DE for the uniform load as in fig. 71; and upon DE as a sloping base, construct the loci EF and DG as in fig. 79. The figure $BFLCGKB$ is the combined diagram, and it gives the positive and negative maxima at each point for a transit of the rolling load. From B to K the stress is always positive, from L to C it is always negative, and from K to L it has both positive and negative values; between K and L the range is the double ordinate, it is constant at each point, and is equal to R the rolling load.

Beam under a fixed uniform load together with an advancing load of uniform intensity.—Fig. 85. Let u be the intensity of the fixed uniform load, and w the intensity of the advancing load in terms of its equivalent dead load; as in fig. 71, draw DE for the uniform dead load, and from DE as a sloping base plot the ordinates of the two parabolas, fig. 82, up and down respectively; the two loci FLE and DKG give the maximum and minimum, or the positive and negative maximum, shearing force at each point.

The ordinates to DKN, the portion of the curve which extends over the left half of span, are derived by taking the ordinates of the slope OD fig. 71, and subtracting the ordinates of BN fig. 82; or, what is the same thing, by adding the ordinates of the slope OD fig. 71, and of the parabolic segment JBN fig. 82, and then subtracting the constant quantity ON. When the ordinates of the slope and of the segment are added by the theorem at fig. 35, the resulting locus is a portion of the *same* parabola as BN, but with its apex to the left of B at a distance

$$BS = a = \frac{BD}{2BJ} . OB = \frac{\frac{1}{2}U}{2 \times \frac{1}{8}W} . c = \frac{u}{w} . 2c ; \quad \text{........(1.)}$$

When the constant quantity ON is subtracted, the locus unaltered in form moves vertically downwards through that distance; the apex still remains to the left of B at the distance a; the curve passes through the point D whose height is $BD = uc$, and through the point G whose depth is $CG = (u+w)c$. Similarly ELF is symmetrical, left and right, up and down.

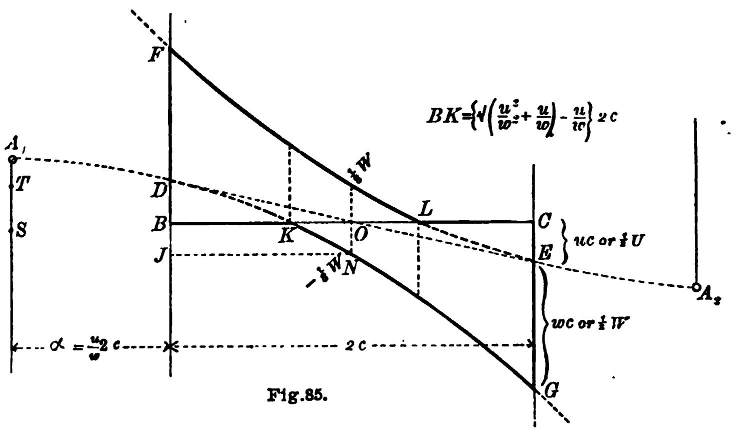


Fig. 85.

Taking the centre of span as origin, the equation to the parabola A_1DG is—

$$y = -\frac{wc}{4} + \frac{2u+w}{2}x - \frac{w}{4c}x^2 \quad \text{(2.)}$$

The point K is found by making $y = 0$, and finding the corresponding value of x; thus—

$$OK = c - 2c\left(\sqrt{\frac{u^2}{w^2} + \frac{u}{w}} - \frac{u}{w}\right) \quad \text{(3.)}$$

The ordinate to the apex A_1 is found by substituting $(c + a)$ for x in equation 2; and is—

$$SA_1 = \left(\frac{u^2}{w} + u\right)c \quad \text{(4.)}$$

Graphical Solution.—Length of advancing load equal to, or greater than, span. Draw two verticals BF and CG at the ends of the span, and draw another vertical to the left of BF at a distance $a = \frac{u}{w}2c$. With *any* parabolic segment draw the arc A_1DG with its axis vertical, its apex on the left vertical, and cutting the other two verticals at the points D and G. From D draw a tangent DE, and bisect it in O; through O draw the base BOC; a check upon the accuracy of the drawing is obtained by observing that $CE : EG :: u : w$. From the point A_2 symmetrical with A_1, with the same parabolic segment draw the curve A_2LF; and construct a vertical scale for shearing forces such that $CG = (u+w)c$; that is, CG represents half the total load, if we consider the advancing load to be equal in length to the span. The line DE is a tangent at D to the parabola, since the ordinates of the curve referred to DE as a sloping axis are proportional to the squares of the abscissæ laid along DE. Todhunter's Conic Sections, § 151.

To the left of K the shearing force is always positive, to the right of L it is always negative, and between K and L there are both positive and negative maxima. The range at the centre is $\frac{1}{4}W$ as for W alone; if u be great compared to w, the range between K and L is nearly constant and equal to $\frac{1}{4}W$, since the apex is then far out and the portion of the parabola over KL is very flat.

Approximate Shearing Force Diagram.—Length of advancing load equal to or greater than span. Lay off BK and CL one from each end towards the centre equal to the above fraction of the span; on the vertical drawn upwards from the left end, lay off $BF = (u+w)c$, the sum of the intensities of the loads into the semi-span, or half the total load; on the vertical drawn downwards from the right end, lay off $CG = BF$, and join FL and KG with straight lines. The diagram thus drawn gives ordinates greater than those in the exact diagram, and therefore on the safe side.

Graphical Solution.—Length of advancing load less than span. Construct the diagram fig. 85 as for the load equal to the span; ink in a portion of the curve DG, extending from D through a horizontal distance equal to $2k$ the length of the load; replace the remainder of the curve by the tangent at the point where the inked line ends; similarly replace a portion of the curve EF by its tangent. By the above construction, fig. 83 is drawn upon DE as a sloping base; that is, DE, fig. 85, corresponds with BC, fig. 83.

Approximate Shearing Force Diagram.—Length of advancing load less than span. If $2k$ be greater than BK, the locus cuts at K just as for the load equal to the span; if $2k$ be less than BK then, since the tangent is drawn from a point between D and K, it cuts the base at K' a point *nearer* the centre than K. Lay off BK and CL the same fraction of the span as for the load equal to the span; at the two ends of the span respectively lay off

$$BF' = CG' = uc + \frac{wk}{c}(2c - k),$$

the value of the supporting force when the end of the advancing load is at the point of support. Join KG' and LF' with straight lines. The figure $BF'LCG'K$, thus obtained, gives ordinates which are either equal to or longer than the ordinates for the exact diagram, and are therefore on the safe side.

Examples.

110. A beam 20 feet span supports a load of 10 tons uniformly distributed. Find the shearing forces at intervals of 5 feet.

Ans. $F_{10} = 5$ tons; $F_5 = 2·5$ tons; $F_0 = 0$. On right half of span, the values are the same, but negative.

111. A beam 20 feet span supports a load of 10 tons concentrated at the centre. Find the shearing forces.

Ans. $F_{10\text{ to }0} = 5$ tons, $F_{0\text{ to }-10} = -5$ tons, and the sign changes at the centre.

112. In the example given on page 32 and shown in fig. 17, find the shearing forces.

Ans.

$$\begin{aligned}
F_{21\text{ to }20} &= P &&= 24 \text{ tons.}\\
F_{20\text{ to }15} &= 24 - W_1 &&= 19 \text{ tons.}\\
F_{15\text{ to }7} &= 19 - W_2 &&= 14 \text{ tons.}\\
F_{7\text{ to }-3} &= 14 - W_3 &&= 3 \text{ tons.}\\
F_{-3\text{ to }-10} &= 3 - W_4 &&= -9 \text{ tons.}\\
F_{-10\text{ to }-21} &= -9 - W_5 &&= -18 \text{ tons} = (-Q).
\end{aligned}$$

See figs. 18 and 68, and note that the shearing force changes sign, and that the bending moment is a maximum, at the point $x = -3$.

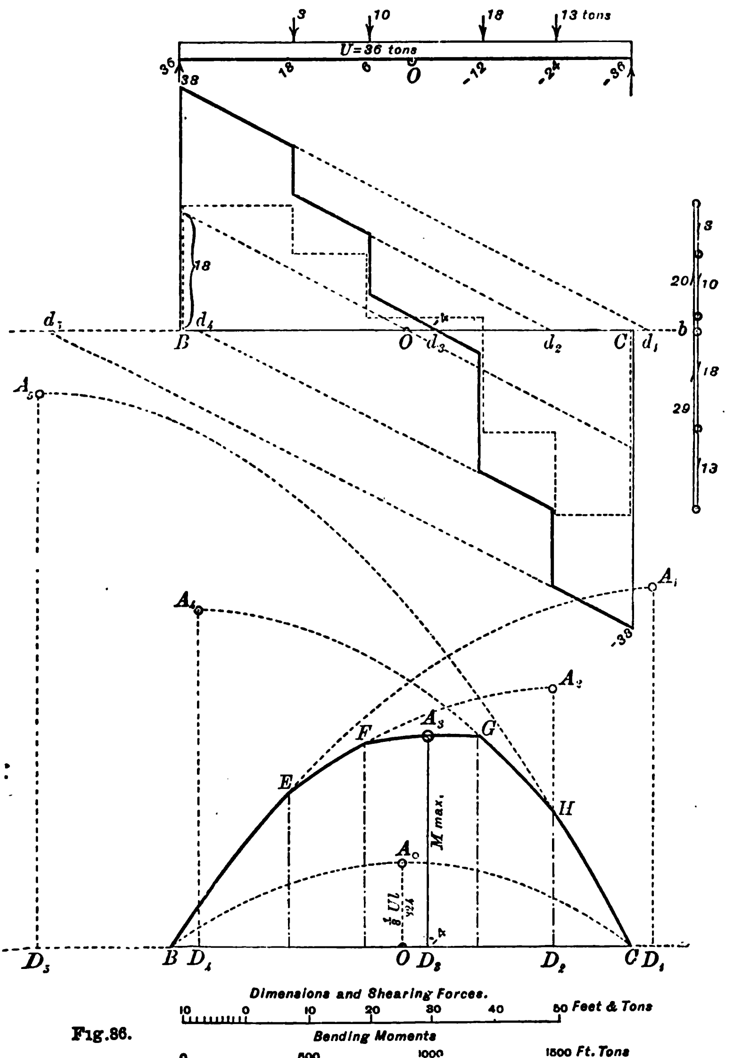


Fig. 86.

113. A beam 72 feet span supports an uniform load of $\frac{1}{2}$ ton per foot run, and fixed loads $W_1 = 8$, $W_2 = 10$, $W_3 = 18$, and $W_4 = 13$ tons at points whose abscissæ measuring from the centre of span are $x_1 = 18$, $x_2 = 6$, $x_3 = -12$, and $x_4 = -24$ feet; fig. 86. Find the shearing forces at intervals of 4 feet.

For the fixed loads alone $F_{36 \text{ to } 18} = P = 20$, $F_{18 \text{ to } 6} = 12$, $F_{6 \text{ to } -12} = 2$, $F_{-12 \text{ to } -24} = -16$, $F_{-24 \text{ to } -36} = -Q = -29$ tons.

For the uniform load alone, $F_{36} = P = 18$ tons, and by proportion the shearing force at each point is half the abscissa of the point.

For the two loads combined, the shearing forces at points 4 feet apart are—

Feet.	Tons.	Feet.	Tons.	Feet.	Tons.	Feet.	Tons.
36	38	16	20	−4	0	−24	−41
32	36	12	18	−8	−2	−28	−43
28	34	8	16	−12	−22	−32	−45
24	32	4	4	−16	−24	−36	−47
20	30	0	2	−20	−26		

The shearing force changes sign at the point $x = -4$, and at this point the bending moment (see fig. 39) is a maximum.

After having drawn the diagram for shearing forces, the bending moment diagram is constructed with greater facility than by the general method, fig. 39, since the points d_1, d_2, &c., are determined by producing the slopes of the shearing force diagram to meet the base; this saves the trouble of drawing the bending moment diagram for the fixed loads alone, and allows *any* scale to be chosen for shearing forces and loads, irrespective of the scale for bending moments. The slopes determine the points d_1, d_2, &c., because (theorem, page 181) if the shearing force for the remainder of the span varied as for the first interval, it would change sign at the intersection d_1; hence if the bending moments varied for the rest of span as for the first interval, the maximum would be at D_1; that is, the apex of the parabola governing the first field is on the vertical through D_1; it is evident that there is only one such maximum agreeing with what is shown at fig. 39; similarly for the other fields.

114. A cantilever 12 feet long supports an uniform load

of one ton per foot, which extends from the left end to the middle. Find the shearing forces at intervals of 4 feet.

$$F_{12} = 0,\ F_8 = 4,\ F_{6\text{ to }0} = 6 \text{ tons.}$$

115. A beam 72 feet span supports an uniform load of 2 tons per foot run, extending from the left end to the point $x = -12$, and an uniform load of 1 ton per foot run extending from the right end to the point $x = -12$. Find the shearing forces at intervals of 12 feet. See example No. 47, and figs. 36 and 78.

$P = 68$ tons; F decreases at the rate of 2 tons per foot for the left segment, and at the rate of 1 ton per foot for the right segment.

Ans. $F_{36} = 68$; $F_{24} = 44$; $F_{12} = 20$; $F_0 = -4$; $F_{-12} = -28$; $F_{-24} = -40$; $F_{-36} = -52$ $(= -Q)$.

F changes sign at $x = 2$.

116. A beam 20 feet span is subject to the transit of 5 weights, each 2 tons, and fixed at intervals of 3 feet. Find the maximum positive and negative shearing force at each foot of span during the transit. See example 106.

The slopes in fig. 80 are—1st link, $\frac{1}{10}$ vertical to 1 horizontal; 2nd, $\frac{2}{10}$ to 1; 3rd, $\frac{3}{10}$ to 1; 4th, $\frac{4}{10}$ to 1; and 5th, $\frac{5}{10}$ to 1; hence upward ordinates to *Aa*..., foot by foot from left end, are—

1st link *Aa*... 0, ·1, ·2, ·3; 2nd link *aa*... ·3, ·5, ·7, ·9;
3rd „ *aa*... ·9, 1·2, 1·5, 1·8; 4th „ „ ... 1·8, 2·2, 2·6, 3·0;
5th „ „ ... 3·0, 3·5, 4·0, 4·5, 5·0, 5·5, 6·0, 6·5, 7·0.

Hence the maxima are—

$$\left\{\begin{array}{lll} F_{10} = 7 \text{ and } 0, & F_9 = 6{\cdot}5 \text{ and } -{\cdot}1, & F_8 = 6 \text{ and } -{\cdot}2, \\ F_{-10} = 0 \text{ and } -7, & F_{-9} = {\cdot}1 \text{ and } -6{\cdot}5, & F_{-8} = {\cdot}2 \text{ and } -6, \end{array}\right.$$

$$\left\{\begin{array}{lll} F_7 = 5{\cdot}5 \text{ and } -{\cdot}3, & F_6 = 5 \text{ and } -{\cdot}5, & F_5 = 4{\cdot}5 \text{ and } -{\cdot}7, \\ F_{-7} = {\cdot}3 \text{ and } -5{\cdot}5, & F_{-6} = {\cdot}5 \text{ and } -5, & F_{-5} = {\cdot}7 \text{ and } -4{\cdot}5, \end{array}\right.$$

$$\left\{\begin{array}{lll} F_4 = 4 \text{ and } -{\cdot}9, & F_3 = 3{\cdot}5 \text{ and } -1{\cdot}2, & F_2 = 3 \text{ and } -1{\cdot}5, \\ F_{-4} = {\cdot}9 \text{ and } -4, & F_{-3} = 1{\cdot}2 \text{ and } -3{\cdot}5, & F_{-2} = 1{\cdot}5 \text{ and } -3, \end{array}\right.$$

$$\left\{\begin{array}{l} F_1 = 2{\cdot}6 \text{ and } -1{\cdot}8, \\ F_{-1} = 1{\cdot}8 \text{ and } -2{\cdot}6, \end{array}\right. \text{ and } F_0 = 2{\cdot}2 \text{ and } -2{\cdot}2.$$

117. Calculate the maxima positive and negative shearing forces at intervals of one foot, during the transit of the load system shown in fig. 81; see also figs. 56—65.

Here $2c = 42$ feet; $W_1 = 5$, $W_2 = 5$, $W_3 = 11$, $W_4 = 12$, and $W_5 = 9$ tons; the distances between each pair of weights beginning at the left end are respectively 5, 8, 10, and 7 feet.

The slopes of the links are 5, 10, 21, 33, 42; 37, 32, 21, and 9 tons per 42 feet; 42 feet being the span.

The co-ordinates of the joints of the links are—

	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.	Ft.	Tons.
For link *Aaa*...	–21,	0;	–16,	0·6;	–8,	2·5;	2,	7·5;	9,	13;	21,	25.
Bbb....	–21,	– 4·4;	–13,	–2·5;	–3,	2·5;	4,	8;	16,	20;	21,	24·4.
Ccc....	–21,	– 7·5;	–11,	–2·5;	–4,	3;	8,	15;	13,	19·4;	21,	25·5.
Ddd...	–21,	–13·5;	–14,	–8;	–2,	4;	3,	8·4;	11,	14·5;	21,	19·5.
E'e'e'...	–21,	–29;	– 9,	– 17;	–4,	–12·6;	4,	–6·5;	14,	–1·5;	21,	0.

For positive maxima, the locus *Ccc*... is to be taken between the left end of span and the point $x = -1$, and the locus *Aaa*... from that point to the right end of span; for negative maxima, the locus *E'e'e'*... is to be taken for the whole span.

118. An advancing load in length not less than the span, and of uniform intensity 3 tons per foot, passes over a beam 42 feet span. Find the maxima shearing forces positive and negative at the points $x = 21$, 14, 7, and 0.

		Positive. Tons.	Negative. Tons.	Range. Tons.
Ans.	F_{21}	... 63.0	... 0·0	... 63·0.
	F_{14}	... 43·75	... 1·75	... 45·5.
	F_7	... 28·0	... 7·0	... 35·0.
	F_0	... 15·75	... 15·75	... 31·5.

Note.—If the actual load be 3 tons per foot, these results are in terms of a moving load; if, however, we have increased the actual load to get its equivalent dead load, the results are in terms of a dead load. Since the span is over 40 feet, the difference is not very great; probably if the actual load were $2\frac{1}{2}$ tons per foot, the equivalent dead load would be 3 tons per foot, and in this case the results above would be for the dead load equivalent to an actual advancing load of $2\frac{1}{2}$ tons per foot.

119. In the previous example, find the maximum value of the positive and negative shearing forces at the point $x = 7$, directly.

Let the segment to the right of the point be loaded; the amount of load will then be $28 \times 3 = 84$ tons; suppose this concentrated at the centre of the loaded segment and

$$+F_7 = P = (84 \div 4) \times 14 = 28 \text{ tons};$$

again let the segment to the left of the point be loaded, and

$$-F_7 = Q = (42 \div 42) \times 7 = 7 \text{ tons}.$$

120. The beam of example 118 is subject to an advancing dead load of 3 tons per foot, and an uniform dead load of 1 ton per foot of span. Find the maximum positive and negative value of the shearing force at intervals of 7 feet for the combined load, and the range at these points.

For the uniform load alone, the shearing forces at these points beginning at the left end are—

$$21, 14, 7, 0, -7, -14, -21 \text{ tons}.$$

Taking the algebraic sums of these quantities and the results given for example 118, we find that—

Ans. $F_{21} = 84$, $-F_{21} = 0$; $F_{14} = 57\frac{3}{4}$, $-F_{14} = 0$; $F_7 = 35$, $-F_7 = 0$; $F_0 = 15\frac{3}{4}$, $-F_0 = 15\frac{3}{4}$; $F_{-7} = 0$, $-F_{-7} = 35$; $F_{-14} = 0$, $-F_{-14} = 57\frac{3}{4}$; $F_{-21} = 0$, $-F_{-21} = 84$ tons.

Between the two points $x = \pm 7$, we have both a positive and negative maximum; to the left of $x = 7$ the stress is always positive, to the right of $x = -7$ it is always negative.

The range of the shearing force at each of these points is respectively—

$$84, 57\tfrac{3}{4}, 35, 31\tfrac{1}{2}, 35, 57\tfrac{3}{4}, \text{ and } 84 \text{ tons}.$$

121. Find for the previous example the points between which the shearing force is sometimes positive and sometimes negative; and find the co-ordinates of the apex A_1. See fig. 85.

Ans. $OK = 21 - 42\{\sqrt{(\frac{1}{9}+\frac{1}{3})} - \frac{1}{3}\} = 7$ feet; $OL = -7$ feet.

For the apex A_1,—

$$OS = x = c + \frac{u}{w}2c = 35 \text{ feet on horizontal scale;}$$

$$SA_1 = y = \left(\frac{u^2}{w} + u\right)c = 28 \text{ tons on vertical scale.}$$

By using the co-ordinates of A_1, the graphical construction can be made with greater accuracy, more especially if the points A_1 and D are situated *near* each other.

122. A beam 24 feet span bears a load of 6 tons uniformly distributed, and is subject to a rolling load of 7 tons. Find the amounts and the range of the shearing forces at intervals of 2 feet.

Since the span is a little greater than 20 feet, the dead rolling load equivalent to the actual rolling load of 7 tons will be, say, 12 tons.

For the uniform load, F varies uniformly from 3 tons at the left end of span, to -3 tons at the right; for the rolling load, F varies uniformly from 12 tons to 0, positive from left to right, and negative from right to left; and we have for both loads—

$$F_{12} = 15,\ -F_{12} = 0;\quad F_{10} = 13{\cdot}5,\quad -F_{10} = 0;\quad F_8 = 12,$$
$$-F_8 = 0;\quad F_6 = 10{\cdot}5,\ -F_6 = 1{\cdot}5;\qquad F_4 = 9,\ -F_4 = 3;$$
$$F_2 = 7{\cdot}5,\ -F_2 = 4{\cdot}5;\quad F_0 = 6,\ -F_0 = 6 \text{ tons.}$$

The results for the right half of the span are similar to the above, the signs alone being changed.

The range at intervals of 2 feet for the left half of the span, beginning at the point of support, is—

15, 13·5, 12, 12, 12, 12, and 12 tons.

123. A beam 24 feet span bears an uniform dead load of 100 lbs. per foot, and is subject to an advancing load as long as the span, and of intensity 500 lbs. per foot. Find the shearing forces, and their range, on the left half of span, at intervals of 2 feet. Find the *critical* point K, and the co-ordinates of the apex A_1. See fig. 87.

Since the span is a little greater than 20 feet, the dead advancing load equivalent to the actual advancing load of 500 lbs. per foot will be, say, 800 lbs. per foot.

Taking this value, we have—

$$u = 100, \text{ and } w = 800 \text{ lbs}; \; c = 12 \text{ feet}.$$

Substituting in equation (2), page 198;—

$$F_{12} = 10800, -F_{12} = 0; \; F_{10} = 9067, -F_{10} = 0; \; F_8 = 7467, -F_8 = 0; \quad F_6 = 6000, -F_6 = 0; \quad F_4 = 4667, -F_4 = 667; \; F_2 = 3467, -F_2 = 1467; \; F_0 = 2400, -F_0 = 2400 \text{ lbs.}$$

The range at intervals of 2 feet beginning at left end of span is 10800, 9067, 7467, 6000, 5334, 4934, and 4800 lbs.

$$OK = 6 \text{ feet}; \; OS = 15 \text{ feet}; \; SA_1 = 1350 \text{ lbs.}$$

Application to Framed Beams of Uniform Strength.

Framed beams are built of pieces either freely jointed, or so slightly connected at their joints that they may be considered as freely jointed. If the load be applied at the joints, each piece is either a strut or a tie; and if the load be applied at intermediate points on, or distributed over a portion of, a piece, it (the load) is to be replaced by a pair of equivalent forces at the ends, evidently equal and opposite to the supporting forces of that piece looked upon as a short beam. Such a piece has two duties to perform, viz., to resist the bending moments and shearing forces as a small beam, and to act as a strut or tie in the built beam considered as loaded at the joints only; by making such pieces short enough, the duty they have to discharge as beams can be made so small compared to what they have to perform as struts or ties, that when designed to fulfil the latter duty they will also be able to fulfil the former; by making the pieces continuous at their joints, they are greatly strengthened for their duties as short beams.

If a cross section cuts not more than three pieces, the

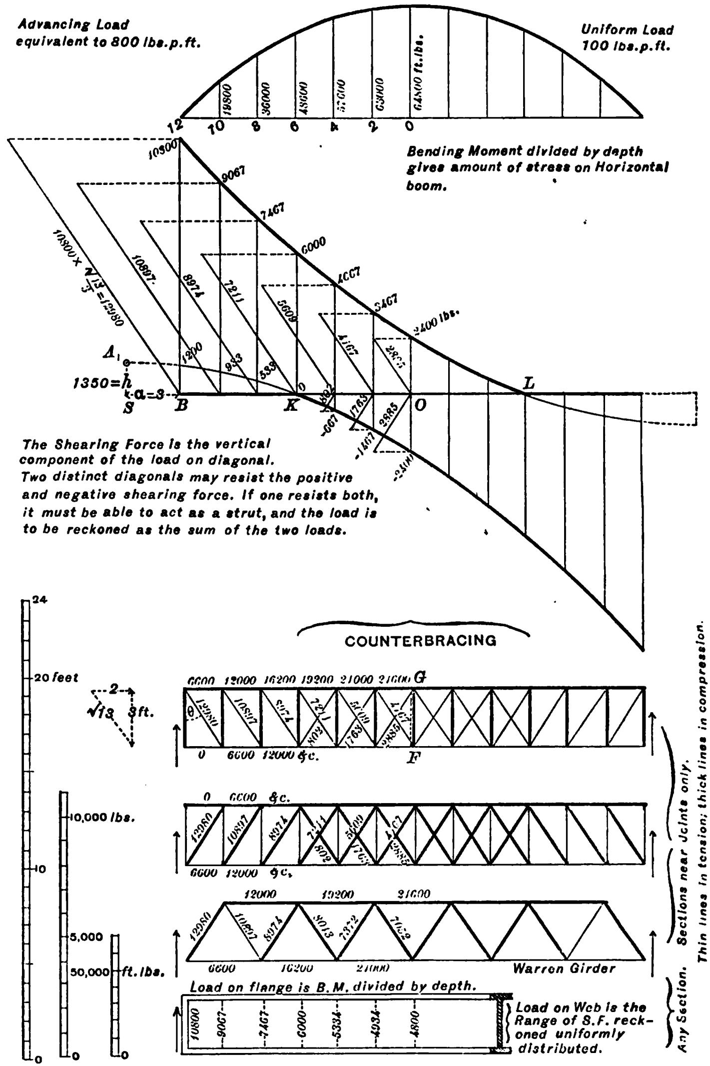

Advancing Load
equivalent to 800 lbs.p.ft.
Uniform Load
100 lbs.p.ft.
Bending Moment divided by depth
gives amount of stress on Horizontal
boom.
1350=h
a=3
The Shearing Force is the vertical
component of the load on diagonal.
Two distinct diagonals may resist the positive
and negative shearing force. If one resists both,
it must be able to act as a strut, and the load is
to be reckoned as the sum of the two loads.
COUNTERBRACING
20 feet
10,000 lbs.
5,000
50,000 ft.lbs.
Warren Girder
Load on flange is B.M. divided by depth.
Load on Web is the
Range of S.F. reck-
oned uniformly
distributed.
Any Section.
Sections near Joints only.
Thin lines in tension; thick lines in compression.

Fig. 87.

unknown stresses are not more than three in number, and the three conditions of equilibrium, page 17, enable us to calculate the stress on each; or using the bending moment and shearing force diagrams, we can readily apply the conditions given on page 18 to as many sections as necessary, and so design the whole beam. The sections chosen for this purpose should lie just to one side of the joints.

For any given system of loading, a beam is said to be of uniform strength when the cross section of each piece is such, that the ratio of the ultimate or proof resistance of the material and the tension or compression which it has to bear is constant; an allowance being made, if necessary, for pieces which also act as small beams.

The term Flanged Girder is employed to denote all girders consisting of a web and one flange, or of two flanges connected together either by a continuous web or by open lattice work; in bridges one at least of the flanges is usually straight, and also horizontal.

In figs. 8 and 9, page 16, the stress at A is horizontal and is denoted by p_a; if the flange is thin, as is often the case in iron bridges, this stress is sensibly constant; and since the intensity of the stress to which a piece is exposed should not exceed the strength of its material, we have

$$p_a \leqq f \text{......................} (1.)$$

where f is the working or proof stress as may be desired; and if t = amount of stress on the horizontal flange, and S = the cross sectional area of that flange, then

$$t \leqq S.f \text{......................} (2.)$$

When there is only one set of triangles, as is shown in the lowest of the three systems of bracing, fig. 87, and also in the two upper systems if we neglect the counter-bracing, the amount of stress on the straight boom may be found thus;—Take a cross section at a point, say F, just on either side of a joint in the boom opposite the straight boom (in fig. 87, both booms are straight); take moments round the point F, and since we neglect the counter-brace, the only member not passing through the point F is the upper boom; the product $t.h$ gives the moment where h is the depth of the beam at the point; this is equal to the bending moment, and we have by substituting the value of t given in equation 2,

$$t.h = S.f.h = M \quad \text{......................} (3.)$$

The quantity M is different at different points of the beam, and f is a constant quantity; if the above equation is to be fulfilled for every point, we make $S.h$ vary as M; in practice one of these two factors is generally kept constant, and the other is made to vary.

If we make h *constant*, then both booms are horizontal, and S varies as the bending moment; hence the bending moment diagram gives, upon a suitable scale, the area of the boom at each point. The vertical component of the stress on any diagonal is the amount of the shearing force at that end of the diagonal where the shearing force is greatest; hence if τ be the stress on a diagonal, F the shearing force at the end of the diagonal where it is greatest and as given by the shearing force diagram, and θ the angle made by the diagonal with the vertical, then

$$\tau = F \sec \theta \quad \text{......................} (4.)$$

The depth h is chosen from $\frac{1}{8}$th to $\frac{1}{14}$th of the span to ensure stiffness; in fig. 87, h is taken at 3 feet, that is $\frac{1}{8}$th of span, and that figure shows how the stresses on the booms and on the diagonals are found. Between the points K and L, counter-bracing is required; this is accomplished in the two upper girders by introducing *pairs* of diagonals between these points, both being ties or both struts as the case may be; the one diagonal resists the positive, the other resists the negative shearing force. In the Warren girder, the third in the figure, one diagonal resists both the positive and negative shearing force, the one being applied suddenly after the other; each diagonal should be designed to bear the stress due to the sum of these stresses. The lowest girder shown in the figure is one with thin flanges and a thin continuous web; part of the bending moment is resisted by the web and part of the shearing is resisted by the flanges, these parts however are small; practically the flanges are considered, as in the case of open beams, to resist the whole of the bending moment, and the web is considered to resist the whole of the shearing force; an approximate result is obtained if we consider that the shearing force is uniformly distributed over the cross sectional area of the web.

If we make S *constant*, then h varies as M; and, fig. 88,

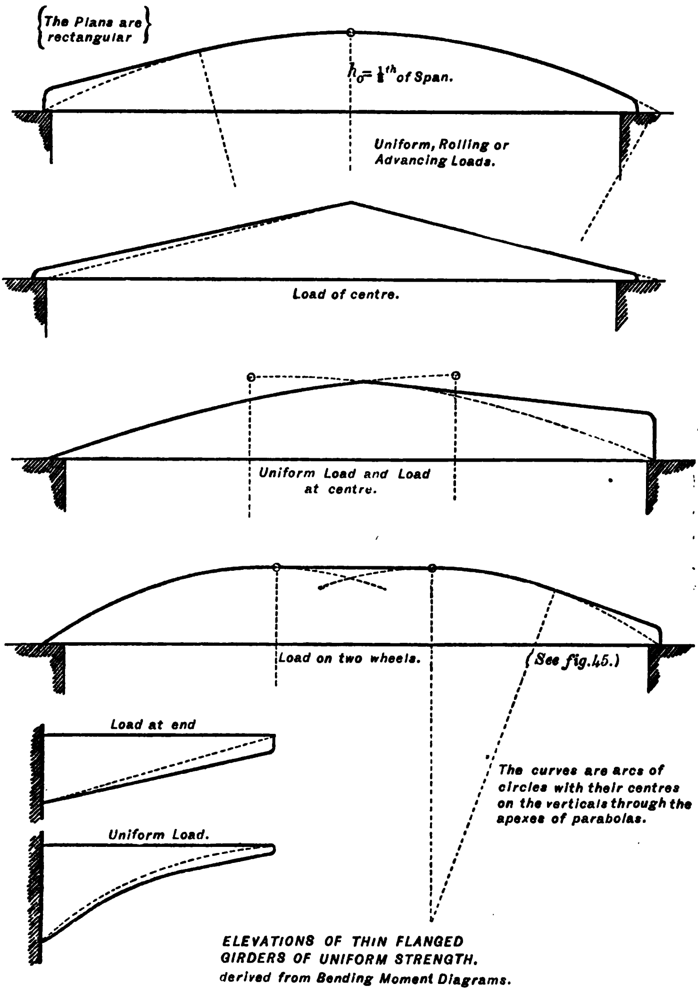


Fig. 88.

the elevation of the beam will correspond with the bending moment diagram; h_0 the depth of the beam at the centre is, as in the previous case, to be taken sufficient to ensure stiffness. The curved boom will bear a share of the shearing force; this compounded with the stress on the horizontal boom at the same section will give the resultant stress on the curved boom. It is usual in practice to make the area of the curved and of the straight booms uniform, and to make the diagonals sufficiently strong to resist the whole of the shearing force as in the previous case. Where the curved member slopes considerably, as in the Bowstring Girder, it is made sufficiently strong to bear the whole shearing force, the diagonals being intended for another purpose, viz.: to distribute partial loads in a sensibly uniform manner.

The theoretical elevations reduce to a height zero at the ends, and so give no material to resist the shearing force at the point where it is greatest; sufficient material is generally allowed at the ends either by making the span of the girder exceed the clear span, or by departing from the theoretical form along a tangent near the end. Further, whatever the curves may be, and, as we have seen, they are generally parabolas, they are usually replaced by circles which nearly coincide therewith; when the figure passes from one curve to another, the passage is made along a tangent as will be seen on some of the figures. Approximate forms consisting entirely of straight lines enveloping the bending moment diagram, are sometimes adopted.

Moment of Resistance to Bending of Rectangular and Triangular Cross Sections.

The Moment of Resistance to Bending we have defined as the moment of the total stress upon the cross section about any point in it; and this we have shown, figs. 7, 8, 9, to be equal to the couple which is the moment of the normal stress on the cross section.

The stress, fig. 8, might be artificially produced by building on the portion $O'A$, columns of a material tending to gravitate towards the left; and on the portion $O'B$, columns gravitating towards the right. These columns standing on very small bases, being of uniform density, and of the proper height to produce the intensity at each point, will, if we suppose them to become one solid, form a wedge with a stepped or notched sloping surface; the more slender the columns are, the more accurately do they give the stress at each point, and the smaller are the notches on the wedge; hence *two right wedges* exactly represent the normal stress on the cross section. Such a stress is called an *uniformly varying stress.* Taking the density of the wedges as unity, the height of one will be expressed by p_a, and the other by p_b; the volumes of the wedges will give the two normal forces, fig. 10, the resultants of the thrusts and tensions respectively; these forces are equal, the volumes of the two wedges must therefore be equal, and the position of the neutral axis of the cross section is thus determined. Further, each wedge, instead of distributing its weight over its base, may be supposed to stand on the point below its centre of gravity; this enables us to find the positions of the normal forces, fig. 10, and gives us the arm of the couple; if we multiply the volume of either wedge by this arm, we have **M** the moment of resistance to bending. For a

Rectangular cross section, the neutral axis is at the centre since the wedges are equal, and p_a equals p_b; and if the common volumes of the right wedges be represented by V, then

$$V = \tfrac{1}{2} p_a \times \tfrac{1}{2} bh = \tfrac{1}{4} p_a bh;$$

where b is the breadth and h is the depth of the beam, as shown in fig. 1. Each wedge stands on a rectangular base, so that the point on the cross section below its centre of gravity is distant from O' by two-thirds of $O'A$, or $\frac{1}{3}h$; hence the arm of the couple is $\frac{2}{3}h$, and

$$\mathbf{M} = \tfrac{1}{4} p_a bh \times \tfrac{2}{3}h = \tfrac{1}{6} p_a bh^2.$$

By increasing the bending moment we can increase **M** till p_a becomes equal to f, the resistance of the material to direct tension or thrust, but no further; because if p_a becomes greater than f, the fibres at the skin will be injured; hence

$$\mathbf{M} = \tfrac{1}{6} fbh^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

is the ultimate, proof, or working resistance to bending, according as f is the ultimate, proof, or working strength of the material. Since f is generally expressed in tons or lbs., per sq. *inch*, it is necessary to express b and h also in *inches*; in which case **M** will be in *inch* tons or lbs.; on the other hand, M is generally expressed in *foot* tons or lbs.; and it is well to observe that, if such be the case, M requires to be multiplied by 12 to reduce it to *inch* tons, or lbs., before equating to **M**.

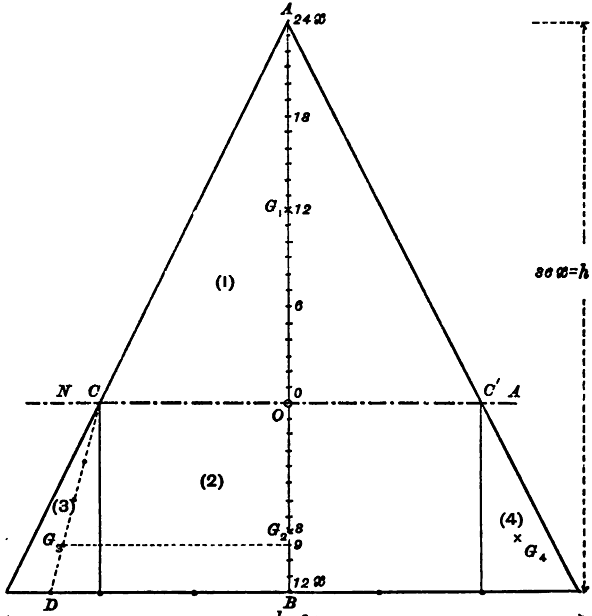
For all cross sections, if b and h are the dimensions of the circumscribing rectangle, the equation giving the moment of resistance is of the same form, but the numerical coefficient assumes values other than $\frac{1}{6}$, as will be proved hereafter. Rankine uses n for the value of this constant, which he calls the numerical coefficient of the moment of resistance to bending of any cross section; and we may put

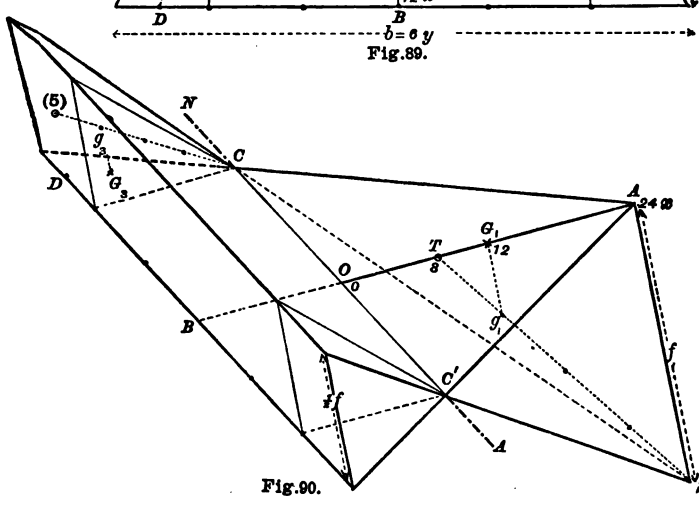
$$\mathbf{M} = nfbh^2.$$

We shall now verify this for a triangular cross section; an isosceles triangle is taken, but exactly the same result would be obtained for any triangle; the angle A, however, must not be too acute.

Triangular cross section. Figs. 89 and 90. Assume that the neutral axis passes through O the centre of gravity of the triangle, and divide the part below into three areas as shown; put S and V with suffixes for the areas and volumes of the different parts, put l for the leverages about the neutral axis of their centres of gravity, and for convenience suppose $h = 36x$, and $b = 6y$.

V_1 is a pyramid on S_1 as a base, and let f be its height; V_2 is a right prism of height $\frac{1}{2}f$; V_3 is a pyramid on S_3 as base and of height $12x$. Now the centre of gravity of a pyramid is on the line joining the apex with the centre of


A
24 x
18
G1
12
(1)
6
0
N
C
C'
A
O
(2)
(3)
G3
G2
8
9
12 x
(4)
G4
36 x=h
D
B
b=6 y
Fig. 89.


(5)
N
g3
G3
C
D
A
24 x
G1
T
12
O
8
0
B
g1
f
C'
f
f
A
S
Fig. 90.

base, and at three-quarters of the length of that line from the apex; and

$$S_1 = \tfrac{1}{2}.24x.4y = 48xy;\ S_2 = 12x.4y = 48xy;\ S_5 = \frac{f}{2}y.$$

$$V_1 = \tfrac{1}{3}fS_1 = 16fxy; \qquad V_2 = \tfrac{1}{2}.\frac{f}{2}.S_2 = 12fxy;$$

$$V_3 = \tfrac{1}{3}.12x.S_5 = 2fxy = V_4;$$

hence $V_1 = V_2 + V_3 + V_4$, and therefore the assumption as to the position of the neutral axis is correct.

Now $l_1 = 12x, l_2 = 8x, l_3 = l_4 = 9x$; and

$$\begin{aligned} \mathsf{M} &= V_1l_1 + V_2l_2 + V_3l_3 + V_4l_4 \\ &= (192 + 96 + 18 + 18)fyx^2 = 324fyx^2. \end{aligned}$$

Substituting $y = \frac{b}{6}$, and $x^2 = \left(\frac{h}{36}\right)^2 = \frac{h^2}{1296}$, we have

$$\mathsf{M} = \frac{1}{24}fbh^2 \quad \text{.................................} (2.)$$

so that for a triangular section, $n = \frac{1}{24}$.

Solid beams are sometimes made of *uniform section;* that is, at the point of maximum bending moment the section is made sufficient to resist the bending moment, and this section is adopted along the entire length; at every other section, therefore, the beam is too strong. This is frequently done with small timber beams cut out of one piece, because the material in excess, even if cut away, would be lost; the weight of this excess is very little since timber is light, and probably the cutting of it away would add to the expense of the beam.

When the section of an uniform beam is designed to resist the maximum bending moment, it is generally many times more than sufficient to resist the maximum shearing force.

Solid rectangular beams are designed of uniform strength to economise material, and to reduce the weight when the material itself is heavy.

Examples.

124. Find the working moment of resistance to bending of a rectangular section, 10 inches deep and 3 inches broad, the working strength of the material being 4 tons per sq. inch.

$$\mathbf{M} = nfbh^2 = \tfrac{1}{6} . 4 . 3 . 100 = 200 \text{ inch-tons.}$$

125. Find the same for an isosceles cross section inscribed in the above rectangle, and with the base horizontal.

Ans. $\mathbf{M} = nfbh^2 = \frac{1}{24} . 4 . 3 . 100 = 50$ inch-tons.

126. Find suitable dimensions for a cast iron beam 20 feet span, of uniform and rectangular cross section, and subject to a load of 10 tons at the centre.

Taking $h = 20$ inches, a twelfth of the span, to ensure stiffness; $f = 2$ tons per sq. inch; and $\mathbf{M} = M_0 = mWl$, the max. bending moment;

Ans. $\frac{1}{6} . 2 . b . 20^2 = \frac{1}{4} . 10 . (12 \times 20)$. $\therefore b = 4{\cdot}5$ inches.

127. If the breadth be taken at 6 inches, what depth would give sufficient strength?

Ans. $\frac{1}{6} . 2 . 6 . h^2 = \frac{1}{4} . 10 . (12 \times 20)$. $\therefore h = 17{\cdot}3$ inches.

128. If the load is uniformly distributed, and the cross section is a triangle whose base is horizontal and 8 inches broad; find the height of the triangle.

Ans. $\frac{1}{24} . 2 . 8 . h^2 = \frac{1}{8} . 10 . (12 \times 20)$. $\therefore h = 21{\cdot}2$ ins.

129. Find the greatest cross section for a wrought iron beam of rectangular section and 15 feet span, to bear a load of 20 tons uniformly distributed, together with a load of 5 tons at the centre. Take $f = 4$ tons per sq. inch, and $h = 15$ ins. to give sufficient stiffness.

Ans. $\frac{1}{6} . 4 . b . 15^2 = (\frac{1}{8} . 20 . 15 + \frac{1}{4} . 5 . 15) \times 12$.
$\therefore b = 4{\cdot}5$ ins.

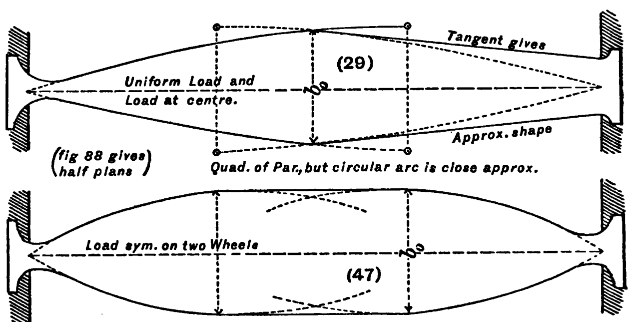
130. Taking the depth one-twelfth of the span, and $f = 4$ tons per sq. inch, find the breadth for a wrought iron beam of rectangular section, to resist the maximum bending moment in example 79.

Ans. $\frac{1}{6} \times 4 \times b \times 20^2 = 32 \times 12$. $\therefore b = 1{\cdot}44$ in.

Rectangular beam of uniform strength and uniform depth.—In this case the breadth is varied so as to make the cross section at every point just sufficient to resist the bending moment. It is evident that the elevation is a rectangle, while the plan will vary according to the load; the plan however is to be symmetrical about a centre line. The theoretical shape of the plan so designed has to be departed from near the ends, so as to make the end sections large enough to resist the shearing force; and some additional material is required at the ends of the beam to give it lateral stability; now

$$nfbh^2 = M; \therefore b = \frac{M}{nfh^2},$$

or b is proportional to M, since n, f, and h are constant



PLANS OF RECT. BEAMS OF UNIFORM STRENGTH & UNIFORM DEPTH SAME AS B. M. DIAGRAMS.
Fig. 91.

quantities. Hence the theoretical half plan is the bending moment diagram reduced so that its highest ordinate is equal to $\frac{1}{2}b_0$, the half breadth of the cross section which can resist the maximum bending moment; see figs. 88 and 91. As has previously been explained, the parabolas being very flat are readily replaced by arcs of circles with their centres on the axes of the parabolas.

In designing such beams, then, the uniform depth is fixed as a fraction of the span from an eighth to a fourteenth to ensure the required degree of stiffness; b_0, the breadth of the cross section where the bending moment is a maximum, is calculated to make that section sufficiently strong, exactly as in the preceding examples; then the bending moment diagram is reduced till its highest ordinate is $\frac{1}{2}b_0$, and drawn on both sides of a central line. All the curves may now be replaced by circular arcs, and all sudden changes of breadth bridged over by tangents; the curves are to be departed from, near the ends, to make the sections there strong enough to resist the shearing force. Otherwise, the breadths may be calculated at a number of sections and plotted, and a fair curve drawn through them; &c.

Definition.—If a locus be drawn whose ordinates are proportional to the square roots of those of a given locus, the new locus is the given locus *degraded.*

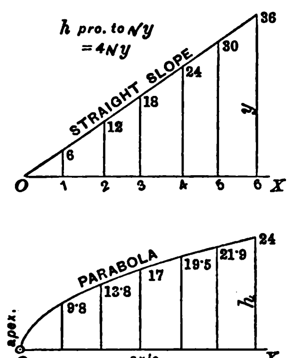


Fig.92.

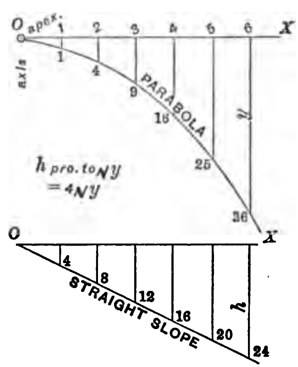


Fig.93.

Theorem.—Fig. 92. If a locus which is a straight slope be degraded, the new locus is a parabola with its axis horizontal, and its apex at the point where the loci cross the horizontal base.

For the straight slope, $y = ax$, if we take the origin at the point where it cuts the base; if h be the corresponding ordinate of the new locus,

$$h^2 \propto y \text{ or } ax; \text{ that is, } x \propto h^2,$$

a parabola with axis parallel to OX, and apex at the origin.

Theorem.—Fig. 93. If a locus, which is a parabola with axis vertical and apex on the base, be degraded, the new locus is a straight slope crossing the base at the apex.

For a parabola with origin at apex, $y = mx^2$; but $h^2 \propto y$; therefore

$$h \propto \sqrt{m}\, x,$$

a straight line passing through the origin.

Theorem.—Fig. 94. If a locus which is a parabolic segment be degraded, the new locus is an ellipse with its centre at the middle of the base, the base being a major or minor diameter.

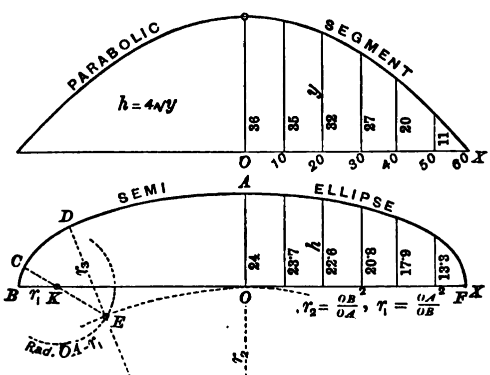


Fig. 94.

Taking the origin at the centre of base, we have for the equation to a parabola, $y = m(c^2 - x^2)$, where $2c$ is the base. Let $h' = \mu h$, where μ is a quantity such that $h'^2 = y$; then since $h^2 \propto y$, we have

$$\frac{h'^2}{m} = c^2 - x^2, \text{ or } \frac{h'^2}{m} + x^2 = c^2;$$

$$\therefore \frac{\mu^2 h^2}{mc^2} + \frac{x^2}{c^2} = 1,$$

the central equation to an ellipse whose semi-diameters are c and $\frac{c\sqrt{m}}{\mu}$.

Rectangular beam of uniform strength and uniform breadth.—It is evident that the plan is a rectangle; and since $nfbh^2 = M$, we have

$$h \text{ proportional to } \sqrt{M}.$$

Hence the theoretical elevation of the beam is obtained by degrading the bending moment diagram, so that the derived figure has its highest ordinate equal to h_0, the depth required to ensure stiffness; b is then to be made sufficient to resist the maximum bending moment.

The figures in brackets refer to the corresponding bending moment diagrams.

Fig. 95 shows what the bending moment diagrams (22), (23), (25), (27) become when degraded by the preceding theorems. The elliptical elevation of a beam for an uniform load is readily struck from three centres, as shown in fig. 94; AD from a centre on AO and with a radius $r_2 = \frac{OB^2}{OA} = 60^2 \div 24 = 150$; BC from a centre K with radius $r_1 = \frac{OA^2}{OB} = 24^2 \div 60 = 10$ nearly; E is found by drawing OE from the first centre, and a circle about K with radius $r_3 = h_0 - r_1 = 24 - 10 = 14$; further, if we choose we may retain the single circular arc AD, and depart along the tangent at D. For a beam with the load at the centre, the two parabolas may be replaced by their tangents; this gives an approximate form, and the beam will now consist of two straight portions tapering so that the depth at each side is $\frac{1}{2}h_0$, h_0 being the depth at the middle, see fig. 14; the area of the theoretical elevation is two-thirds, while that of the approximate elevation is three-quarters, of a rectangle of height h_0; these areas are as 8 to 9, hence the volume of the approximate form is only one eighth in excess of the theoretical one, and the additional material is well placed to resist shearing; the approximate form is in some cases preferable, since it has the great advantage of straight boundaries. The same remarks apply to the elevation of a

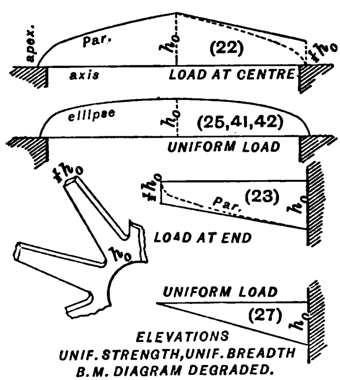


Fig. 95.

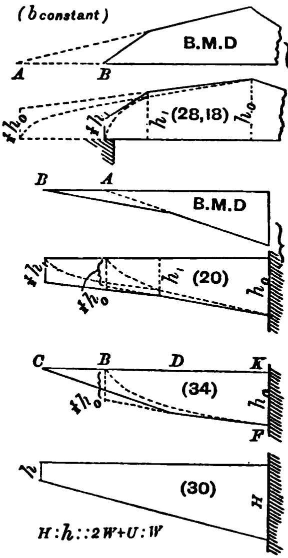


Fig. 96.

cantilever with the load at the end; on this principle spokes of wheels, when of uniform thickness, taper to half the depth from boss to tyre.

In fig. 96 are shown the effects of degrading such bending moment diagrams as (18), (20), (28), diagrams consisting of straight slopes. Producing the slopes to meet the base as at *A*, *B*, &c., the theoretical elevation for beams of uniform strength and uniform breadth will consist of a series of parabolas with their apexes at A, B, &c., and intersecting on the lines of action of the concentrated weights.

An approximate elevation is to be derived thus:—Lay up h_0 at the point of greatest bending moment, the proper fraction of the span to ensure stiffness, and calculate the breadth of that cross section as in the preceding examples, so as to give the proper resistance to bending there; from the top of h_0, draw a tangent to the parabola whose apex is A, that is draw the line which intercepts $\frac{1}{2}h_0$ on the vertical through A, and it will cut off h_1 on the line of nearest weight; from the top of h_1 draw a slope to cut off $\frac{1}{2}h_1$ on the vertical through B, the apex of second parabola and the point where the second slope of the bending moment diagram meets the base, this will be parallel to the tangent to that parabola; &c., &c.

It is evident then that these approximate elevations for concentrated loads consist of straight lines, each sloping at *half the rate* of the corresponding side of the bending moment diagram; from this fact they are readily drawn thus—From the highest point in the bending moment diagram draw a line at half the slope of the adjacent side till it cuts the line of the weight nearest to that point; from the point thus found draw a line to cut the next weight at half the slope of the corresponding side of the bending moment diagram; &c.; lastly, reduce the ordinates so that the highest is h_0.

When fig. 34 is degraded, from K to D it will be a portion of the parabola whose apex is at B the centre of load, and from D to C a straight slope; the approximate form is a straight slope to D the end of load, and which when continued tapers to $\frac{1}{2}h_0$ at B; then from D a straight slope tapering to zero at the free end. On degrading the two parts of fig. 30 separately, that for U will be a taper from a at the fixed end to zero at the free end; while the approximate form for W will be a taper from b at the fixed end to $\frac{1}{2}b$ at the free end; hence the approximate elevation for the combined load is a taper from $(a + b)$ at the fixed end to $\frac{1}{2}b$ at the free end; substituting for a and b their values, we have $H : h :: 2W + U : W$.

Fig. 97 shows (29) and (47) degraded, the quadrants of parabolas becoming quadrants of ellipses.

In those bending moment diagrams which consist of a series of intersecting parabolic segments all of one modulus, the degraded figures will consist of a series of intersecting semi-ellipses on the same base line, intersecting on the same

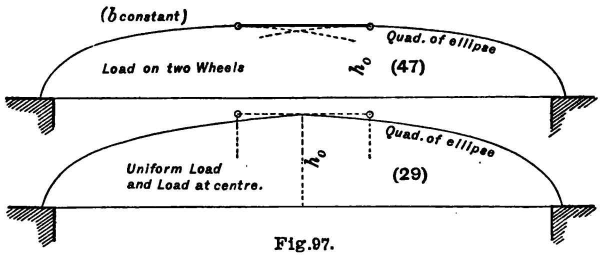


Fig. 97.

verticals, and having their heights proportional to their bases; this follows from theorem at fig. 94, or at once from the fact that the heights of the parabolic segments were proportional to the squares of their bases; in some figures it will be necessary to bridge over gaps at junctions of pairs of ellipses. The following is an easy method, fig. 98, of degrading these diagrams, and as an example we will take fig. 51; from S_1 with radius S_1B, describe the circular arc BA_1D; from S_2 with radius S_2D, draw arc DA_2C; draw

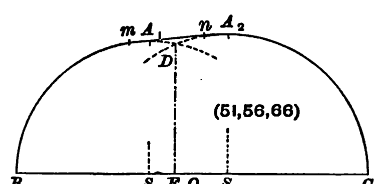


Fig. 98.

$\overline{mn}$ the common tangent to the two circles, and $BmnC$ is the degraded figure; it only remains now to reduce the ordinates of m, A_1, n, A_2, so that the highest is h_0, when $\overline{mn}$

will still be a straight line and a common tangent to $\overline{mB}$ and $\overline{nC}$, which will be arcs of ellipses whose centres are S_1 and S_2.

The same method applies to fig. 56, and in drawing the intermediate elliptic arcs, they will be struck from one centre as they will not extend past the portion AD in fig. 94; the end arcs will be struck from three centres, as also shown in fig. 94. In fig. 98 we may depart along tangents to the circles near B and C; in which case, when we reduce the ordinates, *all* the elliptic arcs may be struck from one centre, and the elevation will consist of straight lines and circular arcs; the radii of the circular arcs being calculated like r_2 in figure 94, and their centres lying in the axes of the parabolas in the bending moment diagram.

The same construction degrades fig. 39, and here no bridging over is required; thus from D_1 with radius D_1B describe a circular arc BE; from D_2 with radius D_2E describe arc EF; &c., &c., and reduce the ordinates so that the highest is h_0.

When fig. 33 is degraded, BAE becomes an elliptic arc, and EC a parabola with its apex at C and its axis horizontal; the two curves have a common tangent at E; the parabola EC will be replaced by the tangent at E, which will cut off on the vertical at C one-half of FE; the ordinates of the figure are readily reduced. Otherwise, from D with radius DB describe a circular arc BAE; draw a tangent from E to meet the vertical through C, and reduce the ordinates.

In the same way fig. 32 can be degraded.

In degrading such figures as (36), (52), (67), it is to be remembered that the parabolas have not a common modulus. For instance, in degrading (52) each of the parabolas 1 and 2 is to be replaced by a semi-ellipse, whose height is proportional to the product of the base of the corresponding parabolic quadrant into the square root of its modulus; while BEC is to be replaced by a semi-ellipse, whose height is proportional to the product of its base into the square root of *its* modulus. Now the common modulus of 1 and 2 is $\frac{R}{2}$, while that of BEC is $\frac{W_2}{2c}$; hence for each semi-ellipse we have—

Height of No. 1 : height of No. 2 : height of ellipse BEC
$:: R . BS_1 : R . CS_2 : W_2 . OB.$

Rectangular beam of uniform strength and similar cross section. —In this case both b and h vary, but they bear to each other a constant ratio; it is evident that the plan and elevation will have the same form. The plan is always to be symmetrical about a centre line; the elevation may either have one straight boundary, or be symmetrical about a centre line.

Since $b \propto h$, then $bh^2 \propto b^3$ or h^3;

now,
$$nfbh^2 = M,$$
$$bh^2 \propto M;$$
$$\therefore\ b \text{ and } h \propto \sqrt[3]{M}.$$

That is, both plan and elevation are derived by drawing a

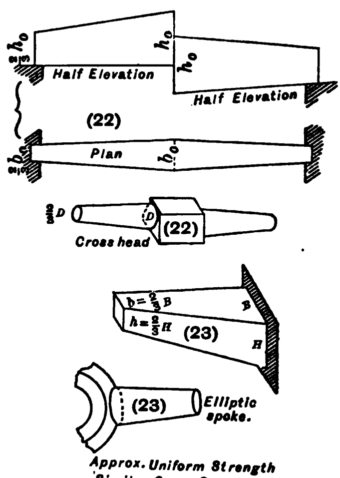


Fig.99.

locus whose ordinates are proportional to the cube roots of those of the bending moment diagram.

For bending moment diagrams with straight slopes, that is, for concentrated loads, the degraded figures were parabolas with axes horizontal, &c.; in the same way, when the new figure is made with its ordinates proportional to the cube roots of the ordinates of the straight slopes, it becomes what is called a cubic parabola; a property of this curve is that the tangent cuts off *two-thirds* of the ordinate upon the vertical through the apex, instead of *one-half* as in the case of the common parabola. Hence figure 95 and figure 96 give approximate elevations and half plans of beams of uniform strength and similar cross sections, if we use $\frac{2}{3}h_0$ instead of $\frac{1}{2}h_0$ in making the construction; or if we draw the tapers at two-thirds of the slopes of the bending moment diagram instead of one-half. Observe that in this case there is a double taper in planes at right angles to each other.

On this principle the crosshead of a piston rod may have a conical taper, so that the diameter at each end may be two-thirds of the diameter at the centre; and the spokes of wheels may have a conical taper from the boss to two-thirds (linear dimensions) at the tyre.

Examples.

131. Design a rectangular cantilever 10 feet long of approximately uniform strength and of uniform breadth, of timber whose working strength is 1 ton per sq. inch; the load is 2 tons at the free end, and $h_0 = 15$ ins., an eighth of the length.

Ans. $nfbh_0^2 = m \,.\, W \,.\, l$

$\frac{1}{6} \,.\, 1 \,.\, b \,.\, 15^2 = (1 \,.\, 2 \,.\, 10)\ 12$, gives $b = 6{\cdot}4$ ins. for uniform breadth; the depth tapers from 15 ins. at fixed end to 7·5 ins. at free end.

132. If an additional uniform load of 4 tons be added, and h_0 be still retained 15 inches; find the value now of b, and of h at the free end.

Ans. At the fixed end the bending moment will be double its former amount, so that b will be doubled; that is, $b = 12{\cdot}8$ ins.; and

$$h_0 : h :: 2W + U : W$$
$$15 : h :: 2 \times 2 + 4 : 2; \therefore h_{10} = 3{\cdot}75 \text{ ins.}$$

133. A wooden cantilever 12 feet long bears 3 tons uniformly distributed on the half next the free end. Design an approximate elevation, supposing the breadth to be uniform, and $f = 1$ ton per square inch; take depth at fixed end as 18 ins.

Ans. $M_0 = 27$ ft.-tons $= 324$ inch-tons.

$\frac{1}{6} . 1 . b . 18^2 = 324; \therefore b = 6$ ins. constant.

Between the fixed end and the end of the load next thereto, the elevation will taper at the rate of one inch per foot, and thence to zero at the free end; that is, $h_0 = 18$, $h_6 = 12$, and $h_{12} = 0$ inches.

134. Design a cantilever for Ex. 131, supposing its section to be a square, taking dimensions to the nearest whole number in inches.

Ans. Put $b_0 = h_0 =$ side of square at fixed end, and equating $nfb^3 = m . W . l$, gives $b_0 = h_0 = 11{\cdot}3$, say 12 inches. The side of the square at the free end is $b_{12} = h_{12} = 8$ inches.

135. Design a beam whose span is 20 feet, of timber whose strength is 1 ton per sq. inch; the beam to be of uniform breadth and approximately uniform strength; the loads are 4, 3, and 4 tons at points 5, 12, and 16 feet respectively from the left end.

Ans. $P = 5$, $Q = 6$ tons; in the bending moment diagram, beginning at the left end, we have the slopes as follows:—

		Vert.		Horiz.	Up to Right.	
$s_1 = P,$	=	5	to	1	,,	,,
$s_2 = P - W_1$	=	1	,,	,,	,,	,,
$s_3 = P - W_1 - W_2$	=	2	,,	,,	down	,,
$s_4 = P - W_1 - W_2 - W_3$	=	6	,,	,,	,,	,,

hence $M_{12} = 32$ ft.-tons maximum.

In the degraded figure let $h_{12} = 32$ parts; from the top of this ordinate the slope down to the left will be half of the corresponding slope in the bending moment diagram; that is, ·5 vert. to 1 hor., and it extends over 7 feet; hence $h_5 = h_{12} - \cdot 5 \times 7 = 28{\cdot}5$ parts. The next slope from this down to left is at the rate 2·5 to 1, and extends over 5 feet; so that $h_0 = h_5 - 2{\cdot}5 \times 5 = 16$ parts. Beginning again at top of h_{12} the slope down to right is at the rate 1 and extends over 4 feet, so that $h_{16} = h_{12} - 1 \times 4 = 28$ parts. The next slope down to right is at the rate 3 and extends over 4 feet; so that $h_{20} = h_{16} - 3 \times 4 = 16$ parts; thus—

$$h_1 : h_5 : h_{12} : h_{16} : h_{20} :: 16 : 28{\cdot}5 : 32 : 28 : 16.$$

Putting $h_{12} = 20$ ins. a twelfth of the span for stiffness, and reducing the others in the proportion of 20 to 32, we have on the elevation

$$h_1 = 10,\ h_5 = 18,\ h_{12} = 20,\ h_{16} = 17{\cdot}5,\ h_{20} = 10 \text{ inches.}$$

To find the uniform breadth put

$$\tfrac{1}{6} \times 1 \times b \times 20^2 = 384 \text{ inch-tons; and } \therefore b = 5{\cdot}76 \text{ inches.}$$

Area, Geometrical Moment, and Moment of Inertia.

The functions of a plane surface which we require for our investigations regarding the moment of resistance to bending of a cross section in general, are the Area, Geometrical Moment, and Moment of Inertia.

The *Area* of a rectangle is the product of its two adjacent sides; the area of any other surface is the sum of all the elementary rectangles into which it may be divided. We take it for granted that the area of the triangle, the circle, the ellipse, and parabolic quadrant are respectively half the product of the height into the base, π into radius squared, π into the product of the semi-major and semi-minor diameters, and two-thirds of the product of the circumscribed rectangle.

The area of any figure is quite independent of the position of the axis to which the figure is referred; and the word *area* is conveniently used as a name for a finite plane surface.

DEFINITION.—The *Geometrical Moment* of a surface about any line in its plane as axis, is the sum of the products of each elemental area into its leverage or perpendicular distance from that axis; the leverages which lie to one side of the axis being reckoned positive, and those to the other side negative.

Each elemental area is to be so small that the distance from the axis to every point in it is sensibly the same. If the surface were a plate of unit thickness and unit density, it is evident that the statical moment of the plate would be exactly the geometrical moment of the surface. Suppose the axis, for instance, to be a knife edge upon which the plate rests, the weights of the portions on different sides tend to cause the plate to rotate in opposite directions, and their statical moments are of different sign; the definition shows the geometrical moment of these portions of the surface also to be of opposite sign.

It will be convenient for us always to choose a *horizontal* axis; and if we consider leverages *up* to be positive, then, when the axis is below the area, the geometrical moment is positive; when above it, negative; when the axis cuts the area, it will be positive or negative according as the axis is near one or other edge; and for one position of the axis, cutting the area, the positive and negative products will destroy each other, and the geometrical moment will be zero.

An axis about which the geometrical moment of an area is zero passes through a point called the *Geometrical Centre* of the area. From this it appears that the geometrical centre of an area corresponds with the centre of gravity of a thin plate of uniform thickness and of that area; and for this reason the geometrical centre of an area is often called its centre of gravity.

Theorem.—The Geometrical Moment of a surface about any axis in its plane is equal to the area multiplied by the distance of the geometrical centre from the axis. Fig. 100.

Suppose G the geometrical centre of the surface to be known; through G draw OO' parallel to the axis AA; let s and s' be a pair of elemental areas, one on each side of OO', such that the sum of their geometrical moments is zero; that is $s'.a=s.b$. It is evident that the whole area can be divided into such pairs from the definition of the geometrical centre. The geometrical moment of s' about AA is $s'(d+a)$, that of s is $s(d-b)$, and their joint moment is $(s'+s)d+(s'a-sb)=(s'+s)d$, since the second term is zero. In the same way the moment of each pair is their sum multiplied by d; hence the geometrical moment of the area about AA is the sum of all the pairs into d, that is, the area multiplied by the distance of the geometrical centre from the axis.

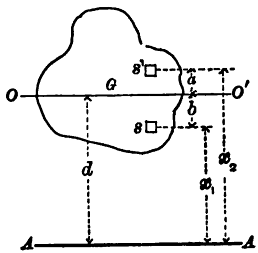


Fig. 100.

Corollary.—The geometrical moment of an area which can be divided into simpler figures whose geometrical centres are known, may be found by multiplying the area of each such figure by the distance of the axis from its geometrical centre and summing algebraically.

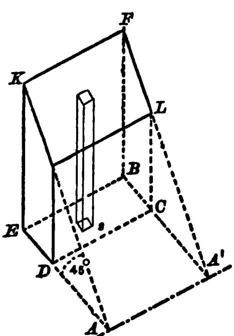


Fig. 101.

Theorem.—The Geometrical Moment of an area about an axis in its plane is expressed by the number which denotes the volume of that portion of a right-angled isosceles wedge whose sloping side passes through the axis, and which stands on the area as base.

Let $AEKFA'$, fig. 101, be the wedge, and AF its sloping side passing through the axis AA'; the angle at A is 45° and that at E is 90°; let $BCDE$ be the area, then the geometrical moment of $BCDE$ relatively to the axis AA' is represented by the volume of $DBCKL$.

If s be an elemental portion of the area $BCDE$, its geometrical moment about AA' is s multiplied by its distance from AA'; but the column of the wedge standing on s is sensibly a parallelipiped whose height is the same as the distance of s from AA', and the volume of that column expresses the geometrical moment of s; hence the volume of $DBCKL$, a portion of the isosceles wedge, is the geometrical moment of $BCDE$ about AA'.

When the axis cuts the area, the plane sloping at 45° will form a wedge *above* one portion and *below* the other; by considering these of different signs, their algebraic sum is still the geometrical moment. For example, if we wish to find the geometrical moment of the triangle, fig. 90, about the axis NA; let $f = \frac{2}{3}h$, then the plane will slope at 45°; in the figure, the volume on one side of the axis is equal to the volume on the other, that is, the geometrical moment is zero; the axis NA must pass therefore through the geometrical centre.

To find an axis passing through the geometrical centre of a plane area then, it is only necessary to draw a plane sloping at 45° which will cut off an equal volume of the wedge above and below; the intersection of this plane with the area will be the axis required.

By similar triangles a plane through that axis at *any* slope will cut off equal volumes above and below; the wedges which represent the normal stress, fig. 8, require to be equal, hence—*the neutral axis of a cross section passes through the geometrical centre (or centre of gravity) of the cross section.*

The distance of the neutral axis from the furthest away skin is, in each cross section, a definite fraction of h the depth of the circumscribing rectangle; for instance, for a triangular cross section, figs. 89, 90,

$$OA = \tfrac{2}{3}h.$$

Rankine expresses this generally thus—

$$OA = m'h$$

where *m' is the fraction which the distance from the neutral axis to the farthest away skin is of the depth;* for all cross sections, symmetrical above and below, as a rectangle, ellipse, hollow rectangle, &c., $m' = \frac{1}{2}$.

DEFINITION.—The *Moment of Inertia* of a surface, about a line in its plane as axis, is the sum of the products of each elemental area into the square of its distance from the axis.

Whether the horizontal axis intersects, or is below or above the area, the moment of inertia will be positive; for, though the leverage be negative, the square of that quantity is always positive; it is also impossible that the sum can ever be zero. When the horizontal axis is at a great distance below, the moment of inertia is very great, since the leverage of each element is great; as the axis approaches, the moment of inertia decreases; when the axis has passed above the area and recedes, the moment of inertia again increases; hence for one position of the horizontal axis, the moment of inertia was less than when that axis was in any other position.

Theorem.—The moment of inertia of a surface about any axis in its plane equals that about a parallel axis through its geometrical centre, together with the product of the area into the square of the deviation of the axis from the centre.

Let s and s', fig. 100, be a pair of elemental areas, the sum of whose geometrical moments about OO' is zero; and let x_1 and x_2 be their leverages about AA respectively. The sum of their moments of inertia about AA is

$$\begin{aligned} s'x_2^2 + sx_1^2 &= s'(d + a) + s(d - b)^2 \\ &= (s' + s)d^2 + 2(s'a - sb)d + (s'a^2 + sb^2) \\ &= (s' + s)d^2 + (s'a^2 + sb^2), \text{ since } (s'a + sb) = 0. \end{aligned}$$

Summing the left side for all pairs we have the moment of inertia of the area about AA. The first term on the right side is the area of each pair into the square of the deviation; and the sum of these for all pairs is the area into the square of the deviation; the second term on the right side is the moment of inertia of s and s' about OO', the sum will be the moment of inertia of the whole area about OO'. If I_A and I_O represent the moments of inertia round the axes A and O respectively, then

$$I_A = S.d^2 + I_O,$$

where S is the area of the figure.

COROLLARY.—For any set of parallel axes, the moment of inertia about that axis which passes through the geometrical centre is a minimum; and those axes which give minima values for the moment of inertia intersect at a point.

When we speak of the moment of inertia of a body without specifying with regard to the axis, it is to be understood that the axis is horizontal and passes through the geometrical centre. The point of intersection for those axes which give minima values for the moments of inertia is called the *Centre of Inertia* of the area; this point coincides with the geometrical centre of the area, and with the centre of gravity of a thin plate of that area. Hence we may say shortly—The neutral axis of a cross section of a beam passes through the centre of the section; this centre being called the geometrical centre, the centre of inertia, or the centre of gravity.

Theorem.—The moment of inertia of a plane area about an axis in its plane is expressed by the number which denotes the statical moment of that portion of a right-angled isosceles wedge of unit density whose sloping side passes through the axis, and which stands on the area as base.

To obtain the moment of inertia of the elemental area s, fig. 101, we multiply its area by the square of its distance from AA'; but the height of the column standing on s is equal to that distance. Multiplying s by that equivalent we have the volume of the column, and multiplying again by the leverage we have the statical moment of that column about AA'. Hence summing for all elemental areas we have the moment of inertia of $BCDE$ about AA' equal to the statical moment of $DBCKL$, a portion of the isosceles wedge, standing on $BCDE$, reckoning its density unity. Now $DBCKL$ instead of extending all over its base, may be supposed to stand on the point of $BCDE$ directly below its centre of gravity; the moment of inertia is thus readily found, if we know the volume of the isosceles wedge and the position of its centre of gravity.

We generally wish to find I_0, since we can readily find I_A from it by the theorem on the preceding page. Now for I_0, one portion of the wedge stands below and one above, and on taking statical moments these must be supposed to gravitate in opposite directions.

COROLLARY.—If the plane area be divided into parts, the moment of inertia for the whole will be the sum of the moments for the parts; thus for fig. 89, making $f = \frac{2}{3}h$ so as to make the wedges isosceles, we have

$$I_o = 16fxy \cdot 12x + 12fxy \cdot 8x + 4fxy \cdot 9x = \tfrac{1}{36}\, bh^3,$$

for a triangle about an axis parallel to b, and passing through the *centre.*

For every cross section, I_o will be of the same form, a constant multiplied by the breadth and multiplied by the cube of the depth of the circumscribing rectangle. Rankine puts generally

$$I_o = n'bh^3$$

where n' is the numerical coefficient of the moment of inertia of the cross section about its neutral axis, the other factors being the breadth and cube of the depth of the circumscribing rectangle.

If through the neutral axis of a cross section we draw a plane sloping at 45°, it will form two isosceles wedges, or two portions, one on each side of the plane; the sum of the products got by multiplying the volume of each by the distance of the point *under* its centre of gravity from the neutral axis gives I_o for the cross section; for this purpose we may take each portion as a whole or subdivide it into a number of parts if such is more convenient.

For a rectangular wedge $AEKFBA'$, fig. 101, let $S =$ area of base $AEBA'$, $f =$ height EK, $V =$ volume, $x_0 =$ distance from AA' to the point which is *under* the centre of gravity, then

$$V = \tfrac{1}{2}S.f;\ x_0 = \tfrac{2}{3}\overline{AE}.$$

For an isosceles-triangular wedge $ACC'S$ (fig. 90), let $S =$ area of base ACC', $f =$ height $\overline{AS}$, $x_0 =$ distance from CC' of the point which is *under* the centre of gravity, then

$$V = \tfrac{1}{3}S.f;\ x_0 = \tfrac{1}{2}\overline{OA}.$$

If *any* sloping plane be drawn through the neutral axis, it will cut off two wedges; and since the volumes of all such wedges are proportional to their heights, we have

$$V' : V :: f' : f :: f : m'h$$

where V' is the volume corresponding to f' the new value of EK in fig. 101, and of AS in fig. 90; and $m'h$ is the height of the isosceles wedge, that is the distance of the skin from the neutral axis. Since the leverages are the same as before, the moments of inertia are also in the above ratio.

Moment of Resistance to Bending of Cross Sections in General.

We see then that I_0 the moment of inertia of the cross section about its neutral axis is represented by the statical moment of the isosceles wedges made by a plane sloping at 45° and passing through the neutral axis; that the highest point of these wedges is $y_{a \text{ or } b} = m'h$, the distance from the neutral axis to the further skin; while **M**, the moment of resistance to bending, is represented by the statical moment of the wedges made by a plane passing through the neutral axis, and sloping so that the height of the highest point of these wedges is $p_{a \text{ or } b} = f$, the stress on the skin A or B, whichever is further from the neutral axis; hence—

$$\mathbf{M} = \frac{\text{Normal stress on skin furthest from neutral axis}}{\text{Distance of skin from neutral axis}} \times I_0$$

$$= \frac{p_a}{m'h} I_0 = \frac{p_b}{m'h} I_0 = \frac{f}{m'h} I_0$$

is the ultimate, proof, or working moment of resistance according as f is the ultimate, proof, or working strength of the material supposed to be the same for both skins, that is, for thrust and tension.

Rectangular cross section and sections which can be divided into rectangles.

Rectangular Section.—Fig. 102. Let $BCDE$ be a rectangle. Its moment of inertia about DC one side is found thus: The height of the isosceles wedge is zero at C and BC at B; the average height is therefore $\frac{1}{2}BC$; its volume is $\frac{1}{2}EB.BC^2$, and since the point below the centre of gravity of the wedge is two-thirds of BC from DC, we have

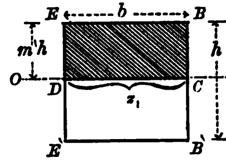


Fig.102.

$$I_{DC} = \frac{1}{2}EB.BC^2 \times \frac{2}{3}BC$$
$$= \frac{1}{3}EB \times BC^3$$

as the moment of inertia of a rectangle about a side.

Now if BE' be a rectangular cross section of which DC is the neutral axis, then its moment of inertia about that axis will be double the above; and thus, the moment of inertia of a rectangle about an axis through the centre and parallel to a side is

$$I_o = 2 \times \frac{1}{3}.EB.BC^3 = \frac{1}{12}bh^3.$$

In this case $m' = \frac{1}{2}$, so that

$$\mathsf{M} = \frac{f}{m'h}I_o = \frac{f}{\frac{1}{2}h} \times \frac{1}{12}bh^3 = \frac{1}{6}fbh^2.$$

Thus, for a rectangle, $m' = \frac{1}{2}$, $n' = \frac{1}{12}$, and $n = \frac{1}{6}$.

Hollow Rectangular Section.—For a hollow rectangular section, symmetrical above and below the neutral axis, that is, when the whole rectangle and rectangle removed have their centres on the same horizontal axis, the moment of inertia is the difference of the moments of the two rectangles.

Let H, h, and B, b be the outer and inner dimensions respectively, so that the area is $BH - bh$; then

$$I_o = \frac{1}{12}BH^3 - \frac{1}{12}bh^3 = \frac{1}{12}\left(1 - \frac{bh^3}{BH^3}\right)BH^3;$$

$$\mathsf{M} = \frac{f}{m'H}I_o = \frac{1}{6}\left(1 - \frac{bh^3}{BH^3}\right)fBH^2.$$

Hence, for a hollow rectangle, $m' = \frac{1}{2}$, $n' = \frac{1}{12}\left(1 - \frac{bh^3}{BH^3}\right)$, and $n = \frac{1}{6}\left(1 - \frac{bh^3}{BH^3}\right)$; and for the dimensions given in fig. 104, viz., $H=30$, $B=10$, $h=24$, $b=6$ inches, we obtain

$$I_o = 15588; \text{ and } \mathsf{M} = 1039{\cdot}2f \text{ inch-lbs., if } f \text{ be in lbs.}$$

Tabular Method.—The following is a convenient form for expressing the area, geometrical moment, and moment of inertia of a rectangle:—Let $BCDE$, fig. 103, be a rectangle, and AA' an axis parallel to DC, and let Y and y be the distances of its sides EB and DC from AA'. The area is

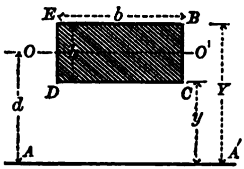


Fig.103.

$$S = b(Y - y), \ldots\ldots\ldots(1)$$

the breadth into the difference of the ordinates. For EL, part of the isosceles wedge, fig. 101, $EK = Y$, $CL = y$; and the geometrical moment of the rectangle $BCDE$ about AA' is

$$G_A = \text{volume of part of isosceles wedge} = b(Y-y) \times \tfrac{1}{2}(Y+y)$$
$$= \tfrac{1}{2} \,.\, b(Y^2 - y^2), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

one-half the breadth into the difference of the squares of the ordinates of the sides parallel to the axis. By theorem on page 233 we have the moment of inertia of the rectangle $BCDE$ about AA'

$$I_A = I_o + Sd^2 = \tfrac{1}{12}EB \,.\, BC^3 + EB \,.\, BC\left(\frac{Y+y}{2}\right)^2$$
$$= \tfrac{1}{12}b(Y-y)^3 + b(Y-y)\frac{(Y+y)^2}{4}$$
$$= \tfrac{1}{3} \,.\, b \,.\, (Y^3 - y^3), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

one-third of the breadth into the difference of the cubes of the ordinates of the sides parallel to the axis.

To apply this to the case of a hollow rectangle and a symmetrical double-T section, which give similar results, fig. 104—Choose OO the neutral axis in the centre from symmetry; and if only the upper half of the section be considered, it consists of two rectangles, the ordinates to the edges being 0, 12, 15; for one rectangle $Y=15$, $y=12$; for the other $Y=12$, and $y=0$; and for each (eqn. 3), $I_o = \frac{1}{3}b(Y^3 - y^3)$.

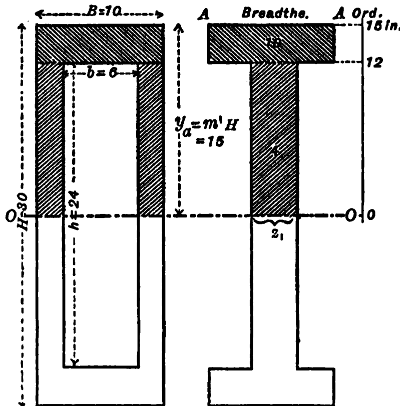


Fig.104.

b	Y	Y^3	Y^3-y^3	$\frac{1}{3}b(Y^3-y^3)$
	15	3375		
10			1647	5490
	12	1728		
4			1728	2304
	0	0		

7794

Double for both halves 2

$$I_o = 15588$$

This tabular method applies to any section made up of rectangles and which is symmetrical above and below the neutral axis. As we require G' the geometrical moment of the semi-section relatively to the neutral axis when we come to resistance to shearing, it is convenient in making

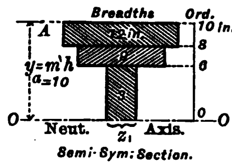


Fig.105.

the table to find G'; thus for the symmetrical section, one half of which is shown in fig. 105, we have—

b	Y	Y^2	Y^2-y^2	$\frac{1}{2}b(Y^2-y^2)$	Y^3	Y^3-y^3	$\frac{1}{3}b(Y^3-y^3)$
	10	100			1000		
12			36	216		488	1952
	8	64			512		
9			28	126		296	888
	6	36			216		
3			36	54		216	216
	0	0			0		

For semi-section $G' = 396$ 3056

2

$I_o = 6112$

Suppose the working strength $f = 4$ tons per sq. inch, then the distance from OO the neutral axis to the skin is 10 inches; hence

$$\mathbf{M} = \frac{f}{m'h} I_o = \frac{4}{10} \times 6112 = 2444{\cdot}8 \text{ inch-tons.}$$

The use of G' will be shown at the proper place.

Symmetrical sections are suitable for materials for which the strengths to resist thrust and tension are equal, as p_a and p_b become f simultaneously, fig. 8. For materials whose strengths are unequal, f_a is put for the greatest value of p_a

and f_b for that of p_b; for wrought iron, for instance, the working resistance to thrust is $f_a = 4$ tons per sq. inch, while to tension it is $f_b = 5$ tons per sq. inch. For symmetrical sections in such materials the value of f employed must obviously be the smaller. Because of this property of materials, cross sections are made unsymmetrical above and below.

Unsymmetrical Section.—Let the unsymmetrical section, fig. 106, be given, and let $f_a = 4$, and $f_b = 5$ tons per square inch. In order to determine its resistance to bending we find I_o; we will also find the area of the section, and G' the geometrical moment of the portion of the section lying to one side of the neutral axis whose position is not yet known.

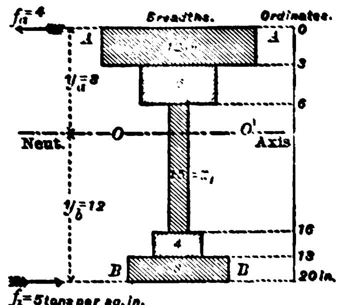


Fig. 106.

Choose *any* line as axis, say the upper skin; the diagram and table show the breadths, and ordinates laid down from this line.

b	Y	$Y-y$	*Area.*	Y^2	Y^2-y^2	$\frac{1}{2}b(Y^2-y^2)$	Y^3	Y^3-y^3	$\frac{1}{3}b(Y^3-y^3)$
	0			0			0		
12		3	36		9	54		27	108
	3			9			27		
6		3	18		27	81		189	378
	6			36			216		
1·5		10	15		220	165		3880	1940
	16			256			4096		
4		2	8		68	136		1736	2315
	18			324			5832		
8		2	16		76	304		2168	5781
	20			400			8000		

$S = 93$ $\qquad G_A = 740$ $\qquad I_A = 10522$

Now $y_a = \dfrac{G_A}{S} = \dfrac{740}{93} = 8$ inches, by theorem at fig. 100.

$\therefore \; y_b = 20 - 8 = 12$ inches.

Again $I_0 = I_A - Sy_a^2 = 10522 - 93 \times \left(\dfrac{740}{93}\right)^2 = 4634.$

The neutral axis divides the depth almost exactly as 2 to 3; and if we had $f_a : f_b :: 2 : 3$, then both skins would come to their working stress simultaneously, and we might obtain M by multiplying I_0 either by $\dfrac{f_a}{y_a}$ or by $\dfrac{f_b}{y_b}$; but since $f_a : f_b :: 4 : 5$, it is evident that both skins cannot come to their working strength at once. In this example when the skin *B* comes to $f_b = 5$ tons, the skin *A* will be at $\frac{2}{3} f_b$ or $3\frac{1}{3}$ tons per square inch and so is not at its full strength 4, yet the stresses are now at their greatest; on the other hand the skin *A* cannot come to its strength $f_a = 4$, because then the skin *B* would be at $\frac{3}{2} \times 4$, or 6 tons, and so be overtaxed; hence M is to be obtained by multiplying I_0 by the ratio $\dfrac{f_b}{y_b}$, not by $\dfrac{f_a}{y_a}$ which would give too much.

Of the two ratios $\dfrac{f_a}{y_a} = \dfrac{4}{8} = \cdot 5$, and $\dfrac{f_b}{y_b} = \dfrac{5}{12} = \cdot 42$, select the less; and

$$\mathsf{M} = \frac{f_b}{y} \times I_0 = \cdot 42 \times 4634 = 1931 \text{ inch tons.}$$

Taking the neutral axis now as origin, the geometrical moment for the part of the section lying above or below, is $G'_0 = 296$; a quantity to be used in calculating the resistance to shearing.

b	Y	Y^2	$Y^2 - y$	$\frac{1}{2}b(Y^2 - y^2)$
	0	0		
1.5			64	48
	8	64		
4			36	72
	10	100		
8			44	176
	12	144		

G'_0 = 296

Graphical Solution.—Fig. 107. (For the same data as fig. 106.) Replace the areas of the rectangles by the forces (1), (2), (3), (4), (5) acting at their centres of gravity, the amount of each force being the same as the number of units in the areà corresponding; draw the first link polygon as in fig. 19, and L, the intersection of the end links, gives the centre of the forces and therefore the centre of gravity and neutral axis of the cross section. The intercepts on OO are the geometrical moments of the areas respectively, that is the product of each area into its leverage about OO; the scale is found by subdividing the scale for areas by 10, since 10 on the scale for dimensions was taken as the polar distance.

Considering these intercepts as magnitudes of forces acting in the same lines as before, and drawing a second link polygon, its intercepts on OO are the products of these forces each into its leverage about OO; or these intercepts are the areas each into the square of its leverage about OO; the scale is derived from the previous one by again subdividing by the polar distance, 10 on the scale for dimensions. The proof is given at fig. 19.

The sum of the intercepts made by the second link polygon is nearly I_0; being deficient by one-twelfth of the sum of the products of the breadth of each rectangle into the cube of its depth.

Also KL the geometrical moment of areas (1) and (2) is nearly $G_0^{\backprime}$, the deficiency being that of the rectangle lying between the area (2) and the neutral axis; this rectangle, however, is small and has a short leverage.

There is also shown in fig. 108 a link polygon drawn for the three areas (1), (2), (3`), which constitute the portion of the section lying above the neutral axis; the sum of the intercepts is $G_0^{\backprime}$.

Corrections.—The manner of correcting is shown in figs. 107 and 108. Take the lines of action of the areas first along their (say) *upper* edges, and construct the first link polygon; treating the intercepts on OO as forces acting along the *under* edges of the rectangles, construct a second link polygon; then to the sum of the intercepts made by the second

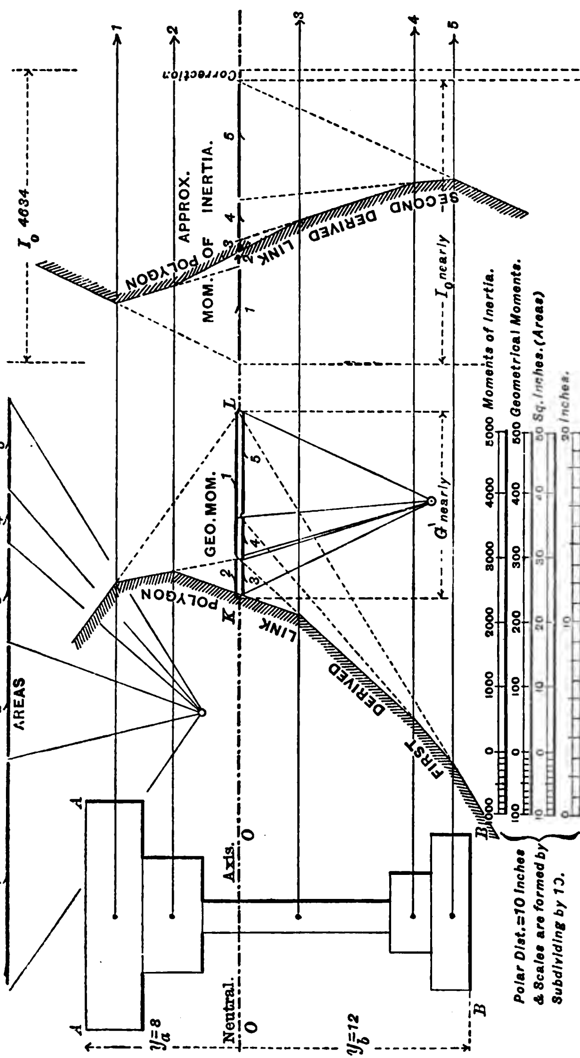

AREAS
GEO. MOM.
APPROX. MOM. OF INERTIA.
FIRST DERIVED LINK POLYGON
SECOND DERIVED LINK POLYGON
Correction
Neutral. Axis.
Moments of Inertia.
Geometrical Moments.
Sq. Inches. (Areas)
Inches.
Polar Dist.=10 Inches & Scales are formed by Subdividing by 10.

Fig. 107.

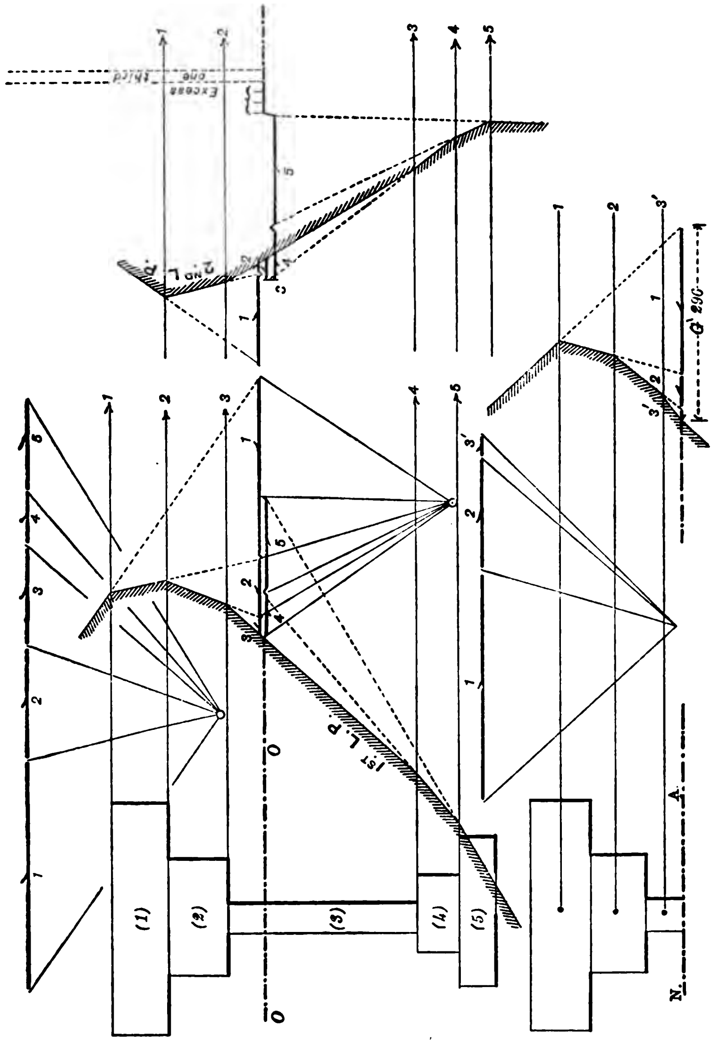


Fig.108.

link polygon in fig. 107, add, as a correction, one third of its excess above the sum of the intercepts made by the second link polygon in fig. 108.

The proof of the correction on I_0, which is absolute, is shown thus:—For any rectangle

$$I_A = \tfrac{1}{3}b(Y^3 - y^3) = \tfrac{1}{3}b(Y - y)(Y^2 + Yy + y^2)$$
$$= \tfrac{1}{3}bh\{(Y+y)^2 - Yy\} = S\left\{\left(\frac{Y+y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y+y}{2}\right)^2 - \frac{Yy}{3}\right\}$$
$$= S\{d^2 + \tfrac{1}{3}(d^2 - Yy)\} = Sd^2 + \tfrac{1}{3}(Sd^2 - SYy);$$

where S is the area of the rectangle under consideration. The smaller h becomes, the more nearly does the value of Yy approach d^2, and the smaller is the correction. The greater the number of rectangles into which the cross section is divided, the more nearly accurate will be the approximation given by fig. 107 alone.

In the example the correction on I_0 is about 4 per cent; in the rectangle $EBB'E'$, fig. 102, the areas of the two halves each into the square of its leverage about OO is less than I_0 by one-twelfth, or about 8 per cent.

Any cross section can be blocked out into rectangles, and I_0 and G_0' easily calculated for it by the tabular method, if we have a table of squares and cubes. If the cross section has a very irregular outline, it may require to be blocked into a great many rectangles, and the construction, fig. 107, will probably give a sufficiently close approximation.

Definition. A cross section for which the neutral axis divides the depth in the same ratio as the strengths of a given material to resist tension and thrust is called a *Cross Section of Uniform Strength* for that material.

The cross section, fig. 106 for instance, would be of uniform strength for a beam of a material whose resistance to tension and thrust is as 3 to 2, AA being the compressed skin and BB the stretched skin; for a cantilever of the same material it would be turned upside down.

Triangular cross section and sections which can be divided into triangles.

Triangular Section.—On making $f = \tfrac{2}{3}h$, the wedges, figs. 89 and 90, become isosceles, and we have by substituting in the expression given thereat—

$$G' = \text{vol. of wedge on triangular portion above } N.A.$$
$$= \tfrac{4}{81}bh^2;$$

and $I_0 = \tfrac{1}{36}bh^3$.

In the following way, I_0 may be derived from the rectangle, fig. 109. The moment of inertia of the shaded triangle about the central axis CC is half that of the rectangle; for the triangle, then

$$I_C = \tfrac{1}{2}(\tfrac{1}{12}bh^3) = \tfrac{1}{24}bh^3.$$

$$I_0 = I_C - Sd^2$$

$$= \tfrac{1}{24}bh^3 - \frac{bh}{2} \times \left(\frac{h}{6}\right)^2 = \tfrac{1}{36}bh^3.$$

$$\mathrm{M} = \frac{f}{\frac{2}{3}h}I_0 = \tfrac{1}{24}f\,bh^2.$$

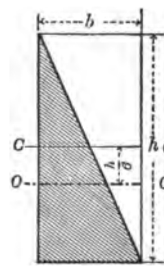


Fig.109.

Hence for a triangular section—

$$m' = \tfrac{2}{3},\ n' = \tfrac{1}{36},\ n = \tfrac{1}{24}.$$

Hexagonal Section.—As an example of a built figure, we will take a hexagon about a diameter as axis. Fig. 110.

Taking a quadrant, we can divide it into three equal triangles; the breadth of each is $\tfrac{1}{4}b$, and the height $\tfrac{1}{2}h$; hence the area of each is $\tfrac{1}{16}bh$, and the moment of inertia of each about its own neutral axis is $\tfrac{1}{36} \times \tfrac{1}{4}b \times \tfrac{1}{8}h^3 = \tfrac{1}{1152}bh^3$. If I_0 be the moment of inertia of the hexagon about OO, we have for a quadrant

$\tfrac{1}{4}I_0$ = moment of each of the three triangles about its own neutral axis, together with the area of each into the square of the distance of its centre from OO,

$$= 3 \times \frac{bh^3}{1152} + \frac{bh}{16}\left\{\left(\frac{h}{6}\right)^2 + \left(\frac{h}{6}\right)^2 + \left(\frac{h}{3}\right)^2\right\} = \tfrac{5}{384}bh^3.$$

$$\therefore\ I_0 = \tfrac{5}{96}bh^3.$$

Or, taking another quadrant divided into a rectangle and a triangle—

$$\left.\begin{array}{l}\frac{1}{4}I_0 = \frac{1}{12}\left(\frac{b}{4}\right)\left(\frac{h}{2}\right)^3 + \frac{bh}{8}\left(\frac{h}{4}\right) \text{ for rectangle} \\ \quad + \frac{1}{36}\left(\frac{b}{4}\right)\left(\frac{h}{2}\right)^3 + \frac{bh}{16}\left(\frac{h}{6}\right)^2 \text{ for triangle}\end{array}\right\} = \frac{5}{384}bh^3;$$

$$\therefore \quad I_0 = \frac{5}{96}bh^3.$$

Dividing the semi-hexagon above the axis into any set of convenient figures, and multiplying the area of each by the deviation of its centre of gravity, we have

$$G_0` = \frac{1}{12}bh^2.$$

Also $$\mathsf{M} = \frac{f}{m'h}I_0 = \frac{f}{\frac{1}{2}h} \times \frac{5}{96}bh^3 = \frac{5}{48}fbh^2.$$

Hence for a hexagonal section about a diameter as axis,

$$m' = \tfrac{1}{2},\ n' = \tfrac{5}{96} = \cdot 05208,\ n = \tfrac{5}{48} = \cdot 10417.$$

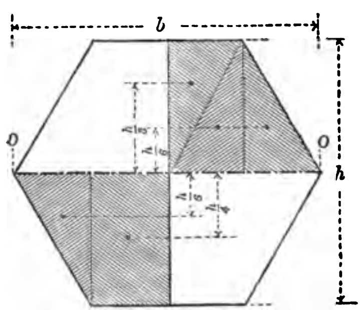


Fig.110.

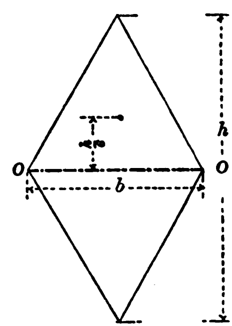


Fig.111.

Rhomboidal Section.—As another example, we will take a section in the form of a rhombus, with diagonal as neutral axis. Fig. 111.

For upper half section, the area is $\frac{bh}{4}$, the moment of inertia about its own neutral axis is $\frac{1}{8}b\left(\frac{h}{2}\right)^3 = \frac{bh^3}{288}$, and the deviation of its centre of gravity is $\frac{h}{6}$; hence

$$\tfrac{1}{2}I_0 = \frac{bh^3}{288} + \left(\frac{bh}{4}\right)\left(\frac{h}{6}\right)^2 = \tfrac{1}{96}bh^3.$$

$$\therefore \quad I_0 = \tfrac{1}{48}bh^3, \text{ and } \mathsf{M} = \frac{f}{\frac{1}{2}h}I_0 = \tfrac{1}{24}fbh^2.$$

$$G' = \left(\frac{bh}{4}\right)\left(\frac{h}{6}\right) = \tfrac{1}{24}bh^2.$$

Hence for a rhomboidal section with diagonal as neutral axis,

$$m' = \tfrac{1}{2},\ n' = \tfrac{1}{48},\ n = \tfrac{1}{24}.$$

Square Section.—A square section lying with its diagonal horizontal is a particular case of this.

In order to compare the strengths of a square section when lying with a side horizontal, and when lying with a diagonal horizontal; let a be the side of the square, then $d = a\sqrt{2}$ is the diagonal; for b and h substitute a in the one case, and $a\sqrt{2}$ in the other; then

$$\mathsf{M} = \tfrac{1}{6}fa^3;\ \mathsf{M}' = \tfrac{1}{24}f(a\sqrt{2})^3 = \frac{\sqrt{2}}{12}fa^3;$$

so that $\mathsf{M} : \mathsf{M}' :: \sqrt{2} : 1$, being stronger when the side is horizontal.

Trapezoidal Section.—Observe that a triangular section is of *uniform strength* for a material whose strengths to resist tension and thrust are in the ratio 1 : 2, or 2 : 1. It is evident, then, that for a material, the ratio of whose strengths is between 1 and 2, a cross section of uniform strength can be made by selecting the proper frustum of a triangle; that is, a trapezoid with the parallel sides horizontal. Suppose the strengths are as $S : s$, not greater than 2 : 1; then to find the ratio of the parallel sides B and b, fig. 112:

a well-known construction for finding G is to lay off B in a continuation of b, and b in a continuation of B, one in each direction; the line joining the extremities cuts the medial line at G; hence

$$B+\frac{b}{2} : b+\frac{B}{2} :: y_a : y_b :: S : s.$$

$$\therefore \quad \frac{b}{B}=\frac{2s-S}{2S-s}.$$

For instance, if the strengths are as 3 : 2, then the parallel sides ought to be in the ratio

$$\frac{b}{B}=\frac{2\times 2-3}{2\times 3-2}=\tfrac{1}{4},$$

when the neutral axis will divide the depth as 3 : 2.

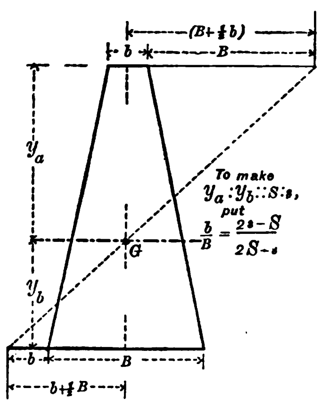


Fig.112.

Though such sections are thus far economical, still they are not the most economical, as too much of the material lies near the centre of the section where it cannot act effectively in resisting bending.

Circular cross section and derived figures.

Circular Section.—In the circle, fig. 113, inscribe a hexagon with a diameter horizontal; for the hexagon $b = d$, $h = d.\frac{\sqrt{3}}{2}$, and

$$I_0 = \tfrac{5}{96}(d)\left(d\frac{\sqrt{3}}{2}\right)^3 = \cdot 0338d^4,$$

an approximation on the small side to that for the circle about a diameter.

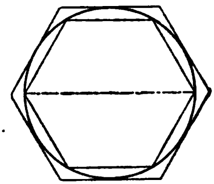


Fig.113.

Again, circumscribe a hexagon with a diameter horizontal; for this hexagon

$$h = d,\ b = d.\frac{2}{\sqrt{3}} = \frac{2d}{3}\sqrt{3}, \text{ and}$$

$$I_0 = \tfrac{5}{96}\left(\frac{2d}{3}\sqrt{3}\right)(d)^3 = \cdot 0602d^4,$$

an approximation on the large side.

Taking the average of these, we have for the circle

$$I_0 = \cdot 047d^4.$$

where d is the diameter of the circle, and the breadth and depth of the circumscribing rectangle.

It will be seen that $n' = \cdot 047$ is a close approximation, being correct to 2 decimal places.

In the same way an approximation to G_0^4 may be found.

The exact value of I_0 is found thus,—From O the centre of the circle, fig. 114, draw three axes OX, OY diameters at right angles, and OZ normal to the plane of the paper. Let I, J, K be the moments of inertia of the circle about these axes respectively; and let s be any small elementary area. Now $\overline{OA}$, $\overline{OB}$, and $\overline{OS}$ are its leverages about the three axes respectively, and by definition—

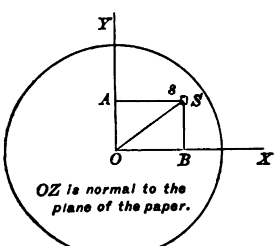


Fig.114.

$I = (s \times \overline{OA}^2)$ sumd. for each elemt.

$J = (s \times \overline{OB}^2)$ „ „

$K = (s \times \overline{OS}^2)$ „ „

but by Euclid I., 47, $\overline{OS}^2 = \overline{OA}^2 + \overline{OB}^2$ for each element, hence

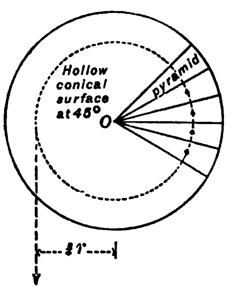
$$K = I + J;$$

and further, since I and J are equal, each being the moment of inertia about a diameter,

$$K = 2I;$$

so that if we find K, the value of I is at once obtained.

On the element s build a column of material of unit density, and whose height is $\overline{OS}$; suppose this column to gravitate *tangentially*, that is, at right angles to $\overline{OS}$, then its statical moment about OZ, that is its volume into $\overline{OS}$, gives the moment of inertia of the element about OZ; all these columns will form a solid standing on the circle as base, whose height at the circumference is r, and whose upper surface is a conical surface, apex at O, and sloping at 45° to the plane of the circle. The volume of this solid is that of a cylinder of height r standing on the circle, minus a cone of equal height and base; its volume is therefore two-thirds of the cylinder, viz.,



V = Vol. of Cyl. − Vol of Cone = $\frac{2}{3}\pi r^3$
Fig. 115.

$$V = \tfrac{2}{3} \times \pi r^2 \times r = \tfrac{2}{3}\pi r^3,$$

every particle gravitating tangentially and in the same direction. If we cut this solid, fig. 115, into slices by planes at right angles to the plane of the circle, and passing through consecutive radii, the slice between two adjacent planes will be a pyramid with its apex at O, and having its base on the cylindrical surface; one dimension of the base, therefore, is r, and the other an arc of the circle. Now by taking the slices thin enough, the base of each pyramid becomes in the limit a rectangle, and the points in the circle below the centres of gravity of these pyramids will form a circle of radius $\frac{3}{4}r$; the whole weight may be supposed to be applied by a cord on a pulley of radius $\frac{3}{4}r$, and hence

$$K = \text{Vol. of solid} \times \text{rad. of pulley}$$
$$= \tfrac{2}{3}\pi r^3 \times \tfrac{3}{4}r = \tfrac{1}{2}\pi r^4.$$

$$\therefore \qquad I_0 = \tfrac{1}{4}\pi r^4 = \frac{\pi}{64}d^4, \text{ or } \frac{\pi}{64}bh^3;$$

and $$\mathbf{M} = \frac{f}{\frac{1}{2}h}I_0 = \frac{\pi}{32}fd^3, \text{ or } \frac{\pi}{32}fbh^2.$$

Hence for a circular section,

$$m' = \tfrac{1}{2},\ n' = \frac{\pi}{64} = \cdot 049,\ n = \frac{\pi}{32} = \cdot 098.$$

To find G_0. Suppose a wedge standing on the quadrant, fig. 116, formed by a 45° sloping plane passing through OX; the geometrical moment of a semicircle about a diameter will be twice this volume. Cut the wedge into slices by planes at right angles to the plane of the circle, and passing through consecutive radii. Each slice is a pyramid with apex at O, and of height r; its base is part of a cylindrical surface, and the upper edge of the base is sloping. Let a be the short arc as shown in the figure, and δ its projection on OX; then in the limit when the slices are thin, the base becomes a plane rectangle whose dimensions are a and the height of the sloping plane at K; but height of $K = LK \doteq r \cos \theta$; hence

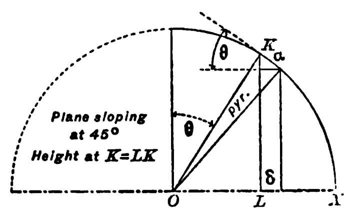


Fig.116.

$$\text{Vol. of pyramid} = \tfrac{1}{3} \times r \times a \,.\, r \cos \theta = \tfrac{1}{3}r^2.\delta,$$
since $\delta = a \cos \theta$.

For each pyramid $\frac{1}{3}r^2$ is constant; so that the volume of wedge on the quadrant is $\frac{1}{3}r^2$ multiplied by the sum of the quantities δ; this sum is r whether the slices be thick or indefinitely thin; hence the

$$\text{Volume of wedge} = \tfrac{1}{3}r^3;$$

and doubling, we have for semi-circle,

$$G_0 = \tfrac{2}{3}r^3.$$

If we divide G_0 by the area of the semi-circle, we obtain the distance from the centre of the circle to the centre of gravity of the semi-circle,

$$\bar{y} = \frac{4r}{3\pi}.$$

Elliptic Section.—The Ellipse is immediately derived from the circle thus:—Let the ellipse, fig. 117, have the same minor diameter as the circle above it, so that $b = 2r$; all the vertical dimensions of the circle are, however, to be altered in the constant ratio $a : r$, since the vertical radius of the circle is altered to a. Let the circle be divided into rectangular elements, each with an edge lying on the neutral axis; when the circle is changed into the ellipse, each element remains of the same breadth b' as before, but its vertical dimension is changed from y to Y, where $Y : y :: a : r$.

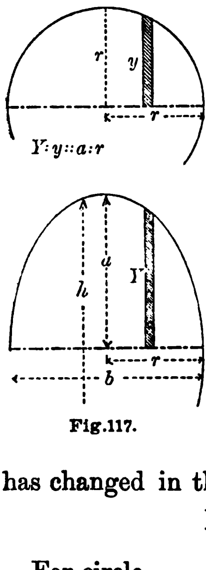


Fig.117.

For each elmt. of circle $I = \tfrac{1}{3}b'(y^3 - 0^3)$.

" " ellipse $I = \tfrac{1}{3}b'(Y^3 - 0^3)$.

For each element, and therefore for the whole figure, the moment of inertia has changed in the ratio

$$Y^3 : y^3 = a^3 : r^3.$$

For circle, $$I_0 = \frac{\pi}{4}r^4;$$

hence for ellipse, $$I_0 = \frac{a^3}{r^3}\left(\frac{\pi}{4} \cdot r^4\right) = \frac{\pi}{4}ra^3 = \frac{\pi}{64}bh^3,$$

putting $\tfrac{1}{2}b$ for r, and $\tfrac{1}{2}h$ for a;

$$\mathrm{M} = \frac{f}{\tfrac{1}{2}h}I_0 = \frac{\pi}{32}fbh^2.$$

Hence for elliptic section,—

$$m' = \tfrac{1}{2},\ n' = \frac{\pi}{64},\ n = \frac{\pi}{32}.$$

For a semi-ellipse about a diameter as axis, we have

$$G'_0 = \tfrac{2}{3} r a^2;\ \text{and}\ \bar{y} = \frac{4a}{3\pi}.$$

Hollow Circular Section.—For a hollow circle or ellipse the reduction is the same as for the hollow rectangle, page 237, only $\frac{\pi}{32}$ replaces $\frac{1}{6}$; hence

$$\mathsf{M} = \frac{\pi}{32}\left(1 - \frac{bh^3}{BH^3}\right) fBH^2.$$

Examples.

136. Find the moment of resistance to bending of the symmetrical section, the upper half of which is shown in fig. 118; the material is wrought iron, for which the weaker working strength is $f_a = 4$ tons per square inch.

Consider the two triangles as one of breadth 8 inches; for a triangle about its own neutral axis, $I = \frac{1}{36}bh^3 = 48$; hence for semi-section

$$\tfrac{1}{2}I_0 = (48 + 24 \times 8^2) \text{ for triangle} + \tfrac{1}{3} \times 2(10^3 - 0^3) \text{ for rectangle.}$$

$$\therefore\ I_0 = 4501;\ \text{and}\ \mathsf{M} = \frac{f}{\frac{1}{2}h} I_0 = \tfrac{4}{10} \times 4501 = 1800 \text{ inch tons.}$$

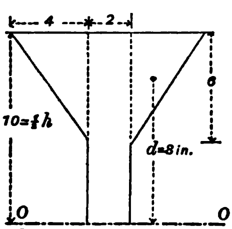


Fig.118.

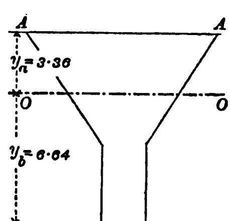


Fig.119.

137. For the cross section shown in fig. 119, and which is the semi-section shown in fig. 118, find the working value of M, if the working strengths of the material be $f_a = 4$, and $f_b = 5$ tons per square inch.

Choose *BB* as an axis of reference; then

$$S = 24 + 20 = 44; \; G_B = (24 \times 8) + \tfrac{1}{2} \times 2(10^2 - 0^2) = 292.$$

$$I_B = \tfrac{1}{2} I_0 \text{ of example 136, } = 2251.$$

$$y_b = \frac{G_B}{S} = \frac{292}{44} = 6\cdot64, \text{ and } y_a = 10 - 6\cdot64 = 3\cdot36;$$

$$I_0 = I_B - S \, . \, y_b^2 = 315,$$

$$\frac{f_a}{y_a} = \frac{4}{3\cdot36} = 1\cdot19, \text{ and } \frac{f_b}{y_b} = \frac{5}{6\cdot64} = \cdot752;$$

taking the smaller value, we have

$$M = \cdot752 \times I_0 = 237 \text{ inch-tons.}$$

138. Find M for the section, the upper half of which is shown in fig. 120; the material is cast iron, for which $f_b = 2$ tons per square inch.

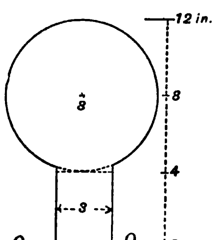


Fig.120.

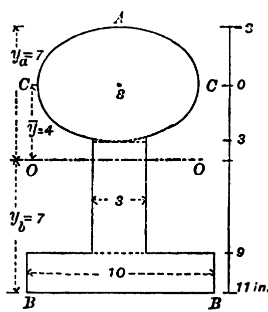


Fig.121.

For the circle about its own neutral axis,

$$I = \frac{\pi}{64} bh^3 = \cdot049 \times 8^4 = 201; \text{ and } S = 50\cdot26.$$

For the cross section,

$$\tfrac{1}{2}I_0 = (201 + 50{\cdot}26 \times 8^2) + \tfrac{1}{3} \times 3(4^3 - 0^3).$$

$$\therefore\ I_0 = 6964; \text{ and } \mathsf{M} = \frac{f_b}{\frac{1}{2}h} I_0 = \tfrac{2}{12} \times 6964 = 1161 \text{ inch-tons.}$$

139. Find M for the section shown in fig. 121, the working strengths f_a and f_b being 4 and 5 tons per square inch respectively.

It is convenient to choose CC the diameter of the ellipse as the axis of reference, and we have—

$$\begin{array}{lccccc} & \text{Ellipse.} & & \text{Middle rect.} & & \text{Lower rect.} \\ S = & \pi \times 4 \times 3 & + & 3 \times 6 & + & 10 \times 2 = 75{\cdot}7; \\ G_C = & 0 & + & \tfrac{1}{2} \times 3(9^2 - 3^2) & + & \tfrac{1}{2} \times 10(11^2 - 9^2) = 308; \\ I_C = & \frac{\pi}{64} \times 8 \times 6^3 & + & \tfrac{1}{3} \times 3(9^3 - 3^3) & + & \tfrac{1}{3} \times 10(11^3 - 9^3) = 2793; \end{array}$$

$\bar{y} = \dfrac{G_C}{S} = \dfrac{308}{75{\cdot}7} = 4$ inches sensibly; so that $y_a = 7$, and $y_b = 7$ inches.

$$I_0 = I_C - S\bar{y}^2 = 1582.$$

The neutral axis being sensibly in the middle, take $f_b = 4$ the smaller strength, and $\mathsf{M} = \frac{4}{7}I_0 = 904$ inch-tons.

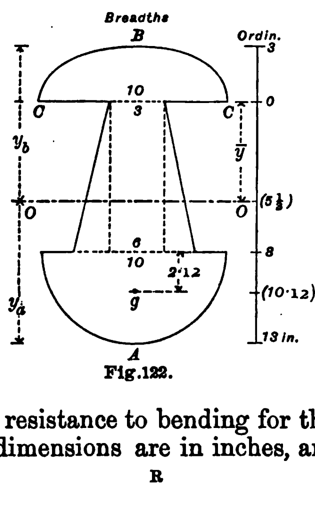


Fig.122.

140. Find the resistance to bending for the section shown in fig. 122; the dimensions are in inches, and the strengths

of the material are $f_a = 4$ tons (thrust), and $f_b = 5$ tons (tension) per square inch.

Choose CC the diameter of the ellipse as an axis of reference; and for semi-ellipse—

$$S = 23{\cdot}6; \quad G_c = -30; \quad I_c = 53.$$

For semi-circle about its diameter, $I = 245$, and distance from centre to g its centre of gravity is 2·12 inches; hence about its own neutral axis, that is an axis through g, we have $I_0 = I - S(2{\cdot}12)^2 = 69$; and for semi-circle—

$$S = 39{\cdot}3; \quad G_c = 398; \quad I_c = 4093.$$

For the two triangles, reckoned as one about its own neutral axis $I_0 = 42{\cdot}7$; hence for double triangle—

$$S = 12; \quad G_c = 64; \quad I_c = 384.$$

For rectangle—

$$S = 24; \quad G_c = 96; \quad I_c = 512.$$

For total section—

$$S = 98{\cdot}9; \quad G_c = 528; \quad I_c = 5042.$$

$$\bar{y} = \frac{G_c}{S} = \frac{528}{98{\cdot}9} = 5{\cdot}3; \text{ hence}$$

$y_b = 8{\cdot}3$ ins., and $y_a = 7{\cdot}7$ ins.; and $I_0 = I_c - S\bar{y}^2 = 2250$.

$$\frac{f_a}{y_a} = \frac{4}{7{\cdot}7} = {\cdot}519, \text{ and } \frac{f_b}{y_b} = \frac{5}{8{\cdot}3} = {\cdot}602;$$

taking the smaller value, we have—

$$\mathsf{M} = {\cdot}519 I_0 = 1167 \text{ inch-tons.}$$

As shown in the figure, the section is lying in a position suitable for a cantilever; if intended for a beam, it should be turned upside down; should it be kept as it stands, however, then for a beam we take the smaller of the two ratios, $\frac{5}{7{\cdot}7}$ and $\frac{4}{8{\cdot}3}$, viz., ·482; multiplying I_0 by this quantity gives a value of M, which is less than the above.

141. Find the resistance to bending of the wrought iron section, fig. 123; the dimensions are in inches, and the metal everywhere is 1 inch thick.

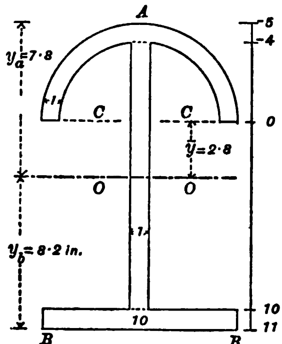


Fig. 123.

Choose CC the diameter of the hollow semi-circle as an axis of reference; then for

	S	G_c	I_c
Hollow semi-circle,.........	14	−41	144
Rectangle above CC,......	4	−8	21
1st rectangle below CC,...	10	50	333
2nd ,, ,, ...	10	105	1103
Whole section,.............	38	106	1601

For whole section—

$$y = \frac{G_c}{S} = \frac{106}{38} = 2{\cdot}8 \quad \therefore \quad y_a = 7{\cdot}8, \; y_b = 8{\cdot}2.$$

$$I_0 = I_c - Sy^2 = 1601 - 38 \times 2{\cdot}8^2 = 1305.$$

$$\frac{f_a}{y_a} = \frac{4}{7{\cdot}8} = {\cdot}512, \text{ and } \frac{f_b}{y_b} = \frac{5}{8{\cdot}2} = {\cdot}609;$$

$$\therefore \quad \mathsf{M} = {\cdot}512 I_0 = 668 \text{ inch-tons.}$$

142. For a wheel, design an elliptical spoke of approximately uniform strength, of a material whose smaller strength is 2 tons per square inch; length of spoke 3 feet, load at end due to a force applied to the circumference of wheel $\frac{1}{20}$th of a ton, and at each cross section the depth is to the breadth in the ratio 3 : 1.

$$M_{\text{max.}} = Wl = \tfrac{3}{20} \text{ ft.-tons} = 1{\cdot}8 \text{ inch-tons.}$$

$$\mathsf{M} = nfbh^2 = \frac{\pi}{32} \times 2 \times b(3b)^2 = 1{\cdot}76b^3 \text{ inch-tons.}$$

Equating these values of M and M, we have

$$1{\cdot}76b^3 = 1{\cdot}8; \quad \therefore \quad b^3 = 1.$$

Therefore $b = 1$, and $h = 3$ inches, are the diameters of the elliptic section at boss, while two-thirds of these are the diameters at tyre. See fig. 99.

143. Find the thickness of a cast-iron pipe whose external diameter is 2 feet, that it may have a working moment of resistance to bending of 800 inch-tons; the smaller working strength of cast-iron is 2 tons per square inch.

Put $D = 24$ inches, and let d be the inside diameter;

then $$I_0 = \frac{\pi}{64}(D^4 - d^4), \text{ and } \mathsf{M} = \frac{f}{\frac{1}{2}D} I_0.$$

$$\therefore \quad 800 = {\cdot}00818(D^4 - d^4), \text{ or } (D^4 - d^4) = 97800.$$

Ans. $d = 22$ inches; and for the thickness of the metal, we have $t = 1$ inch.

144. Find the relative strengths of a regular hexagonal section to resist bending when lying with a diameter horizontal, and when lying with a diameter vertical. Find also G'_0 for semi-section in the latter case.

For the diameter horizontal, page 248, we had $\mathsf{M} = \frac{5}{48} fbh^2$;

if a be put for the side of the hexagon, then $h = a\sqrt{3}$, and $b = 2a$; this gives

$$\mathrm{M} = \tfrac{5}{48}f \,.\, 2a \,.\, (a\sqrt{3})^2 = \tfrac{5}{8}fa^3.$$

When the diameter is vertical, the half-section consists of a rectangle and a triangle; the rectangle is of breadth b, one edge is against the neutral axis, and the other is at a distance $\frac{1}{4}h$ therefrom; the triangle is of breadth b, its height is $\frac{1}{4}h$, and its centre of gravity is $\frac{1}{3}h$ distant from the neutral axis; hence

$$\tfrac{1}{2}I_0 = \tfrac{1}{3}b\left\{\left(\frac{h}{4}\right) - 0^3\right\} + \left\{\tfrac{1}{36}b\left(\frac{h}{4}\right)^3 + \left(\tfrac{1}{2}.b.\frac{h}{4}\right)\left(\frac{h}{3}\right)^2\right\};$$

$$\therefore\ I_0 = \tfrac{5}{128}bh^3; \text{ and } \mathrm{M} = \frac{f}{\frac{1}{2}h}I_0 = \tfrac{5}{64}fbh^2.$$

In this case $h = 2a$, and $b = a\sqrt{3}$; so that

$$\mathrm{M} = \tfrac{5}{64}f.\,(a\sqrt{3})\,(2a)^2 = \frac{5\sqrt{3}}{16}fa^3.$$

Ans. The strengths are as $2 : \sqrt{3}$, being greater when the diameter is horizontal; and

$$G'_0 = \tfrac{1}{2}b\left\{\left(\frac{h}{4}\right)^2 - 0^2\right\} + \left(\tfrac{1}{2}\,.\,b\,.\,\frac{h}{4}\right)\left(\frac{h}{3}\right) = \tfrac{7}{96}bh^2.$$

Resistance of Cross Sections to Shearing, and Distribution of Shearing Stress on a Cross Section.

In fig. 124 let AB be the cross section shown in figs. 6, 7, 8, 9, and 10, and let $A'B'$ be another section lying at a small distance δx to the left thereof; in figs. 6 and 7, we assume AB to be in a position such that P is greater than $W_1 + W_2$, that is, AB lies to the left of the section of maximum bending moment; hence the bending moment on $A'B'$ will be less than that on AB.

Consider the *horizontal* equilibrium of $AHH'A'$, fig. 124, part of the slice of the beam between these sections, and bounded below by the horizontal face HH' a portion of the plane CD, fig. 6. There is no stress on the free surface.

On AH the horizontal stress is indicated by arrows p_a, &c.; and on $A'H'$ by the shorter arrows p_a', &c. (see fig. 8), shorter because the bending moment on $A'B'$ is less; in so far as these affect the *horizontal* equilibrium of $AHH'A'$, they may be replaced by arrows $(p_a - p_a')$ &c., on the face AH alone. For *horizontal* equilibrium, there must be a tangential stress q acting towards the right on the remaining

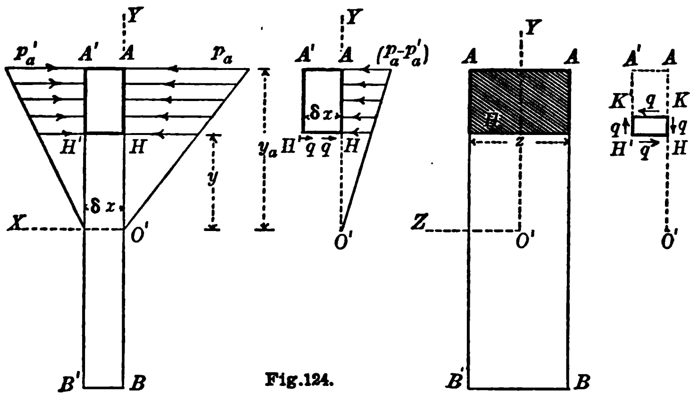


Fig.124.

face $H'H$; and since $H'H$ is very small, the stress on it will be of constant intensity q; therefore

$q \times$ area of plane $H'H$ = sum of the arrows $(p_a - p'_a)$, &c., on plane AH;

or $q.z.\delta x$ = volume of frustum of wedge standing on HAA the shaded part of section as base, and of height $(p_a - p'_a)$.

Now if G stands for the geometrical moment of the shaded part of the cross section relatively to the neutral axis, that is, for the volume of the frustum standing on HAA of the isosceles wedge made by a plane sloping at 45° and passing through the neutral axis, then the frustum of the wedge above is the same fraction of G, that $(p_a - p'_a)$ its height is of y_a the height of the isosceles wedge; and

$$q.z.\delta x = \frac{p_a - p'_a}{y_a} G.$$

Now on the cross section AB, the bending moment equals the moment of resistance, that is

$$M = \mathsf{M} = \frac{p_a}{y_a} I_0;$$

$$\therefore \qquad \frac{p_a}{y_a} = \frac{M}{I_0};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

For the sections AB and $A'B'$, y_a and the moments of inertia are the same, because in the limit when δx is indefinitely small the sections coincide; so that if M' be the bending moment at $A'B'$, then

$$M' = \frac{p'_a}{y_a} I_0; \quad \therefore \quad \frac{p'_a}{y^a} = \frac{M}{I_0}.$$

and
$$\frac{p_a - p'_a}{y_a} = \frac{M - M'}{I_0} = \frac{\delta M}{I_0},$$

where δM stands for the small difference of the bending moments on the sections AB and $A'B'$ at the small distance δx apart.

$$\therefore \qquad q . z . \delta x = \frac{\delta M}{I_0} G;$$

or
$$q = \frac{\left(\frac{\delta M}{\delta x}\right)}{I_0} \cdot \frac{G}{z},$$

In the limit when δx becomes indefinitely small, so does δM; but their ratio $\frac{\delta M}{\delta x}$ becomes F the shearing force at the section AB, by theorem, page 181, and therefore

$$q = \frac{F}{I_0} \cdot \frac{G}{z}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

since H' now coincides with H, we are warranted in assuming q constant over $H'H$.

Now q is the intensity of the shearing or tangential stress at the point H in the horizontal plane CD, fig. 6; but it is *also* the intensity of the shearing stress at the point H in the vertical plane AB (Part I., pages 42 and 67); hence (fig. 9) the intensity of the shearing stress at H, any point of the cross section AB, is

$$q = \frac{F}{I_0} \cdot \frac{G}{z} ; \qquad (3.)$$

where F is the shearing force on the cross section; I_0 is the moment of inertia of the cross section about the neutral axis; G is the geometrical moment of the portion of the cross section beyond the point, about the neutral axis; and z is the breadth of the cross section at the point.

On the cross section AB, it will be seen that q acts downwards at *any* point H; if we choose K at an indefinitely small distance *above* H, the tangential stress on $K'K$ will also be q since it is indefinitely near $H'H$; and on $K'K$ the *lower* surface of $AKK'A'$, q will act to the right just as on $H'H$, but on $K'K$ the *upper* surface of the small prism $KHH'K'$, q acts to the left. The horizontal stresses q form a couple tending to turn the small prism in the left-handed direction; hence for equilibrium, the vertical stress q on the faces KH and $K'H'$ tend to turn it in the opposite direction; so that q on the face KH acts downwards, an assumption made in fig. 9, which is now proved.

On the other hand, if P were less than $W_1 + W_2$, &c., then the bending moment on $A'B'$ would be the greater; q on face $H'H$ would act towards the left, the arrows q all round the prism would be reversed, and so q on the face HK would act upwards; that is, q at *any* point H of the cross section would act upwards.

Observe that if q be evaluated for two different points of a cross section of a beam loaded in any manner, F and I_0 will be the same in both, and the two values of q will therefore be to each other directly as the geometrical moments of the parts of the section beyond the points, and inversely as the breadths at the points, respectively. But for any given form of cross section, as rectangular, circular, elliptical, &c., the ratio of the geometrical moments of the portions beyond two points definitely situated in the section with respect to each other, is the same whether the section be large or small; whether, for instance, it be a large circle or a small one; so also is the ratio of the breadths; and hence the distribution of the shearing stress on a cross section, or the manner in which q varies from point to point in the section, depends *only* upon the *form* of the cross section. It is instructive to know how the shearing stress is distributed on cross sections of various forms employed in practice; and it is of the greatest practical importance to know *where* the intensity is a maximum, the amount of that intensity, and the ratio of its maximum and average values. This ratio is an abstract quantity, and depends only upon the form of the cross section.

In the graphical solutions it is inconvenient to draw the tangential arrows q as in fig. 9, since they interfere with each other; we will therefore draw them at right angles to AB, when their extremities will give a locus which specifies the distribution. For such a locus the origin will always be at O the neutral axis, the abscissæ y are measured on the vertical axis, and the ordinates q are measured on the horizontal axis; and at any point H whose abscissa y is given, the ordinate q gives the intensity of the shearing stress.

Rectangular cross section.—Fig. 125. Since $z = b =$ constant, q will vary as G the geometrical moment of the shaded portion of the section, that is as the isosceles (frustum) wedge standing on that portion; now this wedge will be greatest when $y = 0$, for then the shaded portion will be the whole isosceles wedge on the semi-section. If y be negative, we have more than the semi-section shaded, but the portion lying below the neutral axis gives a negative geometrical moment, and q is again less than q_0; hence q is a maximum at the neutral axis, it has equal values for equal values of y above and below O, and is zero

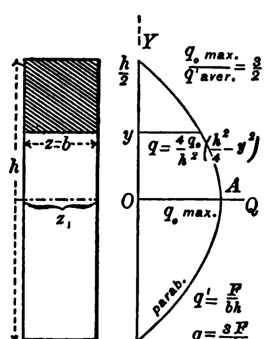


Fig.125.

at each skin. The maximum value is thus:—

$$q_0 = \frac{F}{I_0} \cdot \frac{G_0}{b} = \frac{F}{\frac{1}{12}bh^3} \frac{\frac{1}{2}b\left\{\left(\frac{h}{2}\right)^2 - 0^2\right\}}{b}$$

$$= \frac{3F}{2bh}.$$

Graphical Solution.—Lay off $\overline{OA} = q_0$; and if we take A as origin, $q_0 - q$ the ordinate of any point will be proportional to the volume of the isosceles wedge on the semi-section minus that on the shaded part, that is to the isosceles wedge on the part of the section from O to y; but the breadth being constant, the volume of that wedge is proportional to $\overline{Oy}^2$; so that from A as origin, the ordinate of any point on the curve is proportional to y^2, and the curve is a parabola.

The modulus of the parabola is $\frac{q_0}{(\frac{1}{2}h)^2} = \frac{4q_0}{h^2}$;

and the equation with O as origin is $q = \frac{4q_0}{h^2}\left(\frac{h^2}{4} - y^2\right)$.

Let $q'_{\text{aver.}}$ represent the average value of the intensity of the shearing force on the cross section, then

$$q'_{\text{aver.}} = \frac{\text{shearing force}}{\text{area}} = \frac{F}{bh};$$

and we have for the ratio of the maximum and average intensity

$$\frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{3}{2}.$$

Hollow rectangular cross section, or symmetrical double-T section.—Fig. 126. For values of y from $\frac{1}{2}H$ to $\frac{1}{2}h$, q varies as G exactly as in the previous case, so that the locus from Y to L is a parabola whose apex is A_1; the modulus of this parabola is greater than that for the solid rectangle since the constant divisor I_0 is now less. For values of y between $\frac{1}{2}h$ and 0, consider the effect upon each of the items that make up q. If the hollow were filled up, SA_2 would be part of the parabola in fig. 125; I_0 is now less, however, and allowing for that fact alone SA_2 would coincide with LA_1. The removal of the centre decreases z from B to z_1; this increases each ordinate in the same proportion, so that SA_2 is still a parabola. Lastly, for values of y between $\frac{1}{2}h$ and 0, the removal of the centre alters the value of G from what it would be for the

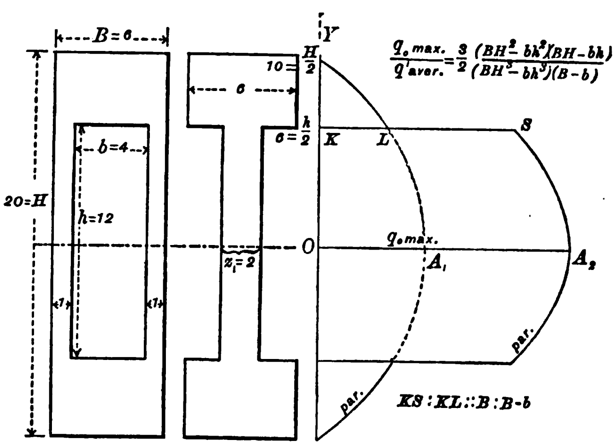


Fig. 126.

solid figure, by the geometrical moment of the part of the hollow beyond y; now the geometrical moment of that part of the hollow is $\frac{1}{2}b\{(\frac{1}{2}h)^2 - y^2\}$; this leaves the equation consisting of a term in y^2 and a constant term, so that SA_2 is finally a parabola with apex at A_2, but with a modulus different from that of LA_1; and hence

$$q_0 = OA_2, \text{ is the maximum.}$$

For semi-section, $G'_0 = \frac{1}{8}(BH^2 - bh^2)$,

For total section, $I_0 = \frac{1}{12}(BH^3 - bh^3)$, Area $S = (BH - bh)$, and $z_1 = (B - b)$ is the breadth at the neutral axis;

$$\therefore \quad q_0 = \frac{F}{I_0} \cdot \frac{G_0}{z_1} = \frac{3F}{2} \cdot \frac{BH^2 - bh^2}{(BH^3 - bh^3)(B - b)};$$

$$q'_{\text{aver.}} = \frac{F}{S} = \frac{F}{(BH - bh)};$$

$$\therefore \quad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{3}{2} \frac{(BH^2 - bh^2)(BH - bh)}{(BH^3 - bh^3)(B - b)}.$$

For the dimensions given in fig. 126, $q_{0\text{max.}} = {\cdot}033F$,

$$\frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = 2{\cdot}4, \text{ and } KS : KL :: 6 : 2.$$

The locus, fig. 126, gives the shearing stress on the horizontal layers as well as on the cross section. Now, as you pass from the horizontal layer K cutting the web to another cutting the flange, there is a sudden change of intensity from KS to KL; this change cannot be supposed to take place altogether at K, as the free overhanging surface of the section at that point does not bear any share of the shearing at all, and for a small distance just above K the portion overhanging does not bear its proper share; in the vicinity of K, therefore, the stress is not constant on the horizontal on the cross section. The consequence of this is to introduce shearing stress on the *vertical* plane through K. In order that the intensity of the stress should change from KL to KS absolutely at K, there would require to be an infinite amount of shearing force on the horizontal plane at K; and since the intensity changes from KL to KS in passing through a *small* vertical distance at K, there must be a *great* amount of shearing force on the horizontal plane at K. Hence sections of this shape very readily give way by shearing at K; cast iron sections being specially liable to do so under the shearing force developed by irregularities in cooling. Re-entrant angles, as that at K, are to be rounded off in castings and rolled plates, and filled in with angle irons in built sections; this allows the breadth to change *gradually* from B to $(B - b)$, and the intensity of shearing stress to change *gradually* from KL to KS.

It follows as a corollary from fig. 126 that, for symmetrical sections made up of rectangles with breadths diminishing towards the centre, q_0 is the maximum.

Symmetrical section of three rectangles.—When the middle rectangle is the narrowest, q_0 is the maximum as we saw in the previous case.

In fig. 127 the middle rectangle is the broadest; and in

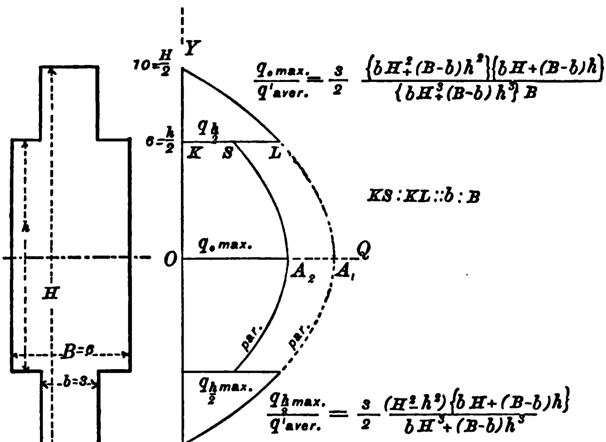


Fig.127.

this case, the intensity of the shearing force has two maxima values, one at $y = 0$, the other at $y = \frac{h}{2}$.

For the whole section, $S = bH + (B - b)h$;

$$I_0 = \frac{bH^3 + (B - b)h^3}{12}; \text{ and } q'_{\text{aver.}} = \frac{F}{bH + (B - b)h}.$$

For the middle portion of the section, the maximum value of q is

$$q_0 = \frac{F}{I_0} \frac{G_0}{B} = \frac{3F}{2B} \frac{bH^2 + (B - b)h^2}{bH^3 + (B - b)h^3}.$$

For the portion of the cross section beyond K,

$$G=\frac{b}{8}(H^2-h^2);$$

and the maximum value of q is

$$q_{\frac{h}{2}}=\frac{3F}{2}\frac{H^2-h^2}{bH^3+(B-b)h^3}.$$

To find the ratio between the maximum and the average values of q, we have the two equations—

$$\frac{q_{0\text{max.}}}{q'_{\text{aver.}}}=\frac{3}{2}\frac{\{bH^2+(B-b)h^2\}\{bH+(B-b)h\}}{B\{bH^3+(B-b)h^3\}};$$

and

$$\frac{q_{\frac{h}{2}\text{max.}}}{q'_{\text{aver.}}}=\frac{3}{2}\frac{(H^2-h^2)\{bH+(B-b)h\}}{bH^3+(B-b)h^3};$$

in any example the greater of these two ratios is to be used.

In the numerical example shown in fig. 127,

$$q_0=\cdot014F,\text{ and } q_{\frac{h}{2}}=\cdot013F;$$

and $$\frac{q_{0\text{max.}}}{q'_{\text{aver.}}}=1\cdot3;\quad \frac{q_{\frac{h}{2}\text{max.}}}{q'_{\text{aver.}}}=1\cdot25.$$

Triangular section.—Fig. 128. Let y, which includes its sign, be the ordinate of the base of any portion such as is shaded in the diagram; then for that portion

$$G_0=\tfrac{1}{2}\left(\frac{2h}{3}-y\right)z\left\{y+\tfrac{1}{3}\left(\frac{2h}{3}-y\right)\right\}$$

$$=\frac{z}{27}(2h-3y)(h+3y).$$

Now $$q=\frac{F}{I_0}\frac{G_0}{z},\text{ and since }\frac{F}{I_0}\text{ is constant,}$$

$$q\propto\frac{G_0}{z},\text{ or } q\propto(2h-3y)(h+3y).$$

The sum of these factors is constant, and their product is therefore greatest when they are equal; that is when

$$(2h-3y) = (h+3y), \text{ or } y = \frac{h}{6};$$

hence q is a maximum at middle of depth h, and the locus is a parabola with its axis horizontal.

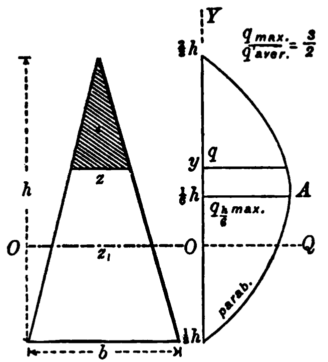


Fig. 128.

For the portion above this central point—

$$G_0 = \frac{1}{2}\frac{h}{2}\cdot\frac{b}{2} \times \left(\frac{h}{6} + \frac{1}{3}\frac{h}{2}\right) = \frac{bh^2}{24}, \text{ and } z = \frac{b}{2}.$$

For the whole section $I_0 = \frac{1}{36}bh^3$; hence

$$q_{\frac{h}{6}} \text{ max.} = \frac{FG_0}{I_0 \cdot z} = \frac{F \cdot \frac{bh^2}{24}}{\frac{bh^3}{36} \times \frac{b}{2}} = \frac{3F}{bh};$$

$$q \text{ aver.} = \frac{F}{S} = \frac{F}{\frac{1}{2}bh} = \frac{2F}{bh}.$$

$$\therefore \quad \frac{q_{\frac{h}{6}} \text{ max.}}{q' \text{ aver.}} = \frac{3}{2}.$$

The equation to the parabola is easily found.

Rhomboidal section.—Fig. 129. For a portion such as is shaded, in the upper half of section, that is for positive values of y,

$$G_0 = \tfrac{1}{2}\left(\frac{h}{2} - y\right)z\left\{y + \tfrac{1}{3}\left(\frac{h}{2} - y\right)\right\} = \frac{z}{24}\left\{h^2 + 8y\left(\frac{h}{4} - y\right)\right\}.$$

Now

$$q \propto \frac{G}{z}$$

$$\propto h^2 + 8y\left(\frac{h}{4} - y\right),$$

so that the locus is a parabola with its axis horizontal. The sum of the factors $y\left(\frac{h}{4} - y\right)$ is constant, and the product is greatest when these are equal, so that when $y = \frac{h}{8}$, q is a maximum. The locus for the upper half is a parabola whose axis is $\frac{1}{8}h$ above the neutral axis, and for the under half a similar parabola symmetrical below.

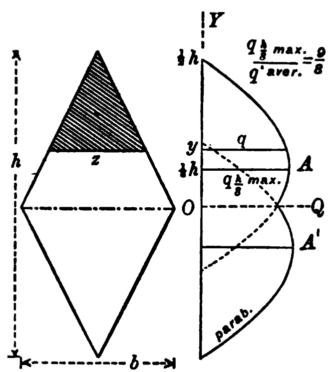


Fig. 129.

For that part of the section above $y = \frac{h}{8}$,

$$G'_0 = \tfrac{1}{2}\cdot\frac{3h}{8}\cdot\frac{3b}{4} \times \left\{\frac{h}{8} + \tfrac{1}{3}\frac{3h}{8}\right\} = \tfrac{9}{256}bh^2; \text{ and } z = \frac{3b}{4}.$$

For the cross section—

$$\tfrac{1}{2}I_0 = \tfrac{1}{36}b\left(\frac{h}{2}\right)^3 + \tfrac{1}{2}b\frac{h}{2} \times \left(\tfrac{1}{3}\cdot\frac{h}{2}\right)^2 = \frac{bh^3}{96}; \quad \therefore\ I_0 = \frac{bh^3}{48};$$

$$\therefore\ q_{\frac{h}{8}\text{max.}} = \frac{FG_0}{I_0 . z} = \frac{9}{4}\,\frac{F}{bh};$$

$$q'_{\text{aver.}} = \frac{F}{\frac{1}{2}bh} = \frac{2F}{bh};$$

$$\therefore\ \frac{q_{\frac{h}{8}\text{max.}}}{q'_{\text{aver.}}} = \frac{9}{8}.$$

Regular hexagonal section, diameter vertical.—Fig. 130. If we suppose the sloping sides above and below produced to meet the neutral axis, we have a form like fig. 129; and for values of y from $\frac{1}{4}h$ to $\frac{1}{2}h$, everything so far as regards q is

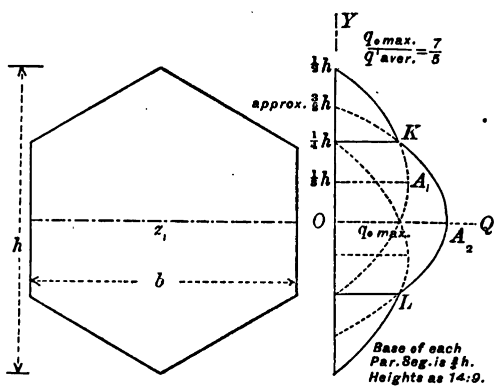


Fig. 130.

the same as if those sides were produced, except the constant I_0 which is less; hence for those values of y, the locus is a parabola with its apex A_1 on the horizontal $\frac{1}{8}h$ above the centre as in the previous case. For values of y from 0 to $\frac{1}{4}h$, suppose the vertical sides produced and the rectangle completed; the four triangles which require to be added on to complete that rectangle increase the constant I_0, but so far as that affects q the locus is still a parabola with its apex on the neutral axis; they also increase G_0 for every value of y, augmenting both the constant term and the term in y^2 in the expression for q; the locus, however, is still a parabola with its axis coinciding with the neutral axis, but the modulus is altered.

Now we know that this parabola intersects the other pair at K and L, points beyond their apexes, hence OA_2 is greater than OA_1; and

$$q_0 = OA_2 \text{ is the maximum.}$$

From example No. 144, page 260, we have

$$G_0' = \tfrac{7}{96}bh^2;\ I_0 = \tfrac{5}{128}bh^3;\ z_1 = b;\ \text{and } S = \frac{3bh}{4};$$

$$\therefore \quad q_{0\text{max.}} = \frac{FG_0'}{I_0 z} = \frac{28F}{15bh};$$

$$q'_{\text{aver.}} = \frac{F}{\frac{3}{4}bh} = \frac{4}{3}\frac{F}{bh};$$

$$\therefore \quad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{7}{5}.$$

Circular cross section.—Figs. 131 and 132. On the shaded sector, fig. 131, suppose an isosceles wedge standing, made by a plane sloping at 45° and intersecting the horizontal radius, and cut up into pyramids as in fig. 116. In this case the sum of the small quantities δ is $OC = r \sin\theta$, and the volume of the wedge is $\frac{1}{3}r^2 \times r\sin\theta$; for the sector, the geometrical moment about OX, $G = \frac{1}{3}r^3 \sin\theta$; and deducting for triangle ODB, we have for the geometrical moment of ABD about OX as axis—

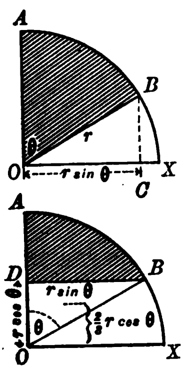


Fig.131.

$$G = \tfrac{1}{3}r^3 \sin\theta - (\tfrac{1}{2}.r\sin\theta . r\cos\theta) \times \tfrac{2}{3}r\cos\theta$$

$$= \frac{r^3}{3}\sin^3\theta.$$

Hence for the shaded part of the circle in fig. 132—

$$G = \tfrac{2}{3}r^3 \sin^3\theta\,;\ z = 2r \sin\theta\,;$$

and $\dfrac{G_0}{z} = \dfrac{r^2}{3}\sin^2\theta$, where θ is half the angle subtended at centre.

But $y = r\cos\theta\,;\ \therefore\ \cos\theta = \dfrac{y}{r};\ \sin^2\theta = \dfrac{r^2 - y^2}{r^2};$

and $$\frac{G_0}{z} = \tfrac{1}{3}(r^2 - y^2)\,;$$

so that $q \propto \frac{1}{3}(r^2 - y^2)$, and the locus is a parabola with its axis on OQ.

Hence q_0 is the maximum; and since

$$G_0' = \frac{2}{3}r^3;\ I_0 = \frac{\pi}{4}r^4;\ S = \pi r^2;\ \text{and } z_1 = 2r.$$

$$\therefore\quad q_0\text{max.} = \frac{FG_0'}{I_0 z} = \frac{4}{3}\,\frac{F}{\pi r^2};\quad q'\text{aver.} = \frac{F}{\pi r^2};$$

$$\frac{q_0\text{max.}}{q'\text{aver.}} = \frac{4}{3}.$$

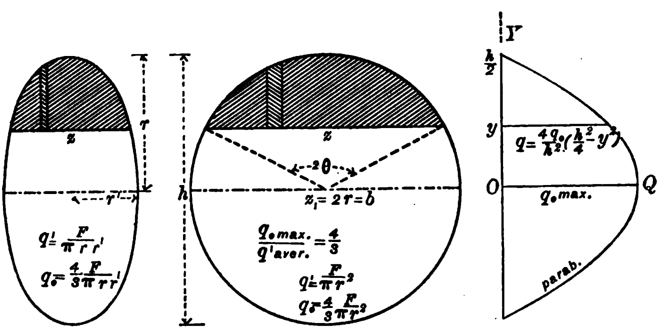


Fig. 132.

Elliptical section.—Fig. 132. Let the ellipse shown in the figure be derived from the circle by altering in a constant ratio the breadth z of the circle at any point, to z' the breadth of the ellipse at the point corresponding; that is, let

$$z = nz'.$$

Suppose the shaded part of the circle to be divided by vertical lines into elementary rectangles, and let the ellipse be correspondingly divided; by considering each of these elementary rectangles, it is easily seen that I, G, S, and z' for the ellipse are derived from the corresponding quantities for the circle by multiplying by n; for the circle we had

$$q = \frac{F}{I}\frac{G}{z};$$

so that for the ellipse

$$q = \frac{F}{nI}\frac{nG}{nz'} = \frac{F}{I}\frac{G}{z'};$$

$$q_0 = \frac{4}{3}\frac{F}{\pi r r'}; \quad q'_{\text{aver.}} = \frac{F}{\pi r r'};$$

$$\frac{q_0\text{max.}}{q'_{\text{aver.}}} = \frac{4}{3}, \text{ the same as for the circle.}$$

Regular hexagonal section with diameter horizontal.—Fig 133. Let z be a horizontal line at any distance y from the neutral axis. Suppose the two shaded triangles joined together; they will form an equilateral triangle, for which

$$S = \tfrac{1}{2}\left(\frac{h}{2} - y\right)\frac{2}{\sqrt{3}}\left(\frac{h}{2} - y\right) = \frac{1}{4\sqrt{3}}(h - 2y)^2;$$

$$G_0 = \frac{1}{4\sqrt{3}}(h - 2y)^2\left\{y + \tfrac{1}{3}\left(\frac{h}{2} - y\right)\right\} = \frac{1}{24\sqrt{3}}(h - 2y)^2(h + 4y);$$

$$z = \frac{h}{\sqrt{3}} + \frac{2}{\sqrt{3}}\left(\frac{h}{2} - y\right) = \frac{2}{\sqrt{3}}(h - y).$$

For shaded rectangle—

$$G_0 = \tfrac{1}{2}\cdot\frac{h}{\sqrt{3}}\left\{\left(\frac{h}{2}\right)^2 - y^2\right\} = \frac{h}{8\sqrt{3}}(h^2 - 4y^2).$$

For the total shaded part, summing and reducing,—

$$G_0 = \frac{1}{6\sqrt{3}}(h-2y)(h^2+2hy-2y^2);$$

$$\frac{G_0}{z} = \frac{1}{12}\frac{(h-2y)(h^2+2hy-2y^2)}{h-y};$$

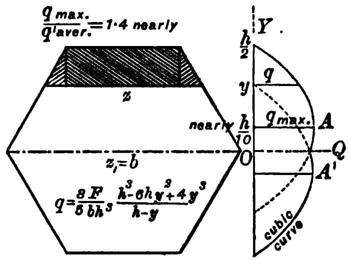


Fig. 133.

and since $\frac{F}{I}$ is a constant quantity, we have

$$q \propto \frac{h^3-6hy^2+4y^3}{h-y}.$$

a cubic curve; the maximum value occurs when $y = \frac{h}{10}$ nearly; and evaluating for $q_{\frac{h}{10}\text{max.}}$ and $q'_{\text{aver.}}$, we find the ratio is $\frac{q_{\frac{h}{10}\text{max.}}}{q'_{\text{aver.}}} = 1{\cdot}26.$

When several points for this equation are plotted, they give, as shown in the figure, a pair of curves resembling parabolas.

This case shows that the locus giving values of q is not necessarily a parabola.

Examples.

145. The section of a beam is a rectangle 2 inches broad by 6 inches deep; the material is wrought iron whose

working resistance to shearing is $f = 5$ tons per square inch. Find the working resistance to shearing of the cross section.

F will be such that q_0 the maximum intensity shall attain to $f = 5$ tons per square inch.

$$\therefore \quad q_0 = 5\text{; but } q'_{\text{aver.}} = \tfrac{2}{3}q_0 = \tfrac{10}{3} \text{ tons.}$$

$$\text{F} = \text{average intensity} \times \text{area}$$

$$= q'.bh = \tfrac{10}{3} \times 12 = 40 \text{ tons.}$$

146. Compare the shearing force and the resistance to shearing of the cross section at the fixed end of the cantilever of example No. 131, taking the working strength of the timber to resist shearing as $\frac{1}{10}$ of a ton per square inch.

$F = 2$ tons.

$q_0 = \frac{1}{10}$ ton per sq. inch; $\therefore\ q' = \frac{2}{3} \times \frac{1}{10} = \frac{1}{15}$ ton per sq. inch.

$\text{F} = q' \times \text{area} = \frac{1}{15} \times 6{\cdot}4 \times 15 = 6{\cdot}4$ tons.

So that in providing for the bending moment at the fixed end, there is three times the necessary strength to resist shearing; but near the free end F is still 2 tons, while F will only be half as much as before, namely 3·2 tons.

147. Find the resistance to shearing of the cross section, fig. 106, taking $f = 4$ tons per sq. inch.

q_0 is the maximum value.

From the tabular form or graphical solution we had

$$I_0 = 4634,\ G_0' = 296, \text{ and } z_1 = 1{\cdot}5\text{;}$$

$$\text{but} \quad q_0 = \frac{\text{F}G_0'}{I_0 z_1}, \text{ or } 4 = \frac{\text{F} \times 296}{4634 \times 1{\cdot}5}; \ \therefore\ \text{F} = 94 \text{ tons.}$$

148. Find the resistance to shearing of the section shown in fig. 105, taking f the resistance of the material to shearing at 4 tons per sq. inch.

$$q_0 = \frac{\text{F}G_0'}{I_0 z_1},\ 4 = \frac{\text{F} \times 396}{6112 \times 3}; \ \therefore\ \text{F} = 185 \text{ tons.}$$

149. In section fig. 105, find the ratio of the maximum and average intensity of the shearing stress.

$$q_0 = \frac{FG_0'}{I_0 z_1}, \text{ and } q' = \frac{F}{S};$$

$$\therefore \quad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{SG_0'}{I_0 z_1} = \frac{120 \times 396}{6112 \times 3} = 2{\cdot}6.$$

150. Find F for fig. 126, taking $f = 4$ tons per sq. inch; and find the ratio of the maximum and average intensity of the shearing force.

$$\tfrac{1}{2}I_0 = \tfrac{1}{3} \times 6(10^3 - 6^3) + \tfrac{1}{3} \times 2(6^3 - 0^3); \quad \therefore \quad I_0 = 3424.$$

$$G_0' = \tfrac{1}{2} \times 6(10^2 - 6^2) + \tfrac{1}{2} \times 2(6^2 - 0^2) = 228.$$

$$\tfrac{1}{2}S = 6(10 - 6) + 2(6 - 0); \quad \therefore \quad S = 72.$$

$z_1 = 2$, and we wish to have $q_0 = 4$; but

$$q_0 = \frac{\mathsf{F}G_0'}{I_0 z_1}; \quad \therefore \quad \mathsf{F} = 120 \text{ tons.}$$

$$q'_{\text{aver.}} = \frac{\mathsf{F}}{72} = \tfrac{5}{3} \text{ tons; and } \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = \frac{4}{\tfrac{5}{3}} = 2{\cdot}4.$$

151. Find the ratio of the maximum and the average intensity of the shearing stress for fig. 121.

Take the under half which consists entirely of rectangles, and find $G_0' = 157{\cdot}5$. Now $I_0 = 1582$, $S = 75{\cdot}7$, and $z_1 = 3$.

$$\textit{Ans.} \quad \frac{q_{0\text{max.}}}{q'_{\text{aver.}}} = 2{\cdot}5.$$

152. Find the working moment of resistance to bending, and the working resistance to shearing of the section, of which the upper half is shown in fig. 134.

In wrought iron built sections the piercing of holes for rivets reduces the effective area to resist tension but not to resist thrust; the area as thus reduced is to the original area in the ratio of about 4 : 5, which is the same as the ratio of the strengths; hence it is usual to make such built sections symmetrical above and below, to consider the working resistance to tension and thrust as each equal to 4 tons per sq. inch, and then to neglect the fact that the holes diminish the effective sectional area; the working resistance to shearing may also be taken at 4 tons per sq. inch.

b	Ord.	Dif.	Dif. × b	Ord.²	Dif.	Dif. × $\frac{b}{2}$	Ord.³	Dif.	Dif. × $\frac{b}{3}$
	10			100			1000		
12		2	24		36	216		488	1952
	8			64			512		
3		4·5	13·5		52	78		469	469
	3.5			12·3			43		
1		3·5	3·5		12	6		43	14
	0			0			0		
			41		$G_0' = 300$				2435
			2						2
			$S = 82$						$I_0 = 4870$

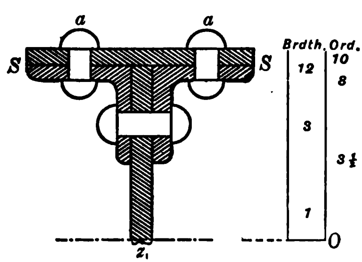


Fig.134.

$$M = \frac{f}{\frac{1}{2}h} I_0 = \tfrac{4}{10} \times 4870 = 1948 \text{ inch-tons.}$$

$$q_0 = \frac{F G_0'}{I_0 z_1}; \quad \therefore \quad F = \frac{q_0 I_0 z_1}{G_0'} = \frac{4 \times 4870 \times 1}{300} = 65 \text{ tons.}$$

153. If the rivets a, a, fig. 134, be pitched at 4 inches apart; find the diameter necessary for each rivet.

For part of section beyond SS, $G_0 = \frac{1}{2} \times 12(10^2 - 9^2) = 114$;

$$\therefore \quad q_0 = \frac{FG_0}{I_0 z} = \frac{67 \times 114}{4870 \times 12} = \cdot 13 \text{ tons per sq. inch}$$

is the intensity of the shearing stress on the horizontal plane SS at the cross section, and it will be sensibly constant on SS for a few inches on either side of cross section. There is one rivet for each 24 sq. inches of SS, hence a rivet has to bear $\cdot 13 \times 24 = 3{\cdot}12$ tons of shearing force. If the rivet be very tight, this shearing force would be uniformly distributed on its section, and the area required would be found by dividing by $f = 4$. If the rivet be not perfectly tight, there will be a bending moment on it, and we must consider the shearing stress distributed as in fig. 132; in which case

$$q_0 = 4; \quad \therefore \quad q' = \tfrac{3}{4} \times 4 = 3 \text{ tons per sq. inch average intensity.}$$

$$\therefore \quad \text{area} = \tfrac{3 \cdot 12}{3} = 1{\cdot}04, \text{ which gives a diameter } d = 1{\cdot}2 \text{ ins.}$$

Taking 1·2 inches as the diameter of the rivets, the six holes reduce the area of the cross section by 16·8 sq. inches, almost exactly a fifth as we supposed it would do.

This is on the supposition that the section bears the full shearing force that it can resist; at other cross sections where the shearing force is less, the rivets might either be made smaller in diameter or be more widely pitched.

154. Find the resistance to shearing of a cross-formed section; width of each pair of wings 5 inches, thickness of metal 1·5 inches; $f = 4$ tons per sq. inch. Fig. 135; see also fig. 127.

For semi-section—

$$G_0' = \tfrac{1}{2} \times 1{\cdot}5(2{\cdot}5^2 - \cdot 75^2) + \tfrac{1}{2} \times 5(\cdot 75^2 - 0^2) = 5{\cdot}7;$$

and $$\frac{G_0'}{z_1} = \frac{5{\cdot}7}{5} = 1{\cdot}14 \text{ is a max.}$$

$$\tfrac{1}{2} I_0 = \tfrac{1}{3} \times 1{\cdot}5(2{\cdot}5^3 - \cdot 75^3) + \tfrac{1}{3} \times 5(\cdot 75^3 - 0^3); \quad \therefore \quad I_0 = 16{\cdot}6.$$

For portion beyond K—

$$G_0 = \tfrac{1}{2} \times 1{\cdot}5(2{\cdot}5^2 - {\cdot}75^2) = 4{\cdot}27\,;$$

and $$\frac{G_0}{z} = \frac{4{\cdot}27}{1{\cdot}5} = 2{\cdot}84 \text{ is therefore the max.}$$

$$q_{\cdot 75} = \frac{\mathsf{F}}{I_0}\,\frac{G_0}{z};\ 4 = \frac{\mathsf{F}}{16{\cdot}6} \times 2{\cdot}84\,;\quad \therefore\quad \mathsf{F} = 23{\cdot}4 \text{ tons.}$$

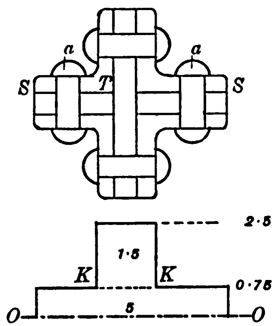


Fig.135.

155. If the section, fig. 135, be built of half-inch plates as shown, and fixed with rivets a, a, one-inch diameter; find what should be the pitch near the cross section which is under $\frac{1}{3}$ of the full working shearing force F, if $f = 5$ tons per sq. inch.

For the part beyond SS—

$$G_0 = \tfrac{1}{2} \times 1{\cdot}5(2{\cdot}5^2 - {\cdot}75^2) + \tfrac{1}{2} \times 5({\cdot}75^2 - {\cdot}25^2) = 5{\cdot}5.$$

$$\therefore\quad q_{\cdot 25} = \frac{\frac{1}{3}\mathsf{F}}{I_0}\,\frac{G_0}{z} = \frac{\frac{1}{3} \times 23{\cdot}4 \times 5{\cdot}5}{16{\cdot}6 \times 5} = {\cdot}5 \text{ tons per sq. inch}$$

is the intensity of the shearing stress on the horizontal plane SS.

If the working strength of the rivets be 5 tons per sq. in.; the average resistance to shearing will be $\frac{3}{4} \times 5 = 4$ tons per sq. inch nearly; hence $\cdot 5 \div 4$, that is $\frac{1}{8}$th of the area along

ST, and normal to the paper, must be pierced with holes. But $ST = 2$ inches; so that for every four inches measured on ST normal to the paper, there should be one sq. inch pierced; that is, rivets about one inch in diameter should be pitched four inches apart.

156. A triangular section, as shown in fig. 128, is 12 inches deep, and the resistance to shearing is $f = 4$ tons per sq. inch. Find what the breadth should be in order to make $\mathsf{F} = 100$ tons.

$$q_0 = 4; \quad \therefore \quad q' = \tfrac{2}{3} \times 4 = \tfrac{8}{3}; \; S = \tfrac{1}{2}bh = 6b; \; \mathsf{F} = 100.$$

$$\mathsf{F} = q'S, \text{ or } 100 = \tfrac{8}{3} \times 6b. \quad \therefore \quad b = 6{\cdot}25 \text{ inches.}$$

157. In the chain of a suspension bridge five flat links dove-tail with four alternately, and a cylindrical pin passes through the eyes. If the pull on the chain be 200 tons, find the area of the pin, supposing that $f = 6$ tons per sq. inch.

As the pin would require to shear at 8 sections simultaneously, $200 \div 8 = 25$ tons is the shearing force on the section,—

$$q_0 = f = 6, \text{ and } q' = \tfrac{3}{4}q_0 = 4{\cdot}5; \; \mathsf{F} = q'.S, \text{ or } 25 = 4{\cdot}5 \; S.$$

$\therefore$ area of pin $= 5{\cdot}6$ sq. inches, or diam. $= 2{\cdot}7$ inches.

Distribution of Shearing Stress—Approximate Method.

For a cross section such as is shown in fig. 126, the web bears the greater share of the shearing stress; and, moreover, the stress is nearly uniform in its distribution. A close approximation to the resistance to shearing for such a cross section will therefore be obtained by multiplying the area of the web into $q_0 = f$, f being the shearing strength of the material. This is equivalent to considering that the web bears *all* the shearing stress uniformly distributed over it; or that the central parabola, in such a diagram as fig. 126, is replaced by a rectangle of height h and breadth q_0, and that the upper and lower portions of the diagram are left out of consideration. The area of the web required for a double-T section is readily found by the converse of the above, and is given by the equation

$$S = \frac{F}{f},$$

where S is the area of web in sq. inches; F is the amount of the greatest shearing force in lbs. at the section; and f is the resistance of the material to shearing in lbs. per sq. inch.

Examples.

158. Compare the resistance to shearing for the cross section shown in fig. 104, as obtained by the exact and approximate formulæ respectively, taking $f = 5$ tons per sq. inch.

$$F = \frac{I_0 z_1 q_0}{G} = \frac{15588 \times 4 \times 5}{693} = 450 \text{ tons (exact.)}$$

$$F = fS = 5 \times 96 = 480 \text{ ,, (approx.)}$$

159. Suppose the web of the cross section shown in fig. 105 to extend to the under side of the upper plate, take $f = 4$ tons per sq. inch, and find approximately the resistance to shearing. Compare the approximate result with that obtained for example No. 148.

$$F = 4 \times 48 = 192 \text{ tons.}$$

160. Suppose the web of the cross section shown in fig. 106 to extend to the inner faces of the outer plates, and find the approximate resistance to shearing, supposing $f = 4$ tons per sq. inch; compare this with the result of example No. 147.

$$F = 4 \times 15 \times 1{\cdot}5 = 90 \text{ tons (approx.)}$$

161. The depth of a girder is 20 inches, the upper and lower flanges are each 1 inch in thickness, and the shearing force to be resisted is 65 tons. Find the thickness required for the web, supposing $f = 4$ tons per sq. inch.

$$S = \tfrac{65}{4} = 16\tfrac{1}{4} \text{ sq. inches;}$$

the depth of the web is 18 inches, so that the thickness required is a little over nine-tenths of an inch.

The resistance to shearing of the section shown in fig. 134 is (example 152) 65 tons by the exact formula, and the web is 1 inch thick.

STRESS AT AN INTERNAL POINT OF A BEAM.

Returning to figure 6, we have now found the intensity and obliquity of the stresses at the point H, on AB and CD the pair of rectangular planes through it, viz. :—On CD the total stress tangential and given by q in such diagrams as fig. 125, and on AB the total stress of intensity r at obliquity γ, fig. 136, given in terms of its normal component p, and its tangential component q on such diagrams as fig. 125; for reference, figs. 7, 8, and 9 are reproduced, and form part of fig. 136. The planes of principal stress at H are to be found as in Part I., fig. 47; making that construction, and noting that γ' is a right angle, we have, fig. 136, $OM = \frac{1}{2}p$, and $MR = \sqrt{\frac{p^2}{4} + q^2}$. These are readily calculated and become known quantities; we also have

$$\tan 2\theta = \frac{RL}{ML} = \frac{\text{arrow } q\text{, fig. 125}}{\text{half-arrow } p\text{, fig. 8}},$$

giving θ the angle which the plane of greater principal stress makes with the plane of cross section AB, both planes being normal to the paper.

Further—

$OM + MR =$ intensity of greater principal stress (of same kind as arrow p, fig. 8).

$OM - MR =$ intensity of smaller principal stress (of opposite kind from arrow p, fig. 8).

and $MR =$ intensity of greatest tangential stress, being on the planes inclined at 45° to the planes of principal stress;

that is, we have the maximum value of the thrust, tension, and shearing stress at the point H.

That the two principal stresses are of opposite kind, follows from the fact that the stress on the plane *CD* is wholly tangential; Part I., pages 72 and 86.

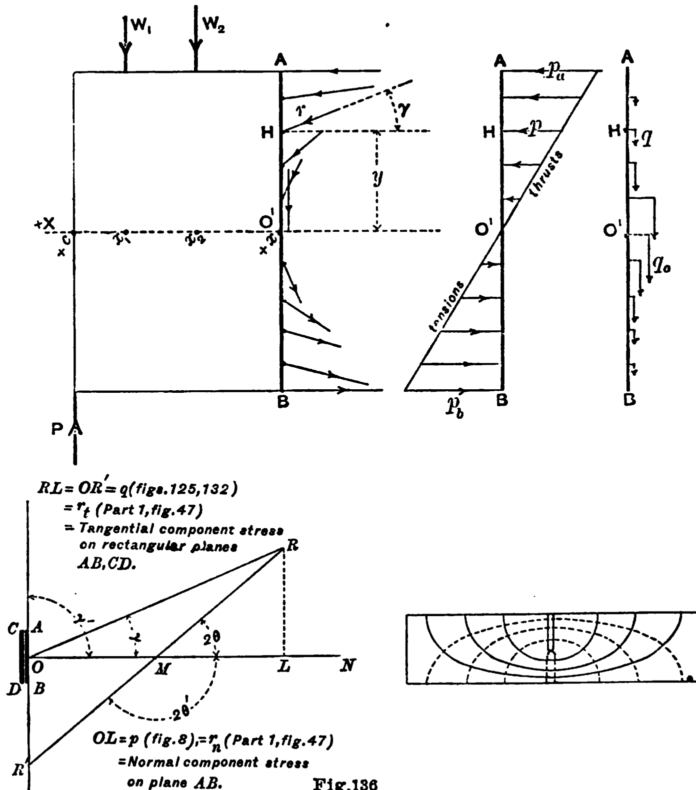


Fig. 136

If the principal planes be drawn through *H* for a short distance on each side, that upon which the stress is thrust by a full line, and that upon which it is tension by a dotted line, and if this be done for a number of points *H* on the elevation; then the full lines and the dotted lines will each form a series of polygons with short sides, and if we take the points H close enough, each polygon becomes a curve. These curves are called lines of principal stress; and the tangents thereto at any point H, where a dotted-line and a full-line curve intersect, are the planes of principal stress at H. See "Rankine's Applied Mechanics," sect. 310, and his "Treatise on Shipbuilding."

These curves have the following properties:—They cut the neutral axis at 45°; for, considering a point there, $p = 0$, and in fig. 136 R will fall on the line $R'O$ produced upwards, $2\theta = 90°$, and $\theta = \theta' = 45°$. Similarly all lines that meet the end section, that is the section over the point of support, meet it at an inclination of 45°; for, since the bending moment is zero, we again have $p = 0$. All lines meet the upper and under skin at right angles, since $q = 0$ at A and B, fig. 9.

In the elevation of the same beam, these curves will be different for different loads, except when the loads are kept in the same positions on the beam and altered in a fixed ratio; thus, for a rectangular beam,—if the load be uniform, then for positions of H ranged on a horizontal line, the arrows p and q, figs. 8 and 9 will both vary, since the bending moment and shearing force alter for each cross section; if the load be at the middle of span, p will vary, but q will not since the shearing force is the same at each cross section.

The planes of principal stress will differ in the elevations of two beams which are loaded alike, and whose cross sections are different but uniform throughout in each case; for instance, if the cross sections be a rectangle and a triangle respectively, we have $q_{\text{max.}}$ at the centre of the cross section in both cases; the neutral axis, however, is at the centre in one case but not in the other; so that, though both be loaded alike, the planes of principal stress will differ in the two elevations.

In designing beams, it will be seen that we have followed the usual practice of considering p_a and p_b at the section of maximum bending moment to be the greatest value of thrust and tensile stress respectively, and q_0 for the section over the greater supporting force to be the greatest intensity of shearing stress. Strictly the points at which these maxima occur are to be defined thus:—Let x, y, fig. 6, be the co-ordinates of any point H referred to O the centre of the neutral axis as origin; then for the cross section at x—

$$M_x = np_a bh^2, \text{ and } p_a = \frac{M_x}{nbh^2};$$

also, $$p_y : p_a :: y : m'h; \therefore p_y = \frac{yM_x}{m'nbh^3}.$$

We also have—

$$q' = \frac{F_x}{S}, \text{ and } q_0 = \frac{kF_x}{S};$$

k being the ratio of the maximum and the average intensity of the shearing stress for such cross section; q_y is readily derived by considering the manner in which the stress is distributed. Then, fig. 136,

$$MR^2 = q_y^2 + (\tfrac{1}{2}p_y)^2,$$

and by finding the values of x and y which make this a maximum, we get the point at which the intensity of the shearing stress is greatest; also

$$(OM \pm MR) = \tfrac{1}{2}p_y \pm \sqrt{q_y^2 + (\tfrac{1}{2}p_y)^2},$$

and by finding the values of x and y which make this sum and difference a maximum respectively, we get the maximum value of the intensity of the thrust and of the tension.

To find these maxima is extremely difficult even in the easiest cases; for instance, for a beam of uniform rectangular cross section, loaded at the centre, we have—

$$p_a = \frac{\frac{1}{2}W(c-x)}{nbh^2} = \frac{3W(c-x)}{bh^2}; \; p_y = \frac{y}{\frac{1}{2}h}p_a = \frac{6W(c-x)y}{bh^3};$$

$$q' = \frac{\frac{1}{2}W}{bh} = \frac{W}{2bh}, \text{ and } q_0 = \tfrac{3}{2}q' = \frac{3W}{4bh};$$

$$q_y = \frac{4q_0}{h^2}\left(\frac{h^2}{4} - y^2\right) = \frac{3W}{4bh^3}(h^2 - 4y^2), \text{ and}$$

$$MR^2 = \left(\frac{3W}{4bh^3}\right)^2 \{(h^2 - 4y^2)^2 + 16y^2(c-x)^2\};$$

for every value of y, $\overline{MR}^2$ is greatest when $x = 0$, and then—

$$\overline{MR}^2 = \left(\frac{3W}{4bh^3}\right)^2 \{(h^2 - 4y^2)^2 + 16c^2y^2\}$$

$$= \left(\frac{3W}{4bh^3}\right)^2 \{(2c^2 + 4y^2 - h^2)^2 - 4c^2(c^2 - h^2)\},$$

the greatest value of this is obtained by putting $y = \pm\frac{h}{2}$, then

$$\overline{MR} = \frac{3Wc}{2bh^2}$$

at the surface of the beam at centre of span.

The quantity p_y is greatest when $x = 0$, and $y = \frac{h}{2}$; then

$$p_y = \frac{6W(c - x)y}{bh^3} = \frac{3Wc}{bh^2};$$

and since p_y and $\overline{MR}$ thus have their greatest values at the same point, viz., at the surface at the middle of the beam, the greatest principal stress is there situated, and its amount is—

$$\overline{OM} + \overline{MR} = \frac{3Wc}{bh^2}.$$

Example.

162. A beam of constant rectangular section 9″ broad, 20″ deep, and 20 feet span, bears a load of 96 tons uniformly distributed. At a point in the section half-way between the centre and left end, and half-way between the neutral axis and upper skin, find the greatest intensities of thrust, of tension, and of shearing stress; and find the inclinations of the planes of principal stress at the point.

$M_5 = 2160$ inch-tons; $\frac{1}{6}p_a bh^2 = M_5$; $\therefore\ p_a = \frac{6 \times 2160}{9 \times 20^2} = 3{\cdot}6$,

and $\qquad p_y = \frac{1}{2}p_a = 1{\cdot}8$ tons per sq. inch.

$F_5 = 24$ tons; $q' = \frac{F}{S} = \frac{24}{9 \times 20} = \frac{2}{15}$; $q_0 = \frac{3}{2}q' = {\cdot}2$; and

$$q_y = \tfrac{3}{4}q_0 = {\cdot}15 \text{ ton per sq. in.}; \; MR^2 = {\cdot}15^2 + {\cdot}9^2.$$

$\therefore\ MR = {\cdot}91$ ton per sq. inch max. int. of shearing stress.

$OM + MR = \cdot 9 + \cdot 91 = 1{\cdot}81$ tons per sq. in. max. (thrust);
$OM - MR = \quad - \cdot 01$ „ „ (tension).

$$\text{Tan } 2\theta = \frac{RL}{ML} = \frac{\cdot 15}{\cdot 90} = 0{\cdot}1\dot{6}; \quad \therefore \quad 2\theta = 9° \, 28';$$

$\theta = 4° \, 44'$, the inclination of the plane of greater principal stress to the cross section.

Curvature, Slope, and Deflection.

Curvature.—At any point in a plane curve, the *direction* of the curve is that of the tangent at the point; and the *curvature* is the rate of change of direction at the point.

If two points be taken on a circle, the change of direction as you pass along the arc from one to the other is the angle which the tangent at one of the points makes with the tangent at the other; this angle is equal to the angle at the centre subtended by the arc between the points; since this change of direction takes place uniformly, the rate of change is found by dividing the total change by the arc; the total change as just stated is the angle at the centre, and this angle when expressed in circular measure is the ratio of the arc to the radius; dividing this ratio by the arc, we then have for every point of a circle—

$$\text{Curvature} = \frac{1}{r}; \qquad \text{(1.)}$$

If we take a point in any plane curve, we can find a circle which coincides with the curve for a short arc in the vicinity of the point; and if ρ be the radius of that circle, then the curvature of the circle everywhere being $1 \div \rho$, it is clear that for that particular point of the plane curve,

$$\text{Curvature} = \frac{1}{\rho}; \qquad \text{(2.)}$$

For any point in the neutral axis of a beam (or cantilever), we have, page 13, fig. 5—

$$\text{Curvature} = \frac{1}{\rho} = \frac{\left(\frac{\gamma}{ds}\right)}{y} = \frac{\text{strain on any horizontal layer}}{\text{dist. from neut. axis to layer}}.$$

$$= \frac{\left(\frac{a}{ds}\right)}{y_a} = \frac{\left(\frac{\beta}{ds}\right)}{y_b}$$

$$= \frac{\text{strain on either skin}}{\text{dist. of skin from neut. axis}}.$$

Now the strain on a horizontal layer is equal to the longitudinal stress on the layer divided by E the modulus of elasticity of the material; see Part I., page 9; hence

$$\frac{1}{\rho} = \frac{\text{stress on any horizontal layer}}{\text{dist. of layer from neut. axis}} \times \frac{1}{E}$$

$$= \frac{p}{y} \cdot \frac{1}{E} = \frac{p_a}{y_a} \cdot \frac{1}{E};$$

p_a being the normal stress at the skin on the cross section, and y_a the distance of the skin from the neutral axis as shown in fig. 8; but page 236, we have

$$\frac{p_a}{y_a} = \frac{M}{I}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

hence at any point of the neutral axis, the curvature due to any load which induces the bending moment M on the cross section passing through the point, is

$$\frac{1}{\rho} = \frac{1}{E} \frac{M}{I}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

I being the moment of inertia of the cross section, and E the modulus of elasticity of the material. Choosing as origin that point where the neutral axis crosses the section of greatest bending moment, we have the curvature at that point for a beam or a cantilever,—

$$\frac{1}{\rho_0} = \frac{1}{E} \cdot \frac{M_0}{I_0}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4a)$$

and if we wish this to correspond to the proof *or* working load, it is only necessary to make M_0 the bending moment due to the one *or* the other. These values of M_0 can be easily obtained from equation 3, by substituting for p_a the value f corresponding to the proof *or* working strength of the material. For this cross section of greatest bending moment, putting y_0 instead of y_a—

$$\frac{1}{\rho_0} = \frac{1}{E}\cdot\frac{f}{y_0} = \frac{1}{E}\cdot\frac{f}{m'h};$$

so that
$$\frac{1}{E}\cdot\frac{M_0}{I_0} = \frac{1}{E}\cdot\frac{f}{y_0}; \text{ or } \frac{f}{y_0}\cdot\frac{I_0}{M_0} = 1.$$

Multiplying the value for $\frac{1}{\rho}$ by this quantity, which being unity will not alter the value, we have—

$$\frac{1}{\rho} = \left(\frac{1}{E}\cdot\frac{M}{I}\right)\left(\frac{f}{y_0}\cdot\frac{I_0}{M_0}\right)$$
$$= \frac{f}{Ey_0}\cdot\frac{MI_0}{M_0I}; \quad\text{.........................} (5.)$$

The slope of a beam or cantilever at any point whose abscissa is x_1, the origin being at the point where the neutral axis is horizontal, is found as follows:—Let S be any point between the origin and that point; then at fig. 5 we had for the increment of slope between T and S, supposing these points to be indefinitely close,

$$di = \frac{1}{\rho}\times dx; \quad\text{.....................} (6)$$

if we add these increments di from point to point between the origin where the slope is zero and the point x_1, it gives us the slope at that point; and

$$i_{x_1} = \int_0^{x_1}\left(\frac{1}{\rho}\right)dx; \quad\text{.....................} (7)$$

the right hand side expresses, in the language of the Integral Calculus, the summation of the products equal respectively to these increments.

In the case of a beam symmetrically loaded, the origin will be at the centre of the span, and in practice, for any load, the origin will be sensibly in the same position; in the case of a cantilever, the origin will be at the point of support.

At the point of support of a beam, and at the free end of a cantilever, we thus have *the slope*,

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx, \text{ .... } (7a)$$

that is the integral with respect to x of the general expression for the curvature between the limits c and 0.

Deflection of the neutral axis of a beam or cantilever at any point whose abscissa is x_2. Let x_1 be the abscissa of S, figs. 4 and 5; in the case of a beam let u be the height of S above the lowest point of the neutral axis, in the case of a cantilever let u be the depth below the highest point, and let du be the small difference of level between S and T, two points indefinitely close; then

$$\frac{du}{dx_1} = \tan i_{x_1} = i_{x_1} \text{ (the angle being small)},$$

or the difference of height of S and T is

$$du = i_{x_1} . dx_1 ; \text{ .... } (8)$$

and summing all these increments from point to point, between the point $x = 0$ (the lowest point) and the point x_2, we have the height of x_2 above 0—

$$u_{x_2} = \int_0^{x_2} (i_{x_1}) dx_1, \text{ .... } (9)$$

the right hand side expressing, in the language of the Integral Calculus, the summation of the products equal respectively to those increments.

The height of the end above the centre of a beam, or above the free end of a cantilever, is given by the equation

$$u_c = \int_0^c (i_{x_1}) dx_1 ; \text{ .... } (9a)$$

this in the case of a beam is the deflection of the centre below the points of support, and in the case of a cantilever is the deflection of the free end below the fixed end; so that *the deflection*

$$v_0 = \int_0^c (i_{x_1}) dx_1 . \text{ .... } (10.)$$

Again the deflection at any point x_2 is

$$v_{x_2} = v_0 - u_{x_2} = v_0 - \int_0^{x_2} (i_{x_1}) dx_1 ; \text{......} \quad (10a.)$$

For i_{x_1} we may substitute the value in eqn. 7, and

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \,.\, dx_1 ; \text{..................} \quad (11)$$

$$v_{x_2} = v_0 - \int_0^{x_2} \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \,.\, dx_1 . \text{.........} \quad (11a.)$$

Beam of uniform section, load at centre.—For all cases of *uniform* section I is constant; and for load at centre, we have, fig. 22,—

$$\frac{M}{M_0} = \frac{W}{2}(c-x) \div \tfrac{1}{2} Wc = \left(1 - \frac{x}{c}\right);$$

$$\frac{1}{\rho} = \frac{f}{Ey_0}\left(1 - \frac{x}{c}\right) \text{.................................} \quad (1.)$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0} \int_0^c \left(1 - \frac{1}{c} x\right) . dx$$

$$= \frac{f}{Ey_0}\left[x - \frac{1}{c} \cdot \frac{x^2}{2}\right]_0^c = \frac{f}{Ey_0}\left(c - \frac{1}{c}\,\frac{c^2}{2} - 0\right)$$

$$= \tfrac{1}{2} \cdot \frac{f}{E} \cdot \frac{c}{y_0}, \text{..................................} \quad (2.)$$

In every case the expression for i_c *the slope* can be arranged into these three factors; a numerical factor depending on the form of the cross section, in this case $\frac{1}{2}$, and for which Rankine puts m''; a factor $\frac{f}{E}$ depending on the material, and a factor $\frac{c}{y_0}$ depending on the dimensions. The general expression is

$$i_c = m'' \frac{f}{E} \cdot \frac{c}{y_0} ; \text{..............................} \quad (2a)$$

and m'' is called the numerical co-efficient for the slope.

The deflection

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \, . \, dx_1 = \frac{f}{Ey_0} \int_0^c \int_0^{x_1} \left(1 - \frac{1}{c} x\right) dx \, . \, dx_1$$

$$= \frac{f}{Ey_0} \int_0^c \left[x - \frac{1}{c} \frac{x^2}{2} \right]_0^{x_1} dx_1 = \frac{f}{Ey_0} \int_0^c \left(x_1 - \frac{1}{c} \frac{x_1^2}{2} - 0 \right) dx_1$$

$$= \frac{f}{Ey_0} \left[\frac{x_1^2}{2} - \frac{1}{2c} \frac{x_1^3}{3} \right]_0^c = \frac{f}{Ey_0} \left(\frac{c^2}{2} - \frac{1}{2c} \cdot \frac{c^3}{3} - 0 \right)$$

$$= \tfrac{1}{3} \cdot \frac{f}{E} \frac{c^2}{y_0}; \quad \text{(3.)}$$

In every case the expression for *the* deflection can be arranged into these three factors; a numerical factor depending on the form of the section, in this case $\frac{1}{3}$, and for which Rankine puts n''; a factor $\frac{f}{E}$ depending on the material, and a factor $\frac{c^2}{y_0}$ depending on the dimensions. The general expression is

$$v_0 = n'' \frac{f}{E} \frac{c^2}{y_0}; \quad \text{(3a)}$$

and n'' is called the numerical coefficient for *the* deflection.

In this case, the numerical coefficients are—

$$m'' = \tfrac{1}{2}; \; n'' = \tfrac{1}{3}.$$

Observe that f is the working or proof strength of the material, according as you may want the working or proof values of i_c and v_0. If f be given, as is generally the case, in lbs. per sq. inch, then E is to be in lbs. per sq. inch, and y_0 and c in inches; if we then calculate ρ or ρ_0, v or v_0, they will be in inches also.

From equation (1), if we put $x = 0$, we get

$$\rho_0 = \frac{Ey_0}{f}; \quad \text{(4.)}$$

This will be the expression for ρ_0 in every case, and we shall not repeat it.

The inclinations i, when calculated from these formulæ, are in circular measure.

Beam of uniform section, uniform load.—From fig. 25—

$$\frac{M}{M_0} = \frac{W}{4c}(c^2 - x^2) \div \tfrac{1}{4} Wc = \left(1 - \frac{x^2}{c^2}\right).$$

$$\frac{1}{\rho} = \frac{f}{Ey_0} \cdot \frac{MI_0}{M_0 I} = \frac{f}{Ey_0}\left(1 - \frac{x^2}{c^2}\right); \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.)}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0}\int_0^c \left(1 - \frac{1}{c^2}x^2\right) dx$$

$$= \frac{f}{Ey_0}\left[x - \frac{1}{c^2}\frac{x^3}{3}\right]_0^c = \frac{f}{Ey_0}\left(c - \frac{1}{c^2}\frac{c^3}{3} - 0\right)$$

$$= \tfrac{2}{3}\frac{f}{E} \cdot \frac{c}{y_0}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.)}$$

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \,.\, dx_1 = \frac{f}{Ey_0}\int_0^c \int_0^{x_1} \left(1 - \frac{1}{c^2}x^2\right) dx \,.\, dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \left[x - \frac{1}{c^2}\frac{x^3}{3}\right]_0^{x_1} dx_1 = \frac{f}{Ey_0}\int_0^c \left(x_1 - \frac{1}{c^2}\frac{x_1^3}{3} - 0\right) dx_1$$

$$= \frac{f}{Ey_0}\left[\frac{x^2}{2} - \frac{1}{3c^2}\frac{x^4}{4}\right]_0^c = \frac{f}{Ey_0}\left(\frac{c^2}{2} - \frac{1}{3c^2}\frac{c^4}{4} - 0\right)$$

$$= \tfrac{5}{12}\frac{f}{E} \cdot \frac{c^2}{y_0}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.)}$$

The numerical co-efficients are

$$m'' = \tfrac{2}{3}; \; n'' = \tfrac{5}{12}.$$

Cantilever of uniform section, load at end.—From fig. 23—

$$\frac{M}{M_0} = W(c - x) \div Wc = \left(1 - \frac{x}{c}\right)$$

exactly as in the case of a beam loaded at the centre; and the numerical co-efficients are—

$$m'' = \tfrac{1}{2}; \; n'' = \tfrac{1}{3}.$$

Cantilever of uniform section, uniform load.—I cancels I_0; from fig. 27—

$$\frac{M}{M_0} = \frac{W}{2c}(c-x)^2 \div \tfrac{1}{2}Wc = \left(1-\frac{x}{c}\right)^2.$$

$$\frac{1}{\rho} = \frac{f}{Ey_0}\frac{MI_0}{M_0 I} = \frac{f}{Ey_0}\left(1-\frac{x}{c}\right)^2; \text{ .... (1.)}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right)dx = \frac{f}{Ey_0}\int_0^c \left(1-\frac{2}{c}x+\frac{1}{c^2}x^2\right)dx$$

$$= \frac{f}{Ey_0}\left[x-\frac{2}{c}\frac{x^2}{2}+\frac{1}{c^2}\frac{x^3}{3}\right]_0^c = \frac{f}{Ey_0}\left(c-\frac{2}{c}\frac{c^2}{2}+\frac{1}{c^2}\frac{c^3}{3}-0\right)$$

$$= \tfrac{1}{3}\cdot\frac{f}{E}\cdot\frac{c}{y_0}; \text{ .... (2.)}$$

$$v_0 = \int_0^c \int_0^{x_1}\left(\frac{1}{\rho}\right)dx\,.\,dx_1 = \frac{f}{Ey_0}\int_0^c \int_0^{x_1}\left(1-\frac{2}{c}x+\frac{1}{c^2}x^2\right)dx\,.\,dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \left[x-\frac{2}{c}\frac{x^2}{2}+\frac{1}{c^2}\frac{x^3}{3}\right]_0^{x_1}.\,dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \left(x_1-\frac{1}{c}x_1^2+\frac{1}{3c^2}x_1^3-0\right)dx_1$$

$$= \frac{f}{Ey_0}\left[\frac{x_1^2}{2}-\frac{1}{c}\frac{x_1^3}{3}+\frac{1}{3c^2}\frac{x_1^4}{4}\right]_0^c$$

$$= \frac{f}{Ey_0}\left(\frac{c^2}{2}-\frac{1}{c}\frac{c^3}{3}+\frac{1}{3c^2}\frac{c^4}{4}-0\right)$$

$$= \tfrac{1}{4}\frac{f}{E}\cdot\frac{c^2}{y_0}; \text{ .... (3.)}$$

The numerical co-efficients are

$$m'' = \tfrac{1}{3};\; n'' = \tfrac{1}{4}.$$

Beam of uniform section, bending moment constant.—If a beam be symmetrically placed on, and extend beyond its two points of support; and if the two projecting parts be loaded symmetrically, while the intermediate portion is unloaded;

the bending moment on the portion of the beam between the points of support is constant, and we have I and I_0, M and M_0, cancelling each other.

$$\frac{1}{\rho} = \frac{f}{Ey_0}; \quad \text{(1)}$$

a constant quantity, so that the neutral axis is circular.

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{f}{Ey_0}\int_0^c dx = \frac{f}{Ey_0}[x]_0^c = \frac{f}{Ey_0}(c-0)$$

$$= \frac{f}{E}\cdot\frac{c}{y_0}; \quad \text{(2.)}$$

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx \, . \, dx_1$$

$$= \frac{f}{Ey_0}\int_0^c \int_0^{x_1} dx \, . \, dx_1 = \frac{f}{Ey_0}\int_0^c [x]_0^{x_1} dx_1$$

$$= \frac{f}{Ey_0}\int_0^c (x_1 - 0) dx_1 = \frac{f}{Ey_0}\left[\frac{x_1^2}{2}\right]_0^c = \frac{f}{Ey_0}\left(\frac{c^2}{2} - 0\right)$$

$$= \tfrac{1}{2} \, . \, \frac{f}{E}\cdot\frac{c^2}{y_0}; \quad \text{(3.)}$$

The numerical co-efficients are

$$m'' = 1; \; n'' = \tfrac{1}{2}.$$

Beam of uniform section, loaded with two equal weights at equal distances from the centre of span.—Let W be the total load, x_r and $-x_r$ the abscissæ of the loads. $P = \frac{1}{2}W$; $M_0 = \frac{1}{2}W(c - x_r)$, and the bending moment is constant along the central portion of span; for values of x between x_r and c, $M_x = \frac{1}{2}W(c-x)$. For the central portion of span, $\frac{M_x}{M_0} = 1$; for the end portion $\frac{M_x}{M_0} = \frac{c-x}{c-x_r}$.

$$\frac{1}{\rho} = \frac{f}{Ey_0}\frac{M}{M_0}$$

$$\frac{1}{\rho} = \frac{f}{Ey_0}, \quad \text{(1)}$$

for values of x from 0 to x_r.

$$\frac{1}{\rho} = \frac{f}{Ey_0}\,\frac{1}{c - x_r}(c - x), \quad \text{(1a)}$$

for values of x from x_r to c.

$$i_c = \int_0^c \left(\frac{1}{\rho}\right).\,dx = \frac{f}{Ey_0}\left\{\int_0^{x_r} dx + \frac{1}{c - x_r}\int_{x_r}^{c}(c - x)dx\right\}$$

$$= \frac{f}{Ey_0}\left\{[x]_0^{x_r} + \frac{1}{c - x_r}\left[cx - \frac{x^2}{2}\right]_{x_r}^{c}\right\}$$

$$= \frac{f}{Ey_0}\left\{(x_r - 0) + \frac{1}{c - x_r}(c^2 - \tfrac{1}{2}c^2 - cx_r + \tfrac{1}{2}x_r^2)\right\}$$

$$= \tfrac{1}{2}\left(1 + \frac{x_r}{c}\right)\frac{f}{E}\,\frac{c}{y_0}; \quad \text{(2.)}$$

$$v_0 = \int_0^c \int_0^{x_1} \frac{1}{\rho}.\,dx.\,dx_1$$

$$= \int_0^{x_r} \int_0^{x_1} \frac{1}{\rho}\,dx.\,dx_1 + \int_{x_r}^{c} \int_0^{x_1} \frac{1}{\rho}\,dx.\,dx_1$$

$$\frac{Ey_0}{f}\,v_0 = \int_0^{x_r} [x]_0^{x_1} dx_1 + \int_{x_r}^{c}\left\{x_r + \frac{1}{c - x_r}[cx - \tfrac{1}{2}x^2]_{x_r}^{x_1}\right\}dx_1$$

$$= \int_0^{x_r} [x_1 - 0]dx_1 + \frac{1}{c - x_r}\int_{x_r}^{c}(cx_1 - \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_r^2)dx_1$$

$$= [\tfrac{1}{2}x_1^2]_0^{x_r} + \frac{1}{c - x_r}\left[\frac{c}{2}x_1^2 - \tfrac{1}{6}x_1^3 - \tfrac{1}{2}x_r^2 x_1\right]_{x_r}^{c}$$

$$= \tfrac{1}{2}x_r^2 + \frac{1}{c - x_r}\left[\frac{c^3}{2} - \frac{c^3}{6} - \frac{c}{2}x_r^2 - \frac{c}{2}x_r^2 + \tfrac{1}{6}x_r^3 + \tfrac{1}{2}x_r^3\right]$$

$$v_0 = \left\{\frac{2c^3 - 3cx_r^2 + x_r^3}{6(c - x_r)}\right\}\frac{f}{Ey_0};$$

$$= \left\{\tfrac{1}{3} + \frac{x_r}{3c} - \tfrac{1}{6}\frac{x_r^2}{c^2}\right\}\frac{f}{E}\,\frac{c^2}{y_0}.$$

$$v_0 = \left\{ \tfrac{1}{2} - \tfrac{1}{6}\left(1 - \frac{x_r}{c}\right)^2 \right\} \frac{f}{E}\frac{c^2}{y_0}; \text{..........................} (3.)$$

Thus, for an uniform beam, loaded at two points which are equidistant from the centre, and so that the stress on the outer fibres of the middle cross section is f, we have—

$$m'' = \tfrac{1}{2}\left(1 + \frac{x_r}{c}\right); \; n'' = \tfrac{1}{2} - \tfrac{1}{6}\left(1 - \frac{x_r}{c}\right)^2.$$

Beam or cantilever of uniform strength and uniform depth.—For the proof or working load, the bending moment at each section equals the proof or working moment of resistance to bending there; since, however, the depth is constant, the moment of resistance to bending at each cross section is proportional to the breadth, that is the breadth at such sections is proportional to M; and since the depth is constant, I for each section is proportional to the breadth;

hence $\frac{M}{I}$ is constant at every section, $\frac{MI_0}{M_0 I} = 1$, and

$$\frac{1}{\rho} = \frac{f}{Ey_0}; \text{....................................} (1)$$

as in the case of a beam of uniform section and bending moment constant, page 298; so that for beams or cantilevers to resist *any load*, and made of uniform strength by varying the breadth only, the numerical co-efficients for the slope and deflection are

$$m'' = 1; \; n'' = \tfrac{1}{2}.$$

Beam of uniform strength and uniform breadth, loaded at the centre.—See text at figs. 92, 97, &c., where it is explained that the elevation is obtained by degrading the bending moment diagram.

Since M is proportional to h^2, and I to h^3, we have $\frac{I}{M}$

proportional to h; and

$$\frac{MI_0}{M_0I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \sqrt{\frac{c}{c-x}},$$

since from fig. 22 we have

$$\frac{M_0}{M} = \tfrac{1}{2}Wc \div \frac{W}{2}(c-x).$$

$$\frac{1}{\rho} = \frac{f}{Ey_0}\,\frac{MI_0}{M_0I} = \frac{f\sqrt{c}}{Ey_0}(c-x)^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = -\frac{f\sqrt{c}}{Ey_0}\int_0^c (c-x)^{-\frac{1}{2}}(-dx)$$

$$= -\frac{f\sqrt{c}}{Ey_0}\left[\frac{(c-x)^{\frac{1}{2}}}{(\frac{1}{2})}\right]_0^c = -\frac{2f\sqrt{c}}{Ey_0}\left[(c-x)^{\frac{1}{2}}\right]_0^c$$

$$= -\frac{2f\sqrt{c}}{Ey_0}\left\{0^{\frac{1}{2}} - c^{\frac{1}{2}}\right\}$$

$$= 2\,.\,\frac{f}{E}\cdot\frac{c}{y_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

$$v_0 = \frac{f\sqrt{c}}{Ey_0}\int_0^c\int_0^{x_1}(c-x)^{-\frac{1}{2}}(-dx)(-dx_1)$$

$$= \frac{f\sqrt{c}}{Ey_0}\int^c\left[\frac{(c-x)^{\frac{1}{2}}}{(\frac{1}{2})}\right]_0^{x_1}(-dx_1) = \frac{2f\sqrt{c}}{Ey_0}\int_0^c\left[(c-x)^{\frac{1}{2}}\right]_0^{x_1}(-dx_1)$$

$$= \frac{2f\sqrt{c}}{Ey_0}\int_0^c \left\{(c-x_1)^{\frac{1}{2}} - c^{\frac{1}{2}}\right\}(-dx_1)$$

$$= \frac{2f\sqrt{c}}{Ey_0}\left\{\int_0^c (c-x_1)^{\frac{1}{2}}(-dx_1) + c^{\frac{1}{2}}\int_0^c dx_1\right\}$$

$$= \frac{2f\sqrt{c}}{Ey_0}\left[\frac{(c-x_1)^{\frac{3}{2}}}{(\frac{3}{2})} + c^{\frac{1}{2}}x_1\right]_0^c = \frac{4f\sqrt{c}}{3Ey_0}\left[(c-x_1)^{\frac{3}{2}} + \tfrac{3}{2}c^{\frac{1}{2}}x_1\right]_0^c$$

$$= \frac{4f\sqrt{c}}{3Ey_0}\left\{(0 + \tfrac{3}{2}c^{\frac{3}{2}}) - (c^{\frac{3}{2}} + 0)\right\}$$

$$= \tfrac{2}{3}\,\frac{f}{E}\frac{c^2}{y_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

The numerical co-efficients are

$$m'' = 2\,;\ n'' = \tfrac{2}{3}.$$

Cantilever of uniform strength and uniform breadth, loaded at the end.

$$\frac{MI_0}{M_0 I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \sqrt{\frac{c}{c-x}},$$

since from fig. 23 we have

$$M_0 \div M = Wc \div W(c-x).$$

The value of $\frac{1}{\rho}$ is the same as for the preceding case, and the co-efficients for the slope and deflection are

$$m'' = 2\,;\ n'' = \tfrac{2}{3}.$$

Beam of uniform strength and uniform breadth, uniformly loaded.

$$\frac{MI_0}{M_0 I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \frac{c}{\sqrt{c^2 - x^2}},$$

since from fig. 24, we have

$$M_0 \div M = \tfrac{1}{4} Wc \div \frac{W}{4c}(c^2 - x^2).$$

$$\frac{1}{\rho} = \frac{f}{Ey_0} \cdot \frac{MI_0}{IM_0} = \frac{fc}{Ey_0} \frac{1}{\sqrt{c^2 - x^2}} \quad \text{.........................(1.)}$$

$$i_c = \int_0^c \left(\frac{1}{\rho}\right) dx = \frac{fc}{Ey_0} \int_0^c \frac{dx}{\sqrt{c^2 - x^2}} = \frac{fc}{Ey_0} \left[\sin^{-1} \frac{x}{c}\right]_0^c$$

$$= \frac{fc}{Ey_0} \{\sin^{-1}(1) - \sin^{-1}(0)\}$$

$$= \frac{\pi}{2} \cdot \frac{f}{E} \cdot \frac{c}{y_0} \quad \text{...(2.)}$$

$$v_0 = \int_0^c \int_0^{x_1} \left(\frac{1}{\rho}\right) dx\, dx_1 = \frac{fc}{Ey_0} \int_0^c \int_0^{x_1} \frac{dx}{\sqrt{c^2 - x^2}} dx_1$$

$$v_0 = \frac{fc}{Ey_0}\int_0^c \left[\sin^{-1}\frac{x}{c}\right]_0^{x_1} dx_1$$

$$= \frac{fc}{Ey_0}\int_0^c \left\{\sin^{-1}\frac{x_1}{c} - \sin^{-1}0\right\} dx_1$$

$$= \frac{fc}{Ey_0}\int_0^c \sin^{-1}\left(\frac{x_1}{c}\right) dx_1; \text{ integrating by parts}$$

$$= \frac{fc}{Ey_0}\left[x_1 \sin^{-1}\left(\frac{x_1}{c}\right) - \int x_1 \cdot \frac{dx_1}{\sqrt{c^2 - x_1^2}}\right]_0^c$$

$$= \frac{fc}{Ey_0}\left[x_1 \sin^{-1}\left(\frac{x_1}{c}\right) + \tfrac{1}{2}\int (c^2 - x_1^2)^{-\frac{1}{2}}(-2x_1 dx_1)\right]_0^c$$

$$= \frac{fc}{Ey_0}\left[x_1 \sin^{-1}\left(\frac{x_1}{c}\right) + \tfrac{1}{2}\frac{(c^2 - x_1^2)^{\frac{1}{2}}}{(\frac{1}{2})}\right]_0^c$$

$$= \frac{fc}{Ey_0}\left[x_1 \sin^{-1}\left(\frac{x_1}{c}\right) + (c^2 - x_1^2)^{\frac{1}{2}}\right]_0^c$$

$$= \frac{fc}{Ey_0}\{(c \cdot \sin^{-1}1 + 0) - (0 + c)\} = \frac{fc}{Ey_0}\left(\frac{\pi}{2}c - c\right)$$

$$= \left(\frac{\pi}{2} - 1\right)\frac{f}{E}\frac{c^2}{y_0} \quad \text{.......... (3.)}$$

The numerical co-efficients are

$$m'' = \frac{\pi}{2} = 1{\cdot}5708; \; n'' = \left(\frac{\pi}{2} - 1\right) = 0{\cdot}5708.$$

Cantilever of uniform strength and uniform breadth, uniformly loaded.

$$\frac{MI_0}{M_0 I} = \frac{h_0}{h} = \sqrt{\frac{M_0}{M}} = \frac{c}{c-x},$$

since from fig. 27, we have

$$\frac{M_0}{M} = \tfrac{1}{2}Wc \div \frac{W}{2c}(c-x)^2.$$

$$\frac{1}{\rho} = -\frac{f}{Ey_0}\frac{MI_0}{M_0 I} = \frac{fc}{Ey_0} \cdot \frac{1}{c-x}; \quad \text{.......... (1.)}$$

$$i_c = -\frac{fc}{Ey_0}\int_0^c \frac{-dx}{c-x} = -\frac{fc}{Ey_0}\Big[\log(c-x)\Big]_0^c$$

$$= -\frac{fc}{Ey_0}\{\log 0 - \log c\} = \frac{fc}{Ey_0}\{\log c - \log 0\}$$

$$= \frac{fc}{Ey_0}\{\log c + \infty\}$$

$$= \infty\,\frac{f}{E}\frac{c}{y_0}; \qquad \text{(2.)}$$

$$v_0 = -\frac{fc}{Ey_0}\int_0^c\int_0^{x_1}\frac{-dx}{(c-x)}dx_1 = -\frac{fc}{Ey_0}\int_0^c\Big[\log(c-x)\Big]_0^{x_1}dx_1$$

$$= -\frac{fc}{Ey_0}\int_0^c\{\log(c-x_1) - \log c\}dx_1$$

$$= \frac{fc}{Ey_0}\int_0^c\left\{-\log\left(1-\frac{x_1}{c}\right)\right\}dx_1$$

$$= \frac{fc}{Ey_0}\int_0^c\left\{\frac{x_1}{c} + \frac{1}{2}\frac{x_1^2}{c^2} + \frac{1}{3}\frac{x_1^3}{c^3} + \&c.\right\}dx_1$$

$$= \frac{fc}{Ey_0}\left[\frac{1}{c}\frac{x_1^2}{2} + \frac{1}{2c^2}\frac{x_1^3}{3} + \frac{1}{3c^3}\frac{x_1^4}{4} + \&c.\right]_0^c$$

$$= \frac{fc}{Ey_0}\left\{\left(\frac{1}{c}\frac{c^2}{2} + \frac{1}{2c^2}\frac{c^3}{3} + \frac{1}{3c^3}\frac{c^4}{4} + \&c.\right) - (0)\right\}$$

$$= \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c.\right)\frac{f}{E}\cdot\frac{c^2}{y_0}$$

$$= \{(1-\tfrac{1}{2}) + (\tfrac{1}{2}-\tfrac{1}{3}) + (\tfrac{1}{3}-\tfrac{1}{4}) + \&c.\}\,\frac{f}{E}\cdot\frac{c^2}{y_0}$$

$$= \frac{f}{E}\cdot\frac{c^2}{y_0}; \qquad \text{(3.)}$$

The numerical co-efficients are

$$m'' = \text{infinity}; \; n'' = 1.$$

Proportion of the greatest depth of a beam to the span.—Putting the working strength of the material as the value of f, we have as a general formula

$$v_0 = n'' \frac{f}{E} \frac{c^2}{y_0};$$

and since $y_0 = m' h_0$, we have

$$\frac{v_0}{2c} = \frac{n''}{4m'} \cdot \frac{f}{E} \cdot \frac{2c}{h_0}.$$

Now $\frac{v_0}{2c}$ is the ratio of the deflection to the span, and its reciprocal $\frac{2c}{v_0}$ represents the stiffness of the beam; $\frac{h_0}{2c}$ is the ratio of depth of beam at centre to span, and we have

$$\frac{2c}{v_0} = \frac{4m'}{n''} \frac{E}{f} \frac{h_0}{2c};$$

or, the stiffness of a beam is proportional to the ratio of the depth at centre to span.

For instance, to give a working stiffness 1000 to a wrought iron beam of uniform symmetrical section uniformly loaded; we have the ratio of depth at centre to span

$$\frac{h_0}{2c} = \tfrac{1}{4}\frac{n''}{m'} \cdot \frac{f}{E} \cdot \frac{2c}{v_0}$$

$$= \tfrac{1}{4}\frac{(\frac{5}{12})}{(\frac{1}{2})} \cdot \frac{10{,}000 \text{ lbs. per sq. in.}}{30{,}000{,}000 \text{ lbs. per sq. in.}} \cdot 1000$$

$$= \frac{1}{14 \cdot 4}.$$

That is to secure the degree of stiffness 1000 required for the wrought iron beam, the depth must not be less than a fourteenth of the span. In the same way it may be shown that, in general, to give to beams the degree of stiffness that practice shows to be necessary, and that is usually prescribed, the depth at centre must bear to the span a ratio varying from $\frac{1}{8}$th to $\frac{1}{14}$th, according to the material, form, and manner of loading.

Slope and deflection under any load less than the proof load.—For the proof load, the stress on the skin furthest from the neutral axis at the cross section of maximum bending moment is f, and for a smaller load it is p_a, fig. 8; now, though the load is less than the proof load, yet being distributed in the same way, the ratio $M : M_0$ is in no way changed; I is the same as before, and we have E and y_0 constant; so that

$$i_c' = m'' \frac{p_a}{E} \frac{c}{y_0}; \; v_0' = n'' . \frac{p_a}{E} . \frac{c^2}{y_0}.$$

But since $p_a = \frac{mWl}{nb_0h_0^2}$, and $y_0 = m'h$, we have—

$$i_c' = \frac{mm''}{m'n} \frac{Wlc}{Eb_0h_0^3}; \; v_0' = \frac{mn''}{m'n} \frac{Wlc^2}{Eb_0h_0^3}.$$

For a beam, $l = 2c$, and for a cantilever $l = c$; so that for any load,

for a beam, $$i_c' = \frac{2mm''}{m'n} \frac{Wc^2}{Eb_0h_0^3}; \; v_0' = \frac{2mn''}{m'n} \frac{Wc^3}{Eb_0h_0^3},$$

and for a cantilever, $$i_c' = \frac{mm''}{m'n} \frac{Wc^2}{Eb_0h_0^3}; \; v_0' = \frac{mn''}{m'n} \frac{Wc^3}{Eb_0h_0^3}.$$

The co-efficients $\frac{mm''}{m'n}$ and $\frac{mn''}{m'n}$ assume values which depend on the cross section and the system of loading.

For similar beams similarly loaded we thus have v_0', the deflection under any load less than the proof load, proportional to W the total load, and to c^3 or the cube of the length; and inversely as b_0 the breadth at centre of span, and h_0^3 the cube of depth at centre of span.

That v_0' is proportional to W does not follow from Hooke's Law, but has been established upon the supposition that the slope is so small that the tangent and circular measure of *the slope* are sensibly equal. In bending small pieces of wood in a machine which registers the load and the deflection as the co-ordinates of a line, it is found that the line is straight even when the piece of wood is bent to a considerable amount; this proves that the formula above is a close approximation, even when the slope is considerable;

it must not be forgotten, however, that the formula is *only* approximate when the slope is great. It has been stated that where such a machine's register ceases to be straight, the elastic limit has been passed; no such conclusion can be drawn; the only just conclusion to draw is that since the slope is visibly great, (say 8° or 10°), the formula above has ceased to be a close approximation. Were a second approximation made, it would be found that v_e' was not exactly proportional to W; and so long as the register did not depart from the curve which is the locus of that new equation, it would be inaccurate to infer that the elastic limit had been passed.

Examples.

163. A beam 24 feet span, of uniform symmetrical section as shown in fig. 134, is made of wrought iron whose working strength $f = 4$ tons per sq. inch, and whose modulus of elasticity $E = 11600$ tons per sq. inch. Find the radii of curvature at intervals of 4 feet, when loaded uniformly with the working load.

Taking equation 1, page 296, we have—

$$\frac{1}{\rho} = \frac{f}{Ey_0}\left(1 - \frac{x^2}{c^2}\right), \text{ where } x \text{ and } c \text{ are to be in one name,}$$

$$= \frac{4}{11600 \times 10}\left(1 - \frac{x^2}{144}\right);$$

$$\rho = 29000 \frac{144}{144 - x^2}, \ x \text{ being in feet.}$$

The radii of curvature are—

$$\rho_0 = 29000; \ \rho_4 = 32625; \ \rho_8 = 52200 \text{ ins.}; \ \rho_{12} = \text{infinity}$$
$$= 2420; \quad = 2720; \quad = 4350 \text{ feet.}$$

The reason that ρ is in inches is because y_0 is in inches, and the proportions derived at fig. 5 show clearly that ρ and y are in one name. We had to put y_0 in inches, because for the material f and E are given in tons on the sq. inch.

164. Calculate *the* slope and deflection in the previous example

$$i_c = \tfrac{2}{3}\frac{f}{E}\frac{c}{y_0} = \tfrac{2}{3}\cdot\frac{4}{11600}\cdot\frac{144}{10} = \cdot 0033\ ;$$

$$\therefore\quad i_c = 0^\circ\, 11'\ \text{nearly};$$

$$v_0 = \frac{5}{12}\frac{f}{E}\cdot\frac{c^2}{y_0} = \frac{5}{12}\cdot\frac{4}{11600}\cdot\frac{(144)^2}{10}$$

$$= 0\cdot 30\ \text{inch} = 0\cdot 025\ \text{foot}.$$

This is nearly one-thousandth of the span, which is about the extreme ratio of working deflection to span allowable in practice, (Rankine's "Applied Mechanics," sect. 302); and this degree of stiffness is secured by making the depth of this wrought iron girder a $\frac{1}{14\cdot 4}$th part of the span.

165. Find the slope and deflection in Example 163 by means of a graphical solution.

As in fig. 137, which is drawn to a scale of 20 feet to an inch, draw verticals; one to represent the vertical through the centre of span; the others to be drawn on each side of the centre, and at 2, 6, 10 and 12 feet therefrom; the last two will be through the extremities of span.

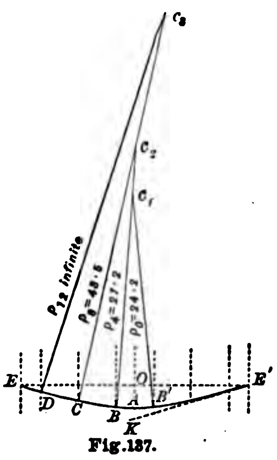


Fig. 137.

From c_1 any point in the central vertical, with radius 24·2 feet, that is a hundredth part of ρ_0, describe the arc $B'AB$ between the verticals through B and B'; produce Bc_1 to c_2, so that $c_2B = 27\cdot 2$, a hundredth part of ρ_4, and about c_2 describe the arc BC; similarly $c_3C = 43\cdot 5$; and from D, DE is drawn at right angles to Dc_3. On the other side of the point A construct $AB'E'$; draw the horizontal chord EOE', and the tangent $E'K$. Then EAE' represents the curve assumed by the neutral axis of the beam when under the proof load, but with its vertical dimensions exaggerated 100 fold.

The deflection $v_0 = \frac{1}{100} \times OA = \frac{1}{100} \times 2{\cdot}5$ (by scale)

$$= {\cdot}025 \text{ foot}$$

$$i_c = \tan i_c = \frac{OK}{100} \div OE' = \frac{4{\cdot}0}{100} \div 124 = {\cdot}003;$$

$$i_c = 0^\circ\, 11'.$$

166. If the beam in example 163 be loaded with one ton per foot of span, find the deflection.

Let W' be the working load, then $mW'l$ equals the working value of M_0; that is $\frac{1}{8}W'(24 \times 12) = 1948$; (see fig. 134.)

$$\therefore \quad W' = 54{\cdot}1 \text{ tons.}$$

Now the load in this example is 24 tons, $\frac{4}{9}$ths of W' the working load; and since deflection is proportional to load, we have—

$$\text{Deflection} = v_0' = \tfrac{4}{9} \times {\cdot}30 = 0{\cdot}13 \text{ inch} = 0{\cdot}011 \text{ foot.}$$

Otherwise; if the working deflection had not been already calculated, we have—

$$\text{Deflection} = v_0' = \left(\frac{2mn''}{m'n}\right)\frac{Wc^3}{Eb_0h_0^3} = \frac{2mn''}{(n'b_0h_0^3)}\,\frac{Wc^3}{E}$$

$$= \frac{2mn''}{I_0}\cdot\frac{Wc^3}{E}; \text{ but } I_0 = 4870 \text{ (fig. 134)}$$

$$= \frac{2(\frac{1}{8})(\frac{5}{12})}{4870}\cdot\frac{24 \times (144 \text{ in.})^3}{11600} = {\cdot}13 \text{ in.} = {\cdot}011 \text{ foot.}$$

167. Find the working deflection for a wrought iron beam of uniform strength and uniform breadth, and loaded uniformly; the span is 24 feet, and the upper half of the cross section at centre of beam is shown in fig. 134; $f = 4$ tons, and $E = 11600$ tons, per sq. inch.

$$v_0 = n''\cdot\frac{f}{E}\cdot\frac{c^2}{y_0} = \left(\frac{\pi}{2}-1\right)\frac{f}{E}\cdot\frac{c^2}{y_0} = {\cdot}57 \times \frac{4}{11600}\cdot\frac{144^2}{10}$$

$$= {\cdot}41 \text{ in.} = {\cdot}034 \text{ ft.}$$

168. A wrought iron rectangular beam 20 feet span, 16 inches deep, and 4 inches broad, is loaded at the centre. Calculate the proof deflection if the proof strength of the iron be 7 tons, and its modulus of elasticity 12000 tons per sq. in.

$$v_0 = n'' . \frac{f}{E} . \frac{c^2}{y_0} = \tfrac{1}{3} . \frac{7}{12000} . \frac{120^2}{8} = \cdot 35 \text{ inch} = \cdot 029 \text{ ft.}$$

169. Find the resilience of the above beam; see Part I., page 19.

Let W be the proof load;—

$$M_0 = \mathbf{M}_0, \text{ or } mWl = nfbh^2;$$

$$\tfrac{1}{4} W.240 = \tfrac{1}{6} \times 7 \times 4 \times 16^2; \quad \therefore \quad W = 20 \text{ tons nearly.}$$

Resilience = $\frac{1}{2}$ proof load × proof deflection.

$$= \tfrac{1}{2} \times 20 \text{ tons} \times \cdot 029 \text{ ft.}$$

$$= \cdot 29 \text{ ft.-tons} = 650 \text{ ft.-lbs.}$$

170. Find a general expression for the resilience of a rectangular beam loaded at the centre.

Let W be the proof load, and v_0 the proof deflection, then—

$$v_0 = \left(\frac{2mn''}{m'n}\right)\frac{Wc^3}{Ebh^3}.$$

$$\text{Resilience} = \tfrac{1}{2} Wv_0 = \left(\frac{mn''}{m'n}\right)\frac{c^3}{Ebh^3} \times W^2;$$

$$\text{but} \quad m.W.2c = nfbh^2; \quad \therefore \quad W = \frac{nfbh^2}{2mc};$$

$$\therefore \text{ Resilience} = \frac{mn''}{m'n} . \frac{c^3}{Ebh^3} . \left(\frac{nfbh^2}{2mc}\right)^2$$

$$= \frac{nn''}{8mm'} . \frac{f^2}{E} . 2cbh.$$

The first factor depends on the form of cross section and the distribution of load, and the second upon the material; the third is the volume of the beam.

Hence for a rectangular beam of a given material loaded at the centre with the proof load, the resilience is directly proportional to its volume; a result corresponding with that obtained for direct stress, Part I., page 20. Suppose for any given material we take a one-inch cube; then, when loaded at the centre as a beam, we have—

$$\text{Resilience per cubic inch} = \left(\frac{nn''}{8mm'}\right)\frac{f^2}{E};$$

for a rectangular section $n = \frac{1}{6}$, $m' = \frac{1}{2}$; for load at centre $m = \frac{1}{4}$, and $n'' = \frac{1}{3}$; $\frac{nn''}{8mm'} = \frac{1}{18}$, and

$$\text{resilience per cubic inch} = \tfrac{1}{18} \cdot f \cdot \frac{f}{E}$$

$$= \tfrac{1}{18} \times \text{proof stress} \times \text{proof strain.}$$

In the case of wrought iron, for which the proof stress $f = 7$, and $E = 12000$ tons per square inch, and remembering that the deflection is in inches—

$$\text{resilience per cubic inch} = \tfrac{1}{18} \times \frac{7^2}{12000} = \cdot 000227 \text{ inch-tons}$$

$$= \cdot 0423 \text{ ft.-lbs.}$$

For the previous example, the volume of the beam is 15360 cubic inches; multiplying this quantity by ·0423 we find the result given there.

171. For a rectangular timber beam of uniform section and uniformly loaded, find the ratio of depth to span, so that the working deflection may be a six-hundredth part of the span. The working strength of the wood is one ton, and its modulus of elasticity is 800 tons, per sq. inch.

$$\frac{h_0}{2c} = \tfrac{1}{4}\,\frac{n''}{m'} \cdot \frac{f}{E} \cdot \frac{2c}{v_0} = \tfrac{1}{4}\,\frac{\frac{5}{12}}{\frac{1}{2}}\,\frac{1}{800} \cdot 600 = \cdot 156.$$

That is, the depth is to be between a sixth and a seventh of the span.

172. What stiffness will be secured for an uniform rectangular beam of the same timber, loaded at centre, by making the depth an eighth of the span.

$$\frac{v_0}{2c} = \tfrac{1}{4}\,\frac{n''}{m'} \cdot \frac{f}{E} \cdot \frac{2c}{h_0} = \tfrac{1}{4} \cdot \frac{\frac{1}{3}}{\frac{1}{2}} \cdot \frac{1}{800} \cdot 8 = \frac{1}{600}.$$

That is, the working deflection will be a 600th part of span.

173. Find m'' and n'', the numerical coefficients for slope and deflection, for a beam of uniform section loaded with two equal weights at points which trisect the span.

In the general formulæ page 299, substitute $\frac{1}{3}c$ for x_r; and—

Ans. $$m'' = \tfrac{1}{2}\left(1 + \frac{\frac{1}{3}c}{c}\right) = \tfrac{2}{3}.$$

$$n'' = \tfrac{1}{2} - \tfrac{1}{6}\left(1 - \frac{\frac{1}{3}c}{c}\right)^2 = \tfrac{23}{54}.$$

174. A beam of uniform cross section is loaded with two equal weights symmetrically placed on the span; the amount of each weight is for each position such that the outer fibres of the middle cross section bear the working stress f. Compare the amount of deflection when the abscissæ of the left weight are as follows:—$x_r = 0$, $x_r = \frac{1}{3}c$, and $x_r = \frac{2}{5}c$.

Ans. For $x_r = 0, n'' = \frac{1}{3}$; for $x_r = \frac{1}{3}c, n'' = \frac{28}{54}$; for $x_r = \frac{2}{5}c, n'' = \frac{22}{50}$.

Deflection of Beam Supported on Three Props.

Let HK be a beam of uniform cross section, bearing an uniform load of amount U; and let it be supported on three props, one at each end and one at the centre. Let W be the reaction of the central prop, and $P = Q$, the reaction of each end prop; then $P + Q + W = U$.

In fig. 138, let $W = U$, then $P = Q = 0$, and OH is a cantilever of uniform cross section fixed at O and uniformly loaded, for which $n'' = \frac{1}{4}$; in fig. 139 let $W = 0$, then $P = Q = \frac{1}{2}U$, and HK is a beam of uniform cross section uniformly loaded, for which $n'' = \frac{5}{12}$; hence—

$$BH \text{ (fig. 138)} : DO \text{ (fig. 139)} :: 3 : 5.$$

Putting $BH = 3z$, then $DO = 5z$.

In fig. 138, if we suppose the end props to be pushed up till they support all the load, then HOK will assume the form LOB, where $HL = 8z$; and in fig. 139, if we suppose the central prop to be pushed up till it just supports all the load, then HOK will assume the form HTK, where $OT = 8z$. Hence we have the following theorems.

Theorem.—If to an uniform cantilever (OH fig. 138) loaded uniformly, we apply at the end a load ($P = \frac{1}{2}U$) equal to that uniform load, it will produce an additional deflection in the direction of the applied load ($HL = \frac{8}{3}BH$) equal to eight-thirds of that due to the uniform load alone.

In the figure, $P = \frac{1}{2}U$ is applied upwards at H, but the theorem holds for P applied downwards, since deflection is sensibly proportional to load; it being understood that the total deflection in no case exceeds the proof deflection.

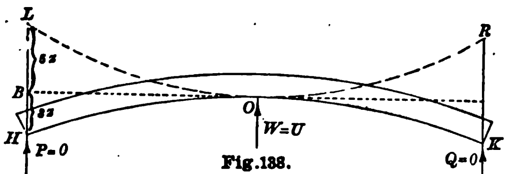


Fig. 138.

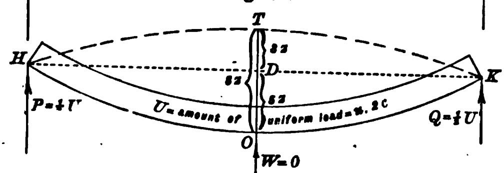


Fig. 139.

Corollary. A load P', applied to the end of an uniform cantilever loaded uniformly, will produce an additional deflection in the direction of P'; and the amount of this additional deflection is proportional to the load P'.

Theorem.—If to an uniform beam (HOK fig. 139) loaded uniformly we apply at the centre a load ($W = U$) equal to the uniform load, it will produce an additional deflection in the direction of the applied load ($OT = \frac{8}{5}DO$) equal to eight-fifths of that due to the uniform load alone.

In the figure, $W = U$ is applied upwards at O, but the theorem holds for W applied downwards, as for the previous theorem.

Corollary. A load W' applied at the centre of an uniform beam loaded uniformly will produce an additional deflection in the direction of W'; and the amount of this additional deflection is proportional to the load W'. See Thomson and Tait's "Natural Philosophy," 1st edition, §§ 618, 619.

Uniform beam uniformly loaded and supported on three props at the same level, one at each end and one at the centre.—Fig. 140. This figure is obtained by pushing up the central prop in fig. 139 till O coincides with D. Then since the reaction of the central prop in fig. 140 has produced the deflection OD, which is $\frac{5}{8}$ths of OT, it follows that $W = \frac{5}{8}U$, and therefore $P = Q = \frac{3}{16}U$.

From fig. 37, $\overline{OS}_1 = \frac{W}{U}\overline{OH} = \frac{5}{8}\overline{OH}$; hence $\overline{OH}_1 = \frac{1}{4}\overline{OH}$.

At centre O .... $M_0 = -\frac{1}{32}Ul$, max.
" " $F_0 = -\frac{5}{16}U$, max.

At S_1 and S_2 the maximum deflection occurs, and as proved in the next article

$$n'' = \frac{1}{4} \times \frac{175}{1024} = \cdot 043.$$

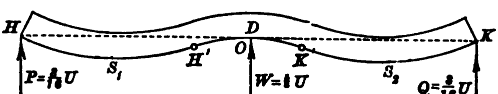


Fig. 140.

The central prop increases the strength four times and the stiffness nearly ten times.

Uniform beam uniformly loaded, fixed at one end and supported at the other.—Fig. 141. The left half of fig. 140 represents such a beam. Let $W = \frac{1}{2}U$ the uniform load, and $l = OH$ the span; then $P = \frac{3}{8}W$; the shearing force at O is the remainder of load, viz., $\frac{5}{8}W$; the shearing force changes sign at S where the bending moment has a maximum value; and the locus of the shearing force diagram is a straight line, since the load is uniform.

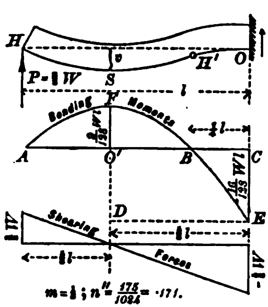


Fig. 141.

From fig. 37 the bending moment diagram is $AFBE$ a parabola with axis vertical and apex on vertical through S.

$$O'F = M_s = \frac{3W}{8}\cdot\frac{3l}{8} - \frac{3W}{8}\cdot\frac{3l}{16} = \frac{9}{128}Wl$$

a positive maximum; and since $AFBE$ is a parabola

$$FD : FO' :: DE^2 : O'B^2 :: 25 : 9;$$

$$\therefore \quad FD = \tfrac{25}{9}FO' \text{ and}$$

$$CE = O'D = FD - FO' = -\tfrac{16}{9}O'F = -\tfrac{1}{8}Wl;$$

that is, $M_0 = -\frac{1}{8}Wl$, the greatest value; this quantity may also be calculated directly by taking moments about O.

This solution is exact, and all the results are readily got by remembering that OH' is a quarter of the span. The approximate solution indicated in Rankine's "Applied Mechanics," sec. 308, assumes H' to be sensibly on the same level as H.

To find the deflection at S.

Considering the beam HH' alone, we have for the deflection at S—

$$v_1 = \frac{5}{12}\frac{f_s}{E}\cdot\frac{\overline{SH}^2}{y_0}; \text{ and since } \overline{SH} = \tfrac{3}{8}l = \tfrac{3}{4}c$$

$$v_1 = \frac{5}{12}\frac{f_s}{Ey_0}\cdot\frac{9c^2}{16} = \frac{15}{64}\frac{f_s}{E}\cdot\frac{c^2}{y_0},$$

where f_s is the intensity of stress at the skin at S.

Next consider the cantilever OH', taking into account its uniform load alone. If the cantilever were of length SH, its deflection would be $\frac{3}{5}v_1$; and since the deflection of a cantilever is proportional to the cube of its length and to the load, we have for the deflection of OH' for uniform load alone—

$$v_2 = \left(\frac{OH'}{SH}\right)^4\cdot\tfrac{3}{5}v_1 = \left(\tfrac{1}{4} \div \tfrac{3}{8}\right)^4\cdot\tfrac{3}{5}v_1 = \tfrac{16}{135}v_1;$$

further;—if a load equal to that on OH' be put at the end, it will produce an additional deflection $\frac{8}{3}v_2$; the load at end of OH', viz., that on SH', will produce a proportionate deflection; and we have for the deflection due to load at end of OH',—

$$v_3 = \left(\frac{SH'}{OH'}\right)\tfrac{8}{3}v_2 = \left(\tfrac{3}{8} \div \tfrac{1}{4}\right)\tfrac{8}{3}v_2 = 4v_2;$$

and the total deflection of $OH' = 5v_2 = \frac{16}{27}v_1$.

The total deflection of the point S is therefore

$$v = v_1 + \tfrac{1}{2}\,\tfrac{16}{27}v_1 = \tfrac{35}{27}v_1 = \tfrac{35}{27}\cdot\tfrac{15}{64}\,\frac{f_s}{E}\cdot\frac{c^2}{y_0} = \tfrac{175}{576}\,\frac{f_s}{E}\,\frac{c^2}{y_0}.$$

Let f_0 be the proof strength of the material; when the beam is loaded with the proof load, then $f_0 =$ intensity of stress on skin at O the fixed end, and

$$f_s : f_0 :: M_s : M_0 :: 9 : 16,$$

hence $f_s = \frac{9}{16}f_0$; substituting this, we have—

$$v = \tfrac{175}{576}\cdot\tfrac{9}{16}\,\frac{f_0}{E}\cdot\frac{c^2}{y_0} = \tfrac{175}{1024}\cdot\frac{f_0}{E}\cdot\frac{c^2}{y_0};$$

so that $m = \frac{1}{8}$; and $n'' = \frac{175}{1024} = \cdot 171$.

Uniform beam uniformly loaded and fixed at both ends.—Fig. 143. Suppose the central of the three props that support HK, fig. 142, to push up till O is above the level of H and K such a distance that H' and K', the points of contrary flexure, shall be on the same level as H and K. Let $x = OK'$ the distance of the point of contrary flexure from O, and let u be the intensity of the uniform load.

Consider OK' alone. It is a cantilever fixed at O, of length x, loaded uniformly with intensity u, and loaded at K' the free end with $\frac{u}{2}(c-x)$, half the load spread on $K'K$. For the uniform load alone, compare OK' with OK, fig. 138, whose deflection is $3z$; their deflections are proportional to the cubes of their lengths and to the loads; hence for uniform load alone the deflection of OK', fig. 142, is

$$\left(\frac{OK'}{OK}\right)^4 \times 3z = \left(\frac{x}{c}\right)^4 . 3z;$$

OK' being shown in fig. 142, and OK in fig. 138. By theorem at fig. 138, a load at the end of this cantilever, and of amount ux, would produce an additional deflection $8\left(\frac{x}{c}\right)^4 z$; the load $\frac{1}{2}u(c-x)$ will produce a proportional deflection; and we have for the deflection due to the load at end of OK',

$$\frac{4(c-x)}{3x}\left(\frac{x}{c}\right)^4 . 3z;$$

and for the total deflection of the point K',

$$OD = \frac{x^3}{c^4}(4c-x)z.$$

To pass from fig. 139 to fig. 142, the central prop has been pushed up through OD fig. 139 together with DO fig. 142; that is through

$$5z + \frac{x^3}{c^4}(4c - x)z = \left\{5 + \frac{(4c - x)x^3}{c^4}\right\}z;$$

and by the converse of corollary to theorem at fig. 139, the reaction on the central prop will be

$$W = \frac{1}{8z}\left\{5 + \frac{(4c - x)x^3}{c^4}\right\}z \times U.$$

Now, $OS_1 = \frac{1}{2}(c + x)$; and since (fig. 37) $OS_1 = \frac{W}{U}c$, we have

$$\tfrac{1}{2}(c + x) = \tfrac{1}{8}\left(5 + \frac{(4c - x)x^3}{c^4}\right)c;$$

solving this equation we find

$$OK' = x = c(2 - \sqrt{3}),$$

which determines the position of the points of contrary flexure. Substituting this for x in the expression for W as given above, we have the reaction of central prop

$$W = \frac{3 - \sqrt{3}}{2}U.$$

If we take $OK'K$, half of this beam, and suppose the end at O fixed, and the end K to be supported by the cantilever $O'K$ similar to OK' in all respects; we have, fig. 143, a beam fixed at both ends and uniformly loaded; its semi-span is

$$c' = OS_1 = \frac{3 - \sqrt{3}}{2}c;$$

but $$OK' = (2 - \sqrt{3})c = \left(1 - \frac{1}{\sqrt{3}}\right)c',$$

hence the distance from the centre to a point of contrary flexure is

$$SK' = \frac{1}{\sqrt{3}}c' = \cdot 289 l',$$

where l' is the span.

Let W' be the total load on the beam, fig. 143, then the shearing force diagram will vary from $\frac{1}{2}W'$ to $-\frac{1}{2}W'$ from left to right end. The bending moment diagram is the same parabola as for the beam supported at the ends, except that it passes through the points of contrary flexure; $FO'' : FD :: O''B^2 : DL^2$, or $O''F$ is one-third, while $O''D$ is two-thirds of the maximum bending moment for the beam supported only; hence at fixed end

$$M_{\max.} = -\tfrac{2}{3}\,\tfrac{1}{8}\,W'l' = -\tfrac{1}{12}\,W'l'.$$

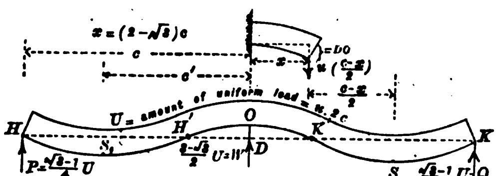


Fig. 142.

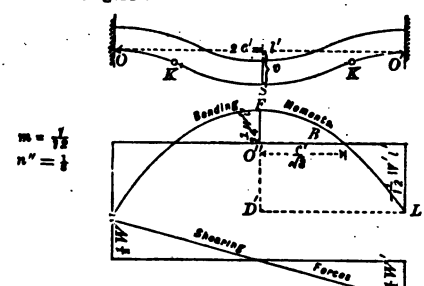


Fig. 143.

To find the deflection;—Consider first the part KK'; for it

$$v_1 = \frac{5}{12}\frac{f_s}{E}\cdot\frac{SK^2}{y_0} = \frac{5}{12}\cdot\frac{f_s}{Ey_0}\frac{c'^2}{3} = \frac{5}{36}\frac{f_s}{E}\cdot\frac{c'^2}{y_0},$$

where f_s is the stress on skin at S. Consider next the cantilever OK', taking into account the uniform load alone; a cantilever of length SK' would deflect $\frac{3}{5}v_1$, and taking account of its length, we have for the deflection of OK' for uniform load alone

$$v_2 = \left(\frac{OK'}{SK'}\right)^4 \cdot \frac{3}{5} v_1.$$

A load at end of OK', equal to the uniform load on it, would produce an additional deflection of $\frac{8}{3}v_2$; but the load at end of OK' is that on $K'S$, and it will produce a proportionate deflection; so that we have for the deflection of OK' for load at end

$$v_3 = \left(\frac{SK'}{OK'}\right) \frac{8}{3} v_2 = \left(\frac{OK'}{SK'}\right)^3 \frac{8}{5} v_1.$$

Hence the total deflection of S is

$$v = v_1 + v_2 + v_3$$
$$= \left\{1 + \frac{3}{5}\left(\frac{OK'}{SK'}\right)^4 + \frac{8}{5}\left(\frac{OK'}{SK'}\right)^3\right\} v_1.$$

Now $$\frac{OK'}{SK'} = \left(1 - \frac{1}{\sqrt{3}}\right) \div \frac{1}{\sqrt{3}} = (\sqrt{3} - 1),$$

$$\left(\frac{OK'}{SK'}\right)^4 = (28 - 16\sqrt{3}), \text{ and } \left(\frac{OK'}{SK'}\right)^3 = (6\sqrt{3} - 10);$$

$$v = \left(1 + \frac{84}{5} - 16\right) v_1 = \frac{9}{5} v_1$$
$$= \frac{1}{4}\frac{f_s}{E}\frac{c'^2}{y_0} = \frac{1}{8}\frac{f_0}{E}\frac{c'^2}{y_0};$$

since M at the end and at the centre are in the ratio of 2 and 1.

We thus have $m = -\frac{1}{12}$; and $n'' = \frac{1}{8}$.

From the above, we see that fixing the ends of an uniform beam which is loaded uniformly increases the

strength in the ratio $\frac{1}{8}$: $\frac{1}{12}$, or 3 : 2;
stiffness " " $\frac{5}{12}$: $\frac{1}{8}$, or 10 : 3.

flexure in order to be able to resist the shearing force at these points; on K', for instance, the shearing force will be the load on SK', and the breadth at K' must be sufficient to resist this amount.

THIN HOLLOW CROSS SECTIONS.

Let t be the uniform thickness of a thin hollow cross section of any form, and let n' and n be the numerical coefficients respectively of the moment of inertia and the moment of resistance to bending of a solid cross section of the same form; let B, H be the breadth and depth of the rectangle circumscribing the section, and b, h the breadth and depth of the rectangle circumscribing the hollow; then—

$$I_0 = n'(BH^3 - bh^3). \qquad \text{(See page 237.)}$$
$$= n'B(H^3 - h^3), \text{ since } (B = b \text{ nearly}).$$
$$= n'B(H - h)(H^2 + Hh + h^2).$$
$$= n'B \,.\, 2t \,.\, 3H^2. \qquad (H = h \text{ nearly}).$$
$$= n' \,.\, 6BH^2t.$$

$$\mathsf{M} = \frac{f}{m'H} I_0 = \left(\frac{n'}{m'}\right) 6f \,.\, B \,.\, H \,.\, t$$
$$= n \,.\, 6fBHt \qquad \text{(1st approximation)} \ldots\ldots\ldots\ldots \text{(1.)}$$

Thin hollow rectangle, $\mathsf{M} = \frac{1}{6} \,.\, 6fBHt = fBHt$;

„ circle, $\mathsf{M} = \frac{\pi}{32} 6fBHt = \cdot 6fd^2t$, ($d$ = diam.)

A closer approximation for any form is obtained thus:—

$$I_0 = n'\{BH^3 - bh^3\} = n'\{(B - b)H^3 + b(H^3 - h^3)\}$$
$$= n'\{(B - b)H^3 + b(H - h)(H^2 + Hh + h^2)\}$$
$$= n'\{2t.H^3 + b \,.\, 2t' \,.\, 3Hh\} \text{ (approx.)}$$
$$= 2n'H\{tH^2 + 3bt'h\} \qquad „$$
$$\therefore \quad \mathsf{M} = 2nf(tH^2 + 3t'bh) \qquad „ \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

where t is the thickness of each side, and t' the thickness of the top or bottom.

To design a thin hollow cross section;—Choose the depth H the proper fraction of the span to give the required degree of stiffness; assume B a suitable fraction of H to give sufficient lateral stiffness, and the above is a simple equation from which to find t; or t may be assumed a multiple of the thickness that the metal plates are usually manufactured, and B found from the formulæ. Now find the moment of resistance to bending, and the resistance to shearing, by the accurate formulæ; and if this differs from M and F, alter t or B for further approximation.

Examples.

175. Find the thickness of metal required for an aqueduct bridge 40 feet span, the waterway being 2 feet square, and the material wrought iron for which $f = 4$ tons per sq. in.

The weight of water supported is $4\frac{1}{2}$ tons, and the weight of metal (assumed) $1\frac{1}{2}$ tons; making a further allowance of 2 tons, we have $W = 8$ tons distributed, which gives $M_0 = 480$ inch-tons. Let $B = 24''$, $H = 24''$, then—

$$t = \frac{M}{fBH} = \frac{480}{4 \times 24^2} = \frac{1}{5} \text{ in. nearly.}$$

Allowing for rivets, the plates might be taken $\frac{1}{4}''$ thick; but as the plates are liable to rust, the thickness would require to be increased still further, and t might be taken as $\frac{3}{8}''$ in an actual case.

176. Find the moment of resistance to bending of a cast iron pipe 18″ external diameter, metal 1″ thick, and $f = 2$ tons per square inch, by means of the exact formula; and compare the result with that obtained by the approximate formulæ 1 and 2.

M = 430 inch tons by exact formula.
= 390 „ approx.
= 429 „ „

177. Find the thickness of metal required for a cast iron pipe 24″ external diameter, so that its moment of resistance to bending may be 50 foot-tons, and its resistance to shearing 10 tons; taking $f = 2$ tons per square inch.

$$600 = \cdot 6 \times 2 \times 24^2 \times t; \quad \therefore \ t = 0''\cdot 87 \text{ nearly.}$$

On checking the calculation by the accurate formula, and taking $t = 0''\cdot 87$, M = 705 inch-tons; so that $t = 0''\cdot 87$ is more than sufficient; and for shearing this thickness will several times too great. On account of the difficulty of casting so as to have the thickness quite uniform, an allowance has to be made; and t would probably be taken at about 1″·25.

178. Find by the approximate formula the moment of resistance to bending of the cross section shown in fig. 104, and compare the result with that given in the text at that figure.

$$M = f\tfrac{1}{3}(2 . 900 + 3 . 3 . 6 . 24) = 1032f.$$

Cross Sections of Equal Strength.

When a beam is made of a material whose strengths to resist tension and thrust are different, the area of the upper flange is made different from that of the lower, fig. 144, in order that both flanges may be brought to the proof or working stress at the same time. In the case of cast iron, the strengths to resist tension and thrust are as 1 and 6; and on this account the area of the upper flange (compressed) is about one-sixth that of the lower flange (extended). This form of cross section was first proposed by Mr. Hodgkinson. On account of the liability of cast iron to crack if unequally cooled, sudden changes of thickness of metal are to be avoided; on this account, the top of the web may be made of the same thickness as the top flange, and the bottom of the web of the same thickness as the bottom flange.

Double-T cross section.—Fig. 144. The position of the Neutral Axis, and the Moment of Inertia about that axis, in terms of the areas and depths of the three rectangles, are found as follows; the notation being,—

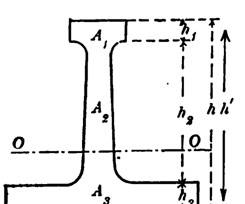


Fig. 144.

	Areas.	Depths.
Upper Flange, ...	A_1,	h_1.
Web, ...	A_2,	h_2.
Lower Flange, ...	A_3,	h_3.
	—	—
Totals, ...	A.	h.

Exact Solution.—The height of the neutral axis above the lower side of the cross section is obtained thus:—

$$Ay_b = A_1\left(\frac{h_1}{2}+h_2+h_3\right)+A_2\left(\frac{h_2}{2}+h_3\right)+A_3\frac{h_3}{2};$$

$$y_b = \frac{h}{2}-\frac{(h_1+h_2)A_3-(h_2+h_3)A_1-(h_3-h_1)A_2}{2A}; \dots\dots (1.)$$

The Moment of Inertia for each rectangle is shown on page 233; and letting Σ as before denote the "sum," the moment of inertia for the cross section is

$$I_0 = \Sigma I+\Sigma A \,.\, x_0^2;$$

$$\Sigma I = \Sigma\frac{bh^3}{12}=\frac{A_1h_1^2+A_2h_2^2+A_3h_3^2}{12};$$

$$\Sigma Ax_0^2 = \Sigma bhx_0^2 = A_1\left[y_b-\left(h_2+h_3+\frac{h_1}{2}\right)\right]^2+A_2\left[y^b-\left(h_3+\frac{h_2}{2}\right)\right]^2 + A_3\left[y_b-\frac{h_3}{2}\right]^2$$

$$= A_1\left\{\frac{-A_2(h_1+h_2)-A_3(h_1+2h_2+h_3)}{2A}\right\}^2 + A_2\left\{\frac{A_1(h_1+h_2)-A_3(h_2+h_3)}{2A}\right\}^2 + A_3\left\{\frac{A_1(h_1+2h_2+h_3)+A_2(h_2+h_3)}{2A}\right\}^2.$$

$$\therefore\ I_0 = \frac{A_1h_1^2+A_2h_2^2+A_3h_3^2}{12}+\frac{1}{4A}\{A_1A_2(h_1+h_2)^2 + A_1A_3(h_1+2h_2+h_3)^2+A_2A_3(h_2+h_3)^2\}; \dots\dots\dots (2.)$$

The Moment of Resistance to bending is as before—

$$\mathbf{M} = \frac{f_bI_0}{y_b}; \dots\dots\dots (3.)$$

Approximate Solution.—(Rankine's "Civil Engineering," § 163, 164.) When h_1 and h_3 are *small* compared with h_2, we may leave them out of the exact formulæ; and we obtain—

$$y_b = \frac{h}{2}-\frac{(A_3-A_1)h_2}{2A}; \dots\dots\dots (4.)$$

$$I_0 = \frac{A_2 h_2^2}{12} + \frac{h_2^2}{4A}(A_1 A_2 + 4A_1 A_3 + A_2 A_3)\,; \quad\ldots\ldots\ldots (5.)$$

Put h' for $h_2 + \frac{h_1 + h_3}{2}$, that is for the distance between the centres of gravity of the flanges; and let A_2 be the area of the cross section of the vertical web measured from centre to centre of the top and bottom flanges; then, *nearly*

$$y_b = \frac{h'}{2} - \frac{(A_3 - A_1)h'}{2A} = \frac{h'}{2}\left(\frac{2A_1 + A_2}{A}\right); \quad\ldots\ldots\ldots (6b.)$$

$$y_a = \frac{h'}{2}\left(\frac{2A_3 + A_2}{A}\right); \quad\ldots\ldots\ldots (6a.)$$

$$I_0 = h'^2\left\{\frac{A_2}{12} + \frac{1}{4A}(A_1 A_2 + 4A_1 A_3 + A_2 A_3)\right\}; \quad\ldots\ldots\ldots (7.)$$

Since $y_a : y_b : h :: f_a : f_b : f_a + f_b$, we have

$y_a : y_b : h' :: f_a : f_b : f_a + f_b$ approx.

From equations (6) we have

$y_a : y_b : h' :: 2A_3 + A_2 : 2A_1 + A_2 : 2A$

$\therefore \quad f_a : f_b : f + f_b :: 2A_3 + A_2 : 2A_1 + A_2 : 2A$ approx.

and $$\frac{2A_3 + A_2}{2(A_1 + A_2 + A_3)} = \frac{f_a}{f_a + f_b}; \quad\ldots\ldots\ldots (8.)$$

From this equation we can eliminate A_1 or A_3, and

$$A_1 = \frac{f_b}{f_a}A_3 + \frac{f_b - f_a}{2f_a}A_2; \quad\ldots\ldots\ldots (9.)$$

$$A_3 = \frac{f_a}{f_b}A_1 + \frac{f - f_b}{2f_b}A_2; \quad\ldots\ldots\ldots (10.)$$

Substituting this value of A_3 in equation (7), we obtain—

$$I_0 = h'^2\left\{\frac{A_1A_2 + A_2^2 + A_2A_3 + 3A_1A_2 + 12A_1A_3 + 3A_2A_3}{12(A_1 + A_2 + A_3)}\right\}$$

$$= h'^2\left\{\frac{4A_1A_2 + 4A_2\left(\frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}A_2\right) + 12A_1\left(\frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}A_2\right) + A_2^2}{12\left(A_1 + A_2 + \frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}A_2\right)}\right\}$$

$$I_0 = \frac{h'^2}{f_a + f_b}\left\{f_a A_1 + (2f_a - f_b)\frac{A_2}{6}\right\}; \qquad (11.)$$

$$\mathsf{M} = \frac{f_b}{y_b} I_0 = \frac{f_a + f_b}{h} I_0 = \frac{f_a + f_b}{h'} I_0$$

$$= h'\left\{f_a A_1 + (2f_a - f_b)\frac{A_2}{6}\right\}; \qquad (12a.)$$

$$= h'\left\{f_b A_3 + (2f_b - f_a)\frac{A_2}{6}\right\}; \qquad (12b.)$$

In designing a beam to resist a given bending moment, the depth h' is taken at a fraction, say $\frac{1}{8}$th to $\frac{1}{14}$th, of the span so as to ensure stiffness; the thickness of the web is then fixed by considerations of practical convenience, and so as to give sufficient resistance to shearing; and the area of the upper and lower flange can then be found by equations 12; having thus fixed the values of A_1 and A_3, we can then choose breadths and depths suitable for the flanges.

For the section as thus fixed, calculate M and F by the tabular method shown for figs. 106, 134, &c.; if h_1 and h_3, the depths chosen for A_1 and A_3, are very small, it will be found that M and F are sensibly what is required, and that the neutral axis sensibly divides the depth of section as f_a and f_b, so that no further calculation is necessary. If, however, one or both of the depths h_1, h_3, be *not* very small, the solution by the tabular method will differ considerably from the data, and further approximation will be necessary. When one of the flanges, as in the case of cast iron, is comparatively deep, the inaccuracy of the results will be considerable, and one or more further approximations may be required. For different examples, the error will be different in amount, and we have no simple means of judging how great this error will be in any particular case.

In the equations 11 and 12 given above, the results for M and I are close when h_1 *and* h_3 are small; the result for y_a will be close although h_3 is not small, and that for y_b will be close although h_1 is not small; thus, suppose that h_3 is small, we have—

$$y_b = \frac{h}{2} - \frac{(h_1 + h_2)A_3 - (h_2 + h_3)A_1 - (h_3 - h_1)A_2}{2A}$$

$$y_b = \frac{h}{2} - \frac{(h_1+h_2)A_3 - h_2A_1 + h_1A_2}{2A}; \quad \text{putting } h' - \tfrac{1}{2}h_1 = h_2, \text{ and } h' = h$$

$$= \frac{h'}{2} - \frac{h_1(A_2+A_3) + (h' - \tfrac{1}{2}h_1)(A_3 - A_1)}{2A}$$

$$= \frac{h'}{2}\left(\frac{A - A_3 + A_1}{A}\right) - h_1\frac{A_1 + 2A_2 + A_3}{4A}$$

$$= h'\frac{2A_1 + A_2}{2A} - \tfrac{1}{4}h_1\frac{A_1 + 2A_2 + A_3}{A_1 + A_2 + A_3}$$

$$= h'\frac{2A_1 + A_2}{2A}, \text{ nearly}; \quad \ldots\ldots\ldots\ldots \quad (13)$$

a result similar to that found previously in equation (6*b*), but y_a cannot now be found by interchanging A_1 and A_3.

Common forms for cast iron beams are shown in fig. 145; the corresponding equations for these T-shaped sections are derived from the above by putting $A_1 = 0$, or $A_3 = 0$, according as f_a or f_b is the greater. Thus for a section of this form, when f_a is greater than f_b, the flange will be required on the extended side; when f_b is greater than f_a, the flange will be required on the compressed side; and we have—

$$A_3 = \frac{f_a - f_b}{2f_b}A_2; \ \ldots (14)$$

$$\text{or } A_1 = \frac{f_b - f_a}{2f_b}A_2; \ \ldots (15)$$

as the case may be.

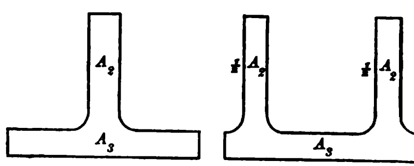


Fig. 145.

Similarly for resistance to bending, we have—

$$\mathsf{M} = h'(2f_a - f_b)\frac{A_2}{6}; \quad \ldots\ldots\ldots\ldots \quad (16a)$$

$$\text{or} \quad = h'(2f_b - f_a)\frac{A_2}{6}; \quad \ldots\ldots\ldots\ldots \quad (16b)$$

as the case may require.

In the case of trough-shaped beams, the same formulæ are applicable, if we consider the web A_2 to consist of the two vertical ribs.

Examples.

179. Design the central cross section for a wrought iron beam, for which $f_a = 4$, and $f_b = 5$ tons per sq. inch, suitable for example 102.

Since the span is 42 feet, we may take $h' = 40$ inches; $M = 405{\cdot}75$ ft.-tons $= 4863$ inch-tons; the web may be taken as $\frac{3}{8}''$ thick.

From equation 12 we have—

$$4863 = 40\{4A_1 + (8 - 5)\tfrac{15}{6}\}; \quad \therefore \quad A_1 = 28{\cdot}5 \text{ sq. ins.}$$

$$4863 = 40\{5A_3 + (10 - 4)\tfrac{15}{6}\}; \quad \therefore \quad A_3 = 21{\cdot}3 \quad ,,$$

Adopting as a first approximation—breadth of flanges, 21 ins.; thickness of top flange, 1·36 in., of bottom do., 1 in.; thickness of web, $\frac{3}{8}$ in., and depth of girder (outside to outside), 41·2 in.; we get by applying the exact method shown at page 241,

$$M = \frac{4 \times 21340}{18{\cdot}35} = 4652 \text{ inch-tons,}$$

a result differing from what is required by only 4 %.

The upper flange may therefore be taken $1\frac{3}{8}$ in. thick, the lower flange 1 in. thick, and the breadth of each 22 in. This does not take into account the angle irons, and the loss by rivet holes.

180. Find suitable dimensions for the top and bottom flanges of a cast iron beam, having given $M = 1200$ inch-tons, $h = 20$ inches, $f_a = 10$ and $f_b = 2$ tons per sq. inch, and web 1 in. thick.

Let $h' = 18$ ins.; then equations 12 and 10,

$$1200 = 18(10A_1 + 54); \quad \therefore \quad A_1 = 1{\cdot}3 \text{ sq. ins. nearly.}$$

$$A_3 = 5A_1 + 36; \quad \therefore \quad A_3 = 42{\cdot}5 \quad ,, \quad ,,$$

Taking the upper flange, 1·3 in. × 1 in., and the lowe flange 14·2 in. × 3 in., and solving by the exact method, we get—

$$y_b = 4·45,\ I_0 = 1720,\ \mathsf{M} = \frac{2 \times 1720}{4·45} = 770 \text{ inch-tons};$$

a result very different from that which is required; we will therefore solve this problem by another approximate method, see example 183.

181. Find suitable dimensions for the top and bottom flanges of a cast iron beam, having given $M = 800$ inch-tons, $h = 20$ inches, $f_a = 8$ and $f_b = 2$ tons per square inch, and web 1 inch thick.

Let the upper flange be $\frac{1}{2}''$, and the lower flange 2″ thick, then $h' = 18''·75$, and we have

$$800 = 18·75\{8A_1 + (16 - 2)3·12\};$$

$$\therefore \quad A_1 = -\tfrac{1}{8},$$

but as negative values are inadmissible, we will take $A_1 = 0$.

$$A_3 = \tfrac{6}{4}\ 18·75 = 28·12 \text{ square inches.}$$

We may therefore take the lower flange 14·1 in. broad and 2 in. deep, and the web 18 in. × 1 in.; solving by the exact equation

$$\mathsf{M} = \frac{2 \times 1595}{5} = 638 \text{ inch-tons,}$$

a result, as in example 180, very different from that required. These three examples show us that unless the flanges are *thin*, the results are not quite satisfactory.

Approximate Solution—Another method.

Problem.—Two heavy particles A_1 and A_3, fig. 146, are placed at a known distance h' apart; and another particle A_2, of known mass, is placed midway between them; to find the mass of A_1 and of A_3, so that the centre of inertia of the three particles may divide the distance h' in a given ratio ρ, and that the sum of the moments of inertia about the common centre of inertia may be of a given amount I_0.

Let $\frac{x}{y} = \rho$, and let k be the distance of the centre of inertia of the three particles from A_2, then

$$x = \frac{\rho h'}{\rho + 1};\ y = \frac{h'}{\rho + 1};\ \text{and } I_0 = A_1 x^2 + A_2 k^2 + A_3 y^2.$$

Now $k = x - \frac{h'}{2} = \frac{\rho h'}{\rho + 1} - \frac{h'}{2} = \frac{h'}{2}\,\frac{\rho - 1}{\rho + 1}$,

$$\therefore \quad \frac{k}{h'} = \tfrac{1}{2}\,\frac{\rho - 1}{\rho + 1}; \qquad (1)$$

if $\frac{1}{2}(A_3 - A_1)$ be taken from A_3 and added to A_1, the common centre of inertia G would coincide with A_2, since the upper and lower particles would then be of equal mass; and we obtain

$$k\,(A_1 + A_2 + A_3) = h' \,.\, \tfrac{1}{2}\,(A_3 - A_1), \text{ or}$$

$$\frac{k}{h'} = \tfrac{1}{2}\,\frac{A_3 - A_1}{A_1 + A_2 + A_3}; \qquad (2.)$$

From equations 1 and 2

$$\frac{A_3 - A_1}{A_1 + A_2 + A_3} = \frac{\rho - 1}{\rho + 1},$$

$$\therefore \quad A_2 = 2\,\frac{A_3 - \rho A_1}{\rho - 1}; \qquad (3)$$

and

$$\rho = \frac{x}{y} = \frac{2A_3 + A_2}{2A_1 + A_2}; \qquad (4.)$$

$$I_0 = A_1\left(\frac{\rho h'}{\rho + 1}\right)^2 + A_3\left(\frac{h'}{\rho + 1}\right)^2 + 2\frac{A_3 - \rho A_1}{\rho - 1}\left(\frac{h'}{2}\,\frac{\rho - 1}{\rho + 1}\right)^2$$

$$= \left(\frac{h'}{\rho + 1}\right)^2 \{A_1\rho^2 + A_3 + \tfrac{1}{2}\,(A_3 - \rho A_1)(\rho - 1)\}$$

$$= \tfrac{1}{2}\,\frac{h'^2}{\rho + 1}\,(\rho A_1 + A_3); \qquad (5)$$

and from equations 4 and 5 we can find A_1 and A_3.

Equation 5 may be written more symmetrically thus—

$$I_0 = \frac{h'}{2}\left\{A_1\frac{\rho h'}{\rho+1} + A_3\frac{h'}{\rho+1}\right\}$$

$$= \frac{h'}{2}\{A_1 x + A_3 y\}$$

$$= h'\{A_1 x + \tfrac{1}{4}(x-y)A_2\}; \quad \ldots\ldots\ldots\ldots\ldots (6)$$

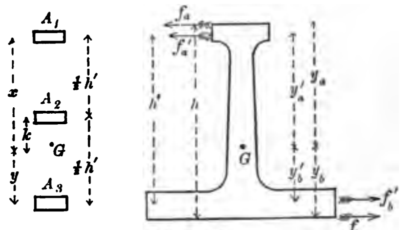


Fig.146.

by substituting from equation 4,

$$A_3 = \tfrac{1}{2}\left\{\frac{x}{y}(2A_1 + A_2) - A_2\right\}; \text{ hence}$$

$$A_1 = \frac{I_0}{xh'} - \tfrac{1}{4}\left(1 - \frac{y}{x}\right)A_2; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

similarly $$A_3 = \frac{I_0}{yh'} - \tfrac{1}{4}\left(1 - \frac{x}{y}\right)A_2; \quad \ldots\ldots\ldots\ldots (8.) \qquad Q.E.F.$$

Suppose now that the particles A_1 and A_3 are replaced by flanges of area A_1 and A_3, whose centres are at these points, and the particle A_2 by the web of area A_2, with its centre sensibly midway between the centres of the two flanges; equation 4 will still hold absolutely; equation 5 will be a close approximation to the moment of inertia of the cross section, the error being that it leaves out of account the moment of inertia of each part about its own neutral axis, a quantity which, for such cross sections as we are treating, is not large.

If we substitute y_a' for x, and y_b' for y in equations 4 and 6

$$y_a' : y_b' : h' :: 2A_3 + A_2 : 2A_1 + A_2 : 2A;$$

$$I_0 = h' \{A_1 y_a' + \tfrac{1}{4}(y_a' - y_b')A_2\} \text{ approx.}$$

and
$$\mathsf{M} = \frac{f_a' I_0}{y_a'} = \frac{f_b' I_0}{y_b'};$$

$$= h' \{f_a' A_1 + \tfrac{1}{4}(f_a' - f_b')A_2\}; \quad \text{(9)}$$

$$= h' \{f_b' A_3 - \tfrac{1}{4}(f_a' - f_b')A_2\}; \quad \text{(10.)}$$

Now
$$f_a' = f_a \frac{y_a'}{y_a} = f_a \frac{y_a - \frac{1}{2}t_a}{y_a}$$

$$= f_a\left(1 - \frac{t_a}{2y_a}\right); \quad \text{(11.)}$$

$$f_b' = f_b\left(1 - \frac{t_b}{2y_b}\right); \quad \text{(12)}$$

where t_a and t_b denote the thickness of the flanges A_1 and A_3 respectively; so that having fixed the thickness of the flanges, we obtain from equations 9 and 10 the area required for A_1 and A_3. Having fixed upon h' from considerations of stiffness, and on A_2 so as to give sufficient resistance to shearing or for other considerations, then equations 9 and 10 will give the areas of the flanges A_1 and A_3, so that the neutral axis shall divide h' in the ratio $f_a' : f_b'$, and, approximately, give M the required moment of resistance to bending; provided that the flanges have their centres at the distance h' apart, and that the web is so disposed that its centre is (sensibly) midway between the centres of the flanges.

When the two strengths of the material, f_a and f_b, are nearly equal, so also are the flanges, and the centre of the web will be sensibly midway. When f_a is much greater than f_b, as is the case with cast iron, then A_3 is much greater than A_1; if the web be made of uniform breadth, its centre will be above the point which is midway between the centres of the flanges; but for practical considerations, it is usual to make the web increase in breadth as we pass from the upper to the lower flange; in which case the centre of the web will be lowered and thus brought so that its centre is sensibly midway between the centres of the two flanges.

Examples.

182. Solve example 179 by the approximate formulæ just given.

In order to fix on t_a and t_b, we will assume f_a' and f_b' equal to f_a and f_b respectively; and then find new breadths to satisfy equations 9 and 10.

$$4863 = 40\{4A_1 - \tfrac{1}{4}(5-4)15\}.$$

$A_1 = 31$, and similarly $A_3 = 24$ sq. inches for a first approximation; we may fix on $t_a = 1{\cdot}25$ and $t_b = 1$ inch, and from these obtain suitable breadths.

$$y_a' = \tfrac{4}{9}40 = 17{\cdot}78 \text{ in., and } y_b' = 22{\cdot}22 \text{ in.}$$

$$y_a = 18{\cdot}4 \text{ in., and } y_b = 22{\cdot}72 \text{ in.}$$

$$\therefore \quad f_a' = 4\left(1 - \frac{1{\cdot}25}{36{\cdot}8}\right) = 3{\cdot}86, \text{ and } f_b' = 4{\cdot}89 \text{ tons.}$$

We have therefore—

$$4863 = 40\{3{\cdot}86A_1 - \tfrac{1}{4}(4{\cdot}89 - 3{\cdot}86)15\}.$$

$$A_1 = 30{\cdot}5 \text{ sq. in., and } A_3 = 25{\cdot}6 \text{ sq. in.}$$

Adopting the following dimensions, upper flange 24·4 in. × 1·25 in., lower flange 25·6 in. × 1 in., and $h = 41{\cdot}12$ in.; and solving by the exact method, we obtain—

$$\mathsf{M} = \frac{4 \times 24250}{19{\cdot}2} = 5040 \text{ inch-tons,}$$

a quantity differing from the required result by less than 4 %.

183. Solve example 180 by the approximate formulæ just given, assuming $t_a = 1$ in., and $t_b = 3$ in. thick.

In this example $y_a' = 15$, and $y_b' = 3$ in.; so that

$$f_a' = 9{\cdot}68, f_b' = 1{\cdot}\dot{3} \text{ tons.}$$

$$1200 = 18\{9{\cdot}68A_1 + \tfrac{1}{4}(9{\cdot}68 - 1{\cdot}33)16\}.$$

$$A_1 = 3{\cdot}4, A_3 = 75 \text{ sq. in.}$$

Adopting the following dimensions, $A_1 = 3{\cdot}4$ in. × 1 in., $A_2 = 16$ in. × 1 in., $A_3 = 25$ in. × 3 in.; and solving by the exact formulæ we obtain—

$$M = \frac{2 \times 2460}{3{\cdot}75} = 1312 \text{ inch-tons},$$

a result 9 % greater than required, but much nearer than that obtained by the method in example 180.

184. Solve example 181 by the approximate formulæ just given, assuming $t_a = \frac{1}{2}$ in., and $t_b = 2$ in. thick.

In this example $y_a' = 15$ in., and $y_b' = 3{\cdot}75$ in.; so that $f_a' = 7{\cdot}87$, and $f_b' = 1{\cdot}58$ tons.

$$800 = 18{\cdot}75\{7{\cdot}87A_1 + \tfrac{1}{4}(7{\cdot}87 - 1{\cdot}58)17{\cdot}5\}.$$

$$A_1 = 1{\cdot}9, \text{ and } A_3 = 44{\cdot}4 \text{ sq. in.}$$

Adopting the following dimensions, upper flange 3·8 in × ½ in., lower flange 22·2 in. × 2 in., and web 17·5 in. × 1 in.; and solving by exact method, we obtain—

$$M = \frac{2 \times 2120}{4{\cdot}75} = 890 \text{ inch-tons},$$

a result again greater than the one required by about 11 %, but much nearer than that obtained by the method in example 181.

Allowance for Weight of Beam.

After having designed a beam which is sufficient to bear a given external load, it is necessary to make an allowance for the weight of the beam itself; especially is this the case for beams of long span, as then the weight of the beam bears a considerable proportion to the amount of the external load.

This allowance is readily made by increasing the breadth of the *provisional* beam sufficient for the external load alone; since the breadth is a dimension which appears in the first power in the expression for the resistance to bending.

Consider the weight of the beam, and the external load reduced to its equivalent dead load, as uniformly distributed.

an approximation sufficiently exact for our present purpose: let b' denote the breadth, and B' the weight of the provisional beam computed for W' the external load alone; let b, B, and W denote the same quantities for the actual beam sufficient to bear the external load and its own weight; then

$$\frac{b}{b'} = \frac{B}{B'} = \frac{W}{W'};$$

$$1 - \frac{B'}{W'} = 1 - \frac{B}{W};$$

$$\frac{W}{W'} = \frac{W - B}{W' - B'} = \frac{W'}{W' - B'};$$

$$\therefore \quad b = \frac{b'W'}{W' - B'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the breadth of beam required.

$$B = \frac{B'W'}{W' - B'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

the weight of beam required.

$$W = \frac{W'^2}{W' - B'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

the gross load.

Examples.

185. A lattice girder 80 feet span bears a load of 100 tons uniformly distributed; depth from centre to centre of flanges 6 feet, and $f = 4$ tons per sq. inch; the breadth of the flange is 1 ft. 9 in., and is constant, the thickness however varies.

Taking the provisional breadth as 1 ft. 9 in., find how much this has to be increased so as to allow for the weight of the beam itself.

$$M_0 = 12{,}000 \text{ inch-tons};\ M_{10} = 11{,}250;\ M_{20} = 9000;$$
$$M_{30} = 5250;\ M_{40} = 0;$$

since the flange is thin, we have $M = fhb't = 4 \times 72 \times 21 \times t$, and we obtain $t_0 = 2$ in., $t_{10} = 2$ in., $t_{20} = 1{\cdot}5$ in., $t_{30} = 1$ in., and $t_{40} = 0 \cdot 5$ in. say; the average being 1·2 inches nearly. Taking this as the thickness of the flanges, and allowing for bracing, we obtain 10 tons nearly as the weight of the provisional girder; so that $b' = 21$ inches, $B' = 10$ tons, and $W' = 100$ tons; from which we readily find $b = 24$ inches, $B = 11$ tons, and $W = 111$ tons for the actual girder.

186. An aqueduct bridge is 60 feet span and 20 feet broad; the water is 6 feet deep and is carried on iron plates supported on cross girders 6 feet apart; the cross girders are supported on the lower flanges of two plate girders; the weight of the water, cross girders, plates and stiffeners is 240 tons, and may be considered uniformly distributed. The flanges of the main girders are to be 80 inches apart centre to centre, the web may be taken as $\frac{3}{8}$th of an inch thick, and $f = 6$ tons per sq. inch.

Find a provisional breadth for the flanges, and also how much this has to be increased so as to allow for the weight of the actual girder.

Here we find $M_0 = 900$, $M_{10} = 800$, $M_{20} = 500$, and $M_{30} = 0$ foot-tons; from this we find that an area of 22·5 sq. inches is required for each flange at the centre of girder, and if b' be taken as 18 ins. throughout, we have $t_0 = 1\frac{1}{4}$, $t_{10} = 1\frac{1}{4}$, $t_{20} = \frac{3}{4}$, and $t_{30} = \frac{1}{2}$ inch. The average thickness of flange will be 1 inch nearly, and allowing for stiffeners, &c., we find the weight of the provisional beam to be 8 tons nearly; hence we have $b' = 18$ inches, $B' = 8$ tons, and $W' = 120$ tons; from which we obtain

$$b = 20 \text{ inches}, B = 9 \text{ tons, and } W = 129 \text{ tons, for each girder.}$$

Resistance to Twisting and Wrenching.

One end of a cylindrical bar is rigidly fixed, and to the other end a couple is applied in a plane at right angles to the axis of the bar; or what is the same thing, as shown in fig. 147, a pair of equal and opposite couples are applied to the ends of the bar; the tendency of these couples is to make the bar rotate about its axis; and if we suppose the bar to consist of fibres originally straight and parallel to the axis, each of these fibres will now have assumed a spiral form. The moment of each couple is called the twisting moment or moment of torsion applied to the bar, and it is constant for each cross section; on account of the bar being uniform, the stress will be similarly distributed on each cross section; and since the bar is circular in section, the stress at all points equidistant from the axis will be the same.

Suppose two cross sections to be taken at the distance dx apart; the twisting moment causes the one section to move relatively to the other through an angle di; and if we consider two points originally opposite to each other, that is in the same fibre, one in each section and at a distance r from the axis; then these points, relatively to each other, move laterally through a distance $r.di$; and since the two sections are dx apart, the rate of twist is—

$$r\frac{di}{dx};\qquad(1)$$

a quantity directly proportional to the distance of the points under consideration from the axis.

We have thus at any point in a cross section, a shearing stress at right angles to the radius drawn to the point, and proportional to that radius in intensity; this may be expressed thus—

$$q = Cr\frac{di}{dx};\qquad(2)$$

where C is the co-efficient of transverse elasticity for the material of the cylinder under consideration; Rankine gives—

For cast iron, $C = 3{,}000{,}000$ lbs. per sq. inch (approx.)

For wrought iron, $C = 9{,}000{,}000$ „ „

The greatest value of q occurs at the surface of the cylinder; and if f represent the resistance of the material to shearing, and r_1 the radius of the cylinder, then we have—

$$q = f\frac{r}{r_1};\qquad(3.)$$

If we consider *s* a small portion of a ring of the cross section, with its middle point at a distance r from the axis, then $r.di$ will be its mean length, and we may

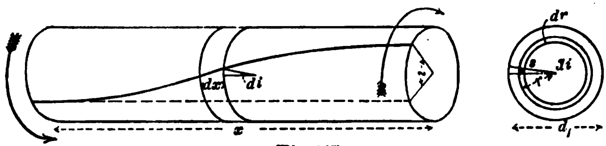


Fig.147.

denote its breadth by dr; its area then is $r.di.dr$; the intensity of the shearing stress at *s* is q; the amount of shearing stress on the small area is therefore—

$$q.r.di.dr = \frac{f}{r_1} r^2.di.dr;$$

and its moment round the axis, found by multiplying this quantity by r, is therefore—

$$\frac{f}{r_1} r^2(r.di.dr).$$

The quantity within brackets is the small area, and r is its distance from the centre; $r^2(r.di.dr)$ is therefore the moment of inertia of the small area about the centre; summing for every such small area, we have the moment of resistance to torsion for the cylinder—

$$\mathsf{M} = \frac{f}{r_1} K = \frac{f}{r_1} 2I_0; \ldots\ldots (4)$$

where K is the moment of inertia of the surface about the centre, and I_0 is the moment of inertia of the same surface about a diameter (see page 252). We therefore have—

$$\mathsf{M} = \frac{f}{r_1} \frac{\pi}{2} r_1^4 = \frac{\pi}{2} f r_1^3$$

$$= \frac{\pi}{16} f d_1^3 = \cdot 196 f d_1^3; \ldots\ldots (5)$$

where d_1 is the diameter of the cylinder.

For a hollow cylinder, let r_0 and r_1, d_0 and d_1, be the internal and external radii or diameters as the case may be; let I_0' and I_0 be the moment of inertia about a diameter of a cylinder equal in radius to r_0 and r_1 respectively, then $I_0 - I_0'$ is the moment of inertia about a diameter of the ring under consideration; we have therefore—

$$\mathbf{M} = \frac{2f}{r_1}(I_0 - I_0') = \frac{2f}{r_1}(\tfrac{1}{4}\pi r_1^4 - \tfrac{1}{4}\pi r_0^4)$$

$$= \frac{\pi f}{2}\frac{r_1^4 - r_0^4}{r_1} = \cdot 196 f \frac{d_1^4 - d_0^4}{d_1}; \qquad (6)$$

Comparing these equations with those on page 253, we find that for equal values of the limiting stress f, the resistance of a cylinder, solid or hollow, to wrenching is double its resistance to breaking across.

The working values of the limiting stress f, suitable for shafts, as given by Rankine, are

For cast iron, $f = 5000$ lbs. per sq. inch.

For wrought iron, $f = 9000$ „ „

For a cross section which is not circular, the above formulæ are inapplicable, since the ratio $\frac{q}{r}$ is no longer constant. For a square shaft M. de St. Venant gives as the moment of resistance to torsion—

$$\mathbf{M} = 0{\cdot}281 f h^3; \qquad (7)$$

Angle of torsion of an uniform cylindrical shaft.—Let x be the length of the shaft, and i the angle in circular measure through which the one end has turned relatively to the other; then since the angle of torsion per unit length is constant, we have from equation 2, page 338,

$$\frac{di}{dx} = \frac{i}{x} = \frac{q}{Cr} = \frac{f}{Cr_1}.$$

$$\therefore \quad i = \frac{fx}{Cr_1} = \frac{2fx}{Cd_1}; \qquad (8.)$$

If f be the working resistance of the material to shearing we have the same angle, whether the shaft be solid or hollow; the values of f and of C for cast and wrought iron have already been stated—

For cast iron, $i = \frac{1}{300} \frac{x}{d_1}$; .... (9a)

For wrought iron, $i = \frac{1}{500} \frac{x}{d_1}$; .... (9b.)

where i is the angle in circular measure through which the one end of a shaft of length x and diameter d_1 has turned relatively to the other end, when the working strain has been produced; the co-efficient for cast iron is somewhat uncertain.

When subjected to M any twisting moment not greater than the proof moment we have for a solid shaft (equation 5),

$$M = \frac{f}{r_1} \frac{\pi}{2} r_1^4 = \frac{q}{r} \frac{\pi}{2} r_1^4;$$

$$\frac{q}{r} = \frac{2M}{\pi r^1};$$

and from equation (8)

$$i = \frac{qx}{Cr} = \frac{2Mx}{\pi r_1^4 C} = 10{\cdot}2 \frac{Mx}{Cd_1^4}; \quad \text{............ (10.)}$$

For a hollow shaft, similarly

$$i = 10{\cdot}2 \frac{Mx}{C(d_1^4 - d_0^4)}; \quad \text{.................... (11.)}$$

If we make $x = 1$, or, what is the same thing, if the distance between the two cross sections which we consider is unity, the stiffness of the shaft will be measured by the reciprocal of i, i being the angle in circular measure through which the two cross sections have turned relatively to each other, when the skin has been brought to the proof strain.

For two shafts of the same length and material, but of different diameters, we see from equation 10 that the twisting moment to be applied to each, in order that both ma be turned through the same angle of torsion, is proportional to the fourth power of the diameter, the proof stress not being in any case exceeded; thus—

$$i = 10{\cdot}2\frac{Mx}{Cd^4} = 10{\cdot}2\frac{M'x}{Cd'^4};$$

or $$\frac{M}{M'} = \frac{d^4}{d'^4}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12.)$$

Examples.

187. A water wheel of 20-horse power makes 5 revolutions per minute; find the diameter suitable for the malleable iron shaft which transmits this force.

For each revolution 132,000 ft.-lbs. of work are performed; this is equivalent to 21,008 lbs. acting on a wheel of radius one foot, and we have $M = 21{,}008$ ft.-lbs. $= 252{,}096$ inch-lbs.

$$252{,}096 = {\cdot}196 \times 9000\, d_1^3 = 1764\, d_1^3; \quad \therefore \; d = 5{\cdot}23 \text{ ins.}$$

188. If this shaft be 12 feet long, what is its angle of torsion when the working moment as above is applied? Take $f = 9000$, and $C = 9{,}000{,}000$.

$$\textit{Ans.} \quad i = \frac{1}{500}\,\frac{144}{5{\cdot}23} = {\cdot}05507 = 3^\circ\ 9'.$$

189. The diameter of one shaft is double that of another of the same material; the smaller gave way when subjected to a twisting moment of 2 ft.-tons. What twisting moment will be required to wrench the other?

Ans. $M = 16$ ft.-tons.

190. A shaft 12 feet long and 6 inches diameter is subjected to a twisting moment of 16 ft.-tons, and the two ends are thus twisted through a certain angle; a second shaft of the same material, 16 feet long and 9 inches diameter, is twisted so that its angle of torsion is exactly the same as that of the first; find the twisting moment required to do this.

$$i = 10{\cdot}2\frac{192 \times 144}{C \times 6^4} = 10{\cdot}2\frac{M \times 192}{C \times 9^4};$$

$$\therefore \frac{192 \times 144}{1296} = \frac{M \times 192}{6561}; \quad \therefore M = 729 \text{ in.-tons} = 60{\cdot}75 \text{ ft.-tons.}$$

191. What thickness of metal is required for a cast iron hollow shaft, 10 inches outer diameter, so as to resist a twisting moment of 10 ft.-tons?

$$\textit{Ans.} \quad M = 120 \times 2240 = {\cdot}196 \times 5000\frac{10^4 - d_0^4}{10}$$

$d_0^4 = 7258$; $\therefore$ $d_0 = 9{\cdot}23$, and the thickness required is 0·4 inch.

192. A malleable iron shaft 20 feet long and 6 inches diameter is subjected to a moment which twists the ends through an angle of 2°; taking C the co-efficient of transverse elasticity as 9,000,000, find f, the stress at the skin.

$$i = {\cdot}0349 = \frac{2f.240}{9 \times 10^6 \times 6}; \quad \therefore f = 3926 \text{ lbs. per sq. inch.}$$

193. The inner and outer diameters of a hollow steel shaft are 10 and 12 inches, and $f = 6$ tons per square inch is the working value of the resistance to shearing. What is the twisting moment this shaft is capable of transmitting?

$$\text{M} = {\cdot}196 \times 6\frac{12^4 - 10^4}{12} = 1052 \text{ inch-tons.}$$

194. The working tensile strength of the steel for the previous example is 12 tons per square inch, and the crushing strength is greater. What is the moment of resistance to bending of this same shaft?

Ans. M = 1052 inch-tons; since the resistance to wrenching is double the resistance to breaking across when the two values of f are equal.

Bending and Torsion Combined.

Let the shaft shown in fig. 148 be acted upon by a bending load and a pair of equal twisting couples; and at the point H let M_1 be the moment of the first, and M_2 the moment of the second; then in order to find the amount and direction of the greatest principal stress, we require to combine the greatest direct stress due to bending with the greatest shearing stress due to twisting; this is done by the method of the ellipse of stress.

At the point H, let p be the intensity of thrust (or tension) due to the bending moment M_1, and q the intensity of shearing stress due to the twisting couple M_2; then we have—

$$p = \frac{4M_1}{\pi r_1^3};\ q = \frac{2M_2}{\pi r_1^3}; \qquad (1.)$$

Let p_1 be the greatest intensity of stress (thrust) at the point, then fig. 136 shows the construction required to find its amount; in that figure $\overline{OL} = p$, $\overline{OR'} = q$, and we have—

$$p_1 = \overline{OM} + \overline{MR} = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}; \qquad (2)$$

the greatest intensity of shearing stress is represented by

$$\overline{MR} = \sqrt{\frac{p^2}{4} + q^2}; \qquad (3)$$

and the angle θ made by the greatest stress p_1 with the axis of the shaft is given by the equation

$$\tan 2\theta = \frac{\overline{RL}}{\overline{ML}} = \frac{2q}{p}; \qquad (4.)$$

By substituting for p and q the values given in eqn. 1, we have—

$$p_1 = \frac{2}{\pi r_1^3}(M_1 + \sqrt{M_1^2 + M_2^2}) \qquad (5)$$

and for the greatest intensity of shearing stress

$$\overline{MR} = \frac{2}{\pi r_1^3}\sqrt{M_1^2 + M_2^2}; \qquad (6.)$$

A very important example of this principle is that of a shaft with a crank attached; in this case we have a force applied to the centre of the crank pin, and resisted by the equal and opposite force at the bearing S. If P represent the force, then the moment of the couple is

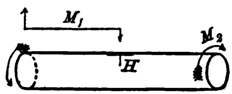
$$M = P . \overline{SP}; \quad \ldots\ldots (7)$$

this couple may be resolved into two couples, one a bending couple—

$$M_1 = P . \overline{NS} = M \cos j; \quad \ldots\ldots (8)$$

the other a twisting couple—

$$M_2 = P . \overline{NP} = M \sin j. \quad \ldots\ldots (9.)$$



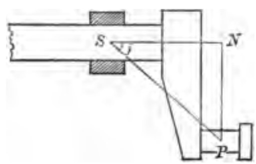


Fig.148.

The greatest intensity of stress is found by eqn. 5.

$$p_1 = \frac{2}{\pi r_1^3}(M \cos j + \sqrt{M^2 \cos^2 j + M^2 \sin^2 j})$$

$$= \frac{2}{\pi r_1^3}(M \cos j + M) = \frac{2}{\pi r_1^3} M(1 + \cos j)$$

$$= \frac{5{\cdot}1}{d^3} M(1 + \cos j); \quad \ldots\ldots (10.)$$

If instead of p_1 we put f the resistance to tension or thrust (the smaller), we get

$$d^3 = \frac{5{\cdot}1}{f} M(1 + \cos j); \quad \ldots\ldots (11)$$

which enables us to calculate the diameter required for the shaft. If we put f for the greatest intensity of the shearing stress, we have—

$$d^3 = \frac{5{\cdot}1}{f} M; \quad \ldots\ldots (12)$$

which also enables us to calculate the diameter required and the greater of the two, one got from eqn. 11, the other from eqn. 12, is to be adopted.

The angle made by the principal stress with the axis of the shaft is given by eqn. 4.

$$\tan 2\theta = \frac{2q}{p} = \frac{M_2}{M_1} = \frac{\sin j}{\cos j} = \tan j;$$

$$\therefore \quad \theta = \frac{j}{2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(13.)}$$

Examples.

195. A shaft 9 inches diameter and 12 feet long is supported at its two ends, and loaded at the two points which divide its length into three equal parts with 4 tons at each point; a twisting moment of 20 foot-tons is applied to one end of the shaft while the other is held fixed. Find the greatest intensity of the thrust, tension, and shearing stress; and the angle that the line of greatest principal stress makes with the axis of the shaft.

At any point between the two loads, the bending moment M_1 = 16 foot-tons = 192 inch-tons; the twisting moment M_2 = 240 inch-tons;

$$p = \frac{4 \times 192}{\pi \times 91 \cdot 1} = 2 \cdot 69 \text{ tons}; \; q = \frac{2 \times 240}{\pi \times 91 \cdot 1} = 1 \cdot 67 \text{ tons}.$$

$$\therefore \quad p_1 = 1 \cdot 35 + \sqrt{\frac{7 \cdot 24}{4} + 2 \cdot 80} = 3 \cdot 5 \text{ tons per sq. inch}$$

is the greatest value of the intensity of the thrust at the upper point, and of the tension at the lowest point of the skin near the middle of the length of the shaft; the greatest intensity of the shearing stress is 2·15 tons per sq. inch, and it is situated at the points just mentioned.

$$\tan 2\theta = \frac{3 \cdot 34}{2 \cdot 69} = 1 \cdot 24; \quad \therefore \; \theta = 25^\circ.$$

196. The crank shaft of an engine is 5 in. diameter; the distance from the centre of the bearing to the point opposite the centre of the crank pin, *NS* in fig. 148, is 12 inches; the half stroke, *NP* in figure, is 16 inches; and the pressure

applied to the crank pin is 5000 lbs. Find the greatest intensity of thrust, tension, and shearing stress; and θ the angle made by the line of principal stress with the axis of the shaft.

$$PS = 20 \text{ inches}; \quad \therefore \ M = 100{,}000 \text{ inch-lbs.};$$

$$M_1 = 60{,}000, \text{ and } M_2 = 80{,}000.$$

$$p_1 = \frac{5 \cdot 1}{125} 100{,}000 (1 + \tfrac{3}{5}) = 6530 \text{ lbs. per sq. inch,}$$

the greatest intensity of thrust and of tension, at the bearing, the one being at the one side and the other being at the other side of the shaft. The greatest intensity of shearing stress is $\frac{5 \cdot 1}{125} 100{,}000 = 4080$ lbs. per sq. inch. The angle $\theta = \frac{j}{2} = 27°$.

Thrust or Tension combined with Torsion.

Let the shaft shown in fig. 149 be acted upon by a thrust (or tension) P and a pair of twisting couples of moment M; the stress due to P is uniformly distributed, and that due to M is greatest at the skin; the greatest intensity of stress will therefore be at the skin. If under thrust, the length of the shaft is to be so short compared with its diameter, that the bending action need not be taken into account. At the point H we have a thrust (or tension) $p = \frac{P}{\pi r_1^2}$, and a shearing stress $q = \frac{2M}{\pi r_1^3}$; proceeding as at fig. 136, we have—

$$\overline{OM} = \frac{p}{2} = \frac{P}{2\pi r_1^2}; \quad \ldots\ldots\ldots\ldots (1.)$$

$$\overline{MR} = \sqrt{\frac{p^2}{4} + q} = \sqrt{\left(\frac{P}{2\pi r_1^2}\right)^2 + \left(\frac{2M}{\pi r_1^3}\right)^2}; \quad \ldots\ldots (2.)$$

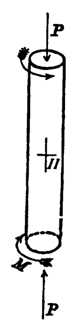


Fig. 149.

The greatest intensity of thrust (or tension) is

$$p_1 = \overline{OM} + \overline{MR}; \quad \ldots\ldots\ldots\ldots (3)$$

the angle θ made by p_1 with the axis of the shaft is given by the equation

$$\tan 2\theta = \frac{\overline{RL}}{\overline{ML}} = \frac{2q}{p}; \qquad \text{(4.)}$$

The greatest intensity of shearing stress is

$$\overline{MR}; \qquad \text{(5.)}$$

Examples.

197. A shaft 8 inches diameter is subjected to a thrust of 100 tons uniformly distributed over its two ends, and a twisting moment of 30 foot-tons. Find the greatest intensity of thrust and shearing stress, and the angle made by the line of principal stress with the axis of the shaft.

$$p = \frac{100}{50{\cdot}26} = 1{\cdot}99 \text{ tons}; \; q = \frac{720}{201{\cdot}04} = 3{\cdot}58 \text{ tons per sq. inch.}$$

$$p_1 = 1{\cdot}0 + \sqrt{{\cdot}99 + 12{\cdot}81} = 4{\cdot}71 \text{ tons per sq. inch,}$$

the greatest intensity of thrust; the greatest intensity of shearing stress is 3·71 tons per sq. inch; and

$$\tan 2\theta = \frac{7{\cdot}16}{1{\cdot}99} = 3{\cdot}6; \; \theta = 37^\circ.$$

198. Find the diameter of a malleable iron shaft capable of bearing a tension of 50 tons, and a twisting couple whose moment is 25 foot-tons; the resistance of the material to tension and shearing being 5 and 4 tons per sq. inch respectively.

$$p = \frac{50}{\pi r_1^2} = \frac{15{\cdot}92}{r_1^2}; \; q = \frac{600}{\pi r_1^3} = \frac{191}{r_1^3}.$$

$$\therefore \quad p_1 = 5 = \frac{7{\cdot}96}{r_1^2} + \sqrt{\frac{63{\cdot}36}{r_1^4} + \frac{36500}{r_1^6}};$$

from which we find $r_1 = 3{\cdot}53$ inches. The greatest intensity of shearing stress $= 4 = \sqrt{\frac{63{\cdot}36}{r_1^4} + \frac{36500}{r_1^6}}$; from which we find $r_1 = 3{\cdot}64$ inches; and since this is greater than the former result, it is to be adopted; that is, the diameter required for the shaft is $7\frac{1}{2}$ inches nearly.

Thrust and Bending.

When thrust is applied to a pillar or strut whose length is great compared with its diameter, it will collapse not by direct crushing but by bending and breaking across.

Let a long thin vertical bar, originally straight, be deflected to an extent not greater than the proof deflection by the application of a horizontal external force applied, say at its middle, while the ends are guided so that they cannot move laterally, and let it be held in that position; it will then have a form such as is shown in fig. 150; let the load P be now applied, then when the restraint is withdrawn, the bar will tend to assume its original vertical form, it will remain neutral, or it will collapse according to the amount of P; that is to say, if the moment of P relatively to the centre of the bar, viz., $P.v$, is less than the moment of resistance of the bar to bending, the bar will tend to right itself.

The stress on the cross section AB consists of one part p' due to the load P, and another part p'' due to the bending which takes place in the direction in which the pillar is most flexible; since P is uniformly distributed, we have—

$$p' = \frac{P}{S}; \qquad \ldots\ldots (1)$$

where S represents the sectional area of the bar; by eqn. given on page 214, we have $M = np''bh^2$; and since $M = Pv$

$$p'' \propto \frac{Pv}{bh^2}; \qquad \ldots\ldots (2)$$

where h is the smaller, and b is the larger diameter, when these are unequal; the proof deflection v, page 295, eqn. ($3a$) is directly proportional to the square of the length and inversely proportional to the depth; that is to say

$$v \propto \frac{l^2}{h};$$

and $$p'' \propto \frac{Pl^2}{bh^3} \propto \frac{Pl^2}{Sh^2},$$

that is $p'' \propto \frac{l^2}{h^2}$; therefore

$$\frac{p''}{p'} \propto \left(\frac{l}{h}\right)^2; \qquad (3)$$

that is, the additional stress due to bending is to the stress due to the direct thrust, as the square of the proportion in which the length of the pillar exceeds the least diameter.

The total intensity of stress $p' + p''$ must not exceed the strength of the material; equating that intensity to f the strength of the material, we have—

$$f = p' + p'' = p' + ap'\frac{l^2}{h^2} = p'\left\{1 + a\left(\frac{l}{h}\right)^2\right\}$$

$$= \frac{P}{S}\left\{1 + a\left(\frac{l}{h}\right)^2\right\}.$$

$$P = \frac{fS}{1 + a\left(\frac{l}{h}\right)^2}; \qquad (4)$$

where a is a constant co-efficient to be determined by experiment. The above investigation is due to Tredgold, and

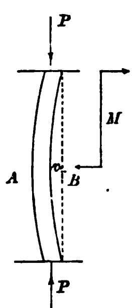


Fig.150.

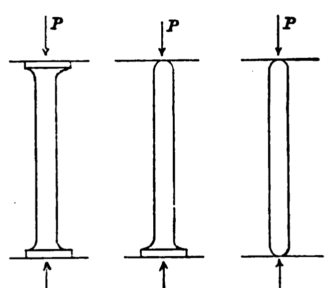


Fig.151.

values of the co-efficient a, as determined by Professor Gordon, are given below.

For a strut or pillar SECURELY FIXED AT THE ENDS—

Wrought iron solid rectangular section, ... $a = \frac{1}{3000}$.

*Angle, channel, cruciform, and T-iron (see fig. 152), $a = \frac{1}{900}$.

Cast iron, hollow cylinder, ... $a = \frac{1}{400}$.

†For timber struts, oak and pine, ... $a = \frac{1}{250}$.

A pillar ROUNDED AT BOTH ENDS is as flexible as one of double the length fixed at the ends; so that

$$P = \frac{fS}{1 + a\left(\frac{2l}{h}\right)^2} = \frac{fS}{1 + 4a\left(\frac{l}{h}\right)^2}; \quad \text{.....................} \quad (5.)$$

The strength of a pillar FIXED AT ONE END AND ROUNDED AT THE OTHER is a mean between that of a beam fixed at both ends and one rounded at both ends.

The values of f in lbs. per sq. inch are given by Rankine as follows:—

	Breaking load.	Proof load.	Working load.
Wrought iron solid rect. section, ...	36,000	18,000	6,000 to 9,000
Wrot. iron cell,.......	27,000	13,500	4,500 to 7,000
Cast iron cylinders,	80,000	26,700	13,300 to 20,000
British oak, dry,	10,000	——	1,000
American „ „	6,000	——	600
Red pine and larch „	5,400	——	550

For green timber the values of f should be halved.

A pillar or strut securely fixed at both ends corresponds with a beam fixed at the ends, fig. 143; the points of fracture are the points where in that figure the bending moment is greatest, viz., at the centre and at the ends. A pillar fixed at one end corresponds with the beam shown in fig. 141; the points of fracture being at the fixed end, and at about one-third of the length from the rounded end. A pillar rounded at both ends corresponds with the beam

* As deduced by Mr. Unwin. † As deduced by Weisbach.

shown in fig. 25; the point of fracture being at the middle of the length.

The following table gives the results of the above formulæ for pillars of wrought and cast iron, whose diameters and lengths are in different proportions, and whose ends are securely fixed;—

$\frac{l}{h}$	Breaking Load, lbs. per sq. in. $=\frac{P}{S}$.		$\frac{l}{h}$	Breaking Load, lbs. per sq. in. $=\frac{P}{S}$.	
	Wrot. Iron solid rectangular sect.	Cast Iron hollow cylinder.		Wrot. Iron solid rectangular sect.	Cast Iron hollow cylinder.
10	34,840	64,000	30	27,700	24,620
15	33,490	51,200	35	25,560	19,700
20	31,765	40,000	40	23,480	16,000
26·4	29,230	29,230	50	19,640	11,030

From this table it is seen that, so far as the ultimate strength of such pillars is concerned, cast iron is stronger than, as strong as, or less strong than wrought iron when the proportion $\frac{l}{h}$ is less than, equal to, or greater than 26·4; this result was first pointed out by Professor Gordon.

For struts in wrought iron framework, such cross sections as are shown in fig. 152 may be chosen on account of their

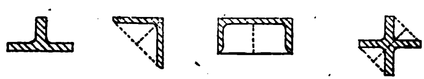


Fig. 152.

stiffness; these are called T, angle, channel, and cruciform iron respectively; in fixing the proportion $\frac{l}{h}$ for such sections, the value of h is to be taken as the *least* diameter; this is marked in the diagram.

Since, however, such cross sections are not made of very large dimensions, struts above a certain size require to be *built;* in this case they usually consist of four thin plates forming a square and connected at the corners by angle irons; of two thin plates held parallel and at a fixed distance apart, by means of a web and angle irons, or by a lattice work of small diagonal bars; or of two T-irons or channel-irons, with the ribs turned towards each other, and held by a lattice work of small diagonal bars as in the case just stated.

Examples.

199. A cast iron column, securely fixed at the ends, external diameter 8 inches, length 20 feet, is to bear a steady load of 30 tons. Find the thickness of metal required.

Here $\frac{l}{h} = 30$, and for that proportion the breaking load is 24,620 lbs. per sq. inch; taking $\frac{1}{6}$ for a factor of safety, we get the working stress $f = 4100$ lbs. $= 1{\cdot}8$ tons per sq. inch; the area of metal required is therefore 17 ins.; this gives 6·7 ins. for the internal diameter, or $\frac{3}{4}$ inch nearly for the thickness of metal. Since an allowance has to be made for slight irregularities in casting, the thickness of metal should be one inch.

200. Find the working load for a cast iron pillar 12 ins. diameter, 40 feet long, metal 1 inch thick, taking 6 as the factor of safety.

Here $\frac{l}{h} = 40$; $S = 34{\cdot}6$ sq. ins.; $f = 1{\cdot}2$ tons per sq. inch; and the steady working load is 42 tons nearly, both ends being securely fixed; but if the load is such as to cause considerable vibration, from one-half to two-thirds of this amount may be taken.

201. What is the crushing load for a malleable iron bar 6 in. × 3 in. × 10 feet long, securely fixed at one end?

$S = 18$ sq. ins.; $\frac{l}{h} = 40$, and the ultimate stress corresponding is 10·5 tons per sq. inch, when fixed at both ends; when rounded at both ends, this reduces to 5·1 tons; when fixed at one end and rounded at the other, the result is the mean of these quantities, viz., 7·8 tons per sq. inch.

The crushing load is therefore 140 tons, steadily applied.

202. Find the working strength of a strut formed of channel irons ½ in. thick, 6 in. broad, width of each flange (outside) 2½ in., and length 6 feet, fixed securely at both ends.

Here $S = 5$ sq. ins.; $\frac{l}{h} = 29$ nearly, and the working strength is 1·4 tons per sq. inch, or 7 tons nearly.

203. What is the working load for a strut of seasoned American oak firmly fixed at the ends, 20 feet long, and 1 foot square?

$$P = 33{,}000 \text{ lbs. nearly, or } 14\tfrac{1}{2} \text{ tons.}$$

204. A strut of red pine whose ends are to be well fixed, is to be 4 ins. thick and 6 feet long; the thrust applied to its ends is calculated to be 4 tons. Find the breadth required.

$$S = P\frac{1 + a\left(\frac{l}{h}\right)^2}{f} = 4\,\frac{1 + \frac{324}{250}}{\frac{1}{4}} = 37 \text{ sq. ins. nearly,}$$

and the breadth required is therefore 9¼ ins. nearly.

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