ELEMENTARY DYNAMICS OF THE PARTICLE AND RIGID BODY

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ELEMENTARY DYNAMICS
OF THE
PARTICLE & RIGID BODY

BY

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special circle has to be thought of in connection with the motion, as is frequently the case in the common method. I have also dealt more fully with simple harmonic motion than usual, and have discussed some cases of systems with two independent coordinates, and also cases of forced oscillations.

(4) I have also dealt more fully than usual with rigid dynamics.

On the other hand I have omitted several special questions where the student loses nothing by waiting until he has a fuller acquaintance with the calculus than he is likely to have in reading this book. Such are chords of quickest descent, motion on a cycloid, and non-circular central orbits.

A number of the examples, particularly those at the ends of the chapters, are taken from Melbourne University papers. Most of the others have been specially constructed by myself, but a few, which I am unable now to specify, may have come from other sources. I can only hope that not many errors will be found in the answers given.

I wish to express my gratitude to Mr. J. H. Michell, F.R.S., for his encouragement and suggestions.

R. J. A. BARNARD.

DUNTROON,
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PART I.

DYNAMICS OF A PARTICLE.

CHAPTER I.

POSITION. VELOCITY. ACCELERATION.

THE science of dynamics is concerned with questions about the motions of bodies. The idea of motion involves two other ideas, those of time and position.

1. Time.

No discussion will be entered upon as to the nature of time, but it is necessary to consider the measurement of time. The unit of time is derived from the rotation of the earth. The interval from the instant when the sun is due north (or due south) of us one day until he is due north (or south) the next day is called a solar day. The solar day varies in length slightly for reasons that need not be discussed here, but the average length of the solar day throughout the year is called a Mean Solar Day, and is divided into 86,400 (or $24 \times 60 \times 60$) seconds.

ERRATA.

p. 125, 10 lines from bottom, *for* ft. per sec. *read* ft./sec^2.

p. 192, Ex. 2, line 6, *for* of *read* if.

p. 228, Diagram, near B, for mg_1 read mg'.

PART I.

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THE science of dynamics is concerned with questions about the motions of bodies. The idea of motion involves two other ideas, those of time and position.

1. Time.

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It is usual in dynamical questions to choose a definite instant as the origin or starting point to measure times from, and to speak of an event as happening at time t, meaning at the instant t seconds after the instant which has been selected as the origin of times. For example, when a stone is thrown in the air from the ground, it is convenient to take the instant leaves the hand as the origin of times, and we can then spe of its position at time $t=3$, $t=5 \cdot 82$, or, in general, at time meaning that 3, 5·82 or t seconds have elapsed since it left tl hand.

2. Position.

In the determination of the position of a point it is necessa to express its position with respect to other points, or lines, planes, and if its position changes with respect to these, v can speak of, and measure, its change of position or displac ment relatively to these points, lines or planes. We cann talk of displacement or motion of a body, unless we have oth objects of some kind to define its position by, and then we ca only speak of its change of position and motion relatively those objects. For the present we will only consider the ca where the position of a point P is referred to a single point and a single line Ox passing through O, on which line the poi P always lies. The point P is then said to be moving in a straig line relatively to the reference point or origin O.

The case is well illustrated by the case of a bicyclist ridir along a straight road Ox. The mile-stones along the road sho the distance from a certain point (O), and his position on tl road at any time may be represented by the distance he is fro O, as determined by the mile-stones. In this case the roa along which the bicyclist is riding is itself in continual an rapid motion, on account of the motions of the earth on it axis, and around the sun, but this does not affect our idea c his motion along the road, and we speak of his motion alon the road exactly as if there were no such thing as a motion o the earth. In other words, we are only concerned with th motion relative to the earth. It will be seen from thi example that all we are concerned with in dynamics is motion relative to some body or bodies, whose motion we may or may not be concerned with. In most of the elementary dynamics the motions we deal with are motions relative to the earth, and the surrounding objects attached to the earth, the motion of the earth being neglected. Cases in which the results are affected by the motion of the earth will appear later.

3. Motion in a Straight Line. Velocity.

Suppose, then, that a point is moving along a straight line, and that at time t_1 the distance from a certain origin is s_1 feet,

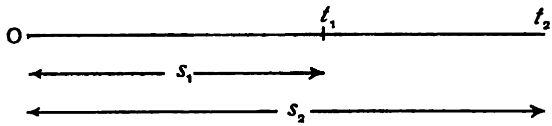


FIG. 1.

and at a time t_2 the distance is s_2 feet, then the ratio $\frac{s_2 - s_1}{t_2 - t_1}$ is called the average velocity (in feet per second) during the interval $t_2 - t_1$. This evidently agrees with the ordinary idea of velocity, for $s_2 - s_1$ is the distance (in feet) travelled in $t_2 - t_1$ seconds.

We shall use the symbols ft./sec. and miles/hr. for feet per second and miles per hour respectively, in speaking of velocities.

If we take the case of the cyclist or of a train, it is obvious that we shall generally get different results for the average velocity for different intervals. Thus, the velocity of the cyclist will be found to be less when he is going up hill than when he is going down. For example, if

$s = 10$ miles when $t = 2$ hours,
$s = 17$,, ,, $t = 3$,,
$s = 30$,, ,, $t = 4$,,

the average velocity during the first of these hours is 7 miles an hour, and during the second hour, 13 miles an hour, while for the two hours the average velocity is 10 miles an hour.

To get a more accurate idea of his velocity, we may take l position at intervals of 5 minutes, from which we could dedu his average velocity during each 5 minutes. Even then l velocity may vary during any 5 minutes, and to be still mo accurate we take a shorter interval still, and though we st get only an average velocity during a short interval, we ca form a much better idea of the way the cyclist has be travelling, than when we only knew the average velocity durin each hour. Thus the shorter the intervals of time for whi the velocity is calculated, the more accurate the knowledge his motion, and we are thus led to the idea of taking an i definitely short interval of time, and calling the velocit obtained then, the velocity *at* the particular *instant*, instea of the average velocity *during* the particular *interval*. Math matically, we may put our definition in the following way :

$$\text{Velocity at time } t_1 = \lim_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1},$$

meaning that we have to take the expression for the averag velocity, and make t_2 approach indefinitely close to t_1; in othe words, make $t_2 - t_1$ indefinitely small, in which case $s_2 - s_1$ als becomes indefinitely small, and the ratio of the two indefinitel small quantities becomes ultimately (or in the *limit*) th velocity at time t_1.

The expression $\lim\limits_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1}$ may be read—the limit as t_2 approaches t_1 of $\frac{s_2 - s_1}{t_2 - t_1}$.

Another notation frequently used to represent small quantities such as occur here is Δs for a small change in s, and Δt for a small change in t. Δs is then a single symbol, and the average velocity during the short interval Δt is expressed by the fraction $\frac{\Delta s}{\Delta t}$, in which the Δ's cannot be cancelled out, because they do not of themselves represent algebraical quantities.

Thus Δs represents the same thing as $s_2 - s_1$, except that Δs is almost always applied to small quantities.

The velocity at an instant will then be represented by $\underset{\Delta t \to 0}{\text{Lim}} \frac{\Delta s}{\Delta t}$. In general, $\underset{\Delta t \to 0}{\text{Lim}} \frac{\Delta s}{\Delta t}$ will differ from $\frac{\Delta s}{\Delta t}$ by a small quantity, say, is greater by an amount ϵ which is small compared with $\frac{\Delta s}{\Delta t}$ itself.

Hence velocity at an instant $= \underset{\Delta t \to 0}{\text{Lim}} \frac{\Delta s}{\Delta t} = \frac{\Delta s}{\Delta t} + \epsilon$.

4. Note on Limiting Values of Fractions.

As limiting values of fractions are involved in the whole of dynamics, a little explanation may be given for the sake of the reader who is unfamiliar with the idea. Take the fraction $\frac{x^2 - 1}{x - 1}$. This is equal to $x + 1$ for all values of x except $x = 1$. If $x = 1$, the numerator and denominator of the fraction are both zero, and the fraction ceases to have a meaning. The fraction, in fact, is not defined when $x = 1$, yet we can find the value when x is nearly equal to 1, the fraction being then nearly equal to 2.

For if $\quad x = 1{\cdot}1, \quad \frac{x^2 - 1}{x - 1} = x + 1 = 2{\cdot}1,$

$$x = 1{\cdot}01, \quad \frac{x^2 - 1}{x - 1} = 2{\cdot}01,$$

$$x = 1{\cdot}001, \quad \frac{x^2 - 1}{x - 1} = 2{\cdot}001,$$

and the nearer x gets to 1, the nearer the fraction approaches the value 2. Hence, though the fraction has no meaning when $x = 1$, we can still talk of it having 2 as its limiting value as x approaches 1, and we write the fact thus :

$$\underset{x \to 1}{\text{Lim}} \frac{x^2 - 1}{x - 1} = \underset{x \to 1}{\text{Lim}} (x + 1) = 2.$$

All cases of calculation of velocity at a given instant from the distances travelled, or, as we shall see immediately, of acceleration from the velocities, are examples of limiting values of fractions, the numerators and denominators being both indefinitely small quantities whose ratio is required.

When the denominator of a fraction approaches zero while the numerator remains finite, the fraction becomes bigger and bigger. Thus $\frac{x}{x-1}$ is defined for all values of x except $x=1$, and for that value the fraction has no meaning. But it is easy to see that the closer x approaches to 1, the bigger the fraction is, and we can make it as big as we like by choosing x near enough to 1. This is expressed by saying that the limiting value of the fraction when x approaches 1 is infinity, or

$$\operatorname*{Lim}_{x \to 1} \frac{x}{x-1} = \infty.$$

Example 1. Find the values of $\frac{x}{x-1}$ when $x = \cdot 9, \cdot 99, 1 \cdot 1, 1 \cdot 01$.

2. Show that the fraction $\frac{x^2-4x+3}{x^2-3x+2}$ is not defined for the values $x=1, 2$. Find its limiting values as x approaches these values. Find also its values when $x = \cdot 99, 1 \cdot 01, 1 \cdot 99, 2 \cdot 01$.

5. Acceleration.

In a great series of experiments carried out in 1590, by dropping cannon balls from the Leaning Tower of Pisa, and again in 1612, Galileo investigated the motion of a body falling to the earth. He shewed for the first time that the velocity was continually increasing in such a way that it received equal increments each second, and was thus led to the idea of acceleration as rate of increase of velocity.

If the velocity v_1 at time t_1, given by the expression $\operatorname*{Lim}_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1}$, is found to be the same at all instants, it is said to be uniform or constant, if otherwise it is variable. When the velocity is variable, we use the term acceleration to denote the rate of change of velocity. Mathematically, the acceleration is determined from the velocity in the same way that the velocity is determined from the distance; thus, if

$$v_1 = \text{the velocity at time } t_1,$$

$$v_2 = \quad " \quad " \quad t_2,$$

the acceleration at time $t_1 = \lim_{t_2 \to t_1} \frac{v_2 - v_1}{t_2 - t_1}$.

We speak of uniform and variable acceleration in the same way as of uniform and variable velocity.

As an acceleration is the rate of change of velocity or change of velocity per second, it is common to express the acceleration as so many feet per second per second. Thus, if a body has a velocity at one instant of 5 feet per second, and 10 seconds later it has a velocity of 25 feet per second, we can say its average acceleration during that time was 2 feet per second per second, but we shall prefer to call it an acceleration of 2 foot-second units, or still better of 2 ft./sec^2.

6. Graphical Representation. Position-Time Graph.

Suppose the position of a point moving in a straight line is represented by a graph giving the position at each instant.

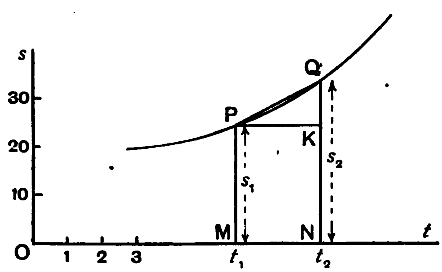


FIG. 2.

In the figure PM represents the distance from a fixed point at time t_1, QN represents the distance at time t_2. If PK is parallel to MN, then

$$QK = QN - PM = s_2 - s_1,$$
$$PK = ON - OM = t_2 - t_1;$$

therefore the average velocity during the interval $t_2 - t_1$ is $\frac{QK}{PQ}$, and is equal to the gradient of the chord PQ. Taking the interval of time very short so that t_2 approaches t_1, MN becomes indefinitely short, and Q approaches indefinitely close to P, and the chord PQ becomes ultimately the tangent at P.

Hence the velocity at time t_1 is represented by the gradient of the tangent at the point P on the graph. It does not matter what scales are used along the two axes, provided the gradient

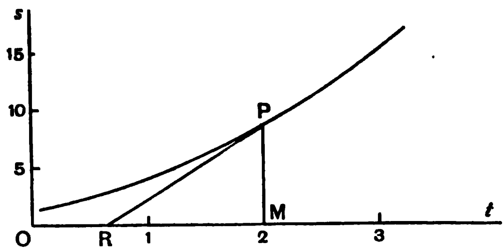


FIG. 8.

is determined from the length of lines measured according to the two scales. Thus, in the figure, if PR is the tangent at P, the velocity, when $t = 2$, is the gradient of PR, or

$$= \frac{PM}{RM} = \frac{8}{1 \cdot 4} = \frac{40}{7} = 5 \cdot 7.$$

7. Velocity-Time Graph.

In exactly the same way, if we draw a graph representing the velocity in terms of the time, the average acceleration during any interval will be represented by the gradient of the chord joining the points on the graph corresponding to the beginning and end of the interval. Also the acceleration at a particular instant will be represented by the gradient of the tangent at the corresponding point on the graph.

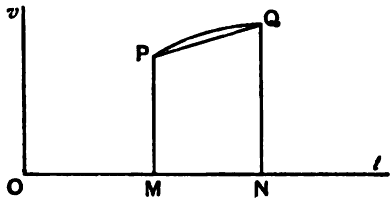


FIG. 4.

8. Positive and Negative Signs.

It must be borne in mind throughout, that the ordinary conventions about positive and negative signs apply to all the above quantities—distance, velocity, and acceleration. From the origin O distances measured in one direction are positive, and in the other negative. If a point is moving in the direction in which the positive distances are measured, its velocity is positive, and if at the same time its velocity is increasing, the acceleration is positive. If, while having a positive velocity, this velocity is diminishing, the acceleration is negative. If it is moving in the opposite direction to the above, the velocity is negative. If, while having a negative velocity, this velocity is increasing numerically, the acceleration is negative, and so on. The expressions above and the graphical representations apply to all cases. Thus, if

$$s_1 = 20 \text{ feet when } t = 2 \text{ secs. } (t_1),$$
$$s_2 = 10 \quad ,, \quad ,, \quad t = 4 \quad ,, \quad (t_2),$$

the average velocity in the interval $= \dfrac{10-20}{4-2} = -5$ ft./sec.

If $\qquad v_1 = -6$, when $t = 3$ secs.,

$\qquad v_2 = \ 4, \quad ,, \quad t = 5 \quad ,,$

the average acceleration in the interval is $\dfrac{4-(-6)}{5-3} = 5$ ft./sec².

9. Examples of the Preceding Results.

To illustrate the use of the above, and to show how they are to be applied to experimental data, suppose we make an experiment of letting a marble roll down a slightly inclined board, say a long table raised a few inches at one end. We can determine the position at any time easily by marking with a chalk the position at each second, using the ticks of a clock or watch to note the seconds. We find, for example, the following results from experiment :

t secs.	s feet.
0	0
1	0·3
2	1·3
3	3·0
4	5·3

Representing these on a graph, we find the curve to be approximately a parabola, and notice that $s = \frac{1}{3}t^2$ represents the results very closely. Let us calculate the velocity and acceleration on this supposition.

$$s_1 = \tfrac{1}{3}t_1^2,$$

$$s_2 = \tfrac{1}{3}t_2^2,$$

$$s_2 - s_1 = \tfrac{1}{3}(t_2^2 - t_1^2),$$

$$\frac{s_2 - s_1}{t_2 - t_1} = \frac{1}{3}\,\frac{t_2^2 - t_1^2}{t_2 - t_1} = \tfrac{1}{3}(t_2 + t_1),$$

$$v_1 = \lim_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1} = \text{Lim}\, \tfrac{1}{3}(t_2 + t_1) = \tfrac{2}{3}t_1,$$

$$v_2 = \tfrac{2}{3}t_2,$$

$$v_2 - v_1 = \tfrac{2}{3}(t_2 - t_1),$$

$$\text{acceleration} = \lim_{t_2 \to t_1} \frac{v_2 - v_1}{t_2 - t_1} = \tfrac{2}{3},$$

that the marble moved with uniform acceleration.

To carry out the work graphically, the graphs would be drawn as follows :

First draw the space-time graph, and draw tangents at each second, as well as possible, by laying a straight edge along the

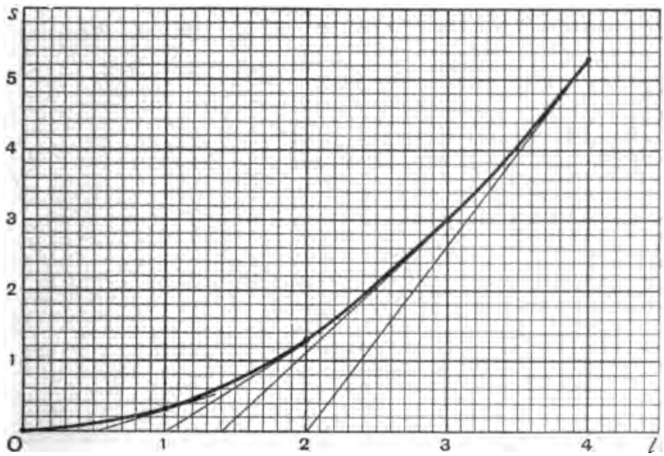


FIG. 5.

curve. Calculate the gradient of each tangent. In the figure these gradients are

$$0{\cdot}6,\ 1{\cdot}3,\ 1{\cdot}87,\ 2{\cdot}65.$$

These therefore represent the velocities at each second.

Next, draw a velocity-time graph to represent these velocities, and it is found that the points lie nearly on a straight line whose gradient is 0·65, shewing that the acceleration is nearly constant and equal to 0·65 approximately.

The case when the distance s is expressed in the form $at^2 + bt + c$, where a, b, c are constants, is very important, from its frequent occurrence in nature. Let us find the velocity and acceleration in this case.

$$s_1 = at_1^2 + bt_1 + c,$$
$$s_2 = at_2^2 + bt_2 + c,$$

$$s_2 - s_1 = a(t_2^2 - t_1^2) + b(t_2 - t_1),$$
$$(s_2 - s_1)/(t_2 - t_1) = a(t_2 + t_1) + b,$$
$$v_1 = \operatorname*{Lim}_{t_2 \to t_1} \{a(t_2 + t_1) + b\} = 2at_1 + b,$$

hence also $v_2 = 2at_2 + b,$

$$v_2 - v_1 = 2a(t_2 - t_1),$$
$$\frac{v_2 - v_1}{t_2 - t_1} = 2a;$$

$\therefore$ the acceleration $f = 2a =$ constant.

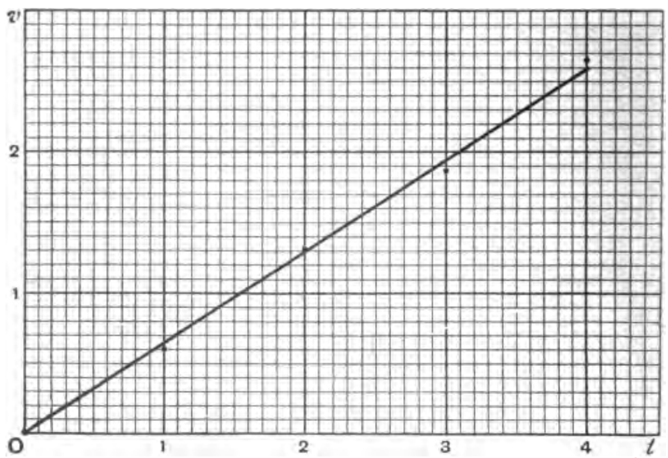


Fig. 6.

Hence, if in any case the distance from a fixed point is given in terms of the time by the formula

$$s = at^2 + bt + c,$$

the acceleration is constant, and $= 2a$, and the velocity is $2at + b$, or

$a =$ half the constant acceleration,

$b =$ the velocity when $t = 0$, usually called the initial velocity,

$c =$ the value of s when $t = 0$, or the distance from the fixed point at the instant from which the times are measured.

If $c=0$, the particle is at a fixed point at the instant from which the times are measured, in other words, is at the origin of distance at the origin of time.

Example. If $s=at^3$ prove that $v=3at^2$, $f=6at$.

10. The Inverse Problem.

In the preceding sections we have shewn how the velocity is to be deduced from a knowledge of the distance travelled,

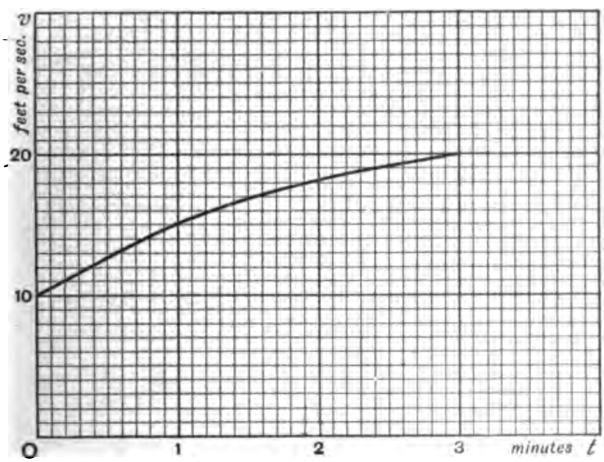


FIG. 7.

and how again the acceleration is to be deduced from the velocity. We now come to the more difficult problems of deducing the change of velocity from a knowledge of the acceleration, and the distance travelled from a knowledge of the velocity. This is the inverse problem. Though it is often impossible to get a complete solution expressed in elementary form, we can always get a graphical representation which will give an approximate solution.

To explain the principles to be followed, we will suppose a train travelling along for three minutes with a varying velocity. The velocity being supposed known, we can draw a velocity-time graph.

Suppose the velocities at intervals of a minute are given by the table:

t mins.	v (feet per sec.).
0	10·0
1	15·0
2	18·2
3	20·0

Now to get an approximation to the distance travelled, we might calculate the distance that would have been travelled if the velocity remained 10 ft./sec. during the first minute, and then suddenly changed to 15·0 ft./sec. and remained at that for the next minute, at the end of which it again suddenly changed to 18·2, remaining at that for the third minute. The distance travelled under these circumstances would be

$$60 \times (10 + 15 + 18{\cdot}2) = 60 \times 43{\cdot}2 = 2592 \text{ ft.}$$

As another approximation, we might calculate the distance that would have been travelled if the velocity was 15 ft./sec. for the whole of the first minute, and then suddenly changed to 18·2, remaining at that for the second minute, and then changed again to 20, at which it remained for the third minute. According to this calculation, the distance travelled would be

$$60 \times (15 + 18{\cdot}2 + 20) = 60 \times 53{\cdot}2 = 3192 \text{ ft.}$$

Naturally neither of these is the true distance travelled by the train, but the true distance might be expected to be between the two. If we take the mean of the two 2892 ft., we would expect to be within 300 ft. of the correct value.

We would get a closer approximation if, instead of taking the velocities at intervals of a minute, we took them at shorter intervals, say 20 secs. We read off from the graph the following :

t	v	t	v
0	10	1 min. 40 sec.	17·2
20 sec.	11·6	2 „	18·2
40 „	13·3	2 „ 20 „	18·9
1 min.	15·0	2 „ 40 „	19·6
1 min. 20 sec.	16·2	3 „	20·0

If we made an approximate calculation as before, assuming the velocity to remain constant during each 20 secs., and equal to the velocity at the beginning of that interval, and to suddenly change at the end of each interval as before, we should find the distance travelled to be

$$20 \times (10 + 11{\cdot}6 + 13{\cdot}3 + 15{\cdot}0 + 16{\cdot}2 + 17{\cdot}2 + 18{\cdot}2 + 18{\cdot}9 + 19{\cdot}6)$$
$$= 20 \times 140 = 2800 \text{ ft.}$$

On the other hand, if we make the similar supposition, with the difference, that the velocity during any 20 secs. is to be the same as that actually possessed by the train at the end of that 20 secs., the distance will be found to be 3000 ft. If we again take the mean of these, we get 2900 ft. as a closer approximation, and do not expect to be more than 100 ft. wrong. By taking the velocity at the end of 5 secs. intervals or 1 sec., we get still closer approximations. The last should give the distance within 5 ft. of the correct value. If we take still shorter intervals of fractions of a second, we will get still closer approximations.

Now let us summarize what we have done up to the present. We have a body moving with a continually changing velocity. We suppose that we know what the velocity is at each instant. We make an approximate calculation of the distance travelled by supposing that the velocity can be treated as constant during certain intervals, and then suddenly changing at the ends of the intervals, instead of changing continuously. In doing this, we try to keep the velocities in the supposed case as near those in the actual case as possible, and the greater the number of intervals into which we divide the whole time, the more closely do the velocities in the true and supposed cases approximate to one another.

We consequently assume that the larger we make the number of intervals (and consequently the smaller the intervals themselves), the smaller will be the error made in the calculation of the distance travelled.

Now let us see what this corresponds to on a velocity-time graph.

Suppose PQRS represents the graph drawn to scale. OP is the initial velocity 10 ft./sec. The distance described in one

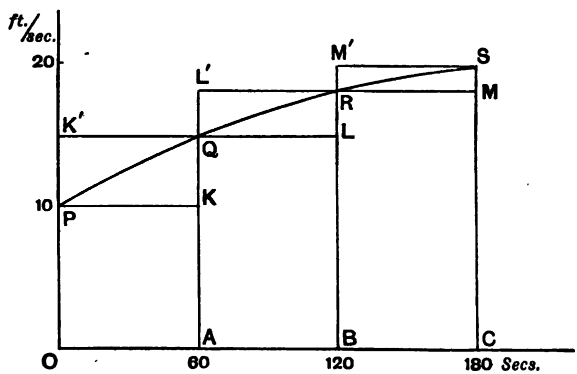


FIG. 8.

minute with this velocity is 10×60, or is represented by the area OPKA. Similarly, the distance described in the next minute with velocity 15 ft./sec. is represented by the area AQLB, and the whole area OPKQLRMCO would represent the 2592 ft. described on the first supposition. The second calculation of 3192 ft. would be represented by the area OK′QL′RM′SCO, and the mean of these is represented by the area bounded by the straight lines PQ, QR, RS, SC, CO, OP.

In the same way, when the intervals were 20 secs. instead of a minute, the two calculations are represented by the areas $OPK_1P_2K_2P_3 \ldots SCO$ and $OK_1'P_2K_2' \ldots SCO$, and the mean of these by the rectilinear area, whose vertices are $OPP_2P_3 \ldots SCO$.

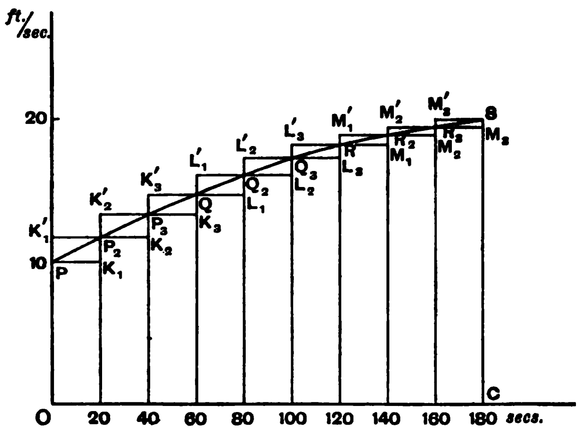


FIG. 9.

Now, evidently this area differs but little from the area bounded by the curve PQRS and the straight lines SC, CO, OP, and the larger the number of intervals or the shorter the chords PP_2, P_2P_3, etc., the more closely will the areas bounded by the chords, and by the curve, approximate to one another.

Hence we deduce that, in any case, when the velocity is represented by a velocity-time graph, the space described in any interval is represented by the area bounded by the graph, the axis of times, and the ordinates at the beginning and end of the interval. It evidently does not matter whether the beginning of the interval is denoted by the time $t=0$ or not. Thus in Fig. 10, if PQ is the velocity-time graph, PABQ represents the space described in the interval of time represented by AB.

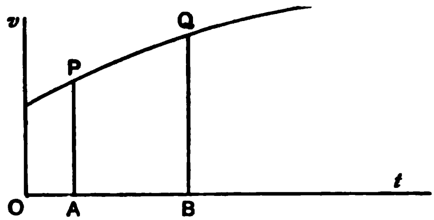


Fig. 10.

In speaking of an area as representing a length, this must be understood in the following way. If an inch along the axis of time represents one minute, and an inch along the axis of v represents 20 ft. per sec., then a square inch represents $60 \times 20 = 1200$ ft., or a square whose side is a tenth of an inch will represent 12 ft., and so on in other cases.

Example 1. If $v=6t$ where v=velocity in ft./sec. and t=time in seconds, draw the v-t graph, and find from the graph the space described from rest in 4 secs.

2. If $v=3t^2$, v and t being in the same units as in question 1, find graphically the distance described in 4 secs. from rest. Work this also by approximate calculations, taking first the velocities at each second, and secondly, the velocities at the end of each half second.

3. If $v=10\sin\frac{\pi t}{30}$, the angle being in circular measure, find the velocity at the end of each 5 secs. up to 30, and deduce approximately the distance described in 30 secs.

Draw the velocity-time graph and give the graphical representation of the calculations made.

Note.—Ex. 3 will shew that where the velocity is sometimes increasing, and sometimes diminishing, the true distance does not necessarily lie between the two distances calculated in the manner explained in this article. Explain graphically why this is so, and shew that it must lie between the two if the velocity is always increasing, or if it is always diminishing.

11. Calculation of Change of Velocity from Acceleration.

As acceleration is derived from velocity in the same way as velocity from distance, so also the velocity generated in any interval can be deduced from the acceleration-time graph, in the same way that the distance described is deduced from the velocity-time graph.

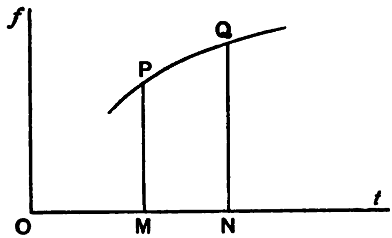


FIG. 11.

Thus, if PQ is the acceleration-time graph, PMNQ represents the change of velocity between the instants represented by M and N.

Example. If f is constant and $=4$, and times are reckoned from the instant when $v=0$, draw the acceleration-time graph, and deduce the velocity at any instant, then draw the velocity-time graph, and deduce the distance described in 6 secs.

12. Mathematical Expressions for the Above Results.

Corresponding to the above graphical representations, we may obtain symbolical expressions for the same results in the following way. The velocity at time t being v, the average velocity during a short interval Δt, beginning at t, is nearly v, say $v+\epsilon$, where ϵ is small compared with v, and vanishes when Δt is indefinitely small. With the former notation,

if Δs is the distance described in the short time Δt,

$$\frac{\Delta s}{\Delta t}=v+\epsilon;$$

$$\therefore \Delta s=(v+\epsilon)\,\Delta t,$$

and the total distance described

$$= \Sigma\Delta s = \Sigma(v+\epsilon)\,\Delta t$$
$$= \Sigma v\,\Delta t + \Sigma\epsilon\,\Delta t,$$

where the summation is intended to take in all intervals from t_1 to t_2.

If now we make these intervals indefinitely short, since ϵ ultimately vanishes compared with v, therefore also $\Sigma\epsilon\,\Delta t$ will ultimately vanish compared with $\Sigma v\,\Delta t$, and in the limit when Δt is infinitely small, may be neglected, and the expression for the distance becomes

$$\operatorname*{Lim}_{\Delta t\to 0} \Sigma v\,\Delta t.$$

In the same way, if f is the acceleration at time t, the change of velocity in the interval $t_2 - t_1$ is

$$\operatorname*{Lim}_{\Delta t\to 0} \Sigma f\,\Delta t,$$

the summation applying to a number of intervals from t_1 to t_2.

13. *Note on Calculus Notation.*

The student who is acquainted with the elements of the calculus will remember that the expression $\operatorname*{Lim}_{t'\to t}\frac{s'-s}{t'-t}$ is the differential coefficient of s with respect to t, and is usually written $\frac{ds}{dt}$; hence $v=\frac{ds}{dt}$, also $f=\frac{dv}{dt}$.

Also the expression $\operatorname{Lim}\Sigma v\,\Delta t$ is the integral of v with respect to t, and is usually written $\int_{t_1}^{t_2} v\,dt$. The integral also represents the area of the graph.

We have therefore

$$s_2 - s_1 = \int_{t_1}^{t_2} v\,dt,$$

also
$$v_2 - v_1 = \int_{t_1}^{t_2} f\,dt.$$

CHAPTER II.

MOTION IN A STRAIGHT LINE; KINEMATICS.

14. Uniform Acceleration.

The case of uniformly accelerated motion in a straight line is of such fundamental importance, that we shall give more than one complete proof of the formulae obtained, especially as we shall thus explain further the methods of the preceding chapter.

The following notation will be used throughout:

u = the initial velocity, or velocity at time zero,
v = the final velocity, or velocity at time t,
f = the acceleration (which is always the same),
s = the distance described in the t secs.

The principal formulae to be proved are:

$$v = u + ft, \quad \text{.......................}(1)$$
$$s = ut + \tfrac{1}{2}ft^2, \quad \text{.......................}(2)$$
$$v^2 = u^2 + 2fs. \quad \text{.......................}(3)$$

Equation (1) follows at once from the definitions, for the average acceleration $= \dfrac{v-u}{t}$, and in this case must equal f, the average acceleration being equal to the acceleration at any instant, since it is always the same.

$$\therefore \frac{v-u}{t} = f;$$
$$\therefore v - u = ft,$$
$$v = u + ft.$$

To prove equation (2), we will first follow the method from which the graphical method is derived. We will calculate the distance that would be described if the interval of time were divided into n equal parts, each of length $\frac{t}{n}$, and the velocity instead of continuously changing, remained constant during each of these short intervals, and changed suddenly at the end of each interval.

If the velocity during the short interval is taken as that which the point actually has at the beginning of the interval, we have the velocities as follows :

At the beginning of the 1st interval, the velocity is u,

,,	,,	,,	2nd	,,	,,	$u+f\frac{t}{n}$,
,,	,,	,,	3rd	,,	,,	$u+f\frac{2t}{n}$, etc.,
,,	,,	,,	rth	,,	,,	$u+\frac{f(r-1)t}{n}$.

There are n of these intervals, and the total distance described on this supposition is

$$\frac{t}{n}\left\{u+\left(u+f\frac{t}{n}\right)+\left(u+f\frac{2t}{n}\right)+\ldots+\left(u+f\frac{\overline{n-1}t}{n}\right)\right\}$$

$$=\frac{t}{n}\left\{nu+\frac{ft}{n}(1+2+\ldots+\overline{n-1})\right\}$$

$$=ut+f\frac{t^2}{n^2}\cdot\frac{n(n-1)}{2}=ut+\tfrac{1}{2}\,\frac{n-1}{n}ft^2.$$

Using the same method, but taking the velocity in any of the short intervals as that actually possessed by the point at the end of the interval, we have

Velocity at end of 1st interval $=u+f\frac{t}{n}$,

,, ,, ,, 2nd ,, $=u+f\frac{2t}{n}$,

Velocity at end of 3rd interval $= u + f\dfrac{3t}{n}$, etc.,

" " " rth " $= u + f\dfrac{rt}{n}$,

and the total distance described

$$= \frac{t}{n}\left\{\left(u + \frac{ft}{n}\right) + \left(u + f\frac{2t}{n}\right) + \dots + \left(u + f\frac{nt}{n}\right)\right\}$$

$$= \frac{t}{n}\left\{nu + \frac{ft}{n}\cdot(1 + 2 + \dots + n)\right\}$$

$$= ut + f\frac{t^2}{n^2}\cdot\frac{n(n+1)}{2} = ut + \tfrac{1}{2}\,\frac{n+1}{n}ft^2.$$

When the number of intervals is made indefinitely large, we get the case where the velocity is uniformly accelerated, and each of the above expressions becomes

$$u + \tfrac{1}{2}ft^2,$$

since $$\lim_{n\to\infty}\frac{n-1}{n} = \text{Lim}\left(1 - \frac{1}{n}\right) = 1,$$

and $$\lim_{n\to\infty}\frac{n+1}{n} = \text{Lim}\left(1 + \frac{1}{n}\right) = 1.$$

Having proved the equations,

$$v = u + ft, \quad\text{.......................................}(1)$$

and $$s = ut + \tfrac{1}{2}ft^2, \quad\text{.................................}(2)$$

equation (3), $v^2 = u^2 + 2fs$, is not an independent equation, but may be deduced from (1) and (2). It is only necessary to eliminate t between the equations (1) and (2) thus:

from (1), $$t = \frac{v-u}{f},$$

substitute in (2), $$s = u\,\frac{v-u}{f} + \tfrac{1}{2}f\left(\frac{v-u}{f}\right)^2$$

$$= \frac{2(uv - u^2) + v^2 - 2uv + u^2}{2f} = \frac{v^2 - u^2}{2f};$$

$$\therefore\ v^2 - u^2 = 2fs. \quad\text{.................................}(3)$$

It will be seen from the above that all the difficulty lies in the proof of (2). It will be a good illustration of the graphical method to give the proof by it.

The acceleration being constant, the gradient of the velocity-time graph is the same everywhere, and therefore the graph is a straight line.

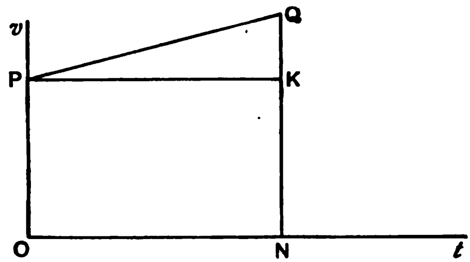


FIG. 12.

In the figure PQ is the graph,

$$OP = u,$$
$$QN = v = u + ft,$$
$$QK = v - u = ft,$$

and the area

$$OPQN = OPKN + PKQ,$$
$$= ut + \tfrac{1}{2}t \, . \, ft,$$
$$= ut + \tfrac{1}{2}ft^2;$$

hence the distance described is $ut + \frac{1}{2}ft^2$.

We will give a third proof which, unlike the others, is not an example of general methods, but depends on special results following from the uniform acceleration.

By (1), the velocity t' seconds after the start is $u + ft'$. If we reckon backwards from the end, the velocity t' seconds before the end will be $v - ft'$ (going backwards from the end, the velocity diminishes by f each second).

The mean of these two is

$$\tfrac{1}{2}(u + ft' + v - ft') = \tfrac{1}{2}(u + v),$$

therefore if the whole time is divided into any number of equal intervals, on taking the instants in pairs we can see that the average velocity of the whole time is

$$\tfrac{1}{2}(u+v).$$

(If there is an unpaired instant at the middle, the velocity then is $\frac{1}{2}(u+v)$. See Art. 15.)

Hence, since average velocity $= \dfrac{\text{total distance}}{\text{total time}}$,

$$\frac{u+v}{2} = \frac{s}{t},$$

$$s = \tfrac{1}{2}(u+v)t = \tfrac{1}{2}(u+u+ft)t$$
$$= ut + \tfrac{1}{2}ft^2.$$

We can get from the above an easier method of deducing (3) for we have

$$s = \frac{u+v}{2}t,$$

$$f = \frac{v-u}{t}.$$

Multiplying these equations

$$fs = \frac{v^2-u^2}{2};$$

$$\therefore\ v^2 - u^2 = 2fs.$$

15. Other Results.

Notice that it is only *because* the acceleration is *uniform* that the average velocity is the mean of the initial and final velocities. Also the average velocity is equal to the velocity at the middle of the interval, for

$$\text{the average velocity} = \frac{u+v}{2} = \frac{u+u+ft}{2}$$
$$= u + \tfrac{1}{2}ft,$$

and the velocity at the middle of the interval

$$= u + f\frac{t}{2} = u + \tfrac{1}{2}ft.$$

The distance described in the n^{th} second can be determined in one or two ways.

First, the distance described in the n^{th} second

$$= \text{distance described in } n \text{ seconds} - \text{distance described in } n-1 \text{ seconds}$$

$$= un + \tfrac{1}{2}fn^2 - \{u(n-1) + \tfrac{1}{2}f(n-1)^2\}$$

$$= u + \tfrac{1}{2}f(2n-1).$$

Secondly, the velocity at the beginning of the n^{th} second is

$$u + f(n-1),$$

and at the end $u + fn$.

Therefore the average velocity during this second is

$$= \tfrac{1}{2}\{u + f(n-1) + u + fn\}$$

$$= \tfrac{1}{2}\{2u + f(2n-1)\}$$

$$= u + \tfrac{1}{2}f(2n-1).$$

Therefore the distance described in this second is

$$= u + \tfrac{1}{2}f(2n-1).$$

If the body starts from rest, so that $u=0$, the equations take the simpler form,

$$v = ft, \quad \text{..............................(1}a\text{)}$$

$$s = \tfrac{1}{2}ft^2, \quad \text{..............................(2}a\text{)}$$

$$v^2 = 2fs. \quad \text{..............................(3}a\text{)}$$

It must be remembered that distances from a point may be positive or negative, and velocities and accelerations may be positive or negative (see Art. 8). Also, that s always means the distance from a definite point at a given instant. Thus if a body moves from A to B and then back to a point P

Fig. 13.

between A and B, at the time the body is at P, s represents the length AP and not the numerical sum of the distances AB, BP travelled in the two directions. Of course, AP is the algebraic sum of AB, BP, treating BP as negative.

The following examples will illustrate these points and the use of the equations.

In all examples care should be taken to use a consistent set of units. Thus it will not do to leave a velocity in miles/hr., a distance in yds., and a time in seconds. It is best usually to express a velocity in feet per second, and then a distance must be expressed in feet, and time in seconds.

Example 1. A train has its velocity increased from 20 to 30 miles/hr. in going 1500 yds. What is the acceleration, supposed uniform, and how long does it take to travel the 1500 yds.?

Here
$$u = 20 \text{ m./hr.} = \frac{88}{3} \text{ ft./sec.},$$
$$v = 30 \text{ m./hr.} = 44 \text{ ft./sec.},$$
$$s = 1500 \text{ yds.} = 4500 \text{ ft.}$$

Using equation (3),
$$2f \times 4500 = 44^2 - \left(\frac{88}{3}\right)^2 = 44^2\left\{1 - \left(\frac{2}{3}\right)^2\right\}$$
$$= 44 \times 44 \times \frac{5}{9};$$
$$\therefore f = \frac{44 \times 44 \times 5}{2 \times 4500 \times 9} = \frac{968}{8100} = 0{\cdot}1195 \text{ ft./sec}^2.;$$

also the average velocity $= \dfrac{u+v}{2}$
$$= 25 \text{ m./hr.}$$
$$= \frac{25}{30} \times 44 \text{ ft./sec.} = \frac{110}{3} \text{ ft./sec.};$$
$$\therefore \text{ time taken} = \frac{\text{distance}}{\text{average velocity}}$$
$$= 4500 \times \frac{3}{110} = 122{\cdot}7 \text{ secs.}$$

2. A particle starting with velocity 15 ft./sec. and moving with uniform acceleration has a velocity of 9 ft./sec. at the end of 3 secs. Find how far it goes before it comes to rest, and the times and velocities when at 50 ft. from the starting point.

Here the velocity is diminishing and the acceleration consequently negative; in fact,

$$u=15 \text{ ft./sec.}$$
$$v=9 \text{ ft./sec.}$$
$$t=3 \text{ secs.}$$
$$f=\frac{9-15}{3}=-2 \text{ ft./sec}^2.$$

The distance travelled from the start until it comes to rest is given by equation (3), when the final velocity is zero;

$$\therefore\ 0=15^2+2(-2)s,$$
$$s=56\tfrac{1}{4} \text{ ft.}$$

The times at which it is 50 ft. from the starting point are given by equation (2):

$$50=15t+\tfrac{1}{2}(-2)t^2,$$
$$t^2-15t+50=0,$$
$$t=5 \text{ or } 10.$$

The velocity there is given by equation (1):

$$v=15-2t$$
$$=5 \text{ or } -5;$$

or otherwise by equation (3):

$$v^2=15^2+2(-2)\times 50$$
$$=25,$$
$$v=5 \text{ or } -5.$$

From the starting point the body travels for $7\frac{1}{2}$ secs. with continually decreasing velocity, coming to rest then and moving back with continually increasing negative velocity.

3. If a train acquires a velocity of 30 miles/hr. in 1 minute from rest, find the acceleration, supposed uniform, and the distance travelled in the minute.

4. A train moving at 10 miles an hour at one instant has a constant acceleration of 2 ft./sec². What distance will it have travelled when the velocity is 20 miles/hr., and how long will it take to do it?

5. If a body travels 30 ft. in the first second it is observed, and 21 in the fourth, what is the acceleration, supposed uniform, and what is the distance travelled in the eighth second?

6. A body moving with uniform acceleration in a straight line has a velocity 10 ft./sec. at a distance of 6 ft. from a point O, and 12 ft./sec. at 17 ft. from O. Find the acceleration and the velocity at O.

7. A particle starts with velocity 20 cms./sec. and travels 400 cms. in 30 secs. Find the acceleration, supposed uniform, and shew that it will come to rest in 45 secs. Find also the times at which it is 200 cms. from the starting point.

8. A body travels 20 ft. in 4 secs., and the next 20 ft. in the next 6 secs. If the acceleration is uniform, find this acceleration, and the further distance and time it will travel before coming to rest.

9. A particle moving with uniform acceleration has a velocity 10 miles/hr. at A, and 30 miles/hr. at B. Find the velocity midway between A and B.

10. A body moving with uniform acceleration has a velocity u at A and v at B. Find the velocity midway between A and B, and shew that it is greater than the mean of the velocities at A and B.

11. At three points, A, B, C, in a straight line such that AB = BC, the velocity of a particle is found to be 8·5, 6·5, 3·5 ft./sec. Is this consistent with uniform acceleration?

If AB = 6 ft., how much further will it go before coming to rest?

16. Falling Bodies.

By a measurement of the distances travelled by a body falling from rest, in different times, it is found that any body let fall in a vacuum describes distances expressed by (2a), Art. 15, so that such a body is falling with uniform acceleration. Further, the acceleration of all bodies let fall in this way is the same at the same point of the earth's surface, this acceleration being about 32·2 ft./sec^2., or in the metric system 981 cm./sec^2. (980 is more nearly the actual value in any part of Australia). In the numerical examples we will generally put $g = 32$ ft./sec^2. It seems certain that if we could carry experiments further we would always find that in a vacuum a body projected vertically upwards or downwards would move with this acceleration. This acceleration is called the acceleration due to gravity, and is always downwards, so that when a body is moving upwards the velocity is diminishing, when it is moving downwards the velocity is increasing.

The equations (1), (2), (3) apply to all such cases, provide care is taken to consistently regard all quantities measured in one direction as positive, and all quantities in the other as negative, whether they are distances, velocities or accelerations.

When, instead of moving in a vacuum, a body is thrown up in the air, it is no longer true that the acceleration is uniform, for, as will be seen later, the effect of the air is to cause an additional acceleration in the direction opposite to the velocity, which is greater at greater speeds than at smaller. For low velocities, not greater, for example, than when a cricket ball is thrown into the air, the equations (1), (2), (3) hold approximately for a heavy body, such as the cricket ball or a stone, but they quite fail to represent the facts when a rifle bullet is fired vertically. So also they fail for a body, such as a sheet of paper, which presents a large surface for its weight to the air.

17. Body Projected Vertically Upwards.

We will now obtain some results for the case of a body projected vertically upwards, as the work will illustrate the use of the positive and negative signs.

Suppose the initial velocity to be upwards. We will take the upward direction as positive and the acceleration will be negative; we will denote it by $-g$, g thus being the acceleration due to gravity. It may be noted that when the velocity of a body is decreasing, it is said to be retarded. In this case, in going upwards, it may be said to have a retardation of g.

Our equations are therefore

$$v = u - gt, \quad \text{.........................} (1)$$

$$s = ut - \tfrac{1}{2} gt^2, \quad \text{.........................} (2)$$

$$v^2 = u^2 - 2gs, \quad \text{.........................} (3)$$

the velocity diminishes at first until $v = 0$, and then

$$u - gt = 0, \text{ or } t = u/g.$$

When $t > \frac{u}{g}$, v is negative.

Hence the body moves upward with continually decreasing velocity for u/g secs., and at the end of this time the velocity vanishes, or the body is instantaneously at rest, and afterwards moves downwards with a velocity increasing numerically, but negative.

The s in the equations is the distance from the starting point, *not* the total distance upward and downward added together; hence we find when the body strikes the ground again, by putting $s=0$.

This gives $$ut-\tfrac{1}{2}gt^2=0;$$
$$\therefore\ t=0, \text{ or } 2u/g.$$

$t=0$ refers to the start, and the other value $2u/g$ shews that the body takes $2u/g$ secs. to return to the starting point. This is called the time of flight. As we have seen that it takes u/g secs. to reach the highest point, we now see that it takes the same time in the descent as in the ascent.

The maximum height reached can be obtained from the fact that it is at the highest point at the end of u/g secs., and is therefore at that time at height

$$ut-\tfrac{1}{2}gt^2=u\frac{u}{g}-\tfrac{1}{2}g\frac{u^2}{g^2}=u^2/2g.$$

But the maximum height can be more quickly obtained from equation (3), for when the body is at the maximum height, its velocity is zero; putting $v=0$,

we get $$u^2-2gs=0,$$
$$s=u^2/2g.$$

To get the velocity at any particular height we can use the equation $$v^2=u^2-2gs,$$

and we see that for a given value of s we have two equal and opposite values of v, one representing the velocity in the upward motion, and the other in the downward. Thus we see that the whole motion in the downward direction is the exact reverse of that in the upward.

To find the time taken to reach a given height s we have the equation (2)

$$s = ut - \tfrac{1}{2}gt^2,$$

or

$$t^2 - \frac{2u}{g}t + \frac{2s}{g} = 0,$$

a quadratic equation for t, giving in general two solutions, one of which gives the time taken to reach the point in the upward motion, and the other the whole time until it reaches the same point again in the descent.

Example 1. Find the condition that the roots of this equation may be real, and interpret the condition.

2. If a stone is thrown vertically upwards with velocity 80 ft. per sec., find the height to which it will rise, and the whole time of flight. (Take $g = 32$.)

Draw the following graphs for the motion in this example (1) v, t; (2) s, t; (3) v, s.

EXAMPLES.

1. A stone is thrown vertically upwards with velocity 64 ft./sec. from the top of a tower 128 ft. high. In what time will it reach the ground?

If we measure in the upward direction so that the initial velocity is positive, we have $s = -128$ when the body strikes the ground, hence we have

$$64t - 16t^2 = -128,$$

$$t^2 - 4t - 8 = 0,$$

$$t = 2 \pm 2\sqrt{3}$$

$$= 5{\cdot}46 \text{ secs.};$$

the negative solution not being applicable to the question.

2. A stone let drop from the top of a tower reaches the ground in $2\tfrac{1}{2}$ secs.; how high is the tower, and with what velocity does the stone strike the ground?

3. A body projected vertically upwards reaches a height of 50 metres. Find the initial velocity and time of flight.

4. With what velocity does a child throw a ball vertically in order to just reach a ceiling 12 feet above the point where the ball leaves the hands?

5. When a balloon is 200 metres above the ground, and rising vertically with a velocity of 6 m./sec., a stone is released from it. How long will it take to reach the ground?

If the balloon has an acceleration, does it affect the result?

6. A body thrown vertically upwards from the top of a tower reaches the ground in 5 secs. If thrown vertically downwards with the same velocity, it reaches it in 3 secs. Find the initial velocity and the height of the tower.

7. A body projected vertically upwards from the top of a tower reaches the ground in t_1 secs. If projected vertically downwards with the same velocity, it reaches it in t_2 secs. Prove that if simply let drop it would reach the ground in $\sqrt{t_1 t_2}$ secs.

8. A body is projected vertically upwards with velocity 80 ft./sec. Find the times at which it is at a height of 40 ft. Find also the time to travel from the height of 40 ft. to 60 ft.

9. If the maximum height reached by a projectile moving vertically is h, find the interval of time between the two instants at which it is at a height kh, k being a proper fraction.

10. If t is the time taken by a projectile to reach a height h, and t' the time from this point to the ground again, prove that $h = \frac{1}{2}gtt'$, and the maximum height is $g(t+t')^2/8$.

11. A person drops a stone into a well, and hears the splash at the end of 4 secs. What is the depth of the well, if sound travels at 1100 ft./sec.?

18. Relative Velocity and Acceleration of Two Bodies.

We have seen that all motion is relative. If two bodies are moving in the same straight line, the relative velocity determines the rate at which they are separating or approaching one another.

Suppose that two cyclists are travelling along the same road, in the same direction, one at 10 miles/hr. and the other at 8 miles/hr. The first is gaining at the rate of 2 miles/hr. from the second. This is the relative velocity of the first with respect to the second.

If the second were moving in the opposite direction to the first, they would be receding from one another (or approaching one another) at 18 miles/hr. The relative velocity would be

18 miles/hr. We can express the facts thus: If A and B are moving in the same straight line through O, and if at any instant

	A has a velocity		u	relative to	O,
and	B ,,	,,	u'	,,	O
then	A ,,	,,	$u-u'$	,,	B,
and	B ,,	,,	$u'-u$	,,	A.

In the same way, if

	A has an acceleration		f	relative to	O,
and	B ,,	,,	f'	,,	O,
then	A ,,	,,	$f-f'$	,,	B,
and	B ,,	,,	$f'-f$	,,	A.

With these velocities and accelerations the distance travelled by A in t secs. is $ut+\frac{1}{2}ft^2$ relative to O,
and by B $u't+\frac{1}{2}f't^2$,, O;
therefore the distance travelled by A relatively to B in t secs. is

$$ut+\tfrac{1}{2}ft^2-(u't+\tfrac{1}{2}f't^2)=(u-u')t+\tfrac{1}{2}(f-f')t^2,$$

or they separate from one another (or approach one another) in exactly the same way as if B was at rest and A was moving with velocity $u-u'$ at a given instant, and with acceleration $f-f'$.

In the case of two bodies thrown vertically upwards, in the same straight line, their accelerations are both g and the relative acceleration is zero. Consequently the relative velocity is constant. If they are projected at the same instant with velocities u and u', then in t secs. they will be $(u-u')t$ ft. apart, exactly the same as if neither was accelerated.

Example 1. A body is projected vertically with velocity 80 ft./sec., and 2 secs. after, a second body is projected vertically upwards from the same point, with velocity 64 ft./sec. Find when and where they will meet.

First Solution. At the end of t secs. from the instant the first starts, it is at height $80t-\frac{1}{2}gt^2=80t-16t^2$.

The second has then been moving for $t-2$ secs., and its height is

$$64(t-2)-\tfrac{1}{2}g(t-2)^2.$$

If they are together at this instant,

$$80t - 16t^2 = 64(t-2) - 16(t-2)^2$$
$$= 64t - 128 - 16t^2 + 64t - 64;$$
$$\therefore\ 48t = 192,$$
$$t = 4.$$

They meet 4 secs. after the first starts at a height of

$$80 \times 4 - 16 \times 16 = 64 \text{ ft.}$$

Second Solution. By relative velocity and acceleration. When the second starts the first is at a height of

$$80 \times 2 - 16 \times 4 = 96 \text{ ft.},$$

and has a velocity of $80 - 32 \times 2 = 16$ ft./sec.

The second has a velocity relative to the first of

$$64 - 16 = 48 \text{ ft./sec.},$$

and the relative velocity remains constant.

The second catches up the 96 ft. in $\frac{96}{48} = 2$ secs.

The second has been moving for 2 secs., and the first for 4 secs. The position is found as before.

2. A body is projected vertically upwards with velocity 72 ft./sec. and 2 secs.; afterwards another is projected upwards from the same point with the same velocity. Find when and where they will meet.

3. A particle is dropped from the top of a tower 144 ft. high, and at the same moment another particle is projected upwards from the bottom. With what velocity is the latter projected, if they meet $\frac{1}{4}$ of the way down, and what is the velocity of each when they meet?

4. Two particles, started as in the last example from the top and bottom of a tower h feet high, meet when the upper one has described $\frac{1}{n}$th of the distance; shew that the velocities when they meet are in the ratio $2 : n-2$, and that the initial velocity of the lower is $\sqrt{\frac{ngh}{2}}$.

5. Particles P and Q start from rest at points A and B 200 cms. apart at the same instant and in the same direction (A to B); if the accelerations are 12, 10 cm./sec^2. respectively, find when and where P will overtake Q.

6. Particles P and Q start from two points A and B 200 cms. apart in opposite directions to meet one another. P has an initial velocity of 10 cm./sec. and acceleration 4 cm./sec^2. Q starts 3 seconds after P, and has initial velocity 8 cm./sec. and acceleration 5 cm./sec^2. When and where will they meet?

7. A particle is let fall from rest at A. When it has been moving for $\frac{1}{2}$ a second, a second is let fall from B, 16 ft. below A. Shew that the first will overtake the second at the end of another $\frac{3}{4}$ sec.

8. A particle is let fall from rest at A, and before it reaches a point B below it, another is let fall from B. Shew that the first necessarily overtakes the second.

9. A particle P starts from O and moves with constant velocity 10 ft./sec.; at the same instant Q starts from rest at O and moves with constant acceleration $\frac{1}{2}$ ft./sec^2. Find when and where Q overtakes P.

10. A body starts from rest at A and moves with constant acceleration f in a straight line. T secs. afterwards a second body starts from A and moves with uniform velocity u in the same line. Prove that the second overtakes the first if

$$u > 2f\mathrm{T},$$

and shew that in this case the first overtakes the second again.

11. A body moves from A with uniform velocity u in a straight line. T seconds afterwards a second body starts from rest at A and moves with constant acceleration f. Shew that the second necessarily overtakes the first, and that the two are together once only.

12. A body starts from A with initial velocity u, and moves in a straight line with constant acceleration f. T seconds afterwards a second starts with velocity u' and acceleration f' in the same line.

Prove that if $f' > f$ the second necessarily overtakes the first, but that if $f > f'$ it only overtakes it if

$$(u' - u)^2 + ff'\mathrm{T}^2 > 2\mathrm{T}(fu' - f'u).$$

Verify that the result of No. 10 is a special case of this.

19. Variable Acceleration.

An important general result about variable acceleration can be deduced from the preceding results for uniform acceleration.

If the acceleration varies we may still regard it as nearly constant during a very short interval, and hence the equation

$$v^2 - u^2 = 2fs,$$

or

$$s = \frac{v^2 - u^2}{2f},$$

holds approximately for a short distance, becoming more and more nearly accurate the shorter the interval is. With the previous notation this would be expressed as

$$f_1 = \operatorname*{Lim}_{s_2 \to s_1} \frac{\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2}{s_2 - s_1},$$

where v_1 is the velocity at distance s_1 from the origin, etc.

Hence, if we know v in terms of s we can draw a graph of $\frac{1}{2}v^2$, and, as in the preceding work, the acceleration is the gradient of the graph so drawn.

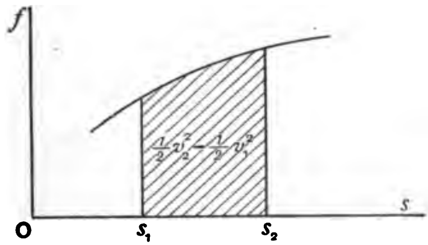


FIG. 14.

Also, if we have the acceleration-distance graph, an area on the graph between two ordinates represents the change in $\frac{1}{2}v^2$.

Example. If $f = -k^2s$, prove that $v^2 - u^2 = -k^2s^2$ where u is the velocity when $s = 0$. Prove also that $v = 0$ where $s = \pm\frac{u}{k}$, and describe the motion.

20. Other Graphical Representations.

Further, since $\quad v_1 = \operatorname*{Lim}_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1},$

$$\frac{1}{v_1} = \operatorname*{Lim}_{s_2 \to s_1} \frac{t_2 - t_1}{s_2 - s_1}.$$

Hence, if a graph is drawn for $\frac{1}{v}$ in terms of s, an area on this graph gives the interval of time required to describe the distance.

A further case is that in which f is given in terms of v.

Since
$$f_1 = \operatorname*{Lim}_{t_2 \to t_1} \frac{v_2 - v_1}{t_2 - t_1},$$
$$\frac{1}{f_1} = \operatorname*{Lim}_{v_2 \to v_1} \frac{t_2 - t_1}{v_2 - v_1}.$$

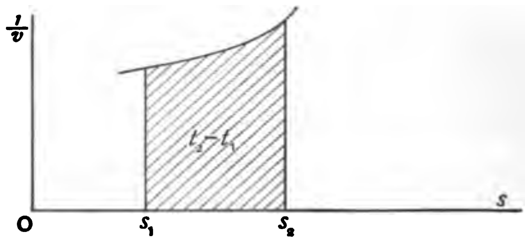


FIG. 15.

Hence, if a graph of $\frac{1}{f}$ is drawn in terms of v, an area of this graph represents the interval of time required to cause a given change in velocity.

21. Hence, summing up the two chapters, we see that we can solve graphically any of the following problems :

If s is given in terms of t, we can find v,

,, v ,, ,, t, ,, ,, f and s,

,, f ,, ,, t, ,, ,, v, and therefore s,

,, v (and therefore v^2) ,, s, ,, ,, f,

,, v (,, ,, $\frac{1}{v}$) ,, s, ,, ,, t,

,, f is given in terms of s, ,, ,, v^2, and therefore v,

,, f ,, ,, v ,, ,, t.

In such a case as the last, we find t for each v, and then can go on to find s in terms of t.

Thus, finally we have the means of finding:

v	and	f,	if	s	is given	in terms	of	t,
s	,,	f,	,,	v	,,	,,		t,
v	,,	s,	,,	f	,,	,,		t,
f	,,	t,	,,	v	,,	,,		s,
v	,,	t,	,,	f	,,	,,		s,
t	,,	s,	,,	f	,,	,,		v,

shewing that any question in which one of the four s, v, f, t is given in terms of another, can be solved completely graphically so as to find the other two.

EXAMPLES.

1. A train passes a station A at 30 miles an hour, and maintains this speed for 7 miles, and then is uniformly retarded, stopping at B, which is 8 miles from A. A second train starts from A at the instant the first passes, and being uniformly accelerated for part of the journey, and uniformly retarded for the rest, stops at B at the same time as the first. Find its greatest speed.

2. Shew graphically or otherwise that if the acceleration of a particle moving in a straight line is always increasing, the average velocity is less than the mean of the initial and final velocities, and greater than the velocity at the middle of the interval.

Give the values of these quantities for the interval from 0 to t when the distance travelled in time t is kt^3.

3. If the distance travelled in t secs. is represented by an expression

$$s = 9t - 6t^2 + t^3,$$

shew that the velocity vanishes when $t=1$ and when $t=3$, and that the acceleration vanishes when $t=2$.

Describe the motion.

4. A particle is moving in a straight line, and its velocity at a distance x from the origin is $k\sqrt{\frac{1}{x}-\frac{1}{a}}$. Find the acceleration, and describe the nature of the motion if the particle is initially at a point $x=\frac{a}{2}$ and is moving away from the origin.

5. Two particles moving in the same straight line are at a certain instant at points A, B, 10 ft. apart and moving in the same direction with velocities 9, 2 ft./sec. in the direction from A to B, and continue to move with constant accelerations 4, 6 ft./sec^2. in the same direction. Prove that A will overtake B and afterwards B will overtake A, and find the times and positions at which the two pass one another.

Shew that they are moving with the same velocity at the instant midway between the instants at which they are together.

6. Two particles moving in the same straight line with constant accelerations f, f' in the positive direction have velocities u, u' at a certain instant, and are then at distances a, a' from the origin. Prove that they cannot pass one another more than twice, and that if they do so twice, the interval T between the two times of passing is given by

$$\tfrac{1}{4}(f-f')^2\mathrm{T}^2=(u-u')^2-2(a-a')(f-f').$$

Hence prove that they cannot pass one another either before or after the given instant if

$$(u-u')^2<2(a-a')(f-f').$$

Interpret the case when

$$(u-u')^2=2(a-a')(f-f').$$

Compare with the numerical results in Question 5.

CHAPTER III.

FORCE AND MASS; MOTION IN A STRAIGHT LINE.

22. So far we have been dealing with motion without consideration of the cause of it. Now we proceed to examine into causes, and we introduce the idea of force. It does not seem possible to give a definition of force, any more than it is possible to give a definition of space or time. We can, however, learn something about the relations of force to other phenomena, and we can recognize the presence of forces by the effects which we find associated with them and which we regard as being produced by them, and we can deduce methods of measuring force, even though we can say nothing about the way in which force acts to produce those effects.

We necessarily get our first ideas of force from our sense of muscular effort. Thus we speak of the force required to lift a weight from the ground. By the exertion of such a force we often see that we alter the motion of a body, as when we push the body along a table. At other times we feel that we are preventing motion, as when we hold a body above the ground; or, again, we imagine we would produce motion in the body, if it were not that some other force opposed it, and prevented the motion, as when we press upon a table, and the table does not move because the floor exerts a force to stop it from moving. Hence we think of the force exerted by us as producing a change in the motion of the body, whether the body was originally at rest or not, or else as tending to produce such a change, and we extend the idea of force to any case where similar results are observed. Thus, in the usual words—

Force is anything that produces or tends to produce change in the state of rest or of motion of a body.

In this statement it is understood that there is change in the motion of a body whenever the motion is not motion in a straight line with uniform velocity. We say, for example, that a falling body is acted on by a force because the velocity continually increases, and even if the body ceases to move, and rests on the ground, we think of it as still acted on by a force in the same way as before, though motion is prevented by a force exerted by the ground pressing up against the body.

23. Measurement of Force.

To obtain accurate ideas about force, we must find some way of measuring force independently of our muscular effort, which can only give rough ideas. One method is to examine, if possible, the weight that the force will support, or will balance on a chemical balance. Another is to examine the extent to which it will elongate a given spiral spring. That these two methods are at any rate approximately consistent with one another, and with the notions derived from muscular effort, may be seen from a number of experiments. Thus two weights that balance on a balance, whether of the same or of different materials, will produce the same elongation in a given spring, and again will produce, as far as we can judge, the same effect on the muscles. It takes, for example, the same muscular effort to raise a 20 lb. lump of iron as it does a 20 lb. block of wood. We find, further, that (conditions of temperature, etc., being unchanged) it will always take the same weight to elongate the spring the same amount. Further, we find that if we take two weights that elongate the spring the same amount separately, the two together would elongate it twice as much, and so on. Hence we may take the elongation of a spiral spring as giving a measure of any force that can be applied to the spring.

Now suppose a little carriage made to run as freely as possible along a horizontal table, and that the carriage can be weighted with different weights. If a spiral spring is now attached to the carriage and pulled along, the force can be determined by the elongation of the spring. It may be measured, for the present, by the number of inches the spring is elongated. We can then determine the relations between the force exerted and the effect produced, and we find the following results :

(1) If we keep the same force applied, that is, see that the spring is always stretched to the same extent, we find that the carriage moves with uniform acceleration.

(2) If we make different experiments with different forces on the same carriage and weights, the acceleration is proportional to the force exerted.

(3) If we keep the force the same in the different experiments, but vary the weight on the carriage, the acceleration is inversely proportional to the total weight of carriage and load.

(4) If we vary the nature of the load, change, for example, iron to wood, but leave the weight the same, no alteration is produced in the acceleration, provided the force remains unchanged also.

We can write these results as follows :

let P = the force exerted,

W = the weight of carriage and load ;

from experiment (1), if a force P acts we get an acceleration f,

„ „ (2) f is proportional to P if W is constant,

„ „ (3) f „ „ „ $\frac{1}{W}$ if P is constant ;

$$\therefore\ f = kP/W,$$

where k is a constant depending only on the units used and by (4), not on the nature of the material used.

Now any body drops to the earth with an acceleration g, shewing that it is acted on by some force. In numberless experiments on falling bodies the only things that remain common to all, as far as we can see, are the presence of the falling body and the earth. We say, therefore, that the earth attracts the body or exerts a force on it which we call the weight of the body. Weight is thus a force, and if we express the weight and the force in the same units, we have for the falling body that the equation

$$f = kP/W,$$

becomes

$$g = kW/W;$$

$$\therefore\ k = g;$$

$$\therefore\ f = gP/W,$$

or

$$P = \frac{W}{g} f. \qquad \text{(i)}$$

This result is the fundamental equation of dynamics, and we suppose it to apply to all cases where force is acting on a body. It is only necessary to remember that in (i) P and W are measured in the same unit of any possible kind, and f, g are also measured in terms of one unit.

The above experiments cannot be carried out very accurately, but the more carefully they are performed the more closely will they be found to hold. In many other experiments we find that the same results hold approximately, and we believe that if we could examine experimentally any case where only one force is acting on a body we should find the same results. In more complicated cases deductions from this equation agree with results found from observation and experiment. Hence we are led to believe it to be true universally.

24. Now we find that the weight of a body varies from one place to another. A body as weighed by a delicate spring balance will weigh slightly more, by about 0·4 per cent., at the poles than at the equator. If we go up a mountain it weighs less, and if we go down a deep mine its weight increases at first (as Airy found), on account of the greater density of the interior of the earth, but further down diminishes towards the centre of the earth. If we could go towards the centre of the earth the weight would continually diminish as shewn by a spring balance, and we could find a place at or near the centre where the weight would be zero. If direct experiments could be accurately carried out, we should always find that for a given body W is always proportional to g, or W/g is constant everywhere for a given body.

It must, however, be noted that if two bodies balance on an ordinary balance at one place, they will balance at any other place, although the weight of each is actually different at the second place from what it was at the first.

When proper units have been selected, the ratio W/g, which we have seen is constant for the same body everywhere, we will call the mass, and denote by M. The mass depends on such things as the volume and constitution of the body, but does not depend on the position in the universe. A given body, if moved about without breakage or other loss, has its mass unchanged however much the weight may change.

Putting $W/g=M$,

equation (i) becomes $P=Mf$, ...(ii)

and in this form we shall usually use the equation.

25. Units.

If the equations $P=Mf$,

and the special case $W=Mg$

are used as representing the laws of motion, the units in which force and mass are measured depend on one another. It is best to take the unit of mass first. In the British systen (or F.P.S. units, ft.-lb.-sec. units) it is the pound ; in th

Metric (C.G.S. units, centimetre-gramme-second units) it is the gramme. Since the mass of a body is proportional to the weight, the number usually used in the British system to denote the weight may also be used to denote the mass in pounds. Thus the mass of a 10-lb. weight is 10 lbs. Similarly in the Metric system, the mass of a 10-gm. weight is 10 gms.

If we first define the unit of mass, the unit of force will be defined by either of the equations

$$P = Mf \quad \text{or} \quad W = Mg,$$

for if $W = 1$, since $g = 32$ ft./sec^2. approximately,

$$M = \tfrac{1}{32} \text{ lb.};$$

or, the unit of force is the weight of a body whose mass is 1/32 lb., or about half an ounce. This force is called a poundal, and is called an absolute unit (in opposition to the gravitational unit, which we will come to later), because it depends only on the units of mass, length and time, and does not vary for any changes of position throughout the universe.

A poundal, then, is the weight of about half an ounce. It may be defined as the force which will produce an acceleration of 1 ft./sec^2. in a mass of a pound; or, again, the force which, acting for 1 sec. on the mass of a pound, will generate in it a velocity of 1 ft./sec.

If we use the fundamental equation in the form $P = \frac{W}{g} f$, it does not matter what units are used for force, provided P and W are expressed in the same units, and the idea of mass does not appear at all. This is common with engineers, and they usually use the pound-weight as the unit of force. The pound-weight is simply the pressure exerted by a standard pound on a horizontal plane. On account of the rotation of the earth, this is not the same as the force with which the earth attracts the standard pound, but differs from it by a small fraction of a pound-weight, as we shall see later. The pound-weight is called a gravitational unit because it depends on the action of gravitation ; it is slightly different at different points of the earth's surface.

A pound-weight is approximately 32 poundals. Of the two units, the poundal is the more convenient for accurate scientific purposes; though not much used now, because the British units are no longer used in this kind of work. For ordinary engineering purposes the pound-weight is used much more frequently; but we feel it is best in working a question in British units to express the forces in poundals, and if the result is a force, to change it afterwards into pounds-weight, and afterwards into tons-weight if required.

Corresponding units will be used in the Metric system. Thus we have the corresponding units :

Unit of	British.	Metric.	Relation.
Mass	pound	gramme	1 lb. = 453·6 gms.
Force	poundal	dyne	1 lbl. = 13825 dynes.
"	lb.-weight	gm.-weight	1 lb.-wt. = 453·6 gms.-wt.

A dyne, for example, is the force which, acting for one second on a mass of a gramme, will generate in it a velocity of one centimetre per second.

To get the relation between the dyne and the poundal we may proceed as follows :

In 1 sec. a force of 1 dyne generates 1 cm./sec. in a mass of 1 gm.

∴ In 1 sec. a force of 30·48 dynes generates 1 ft./sec. in a mass of 1 gm.

∴ In 1 sec. $453{\cdot}6 \times 30{\cdot}48$ dynes generate 1 ft./sec. in a mass of 1 lb.

∴ 1 lbl. $= 453{\cdot}6 \times 30{\cdot}48 = 13825$ dynes (approx.).

The distinction between mass and force must be carefully preserved. They are entirely different kinds of quantities, and cannot be measured in the same units. Masses are only measured in pounds or grammes or multiples or sub-multiples of these. Force cannot be measured in these. It is wrong to speak of a force of 10 lbs., though a force of 10 lbs.-wt. is quite correct. It is, however, common among engineers to speak of a force of 10 lbs., but it must be understood as meaning 10 lbs.-wt., that is, a force equal to the weight of 10 lbs.

26. Facts about Force.

To return now to the nature of force, we find in the first place, in all cases that we observe, that force is exerted by one body on another. The falling body we have spoken of is acted on by a force due to the presence of a second body, the earth. Again, we can only exert pressure with the hand by pressing against some other body. A motor-car cannot start moving without the friction between the wheel and the earth to help it, nor could an aeroplane fly in a vacuum, it cannot move without the exertion of pressure by the air. Thus we only find force exerted as an action of one body on another.

27. Action and Reaction.

Further, we find that in all cases, if one body exerts force on a second, the second exerts force on the first. When we hang a body on to a spring balance, the body pulls at the spring as well as the spring pulling at the body. When we press on the table with the hand, the table presses back, stopping the hand from moving. Even in the case of the falling body, though the effect is not observable, the body exerts an attraction on the earth, in the same way as the earth exerts an attraction on the body. When a horse pulls at a cart, the cart is obviously pulling at the horse, otherwise the horse would rapidly get up a greater velocity.

Further experiments shew that not only does the action of a force on one body by a second involve another force (or a reaction), exerted by the second on the first, but that these two forces are equal in magnitude and opposite in direction.

It is, however, difficult to get simple experiments to shew the equality of action and reaction without bringing in other matters. Some of the simplest are those on the collision of two bodies, and explosion, as in the case of firing a gun, and both of these we will explain soon. If we attach two spiral springs to one another, and exert forces at the two ends, we find that the extensions of the springs shew that the same forces are exerted on the two. That is to say, the force exerted by the first spring on the second is equal to that exerted by the second on the first. It should be noticed that the law of action and reaction has been already assumed in the experiments to illustrate the law $P = Mf$, for the force exerted on the mass by the spiral spring is assumed to be measured by the stretching of the spring which really gives the force exerted by the mass on the spring, and an experiment in which two carriages are connected together by a spiral spring and started moving with any velocities in the line joining them can give no more information than we have already obtained.

28. Laws of Motion.

Summing up the progress so far made, we may say that we have the following laws:

(I) $P = Mf$ for the action of a force on a body, including as a special case the equation $W = Mg$.

(II) A force can only occur in conjunction with an equal and opposite one, or—To every action there is an equal and opposite reaction.

Law (I) includes what are called Newton's first and second that is to say, it is in the opposite direction to the velocity which we have taken as positive.

The magnitude of the resistance is

$$\tfrac{1}{8} \times 800 = 100 \text{ lbls.}$$
$$= 3\tfrac{1}{8} \text{ lbs. wt.,}$$

or 25 times the weight of the bullet.

EXAMPLES.

1. A body of mass 10 lbs. acted on by a constant force attains a velocity of 20 ft./sec. from rest in half a minute. Find the force.

2. A body of mass 4 lbs. is pulled along a smooth horizontal table, and is found to move with uniform acceleration, and to describe 9 ft. from rest in 6 secs. What force is exerted on it ?

3. If a body acted on by a constant force of 2 lbs. wt. moves 10 ft. from rest in 5 seconds, what is the mass of the body ?

4. A mass of 150 gms. is acted on by a force of 10,000 dynes. What distance will it travel in 20 secs. from rest, and what will be its velocity then ?

5. A force of 50 kgms. wt. is exerted on a carriage of mass 500 kgms., capable of moving along horizontal rails without friction. How long will it take to travel 100 metres from rest ?

6. A body of mass 10 lbs. is acted on by a force of 1 oz. wt. for 1 minute. What distance will it have travelled from rest then, and what will be its velocity ?

7. What frictional force will bring a train of mass 100 tons, travelling at 40 miles/hr., to rest in 200 yds. ?

8. A train of weight 100 tons is running at 20 miles/hr., when steam is shut off, and a resistance of 1100 lbs. weight acts continuously until it is brought to rest. Find the distance it travels after steam is shut off, and the time taken to come to rest.

9. A bullet of mass $\frac{1}{2}$ oz. moving horizontally has its velocity reduced from 2000 to 1500 ft./sec. in going 300 yds. Find the air resistance, supposed uniform. Find also the velocity at 150 yds.

10. If, while the velocity of a rifle bullet falls from 1500 to 1200 ft./sec., the average resistance is 20 times the weight, find how far it goes in the interval, regarding the bullet as travelling horizontally.

11. A bullet of mass 1 oz., travelling at 1200 ft./sec., enters a log of wood and penetrates to a depth of 12 inches ; find the resistance, supposed uniform.

If the log had been 8 inches thick, with what velocity would it ...erged ?

12. A railway truck weighing half a ton is pulled along by a constant force, and travels 60 ft. from rest in 12 secs. What is the magnitude of the force ?

13. A force of 8000 dynes acting on a body causes it to get up a velocity of 30 cm./sec. in 6 secs. from rest. What is the mass of the body ? Find, also, the ratio of the force to the weight of the body.

14. A body of mass 10 lbs. is moving along a horizontal table, and its velocity is found to diminish from 12 ft./sec. to 4 ft./sec. in going 5 ft. What is the force acting on it ?

30. Two or More Forces.

In Law I it is also implied that if two forces act on a body, each produces an acceleration proportional to it and independent of the action of the others. We are at present concerned only with the case where the different forces act in one line, but in either the positive or negative direction. The effect of two forces in the same direction on a particle is the same as that of one force equal to the sum of the two, the effect of two forces in opposite directions is the same as that of a force equal to the difference of the two and acting in the direction of the larger of the two. Thus, if a mass of 10 lbs. is acted on by a force of 8 lbs. wt. pulling it in one direction, and 6 lbs. wt. pulling it in the other,

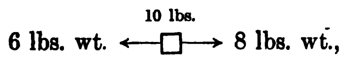


Fig. 16.

it will move with an acceleration given by

$$P = 8 - 6 = 2 \text{ lbs. wt.} = 2 \times 32 \text{ lbls.},$$
$$m = 10;$$
$$\therefore f = \frac{P}{m} = \frac{64}{10} = 6 \cdot 4 \text{ ft./sec}^2.$$

In working any example, however simple, with more than one force, the student should never omit to draw a diagram and mark clearly on it all the forces acting. Also, we advise the student, as a general rule, when he is working a question involving numerical magnitudes of forces, to mark the numerical values of the forces on the diagram. He must not mark accelerations or velocities on the same diagram and in the same manner as forces. If he finds it convenient to mark the acceleration, he may do so by an arrow by the side of the body dealt with, while the forces are marked by arrows passing through the body.

31. Weight carried up by a Lift.

Suppose that a lift moving with upward acceleration f carries a mass m, find the pressure on the floor of the lift.

We cannot form the equation for the lift, since the weight of the lift and the force urging it on are not given, but we can deal with the mass carried. This is moving with upward acceleration f, and is acted on by two forces:

(i) R, the pressure of the lift on the mass upwards (equal and opposite to the pressure of the mass on the lift);

(ii) mg, the weight of the body.

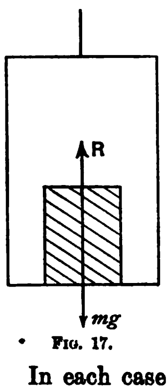


FIG. 17.

Hence we have

$$R - mg = mf,$$

$$R = m(g+f) = mg\left(1+\frac{f}{g}\right).$$

Thus the pressure of the body on the floor of the lift is greater than its weight in the ratio

$$1+\frac{f}{g} : 1.$$

If the acceleration is downwards, the pressure will be less than the weight in the ratio

$$1-\frac{f}{g} : 1.$$

In each case it does not matter in which direction the lift is moving. That is to say, the effect is the same if the lift is ascending with increasing speed and acceleration f or descending with diminishing speed, but the same numerical acceleration for f is upwards in each case.

It is to be specially noticed that it is only the acceleration that affects the pressure, not the velocity.

Example 1. A force of 240 gms. wt. is exerted vertically to raise a body of mass 200 gms. With what acceleration will it move, and what time will be taken in raising it 50 metres? (Allow for the action of gravity.)

2. A train of mass 120 tons is pulled with a constant force of 2 tons weight, while there is a frictional resistance of $\frac{1}{2}$ a ton weight. With what velocity will the train be moving at the end of $1\frac{1}{2}$ mins. from rest, and how far will it have gone then?

3. A train of mass 200 tons starts from rest, the engine exerting a constant force of 2 tons wt. The resistance is 10 lbs. wt. per ton. The train runs for 5 mins., and then steam is shut off. Find (1) the maximum velocity attained, (2) the time it runs without steam before coming to rest, (3) the whole distance travelled.

4. A train of 120 tons starts from rest, and moves against a constant frictional resistance of 11 lbs. wt. per ton, the engine exerting a constant force until steam is shut off. If the acceleration, while steam is on, is 0·14 ft./sec^2., and the train is required to come to rest at the end of 3 miles (without brakes), find (1) the time taken on the journey, (2) the greatest velocity, (3) the distance travelled under steam.

5. How is the reading of a spring balance in a balloon affected by the motion of the balloon?

A mass of 1 lb. is suspended from a spring balance in a balloon; if the pointer reads 1·1 lbs., what is the acceleration of the balloon?

6. A weight of 100 lbs. is on a lift moving downwards. The lift moves at first with acceleration 5 ft./sec^2., then with constant velocity, and finally with retardation 5 ft./sec^2. Find the pressure on the lift exerted by the body in each part of the motion.

7. A string can just support a weight of 5 lbs. at rest. What is the greatest acceleration with which it can raise a weight of 4 lbs.?

32. Air Resistance on a Falling Body.

In the case of a falling body, we have acting besides the force of gravitation the resistance of the air, which is generally assumed to be (for low velocities) proportional to the square of the velocity; hence the resistance can be written kv^2 lbls., where k is some factor depending on the volume and shape of the body and the density of the air, and not on the density of the body; hence the equation for downward motion becomes

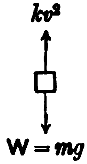


FIG. 18.

$$mg - kv^2 = mf,$$

$$f = g - \frac{k}{m}v^2. \quad \text{.........................(i)}$$

mg is here strictly the weight in air, not the true weight in vacuo. It is less than the true weight by the weight of the air displaced by the body.

As this equation (i) gives the acceleration in terms of the velocity, an approximate solution can be obtained graphically, but an accurate solution cannot be found without the integral calculus. The $\frac{1}{f}$, v graph is of the following shape:

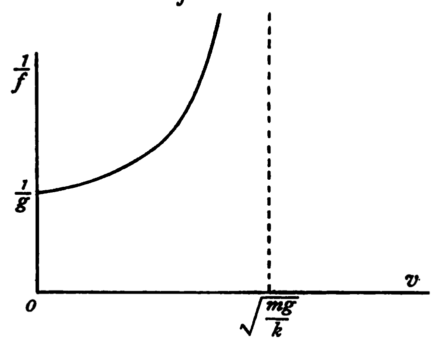


FIG. 19.

The area of this graph by Art. 20 will give the time taken to reach any velocity less than $\sqrt{mg/k}$.

It is important to notice that, as the body falls, the velocity increases and the acceleration diminishes and $\frac{1}{f}$ increases, until ultimately the velocity approaches a limiting value given by

$$g - \frac{k}{m}v^2 = 0,$$

which would make the acceleration zero and $\frac{1}{f}$ infinite; the body cannot, therefore, get up a higher velocity than $\sqrt{\frac{mg}{k}}$, for if the velocity became even slightly higher than this, the acceleration would become negative and the velocity would diminish, while if the velocity diminished below $\sqrt{\frac{mg}{k}}$ again, the acceleration would be positive, and the velocity would increase again. The longer the body falls, the more nearly will the velocity approach the limiting value $\sqrt{\frac{mg}{k}}$. This velocity is often called the terminal velocity.

For example, if $m=10$ and $k=0{\cdot}05$ in F.P.S. units,

$$\sqrt{\frac{mg}{k}}=\sqrt{32\times10\times20}=\sqrt{6400}=80 \text{ ft./sec.},$$

and the body could not get up a greater velocity, starting from rest, than 80 ft./sec., the same velocity that a body falling in vacuo would get up in $2\frac{1}{2}$ secs.

Notice that for upward motion the equation is different, being

$$mf=-mg-kv^2;$$

$$\therefore f=-\left(g+\frac{kv^2}{m}\right);$$

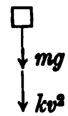


Fig. 20.

and the velocity continually diminishes till the maximum height is reached.

A limiting velocity can be found if the resistance, instead of varying as the square of the velocity, varies as any other power of the velocity, such as the n^{th}. We would then have for the downward motion,

$$f=g-\frac{kv^n}{m},$$

and the limiting velocity $=\sqrt[n]{\frac{mg}{k}}$.

A very important case of this is the fall of small drops of water through the air. The smaller the drop, the larger is the ratio resistance/weight, for diminishing the radius to one-tenth of its former value diminishes the weight to one-thousandth, but the resistance which depends on the area to a hundredth, leaving the ratio 10 times as large as before. The larger consequently becomes k/m, and the smaller the terminal velocity. Minute drops of water consequently fall with exceedingly small velocities, and in this case the resistance is approximately proportional to the velocity. For a small sphere of water, according to Stokes, we have the resistance

$$=6\pi a \times 1\cdot8 \times 10^{-4} v$$

in C.G.S. units if a is the radius, and as the mass is $\frac{4}{3}\pi a^3$, the equation becomes

$$\frac{4}{3}\pi a^3 f = \frac{4}{3}\pi a^3 g - 6\pi a \times 1\cdot8 \times 10^{-4} v,$$

$$f = g - \frac{9}{2} \times \frac{1}{a^2} \times \frac{9}{5} \times 10^{-4} v$$

$$= 980 - \frac{81 \times 10^{-5}}{a^2} v.$$

The limiting velocity is therefore

$$\frac{980a^2}{81 \times 10^{-5}} = 12 \times 10^5 \times a^2.$$

For example, if $a = \cdot001$ cm. $= \frac{1}{100}$ mm.

the terminal velocity 1·2 cm./sec.,

this result only applies if a is very small as we have taken it here. These small limiting velocities account for the suspension of small drops of water and ice crystals in the air, in the form of cloud and fog, and of the minute dust particles which are always present in the air.

33. Collision, Momentum, Impulse.

Suppose that two bodies moving in the same straight line collide and separate from one another again; for example, they may be balls on a horizontal table, such as billiard balls.

They exert force on one another for a very short time only, during which they are in contact, but this force may be very great. During the very short time they are acting on one another, the force between them may (certainly will) vary, but at any particular instant the force will be the same on both, according to the law of action and reaction. Hence again, the rate of change of momentum is the same for both at each instant; hence also, the change of momentum in the interval is the same for both, but in opposite directions.

Hence, let m, m' be the masses,
u, u' be the velocities before collision (measured in the same direction);
v, v' be the velocities after collision.

We may represent the data conveniently thus

	m	m'
	○	○
velocities before impact	$\rightarrow u$	$\rightarrow u'$,
velocities after impact	$\rightarrow v$	$\rightarrow v'$.

FIG. 21.

We have
$$mv - mu = -(m'v' - m'u');$$
$$\therefore \; mv + m'v' = mu + m'u',$$
or the total momentum is unchanged.

As in all other cases, it must be remembered that velocity may have a positive or negative direction. Thus if two balls of masses 6 and 4 lbs. are moving towards one another with velocities 3 and 2 ft./sec. respectively, the total momentum is
$$6 \times 3 + 4 \times (-2) = 10.$$

Many experiments may be made on collision of balls by hanging two balls by strings so that they hang in contact and in the same horizontal. On drawing them aside to any distance, and releasing them they will collide and rebound and their velocities just before and after the collision can be measured, and the above equation verified. A fuller account of this experiment will be given later.

The verification of the equation

$$mv + m'v' = mu + m'u'$$

for various values of the masses and velocities constitutes one of the simplest verifications of the law of action and reaction.

34. Shot and Gun.

An exactly similar case occurs when a gun is fired. Here we have the sudden generation of gases in the barrel, which produce a powerful pressure on the shot, and an equal and opposite pressure on the gun. As in collision, this pressure may be variable, but the total momentum produced in the shot is equal and opposite to the momentum of the gun. As it is usually expressed

$$MV = mv,$$

the large letters referring to the gun and the small to the shot.

(In this form of the equation, V and v are measured in opposite directions.)

Both in collision and in explosion the force is very large, but acts for a very short time, and usually we have to do without a knowledge of the force. It is common in such cases to introduce the term impulse of the force, which is defined in the following way :

When a force is constant, the impulse is the product of the force into the time during which it acts.

If the force is variable, the interval of time during which it acts must be divided up into shorter intervals so short that the force can be regarded as constant during each of the short intervals, and the products of the force into the time for each of these are added together, and the sum called the impulse of the force for the whole time. To get the accurate result, the little intervals must be made indefinitely short, and consequently their number indefinitely great, and the total impulse will then be expressed in the similar notation to that which we have used before as

$$\lim_{n \to \infty} \sum_{r=1}^{r=n} P_r \tau_r,$$

where P_r is the force acting for the r^{th} interval τ_r.

If we draw a force-time curve, it will follow that the impulse is represented on this diagram by the area bounded by the graph the axis of time, and the ordinates at the two instants, between which the impulse is required.

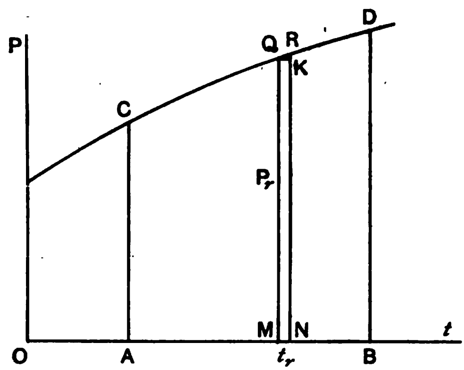


FIG. 22.

In the figure the rectangle QMNK represents $P_r\tau_r$, and the whole impulse between the instants t_1 and t_2 represented by A and B is represented by the area ACDB.

The definition of Impulse can also be expressed thus:

The impulse is equal to the average force multiplied by the time during which it acts.

35. With these definitions the impulse of a force in any interval will be equal to the change of momentum produced by it.

For if the force is constant, the acceleration is constant, and we have with our usual notation

$$v - u = ft,$$
$$mv - mu = mft$$
$$= Pt$$
$$= \text{impulse},$$

but $mv - mu$ is the change of momentum, hence the impulse = the change of momentum.

If the force is variable, we can think, as before, of the time as divided up into the large number of little intervals, and

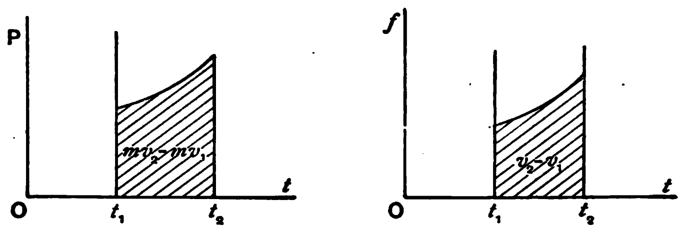


FIG. 23.

for each little interval we have that the impulse during the short time is equal to the change of momentum, hence adding up for the whole time it follows that the total impulse is equal to the total change of momentum. This can also be seen from the graph, for if we consider the acceleration-time graph we know that the area on this graph represents the change in velocity, and on the force-time graph the area represents the impulse. Now, since the force is equal to the mass multiplied by the acceleration, it follows that the ordinates of the force-time graph are all m times as great as the ordinates on the acceleration-time graph, and that therefore the whole area on the first curve is m times the area on the second.

It follows that in the above cases of collision and explosion the impulse can be determined from a measurement of the masses and velocities.

From the above we see that the Newtonian law

$$P = Mf$$

is equivalent to the statement

Impulse = change of momentum,

or $I = mv - mu$ where I stands for the Impulse.

If the pressure in the shot and gun example is required, it is necessary to determine the time t taken by the shot to travel along the barrel as well as the muzzle velocity v, we then have $\qquad I = mv,$

also $\qquad I = Pt,$

or the average pressure $\qquad P = \frac{mv}{t}.$

There is no generally recognized name for the unit of momentum or impulse in either the British or Metric system. We will therefore speak of an impulse in lbl.-sec. units or lb.-wt.-sec. units, or shortly lbl.-secs. or lb.-wt.-secs. A lb.-wt.-sec. will be the impulse of a force of 1 lb. wt. acting for 1 sec., and similarly for a lbl.-sec. A lb.-wt.-sec. is equal to 32 lbl.-secs. Momentum is measured in the same units as impulse. A body of mass 1 lb. moving with a velocity of 1 ft./sec. will possess 1 lbl.-sec. of momentum, or $\frac{1}{32}$ lb.-wt.-sec.

Example 1. Two balls of masses 10 and 5 lbs. moving in opposite directions with velocities 8 and 4 ft./sec. respectively, collide, and the smaller rebounds with a velocity of 5 ft./sec. What is the velocity of the larger after collision? Also find the average force acting between them if they are in contact for 0·01 sec.

Representing the velocities as above,

	(10)	(5)
before collision	$8 \rightarrow$	$\leftarrow 4,$
after	$v \rightarrow$	$\rightarrow 5,$

FIG. 24.

the momentum is unchanged by the collision.

$$10v + 5 \times 5 = 10 \times 8 + 5 \times (-4),$$
$$v = 3 \cdot 5.$$

The change of momentum of the second (and therefore also of the first) is, in magnitude,

$$5 \times 5 - 5 \times (-4) = 45.$$

$$\therefore \text{ the impulse is } 45 \text{ lbl.-secs.} = \tfrac{45}{32} \text{ lb.-wt.-sec.}$$

The force is given by

$$Pt = 45;$$

$$\therefore P = 45 \times 100 \text{ lbls.}$$

$$= 140{\cdot}6 \text{ lbs. wt.}$$

2. A shot of 100 kgms. is fired with velocity 500 metres/sec. from a gun of 6000 kgms. What constant force would be required to stop the recoil of the gun in 2 metres ?

The velocity of recoil is given by

$$MV = mv,$$

or

$$6000V = 100 \times 500.$$

$$V = \frac{25}{3} \text{ metres/sec.} = \frac{2500}{3} \text{ cm./sec.}$$

If the gun stops in 2 m. or 200 cm., the acceleration (negative) is given by

$$v^2 = u^2 + 2fs,$$

$$0 = \left(\frac{2500}{3}\right)^2 + 2f \times 200,$$

$$f = -\frac{62500}{36}.$$

The force required will be Mf,

or

$$6000 \times 1000 \times \frac{62500}{36} \text{ dynes.}$$

$$= 104{\cdot}2 \times 10^8 \text{ dynes nearly}$$

$$= \frac{104{\cdot}2}{980} \times 10^8 \text{ gms. wt.}$$

$$= 1{\cdot}063 \times 10^7 \quad ,,$$

$$= 1{\cdot}063 \times 10^4 \text{ kgms. wt.}$$

$$= 10{,}630 \text{ kgms. wt. nearly.}$$

3. An impulse of 40 lb.-wt.-sec. units is applied to a mass of 12 lbs. at rest. With what velocity does it begin to move ?

4. A stone of weight 2 lbs. lying on ice is struck a horizontal blow of impulse 10 lb.-wt.-sec. units. If the friction is $\frac{1}{6}$ of the weight, how far will the stone go on the ice ?

5. A mass of 2 lbs. moving with velocity 18 ft./sec. overtakes a mass of 3 lbs. moving with velocity 12 ft./sec. If, after collision, the latter mass moves with a velocity of 16 ft./sec., find the velocity of the former and the impulse between them.

6. A mass of 2 lbs. moving with velocity 18 ft./sec. meets a mass of 3 lbs. moving with velocity 6 ft./sec. in the opposite direction. If the second rebounds with velocity 8 ft./sec., find the velocity of the first after collision and the impulse between them.

7. A steel ball of mass 1 lb. drops on a horizontal plate from a height of 10 feet and rebounds to a height of 5 feet. If the ball is in contact with the plate for 0·01 sec., find the average pressure between the sphere and plate while in contact.

8. Two equal steel spheres moving in opposite directions with velocities 15 cm./sec. collide and rebound with velocities 10 cm./sec. The time during which they are in contact is 0·00015 sec. Compare the average pressure between them with the weight of one sphere.

9. If two balls of masses m and m' moving in the same straight line with velocities u and u' collide, and stick together after the collision, prove that the velocity after the collision is

$$\frac{mu + m'u'}{m + m'}.$$

10. A shot of 180 lbs. is discharged from a 12 ton gun with velocity 1260 ft./sec. Find the constant pressure which would be required to stop the recoil of the gun in 6 ft.

11. A shot of 6 lbs. is fired from a gun with velocity 1800 ft./sec. If the barrel is 5 ft. long, find the average pressure exerted on the shot.

12. A cricket ball of mass $5\frac{1}{2}$ oz. has a velocity of 60 ft./sec. before the batsman strikes it, and it is hit back in the same direction with a velocity of 80 ft./sec. What impulse did the batsman give to the ball?

13. Shots, each $\frac{1}{2}$ oz., travelling horizontally at 1500 ft./sec. strike a target at the rate of 50 per minute, and fall dead.

What is the average pressure produced on the supports of the target?

Find the pressure if the shots rebound at 200 ft./sec.

36. Pressure of a Liquid Jet against a Wall.

Suppose that a cylindrical jet of water of diameter d inches, and moving with a velocity of u ft./sec., impinges normally against a fixed vertical wall.

The volume of water arriving at the wall per second is

$$\frac{\pi}{4}\left(\frac{d}{12}\right)^2 u \text{ cub. ft.}$$

its mass is
$$\frac{125}{2} \times \frac{\pi d^2 u}{576} \text{ lbs.,}$$

a cubic foot of water weighing 62·5 lbs.

Before striking the wall its momentum was

$$\frac{125}{2} \times \frac{\pi d^2 u}{576} \times u \text{ F.P.S. units.}$$

If the water does not rebound, this momentum is destroyed by the wall, and consequently an impulse of this magnitude is given to the wall every second.

But since $\quad I = Pt,$

$$P = \frac{125}{1152} \times \frac{22}{7} \times d^2 u^2 \text{ lbls.}$$

$$= \frac{125 \times 22}{1152 \times 7 \times 32} d^2 u^2 \text{ lbs.-wt.}$$

$$= 0{\cdot}0106\, d^2 u^2 \text{ lbs.-wt.}$$

If the water rebounds, the change of momentum, impulse, and pressure will all be greater than this.

Example. Find the pressure with the same notation, if the water rebounds from the wall with velocity v.

37. Work.

In the foregoing work we have been specially concerned with the force exerted on a body and the acceleration produced in it; but the engineer in designing an engine for a railway or steamship is most concerned with such questions as the amount of fuel that will be consumed in a certain journey, or the rate at which the engine will consume the fuel, when going at full speed, and also what that full speed will be. To explain how such questions are to be answered it is necessary first to introduce some fresh ideas and definitions.

Of these the first is Work.

When a body moves under the action of force, the force is said to do Work. Still keeping to the case where the body or particle on which the force acts moves in a straight line, and the force acts along the same straight line, we have the following definition :

When a constant force acting on a particle moves the point of application (the particle), it is said to do work, and the work is measured by the product of the force into the distance the point of application moves in the direction of the force.

When the force acts on a body, the point of application of the force may be supposed to be any point in the line of action of the force. But as the body moves, the same point

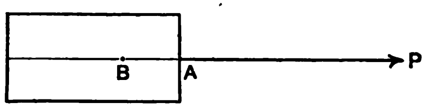


Fig. 25.

in the body must be preserved mentally as the point of application of the force. In other words, the body must actually move for work to be done ; it is not sufficient for the point of application to be simply shifted in the body, such a shift would not cause work to be done. Thus if the body in the figure is being pulled along by a rope, it is quite indifferent whether we regard the force as acting at A or B, or any other point in the line of action of P, as far as the motion produced is concerned, but if we are estimating the work done, it is necessary to think of the same point preserved throughout as the point of application of the force ; whether we choose A or B is, however, still immaterial, since the distance moved by both points in the motions of translation that we are considering are the same.

If the force, instead of being constant, varies, we have to use the same ideas as in the case of the varying velocity.

We would divide the distance the particle travels into a large number n of small distances, the force being supposed to remain constant in each of these short distances, and change suddenly at the end of each, taking, for example, throughout any of the short distances the value it actually has at the beginning of it; we then add up the work done in each of the short distances, and the total when n is made indefinitely large is called the total work done by the varying force.

Mathematically, if d_r is the length of the r^{th} short distance, and P_r is the force at the beginning of this, the total work is

$$\lim_{n \to \infty} \sum_{r=1}^{r=n} P_r d_r.$$

Again, by what we have seen, this is represented graphically by the area of the force-distance graph between two required ordinates.

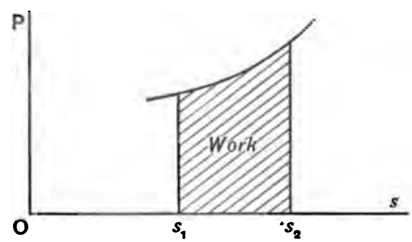


FIG. 26.—Force-Space Diagram.

In the figure the shaded area represents the work done by the variable force as the particle is displaced from the distance s_1 to the distance s_2.

Example. If the force is represented by an expression $a+bs$, shew that the work done between the distances s_1 and s_2 is

$$a(s_2-s_1)+\frac{b}{2}(s_2^2-s_1^2).$$

The work may also be defined as the product of the average force into the whole distance. But it must be remembered exactly what is meant by the use of the term average. Impulse could be defined as the average force multiplied by the time, but the average force in that case had not the same meaning as the average force in the present. In the case of the impulse, it is a time-average; that is, the average could be found by adding the forces for n instants at equal intervals of time apart, and dividing by n afterwards, making n indefinitely large. In the case of work it is a space-average that is required; that is, we have to add the forces for n different positions at equal distances apart, and proceed as before.

Example 1. A body falls for 5 secs. from rest. It describes 400 ft. The time-average of its velocity is 80 ft./sec. Find the space-average of the velocity by averaging the velocity at the distances 0, 100, 200, 300, 400 ft. from the starting point. Find the space-average in the same way by dividing the whole space into 8 equal portions.

2. When the force is expressed by $a+bs$, prove that the space-average for the distance s_2-s_1 is $a+b\frac{s_1+s_2}{2}$, or is equal to the mean of the initial or final forces.

38. The absolute unit of work in the British system is the work done by a force of a poundal when its point of application moves through one foot. This is called a Foot-poundal. There is a gravitational unit used by engineers called the Foot-pound, and defined as the work done when a force of a pound-weight moves its point of application through one foot. In other words, the work done when a weight of one pound is raised one foot against gravity.

It must be noticed that the question of the time required to move the point of application does not come into the definition. Thus the amount of work done by a bricklayer in carrying a hod of bricks of 70 lbs. weight up a ladder 40 ft. high is 70 × 40 ft.-lbs. whether he takes a minute or five minutes to do it. On the other hand, the rate at which the work is done is important in all questions dealing with trains and engines of all kinds.

Since 1 lb.-wt. = 32·2 lbls., it also follows that

1 ft.-lb. = 32·2 ft.-lbls.

The corresponding units in the metric system are the Erg, corresponding to the foot-poundal; and the gramme-centimetre, corresponding to the foot-pound.

Thus an erg is the amount of work done when a force of a dyne moves its point of application through one centimetre; and the gramme-centimetre is the work done in raising a weight of one gramme one centimetre against gravity.

Hence 1 ft.-lb. = $453{\cdot}6 \times 30{\cdot}48$ gm.-cms.

= 13820 gm.-cms. nearly.

A ft.-lbl., being the work done by a force of a lbl. acting through one foot, or 13820 dynes acting through 30·48 cms.,

$= 13820 \times 981$ ergs

$= 4{\cdot}214 \times 10^5$ ergs

and 1 ft.-lb. $= 1{\cdot}356 \times 10^7$ ergs.

Another unit used especially in electrical work is the Joule, which is defined as 10^7 ergs. Hence

1 Joule = 0·737 ft.-lb.

39. Energy.

Connected with Work, we have to introduce the term Energy, usually defined as the capacity for doing work.

Whenever a force does work on a body, and so changes its state either of motion or position or shape, the body is said to have its energy increased by the amount of work done, and the work may frequently be done by the body in returning to its first state. Thus, suppose a body starting from rest is acted on by a constant force P in the line of motion, and after travelling a distance s has a velocity v.

Since the force is constant, the body has a constant acceleration given by $v^2 = 2fs$;

$$\therefore \ \tfrac{1}{2}mv^2 = \tfrac{1}{2}m \,.\, 2fs$$
$$= mfs = \text{P}s,$$

but Ps is the work done by the force,

$$\therefore \ \text{the work done by the force} = \tfrac{1}{2}mv^2.$$

We therefore take $\frac{1}{2}mv^2$ as a measure of the energy possessed by a body of mass m moving with a velocity v.

This is called the Kinetic Energy of the body. If we try to stop the body by exerting a resistance of any magnitude in the opposite direction to the motion, in exactly the same way work $\frac{1}{2}mv^2$ will be done against the resistance, or the body will do this amount of work in coming to rest.

If a constant force P acting on a body changes the velocity from u to v, we have

$$v^2 - u^2 = 2fs;$$
$$\therefore \ \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = mfs = \text{P}s,$$

or the change in kinetic energy is equal to the work done by the force.

In the same way, if a force is exerted to raise a body of mass m (or weight mg) to a height h, the force required is mg or W, and the work done is mgh or Wh. Consequently, the body is then said to possess a quantity Wh of energy more than it did at first. The energy in this case is called Potential. This energy can be recovered in the form of work by letting the body drop. By the time it reaches its original position it has lost its Potential Energy but has gained Kinetic Energy of amount $\frac{1}{2}mv^2$, but $v^2 = 2gh$;

$\therefore$ this Kinetic Energy $= mgh$, or the amount of potential energy it started with. This kinetic energy can be transformed again into work done by the body as in the former case.

From the above examples we see that when there are no resistances from outside, the potential energy of a body can be changed into an equal amount of kinetic energy, and conversely kinetic energy into potential. This is a simple case of what is called the Conservation of Energy; but it will be seen that as far as this simple case is concerned kinetic energy has been defined in such a way that the conservation of energy may hold. However, we find that with these definitions the conservation of energy is of much more far-reaching application, and that in all cases where we find change in energy taking place, we will find that corresponding to any loss of energy of one kind that we observe we will find a gain of the same amount of another kind. We will meet with other forms which energy may take, but in the elementary dynamics we are mainly concerned with the kinetic and potential energy as defined above.

The above results will be extended later to more general cases, and it will be found universally that when a body moves under the action of any forces, the gain in kinetic energy is equal to the work done by the forces. This statement may be regarded as the statement of the conservation of energy in its general form, so far as we are concerned with it in dynamics.

Sometimes when a body is moving against a resistance energy is lost, or rather cannot be restored to its original amount. Thus if a bullet moving horizontally, with a high velocity, strikes a target and sticks in it, it loses all its kinetic energy. There is no change in the potential energy, since the bullet travels horizontally. But we find that the bullet and target were heated by the blow, and we say that the kinetic energy of the bullet has been changed into heat energy.

It must be remembered that the expression $\frac{1}{2}mv^2$ for the kinetic energy gives the energy in foot-poundals if m is in pounds and v in feet per second, and in ergs if m is in grammes and v in centimetres per second. To express in ft.-lbs. or gm.-cms. we must divide by 32 or 980 as the case may be.

40. Positive and Negative Work.

Positive and negative quantities are considered in work as in other physical quantities. If we exert a force W to raise a weight W to a height h, we can express the work done in one or two ways. We can say the force has done Wh units of work, or that Wh units of work have been done against gravitation, or, again, that gravitation has done $-$Wh units. The work done by a force will be positive when the point of application moves in the direction of the force and negative when it moves in the opposite direction. Kinetic energy involving v^2 is necessarily a positive quantity whether v is positive or negative.

41. Power, Horse-power.

In dealing with engines we usually want to know not only the total amount of work done, but also the rate at which it is being done. We therefore introduce the term Power to denote the rate of work of an engine or other agent. We speak of an engine being of such and such a power, meaning that it can do so much work per second. The term Activity is also used frequently in the same sense of rate of work.

We can measure power in foot-poundals per second, or in foot-pounds per second, or in the corresponding C.G.S. units. The usual engineering unit is the Horse-power, which is the power of an agent which does 33000 ft.-lbs. per minute or 550 ft.-lbs. per sec.

In the metric system the Watt is used for a Joule per second, or 10^7 ergs per second. Hence, by art. 38,

$$\begin{aligned} 1 \text{ Kilowatt} &= 1000 \text{ watts} \\ &= 737 \text{ ft.-lbs./sec.} \\ &= 1{\cdot}34 \text{ horse-power.} \end{aligned}$$

A quantity of work is sometimes expressed in horse-power-hours, a horse-power-hour being the amount of work done in one hour by an agent working at one horse-power, hence

$$1 \text{ horse-power-hour} = 550 \times 60 \times 60 \text{ ft.-lbs.}$$
$$= 884 \text{ ft.-tons nearly.}$$

If the force exerted is constant, the work done between two distances s_1 and s_2 is $\mathsf{P}(s_2 - s_1)$;

$$\text{the rate of work} = \operatorname*{Lim}_{t_2 \to t_1} \frac{\mathsf{P}(s_2 - s_1)}{t_2 - t_1}$$

$$= \mathsf{P} \times \operatorname*{Lim}_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1},$$

since P is constant, $= \mathsf{P}v.$

And even if the force is variable, to get the work done we have to divide the distance into very short intervals to get the measure of the work, and from the short interval we still deduce the result that the rate of work $= \mathsf{P}v$.

Example 1. A shot of 6 lbs. is fired from a gun of 5 cwt. with a velocity of 1400 ft./sec. Find the initial kinetic energy of shot and gun.

The K.E. of the shot $= \frac{1}{2}mv^2$

$$= \tfrac{1}{2} \times 6 \times 1400 \times 1400 \text{ ft.-lbls.}$$

$$= \frac{3 \times 1400 \times 1400}{32} \text{ ft.-lbs.}$$

$$= \frac{3 \times 1400 \times 1400}{32 \times 2240} \text{ ft.-tons}$$

$$= 82 \cdot 03 \text{ ft.-tons.}$$

The velocity of the gun is given by

$$\mathsf{M}\mathsf{V} = mv,$$
$$5 \times 112 \times \mathsf{V} = 6 \times 1400,$$
$$\mathsf{V} = 15 \text{ ft./sec.}$$

$$\text{Its K.E.} = \tfrac{1}{2} \times 5 \times 112 \times 15 \times 15 \text{ ft.-lbls.}$$
$$= 0 \cdot 88 \text{ ft.-tons.}$$

2. An engine is developing 200 H.P. and drawing a train at a uniform speed of 30 miles/hr. What is the resistance to the motion?

FIG. 27.

The forces are P exerted by the engine, and F the friction.

As it is moving at constant speed, there is no acceleration;

$$\therefore \quad P - F = 0.$$

Also as the velocity is 44 ft./sec. and the rate of work

$$200 \times 550 \text{ ft.-lbs./sec.,}$$

we have

$$P \times 44 = 200 \times 550,$$

$$P = 2500 \text{ lbs. wt.,}$$

P being in lbs. weight because we have left the rate of work in ft.-lbs./sec. If we had put the rate of work in ft.-lbls./sec., we would have got the force in lbls.

3. At a certain instant a train of 160 tons is travelling on a horizontal line at 20 miles an hour, and has an acceleration of $\frac{1}{4}$ ft./sec^2. If the friction is 12·5 lbs. wt. per ton, find the force exerted by the engine and the horse-power at which it is working. Also find the greatest velocity it could have at the same horse-power and with the same friction.

Here $P - F = mf$, and if m is put in lbs. and f in ft./sec^2., P and F must be in lbls.

$$F = 12{\cdot}5 \times 160 \text{ lbs. wt.}$$

$$= 2000 \times 32 \text{ lbls.}$$

$$P - 2000 \times 32 = 160 \times 2240 \times \tfrac{1}{4},$$

$$P = 2000 \times 32 + 40 \times 2240 \text{ lbls.}$$

$$= 2000 + \frac{40 \times 2240}{32} \text{ lbs. wt.}$$

$$= 2000 + 2800$$

$$= 4800 \text{ lbs. wt.}$$

(Notice that it takes 2000 lbs. wt. to overcome friction, and would take 2800 lbs. wt. to produce the acceleration if there were no friction.)

Since the velocity $= 20$ miles/hr. $= \frac{88}{3}$ ft./sec.,

the rate of work $= Pv$

$$= 4800 \times \frac{88}{3} \text{ ft.-lbs./sec.}$$

$$= 4800 \times \frac{88}{3} \times \frac{1}{550} \text{ H.P.}$$

$$= 256 \text{ H.P.}$$

When at the maximum velocity there is no acceleration, and the force exerted is required to overcome friction only. It will therefore be 2000 lbs. wt. only, and the maximum velocity v' will be given by

$$2000 \times v' = 4800 \times \frac{88}{3},$$

$$v' = \frac{4}{5} \times 88 \text{ ft./sec.}$$

$$= 48 \text{ miles/hr.}$$

4. A shot of mass 56 lbs. is projected with velocity 2000 ft./sec. Find its initial kinetic energy in foot-tons.

5. A shot of mass 20 kgms. is projected with velocity 600 metres/sec. Find its kinetic energy in (1) ergs, (2) kilogrammetres.

6. An impulse of 20 kgm.-wt.-sec. units is applied to a mass of 8 kgms. Find the velocity with which it begins to move and the kinetic energy given to it by the blow.

7. A ball of weight 8 ozs. is thrown vertically upwards with such a velocity that its kinetic energy is initially 32 ft.-lbs. Find the initial velocity and the height to which it will rise.

8. A mass of 6 lbs. is thrown vertically upwards with velocity 100 ft. per sec. Find its kinetic and potential energies after $1\frac{1}{2}$ sec.

9. A body of mass 500 gms. is let fall from a height of 20 metres. What are its kinetic and potential energies at the end of 1 sec.?

10. A 25 ton gun discharges a shot of 56 lbs. at 1400 ft./sec. What is the velocity of recoil of the gun? Find the kinetic energy of shot and gun.

11. When a gun is discharged, shew that the kinetic energies of shot and gun are in the inverse ratio of the masses.

12. A shot of mass m is discharged from a gun of mass M, and the relative velocity is u. Find the velocities of each, and shew that the total kinetic energy generated is

$$\frac{1}{2}\frac{Mm}{M+m}u^2.$$

13. What is the horse-power of an engine which can keep a train of 200 tons weight going at 45 miles an hour against a resistance of 13 lbs.-wt. per ton.

14. What is the greatest speed at which an engine of 270 H.P. can drag a train of mass 180 tons against a resistance of 11·5 lbs.-wt. per ton.

15. Find the resistance if an engine of 120 H.P. drags a train of 100 tons at a maximum velocity of 50 miles/hr. on a level line.

16. Find the maximum velocity that an engine of 240 H.P. and weighing 30 tons can get up in a train weighing (without the engine) 170 tons on a level line, when the resistance is 13·5 lbs.-wt. per ton.

Find also the maximum velocity when extra trucks weighing 100 tons are added to the train.

17. An engine gets up a velocity of 30 miles/hr. in a train of 80 tons in 2 minutes from rest. If the resistance is 12 lbs.-wt. per ton, and the force exerted by the engine constant, find this force, and the horse-power developed by the engine at the end of the 2 minutes.

If this horse-power is maintained, find the maximum velocity attained, assuming the resistance to remain constant.

42. Change of Kinetic Energy due to an Impulse.

If an impulse I in lbl.-sec. units acts on a mass of m lbs., changing its velocity from u ft./sec. to v ft./sec., we have the work done $=\mathsf{E}=\frac{1}{2}mv^2-\frac{1}{2}mu^2=\frac{1}{2}m(v^2-u^2)$,

but $\mathrm{I}=mv-mu=m(v-u)$;

$$\therefore\ \mathsf{E}/\mathrm{I}=\tfrac{1}{2}(v+u)$$

$$\mathsf{E}=\mathrm{I}\frac{v+u}{2}.$$

The kinetic energy produced = Impulse × mean of the initial and final velocities.

43. Loss of Kinetic Energy on Collision.

Suppose that two bodies of masses m, m' are moving in the same line with velocities u and u' and stick together after the collision.

Let us represent the facts thus :

	○	○
	m	m'
velocities before impact	$\rightarrow u$	$\rightarrow u'$
,, after ,,	$\rightarrow v$	$\rightarrow v$.

The velocities are the same after the impact, since the bodies are supposed to stick together afterwards.

By the principle of momentum,

$$mv + m'v = mu + m'u',$$

$$v = (mu + m'u')/(m + m').$$

The loss of kinetic energy on collision is therefore

$$= \frac{1}{2}mu^2 + \frac{1}{2}m'u'^2 - \left(\frac{1}{2}mv^2 + \frac{1}{2}m'v^2\right)$$

$$= \frac{1}{2}\left\{mu^2 + m'u'^2 - (m + m')\left(\frac{mu + m'u'}{m + m'}\right)^2\right\}$$

$$= \frac{1}{2}\left\{mu^2 + m'u'^2 - \frac{m^2u^2 + 2mm'uu' + m'^2u'^2}{m + m'}\right\}$$

$$= \frac{1}{2}\frac{mm'(u^2 + u'^2 - 2uu')}{m + m'} = \frac{1}{2}\frac{mm'}{m + m'}(u - u')^2.$$

$(u - u')^2$ being essentially a positive quantity, the loss of kinetic energy is necessarily positive, or kinetic energy is necessarily lost. We will see later that kinetic energy is always lost, even if the bodies separate after the collision.

Example 1. A mass of 8 lbs. moving with velocity 5 ft./sec. overtakes another of mass 6 lbs. moving with velocity 4 ft./sec. If the two stick together after the collision, find their common velocity, and also the total loss of kinetic energy.

2. If the masses in Question 1 were moving in opposite directions before collision, find the common velocity afterwards, and the loss of kinetic energy.

3. Two bodies moving with velocities 4 and 7 ft./sec. towards one another collide and separate from one another with velocities 3, 4 ft./sec.

Prove that the masses are in the ratio 11 to 7, and that nearly 69 per cent. of the energy is lost in the collision.

4. A nail of weight $\frac{1}{2}$ oz. sticks horizontally into wood, and is struck by a hammer weighing 1 lb. and moving with velocity 20 ft./sec., which drives the nail $\frac{1}{2}$ inch into the wood. Assuming no impulsive pressure (or impulse) between the wood and nail, and that the hammer does not rebound, find the velocity immediately after the blow, the total kinetic energy then, the resistance offered by the wood supposed uniform, and the time the nail is moving.

5. A pile driver of weight 5 cwt. drops 15 ft. on a pile weighing 1 ton. Assuming that the ground is so soft that there is no

impulsive pressure between the pile and the ground, find the initial velocity of the pile, and if it is driven 6 inches into the ground, find the average resistance to the pile.

44. Kinetic Energy of Relative Motion.

In applying the idea of energy to examples, we are using the equation,
$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = Ps,$$
expressing that the change of kinetic energy is equal to the work done. It may seem strange that we are able to use this equation so extensively as we do, seeing that the velocities are usually velocities relative to the earth, which body itself we regard as being in rapid motion, and if this velocity is taken into account, the change in kinetic energy is not the same as the change when calculated from the relative motion only.

It is necessary, therefore, to examine how the calculations are affected by a supposed motion of the origin from which we are measuring our distances.

Suppose, then, that the origin O is moving with uniform velocity, relative to a second origin O′, this velocity being V in the direction in which the initial and final velocities are measured, and suppose the acceleration relative to O to be constant.

Let s be the distance described relative to O in time t, so that according to the usual equations
$$2fs = v^2 - u^2,$$
$$ft = v - u.$$

Now, if s' is the distance and f' the acceleration relative to O′, since the velocities relative to O′ are $V+u$, $V+v$,
$$f't = (V+v) - (V+u) = v - u = ft; \quad \therefore f = f'$$
$$2f's' = (V+v)^2 - (V+u)^2 = 2V(v-u) + v^2 - u^2 = 2Vft + 2fs;$$
$$\therefore s' = s + Vt,$$
and the distance described has simply been increased by Vt, which is the distance travelled by O relative to O′ in the time.

of the string is the same throughout, is true for any case where the weight of the string is neglected, and is true also for an elastic string (that is, a string which is not inextensible) or spring under the same conditions. It is still true that the tension is the same throughout, if the string, instead of being straight, passes round a smooth peg or pulley so that it can slide over the peg without friction.

46. Two Particles connected by a String.

Suppose that two particles are connected by a string, and moving in the line of the string under the action of two forces in the same line thus:

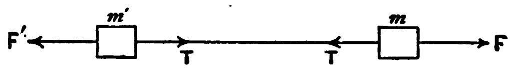


FIG. 30.

Let m, m' be the masses, F, F′ the forces acting in the directions shewn; then the string will remain stretched, and the velocity of the two particles will be the same at a given instant, and so also will be the accelerations. We have already shewn that the tension of the string supposed weightless is the same throughout; hence, if f=acceleration of each (say to the right in the figure), we have the equations

for m, $\quad F - T = mf$

for m', $\quad T - F' = m'f$

hence $\quad F - F' = (m+m')f$

$$f = \frac{F - F'}{m+m'} \quad \text{.......................................(1)}$$

$$T = F' + m'\frac{F - F'}{m+m'} = \frac{F'm + Fm'}{m+m'} \quad \text{...............(2)}$$

Notice that (1) shews that the acceleration is the same as if the two bodies formed one (of mass $m+m'$), and this single body were acted on by the two forces F and F′.

47. Two Particles connected by a String passing over a Smooth Peg.

As in the last example, regarding the string as weightless, the tension is the same throughout. The acceleration is the same for both, but upwards for one and downwards for the other.

Let f be the acceleration of m downwards or the acceleration of m' upwards.

Then the equations of motion are

for m, $$mg - \mathrm{T} = mf,$$

for m', $$\mathrm{T} - m'g = m'f;$$

hence by addition,

$$(m - m')g = (m + m')f,$$

$$f = \frac{m - m'}{m + m'}g$$

$$\mathrm{T} = m'g + m'f = \frac{2mm'}{m + m'}g.$$

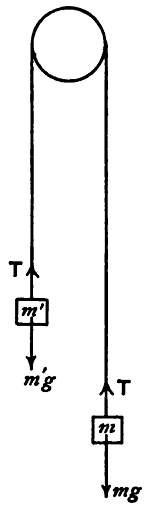


FIG. 31.

Pressure on the Peg.

The peg over which the string passes may be thought of as pulled by the two portions of the string, in other words, it is pulled by a downward force of 2T and requires an equal force of 2T to support it. It will be found that this force is always less than the total load,

for $$mg + m'g - 2\mathrm{T} = mg + m'g - \frac{4mm'}{m + m'}g$$

$$= \frac{g}{m + m'}\{(m + m')^2 - 4mm'\} = \frac{(m - m')^2}{m + m'}g$$

and this is necessarily positive; hence $mg + m'g > 2\mathrm{T}$.

Another way of expressing this is that if an Atwood's machine (see Art. 48) is placed on a weighing machine it will weigh less than the total weight of the machine and weights, unless the latter are equal, and consequently moving without acceleration.

48. Atwood's Machine.

In Atwood's machine two weights are connected as above by a string which passes over a pulley running with as little friction as possible. By means of a ring through which one weight passes, an additional weight may be caught off it in any position. The weights that start moving may be represented as $M+m$ on one side and M on the other. When $M+m$ passes through the ring, m is caught off, and the moving weights are then equal and move on without acceleration, that is, with uniform velocity. This velocity can, therefore, readily be measured, as well as the distance travelled before the weight is caught off, and the time observed.

Consequently the formulae for uniform acceleration starting from rest can be readily verified, namely,

$$v=ft, \quad \text{..........} (1)$$
$$s=\tfrac{1}{2}ft^2, \quad \text{..........} (2)$$
$$v^2=2fs, \quad \text{..........} (3)$$

for we can verify that for given weight and overweight in a number of experiments,

$$v/t=\text{constant},$$
$$s/t^2=\text{constant},$$
$$v^2/s=\text{constant}.$$

In the theory given above the peg is supposed to be perfectly free from friction. When, as in Atwood's machine, the peg is replaced by a pulley, there is usually sufficient friction between the string and the pulley to prevent the string from slipping on the pulley, and even though the pulley may rotate on practically frictionless bearings, the tension will not be the same on the two sides of the pulley, and the result of the last paragraph is affected accordingly. This effect will be considered later (see Part II. Chap. III.). In the following articles and examples the rotation of the pulleys is neglected.

49. Double Atwood's Machine.

In the figure A is a fixed pulley, B a movable pulley of mass m, and strings with weights pass over the pulleys in the way shewn. It is required to find the accelerations of the weights. The forces acting on the different bodies are marked on the figure. The accelerations will be most conveniently represented thus :

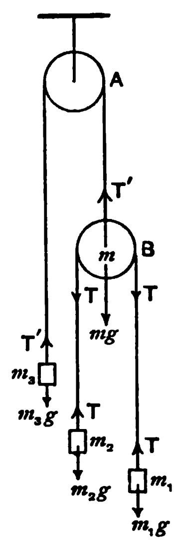


FIG. 32.

Let f = acceleration of m downwards or m_3 upwards,

f' = acceleration of m_1 relative to m downwards

= acceleration of m_2 relative to m upwards,

then the actual accelerations of m_1 and m_2 are $f+f'$ and $f-f'$ respectively (both downwards).

Hence we have the following equations :

for m_1, $m_1g - T = m_1(f+f')$, ...(1)

" m_2, $m_2g - T = m_2(f-f')$, ...(2)

" m, $2T - T' + mg = mf$, ...(3)

" m_3, $T' - m_3g = m_3f$, ...(4)

four equations to find the four unknowns f, f', T, T'. The equations can easily be solved. If we add the four equations, we get

$$(m_1+m_2+m_3+m)f+(m_1-m_2)f'=(m_1+m_2+m-m_3)g, \quad (5)$$

also subtracting (2) from (1),

$$(m_1-m_2)f+(m_1+m_2)f'=(m_1-m_2)g. \quad ....(6)$$

(5) and (6) can now be solved for f and f', and T and T' found substituting in (1) and (4).

50. Body Hanging over the Edge of a Table.

Suppose two bodies of masses m and m' connected by a string passing over a pulley at the edge of a table, the table being supposed smooth, that is, there is no friction tending to stop the motion of a body sliding along the table. The forces on the two bodies are then those marked in the figure.

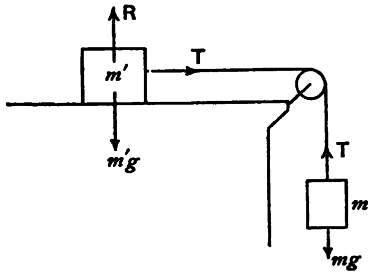


FIG. 33.

On m' there are the normal pressure of the table R, the weight $m'g$, and the tension. But of these the tension alone has any tendency to move the body along the table. Hence, if the acceleration is f, the equations are

for m' $$\mathrm{T} = m'f$$

for m $$mg - \mathrm{T} = mf;$$

$$\therefore f = \frac{mg}{m+m'},$$

$$\mathrm{T} = \frac{mm'g}{m+m'}.$$

Example 1. Masses 240, 250 gms. hang at the two ends of a string passing over a smooth peg. Find the distance in centimetres they will travel in 5 secs., and the velocity attained in that time.

2. The weights on an Atwood machine are 160 and 165 gms., and they are found to move 190 cms. from rest in 5 secs. What is the calculated value of g?

3. The weights on an Atwood machine are 7 and 9 lbs. Find the acceleration and the tension.

4. If the air resistance on each weight of an Atwood's machine is the same and equal to kv^2, find the limiting velocity.

If the masses are 16·5 and 15·5 oz. and $k=0{\cdot}01$ when v is in ft./sec., kv^2 being in lbls., find the limiting velocity.

5. Two scale-pans each of mass 2 oz. are suspended from the ends of a string passing over a smooth peg, and weights 10 and 12 oz. are placed on the scale-pans. Find

(1) the acceleration of the weights;

(2) the tension of the string;

(3) the pressure on each scale-pan.

6. The string of an Atwood's machine can just support a weight of 2 lbs. at rest. If $1\frac{1}{2}$ lbs. hangs on one end of the string, while a heavier weight at the other end rests on a platform, find the maximum weight this heavier one may have without the string breaking when the platform falls.

7. A mass of 4 lbs. hanging vertically is connected by a string with a weight of 8 lbs. on a smooth horizontal table. Find the tension of the string and the acceleration of the weights.

8. A string can just support a weight of 6 lbs. If it is attached to a weight of 12 lbs. on a smooth horizontal table, what is the greatest weight that can be hung at the other end of the string without breaking it?

51. Friction.

The results of the last experiment would agree approximately with the calculated acceleration if the mass were mounted on a smoothly running carriage. But if the body slides along the table, they will be largely affected by friction. Whenever one body slides over a second, a force called friction is called into play which tends to stop the motion. Thus, if the mass m' can slide along and gradually increasing weights are hung to the string, at first there is no motion, shewing that the tension of the string is balanced by the friction brought into play and acting in the opposite direction, but if the weights m are still increased a stage will be reached when the bodies move, shewing that the friction is no longer able to balance the tension. Further, when the bodies once begin to move, they usually move with a finite acceleration, shewing that the friction when moving is less than the force that is required to start them moving; in other words, less than the friction when they were on the point of moving.

Hence we arrive at the results, confirmed by numerous experiments, that there is always a maximum value to the friction that can be brought into play in any particular case, and that the friction when the body is moving is a little less than this maximum friction. Further experiments in this and other cases shew that if we vary the load m' by placing other weights on top of it, that the maximum friction is proportional to the normal reaction between the plane and the body, and that the ratio of the maximum friction to the normal reaction depends only on the nature of the substances in contact (including in nature the degree of polish), and not on the area or shape of the surfaces in contact.

When the bodies move, the friction is, as has been said, rather less than the maximum friction, but the same relations hold, namely, that

$$\frac{\text{friction when moving}}{\text{normal reaction}} \text{ or } \frac{F}{R}$$

is a ratio which depends only on the nature of the surfaces in contact, and is nearly independent of the velocity with which the bodies are moving.

We will, therefore, take as a closely approximate result that for sliding motion $F/R = \mu$, where μ is a constant for a given pair of bodies in contact. μ will be called the coefficient of friction.

The value of μ varies greatly for different pairs of materials. For smooth blocks of wood μ may lie between 0·25 and 0·50. For wood on glass from 0·20 to 0·40. But there is no limit to the values which μ may have in different cases ; it may have any value from zero upwards.

Example. If we return to the body on the table, the forces act as shewn. Since the body does not move vertically, or there is no vertical acceleration,

$$R - Mg = 0,$$

or

$$R = Mg;$$

$$\therefore \; F = \mu R = \mu Mg;$$

hence if f is the acceleration, the equations are

for M, $$T - \mu Mg = Mf,$$

for m, $$mg - T = mf;$$

$$\therefore\ (M+m)f = (m - \mu M)g,$$

$$T = uMg + M\frac{m - \mu M}{M+m}g$$

$$= \frac{Mm(1+\mu)}{M+m}g.$$

It should be noticed that whatever the values of M and m, f can never be negative; in other words, the body only moves and the above equations hold only if $m > \mu M$.

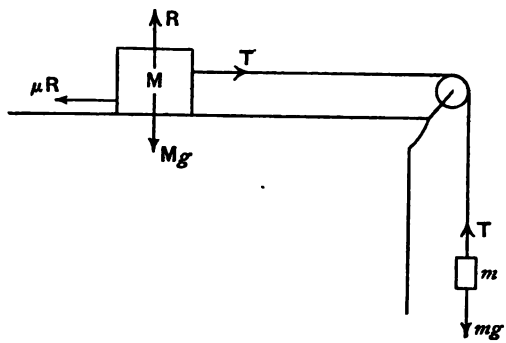


FIG. 34.

If $m < \mu M$, the maximum friction is not required to be brought into play to prevent motion, and consequently the friction will be less than μMg.

Example 1. If in Question 7, Art. 50, the table is rough, and the coefficient of friction between the table and the weight is 0·4, find the acceleration and tension.

2. A mass of 200 gms. hanging vertically drags a mass of 400 gms. along a horizontal table. If the coefficient of friction is 0·4 and the falling weight strikes the floor after moving 150 centimetres, how far will the mass on the table move afterwards?

3. A hanging weight of 200 gms. drags a mass of 500 gms. along a rough table 3 metres from rest in 3 secs. What is the coefficient of friction?

52. Horse and Cart.

The law of action and reaction when applied to such a case as a horse and cart states that the force with which the horse pulls the cart is exactly the same as the force with which the cart pulls the horse. Beginners sometimes ask, How, then, can the horse and cart progress? The simple answer is that they could not, if there were no other forces but these two acting. If there were no friction between the horse's feet and the ground, there would be no possibility of the horse getting along. The difficulty of progressing becomes very much greater on a level sheet of ice or frozen road, where the friction is very much less than on the ordinary ground.

The horse in trying to progress pushes backwards with his feet, and the ground exerts the equal and opposite reaction on the horse, so that it is really this force, the friction acting forwards, which moves the horse on. Friction acts on the cart too, but in the opposite direction, tending to stop the motion of the cart, but this is much smaller, on account of the use of wheels, than the friction at the horse's feet. There are thus two external forces acting on the horse and cart, forces, that is to say, from outside, and which alone can cause the horse and cart to move; these are the friction forwards at the horse's feet and the friction backwards on the cart. The former of these is the greater, and therefore the horse and cart move forwards.

53. Trains.

It is quite similar with trains. The wheels on a train are of two kinds—the large driving wheels on the engine, themselves driven by the piston and crank, and corresponding to the legs of the horse, and all the other wheels on the train, which are simply for diminishing the backward friction, and correspond to the wheels of the cart. The driving wheels being made to rotate by the mechanism, the friction, which necessarily tends to stop relative motion, will be found to act on them in the forward direction, and if there is insufficient friction the wheels skid and the train does not move. The other wheels are set in motion by the friction which will similarly be found to act backwards. Consequently the difference between the friction on the driving wheels and on the rest of the wheels is the force which moves the train along.

The frictional force exerted by the rails on the driving wheels is usually called the force exerted by the engine. The friction on the other wheels and the air resistance are generally classed together and called the frictional resistance, or simply friction or resistance.

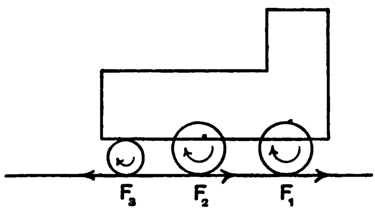


FIG. 35.

54. The quantities involved in questions concerning the motion of trains along a straight horizontal track are the following:

P = force exerted by the engine,

F = frictional resistance,

f = acceleration,

v = velocity,

H = rate of work of the engine,

m = mass of the train;

and the relations between them are

$$P - F = mf,$$

$$H = Pv.$$

If the train is travelling at a uniform speed $f=0$, and

$$\therefore \quad P=F.$$

This holds in particular when the train is going at its greatest speed, for then it can have no acceleration. For if it was being accelerated its velocity would be increasing, and therefore could not be at a maximum; and similarly, if it was being retarded the velocity would be diminishing, and it would have been greater previously.

Hence, at the maximum speed,

$$P=F,$$

$$H=Pv=Fv.$$

When we speak of an engine having a horse-power of 200 we mean that it has been designed to work continously at that rate. It may work sometimes at a higher rate, but cannot be expected to continue working long satisfactorily or economically at the higher rate.

In some of the examples on trains the engine is supposed to exert a constant force P, so that if the resistance R is constant the velocity v, and rate of Pv, increase uniformly with the time.

Actually an engine is not likely to behave in this way. In the first place R increases as the velocity increases, so that even if P is constant, v will not increase uniformly, but the acceleration f diminishes gradually till the maximum velocity is reached, the rate of work being proportional to the velocity.

Further, it is unlikely that as v increases P will remain constant. It is more likely to diminish, and hence f will diminish more rapidly, and v and Pv increase less rapidly, than if P were constant. Also the rate of work will increase less rapidly than the velocity.

Example 1. An engine gets up a velocity of 30 miles/hr. in 1 min. 50 secs. from rest in a train of 200 tons weight on a level line. If the resistances are 14 lbs.-wt. per ton, and the velocity is uniformly accelerated, find the greatest horse-power the engine is developing, and also the greatest velocity it can get up, if this horse-power is maintained.

$$30 \text{ miles/hour} = \frac{30 \times 5280}{60 \times 60} \text{ ft./sec.} = 44 \text{ ft./sec.},$$

$$f = \frac{44}{110} = \frac{2}{5} \text{ ft./sec}^2.,$$

$$\mathsf{F} = 14 \times 200 \text{ lbs.-wt.} = 2800 \times 32 \text{ lbls.},$$

$$\mathsf{P} - \mathsf{F} = mf;$$

$$\therefore \ \mathsf{P} = 2800 \times 32 + 200 \times 2240 \times \frac{2}{5}$$

$$= 2800 \times 32 + 400 \times 448 \text{ lbls.}$$

$$= 2800 + 400 \times 14 \text{ lbs.-wt. (dividing by 32)}$$

$$= 8400 \text{ lbs.-wt.}$$

The rate of work increases gradually from zero to

$$8400 \times 44 \text{ ft.-lbs./sec.};$$

$$\therefore \text{ the horse-power required} = 8400 \times 44 \times \frac{1}{550} = 672.$$

To find the maximum velocity possible we have only to use the same equations, with $f = 0$.

The force exerted now is 2800 lbs.-wt. only, and we have

$$672 \times 550 = 2800 \times v,$$

$$v = 132 \text{ ft./sec.} = 90 \text{ miles/hr.}$$

The large result obtained shews that the friction has been underestimated at the high velocities. While it is 2800 lbs.-wt. at 30 miles/hr., it would be considerably larger before the velocity reached 90 miles/hr.

2. If an engine exerts a uniform force of 2 tons weight on a train of 150 tons, when there is a resistance of 14 lbs.-wt. per ton, find the time required to get up a velocity of 20 miles per hr. from rest and the greatest horse-power developed in doing this.

3. Shew that the maximum velocity with which a train of 100 tons weight can be drawn by an engine of 80 H.P. against a resistance of 12·5 lbs.-wt. per ton is 24 miles/hr.

If the engine is working at this rate when the velocity is 12 miles/hr., find the force exerted then and the acceleration.

4. A train of 150 tons wt. gets up a velocity of 24 miles/hr. at a constant acceleration in $2\frac{1}{2}$ mins. from rest against a resistance of 14 lbs.-wt. per ton. Find the greatest horse-power at which the engine is working, and assuming the engine to continue working at this rate, and the resistance to remain constant, find the maximum velocity attained.

5. An engine exerting a uniform force gets up a velocity of v miles/hr. in a train of m tons in t secs. from rest. If the resistance is n lbs.-wt. per ton, find the force exerted by the engine and the maximum horse-power developed.

55. Extensible Strings.

If an elastic string or wire is fixed at one end and a weight hung at the other, the string is found to increase in length, and the extension of the string is proportional to the weight applied, that is, to the tension of the string. The extension for different strings of the same material is also found to be proportional to the length of the string and inversely proportional to the cross section.

Thus the extension is proportional to $\frac{Pl}{A}$, where P is the force applied, A the area of cross section, l the length when unstretched.

If we call s the extension, we can therefore write

$$s = \frac{Pl}{EA},$$

where E is a constant for the material, and is called Young's modulus.

Writing this $$P = \frac{EA}{l} s,$$

we may also write it $$P = \lambda s/l,$$

or $$P = ks,$$

where $\lambda = EA$, and is frequently called the modulus of the string, and $k = EA/l$.

If we put $A = 1$ and $s = l$, we obtain $P = E$, or E is numerically equal to the force which will stretch a string of unit cross section to double its natural length.

We have already seen that the extension of a spiral spring follows the same law in so far as it is proportional to the force applied, and if in this case we put

$$P = ks,$$

k may be called the stiffness of the spring.

Example. A brass wire of length 1000 cms. and diameter 1 mm. is found to be extended 1 mm. by a weight of 6 kgms. Find Young's modulus for the wire.

(Note that the force P should be expressed in dynes in this example to give the result in absolute C.G.S. units.)

56. Work done in Stretching a String.

Since the force acting when the string is elongated an amount s above its natural length is ks, if we draw the force-space diagram, the graph is a straight line and the work done in stretching to a length s where

$$\mathrm{ON} = s,$$
$$\mathrm{AN} = ks,$$

is

$$\frac{1}{2}\mathrm{ON} \, . \, \mathrm{AN} = \frac{1}{2}ks^2$$
$$= \frac{1}{2}\mathrm{P}s,$$

or we may write it $\frac{1}{2}\frac{\lambda}{l}s^2$ or $\frac{1}{2} \cdot \frac{\mathrm{EA}}{l} \cdot s^2$.

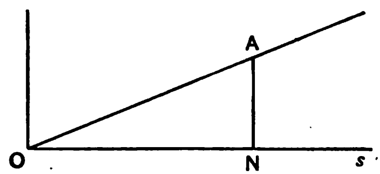


FIG. 36.

This result applies to all cases of strings and also to springs, and in the latter case both for extension and compression of the spring, but in the spring k no longer denotes EA/l.

57. Impulsive Tension in a String.

If, in an arrangement like Atwood's machine, an extra weight is suddenly brought into motion, there is produced a sudden change in the velocity of the weights, and an accompanying impulse exerted by the string. This is usually called an impulsive tension.

Experimentally we can produce the effect by making the ascending weight pass through a ring on which is placed an extra weight to be caught up; or a loose thread may be attached to the ascending weight, and to a weight placed at rest vertically under it. When the ascending weight rises to a sufficient height, the loose string suddenly tightens, and the extra weight is jerked into motion. The result is the same in either arrangement. Thus let m, m' be the two masses of which m' is ascending and m descending with velocity v at the instant the extra weight M is brought into motion, and let v' be the velocity immediately after M begins to move.

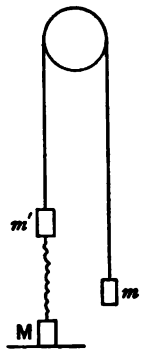


FIG. 37.

There will be an impulse I in the upper string (in the figure) which will be the same throughout, as in an ordinary tension. This impulse is equal to the change of momentum in each case. Hence the equations are:

for m' and M together $\quad I = (M + m')v' - m'v$ ....(1)

for m $\quad -I = mv' - mv.$ ....(2)

The negative sign in the latter case comes from the fact that $mv' - mv$ is the change of momentum reckoned downwards while I is the impulse upwards.

Adding (1) and (2), we have

$$(M + m' + m)v' = (m + m')v \quad \text{.....................(3)}$$

$$v' = \frac{m + m'}{M + m + m'}v$$

$$I = \frac{Mm}{M + m + m'}v.$$

The impulsive tension in the lower string is

$$Mv' = \frac{M(m + m')}{M + m + m'}v.$$

Example 1. An Atwood's machine has weights 4 and 6 lbs. on the two ends of the string. If the mass 6 strikes the ground and comes to rest after travelling 5 ft., find

(1) how much farther the 4 lb. wt. will rise;

(2) the velocity with which it begins to raise the 6 lb. wt. again;

(3) the impulse in the string when it becomes taut.

2. A mass P of 4 lbs. hanging over the edge of a table is attached by a string to a mass Q of 12 lbs. on the table, and the latter by another string 4 ft. long to a mass R of 8 lbs. The coefficient of friction between the table and each weight is 0·3. The weights Q and R are initially held at rest close together, and then Q is released. Find

(1) the time until R begins to move;

(2) its initial velocity;

(3) the distance moved by R before coming to rest;

(4) the impulsive tension in each string when R begins to move.

3. An Atwood machine has weights $7\frac{1}{2}$ and $8\frac{1}{2}$ lbs. on the ends of the string. After moving from rest through 4 ft., the ascending weight begins to raise a mass of 1 lb. attached to it by a string. Find the impulse in the two strings and the velocity immediately after the lower string becomes taut.

58. System of Pulleys.

The following example will illustrate different methods of treatment of questions on pulleys:

M, m are the "Weight" and "Power" in the system of frictionless pulleys shewn, and m_1, m_2 are the masses of the pulleys. Find the accelerations of the bodies.

First solution, by considering the forces acting.

If M moves a distance x upwards, then m_2 moves a distance

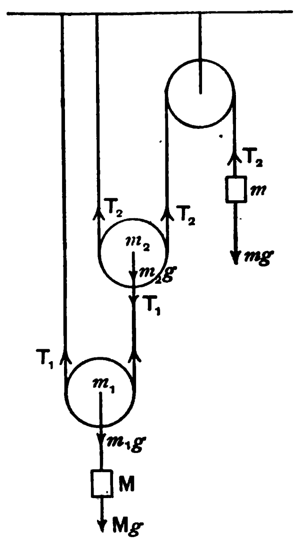


FIG. 38.

$2x$, and m moves a distance $4x$ downwards. The velocities of the bodies are in the same ratio, and so are the accelerations.

Let f = the acceleration of M and m_1 upwards,
then $2f$ = „ „ „ m_2 upwards,
and $4f$ = „ „ „ m downwards

Hence the equations are

for m, $mg - T_2 = m \times 4f$, ...(1)
for m_2, $2T_2 - T_1 - m_2 g = m_2 \times 2f$, ...(2)
for $M + m_1$, $2T_1 - m_1 g - Mg = (M + m_1)f$. ....(3)

Multiply (1) by 4, (2) by 2, and add the three equations

$$4mg - 2m_2 g - m_1 g - Mg = (16m + 4m_2 + M + m_1)f.$$

Second solution, by energy.

As before, if M goes up a distance x and in so doing gets up a velocity v, then m_2 goes up a distance $2x$ and gets up a velocity $2v$, and m goes down a distance $4x$ and gets up a velocity $4v$; therefore the gain in potential energy is

$$(M + m_1 + 2m_2 - 4m)gx,$$

and the gain in kinetic energy is

$$\tfrac{1}{2}(M + m_1 + 4m_2 + 16m)v^2.$$

The gain in kinetic energy is equal to the loss of potential energy;

$$\therefore\ \tfrac{1}{2}(M + m_1 + 4m_2 + 16m)v^2 = (4m - 2m_2 - m_1 - M)gx.$$

Comparing this with the equation for uniform acceleration,

$$v^2 = 2fx,$$

where f is the acceleration of M,

we see that $$f = \frac{4m - 2m_2 - m_1 - M}{16m + 4m_2 + m_1 + M} g.$$

Example 1. A string passing round a smooth peg supports pulleys each of weight 20 gms., and strings pass round these and support weights of 120 and 100 at the ends of one, and 180 and 160 at the ends of the other. Find the accelerations.

2. In a system of pulleys (the second) in which there is one string passing round two blocks, there are three portions of the

string passing between the two blocks. A mass of 4 lbs. is hung to the movable block and 2 lbs. to the end of the string. Find the accelerations and the tension, neglecting the mass of the blocks.

3. A string, fixed at one end at A, passes over a pulley B and a mass of 2 lbs. is attached to the other end. A mass of 5 lbs. can slide on the string between A and B. If all portions of the string are vertical, find the accelerations and tension.

4. A string passes over two pulleys close together, and hangs in a loop between. Masses 150 and 200 gms. are attached to the ends, and 300 gms. can slide freely on the loop. Find the accelerations and tension.

Explain thé difference caused if the 300 gm. mass is tied to the string, and find the tensions and accelerations in this case.

5. A string has a mass of 200 gms. attached to one end, and passes over a fixed pulley, and has its other end attached to a movable pulley over which passes a second string with a weight of 150 gms. at one end, and with the other end attached to a fixed point. If all the strings are vertical find the accelerations and tensions:

(1) if the movable pulley is weightless;

(2) if it weighs 25 gms.

59. Units and Dimensions.

We have in the previous work used three units which may be called fundamental, namely, those of mass, length, and time. Any other dynamical quantity may be expressed in terms of these, and we have seen that when we have settled these units, such units as those of velocity and acceleration naturally follow, if we wish to make a consistent system. Thus, if the units of length and time are 1 foot and 1 sec. respectively, the unit of velocity will be 1 ft./sec. and of acceleration 1 ft./sec². Thus again, force being the product of mass and acceleration, a force of a dyne may be written as 1 gm. × cm./sec²., and so for any other quantities.

If, consequently, we denote the units of length, mass, and time by L, M, T, the unit of velocity will be $\frac{L}{T}$;

$$\text{the unit of acceleration} = \frac{\text{unit of velocity}}{\text{unit of time}} = \frac{L}{T^2},$$

the unit of force = unit of mass × unit of acceleration

$$=\frac{\mathsf{ML}}{\mathsf{T}^2},$$

the unit of work = unit of force × unit of distance

$$=\frac{\mathsf{ML}}{\mathsf{T}^2}.\mathsf{L}=\frac{\mathsf{ML}^2}{\mathsf{T}^2}.$$

The unit of angle (the radian) is independent of the three units or is a number only, and consequently the unit of angular velocity (see Art. 113)

$$=\frac{1}{\text{unit of time}}=\frac{1}{\mathsf{T}}.$$

The unit of angular acceleration $=\frac{1}{\mathsf{T}^2}$ and so on for other units.

The above expressions are frequently called the dimensions of the physical quantity, thus the dimensions of force are said to be $\frac{\mathsf{ML}}{\mathsf{T}^2}$, or frequently the dimensions are understood to mean the powers to which the units are raised, thus force is said to be of dimensions 1, 1, -2 in mass, length, and time respectively.

Example. Write down the dimensions of Impulse, Rate of Work, Fluid Pressure (force per unit area), Volume, Density (mass per unit volume).

It will be carefully noticed that dimensions only express the manner in which the unit of the quantity concerned depends on the three fundamental units. Thus the kinetic energy of a body is expressed by $\frac{1}{2}mv^2$ and the dimensions of kinetic energy are $\frac{\mathsf{ML}^2}{\mathsf{T}^2}$, the factor $\frac{1}{2}$ not appearing in the expression for the dimensions, which only denote that the unit of kinetic energy (or the erg) can be thought of as a mass of 1 gm. multiplied by the square of a velocity of 1 cm./sec. That a body of mass a gram moving with a velocity of 1 cm./sec. has kinetic energy of half this amount does not affect the idea of the dimensions.

60. There are two important uses of dimensions.

First. In any physical expression or equation all terms must be of the same dimensions in the fundamental units. Just as it is impossible in ordinary arithmetic to add, say, an amount of money to a length, so it is impossible in any case to add two terms which are not of the same dimensions. Thus, taking as a simple case the equation,

$$s = ut + \tfrac{1}{2}ft^2,$$

the dimensions of s are L,

" ut are $\frac{\mathsf{L}}{\mathsf{T}}\mathsf{T} = \mathsf{L}$,

" $\frac{1}{2}ft^2$ are $\frac{\mathsf{L}}{\mathsf{T}^2}\mathsf{T}^2 = \mathsf{L}$,

so that all terms are of the same dimensions, namely L.

It is obvious that this may form a frequent check on the accuracy of a dynamical formula, but it must be noticed that the check is only useful if all the quantities are expressed algebraically. For example, if we write the equation for the falling body and put $s = 16t^2$,

the check is no longer possible, for the acceleration with its dimensions $\frac{\mathsf{L}}{\mathsf{T}^2}$ has been replaced by a number which apparently has no dimensions.

Secondly. The second important use of the dimensional formula is to change from one set of units to another. A couple of examples will be sufficient to show how this is done.

Example. Find the number of dynes in a poundal.

The dimensions of force are $\frac{\mathsf{ML}}{\mathsf{T}^2}$; hence if there are x dynes in a 1bl.,

$$x\frac{(1 \text{ gm.}) \times (1 \text{ cm.})}{(1 \text{ sec.})^2} = \frac{(1 \text{ lb.}) \times (1 \text{ ft.})}{(1 \text{ sec.})^2};$$

$$\therefore \; x = \frac{1 \text{ lb.}}{1 \text{ gm.}} \times \frac{1 \text{ ft.}}{1 \text{ cm.}} = 453{\cdot}6 \times 30{\cdot}48$$

$$= 13{,}820.$$

If one or both of the quantities are given in gravitational units, care must be taken to express them in absolute units before the transformation is made.

Example. Find the number (x) of ergs in a foot-pound.

$$1 \text{ ft.-lb.} = 32 \cdot 17 \text{ ft.-lbls.}$$

and the dimensions of work are $\frac{ML^2}{T^2}$;

$$\therefore\ x\frac{(1 \text{ gm.}) \times (1 \text{ cm.})^2}{(1 \text{ sec.})^2} = 32 \cdot 17 \times \frac{(1 \text{ lb.}) \times (1 \text{ ft.})^2}{(1 \text{ sec.})^2},$$

$$x = 32 \cdot 17 \times \frac{1 \text{ lb.}}{1 \text{ gm.}} \times \left(\frac{1 \text{ ft.}}{1 \text{ cm.}}\right)^2$$

$$= 32 \cdot 17 \times 453 \cdot 6 \times (30 \cdot 48)^2$$

$$= 1 \cdot 356 \times 10^7.$$

Example. Find in this way the number of

(1) ergs in a ft.-lbl.;

(2) watts in a horse-power (a watt $= 10^7$ ergs per sec.);

(3) horse-power in a watt.

EXAMPLES.

1. A tramcar starts from rest, and the velocities at intervals of 5 secs. are given as follows:

Time in secs. -	0	5	10	15	20	25	30
Velocity in miles/hr. -	0	8·1	11·8	14·6	16·3	17·7	19·0

Find graphically the distance travelled in yards in the above time, also, if the car weighs 8 tons, find the effective pull exerted on the car at the end of 20 secs.

2. An engine of 30 tons exerting a force of $1\frac{1}{2}$ tons-wt. draws three trucks of 10 tons each. If the resistance is 15 lbs.-weight per ton, find the acceleration of the train and the tension in each coupling.

3. A hailstone acted on by a resistance proportional to the square of the velocity has a terminal velocity of 15 ft./sec. What will its acceleration be when moving with a velocity of 5 ft./sec, and what will the velocity be when the acceleration is $g/2$?

4. If an athlete weighing 160 lbs. runs 100 yds. in 10 secs., calculate the average horse-power he exerts if he takes a stride of 5 ft. and during each stride he raises his centre of mass 2 inches, air resistance being neglected.

5. A girl weighing 100 lbs. skips 40 times in half a minute, raising her centre of mass 4 inches at each jump. Find her average rate of work.

6. A mass of m lbs. is moving with a velocity of u ft./sec., and is brought to rest by a constant resistance in t secs. Find the work done against the force in ft.-lbs. and its initial rate of work in horse-power.

7. Find the maximum horse-power at which a locomotive works in bringing a train of 300 tons weight to a velocity of 30 miles/hr. from rest in 90 secs., assuming the propelling force constant and the resistance 10 lbs.-wt. per ton.

8. A train running on the level at full speed of 40 miles/hr. slips a carriage weighing 20 tons, and the full speed increases to 45 miles/hr. Find the horse-power and the mass of the train if the resistance due to friction is 15 lbs.-wt. per ton.

9. The resistance to a train of M tons at any velocity is given to differ from the resistance when it is just moving by a quantity proportional to the square of the velocity. The resistance at the lowest velocity is 5 lbs.-wt. per ton, and at 30 miles/hr. is 10 lbs.-wt. per ton; what is the total resistance in tons weight at v ft./sec.?

If $M = 100$ tons, and there is a constant driving force of 2 tons wt., find the maximum speed and the horse-power expended at that speed.

10. By means of a chain, an engine working always at horse-power H is raising a weight of M lbs. vertically. Find

(1) the maximum velocity v attainable by the weight;

(2) the tension of the chain and the acceleration when the velocity is $v'(<v)$.

11. If the mass of a train is M tons, the engine works at horse-power H, and the resistance $a+bv^2$ lbs.-wt. per ton, where v is given in miles per hour and a and b are constants, find the acceleration when the velocity is v miles per hour.

12. A train of 200 tons weight is drawn by an engine of 300 H.P. If the resistance is $12+0{\cdot}005v^2$ lbs.-wt. per ton, where v is the velocity in ft./sec., shew that the maximum velocity is given by

$$v^3 + 2400v - 165{,}000 = 0.$$

Shew that the maximum velocity is nearly 40 ft./sec., but if the resistance were constant (12 lbs.-wt. per ton) at all velocities, the maximum velocity would be about 68 ft./sec.

13 The resistance to a train of 200 tons wt. is $a+bv^2$ lbs.-wt. per ton, where v is given in ft./sec., and it is found that the horse-power developed is 240 when the train is travelling at a constant speed of 30 miles/hr., and is 426 at 45 miles/hr. Find a and b.

14. If 1 lb. of coal on being burnt gives out 10 million foot-pounds of energy, how much coal will be consumed per hour by an engine working at 200 horse-power if only one-tenth of the energy derived from the coal is utilized?

15. If 1 lb. of coal on being burnt gives out 10 million foot-lbs. of energy, and an engine working at 1200 H.P. burns 1 ton of coal per hour, find the ratio of the work done by the engine to the energy supplied (that is, the efficiency of the engine).

16. An engine of 5 H.P. is used for lifting a weight of $\frac{1}{2}$ ton. If 40 per cent. of the work done by the engine is wasted, what is the maximum velocity attainable by the weight?

If the engine is working at the same rate, and the velocity of the weight is 1 ft./sec. at a given instant, find the force exerted on the weight then, and its acceleration.

17. A mass of M tons is to be raised and brought to rest at a height of h feet above its former position by a rope whose tension is limited to M′ tons wt.

Shew that the shortest time in which the operation can be performed is

$$\sqrt{\frac{M'}{M'-M}\cdot\frac{h}{16}} \text{ secs.,}$$

and that the rope exerts the force M′ tons wt. for

$$\sqrt{\frac{M^2}{M'(M'-M)}\,\frac{h}{16}} \text{ secs.}$$

18. Prove that the shortest time from rest to rest in which a chain which can bear a stress of 5 cwt. can lift a weight of 3 cwt. a vertical distance of 40 ft. is $2\frac{1}{2}$ secs., and shew that the engine driving the chain must be capable of working at 32·6 H.P.

19. A man falls down a lift well on to the top of a lift which is descending with uniform velocity v, and is at a distance h below him when he begins to fall. Shew that, as far as the shock is concerned, supposing it measured by the change of momentum, his fall is equivalent to a fall on a fixed object through a distance

$$h+\frac{v^2}{2g}.$$

20. After a body of mass m lbs. has fallen through h feet, it is required to bring it to rest by a force not exceeding P lbs.-wt. Shew that it must be exerted through a distance $\frac{h}{P/m-1}$ at least.

21. A shot of 56 lbs. is discharged from a gun of 4 tons with velocity 1800 ft./sec. Find the charge of powder required, supposing that the explosion of 1 lb. of powder releases 135,000 ft.-lbs. of energy.

22. A gram of powder contains 45 kgm.-metres of energy; find the weight of the charge necessary to produce a velocity of 400 metres per sec. in a projectile of 200 kgms., neglecting rotation of the shot and recoil of the gun.

23. A weight of 46 oz. is dropped from a height of 2 ft. 3 in. on the head of a vertical nail weighing 2 oz. already driven some distance into a board. Shew that if the nail is driven a further distance of $\frac{1}{2}$ inch, the resistance of the board, supposed constant, is approximately 152 lbs. wt.

24. A shell of mass M travelling with velocity V suddenly breaks into two fragments m and $M-m$, which continue to travel in the same line. If the velocity of m is v, find that of the other fragment, and the amount of energy generated by the explosion.

25. A 100 lb. shell travelling at 1500 ft./sec. bursts into two equal portions which continue to travel in the same line. If 200 ft.-tons of energy are generated by the explosion, find the subsequent velocities.

26. Find the energy in horse-power-hours required to propel a shot of 1 cwt. with a velocity of 2000 ft./sec.

27. A shot of mass m lbs. is fired horizontally from a gun of mass M lbs., and the kinetic energy due to the explosion is E ft.-lbs. Find the velocity of the shot and gun, the latter being supposed free to recoil.

28. A gun fires a shot weighing 1400 lbs. and possessing energy of 64,000 ft.-tons when it issues from the muzzle.
What is the muzzle velocity of the shot?

29. A gun of mass M lbs. firing a shot of mass m lbs. recoils with velocity V ft./sec. Shew that if the mass of the shot is increased to $2m$, the kinetic energy of the explosion remaining the same, the velocity of recoil becomes

$$V\sqrt{\frac{2(M+m)}{M+2m}}.$$

30. A shell travelling with velocity V bursts into two pieces of masses m_1, m_2, which continue to travel in the same line. If the kinetic energy produced by this explosion is E, find the velocities immediately after the explosion. If the shell breaks into equal masses, and the kinetic energy generated is equal to the kinetic energy of the shell just before bursting, shew that one-half is brought to rest and the other moves with velocity $2V$.

31. A shot of mass m is fired with velocity V point blank at a target of thickness h and mass M, and emerges with velocity v. If the target is free to move, find its velocity, and the time taken by the shot to traverse it, supposing the resistance uniform, and that there is no impulse on striking the target.

32. A bullet of mass m is fired into a block of wood of mass M which is free to move on a smooth horizontal table, and penetrates it to a depth a.

Shew that at the instant when the bullet comes to rest relatively to the block, the block has moved a distance $ma/(M+m)$, the stress between the bullet and the block being assumed constant as long as there is relative motion.

33. A train consists of an engine of mass M tons, and two carriages each of mass m tons. Initially the train is at rest, and the buffers are in contact, but when the coupling chains are tight the buffers are a feet apart. If the engine exerts a constant force of F tons-wt. and friction is neglected, prove that the velocity with which the second carriage starts is given by

$$V^2 = 2Fga(2M+m)/(M+2m)^2.$$

34. Rain falls uniformly so that 5 inches fall in 12 hrs. Find the pressure on the ground at any instant produced by the falling rain in tons-wt. per acre, assuming the drops indefinitely small and the terminal velocity to be 10 ft./sec.

35. A target of 4 sq. metres in area is struck perpendicularly every second by 50 bullets, each weighing 20 gms. and travelling at a velocity of 500 metres per sec. Supposing the bullets not to rebound, find the steady pressure on the target that would produce the same average stress on the supports.

If the bullets rebound with a velocity of 50 metres per sec., how would the equivalent pressure be affected ?

36. A mass of 10 lbs. is at relative rest on the rough horizontal floor of a car moving at the instant with velocity 40 ft./sec. and with retardation 5 ft./sec^2. Find the friction between the mass and the car, and infer a lower limit for the coefficient of friction between them.

If the coefficient of friction is 0·3 and, when moving at the same velocity as before, the retardation is suddenly increased to 10 ft./sec^2., and continues so until the carriage is brought to rest, prove that relative motion ensues, and find how far the mass moves in the carriage before it is finally at rest.

37. A man descends from a balloon by means of a parachute. If the resistance to the parachute is $2v^2$ lbs. wt. when the velocity is v ft./sec., and the parachute and man together weigh 400 lbs., find the maximum velocity attainable. Draw a graph to give $\frac{1}{f}$ in terms of v from $v=0$ to $v=14{\cdot}1$. Deduce from it the time taken to get up a velocity of 10 ft./sec.

Compare the latter time with the time taken when no resistance is acting.

38. The weight of a train is 200 tons, the part of the weight of the engine supported by the driving wheels is 25 tons, and the coefficient of friction between driving wheels and rails is 0·18. Prove that at the end of 1 minute after starting on the level the velocity cannot be as great as $29\frac{5}{11}$ miles/hr.

39. Two masses m_1, m_2 are free to move in a horizontal line. They are projected towards each other with velocities u, v, and there is a constant force F between them opposing the relative motion. Find the change of distance between them when the relative motion ceases, and the common velocity they then have.

40. A hanging weight of 2 lbs. drags a weight of 6 lbs. along a table a distance of 6 ft. 6 in. in 4 secs. If the weights are changed to 5 and 13 lbs. respectively, the weights move 9 ft. in 3 secs. Find the coefficient of friction and the value of g.

41. A weight Q hanging over the edge of a smooth table drags P a distance of a ft. on the table in t secs. If P hung and Q were on the table they would move $2a$ ft. in t secs. Shew $P=2Q$, and compare the tensions in the two cases.

42. Shew that when P and Q are connected by a string over the edge of a smooth table the tension is the same whether P hangs and Q is on the table, or Q hangs and P is on the table.

43. A weight of 2 lbs. hanging over the edge of a smooth table is attached by two strings to two weights of 1 and 3 lbs. on the table, the strings lying in the same vertical plane perpendicular to the edge of the table. Find the tension of each string and the acceleration of the weights.

44. A particle of mass m is attached by an inelastic string of length l to a particle of mass m'. From the point where m' lies on the ground m is projected upwards with velocity $v > \sqrt{2gl}$. Find the height to which each rises, and the velocity with which each reaches the ground again.

45. A string passes over two smooth pegs close together, and hangs in a loop between them. A mass m is tied to one end of the string and a mass M is slung on the loop (so that the string runs through a smooth ring on M). The other end of the string is pulled with an acceleration f. Find the accelerations of the weights and the tension of the string.

If M is tied to the string, find the accelerations of the masses and the tensions of the two parts of the string.

46. A string passing over a smooth peg carries a mass m at one end and a frictionless pulley of mass M at the other. Over the latter another string passes, one end of which is fixed and the other carries a weight m'. Find the accelerations of the masses, and tensions of the strings.

47. Two men of masses m, m' sitting on a smooth table hold the ends of a stretched rope. If at a certain instant the first is pulling the rope through his hands at a rate v ft./sec. and acceleration f, find the motion of the men, and the tension of the rope.

48. A mass of 6 lbs. can move on a smooth horizontal table. Two strings attached to it on opposite sides pass over pulleys at the two ends of the table, and weights of 8 and 4 lbs. hang from them respectively. Find the accelerations and tensions of the two strings.

If the 8 lbs. mass, after moving for 4 ft., comes to rest by striking the ground, how much further will the other masses move before coming to instantaneous rest?

49. Solve the same question as No. 48, supposing the table to be rough with coefficient of friction 0·25 between the weight and the table.

50. If there are n portions of the string between the blocks in a system of pulleys of the second order, and a mass m is attached to the end of the string and $M(>nm)$ to the block, find the tension and accelerations, assuming the block weightless.

If M strikes the ground when moving with velocity v and remains at rest, find how long m will move before the string is stretched again, and find the impulsive tension in the string then, and the velocity of M immediately afterwards.

51. In the system as in Ex. 50, if $M < nm$ and M moves upwards for t secs., and at the end of that time a second mass M attached to the first by a string is suddenly brought into motion, find

(1) the accelerations and tension before and after the second M begins to move;

(2) the velocities just before and after the change;

(3) the impulsive tension in each string.

52. A mass M of 5 lbs. hanging vertically drags a mass m of 8 lbs. along a rough horizontal table. A third mass m' of 4 lbs. is connected to m by a string 4 ft. long, and m and m' are held initially close to one another. If the coefficients of friction between the masses and the table are each 0·5, find

(1) the velocities just before and after m' begins to move;

(2) the impulses in the strings at the instant m' begins to move;

(3) the distance m' moves before coming to rest.

53. On one end of a string of an Atwood machine are weights of 120 and 20 gms. and on the other 130 gms. After moving from rest for 3 secs., the descending weight passes through a ring and the 20 gms. is caught off. How much further will the weights go before first coming to rest, and after what time will the first pass through the ring again?

54. The weights on an Atwood machine are 170 and 150 gms. When they have been moving from rest through 4 ft. the ascending weight passes through a ring and catches up a weight of 40 gms.

This weight is dropped again when the weight passes downwards through the ring, and so on. Find the time between the n^{th} time the weight is caught up and the $(n+1)^{th}$, and the whole distance (up and down) travelled in the interval.

Find the whole time and whole distance travelled until they finally come to rest.

55. The masses on an Atwood machine are P and Q $(P > Q)$. They move a feet from rest, and then P strikes the ground and remains at rest. Afterwards P starts to move upwards and there is an impulsive tension in the string, and later P strikes the ground again, and so on. Find

(1) the time between the n^{th} and $(n+1)^{th}$ impacts that Q is moving by itself, and the time that both P and Q are moving;

(2) the impulse in the string in the same interval;

(3) the total time before the system comes to rest.

56. Weights of 4 lbs. and 10 lbs. hang by strings respectively, passing round a wheel and axle, the former of diameter 12 inches and latter 3 inches. If friction is neglected, and also the mass of the wheel and axle, prove, by writing down the kinetic and potential energy in any position, that the acceleration of the 4 lb. wt. is $\frac{1}{3}\frac{3}{7}g$.

57. A weight of 5 lbs. hanging over the edge of a table draws along a weight of 10 lbs. on the table. If the coefficient of friction between the table and the weight is 0·3, find the acceleration and tension.

If the string breaks after the masses have been moving for $2\frac{1}{2}$ secs., find how far the 10 lb. mass moves altogether before coming to rest.

58. Three masses are such that $M'+m > M > M'$. M is hung at one end of a string passing over a smooth peg and $M'+m$ at the other in such a way that m' can be detached as in Atwood's machine when $M'+m$ is moving down, and picked up again when M' is moving up. If the masses move a distance h from rest before m is caught off, find the next two positions in which the masses are instantaneously at rest.

59. A particle of mass m moves in a straight line, and is only acted on by a series of impulses at equal intervals of time τ, the n^{th} impulse being $-k\tau v_n$ where k is a constant and v_n is the velocity just before the impulse. Prove that if v_1 is the initial velocity,

$$v_n = v_1\left(1 - \frac{k\tau}{m}\right)^{n-1},$$

and

$$x_n = \frac{v_1 m}{k}\left\{1 - \left(1 - \frac{k\tau}{m}\right)^n\right\},$$

where x_n is the distance described to the time of the n^{th} impulse.

Deduce that if a particle is moving in a straight line under a resistance kv,

$$v = v_1 e^{-\frac{kt}{m}},$$

$$x = \frac{v_1 m}{k}\left(1 - e^{-\frac{kt}{m}}\right),$$

where v and x are the velocity and distance described at time t.

60. A number of masses are attached to a light string AB suspended at A over a table which is at a distance h below the lowest mass B. The end A being suddenly released, the masses are heard to hit the table at equal intervals of time τ. Find the distances apart of the successive masses.

CHAPTER IV.

FORCES IN TWO DIMENSIONS.

61. So far we have been dealing with motion of a point whose position was fixed relatively to a point of reference by a single measurement along a line passing through the point of reference. Now we have to consider the next case in order of difficulty, where we have two lines of reference Ox, Oy meeting in a point O, and the moving point P always

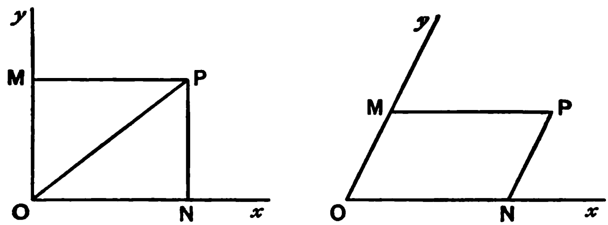


FIG. 39.

lies in the plane determined by Ox and Oy. The plane of reference may be moving in any way. Thus the point P may be a point describing any path on a plane sheet of card on which straight lines Ox, Oy are drawn, while the piece of card itself may be moving in any way. The latter motion will be neglected for the present, and it is only the relative motion in the plane that we have to deal with. It is generally most convenient, though not necessary, to take the lines Ox, Oy at right angles, and we shall assume them to be so unless otherwise stated.

The position of P in the plane is determined in one or other of two ways:

(1) It may be determined by the lengths of the lines PM,

PN drawn parallel to the axes Ox, Oy and meeting Oy and Ox; these are denoted by x, y, thus:

$$\text{PM} = x,$$
$$\text{PN} = y.$$

(2) It may be determined by the length OP and the angle POx; these we denote thus:

$$\text{OP} = r,$$
$$\text{PO}x = \theta.$$

Obviously, when Ox, Oy are at right angles,

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

62. In the second method the position is determined by the length and direction of a straight line, and if we want to represent the position on a sheet of paper we can do so by taking a line O$'x'$ on the paper and drawing a line O$'$P$'$, representing OP on a certain chosen scale and making an angle P$'$O$'x'$ equal to the angle POx. Any quantity which involves magnitude and direction in this way, and is hence capable of being represented by a straight line of a certain length and drawn in a certain direction, is called a vector quantity. Thus a velocity is a vector, for it is only determined completely when its direction as well as its magnitude is given.

Quantities which do not involve direction, but only magnitude, are called scalars. Of the quantities we have been dealing with it will be found that the following are

Scalar—mass, work and energy, power;

Vector—displacement, velocity, acceleration, force, momentum, impulse.

That kinetic energy ($\frac{1}{2}mv^2$) is a scalar while velocity (v) is a vector is suggested, though not proved, by the fact that kinetic energy cannot be negative, and that if v is changed into $-v$ no change is made in the kinetic energy.

63. In dealing with the rules relating to vectors it will be best to begin with the simplest—the vector denoting the distance of one point from another

Suppose, then, there are three towns O, P, Q, and we are told that P is 3 miles north-east of O and Q 2 miles north of P. We can draw a figure to scale on paper and find the distance of Q from O.

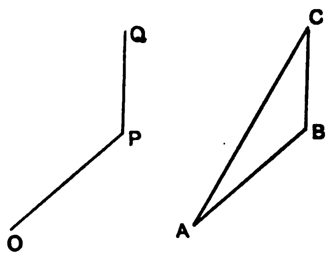


FIG. 40.

We may express the result more fully thus:

If AB *is drawn to represent in magnitude and direction the distance of* P *from* O, *and* BC *to represent on the same scale the distance of* Q *from* P, *then the distance of* Q *from* O *is represented by* AC.

64. Change of Position or Displacement.

We may imagine in Fig. 40 P is a moving point and Q a second position of P. Then the first position as measured from O is represented by OP as before (or AB); the second is represented by OQ (or AC), and PQ (or BC) may be called the change in position or displacement.

Hence, if the position of a point relatively to O is represented at one time by AB and at another by AC, the change of position is represented by BC.

65. Composition of Displacements.

The following law will now hold for displacements,—

If AB *represents the displacement in any time of* P *relative to* O, *and* BC *represents the displacement of* Q *relative to* P *in the same time, then the displacement of* Q *relative to* O *is represented* by AC.

For, let O, P, Q represent the original positions of the points, and suppose P displaced to P′, PP′ (or AB) is the displacement of P relative to O. If Q were not displaced *relatively to* P, its distance from P′ after the displacement would be the same in direction and magnitude as it was from P before the displacement, and the displacement of P have brought Q to Q_1, where $P'Q_1$ is equal and parallel to PQ. But now Q is displaced relatively to P′ so that it comes to Q′, and Q_1Q' is the displacement relatively to P. BC is therefore drawn to represent Q_1Q'. Now the total displacement

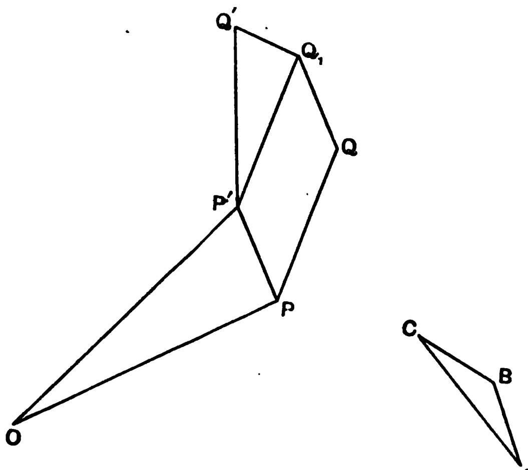


Fig. 41.

of Q relative to O is represented by QQ′, and this will be seen at once to be the same as AC for AB, BD are equal and parallel to QQ_1, Q_1Q', and the law for the composition of displacements is therefore proved.

66. Composition of Velocities.

Velocities in like manner require direction as well as magnitude to completely determine them. From the composition of displacements we can deduce at once the composition of velocities which should be stated in the following form:

If AB *represents in direction and magnitude the velocity of* P *relative to* O, *and* BC *represents the velocity of* Q *relative to* P, *then the velocity of* Q *relative to* O *is represented by* AC.

For it is only necessary to think of the displacements as taking place in a time *t*, and the average velocities during the interval of P relative to O, Q relative to P, and Q relative to O, are

$$\frac{AB}{t}, \frac{BC}{t}, \frac{AC}{t} \text{ respectively,}$$

and therefore the lines which represent the displacements, represent correctly, on some definite scale, the average velocities. Hence the parallelogram law applies to the average velocities, and therefore also, by thinking of the interval as reduced indefinitely, to the velocities at an instant.

Example 1. A train is travelling at 15 miles/hr. to the east, and drops of rain appear to a passenger to make an angle 30° with the vertical towards the west. When the velocity of the train is increased to 30 miles an hour, the drops appear to make an angle 60° with the vertical. Find the true velocity of the drops in direction and magnitude.

Graphical Solution.

If OA represents the velocity of 15 miles/hr. (or 22 ft./sec.) of the train, and OB represents the true velocity of the drops, AB represents the velocity of the drops relative to the train, and consequently we have

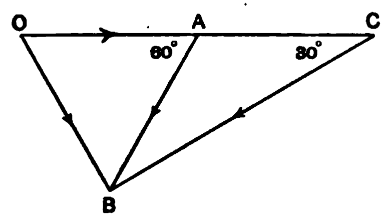


FIG. 42.

OA = 22 ft./sec.,

∠OAB = 60°,

but B is as yet not fixed, but if OC represents the velocity of the train in the second case,

OCB = 30°;

and since OC = 2OA,

we have ∠ABC = 30°;

∴ AB = AC = OA;

∴ OB = OA,

and ∠AOB = 60°;

∴ the true velocity of the drops is 22 ft./sec., making 30° with the vertical and towards the east.

2. What velocity would the train have if the drops appear to come down vertically?

66a. Change of Velocity.

In Fig. 41 OP, OP′ represent the positions of a point relative to a given point at two instants, and PP′ represents the change of position. In exactly the same way, if OP, OP′ represent velocities relative to a given point at two instants t secs. apart, PP′ must be regarded as representing the change of velocity in the interval, and $\frac{PP'}{t}$ will represent both in direction and magnitude the average acceleration during the interval, from which the acceleration at any instant can be deduced by making the interval t indefinitely short.

This may be regarded as a definition of change of velocity when the direction of the velocity is changing; and the importance of regarding change of velocity in this way may be seen if we think of the relation being impulse and momentum. Impulse was equal to change of momentum in motion in a straight line, and the same thing will still be true in any motion if we understand change of momentum correctly.

If a particle is moving at one instant with a velocity represented by OP, and a blow is given to it which changes the magnitude and direction of the velocity, so that it becomes OP′, the change in velocity is PP′, and the change in momentum is $m \times$ PP′, and the impulse given to the body to produce this change of momentum will be represented by $m \times$ PP′ in magnitude and must be parallel to PP′.

In fact, if change of velocity is understood in this way and acceleration deduced from it as above, all the statements made in the last chapter about the relations between mass, acceleration, force, momentum and impulse will still hold for a particle moving in any way.

Example. A mass of 2 lbs. is moving with a velocity of 10 ft./sec. to the north, and is given a blow which changes its velocity to 10 ft./sec. to the east. Show that the impulse must be 1·7 lb.-wt.-secs. to the south-east.

67. Composition of Accelerations.

In exactly the same way as the composition of velocities was deduced from that of displacements, so the composition of accelerations can be deduced from that of velocities, and the statement of the composition of accelerations is the same as that of velocities with the substitution of the word acceleration for velocity throughout, thus :

If AB *represents in direction and magnitude the acceleration of* P *relative to* O, *and* BC *represents the acceleration of* Q *relative to* P, *then the acceleration of* Q *relative to* O *is represented by* AC.

This is often expressed by saying that the point Q has two independent accelerations AB, BC (one on account of the motion of P, the other on account of the motion relative to P), and that these two are equivalent to a single acceleration AC, and the composition of velocities is often expressed in a similar way.

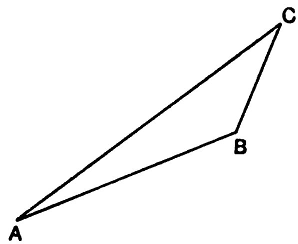


FIG. 43.

AC is called the resultant of the two AB and BC, and AB, BC are called the components of AC. We will discuss these terms more fully in connection with forces.

68. Forces.

When we pass to the case of forces, a new idea comes in, namely, the **Independence of forces**. This may be supposed to be understood in the laws of motion, and may be expressly stated thus :

If two or more forces act on a particle at the same time, each produces an acceleration exactly as if it acted alone.

Further, we must regard it as an experimental result that two accelerations produced by two forces may be compounded into a single acceleration by the same law as given above, and that the particle moves with the acceleration thus determined. The effect of the two forces is therefore to produce a definite acceleration, which again could be regarded as being produced by a single force, and this single force would therefore be equivalent in its action to the two forces acting separately. We are thus led to the idea of replacing two forces by a single one which produces the same effect as the two, and we can prove that this single force can be found from the two original forces by a similar law, which is known as the

Parallelogram of Forces.

If two forces acting on a particle are represented in magnitude and direction by two straight lines AB, AC, *they are equivalent in their effect to a single force represented by* AD, *the diagonal of the parallelogram of which* AB, AC *are adjacent sides.*

To prove this, let AB, AC represent the two forces whose magnitudes are P and Q, so that

$$\frac{P}{Q}=\frac{AB}{AC}.$$

Now, on the principle of the independence of forces, these will produce accelerations, say p, q, where

$$p=\frac{P}{m}, \quad q=\frac{Q}{m},$$

m being the mass of the particle.

If these accelerations are represented by ab, ac it will follow that

$$\frac{ab}{ac}=\frac{p}{q}=\frac{P}{Q}=\frac{AB}{AC},$$

and also the directions of AB, AC are the same as those of ab, ac.

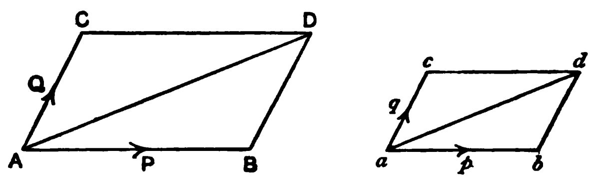


FIG. 44.

Therefore the parallelogram $abdc$ is similar to the parallelogram ABDC, and

$$\frac{AD}{ad}=\frac{AB}{ab}=\frac{P}{p}=m,$$

$$AD=m\,.\,ad,$$

and AD is in the same direction as ad. Now, by the composition of accelerations ad is the acceleration equivalent to the two accelerations p, q, or is the acceleration produced as the result of the two forces. But we see that since

$$AD = m \cdot ad,$$

a single force AD would produce the acceleration ad.

∴ the two forces represented by AB and AC are equivalent in their effect to the single force represented by AD.

69. It will now be unnecessary to prove the parallelogram law for other vectors. The student will find no difficulty in proving it for momenta or impulses. All other vectors which are used in this book will likewise obey the parallelogram law, though there are quantities of which angular displacement is the most important, which require magnitude and direction to express them, but which do not follow the parallelogram law. Such quantities are called vectors by some writers, though others refuse them the name.

70. Resultant and Components.

The force that is equivalent to two given forces is called the *resultant* of the two, and the two are called the *components*. From what has gone before it will be seen that if two forces are given, there is only one possible resultant; but if the resultant is given, it can be replaced by two components in an infinite number of ways.

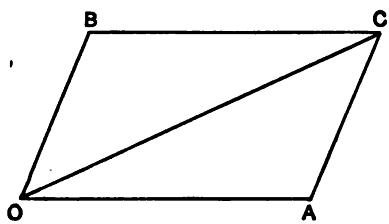


FIG. 45.

Thus if OC represents a force, it is only necessary to draw any two lines through O and parallels through C, and we get components OA, OB together equivalent to OC.

The most important case is where the components are at right angles, the components being then usually called *resolved parts* or *resolutes* of the force.

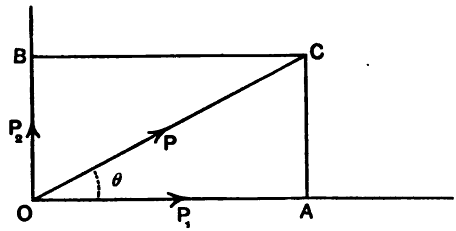


FIG. 46.

If P is the magnitude of the resultant force, and suppose it is required to find the resolved parts when one of them makes an angle θ with the resultant.

Let P_1, P_2 be the resolved parts. Drawing the parallelogram OACB, which is now a rectangle, we have

$$\frac{OA}{OC} = \cos\theta,$$

but OC represents P, and OA represents P_1;

$$\therefore\ P_1 = P\cos\theta, \quad \text{.............................(1)}$$

similarly,

$$P_2 = P\sin\theta, \quad \text{.............................(2)}$$

by squaring and adding (1) and (2), we have

$$P^2 = P_1^2 + P_2^2, \quad \text{...........................(3)}$$

and by dividing (2) by (1),

$$\tan\theta = \frac{P_2}{P_1} \quad \text{...............................(4)}$$

(3) and (4) also follow at once geometrically.

(1) and (2) give the resolved parts in two directions at right angles when the magnitude and direction of the resultant are given, (3) and (4) give the magnitude and direction of the resultant when the resolved parts in two directions at right angles are given.

71. When the components make any angle α with one another, the equations are rather more complicated. Using the other letters as before, we have from the figure:

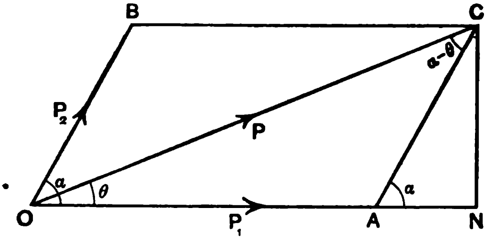


FIG. 47.

$$P^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\alpha \quad \text{.............(5)}$$

$$\tan\theta = \frac{P_2\sin\alpha}{P_1 + P_2\cos\alpha}, \quad \text{.............(6)}$$

$$P_1 = \frac{P\sin(\alpha-\theta)}{\sin\alpha}, \quad \text{.............(7)}$$

$$P_2 = \frac{P\sin\theta}{\sin\alpha}, \quad \text{.............(8)}$$

all following from elementary trigonometry. They should be verified by the student.

Example. Shew that equations (5)-(8) reduce to (1)-(4) if α is a right angle.

72. It is important to observe that the resolved part of a force in any direction is obtained by multiplying the force by the cosine of the angle between it and the given direction.

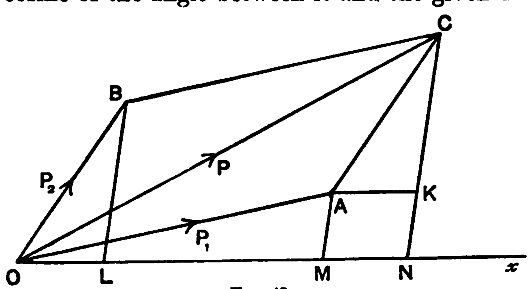


FIG. 48.

The sum of the resolved parts of two forces in a given direction is equal to the resolved part of the resultant in the same direction.

For in the figure OM is the resolved part in the direction Ox of P_1; $MN = AK = OL$ and is the resolved part of P_2; while ON is the resolved part of P in the same direction, and

$$OM + MN = ON.$$

73. More than two Forces.

When we consider the action of more than two forces we can proceed by steps. Thus, if OA, OB, OC represent three forces acting at a point, OA and OB can be combined into a single force OE, and then OE and OC into a single force OF, and evidently OF is equivalent in its action to (or is the resultant of) the original three.

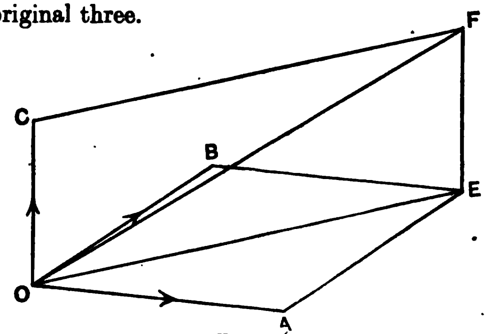


Fig. 49.

But the best way to deal with three or more forces is to extend the theorem of the last article. It will be easy to see that the following result must hold:

The sum of the resolved parts in any direction of a number of forces or other vectors is equal to the resolved part, in the same direction, of their resultant.

74. Application of the foregoing Principles.

In applying the above to examples it is usually most convenient to replace each force by its resolved parts in two chosen directions.

Thus, if we have a number of forces P making an angle α with Ox, Q making an angle β, etc., these forces can be replaced by resolved parts :

$$P \cos \alpha + Q \cos \beta + \text{etc. along } Ox,$$

and $$P \sin \alpha + Q \sin \beta + \text{etc. along } Oy.$$

Under the influence of such forces a particle of mass m will have an acceleration whose components f_1, f_2 along Ox and Oy are given by

$$\Sigma P \cos \alpha = mf_1,$$
$$\Sigma P \sin \alpha = mf_2.$$

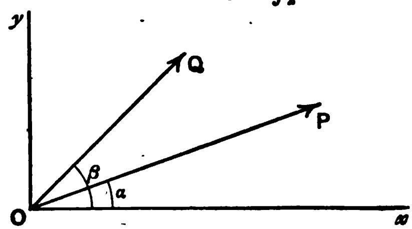


FIG. 50.

Having obtained each component of the acceleration, the components of the velocity and the distances described in each of the two directions are to be obtained, if possible, by the methods of the previous chapters.

In the following examples the particle moves in a straight line, and it is best to choose the directions for resolving the forces, one along the line and the other perpendicular to it.

EXAMPLES.

1. A body moves along a line of greatest slope of a smooth inclined plane under the action of gravity alone. Find the acceleration, and also the pressure on the plane.

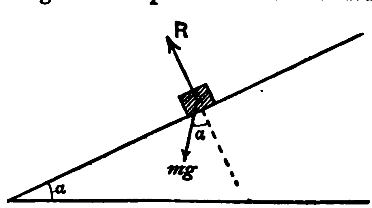


FIG. 51.

In this case the only forces are the weight mg acting vertically and the pressure R of the plane, which is perpendicular to the plane, since this is smooth. If the plane makes an angle α with the horizontal, the force mg can be resolved into $mg\cos\alpha$ perpendicular to the plane, and $mg\sin\alpha$ down the plane.

Hence, if f = acceleration down the plane,

$$mf = mg\sin\alpha;$$

$$\therefore\ f = g\sin\alpha.$$

Also, taking the forces perpendicular to the plane, since there is no motion and therefore no acceleration in this direction,

$$R - mg\cos\alpha = 0;$$

$$\therefore\ R = mg\cos\alpha.$$

2. Suppose in the previous example the plane is not smooth, and the coefficient of friction between the plane and particle is μ, find the motion.

The result will depend on whether the particle is moving up or down the plane, for in one case the friction is downwards and in the other upwards.

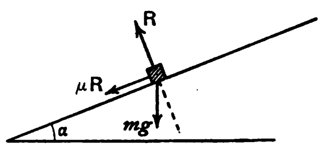


FIG. 52. (Motion up the plane.)

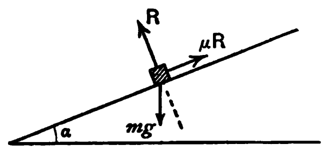


FIG. 53. (Motion down the plane.)

The forces are consequently as marked in the Figs. 52 and 53.

In 52 let f = acceleration *down* the plane, we have resolving along the plane,

$$\mu R + mg\sin\alpha = mf;$$

perpendicular to the plane,

$$R - mg\cos\alpha = 0;$$

$$\therefore\ mf = \mu mg\cos\alpha + mg\sin\alpha;$$

$$\therefore\ f = g\sin\alpha + \mu g\cos\alpha.$$

In Fig. 53 in the same way the acceleration down the plane will be

$$g\sin\alpha - \mu g\cos\alpha.$$

3. A train of 160 tons weight is drawn from rest down an incline of 1 in 200 against a frictional resistance of 11·4 lbs.-wt. per ton, and gets up a velocity of 36 miles an hour in 2 mins. If the force exerted by the engine is constant, find the force and also the greatest horse-power developed. Find also what horse-power would be developed in drawing the same train up the incline with a constant velocity of 36 miles per hour.

An incline is said to be 1 in 200 when there is a rise of 1 ft. vertically for 200 travelled along the incline, thus the sine of the angle of inclination is $\frac{1}{200}$.

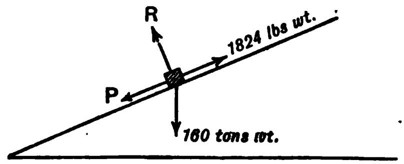


FIG. 54.

The forces acting in the first case are:

(1) the weight 160 tons or $160 \times 2240 \times 32$ lbls.,

(2) the friction $160 \times 11{\cdot}4$ lbs. wt. $= 1824 \times 32$ lbls.,

(3) the normal reaction R,

(4) the force P exerted by the engine.

There is a constant acceleration, and the velocity of 36 miles per hour, or $36 \times \frac{44}{30}$ ft. per sec., being generated in 120 secs., the acceleration is

$$36 \times \frac{44}{30} \times \frac{1}{120} = \frac{44}{100} \text{ ft. per sec.}$$

Hence, resolving the forces along the plane,

$$P + 160 \times 2240 \times \frac{1}{200} \times 32 - 1824 \times 32 = 160 \times 2240 \times \frac{44}{100}.$$

$$P = 16 \times 224 \times 44 + 1824 \times 32 - 1792 \times 32 \text{ lbls.}$$

$$= \frac{16 \times 224 \times 44}{32} + 1824 - 1792 \text{ lbs. wt.}$$

$$= 4928 + 1824 - 1792$$

$$= 4960 \text{ lbs. wt.}$$

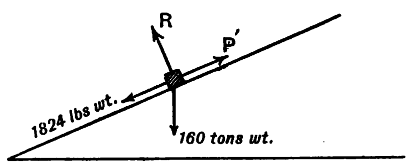


FIG. 55.

The horse-power will be greatest when the velocity is greatest, and then equals

$$4960 \times 36 \times \frac{44}{30} \times \frac{1}{550} = 476 \text{ nearly.}$$

If the train were travelling up the plane with constant velocity, the forces would be similar to the preceding, but if P' is the force up the plane,

$$P' = 160 \times 2240 \times \frac{1}{200} + 1824 \text{ lbs. wt.}$$

$$= 1792 + 1824 = 3616 \text{ lbs. wt.}$$

$$\therefore \text{H.P.} = 3616 \times \frac{36 \times 44}{30} \times \frac{1}{550} = 347 \text{ nearly.}$$

4. A body is projected up a smooth plane of inclination 30° with velocity 16 ft./sec., how far up the plane will it go?

5. A body slides 12 ft. from rest down a smooth incline of 1 in 5. Find the time taken and the final velocity. (The incline rises 1 ft. in going 5 ft. along the plane.)

6. A train runs from rest without steam for half a mile down an incline of 1 in 112. If the friction is 12 lbs.-wt. per ton, find the velocity generated and the time taken.

7. A body is projected up an incline of 30° with velocity 20 ft./sec. If the friction is always $\frac{1}{4}$ of the weight of the body, find the distance the body will go up the plane, the velocity with which it returns to the starting point, and the time of ascent and descent.

8. A body is projected up an incline of 20° with a velocity of 30 ft./sec. If the coefficient of friction between the body and the plane is 0·25, find the distance it goes up the plane and the velocity with which it returns to the starting point.

9. Particles slide down a series of smooth wires starting at the same point and ending in the same horizontal line. Shew that the time taken is proportional to the length of the wire, and the velocity generated is the same for all.

10. Particles slide down a series of smooth wires starting at the same point and ending in the same vertical line. Find an expression for the time taken and shew that it is least for the chord which makes an angle 45° with the vertical.

11. A train runs from rest without steam down an incline of 1 in 112 for half a mile and then comes to a level line. How far will it run on the level if the resistance is 15 lbs.-wt. per ton throughout?

12. A train of 180 tons is travelling on an incline of 1 in 120 at 40 miles/hr. What force will be required to stop it in 200 yds.,

(1) when the motion is uphill,

(2) when it is downhill?

13. A train of 90 tons is drawn up an incline of 1 in 150 by an engine exerting a constant force of 2 tons weight, the frictional resistance being 15 lbs.-wt. per ton. After 3 minutes from rest steam is shut off. Find

(1) the maximum velocity,
(2) the time to come to rest,
(3) the total distance travelled.

14. A train runs from rest for a mile down an incline of 1 in 140, and then comes to an upward incline also of 1 in 140. How far will it run up this incline if the friction is 12 lbs.-wt. per ton, steam being shut off the whole time?

15. A train of 240 tons weight is drawn up an incline of 1 in 100 at the rate of 20 miles/hr. If the friction is 11·5 lbs.-wt. per ton, what is the horse-power of the engine?

16. What is the maximum velocity with which a train of 180 tons can be drawn up an incline of 1 in 160 if the horse-power of the engine is 225 and the friction 12·5 lbs.-wt. per ton?

17. Compare the maximum velocity up and down an incline of 1 in 200 when the friction is 14 lbs.-wt. per ton and the horse-power is given.

18. If the maximum velocity of a train up an incline α is v, and up an incline α' is v', prove that

$$\frac{v'}{v}=\frac{1+n\sin\alpha}{1+n\sin\alpha'},$$

the friction being $\frac{1}{n}$ of the weight.

19. If the maximum velocity for a train of 200 tons on the horizontal is 50 miles/hr., and up an incline of 1 in 80 is 15 miles an hour, find the horse-power and the resistance.

20. A train of 200 tons wt. gets up a velocity of 30 miles per hour in 2 mins. from rest travelling down an incline of 1 in 280, and friction is 13 lbs.-wt. per ton. If the force exerted is constant, find its magnitude. What would be the least horse-power capable of obtaining the above result?

21. If an engine can draw a train of 200 tons weight at a maximum velocity of 35 miles an hour on the level, at what rate would it draw it up an incline of 1 in 80, friction being 12 lbs.-wt. per ton in each case?

22. A train of m tons weight gets up a velocity v miles/hr. on an incline α in t secs. from starting. If the friction is n lbs.-wt. per ton, find the force exerted by the engine (supposed constant) and the horse-power developed at the end of the t secs.

23. A train running at 40 miles/hr. comes to the foot of an incline of 1 in 160 with no steam on. If the resistance is 11 lbs.-wt. per ton, find the distance it will ascend the incline, the velocity with which it returns to the foot of the plane, and the time of ascent and descent.

24. A mass of 20 lbs. is drawn up a smooth plane of inclination 20° by a mass of 15 lbs. hanging vertically and attached to the first by a string passing over the pulley at the top of the plane. Find the acceleration.

25. A weight P hanging vertically draws Q up a smooth incline of 30° in half the time that Q hanging vertically would draw P up it. Shew that

$$P = \tfrac{3}{2}Q.$$

26. A mass of 6 lbs. hanging vertically draws a mass of 9 lbs. up a rough plane of inclination 1 in 3. Find the acceleration if the coefficient of friction is 0·3.

27. A mass of 10 lbs. is drawn up an inclined plane whose height and base are in the ratio 3 to 4 by a weight of 10 lbs. hanging vertically. If the mass moves 5 ft. in $2\frac{1}{2}$ secs. from rest, find the coefficient of friction.

28. A mass of 10 lbs. is drawn up a rough incline of 30° by a weight of 9 lbs. hanging vertically. The coefficient of friction is 0·4, and after the bodies have been moving for 3 secs. the string breaks. Find

(1) the total distance travelled up the plane,

(2) the time till the body comes to rest,

(3) the time it takes to descend the plane again to the starting point.

29. If P hanging vertically draws Q up a smooth incline of 15° with acceleration $g/3$, what will be the acceleration if the inclination is increased to 25°?

30. Two masses 10 and 5 lbs. wt. respectively on smooth inclines of 30° and 45° placed back to back are connected by a string passing over a pulley at the top of the planes. Find the acceleration.

31. Two masses, 10 and 20 lbs. wt., are placed on two smooth planes back to back, the sines of whose inclinations are 0·6 and 0·8 respectively. They are connected by a string passing over the intersection of the planes and perpendicular to it. Find the acceleration of the weights, and tension of the string.

Find also the acceleration if there were friction and the coefficients of friction between the planes and weights were respectively 0·3 and 0·4.

32. Two particles are connected by a string passing over a pulley at the top of two smooth inclined planes placed back to back. If the masses are m, n, and the inclinations of the planes α, β, find the acceleration of each.

33. A particle of mass m slides from rest down the smooth face (of inclination α) of a wedge of mass M resting on a smooth horizontal table. Find the motion of the particle and of the wedge.

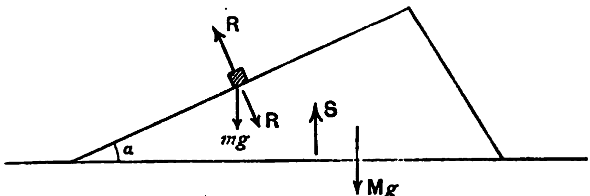


FIG. 56.

The forces on the particle are:

(i) its weight mg vertical,
(ii) the pressure R of the wedge perpendicular to the face.

The forces on the wedge are:

(i) its weight Mg,
(ii) the pressure R of the particle on the wedge,
(iii) the pressure S of the table (vertical because the table is smooth).

Let f = acceleration of the wedge to the right,
f' = " " " particle *relatively to the wedge.*

Then the acceleration of the particle is the resultant of f and f', and its components are $f'-f\cos\alpha$ along the face of the wedge downwards, and $f\sin\alpha$ perpendicular to the face; therefore the equations for the particle are along the face

$$m(f'-f\cos\alpha)=mg\sin\alpha, \quad\text{.....(1)}$$

perpendicular to the face

$$mf\sin\alpha=mg\cos\alpha-R. \quad\text{.....(2)}$$

For the wedge horizontally,

$$Mf=R\sin\alpha. \quad\text{.....(3)}$$

From (1), $$f'-f\cos\alpha=g\sin\alpha. \quad\text{.....(4)}$$

From (2) and (3), eliminating R,

$$f(M+m\sin^2\alpha)=mg\sin\alpha\cos\alpha; \quad\text{.....(5)}$$

$$\therefore f=\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha},$$

$$f'=f\cos\alpha+g\sin\alpha=\frac{M+m}{M+m\sin^2\alpha}\cdot g\sin\alpha.$$

34. A wedge of mass M, whose faces are inclined at angles α, β to the horizontal, moves on a smooth horizontal table. Two particles of masses m, m' move on its smooth faces, being connected by a string passing over a light pulley at the top of the plane, and lying in a plane perpendicular to the line of intersection of the two faces. Find the motion of the particles and wedge.

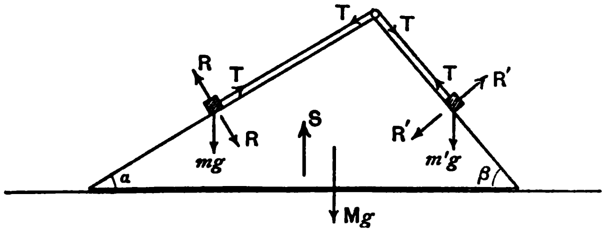


FIG. 57.

The forces acting are as marked in the figure,

(1) on m, mg, R, T;

(2) on m', $m'g$, R', T;

(3) on M, R, R' (the reactions to the above),

T, T (reactions to the former T and T, and acting on the pulley, which may be considered as a part of the wedge. These forces are represented by the arrow-heads close to the pulley).

Also Mg and S.

Let f = acceleration of M to the right,

f' = " " m relatively to the wedge down the first face;

$\therefore$ f' = acceleration of m' relatively to the wedge up the second face.

$\therefore$ the component accelerations of m are

$f' - f\cos\alpha$ along the first face downwards,

$f\sin\alpha$ perpendicular to the first face and towards the wedge; the component accelerations of m' are

$f' - f\cos\beta$ along the second face upwards,

$-f\sin\beta$ perpendicular to the second face towards the wedge.

Hence the equations are:

for m along the face,

$$m(f' - f\cos\alpha) = mg\sin\alpha - \text{T}; \qquad \ldots\ldots(1)$$

perpendicular to the face,

$$mf \sin \alpha = mg \cos \alpha - R; \quad \ldots\ldots (2)$$

for m' along the face,

$$m'(f' - f \cos \beta) = T - m'g \sin \beta; \quad \ldots\ldots (3)$$

perpendicular to the face,

$$-m'f \sin \beta = m'g \cos \beta - R'. \quad \ldots\ldots (4)$$

For M, $\quad Mf = R \sin \alpha - R' \cos \beta + T \cos \beta - T \cos \alpha. \quad \ldots\ldots (5)$

Giving five equations for the five unknowns f, f', R, R′, T, which can consequently all be found.

35. A particle of mass 4 lbs. slides from rest down the smooth face of inclination $\sin^{-1}\frac{3}{5}$ of a wedge of mass 12 lbs. which can slide without friction on a horizontal table. Find the accelerations of the particle and wedge.

75. Work.

We have defined work done by a constant force as the produce of the force into the distance the point of application moves in the direction of the force. This definition is still complete in any case if we understand *the distance the point of application moves* to be the same as *the resolved part, in the direction of the force, of the displacement of the point of application.*

Thus, in the figure, if the point of application moves from A to B while the force keeps its magnitude and direction unchanged, the work is estimated by the product

$$P \cdot AN = P \cdot AB \cos \theta.$$

FIG. 58.

This can also be expressed as the displacement multiplied by the resolved part of the force in the direction of the displacement.

Work, it should be noticed, is a scalar quantity not involving direction in its specification.

76. Case of several Forces.

The work done by the resultant of two forces is the sum of the amounts of work done by each force separately. For if P, Q are two forces acting on a particle and the particle is displaced from A to B, the sum of the amounts of work done by P and Q is

$$AB(P\cos\alpha+Q\cos\beta).$$

where α, β are the angles P and Q make with AB.

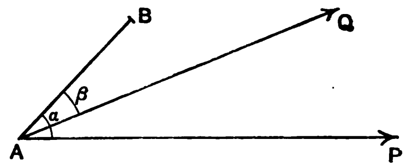


FIG. 59.

But $P\cos\alpha+Q\cos\beta$ is the sum of the resolved parts of P and Q along AB, and therefore equals the resolved part of the resultant along AB (Art. 73);

∴ the above amount of work is equal to the work done by the resultant.

The same result evidently holds for any number of forces.

77. Similarly, if a force remains constant in magnitude

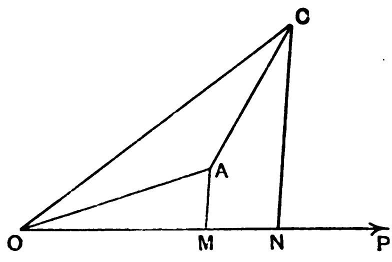


FIG. 60.

and direction while the point of application undergoes two successive displacements, the total work done is the same as if the point of application had undergone a single displacement equal to the resultant of the two.

For in the figure the work done by the force P in the displacement OA is P . OM, and in the displacement AC, P . MN ;

∴ the total work done is P . ON, which is the work done in the resultant displacement OC.

The same result will follow if the point of application undergoes any number of successive displacements. It will still be true if we have an infinitely large number of infinitely small displacements so that the point of application may move in any curve, and we get the following result :

The work done by a force, constant in magnitude and direction, as the point of application moves from one position to another, is independent of the path described by the point of application; or, in other words, depends only on the initial and final positions of the point of application.

In many cases besides that in which the force is constant in magnitude and direction, the work done depends only on the initial and final positions, and in such cases the forces are said to be conservative. In cases where the work done is different for different paths taken, as is usually the case when friction comes into play, the forces are said to be non-conservative.

78. Work done by a varying Force.

We will now examine the work done when the point of application moves in any path, and the force varies both in magnitude and direction.

Suppose the point of application moves from A to B along any given path. It will be necessary to divide this path up into a large number of very short portions, so short that they may be considered straight, and of which $Q_{r-1}Q_r$ is taken as a type. During this short distance the force may be regarded as constant in magnitude and direction, being P_r, making an angle θ_r with $Q_{r-1}Q_r$, then the work done from Q_{r-1} to Q_r is $P_r . Q_{r-1}Q_r \cos\theta_r = P_r s_r \cos\theta_r$, where $s_r = Q_{r-1}Q_r$, and the total work for the whole path $= \Sigma P_r s_r \cos\theta_r$.

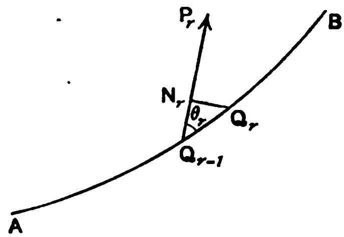


FIG. 61.

This becomes more and more accurate the larger the number of divisions in AB. If the number of parts like $Q_{r-1}Q_r$ is n, the accurate expression for the work is

$$\operatorname*{Lim}_{n \to \infty} \sum_{r=1}^{r=n} P_r s_r \cos\theta_r.$$

There is no difficulty in representing this work graphically. Draw a straight line OC representing the whole length AB, and take it as axis of x. At each point draw an ordinate

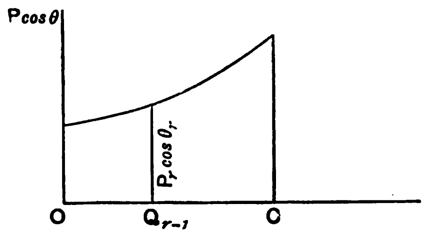


FIG. 62.

representing the resolved part of the force in the direction of the tangent (that is, $P_r \cos\theta_r$) at the corresponding point of the curve AB. We will thus get a graph, the area of which represents the work done between the extreme points.

Example 1. A man of 12 stone weight climbs a hill 220 ft. high in 7 mins. Find the amount of work he does and his average horse-power.

2. If a body just slides down an incline without acceleration, shew that the work done by gravity is equal to the work done against friction.

3. Find the work done in dragging a body of 40 kgms. 5 metres along an incline if the vertical height travelled is 3 metres and the coefficient of friction 0·4.

79. Centre of Mass of a System of Particles.

If two particles P, Q have masses m_1, m_2, the point which divides PQ in the ratio $m_2 : m_1$ is called the centre of mass of the two. Thus, let the positions of two particles P, Q of

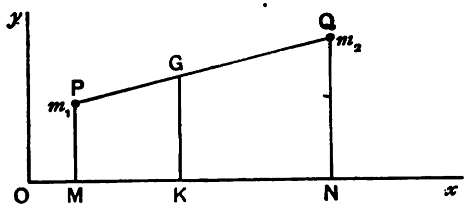


FIG. 63.

masses m_1, m_2 be given by the coordinates x_1, y_1 and x_2, y_2, and let the coordinates of G, their centre of mass, be x, y.

Draw from P, G, Q perpendiculars PM, GK, QN to the axis of x. Then

$$\text{OM} = x_1,$$
$$\text{ON} = x_2,$$
$$\text{OK} = x,$$
$$\text{MK} = x - x_1,$$
$$\text{KN} = x_2 - x,$$

and since

$$\frac{\text{PQ}}{\text{OQ}} = \frac{m_2}{m_1};$$

$$\therefore \frac{\text{MK}}{\text{KN}} = \frac{m_2}{m_1};$$

$$\therefore \frac{x - x_1}{x_2 - x} = \frac{m_2}{m_1};$$

$$\therefore\ m_1(x - x_1) = m_2(x_2 - x);$$

$$\therefore\ x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

Similarly,
$$y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

If there is a third particle R of mass m_3 at x_3, y_3, and H is the centre of mass of a particle $m_1 + m_2$ at G, and m_3 at R, H is called the centre of mass of the three m_1, m_2, m_3.

The x coordinate of H is

$$\frac{(m_1 + m_2)\dfrac{m_1x_1 + m_2x_2}{m_1 + m_2} + m_3x_3}{(m_1 + m_2) + m_3}$$

$$= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3},$$

and for any number of particles we may define the centre of mass as the point whose coordinates (which we shall denote in future by $\bar{x}$, $\bar{y}$) are

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{\Sigma mx}{\Sigma m}$$

$$\bar{y} = \frac{\Sigma my}{\Sigma m}.$$

80. If the particles are moving in any way, the centre of mass will also move (in general), and may be said to have a velocity and acceleration.

Let x_1, y_1 be the coordinate of m_1 at time t;

- u_1, v_1 be the component velocities along Ox and Oy of m_1 at time t;
- f_1, g_1 be the component accelerations of m_1 at time t;
- $\bar{x}$, $\bar{y}$ be the coordinates of the centre of mass at time t;
- $\bar{u}$, $\bar{v}$ be the component velocities of the centre of mass at time t;
- $\bar{f}$, $\bar{g}$ be the component accelerations of the centre of mass at time t;

and denote by dashed letters the corresponding quantities at time t'. Then

$$\bar{x}' - \bar{x} = \frac{\Sigma mx'}{\Sigma m} - \frac{\Sigma mx}{\Sigma m}$$

$$= \frac{\Sigma m(x' - x)}{\Sigma m},$$

$$\bar{u} = \operatorname*{Lim}_{t' \to t} \frac{\bar{x}' - x}{t' - t} = \text{Lim}\, \frac{\Sigma m(x' - x)}{\Sigma m} \frac{1}{t' - t}$$

$$= \text{Lim}\, \frac{1}{\Sigma m} \Sigma m \frac{x' - x}{t' - t}$$

$$= \frac{\Sigma mu}{\Sigma m},$$

$$\bar{u}\Sigma m = \Sigma mu. \qquad \text{..............................(2)}$$

Now Σmu = the total momentum in the x direction of all the particles, and $\bar{u}\Sigma m$ is the momentum in the same direction which a particle of mass Σm would have if moving with the velocity of the centre of mass.

Hence the equation (2) expresses that the total momentum in the x direction of all the particles is the same as that of a particle equal to the total mass and moving with the velocity of the centre of mass.

81. Similarly,

$$\bar{f} = \operatorname*{Lim}_{t' \to t} \frac{\bar{u}' - \bar{u}}{t' \to t}$$

$$= \text{Lim}\left(\frac{\Sigma mu'}{\Sigma m} - \frac{\Sigma mu}{\Sigma m}\right)\frac{1}{t' - t}$$

$$= \text{Lim}\, \frac{\Sigma m(u' - u)}{\Sigma m} \frac{1}{t' - t}$$

$$= \text{Lim}\, \frac{1}{\Sigma m} \Sigma m \frac{u' - u}{t' - t}$$

$$= \frac{\Sigma mf}{\Sigma m},$$

$$\bar{f}\Sigma m = \Sigma mf. \qquad \text{..................................(3)}$$

Now Σmf is the sum of all the x-components of the forces actually acting on the separate particles. Hence equation (3) expresses the fact that the acceleration of the centre of mass in the x-direction is the same as that of a mass equal to the total mass of the particles and acted on by the forces which actually act on the separate particles.

Σmf is the sum of all the components in the x-direction of all the forces acting on all the particles, including forces due to the action between any pairs of the particles. But any forces that exist between a pair of particles are of equal magnitude on the two and act in opposite directions on them, and hence in adding up the resolved parts these cancel one another. These actions and reactions (called internal) between different pairs of particles consequently disappear from the expression, and the only forces that need be considered are the external forces; that is, those due to some external system. If X_1 is the component in the x-direction of the external force acting on m_1, we may write the above equations:

$$\bar{x}\Sigma m = \Sigma mx,$$
$$\bar{u}\Sigma m = \Sigma mu,$$
$$\bar{f}\Sigma m = \Sigma mf = \Sigma X,$$

and ΣX is the sum of the x-components of all the external forces acting on the system.

Also, Σm = total mass of the particles

= M, say,

and the equation $M\bar{f} = \Sigma X$

expresses that the acceleration of the centre of mass in the x-direction is the same as that of a single particle whose mass is the total mass of the particles, and which is acted on by the x-components of the forces.

As the same applies to the y-components, it follows that *the resultant acceleration of the centre of mass is the same as that of a single particle whose mass is the total mass of the* *particles, and which is acted on by forces the same in direction and magnitude as those actually acting on the separate particles.*

A simple case of this has been used extensively already—namely, it has been possible to treat a train as a single particle, though the different forces acting on it may not necessarily act through one point.

The effect of gravity on a body can always be regarded as a single force acting through the centre of mass, and consequently, if a body such as a stick is thrown into the air its centre of mass will move in exactly the same way as if it were a single particle, provided air resistance is neglected; for example, rotation of the stick will not affect the velocity of the centre of mass.

Even if a body, such as a shell, projected in vacuo explodes, the centre of mass of the body continues to move in exactly the same way as it would have done if the explosion had not taken place, for the impulses due to the explosion acting on the different parts of the shell counterbalance one another when taken together, and therefore the velocity of the centre of mass is unaltered by the explosion.

82. If there are no external forces with resolved parts in the x-direction,

$$\Sigma X = 0;$$
$$\therefore \bar{f} = 0;$$
$$\therefore \bar{u} = \text{constant},$$

or the x-component of the velocity of the centre of mass is constant.

$$\therefore \text{ also } \Sigma mu = \text{constant},$$

or the sum of the component momenta of the particles in this direction is constant.

If there are no external forces at all, the centre of mass moves with uniform velocity in a straight line, and the sum of the momenta in any direction is constant.

This result is generally called the principle of the conservation of momentum.

It will be noticed that nothing has been said about the manner in which the particles move among one another. In fact, the theorems are true whether the particles move about like the molecules of a gas, or like the sun and planets in the Solar System, or whether they preserve the same positions relatively to one another as in a heavy body thrown into the air.

Example 1. Two particles of masses m, m' move in a vertical plane on the smooth faces of a double inclined plane, being connected by a string passing over the top. Find the acceleration of the centre of mass of the two.

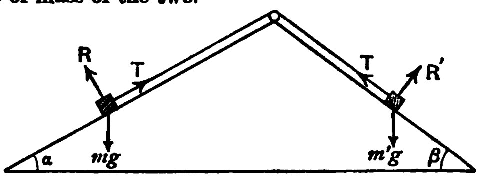


FIG. 64.

The forces are as marked in the diagram. Let f = acceleration of m down the plane,

$$mf = mg \sin \alpha - T,$$
$$m'f = T - m'g \sin \beta,$$
$$f = \frac{m \sin \alpha - m' \sin \beta}{m+m'} g.$$

The horizontal and vertical components of the acceleration of m are

$f \cos \alpha$ to the left,

$f \sin \alpha$ downwards,

and of m', $f \cos \beta$ to the left,

$f \sin \beta$ upwards;

∴ the components of the acceleration of the centre of mass are,—

(i) horizontally to the left,

$$\frac{mf \cos \alpha + m'f \cos \beta}{m+m'}$$
$$= \frac{(m \cos \alpha + m' \cos \beta)(m \sin \alpha - m' \sin \beta)}{(m+m')^2} g,$$

(ii) vertically downwards,

$$\frac{mf \sin \alpha - m'f \sin \beta}{m+m'}$$
$$= \frac{(m \sin \alpha - m' \sin \beta)^2}{(m+m')^2} g.$$

2. The weights in an Atwood machine are m, m'. Find the acceleration of their centre of mass.

3. Two masses of 3 and 2 lbs. wt. travel with uniform velocities 5 and 10 ft./sec. along lines at right angles, find the velocity of their centre of mass in direction and magnitude.

4. Two masses of 6 and 4 lbs. move along two straight lines Ox, Oy at an angle 60° with one another, with constant velocities 10 and 20 ft./sec. When the second is at O the first is 5 ft. from O. Find the direction and magnitude of the velocity of the centre of mass, and also the position of the line along which the centre of mass moves.

EXAMPLES.

1. A steamer can travel 12 miles an hour in still water. She apparently sails due east as shewn by the compass, but there exists a south-east current with a velocity of u miles an hour which causes her true course to be 15° S. of E. Find u and the resultant velocity.

2. A, B are two ships which steam at the same rate, and are initially at points P, Q at a distance a apart. A steams at right angles to PQ. Shew that if B steams so that the least distance between A and B may be ka, its course will make an angle $2\sin^{-1}k$ with A's, and that if it steams so that it may get within a distance ka in as short a time as possible, its course will make an angle $2\tan^{-1}k$ with A's.

3. A point P moves with uniform velocity u along Ox, and Q moves with uniform velocity v along Oy. At one instant OP$=au$, OQ$=bv$. Shew that the least distance between P and Q is

$$(a-b)\sin\alpha\left(\frac{1}{u^2}+\frac{1}{v^2}-\frac{2}{uv}\cos\alpha\right)^{-\frac{1}{2}},$$

where the angle xO$y=\alpha$.

4. A particle of mass 20 lbs. on a smooth horizontal plane is pulled by two horizontal strings at an angle 60° with one another, with forces 3 and 4 lbs.-wt.; if the directions and tensions of the strings are kept constant, find the position and velocity of the weight at the end of 3 secs. from rest.

5. A particle P is moving at a certain instant in the direction PX with a velocity of 30 cm./sec. After 10 secs. it has moved 4 metres along PX and 1 metre perpendicular to PX. If the acceleration is uniform, prove that it is 2·828 cm./sec². at an angle 45° with PX, and the final velocity is 53·85 cm./sec.

6. A particle of mass m describes a regular polygon of n sides with uniform speed v. Find the magnitude and dire[illegible] impulse required to be given to it at each angular poi[illegible]

7. One particle slides down a smooth inclined plane of inclination 30° and a second drops vertically, both starting together from the top of the plane. Find the relative velocity in direction and magnitude at the end of 3 secs.

8. Two points P and Q are describing concentric circles of radii a and b, and centre O, with velocities u and v. Find the velocity of P relative to Q when the angle POQ is θ, and find θ when

(i) the relative velocity is along PQ,
(ii) the component relative velocity along PQ is greatest.

9. An engine exerting a constant force draws a train of 50 tons weight up an incline of 1 in 100 and attains a velocity of 30 miles/hr. in 3 mins. If there is a frictional resistance of 10 lbs.-weight per ton, what is the tension of the coupling of the locomotive?

Discuss the horse-power.

10. If the frictional resistance to a train is $6+\frac{v^2}{240}$ lbs.-wt. per ton where v is the velocity in miles/hr., prove that if the train runs down an incline of 1 in 112 without steam on, the maximum velocity is 58 miles/hr.

11. If the resistance is $6+\frac{v^2}{240}$, as in the last question, and a train of weight 160 tons is pulled up an incline of 1 in 200 by an engine of 360 H.P., shew that the maximum velocity is given by

$$v^3+4128v=202500,$$

and shew that this gives a maximum velocity of 37 miles/hr. approximately.

Find also the maximum velocity down the plane.

12. If the resistance is $a+bv^2$ lbs.-wt. per ton where v is the velocity in miles/hr., find the maximum velocity attained by a train running without steam down an incline of 1 in n.

13. A mass of 20 lbs. moves down a rough inclined plane of elevation 45°, the coefficient of friction being 0·2. The air resistance varies as the square of the velocity, and is 1 lb.-wt. when the velocity is 50 ft./sec. Shew that the terminal velocity is 168 ft./sec. nearly.

14. A body is projected with velocity u up a rough plane of inclination α. If $\mu(<\tan\alpha)$ is the coefficient of friction between the body and the plane, find the distance the body goes up the plane, and the time until it returns to the starting point.

15. Two masses of 4 and 5 lbs. moving with velocities 20 and 30 ft./sec. respectively at an angle 60° with each other collide, and subsequently move as one body.

Shew that the velocity of this body is 22·47 ft./sec.

16. A shot of mass m is fired from a gun of mass M at rest on a horizontal plane. If the barrel is inclined at an angle α to the horizon, and the explosion causes the velocity of the shot relative to the gun on leaving it to be u, find the initial horizontal and vertical components of the velocity of the shot, the velocity of the gun, and the energy due to the explosion.

17. A shell travelling with velocity V breaks into two fragments one twice as heavy as the other. If the explosion adds an amount of kinetic energy equal to the original, and if the larger piece travels immediately after the explosion at an angle 45° with the previous direction of motion, shew that its velocity is $V/\sqrt{2}$ and that of the smaller $V\sqrt{5}$, and that the latter travels at an angle $\sin^{-1} 1/\sqrt{5}$ with the original direction.

18. A shell travelling with velocity V breaks into two portions, one twice as heavy as the other. If the explosion adds an amount of energy equal to the original, and if the larger mass has a velocity v immediately after, prove that the velocity of the smaller becomes $\sqrt{2(3V^2 - v^2)}$, and that if the directions they make with the direction before explosion are α, α',

$$\cos\alpha = (V^2 + 2v^2)/4Vv,$$
$$\cos\alpha' = (5V^2 - 2v^2)/2V\sqrt{2(3V^2 - v^2)}.$$

19. A particle of mass m slides from rest down the rough face (of inclination α) of a wedge of mass M capable of moving on a smooth horizontal plane. Find the distances moved by the particle and wedge in any time, taking the coefficient of friction to be μ.

20. A triangular prism of mass M rests with one face on a smooth horizontal table, the other two (smooth) faces are at right angles to each other. Two particles each of mass m are connected by a string passing over the middle of the upper edge of the prism. Shew that the acceleration of the prism is

$$mg\cos 2\alpha/(2M + 3m - m\sin 2\alpha),$$

where α is the smaller acute angle of the prism.

21. A rectangular block of mass m rests on a smooth horizontal table, and two particles each of mass m are attached to a string which passes over one edge of the block, one of these hanging down, and the other moving on the smooth horizontal face of the block. If the system is allowed to move freely, find

(1) the angle the hanging end of the string makes with the vertical, when this angle is constant,

(2) the tension of the string,

(3) the accelerations.

22. A smooth prism of mass M moves with its edges horizontal down a plane of inclination i, the angles of the prism in contact with the plane being α and β. A string passing over the upper edge of the prism carries masses m, m' at its ends which move on the smooth faces of the prism. Write down the equations of motion.

23. A particle of mass m is on a rough inclined plane of inclination α, the coefficient of friction being μ. A string from the particle passes over a smooth peg, and carries a mass $M(<m)$ at the other end. Supposing the peg is so far from m that the string can always be regarded as vertical, find the accelerations in motion up and down the plane.

24. Two particles on smooth inclined planes of inclination α, β are simultaneously released from the same point on their line of intersection. Find the velocity and acceleration of the one relative to the other in direction and magnitude at any instant.

25. Particles slide down a series of rough wires starting from one point and ending in the same vertical line, the coefficient of friction in all cases being $\mu(=\tan\lambda)$; prove that the time taken is least down a chord which makes an angle $\frac{\pi}{4}-\frac{\lambda}{2}$ with the vertical.

26. Particles slide down a series of smooth chords of a vertical circle starting from rest at the highest point. Shew that the times taken are the same, and the velocities generated are proportional to the length of the chords.

27. A particle slides down a smooth wire from a point O to a line AB in the same vertical plane. Prove that if a is the distance of O from AB, and α is the angle AB makes with the horizontal, and θ the angle the wire makes with the horizontal, the time taken is

$$\sqrt{2a/g \sin\theta \sin(\theta+\alpha)}.$$

Hence shew that the least time from O to any point of AB is $\sqrt{\frac{2a}{g}}\sec\frac{\alpha}{2}$ when the wire makes an angle $\frac{\pi}{2}-\frac{\alpha}{2}$ with the horizontal.

28. Two equal weights slide down smooth wires in the same vertical plane, and start at the same instant from rest at the same point. If the weights are each 2 lbs., and the inclinations of the wires 15° and 75°, find the magnitude and direction of the acceleration of the centre of mass, and find the resultant momentum at the end of 3 secs.

CHAPTER V.

VECTOR METHODS.

83. The work of the last chapter could be considerably shortened by a more extensive use of vector methods which are now so generally used in all the higher mathematics. In the present chapter we intend to give the simplest part of the vector work, and the rules relating to the algebra of vectors, deducing them from the laws obeyed by vectors relating to position. We shall repeat some of the previous chapter in this form, and the present chapter may be regarded as an alternative method of dealing with questions connected with parallelogram laws.

Ordinary algebra, it will be noticed, deals entirely with scalars.

Since we frequently want letters to denote the numerical magnitude of the vector only, we will use any of the following notations :

Vector.	*Magnitude or Modulus of Vector.*
$\overrightarrow{PP'}$	PP′
α, β, etc.	$\lvert\alpha\rvert$, $\lvert\beta\rvert$, etc.
$\mathbf{A}$, $\mathbf{v}$, etc.	$\lvert\mathbf{A}\rvert$ or A, $\lvert\mathbf{v}\rvert$ or v, etc.

In the notation $\overrightarrow{PP'}$ it is understood that the vector is to run in the direction from P to P′.

84. Two vectors are equal if the magnitudes are the same and their directions parallel. They need not be thought of as necessarily lying in the same line.

85. Vector Sum.

When three points O, A, P are in a straight line, if x is the distance of P from A and x' the distance of A from O, then $x+x'$ is the distance of P from O (with the usual convention of signs).

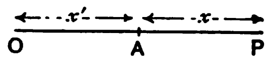


FIG. 65.

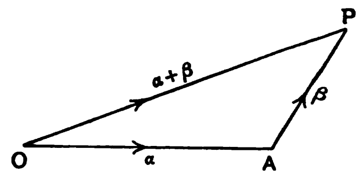


FIG. 66.

In the same way if the position of P relatively to A is given by a vector β and the position of A relatively to O by a vector α, we shall say that the position of P relatively to O is given by a vector $\alpha+\beta$. This consequently defines the meaning of the sum of two vectors.

$\overrightarrow{OP}$ is consequently called the vector sum of $\overrightarrow{OA}$ and $\overrightarrow{AP}$.

We can write this for shortness :

$$\overrightarrow{OP} \equiv \overrightarrow{OA} + \overrightarrow{AP}$$

using the sign $\equiv$ for *is equivalent to.*

Consequently, for any three points O, A, B,

$$\overrightarrow{OA} + \overrightarrow{AB} \equiv \overrightarrow{OB};$$

hence we say $\overrightarrow{AB} \equiv \overrightarrow{OB} - \overrightarrow{OA}$

$$= \beta - \alpha.$$

Similarly, $\overrightarrow{BA} = \alpha - \beta.$

Of course, from the meaning of a vector, we know that

$$\overrightarrow{AB} \equiv -\overrightarrow{BA}.$$

Notice particularly the following results, which the student can prove at once:

$$\alpha+\beta=\beta+\alpha, \quad \text{.........................(1)}$$

$$m(\alpha+\beta)=m\alpha+m\beta, \quad \text{.........................(2)}$$

where m is any *scalar*; for example, a number.

86. If now P moves in the plane considered and comes from a position P_1, whose vector relative to O is α_1 to P_2, whose vector is α_2, the displacement of P is $\overrightarrow{P_1P_2}$, which we

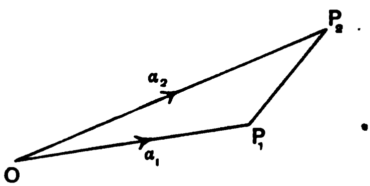


FIG. 67.

have seen is denoted according to our rules by $\alpha_2-\alpha_1$. If the point is at P_1 at time t_1 and at P_2 at time t_2, then the ratio $\dfrac{\overrightarrow{P_1P_2}}{t_2-t_1}$ or $\dfrac{\alpha_2-\alpha_1}{t_2-t_1}$ is the average velocity of the point during the interval (giving both its direction and magnitude). As in the previous work, we get a definite idea of the velocity at a given instant by taking the interval indefinitely short, thus:

$$\mathbf{v}_1=\lim_{t_2\to t_1}\frac{\alpha_2-\alpha_1}{t_2-t_1},$$

$\mathbf{v}_1$ being the vector representing the velocity. In the same way, if the velocity at time t_2 is $\mathbf{v}_2$, the vector denoting the acceleration is

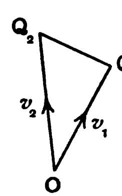


FIG. 68.

$$\mathbf{f}_1=\lim_{t_2\to t_1}\frac{\mathbf{v}_2-\mathbf{v}_1}{t_2-t_1}.$$

If $\mathbf{v}_1$, $\mathbf{v}_2$ are represented by $\overrightarrow{OQ_1}$ and $\overrightarrow{OQ_2}$, $\overrightarrow{Q_1Q_2}$ represents $\mathbf{v}_2-\mathbf{v}_1$.

87. Composition of Velocities.

Now, returning to the case where the position of P is given at time t_1 by a vector β_1 denoting its position relatively to

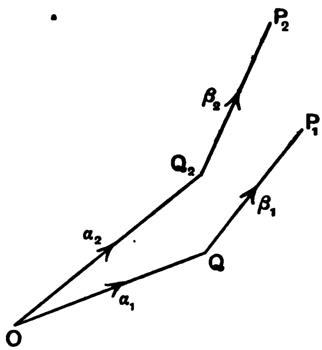


FIG. 69.

Q_1 and the position of Q_1 relatively to O by a vector α_1, and similarly at time t_2:

$$\overrightarrow{OP_1} = \alpha_1 + \beta_1$$

$$\overrightarrow{OP_2} = \alpha_2 + \beta_2 ;$$

$$\therefore \quad \overrightarrow{P_1P_2} = \alpha_2 + \beta_2 - (\alpha_1 + \beta_1)$$

$$= \alpha_2 - \alpha_1 + \beta_2 - \beta_1,$$

$$\mathbf{v}_1 = \operatorname*{Lim}_{t_2 \to t_1} \frac{\alpha_2 - \alpha_1 + \beta_2 - \beta_1}{t_2 - t_1},$$

$$= \operatorname*{Lim}_{t_2 \to t_1} \left(\frac{\alpha_2 - \alpha_1}{t_2 - t_1} + \frac{\beta_2 - \beta_1}{t_2 - t_1} \right) = \dot{\alpha}_1 + \dot{\beta}_1$$

using Newton's notation of a dot to denote the rate of change of a quantity.

Now $\dot{\alpha}_1$ is the velocity of Q relative to O at time t_1,

$\dot{\beta}_1$ is the velocity of P relative to Q at time t_1.

Hence we have, that the velocity of P relative to O is the ector sum of the velocity of P relative to Q and of Q relative o O.

This result is the composition of velocities as given in Art. 66.

It is obvious now that the same result applies to accelerations; thus, if P has an acceleration relative to Q represented by a vector β, and Q has an acceleration relative to O represented by α, then the acceleration of P relative to O is represented by the vector sum $\alpha+\beta$.

The vector sum is the same as the resultant of the two ectors.

The above results can be extended to cases where any umber of vectors occur.

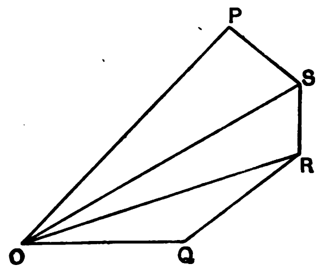


FIG. 70.

Thus, if OQ, QR, RS, SP represent four vectors, OP will note their vector sum, for

$$\overrightarrow{OR} \equiv \overrightarrow{OQ} + \overrightarrow{QR},$$

$$\overrightarrow{OS} \equiv \overrightarrow{OR} + \overrightarrow{RS}$$

$$\equiv \overrightarrow{OQ} + \overrightarrow{QR} + \overrightarrow{RS},$$

$$\overrightarrow{OP} \equiv \overrightarrow{OS} + \overrightarrow{SP}$$

$$\overrightarrow{OQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{SP}.$$

Hence, also,

if the velocity of P relative to S is represented by a vector δ,

,,	,,	S	,,	,,	R	,,	,,	,,	,,	γ,
,,	,,	R	,,	,,	Q	,,	,,	,,	,,	β,
,,	,,	Q	,,	,,	O	,,	,,	,,	,,	α,

then the velocity of P relative to O is represented by a vector $\alpha+\beta+\gamma+\delta$. This result for any number of velocities is called the polygon of velocities.

88. Composition of Forces

What we have called the Independence of forces states that if **P** and **Q** be two forces (as vectors) acting on a particle of mass m, and that they would separately produce accelerations α and β (also vectors), and since

$$\mathbf{P}=m\alpha,$$
$$\mathbf{Q}=m\beta,$$

then when the two act together the acceleration is $\alpha+\beta$.

Hence the effect is the same as that of a single force

$$m(\alpha+\beta).$$

If we call this force **R**, we have

$$\mathbf{R}=m(\alpha+\beta)=m\alpha+m\beta=\mathbf{P}+\mathbf{Q}\ ;$$

that is, **R** is the vector sum of **P** and **Q**.

Hence, if two forces act on a particle at the same time they produce the same effect as (and can therefore be replaced by) their vector sum.

This is the parallelogram of forces.

89. Work.

It will be remembered that the work done by a constant force is defined as the product of the force into the distance the point of application moves in the direction of the force. If the force P moves the point of application from A to B, the work done is P . AB cos θ, where θ is the angle between P and AB.

This is a scalar quantity.

If we denote the force as a vector by **P**, and the displacement as a vector by ρ, the quantity of work done is denoted by $\mathbf{P} \cdot \rho$, and $\mathbf{P} \cdot \rho$ is called the scalar product of **P** and ρ.

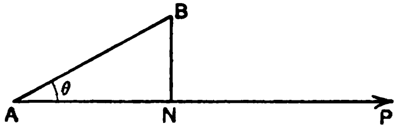


FIG. 71.

So, also, if we denote any two vectors by α and β, and the angle between them is θ, the scalar product of the two is written $\alpha \cdot \beta$, and is defined as

$$|\alpha| \, |\beta| \cos\theta,$$

where $|\alpha|$, $|\beta|$, are the magnitudes of α and β.

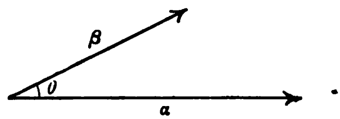


FIG. 72.

Thus the scalar product is the product of the magnitude of one of the vectors into the resolved part of the second in the direction of the first.

It is to be remarked that this is a pure matter of definition, the scalar product being defined in this way because it is a quantity that frequently occurs in physical questions. It must not be called *the product* of the two vectors, because there is another quantity which is called the *vector product* of two vectors, and the product $\alpha\beta$ will have no meaning given to it.

90. The following are the principal laws obeyed by scalar products :

(1) $\alpha \cdot \beta = \beta \cdot \alpha$ from the definition.

(2) If α, β, γ are any three vectors in the same plane,

$$\gamma \cdot (\alpha + \beta) = \gamma \cdot \alpha + \gamma \cdot \beta.$$

For in the figure LN = AK = OM ;

$$\therefore\ \gamma . (\alpha+\beta) = |\gamma|\,\text{ON} = |\gamma|\,(\text{OL}+\text{OM})$$
$$= |\gamma|\,\text{OL} + |\gamma|\,\text{OM}$$
$$= \gamma . \alpha + \gamma . \beta.$$

Proofs of the following, where required, can be supplied by the student.

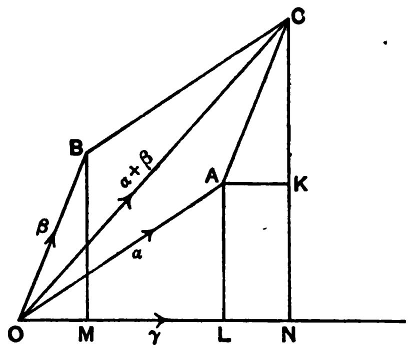


FIG. 73.

(3) If m is a scalar,

$$(m\alpha) . \beta = m(\alpha . \beta).$$

(4) $(\alpha+\beta) . (\gamma+\delta) = \alpha . \gamma + \alpha . \delta + \beta . \gamma + \beta . \delta.$

(5) If α, β are at right angles,

$$\alpha . \beta = 0.$$

(6) Conversely, if $\alpha . \beta = 0$, we have three alternatives :

either $\qquad |\alpha| = 0,$

or $\qquad |\beta| = 0,$

or $\qquad \cos\theta = 0,$

i.e. α and β are at right angles.

(7) The scalar product of a vector by itself is the square of its magnitude, thus :

$$\alpha . \alpha = \{|\alpha|\}^2,$$
$$\mathbf{A} . \mathbf{A} = \text{A}^2.$$

(8) The scalar product is positive if the vectors include an acute angle, negative if they include an obtuse angle. Thus:

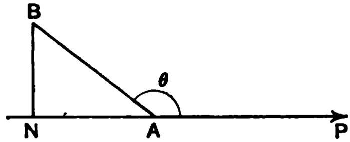


FIG. 74.

in the figure, if the displacement is from A to B, the work done by the force is negative, or work is done against the force.

91. Several Forces.

The work done by the resultant of two forces is equal to the sum of the works done by the two forces separately. This is an example of (2) above, for, if **P**, **Q** are the two forces as vectors, **P** + **Q** is the resultant, and if α is the displacement, the work done by the resultant

$$=(\mathbf{P}+\mathbf{Q}) \, . \, \alpha=\mathbf{P} \, . \, \alpha+\mathbf{Q} \, . \, \alpha$$

=sum of works done by each force separately.

The same result evidently holds for any number of forces.

Similarly, if a force remains constant in magnitude and direction while the point of application undergoes two successive displacements, the total work done is the same as if the point of application had undergone a single displacement equal to the resultant of the two displacements.

As before, if α, β are the two displacements as vectors and **P** the force, the total work is $\mathbf{P} \, . \, \alpha+\mathbf{P} \, . \, \beta=\mathbf{P} \, . \, (\alpha+\beta)$, which proves the result.

The results of this paragraph should be examined in conjunction with Fig. 73.

92. Centre of Mass of a System of Particles.

We have defined the centre of mass of two particles P, Q of masses m_1, m_2 as the point which divides the line joining

PQ in the ratio m_2 to m_1. Thus, if α_1, α_2 are the vectors of P and Q, and G is their centre of mass, we have

$$\overrightarrow{PQ} = \alpha_2 - \alpha_1;$$

$$\therefore\ \overrightarrow{PG} = \frac{m_2}{m_1 + m_2}\overrightarrow{PQ} = \frac{m_2}{m_1 + m_2}(\alpha_2 - \alpha_1);$$

$$\therefore\ \overrightarrow{OG} = \alpha_1 + \frac{m_2}{m_1 + m_2}(\alpha_2 - \alpha_1)$$

$$= \frac{m_1\alpha_1 + m_2\alpha_2}{m_1 + m_2}.$$

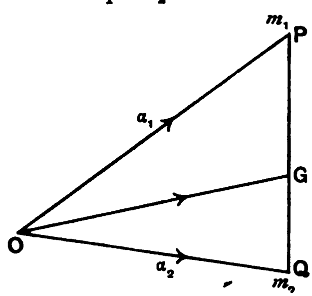


FIG. 75.

Similarly, we can prove (as in Chap IV.) that if there is any number of particles of masses m_1, m_2 .., whose position vectors at time t are α_1, α_2 ..., the vector to the centre of mass is $\frac{\Sigma m\alpha}{\Sigma m}$. Denoting this by $\bar{\alpha}$, so that

$$\bar{\alpha} = \frac{\Sigma m\alpha}{\Sigma m},$$

and supposing that at time t' the positions have become

$$\alpha'_1, \alpha'_2 \ldots,$$

the centre of mass has come to $\bar{\alpha}'$ where

$$\bar{\alpha}' = \frac{\Sigma m\alpha'}{\Sigma m}.$$

Similarly, if $\mathbf{v}_1$, $\mathbf{v}_2$ ... denote the velocities as vectors and $\bar{\mathbf{v}}$ the velocity of the centre of mass at time t,

$$\bar{\mathbf{v}} = \operatorname*{Lim}_{t' \to t} \frac{\bar{\alpha}' - \bar{\alpha}}{t' - t} = \operatorname{Lim} \frac{\Sigma m\alpha' - \Sigma m\alpha}{(t' - t)\Sigma m}$$

$$= \frac{1}{\Sigma m} \operatorname{Lim} \frac{\Sigma m(\alpha' - \alpha)}{t' - t}$$

$$= \frac{1}{\Sigma m} \operatorname{Lim} \Sigma m\mathbf{v}$$

$$= \frac{\Sigma m\mathbf{v}}{\Sigma m}$$

or $$\bar{\mathbf{v}}\Sigma m = \Sigma m\mathbf{v}. \qquad \text{...(1)}$$

Now $m_r\mathbf{v}_r$ is the momentum (as a vector) of the mass m_r, and $\Sigma m\mathbf{v}$ is consequently the vector sum of the momenta of all the particles, or, as we may call it, the resultant momentum of the system, and hence this equation expresses that the resultant momentum of all particles is equal to the momentum of a particle whose mass is the total mass of the particles, and which is moving with the velocity $\bar{\mathbf{v}}$ with which the centre of mass moves.

With similar notation.

Since $$\bar{\mathbf{v}} = \frac{\Sigma m\mathbf{v}}{\Sigma m}$$

$\therefore$ at time t', $$\bar{\mathbf{v}}' = \frac{\Sigma m\mathbf{v}'}{\Sigma m}$$

$$\therefore \bar{\mathbf{f}} = \frac{\Sigma m\mathbf{f}}{\Sigma m};$$

$$\therefore \bar{\mathbf{f}}\Sigma m = \Sigma m\mathbf{f} = \Sigma\mathbf{P}. \qquad \text{...... ....(2)}$$

Where $\mathbf{f}_r$ is the acceleration (as a vector) of m_r,

$\bar{\mathbf{f}}$ is the acceleration (as a vector) of the centre of mass,

$\Sigma\mathbf{P}$ is the vector sum of all the forces that act on all the particles.

Hence (2) expresses the fact that the acceleration of the centre of mass is the same as that of a mass equal to the total mass of the particles and acted on by all the forces which actually act on the separate particles.

This may be interpreted as in the last chapter. For example, if there are no external forces,

$$\Sigma P = 0;$$

$$\therefore \bar{f} = 0;$$

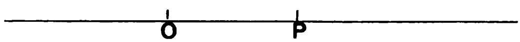
$$\therefore \bar{v} = \text{constant},$$

or the centre of mass moves with uniform velocity in a straight line, and the resultant momentum is constant.

CHAPTER VI.

SIMPLE HARMONIC MOTION.

93. A simple case of motion in a straight line, and one of very frequent occurrence in nature, is the case where the particle is moving under the action of a force directed towards a fixed point in the line of motion, the force being proportional to the distance from the fixed point.



Thus P is the position of the particle of mass m at any time, the force acting is represented by k . OP and always acts towards O.

Taking O as origin and

$$OP = x,$$

$$\text{the force} = -kx,$$

and the equation of motion is therefore

$$mf = -kx, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

k being a positive quantity. Motion of this nature, or closely approximating to it, occurs in the particles of air in a sound wave, the particles of water in a wave on water, the vibrations of a pendulum or of a weight attached to a spiral spring, rocking chairs and rocking stones, vibrating rods, and so on.

Since the force is $-kx$, the work done in any change of position can be calculated from the force-distance curve, which is in this case a straight line.

Thus the work done against the force in the distance P_1P_2 is the area of the quadrilateral $P_1P_2Q_2Q_1$ in the figure, or

$$OP_2Q_2 - OP_1Q_1 = \tfrac{1}{2}OP_2 \cdot P_2Q_2 - \tfrac{1}{2}OP_1 \cdot P_1Q_1$$
$$= \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2$$
$$= \tfrac{1}{2}k(x_2^2 - x_1^2),$$

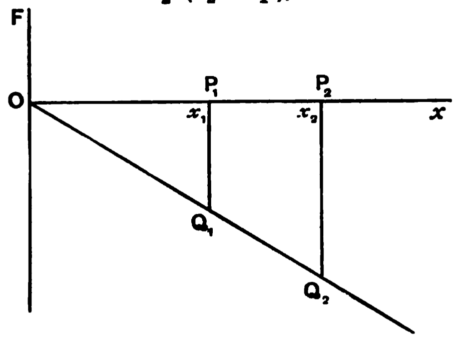


FIG. 76.

and the loss of kinetic energy, $\tfrac{1}{2}m(v_1^2 - v_2^2)$ being the work done against the force,

$$\therefore\ \tfrac{1}{2}m(v_1^2 - v_2^2) = \tfrac{1}{2}k(x_2^2 - x_1^2);$$

$$\therefore\ v_1^2 + \frac{k}{m}x_1^2 = v_2^2 + \frac{k}{m}x_2^2,$$

and hence for all values of x and v,

$$v^2 + \frac{k}{m}x^2 \text{ remains unaltered.}$$

It will be convenient sometimes to replace $\frac{k}{m}$ by a single letter μ (note that the acceleration is now $-\mu \times$ displacement, and μ is positive).

Since $v^2 + \frac{k}{m}x^2$ remains unaltered, as x increases v diminishes, and consequently, if the particle is moving from the centre with any velocity at one instant, sooner or later its velocity must vanish or it comes to rest and begins to move back.

If a is the distance from the origin when the velocity vanishes, we have therefore $v=0$ when $x=a$;

$$\therefore \quad v^2+\frac{k}{m}x^2=0+\frac{k}{m}a^2;$$

$$\therefore \quad v^2=\frac{k}{m}(a^2-x^2)$$

$$v=\pm\sqrt{\frac{k}{m}(a^2-x^2)}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

94. Now draw a velocity position graph in which, however, the ordinate instead of representing v itself represents $-v\sqrt{\frac{m}{k}}$, putting

$$y=-v\sqrt{\frac{m}{k}}$$

the graph is

$$y=\pm\sqrt{a^2-x^2}$$

or

$$x^2+y^2=a^2, \text{ a circle.}$$

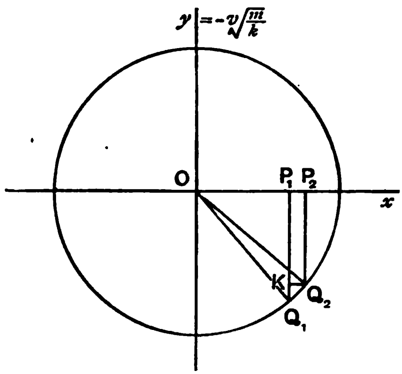


FIG. 77.

Draw ordinates at two adjacent points P_1 and P_2, and let them meet the curve on the negative side of Ox in Q_1 and Q_2. Then, as P moves along the line Ox, think of Q moving in the circle. While P describes the distance

P_1P_2, Q describes the distance Q_1Q_2. Draw Q_2K perpendicular to P_1Q_1.

Then the velocity of $Q = \frac{Q_1Q_2}{P_1P_2} \times$ vel. of P.

But since Q_1Q_2 is ultimately perpendicular to OQ_2,

and Q_1K is perpendicular to OP_1

$\therefore$ $\triangle Q_1Q_2K$ is similar to $\triangle P_1OQ_2$ ultimately;

$$\therefore \frac{Q_1Q_2}{Q_2K} = \frac{OQ_2}{P_2Q_2};$$

$$\therefore \frac{Q_1Q_2}{P_1P_2} = \frac{OQ_2}{P_2Q_2};$$

$$\therefore \text{ velocity of } Q = \frac{Q_1Q_2}{P_1P_2} \times \text{vel. of } P$$

$$= \frac{OQ_2}{P_2Q_2} \times \sqrt{\frac{k}{m}} \times P_2Q_2$$

$$= \sqrt{\frac{k}{m}}\, OQ_2 = \sqrt{\frac{k}{m}}\, a,$$

and is therefore a constant.

$\therefore$ as P moves along the diameter AOA' with simple harmonic motion, Q moves with uniform speed round the circle.

This important property is frequently used as a definition of simple harmonic motion, thus: If a point Q moves with uniform speed in a circle, and perpendiculars QP are drawn from Q to a fixed line, P is said to describe a simple harmonic motion.

Now the circumference being $2\pi a$, Q takes a time

$$2\pi a \Big/ \sqrt{\frac{k}{m}}\, a = 2\pi\sqrt{\frac{m}{k}}$$

to get round the circle.

Consequently, P takes a time $2\pi\sqrt{\frac{m}{k}}$ to get from A to A' and back to A again. This is called a complete oscillation of the particle; and the time is called the periodic time, and it is to be specially noted that this time does not depend on a. For example, the particle might be released from any point A under the action of this force, and the time taken for a complete oscillation would be the same however far off A was.

Equation (2) shews that the velocity is the same at the same distance on each side of the origin (*i.e.* same at distance x as at $-x$).

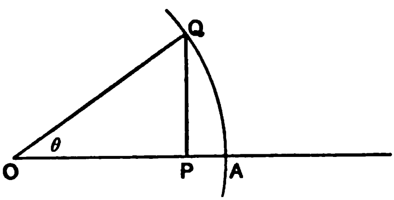


FIG. 78.

As the time taken by Q to describe the whole angle 2π is $2\pi\sqrt{\frac{m}{k}}$, an angle θ is described in $\theta\sqrt{\frac{m}{k}}$ secs., or in t secs. an angle $\sqrt{\frac{k}{m}}\,t$ is described;

$\therefore$ if we measure the times from the moment when P is at A, at the end of t secs.

$$\angle QOA = \sqrt{\frac{k}{m}}\,t;$$

$$\therefore \quad OP = a\cos\sqrt{\frac{k}{m}}\,t.$$

95. Collecting the results, and writing μ instead of $\sqrt{\frac{k}{m}}$, we have that, if in any case,

$$f = -\mu x, \quad \text{.........} \quad (1)$$

then

$$v = \pm\sqrt{\mu(a^2 - x^2)}, \quad \text{.........} \quad (2)$$

$$x = a\cos\sqrt{\mu}t, \quad \text{.........} \quad (3)$$

and the periodic time $= \frac{2\pi}{\sqrt{\mu}}$ ....(4)

a is called the amplitude.

From these we have also

$$v = -\sqrt{\mu}a \sin \sqrt{\mu}t, \quad \text{..................(5)}$$

the negative sign here replaces the ambiguous one, for it will be seen that (5) agrees with the fact that

$$v \text{ is negative from } t=0 \text{ to } t=\frac{\pi}{\sqrt{\mu}},$$

(*i.e.* while P moves from A to A′),

$$\text{and positive from } t=\frac{\pi}{\sqrt{\mu}} \text{ to } t=\frac{2\pi}{\sqrt{\mu}}$$

(while P moves from A′ to A).

96. We can now describe the motion more fully. Starting from rest at A, the particle moves with acceleration towards O, and therefore increasing velocity till it reaches O, where its velocity is a maximum and equals $\sqrt{\mu}a$. After passing O the acceleration is in the opposite direction to the velocity, and the velocity diminishes until the particle comes to rest at the same distance a on the other side of O. It then starts to return and moves in exactly the same way as before, completing the whole oscillation in $\frac{2\pi}{\sqrt{\mu}}$ seconds, each quarter of the path from A to O, or O to A′, etc., taking the same time $\frac{\pi}{2\sqrt{\mu}}$ secs. Notice that at the centre of the path the velocity has its maximum value $\sqrt{\mu}a$, while the acceleration is zero; at the extremity of the path the acceleration has its maximum value μa, while the velocity is zero.

Example 1. A particle starting from rest and moving with S.H.M. of period 18 secs. travels 10 ins. in 3 secs. Find the amplitude, maximum velocity, and velocity at the end of 3 secs.

Here, from the equation (4), we have

$$18=\frac{2\pi}{\sqrt{\mu}};$$

$$\therefore \sqrt{\mu}=\frac{\pi}{9},$$

hence from (3), since $x=a-\frac{10}{12}$ feet when $t=3$ sec.,

$$a-\frac{10}{12}=a\cos\left(\frac{\pi}{9}\times 3\right)=a\cos\frac{\pi}{3}=\frac{a}{2};$$

$$\therefore\ a=\frac{20}{12}\text{ ft.}=20\text{ ins.}$$

$$v_{max}=\sqrt{\mu}a=\frac{\pi}{9}\times\frac{20}{12}=0\cdot582\text{ ft./sec.}$$

and when $t=3$, $$x=\frac{20}{12}-\frac{10}{12}=\frac{10}{12},$$

$$v=\sqrt{\mu(a^2-x^2)}=\frac{\pi}{9}\sqrt{\frac{400-100}{144}}=0\cdot504\text{ ft./sec.}$$

obviously the lengths could have been left in inches equally well.

2. If the velocity of a particle moving in simple harmonic motion is 6 ft./sec. when at a distance of 4 ins. from the centre, and 8 ft./sec. when 3 ins. from the centre, find the amplitude, time of oscillation, and maximum velocity.

3. Draw a velocity-space graph for simple harmonic motion of period 4 secs. and amplitude 12 ins.

4. A particle moves in S.H.M. of amplitude 13 ins., and has a maximum velocity of 13 ft./sec. Find its velocity at 5 and at 12 ins. from the centre.

5. If the amplitude in S.H.M. is 10 inches and the period 4 secs., find the times taken to describe the successive inches of the path from rest.

6. In the last example find the distances travelled in each tenth of a second in the first second from rest.

7. A particle moves in S.H.M. of amplitude 10 ins. and period 5 secs. Find the time taken to travel 5 ins. from rest, and the time to travel the next 5 inches.

8. A particle moving in S.H.M. of period 2π secs. has velocities of 3 ft./sec. and 2 ft./sec. at points 1 ft. apart. Find the positions of the centre and extremities of the motion, and the maximum velocity.

• 9. At three points in a straight line at distances of 1 ft. from one another the velocities of a particle moving with S.H.M. are respectively 8, 7, 4 ft./sec. Find the centre and extremities of the motion, the time of oscillation, and maximum velocity.

10. A body of mass 2 lbs. moving with S.H.M. of period 4 secs. is, at a certain instant, 6 inches from the centre and has a velocity of 3 ft./sec. At this instant an impulse of 4 lbl.-second units is given to it in the direction of the motion. Find the original and new amplitude.

11. A boy weighing 6 stone standing on a plank oscillates vertically in S.H.M. of amplitude 6 inches and period 1 sec. Find the greatest and least pressures exerted on the plank.

12. A shelf oscillates vertically with simple harmonic motion of period $\frac{1}{2}$ sec. Shew that if the amplitude of the oscillation is 2·5 inches, an object on the shelf will leave it when it is nearly at the highest point of its path.

13. A particle vibrating in simple harmonic motion of period 4 secs. is at a given instant at a distance 5 ins. from the centre and moving towards the centre with velocity 1 ft./sec. Find the amplitude and time to reach the centre.

14. Shew that the average speed in simple harmonic motion is 0·637 of the maximum, and that the average acceleration (in magnitude) is 0·637 of the maximum acceleration.

15. An air particle makes 500 oscillations per second of amplitude 10^{-5} cm. Find the greatest velocity of the particle and its greatest acceleration.

97. Other Results.

If the force instead of being kx is given by an expression $h+kx=k\left(x+\frac{h}{k}\right)$, the motion is still simple harmonic, for it is only necessary to mark a point O′ at a distance $-\frac{h}{k}$ from

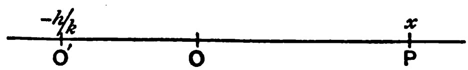


FIG. 79.

the origin and the force becomes $k \times$ O′P, and the motion will be simple harmonic about O′ as centre.

It may not be convenient to measure the time from the instant at which the particle is at its extreme position. If, for example, the origin of times is taken τ secs. after the moment at which it is at the extreme position, the position is given by

$$x = a \cos \sqrt{\mu}(t + \tau),$$

or

$$x = a \cos (\sqrt{\mu}t + \epsilon) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where

$$\epsilon = \sqrt{\mu}\tau.$$

The angle ϵ thus introduced is usually called the epoch. It is evidently the angle θ of Fig. 78 when $t=0$. The whole angle or $\sqrt{\mu}t + \epsilon$ is called the phase. Hence the epoch can also be called the initial phase.

The velocity will now be given by the expression:

$$v = -\sqrt{\mu}a \sin (\sqrt{\mu}t + \epsilon). \quad \ldots\ldots\ldots\ldots (6)$$

These equations may also be conveniently written:

$$x = \mathsf{A} \cos \sqrt{\mu}t + \mathsf{B} \sin \sqrt{\mu}t. \quad \ldots\ldots\ldots\ldots (5a)$$

where

$$\mathsf{A} = a \cos \epsilon$$

$$\mathsf{B} = -a \sin \epsilon$$

and then

$$v = -\sqrt{\mu}\mathsf{A} \sin \sqrt{\mu}t + \sqrt{\mu}\mathsf{B} \cos \sqrt{\mu}t. \ldots (6a)$$

For example, if the particle is projected from a point C at a distance c from the origin and in the direction *from* the origin with velocity u, we can determine a and ϵ, for we have, when $t=0$,

$$\left.\begin{array}{l} c = a \cos \epsilon \\ u = -\sqrt{\mu}a \sin \epsilon \end{array}\right\}$$

whence

$$a = \sqrt{c^2 + \frac{u^2}{\mu}}$$

$$\tan \epsilon = u/c\sqrt{\mu}.$$

With the same initial conditions it will be seen that (5a) becomes

$$x = c \cos \sqrt{\mu}t + \frac{u}{\sqrt{\mu}} \sin \sqrt{\mu}t.$$

EXAMPLES OF SIMPLE HARMONIC MOTION.

98. Spiral Spring.

The simplest case of simple harmonic motion, theoretically, is where a body is on a smooth horizontal table and attached by a spring to a fixed point, and the body is released from any point when the spring is not at its natural length, or projected with any velocity in the direction of the length of the spring.

FIG. 80.

Let O be the fixed end of the spring,
OA the natural length,
P the position of the body at any time,
m the mass,
$AP = x$ (taking A as origin).

In the figure when x is positive the spring is elongated and therefore in tension, and a force kx acts on m towards A, or a negative force.

When x is negative the spring is compressed, and therefore a force kx acts towards A in this case also, or a positive force.

The equation of motion is therefore correctly put

$$mf = -kx,$$

and the motion is simple harmonic with the period $2\pi\sqrt{\frac{m}{k}}$.

An important example of this is when an engine or truck possessing buffers runs against fixed supports at the dead

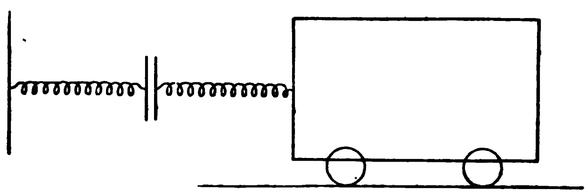


FIG. 81.

end of a line, also provided with buffers of the same stiffness. Here, after the buffers meet, the two springs act as a single one with one end fixed and the other attached to the engine or truck.

Numerical Example. A spring of natural length 1 ft. has one end attached to a point O on a smooth horizontal table, and a mass of 8 lbs. is attached to the other end. The spring is pulled out to a length of 15 inches, and the weight is then started with velocity 12 ft./sec. in the direction away from O. If the spring is of such stiffness that it would be elongated 1 in. by a weight of 5 lbs. hanging vertically, find the time of oscillation and the maximum length of the spring in the subsequent motion, and the maximum force, acceleration, and velocity.

The stiffness k is given by the fact that a force of 5 lbs. wt. elongates it 1 inch, or

$$5 \times 32 = k \times \frac{1}{12},$$

$$k = 60 \times 32 \text{ in absolute units.}$$

The equation of motion is, therefore,

$$8f = -60 \times 32x;$$

$$\therefore \; f = -240x.$$

The motion is therefore simple harmonic of period

$$\frac{2\pi}{\sqrt{240}} = \cdot 42 \text{ secs.}$$

The velocity in any position is given by

$$v = \sqrt{\mu(a^2 - x^2)},$$

and since initially $x = 3$ ins. $= \frac{1}{4}$ ft.,

and $v = 12$ ft./sec.,

we have
$$144 = 240\left(a^2 - \frac{1}{16}\right),$$

giving $a = 0{\cdot}814$ ft. $= 9{\cdot}8$ ins.

This is the maximum extension of the spring, the maximum length being consequently 21·7 ins.

The maximum force $= ka$

$$= 60 \times 32 \times 0{\cdot}814 \text{ lbls.}$$

$$= 48{\cdot}8 \text{ lbs. wt.}$$

The maximum acceleration

$$= \mu a = \frac{ka}{m} = 240 \times 0{\cdot}814$$

$$= 195{\cdot}3 \text{ ft./sec}^2.$$

The maximum velocity (when $x = 0$) is $\sqrt{\mu}a$

$$= \sqrt{240} \times 0{\cdot}814 = 12{\cdot}61 \text{ ft./sec.}$$

99. *Weight hanging by a Weightless Spiral Spring or Extensible String.*

Let OA be the natural length of the spring, then, as before, when the mass is at a point P at a distance x below A the forces are the tension $= kx$ upwards and weight mg downwards, and the equation of motion is therefore

FIG. 82.

$$mf = mg - kx$$

$$= -k\left(x - \frac{mg}{k}\right),$$

and by Art. 97 the motion is still simple harmonic with period

$$2\pi\sqrt{\frac{m}{k}},$$

but the centre is at a distance $\frac{mg}{k}$ below A.

This may otherwise be put in the following way, OA being the natural length (say l) when a mass m is hung on it would rest at a point B(OB $= l'$), so that at B the tension would be equal to the weight, or

$$k(l' - l) = mg. \qquad \text{.........................(1)}$$

If now the mass is in any other position at a distance x from B, the tension is $k(l' - l + x)$,

and the resultant force downwards

$$= mg - k(l' - l + x) = k(l' - l) - k(l' - l + x) = -kx, \qquad \text{......(2)}$$

and hence the particle describes simple harmonic motion about B as centre and of period $2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l' - l}{g}}$ by (1).

The same results apply to an elastic string, with the exception that the spring can exert pressure while the string cannot. Hence if the string in contracting reaches its natural length, it ceases to be stretched, and no force is exerted by it until the string is stretched again.

100. *Determination of g by a Spring Balance.*

The last example suggests a method of determining the value of g.

Performing the experiment as above, we determine the time of oscillation T, and the elongation $l'-l$ produced by the mass m, when hanging at rest.

Then $$T = 2\pi\sqrt{\frac{l'-l}{g}},$$

or $$g = 4\pi^2(l'-l)/T^2.$$

Using the formula in the form

$$T = 2\pi\sqrt{\frac{m}{k}},$$

k may be conveniently determined by observing what elongation a is produced by *any* extra weight n added to m; then

$$k = \frac{ng}{a},$$

and $$T = 2\pi\sqrt{\frac{ma}{ng}}.$$

101. *Motion of Piston and Crank* (kinematics).

Let OB be a crank of a steam engine turning about O with uniform velocity.

CD the piston rod moving in the line OCD,
BC the connecting rod.

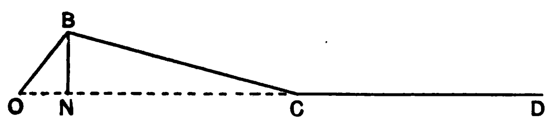


FIG. 83.

Then, if BN is perpendicular to OC, N moves with simple harmonic motion.

If, further, the length of BC is much greater than OB, the angle BCN will always be comparatively small, and hence

NC will differ little from BC, or NC is nearly constant, and C or any point on CD moves in nearly the same way as N. In other words, the motion of the piston is nearly simple harmonic.

Example 1. If OB is 6 ins. and BC is 60 ins., prove that the maximum value of the angle BCN is about 0·1 radian and CN differs from CB by a maximum amount of about 0·3 ins. Hence prove that while C describes a path of length 12 ins., at the instant midway between the instants at which it is at its extreme positions, it is at 0·3 ins. from the centre of the path. Also prove that it is at the mean position at an instant which differs from the mean instant by about 0·008 of the whole period.

2. A particle moving in S.H.M. in which the acceleration is $-\mu x$, has a velocity v from the origin when at a distance c from the origin. Prove that the amplitude is $\sqrt{c^2+v^2/\mu}$, and that the time to the extremity of the oscillation is $\frac{1}{\sqrt{\mu}}\tan^{-1}\frac{v}{c\sqrt{\mu}}$, and hence that the position at time t is given by

$$\sqrt{c^2+v^2/\mu}\cos\sqrt{\mu}\left(t-\frac{1}{\sqrt{\mu}}\tan^{-1}\frac{v}{c\sqrt{\mu}}\right).$$

Shew also that this is equivalent to

$$c\cos\sqrt{\mu}t+\frac{v}{\sqrt{\mu}}\sin\sqrt{\mu}t.$$

3. A mass of m lbs. is attached to one end of a spring, which is of natural length l ft., and is elongated a feet by a tension of T lbs.-wt. The other end of the spring is attached to a point on a smooth horizontal table, and the mass is started with velocity u when the spring is stretched to a length l'. Find the time of oscillation, amplitude, maximum tension and maximum velocity.

4. A mass of 6 lbs. is attached to one end of a spiral spring, the other end of which is fixed to a point on a smooth horizontal table. The spring is of natural length 6 inches, and would be elongated 1 inch by a tension of 5 lbs.-wt. If the body when in its position of equilibrium is struck a blow of impulse 40 lbl.-sec. units in the direction of the length of the spring, find the time and amplitude of the consequent oscillation.

5. A body of mass m is attached by two elastic strings, each of natural length l and modulus λ, to two points A, B on a smooth horizontal table at a distance $2l'(>2l)$ apart. Find the maximum velocity with which the body can be started in the direction of AB from the position of equilibrium without either string becoming slack, and find the time of oscillation.

6. In Question 5, if the strings are of length 1 ft. each, and would be each elongated 3 inches by a tension of 1 lb. wt., and if the mass is 4 oz. and AB 3 ft., find the initial velocity and time of oscillation.

If the initial velocity is greater than that found in this way, describe the ensuing motion.

7. A particle is attached by an elastic string of modulus λ and natural length l to a point O on a smooth horizontal table, and can move in a straight line through O. If the string is stretched to a length $l'(>l)$, and the particle released, describe the motion that ensues and find the time of a complete oscillation.

8. A particle is attached to a spring as in Example 4, but the plane is rough with coefficient of friction between body and plane of $\frac{1}{3}$. Shew that the time of oscillation is unaltered, but the centre of oscillation is shifted at each reversal of the motion. If the mass starts from rest when the spring is of length 1 foot, find the subsequent positions of instantaneous rest and the position where it finally comes to rest.

9. A spring of length 12 ins. would be elongated 4 inches by a tension of 1 lb.-wt. A weight of $\frac{1}{2}$ lb. is attached to it, and the spring is pulled down to a total length of 2 ft. and released. Find the time of oscillation and maximum velocity subsequently.

102. Compositions of S.H.M.'s of the same Period in the same Straight Line.

It has been mentioned that when a sound travels through the air the particles of air undergo simple harmonic motion. If two sounds are travelling in the same direction, and each separately would produce S.H. oscillations of the same period, each will produce its own effect, and the actual motion of the air particles will be the resultant of the two, which we will shew to be also a simple harmonic oscillation.

Another important case in nature is found in the tides. A tide may be regarded as a simple harmonic oscillation in a vertical line, of the surface of the ocean, having a period of 12 hrs. for the tide produced by the sun, and about 20 mins. longer for the moon. The actual tide observed is the resultant of these two tides (with other less important factors).

Supposing, then, a particle acted on by two forces in the same straight line which would each produce separately a simple harmonic vibration of period $\frac{2\pi}{\sqrt{\mu}}$ but of different amplitudes and phases, we can express the displacement due to one as

$$x_1 = a_1 \cos(\sqrt{\mu}t + \epsilon_1),$$

and due to the other,

$$x_2 = a_2 \cos(\sqrt{\mu}t + \epsilon_2),$$

and the actual displacement under the action of the two forces will be given by

$$\begin{aligned} x = x_1 + x_2 &= a_1 \cos(\sqrt{\mu}t + \epsilon_1) + a_2 \cos(\sqrt{\mu}t + \epsilon_2) \\ &= \cos\sqrt{\mu}t\,(a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2) \\ &\quad - \sin\sqrt{\mu}t\,(a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2), \end{aligned}$$

and this can be written

$$x = a\cos(\sqrt{\mu}t + \epsilon), \quad \text{.......................(6)}$$

if

$$a\cos\epsilon = a_1\cos\epsilon_1 + a_2\cos\epsilon_2, \quad \text{.................(6a)}$$

$$a\sin\epsilon = a_1\sin\epsilon_1 + a_2\sin\epsilon_2; \quad \text{.................(6b)}$$

$\therefore$ squaring and adding

$$\begin{aligned} a^2 &= a_1^2 + a_2^2 + 2a_1a_2(\cos\epsilon_1\cos\epsilon_2 + \sin\epsilon_1\sin\epsilon_2) \\ &= a_1^2 + a_2^2 + 2a_1a_2\cos(\epsilon_1 - \epsilon_2), \quad \text{...............(7)} \end{aligned}$$

and

$$\tan\epsilon = \frac{a_1\sin\epsilon_1 + a_2\sin\epsilon_2}{a_1\cos\epsilon_1 + a_2\cos\epsilon_2}. \quad \text{..........................(8)}$$

Now, we can always find a and ϵ to satisfy these equations, for in (7) a^2 is the sum of the squares of the two quantities in (6a) and (6b), and therefore is essentially positive, hence a can be found from (7).

Also, whatever positive or negative value the right hand side of (8) may have, an angle ϵ can be found (between 90° and −90°) whose tangent is equal to the expression.

Hence the resultant motion is found in the form (6), which represents a simple harmonic motion of amplitude a and phase ϵ.

From No. (7) it will be seen that for different values of $\epsilon_1 - \epsilon_2$ the amplitude a of the resultant vibration may take any value from $a_1 + a_2$ when $\epsilon_1 = \epsilon_2$ to $a_1 \sim a_2$ when $\epsilon_1 \sim \epsilon_2 = \pi$.

When $\epsilon_1 = \epsilon_2$, the component oscillations are said to be in the same phase.

When $\epsilon_1 \sim \epsilon_2 = \pi$ they are said to be in opposite phases.

Graphical Representation of the above.

In the following diagrams three simple cases are represented. The components are represented by full lines, and

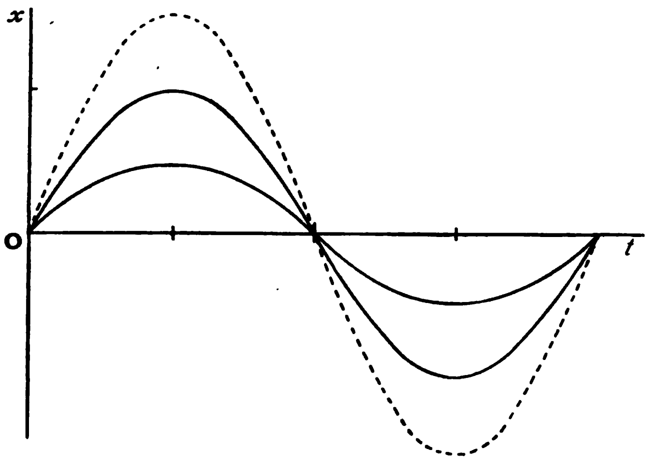


FIG. 84.—Composition of two simple harmonic oscillations in the same phase.

resultant by a dotted line. In each case the amplitude of one component is double that of the other. In Fig. 84 the phases are the same, in Fig. 85 they are opposite or differ by 180°, in Fig. 86 they differ by 90°.

If the periods are different, the resultant of the two cannot be expressed in a single term of the form of (6), and is therefore not a simple harmonic motion. We give one diagram in which the amplitude and period of one oscillation are each double the amplitude and period of the other.

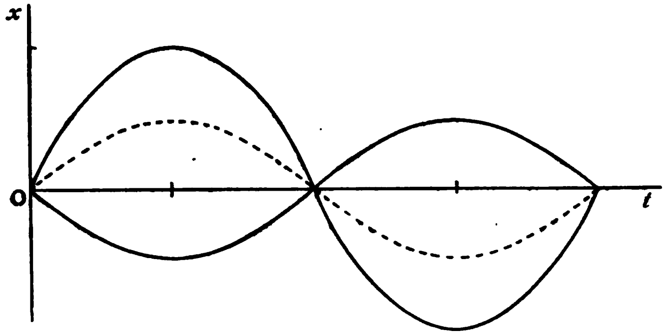


FIG. 85.—Composition of two simple harmonic oscillations in opposite phases

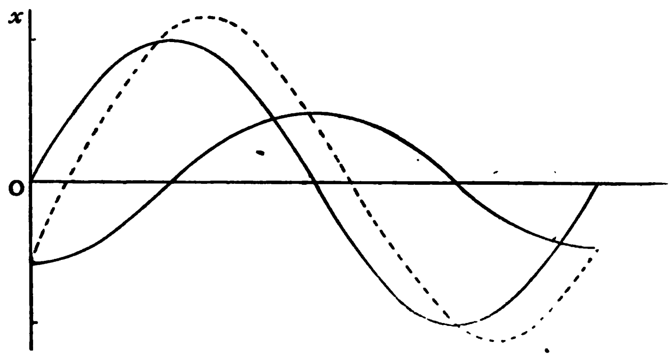


FIG. 86.—Composition of two simple harmonic oscillations in phases differing by 90°.

103. Composition of Two Simple Harmonic Motions at Right Angles.

If a body is acted on by two forces at right angles, each of which by itself would cause it to move with simple harmonic motion, the resultant motion may be found in a similar way.

We shall only consider the case when the periods are the same, and therefore the two vibrations may be represented by

$$x = a_1 \cos(\sqrt{\mu}t + \epsilon_1),$$
$$y = a_2 \cos(\sqrt{\mu}t + \epsilon_2).$$

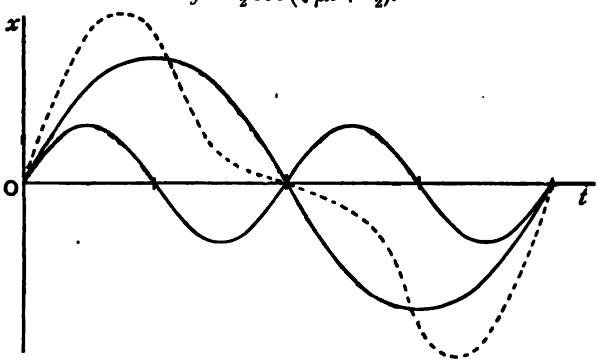


FIG. 87.—Composition of two simple harmonic motions of different periods.

For any values of the constants, the path can be traced by first calculating the values of x and y for a number of different values of t.

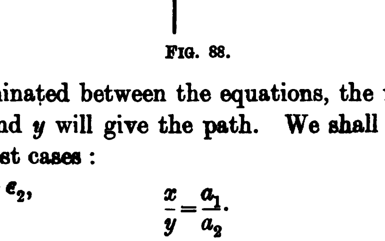


FIG. 88.

If t is eliminated between the equations, the relation found between x and y will give the path. We shall take a couple of the simplest cases :

(1) If $\epsilon_1 = \epsilon_2$, $\frac{x}{y} = \frac{a_1}{a_2}$.

Shewing that if the phases are the same $\frac{y}{x}$ is constant, or the resultant motion is along a straight line through O.

(2) If $\epsilon_2 = \epsilon_1 - \frac{\pi}{2}$ and $a_1 = a_2$,

$$x = a_1 \cos(\sqrt{\mu}t + \epsilon_1),$$
$$y = a_1 \sin(\sqrt{\mu}t + \epsilon_1);$$
$$\therefore \ x^2 + y^2 = a_1^2.$$

Consequently, when the phases differ by a right angle and the amplitudes are equal, the resultant motion is circular.

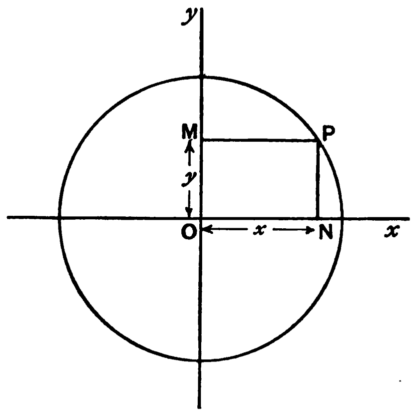


FIG. 89.

Also, since $\qquad \frac{y}{x} = \tan(\sqrt{\mu}t + \epsilon),$

and in the figure, $\quad \frac{y}{x} = \tan\theta;$

$$\therefore \ \theta = \sqrt{\mu}t + \epsilon,$$

or θ increases uniformly with t, or the resultant circular motion is uniform. Conversely, a uniform circular motion can be resolved into two simple harmonic motions at right angles.

(3) In the general case if δ is the difference of the phases, it can be shewn that

$$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}-\frac{2xy}{a_1a_2}\cos\delta=\sin^2\delta,$$

the proof and interpretation of which we will leave to the student.

Example. Prove that a S.H.M. can be resolved into two equal uniform circular motions in opposite directions.

Various methods may be used for shewing the composition of two simple harmonic motions experimentally. Most of them depend on the fact that the bob of a pendulum, or any point rigidly connected with a pendulum, moves with a very close approximation to S.H.M., as we shall prove later. Consequently, a point can be jointed to two pendulums so that it moves with a motion which is the resultant of two simple harmonic motions.

EXAMPLES OF MOTION OF PARTICLES CONNECTED BY SPRINGS.

104. Two particles on a smooth table are connected by a spiral spring. The particles are pulled apart so that the spring is elongated and then released, examine the oscillations.

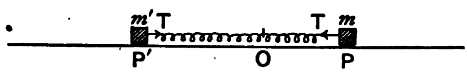


FIG. 90.

Let m, m' be the two masses,

P, P' be their positions at time t.

The particles being released from rest, the centre of mass is also initially at rest, and therefore by the conservation of momentum remains at rest. Now the centre of mass always divides the spring in the same ratio $m' : m$, and hence the [illegible] the centre of [illegible]

[illegible] and the tension (with [illegible]

[illegible]

[illegible] displacement [illegible]

[illegible] would be if the end [illegible]

[illegible] making m very large compared [illegible]

[illegible] of both particles must [illegible] displacements in opposite directions [illegible]

If the particles were started with any velocities in the line joining them, the centre of mass would move with uniform velocity, but the motion of each relative to the centre of mass would be the same as before, the equations being exactly the same as before, since the acceleration of either particle is the same as its acceleration relative to the centre of mass, the latter having no acceleration.

105. The following example, involving two periods of vibration, illustrates a very important method in higher dynamics.

Two particles of equal mass on a smooth table are attached to two similar springs as in the diagram, one spring being attached to a fixed point O. *Determine the oscillations, in the line of the springs.*

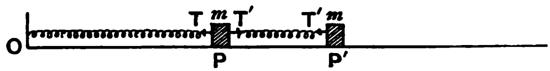


FIG. 91.

Let a be the natural length of each spring, x, x' the displacements to the right at time t of P, P′ from their positions of rest.

The elongations of the springs are, therefore,

$$x, \ x' - x,$$

and the tensions,

$$T = kx,$$
$$T' = k(x' - x).$$

If f, f' are the accelerations to the right, the equations of motion are

$$\left.\begin{aligned} mf &= k(x' - x) - kx = k(x' - 2x) \\ mf' &= -k(x' - x) \end{aligned}\right\} \begin{aligned} &\ldots\ldots(1) \\ &\ldots\ldots(2) \end{aligned}$$

Now we will try to find what possible oscillations can occur in which the period is the same for both particles.

If the period of each is $\dfrac{2\pi}{\sqrt{\mu}}$, we must have

$$f = -\mu x, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$
$$f' = -\mu x', \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and (1) and (2) become

$$\left.\begin{aligned}(m\mu - 2k)x &= -kx' \\ kx &= (m\mu - k)x'\end{aligned}\right\} \quad \begin{aligned}&\ldots\ldots\ldots\ldots(5)\\ &\ldots\ldots\ldots\ldots(6)\end{aligned}$$

hence $$\frac{m\mu - 2k}{k} = \frac{-k}{m\mu - k};$$

$$\therefore \; m^2\mu^2 - 3m\mu k + k^2 = 0;$$

$$\therefore \; m\mu = k\frac{3 \pm \sqrt{5}}{2} \quad \ldots\ldots\ldots\ldots(7)$$

giving two possible periods

$$2\pi\sqrt{\frac{m}{k}\frac{2}{3+\sqrt{5}}} \quad \text{and} \quad 2\pi\sqrt{\frac{m}{k}\frac{2}{3-\sqrt{5}}}.$$

Calling, for shortness, the two values of μ given by (7) μ_1 and μ_2,

(5) gives $$\frac{x}{x'} = -\frac{k}{m\mu_1 - 2k} \quad \text{or} \quad -\frac{k}{m\mu_2 - 2k}, \quad \ldots\ldots\ldots\ldots(8)$$

in other words, for the oscillation given by μ_1 the ratio $\frac{x}{x'}$ is constant or the oscillations must be in the same (or opposite) phases, and similarly for the second possible oscillation.

We will prove immediately that a combination of these two oscillations is likewise a possible motion, so that a more general solution of the equations may be written

$$x = a_1 \cos(\sqrt{\mu_1}t + \epsilon_1) + a_2 \cos(\sqrt{\mu_2}t + \epsilon_2) \quad \ldots\ldots\ldots(9)$$

$$x' = b_1 \cos(\sqrt{\mu_1}t + \epsilon_1) + b_2 \cos(\sqrt{\mu_2}t + \epsilon_2), \quad \ldots\ldots\ldots(10)$$

where the amplitudes are connected by the equation (7), that is

$$\frac{a_1}{b_1} = -\frac{k}{m\mu_1 - 2k} \quad \ldots\ldots\ldots\ldots(11)$$

$$= -\frac{2}{1+\sqrt{5}}$$

$$\frac{a_2}{b_2} = -\frac{k}{m\mu_2 - 2k} \quad \ldots\ldots\ldots\ldots(12)$$

$$= -\frac{2}{1-\sqrt{5}} = \frac{2}{\sqrt{5}-1}$$

and the constants a_1, a_2, ϵ_1, ϵ_2 can be determined when the initial positions and velocities of both particles are given.

The two initial positions and the two initial velocities being given, we get four equations for finding the four constants.

Thus, if the initial values of x and x' are c and c', and the initial velocities u and u', we have the equations :

$$c = a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 \quad \text{.........................(13)}$$

$$c' = b_1 \cos \epsilon_1 + b_2 \cos \epsilon_2 \quad \text{.........................(14)}$$

$$u = -\sqrt{\mu_1} a_1 \sin \epsilon_1 - \sqrt{\mu_2} a_2 \sin \epsilon_2 \quad \text{............(15)}$$

$$u' = -\sqrt{\mu_1} b_1 \sin \epsilon_1 - \sqrt{\mu_2} b_2 \sin \epsilon_2 \quad \text{............(16)}$$

these with the equations :

$$a_1/b_1 = -k/(m\mu_1 - 2k) \quad \text{.........................(11)}$$

$$a_2/b_2 = -k/(m\mu_2 - 2k) \quad \text{.........................(12)}$$

are six equations to give the six constants a_1, b_1, a_2, b_2, ϵ_1, ϵ_2.

They can be solved as follows. Substitute for b_1 and b_2 from (11) and (12) in (14), and (13) and (14) become two simultaneous simple equations for $a_1 \cos \epsilon_1$ and $a_2 \cos \epsilon_2$.

Similarly, from (15) and (16) we have two simultaneous equations for $a_1 \sin \epsilon_1$ and $a_2 \sin \epsilon_2$.

Having solved these, from the values of $a_1 \cos \epsilon_1$ and $a_1 \sin \epsilon_1$ we can deduce by squaring and adding, the value of a_1 and by division $\tan \epsilon_1$, and similarly for a_2 and ϵ_2.

In the above we have found that simple harmonic motions of periods $\frac{2\pi}{\sqrt{\mu_1}}$, $\frac{2\pi}{\sqrt{\mu_2}}$ are separately solutions of the question; that a motion which is the sum of the two is also a solution, that is, that (9), (10), (11), (12) give a solution, may be shewn as follows :

Let $$x = a_1 \cos(\sqrt{\mu_1} t + \epsilon_1) + a_2 \cos(\sqrt{\mu_2} t + \epsilon_2) = x_1 + x_2,$$

and $$x' = x'_1 + x'_2,$$

and therefore $$f = -\mu_1 x_1 - \mu_2 x_2$$

$$f' = -\mu_1 x'_1 - \mu_2 x'_2$$

and substituting in (1) and (2),

$$m(-\mu_1 x_1 - \mu_2 x_2) = k(x'_1 + x'_2 - 2x_1 - 2x_2)$$

$$m(-\mu_1 x'_1 - \mu_2 x'_2) = -k(x'_1 + x'_2 - x_1 - x_2)$$

and these are satisfied because μ_1, μ_2 have been determined, so that

$$-m\mu_1 x_1 = k(x'_1 - 2x_1),$$

$$-m\mu_2 x_2 = k(x'_1 - 2x_2),$$

$$-m\mu_1 x'_1 = -k(x'_1 - x_1),$$

$$-m\mu_2 x'_2 = -k(x'_2 - x_2).$$

Taking the numerical values of (11) and (12), we see that a_1 and b_1 are of opposite signs, shewing that the displacements given by x_1, x_1' are such that the maximum displacements occur in opposite directions at the same moment, or the particles are vibrating in opposite phases; this is the shorter of the two periods. On the other hand, in the longer of the two periods the phases are the same for both particles.

Summing up the results, we see that we have proved the following:

(1) There are two and only two distinct possible simple harmonic vibrations, and that one particle cannot vibrate in either of these periods without the other doing so also (for the amplitudes are connected by definite equations (11) and (12)).

(2) That the phases of the two particles in the corresponding vibrations are either the same or differ by π.

(3) That a motion composed of a combination of the two simple harmonic motions is a possible form of the motion, and that the amplitudes and phases depend on the initial circumstances of the motion and can be calculated when the initial circumstances are known, while the periods depend only on the nature of the springs and the magnitudes of the masses.

It follows that the motion found in this way satisfies all conditions, and, as it is impossible to imagine that two different motions could arise under the same circumstances, we can call this solution—(9) and (10)—*the complete solution* of the question.

It may be added, though the proof cannot be given here, that any periodic motion in a straight line, that is, one which repeats itself at regular intervals, can always be regarded as the resultant of a number of simple harmonic motions.

Example. If one of the quantities a_1 or b_1 is zero and the other is not, then $a_1/b_1 = 0$ or ∞, and one body oscillates in a certain period without the other doing so. Examine if this is possible for any value of k.

106. *A particle is attached to the middle point of an elastic string tightly stretched between two points on a smooth horizontal table, and is projected from the equilibrium position in a direction perpendicular to the string. Find the motion, supposing the greatest displacement small in comparison with the length of the string.*

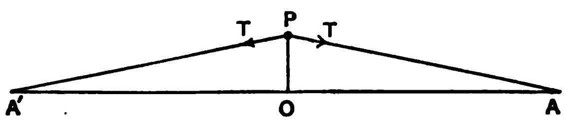


FIG. 92.

In the figure, let

$$OA = l,$$

$$OP = x = \text{displacement at time } t,$$

$$AP = \sqrt{l^2 + x^2} = l\sqrt{1 + \frac{x^2}{l^2}} = l\left(1 + \frac{x^2}{2l^2}\right) \text{ nearly,}$$

hence the string increases in length as the particle goes from O to P by an amount $\frac{x^2}{2l}$.

Now x/l is given as small, so that $x^2/2l$ can be neglected usually, hence the string can be regarded of constant length so long as squares of the small quantities may be neglected.

Hence, also, the tension remains constant to the same degree of approximation.

Resolving the forces along OP, if f is the acceleration,

$$mf = -2 \times T \cos APO = -2T \cdot \frac{x}{AP} = -\frac{2T}{l} x;$$

$\therefore$ the motion is simple harmonic with a period $2\pi\sqrt{\frac{ml}{2T}}$.

The displacement will be given by

$$x = a \cos\left(\sqrt{\frac{2T}{ml}}\, t + \epsilon\right)$$

and the velocity

$$v = -a\sqrt{\frac{2T}{ml}} \sin\left(\sqrt{\frac{2T}{ml}}\, t + \epsilon\right).$$

If the initial velocity is u, we have

$$\text{when } t = 0, \quad x = 0, \quad v = u,$$

$$\therefore \left.\begin{aligned} a \cos \epsilon &= 0 \\ -a\sqrt{\frac{2T}{ml}} \sin \epsilon &= u \end{aligned}\right\};$$

$$\therefore \epsilon = \frac{\pi}{2},$$

$$a = -u\sqrt{\frac{ml}{2T}},$$

$$x = -u\sqrt{\frac{ml}{2T}} \cos\left(\sqrt{\frac{2T}{ml}}\, t + \frac{\pi}{2}\right)$$

$$= u\sqrt{\frac{ml}{2T}} \sin\sqrt{\frac{2T}{ml}}\, t.$$

Note that the above is approximate only, but the smaller the oscillations the more closely do they approach the true simple harmonic motion. Notice, also, that the work would fail altogether if there were no tension in the string when straight, the force then would be due to the elongation when displaced, and would be approximately proportional to x^2.

107. *Two equal particles are attached by three equal and similar, stretched, elastic strings to two points on a smooth table as in the figure. Find the possible oscillations in the line of the string.*

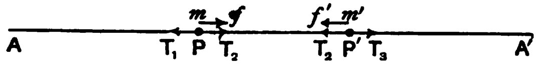


FIG. 93.

Let a be the length of each string in the equilibrium position,
T the tension of each in equilibrium.

Let $a+x$ be the length of AP at time t,

$a+x'$ „ „ A'P' „

$\therefore\ a-x-x'$ is „ PP' „

Let f, f' be the accelerations in the directions marked, then the tensions are

$$T_1 = T + kx,$$
$$T_3 = T + kx',$$
$$T_2 = T - k(x+x'),$$

and the equations of motion are

$$\left.\begin{aligned} mf &= T_2 - T_1 = -k(2x+x') \\ mf' &= T_2 - T_3 = -k(2x'+x) \end{aligned}\right\}$$

putting
$$f = -\mu x,$$
$$f' = -\mu x',$$

we have

$$(-m\mu + 2k)x = -kx' \quad\ldots\ldots(1)$$
$$kx = (m\mu - 2k)x'; \quad\ldots\ldots(2)$$
$$\therefore\ \frac{m\mu - 2k}{k} = \frac{k}{m\mu - 2k}$$
$$(m\mu - 2k)^2 = k^2,$$
$$m\mu - 2k = \pm k,$$
$$m\mu = 3k \text{ or } k.$$

Hence there are two possible periods of oscillation,

$$2\pi\sqrt{\frac{m}{3k}} \quad\text{and}\quad 2\pi\sqrt{\frac{m}{k}}.$$

(i) When $m\mu=3k$ by (1) $x=x'$; hence the particles are always symmetrically placed with regard to A and A′, and the amplitudes are the same.

(ii) When $m\mu=k$, $x=-x'$, and the length of the middle string does not vary, the two particles moving in the same direction at any instant, the amplitudes in this case are also equal.

The complete solution of the equations will be

$$x=a\cos\left(\sqrt{\frac{3k}{m}}t+\epsilon\right)+b\cos\left(\sqrt{\frac{k}{m}}t+\epsilon'\right),$$

$$x'=a\cos\left(\sqrt{\frac{3k}{m}}t+\epsilon\right)-b\cos\left(\sqrt{\frac{k}{m}}t+\epsilon'\right)$$

and the values of a, b, ϵ, ϵ' will have to be found from the initial conditions.

It should be noticed that the tension in the strings in the equilibrium position does not come into the result, which depends only on the changes in tension (involving k).

The two vibrations may be illustrated by the following diagrams shewing their positions at the end of each quarter-period.

FIG. 94.

108. *Two equal particles are attached to tightly stretched strings, as in the last example, and set moving perpendicular to the length of the string; find the motion, supposing the displacements small.*

Let x, x' be the displacements at time t.

As the displacements are small the alterations in length e strings are negligible, as in Art. 106, and the tension e regarded as constant.

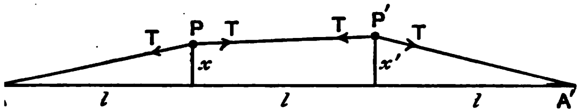


FIG. 95.

e components of the tensions perpendicular to AA′ are quently

$$T\frac{x}{l},\quad T\frac{x'-x}{l},\quad T\frac{x'}{l},$$

he equations of motion,

$$mf = T\frac{x'-x}{l} - T\frac{x}{l} = \frac{T}{l}(x'-2x),$$

$$mf' = -T\frac{x'-x}{l} - T\frac{x'}{l} = \frac{T}{l}(x-2x');$$

, putting

$$f = -\mu x,$$
$$f' = -\mu x',$$

$$\left(-m\mu + \frac{2T}{l}\right)x = \frac{T}{l}x',$$

$$\frac{T}{l}x = \left(-m\mu + \frac{2T}{l}\right)x';$$

$$\left(m\mu - \frac{2T}{l}\right)^2 = \frac{T^2}{l^2};$$

$$\therefore\ m\mu - \frac{2T}{l} = \pm\frac{T}{l};$$

$$\therefore\ m\mu = \frac{3T}{l} \text{ or } \frac{T}{l},$$

the possible periods are

$$2\pi\sqrt{\frac{ml}{3T}} \quad\text{and}\quad 2\pi\sqrt{\frac{ml}{T}}.$$

the first or shorter oscillation $x = -x'$, and the two cles are always on opposite sides of AA′.

In the second or longer $x=x'$, and the particles are always on the same side of AA′; thus:

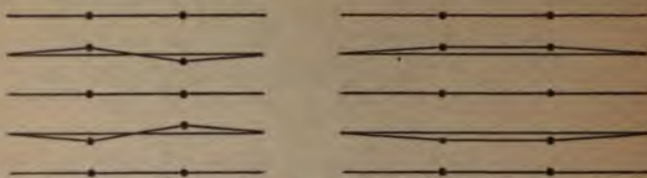


FIG. 96.

109. If we return to the case of one particle, as in Art. 106, we have shewn that we can represent small vibrations at right angles to the line AA′ by an equation

$$y=b\cos\left(\sqrt{\frac{2T}{ml}}\,t+\epsilon\right),$$

(we have for convenience changed x into y and a into b).

Also, we can easily prove that for oscillation in the line of the string the period is

$$2\pi\sqrt{\frac{m}{2k}},$$

and therefore the oscillation can be represented by

$$x=a\cos\left(\sqrt{\frac{2k}{m}}\,t+\epsilon'\right).$$

If $\frac{T}{l}=k$ or $T=kl$, which means that the tension is that required to stretch a string of length l to double its length, in other words, if the natural length of the whole string AA′ is l, the times of the two oscillations are the same, and we will represent them in the former notation as

$$x=a\cos(\sqrt{\mu}t+\epsilon),$$

$$y=b\cos(\sqrt{\mu}t+\epsilon').$$

If the particle, instead of being started moving either along or perpendicular to the string, is started from any point near

O, and in any direction with a small velocity, the motion will be compounded of these two fundamental oscillations, and the particle will consequently describe the elliptic path

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\delta=\sin^2\delta \text{ (see Art. 103)},$$

the point O being the centre of the ellipse. The constants $a, b, \epsilon, \epsilon'$ as before being determined from the initial conditions.

If the periods of the two fundamental oscillations are not the same, the resultant motion is still the result of compounding the two simple harmonic motions at right angles.

110. Free and Forced Oscillations.

The above examples are illustrations of what are called free oscillations. In them the period of oscillation depends on the nature of the vibrating agent with its connections.

But we frequently have to deal with what are called forced oscillations, which occur when an external force acts on the vibrating body and the force itself is periodic, with a period different from that of the free oscillation.

For example, suppose a particle attached to the middle of a spiral spring attached at its ends to two points on a smooth horizontal table. The particle has a period of oscillation depending on its mass and the stiffness of the spring. Suppose this period is T,

$\therefore$ the acceleration is $-\frac{4\pi^2}{T^2}\times$ displacement, and the restoring force due to the string is $-m\frac{4\pi^2}{T^2}\times$ displacement.

Now suppose that an external force also acts on the particle in the line of the spring, and varying harmonically in period T' so that the force P can be expressed as

$$mc\cos\frac{2\pi}{T'}t,$$

where m is the mass and c a constant of the dimensions of acceleration.

Now, obviously PN nearly coincides with PA, and the difference between PA and PN is small compared with either of them,

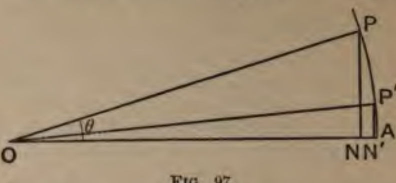


FIG. 97.

in other words, $\frac{PN}{PA}=1$ nearly. Further, if a new point P′ is taken on the arc PA, P′A and P′N′ coincide more nearly than did PA and PN, hence the nearer P comes to A the more nearly does $\frac{PN}{PA}$ approach unity.

Hence, $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}=1,$$

and when θ is small $$\frac{\sin\theta}{\theta}=1 \text{ nearly},$$

or $$\sin\theta=\theta \text{ nearly}.$$

Again, $$\cos\theta=1-2\sin^2\frac{\theta}{2}=1-2\left(\frac{\theta}{2}\right)^2 \text{ nearly}$$
$$=1-\frac{\theta^2}{2} \text{ nearly when } \theta \text{ is small}$$

The importance of this last result lies in the fact that $\cos\theta$ differs from unity by quantities which depend on the second and higher powers of θ, not on terms in θ itself.

114. Angular Velocity.

If a point P describes any curve in a plane, and the line joining the point to a fixed point O turns through an angle θ in time t, $\frac{\theta}{t}$ is called the average angular velocity of P about O, or the average angular velocity of the line OP during the interval t. Following the definition of velocity, if we make *in the positive direction occurring when the greatest force is acting in that direction.*

On the other hand, if the period of the force is less than that of the free oscillation, the displacement is greatest in the positive direction at the instant when the force is greatest in the negative.

If
$$T = T',$$
$$A = \infty,$$
which means that the oscillation, once started, would go on increasing till the amplitude became infinite.

This would be prevented in any actual example either by frictional effects, or by a breakage of connections.

A simple example of forced oscillations in the case of the pendulum will be given later.

111. A very important case of forced oscillations occurs in the theory of the tides. To explain this, suppose the ocean to surround the earth to a uniform depth. The surface would have a definite period of free oscillation. That is to say, if the surface were deformed by heaping up the water at two opposite ends of a diameter and depressing it at points midway between, so that it took a spheroidal shape, and then leaving it to itself, the water surface would oscillate about the mean position with a certain period T.

Now the moon exerts a force, as we shall see, to cause such displacements, and the force at any point depends on the position of the moon, and is approximately of simple harmonic nature whose period T' is about 24 hrs. 50 mins.

There would consequently be high tide under the moon if
$$T' > T,$$
and low tide under the moon if
$$T' < T.$$

The latter actually holds for an ocean of existing depth on the earth; but if the depth were more than 13 miles the reverse would apply.

Similarly, we can define angular acceleration as rate of change of angular velocity.

A line may be spoken of as having an angular velocity, even if it does not always pass through one point. Thus, if a point is moving along any curve, and at each position of the point we draw a tangent, then we can think of the tangent

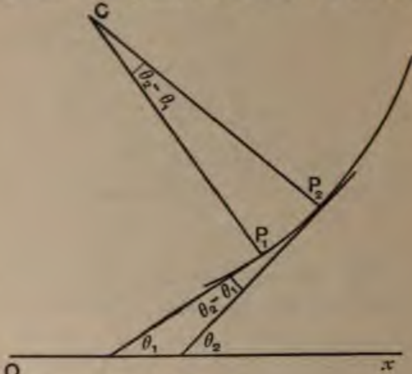


FIG. 100.

moving and turning round. If the tangent at P_1 makes an angle θ_1 with a fixed straight line Ox at an instant t_1, and makes an angle θ_2 with the same line at the instant t_2, then in the interval it has turned through an angle $\theta_2 - \theta_1$, and its average angular velocity during the interval is

$$\frac{\theta_2 - \theta_1}{t_2 - t_1},$$

and the angular velocity at time t_1 is $\lim\limits_{t_2 \to t_1} \dfrac{\theta_2 - \theta_1}{t_2 - t_1}$.

115. Curvature of a Curve.

If s denotes the arc P_1P_2, the $\lim\limits_{s \to 0} \dfrac{\theta_2 - \theta_1}{s}$ is called the curvature of the curve at the point P_1: we will denote it by κ. If the normals at P_1 and P_2 meet at C,

$$\kappa = \lim_{s \to 0} \frac{\theta_2 - \theta_1}{s} = \lim \frac{s}{CP_1} \cdot \frac{1}{s} = \lim \frac{1}{CP_1},$$

to a fixed point O in the plane. If the plane is rough (coefficient μ) and the particle is held initially so that the string is just stretched and lies along the line of greatest slope, find the greatest distance to which it will descend, and find whether it will return up the plane, the inclination of the plane being greater than the angle of friction.

Examine the case when

$$m=10 \text{ lbs.}, \quad \lambda=20 \text{ lbs.-wt.}, \quad \alpha=30^\circ, \quad l=3 \text{ ft.}, \quad \mu=0{\cdot}4.$$

7. If the acceleration due to gravity instead of being assumed constant is represented by an expression $g(1-x/a)$ at height x, shew that an unresisted projectile, projected vertically upwards with velocity v, will reach a height $a-a\sqrt{1-v^2/ga}$.

Shew that this gives the result for the uniform acceleration by making a increase indefinitely.

8. A mass of 10 lbs. is attached to a helical spring, such that a compression of 1 inch requires a force of 20 lbs.-wt. Shew that the period of oscillation is 0 227 secs.

9. A shock of the impact of a moving carriage against a rigid obstacle is diminished by a buffer spring of natural length l, the thrust of which is proportional to its compression and is $\frac{\mathsf{P}}{100}$ lbs.-wt. for a compression of 1 per cent. Shew that, if the mass of the carriage is m, and its velocity when the spring comes into action is v, the greatest thrust on the obstacle is $\sqrt{mv^2\mathsf{P}/l}$.

10. The lower end of an elastic vertical string just unstretched, and 4 ft. long, is attached to a heavy particle resting on a horizontal table. The upper end is then made to move vertically upwards with a constant velocity of 3 ft./sec., and the particle begins to rise after the upper end has risen 2 feet. Find the greatest velocity of the particle and its acceleration at any time.

11. Two equal particles connected by an elastic string which is just taut lie on a smooth table, the string being such that the weight of either particle would produce in it an extension a. Prove that if one particle is projected with velocity u directly away from the other, each will have travelled a distance $\pi u\sqrt{a/8g}$ when the string first returns to its natural length.

12. A mass of 10 lbs. is hung up by an elastic string of natural length 1 ft. and modulus 50 lbs. weight. A mass of 2 lbs. is let fall from the point of suspension of the string, and hitting the 10 lbs. mass becomes fixed to it. Find

(i) the initial velocity of the combined masses,
(ii) the subsequent maximum elongation,
(iii) the period of a vertical oscillation.

13. Two masses m, M slide on a smooth horizontal bar, and a massless helical spring giving a thrust T at unit compression is interposed between the masses. The masses have velocities v, V towards one another before the spring comes into action. Find their velocities after the action ceases, and the time it lasts.

14. Two springs AB, BP, whose tensions at unit extension are K_1, K_2, are connected at one end B of each, and lie along a straight line ABP. The end A is fixed at a point of a smooth horizontal table on which a mass M, fixed to the end P, moves in a straight line. Prove that the motion of M is simple harmonic of period

$$2\pi\sqrt{M\left(\frac{1}{K_1}+\frac{1}{K_2}\right)}.$$

15. Two springs AB, BC of stiffness K, K′ are attached to fixed points A, C and to one another at B in the same line. A mass M is fixed to the springs at B. Shew that the period of oscillation in the line ABC is $2\pi\sqrt{M/(K+K')}$.

16. A mass M has two stretched elastic springs, each of natural length a and modulus λ, proceeding from it in opposite directions. The second end of one spring is fixed, that of the other is constrained to execute a simple harmonic motion $\alpha \sin pt$ in the line of the springs.

Shew that M can execute a simple harmonic motion of the same period and phase, and of amplitude $\lambda\alpha/(2\lambda - Mp^2a)$.

17. A mass M is suspended from a point O by a helical spring of natural length l ft., whose tension is T lbs.-wt. when its length is increased by a feet. The mass is released from a point at a distance l below O and hits a fixed inelastic plate (so that the momentum of M is destroyed) b ft. lower. Find the velocity of the mass when it reaches the plate, and the time elapsed. Find also the time when the mass reaches the plate again, supposing it to leave it for an interval.

CHAPTER VII.

MOTION IN A PLANE CURVE.

112. When a particle is moving in a plane curve it is usually necessary to find the component accelerations in each of two directions, and it is most convenient usually to find the components along and perpendicular to the tangent of the path. Before obtaining expressions for these accelerations it will be necessary to explain some introductory results.

113. Trigonometrical Limits.

From elementary trigonometry we have

$$\sin 0^\circ = 0, \quad \cos 0^\circ = 1,$$

or more accurately expressed,

$$\lim_{\theta \to 0} \sin\theta = 0$$

$$\lim_{\theta \to 0} \cos\theta = 1.$$

Approximations to the values of $\sin\theta$ and $\cos\theta$ when θ is a small angle (which we will suppose given in circular measure) are also given in most trigonometries; we shall prove the following approximations:

$$\sin\theta = \theta \text{ nearly, when } \theta \text{ is small,}$$

$$\cos\theta = 1 - \frac{\theta^2}{2} \text{ nearly, when } \theta \text{ is small.}$$

Taking a small angle θ, then, from the diagram,

$$\theta = \frac{PA}{OP},$$

$$\sin\theta = \frac{PN}{OP}, \quad \frac{\sin\theta}{\theta} = \frac{PN}{PA}.$$

instants, the extremities of the lines will lie on a curve which is called the Hodograph, and it will be easy to prove that the acceleration of the point in the original curve is the same in

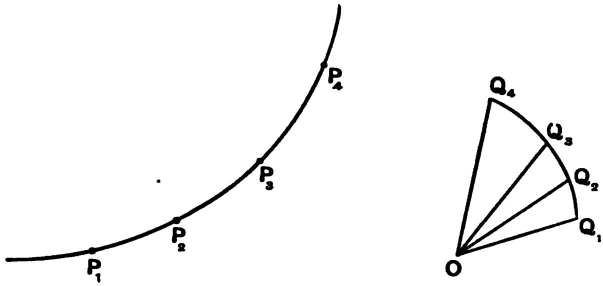


FIG. 103.

direction and magnitude as the velocity of the corresponding point in the hodograph.

Example. Shew that the hodograph of a point moving with uniform speed in a circle is itself a circle.

118. Motion in a Circle.

The most important case of motion in a curve is when the point is moving with a uniform speed in a circle.

Here $$v = \text{constant};$$

$$\therefore \dot{v} = 0,$$

also $$\kappa = \frac{1}{r};$$

$\therefore$ there is no acceleration along the tangent, and the acceleration along the normal is $\frac{v^2}{r}$.

Hence, if a point is moving with uniform speed v in a circle of radius r its acceleration is $\frac{v^2}{r}$ (or $\omega^2 r$) along the radius and directed towards the centre of the circle.

If the motion is not uniform, the acceleration along the radius is still $\frac{v^2}{r}$ if the velocity at the instant is v, and there is also an acceleration along the tangent of

$$\begin{aligned}&\lim_{t \to 0} \frac{v_2 - v_1}{t}\\ &= \text{Lim}\frac{r\omega_2 - r\omega_1}{t}\\ &= r \times \text{Lim}\frac{\omega_2 - \omega_1}{t}\\ &= rA\end{aligned}$$

if A is the angular acceleration of the radius to the point.

If a particle of mass m is moving with uniform speed v in a circle of radius r it is acted on by a force $\frac{mv^2}{r}$ or $m\omega^2 r$ towards the centre. This force is often called consequently a centripetal force.

It is often convenient to speak of the number of revolutions, n, the point makes in a second. It will be seen at once that $\omega = 2\pi n$.

We will illustrate motion in a circle by a number of examples.

Example 1. Find the acceleration of a point on the circumference of the wheel in Question 2, Art. 115.

2. Find the acceleration of a point on the earth's surface in latitude λ due to the earth's rotation.

3. A body having an initial velocity of 30 ft. per second describes a circle of 6 ft. radius with constantly decreasing speed, coming to rest after exactly going round the circle. Find in magnitude and direction the resultant acceleration at each 120° from the initial position.

Uniform motion in a circle is approximately attained in nature in the motion of the earth relative to the sun, and of the moon relative to the earth, and other similar cases in astronomy.

4. Find the acceleration of the earth due to its motion about the sun, supposing it to describe a circle of radius 92,880,000 miles in 365¼ days.

From the result of this example it will be seen that the acceleration of the earth due to its motion round the sun is very small compared with the acceleration due to gravity, and consequently in questions involving accelerations comparable with that due to gravity the earth can be treated as moving with uniform velocity; in other words, the results are the same as if the motion about the sun did not exist.

5. Find the acceleration of the moon relative to the earth, supposing it to describe relative to the earth a circle of radius 238,000 miles in 27 32 days.

This example is important from its historical interest. It shews that the acceleration of a body at the surface of the earth is very nearly 3600 times as great as the acceleration of the moon. Now the moon's distance being very nearly 60 times the radius of the earth, we see that the motion of the moon is accounted for by supposing it acted on by a force of attraction towards the earth similar to the force exerted on a falling body, but less in the ratio of the square of the distance from the centre of the earth. This numerical result was obtained by Newton, and was the first and most direct argument in favour of the **Law of Universal Gravitation**.

119. The Law of Gravitation is as follows: "Any two particles in the universe attract one another with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them."

In symbols, if two particles of masses m, m' are at a distance r apart they attract one another with a force $\lambda \frac{mm'}{r^2}$, λ being a constant depending only on the system of units chosen, and called the constant of gravitation. Thus, if m, m' are in grammes and r in centimetres, the force is given in dynes if $\lambda = 6{\cdot}66 \times 10^{-8}$.

The above formula still holds, as Newton shewed, if the particles are replaced by spheres either uniform or of density depending only on the distance from the centre.

Thus if S, E are the masses of the sun and earth in grammes,

R the distance apart of their centre,

ω the angular velocity of the earth about the sun,

T the time of revolution of earth about sun in seconds, according to the law of gravitation the force on the earth is $\frac{\lambda SE}{R^2}$, but its acceleration is $\omega^2 R$;

$$\therefore \frac{\lambda SE}{R^2} = E\omega^2 R,$$

and
$$\omega = \frac{2\pi}{T};$$

$$\therefore \lambda S = \omega^2 R^3 = \frac{4\pi^2 R^3}{T^2}.$$

Example 1. Taking $\lambda = 6{\cdot}7 \times 10^{-8}$, find the mass of the sun.

The value of λ is to be found experimentally by measuring the attraction acting between two bodies of measured masses and distance apart. This is an extremely delicate experiment, as the force is exceedingly small in any actual case. The principle of one method may, however, be easily explained. Suppose a metal sphere suspended from one arm of a balance and weighed, the weight being found to be M grammes. Then another sphere of weight M′ grammes, usually much heavier than the first, is placed directly under it and close to it. As a consequence the weight of the first is increased by the attraction between the two. This increase in weight is observed, say it is m grammes or mg dynes. The distance d cms. between the centres of the two spheres is also determined, and the constant λ may then be calculated from the equation

$$\lambda \frac{MM'}{d^2} = mg.$$

2. A spherical mass of 20 kgms. is attracted by another of 150 kgms. with a force of 0·23 milligrms.-weight when their centres are 30 cms. apart. Find the constant of gravitation.

3. A body of mass m on the earth is attracted to the earth with a force $= mg$ nearly;

$$\therefore \lambda \frac{Em}{a^2} = mg, \text{ if } a = \text{radius of earth}$$

$$\lambda E = ga^2.$$

Deduce the mass of the earth.

4. If a particle acted on only by the gravitational attraction of a sphere revolves round the sphere close to its surface, shew that the time of revolution of the particle is $\sqrt{\frac{3\pi}{\rho\lambda}}$ where ρ is the density of the sphere. Thus the time depends on the density only and not on the size of the sphere. Find this time in the case of the earth, supposing it to be a uniform sphere of density 5·5.

5. If two planets revolve round the sun in circles of radii r and r', and the times of revolution are T and T′, prove that

$$\frac{r^3}{T^2} = \frac{r'^3}{T'^2}.$$

This result is the statement as far as circular motion is concerned of what is known as Kepler's Third Law. It was discovered empirically by Kepler in 1618, and was later shewn by Newton to follow from the law of gravitation. The law may be stated thus:

The squares of the times of revolution of any two planets about the sun are in the ratio of the cubes of their mean distances.

6. A particle of mass 1 oz. is made to run in a horizontal circular groove of 10 feet radius. If it is started with a velocity of 20 ft./sec., and the friction is always one-tenth of the weight, find

(1) the distance travelled before coming to rest,

(2) the radial and tangential components of acceleration initially and when it has travelled half the distance to rest,

(3) the resultant horizontal reaction in magnitude and direction at the same two points.

7. A particle of mass $\frac{1}{2}$ lb. is tied to a string 3 ft. long, and describes a horizontal circle on a smooth horizontal table about the other end of the string which is fixed. If it makes 6 revolutions a second, what is the tension of the string ?

8. A string a metre long is attached at one end to a fixed point in a smooth horizontal table, and to the other end a weight of a kilogram is attached. If the weight makes 4 revolutions a second, find the tension.

9. A string 3 feet long is attached at one end to a fixed point in a rough horizontal table, and to the other a weight of 2 lbs. is attached. The weight is started with a velocity of 12 ft./sec. perpendicular to the string which is taut. If there is friction of a quarter of the weight, find the tension initially and when it has travelled through a right angle.

10. A string 10 feet long can just support a weight of 2 lbs.; what is the greatest velocity with which a particle of mass $\frac{1}{2}$ lb. attached to one end can revolve about the other on a horizontal table without breaking the string ?

11. A string l feet long can just support a weight of M lbs. What is the greatest number of revolutions per second that a mass of m lbs. tied to it can make without breaking the string ?

12. Two particles of masses 50 and 100 gms. are attached to a string at distances 40 and 60 cms. from one end which is fixed. If the string rotates about the fixed end in a horizontal plane making 5 revolutions per second, find the tension in each portion of the string.

13. Two equal masses are attached to the two ends of a string which passes through a small hole in a smooth horizontal table. With what velocity should the one on the table be projected so that it should describe a circle of 50 cms. radius, the other hanging vertically, and how many revolutions per second will it make ?

14. Two particles A, B of masses 50 and 100 gms. respectively are attached to the ends of a string passing through a small hole on a smooth horizontal table. If B hangs at rest and A describes a circle on the table, making 2 revolutions per second, what is the radius of the circle ?

15. A string of length 10 ft., whose ends are fixed at A and B 8 ft. apart, can bear a maximum tension of 5 lbs.-wt. A mass of 1 lb. weight can slide freely on the string. Neglecting gravity, find the greatest number of revolutions per sec. that the string can make about AB without breaking.

16. A particle of mass m can slide freely on a string ACB whose ends are fixed to points A, B. If $AB=2l$, and the length of the

string $= 2l'$, find the tension when the system makes n revolutions per second, neglecting gravity.

17. A particle of mass m is tied to a string at ACB at C. The ends AB are fixed and the whole is made to rotate about AB with uniform angular velocity ω. Given the lengths of AB, BC, CA, and neglecting gravity, find the tensions of the strings.

120. Conical Pendulum.

Very often uniform motion in a circle is the result of the action of two or more forces whose resultant is along the radius of the circle described. The simplest case is the conical pendulum where a heavy particle is attached by a weightless string to a fixed point. If the bob is drawn aside so that the string is inclined to the vertical, and if a blow is given to it so as to start it moving in a direction perpendicular to the plane containing the string and the vertical, and with the proper velocity (to be determined immediately), the bob may be made to revolve in a horizontal circle with uniform speed.

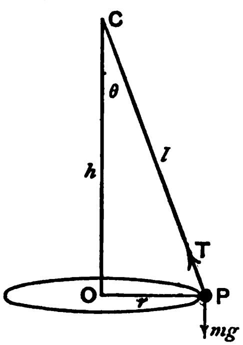


FIG. 104.

Let m be the mass of the particle,
l the length of the string,
h the depth of the plane of the circle described below the fixed point C,
r the radius of the circle described, the centre being O,
θ the angle the string makes with the vertical.

The only forces acting are T, the tension of the string, and mg the weight of the body. These two forces then must be equivalent to a single force $m\omega^2 r$ along the radius PO.

Hence, resolving

horizontally $\quad T \sin \theta = m\omega^2 r,$

vertically $\quad T \cos \theta - mg = 0;$

$$\therefore \tan\theta = \omega^2 r/g;$$

$$\therefore \frac{r}{h} = \frac{\omega^2 r}{g};$$

$$\therefore \omega^2 = \frac{g}{h}$$

$$t = \text{period of revolution} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}} \quad \text{..........(1)}$$

also $$v = \omega r = r\sqrt{\frac{g}{h}}. \quad \text{..........(2)}$$

Example 1. Prove that

$$\cos\theta = \frac{gt^2}{4\pi^2 l} = \frac{g}{l\omega^2}. \quad \text{..........(3)}$$

2. Prove that $$\mathrm{T} = ml\omega^2 = \frac{4\pi^2 ml}{t^2}. \quad \text{..........(4)}$$

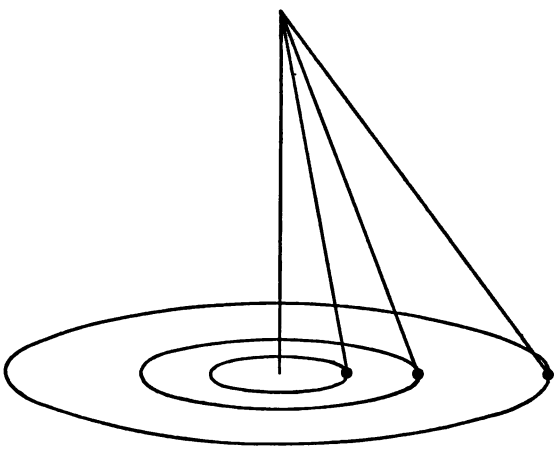


FIG. 105.

The result (1) shews that the time of revolution at a given place depends only on the depth h of the plane of motion below C. Thus, suppose a number of conical pendulums attached to the same point, but with different lengths of string, and suppose the bobs all revolve in the same plane, those describing the larger circles would require to have larger velocities given them, and they would all get round their circles in the same time.

121. Governors.

From equation (3) above we see that if $l\omega^2 > g$ we get a definite value of θ for a given angular velocity, and if the

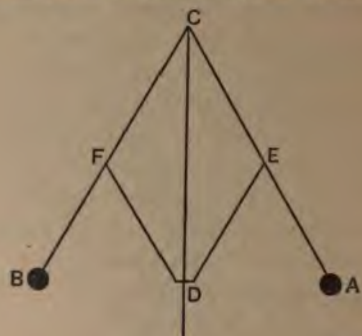


FIG. 106.

velocity increases $\cos\theta$ diminishes and θ increases. This fact is made use of in governors for governing the admission of steam to the cylinders of a steam engine. The principle of the governor will be seen from a diagrammatic sketch.

AC, BC are two rods forming with the heavy bobs A and B a double conical pendulum.

These rods are hinged to a rotating shaft at C. Two other rods FD, ED hinged to BC and AB are also hinged to a collar D which can slide on the shaft. To D a lever is attached which can open or close a valve admitting steam to the cylinders. When the shaft revolves at a normal rate the steam enters the cylinders at its proper rate, but if the rotation becomes more rapid, CA and CB separate further, and D is raised. This moves a lever which operates so as either to contract the opening of the valve admitting steam, or to vary the point when steam is shut off. This will continue till the velocity has dropped again to its normal value.

Example 1. In a conical pendulum prove that

$$r = l\sqrt{1 - g^2/l^2\omega^4}.$$

2. A conical pendulum is 10 ft. long. The bob is of mass 10 lbs., and is held at rest at a distance of 6 ft. from the vertical through the point of support. Find the impulse to be given to the bob that it may continue moving in a horizontal circle, and find the time of a revolution.

3. The string of a conical pendulum is 10 ft. long, and can just support a weight of 10 lbs. If the weight of the bob is 4 lbs., find the greatest velocity it can have without breaking the string, and find the radius of the circle described.

4. It is required to keep a particle moving with given uniform velocity v in a horizontal circle of given radius r by means of a string attached to the particle and to a fixed point vertically above the centre. Shew that this point must be at a distance gr^2/v^2 above the circle.

5. At what angle to the vertical must the string of a conical pendulum be held if it is 3 feet long and an initial velocity of 12 ft./sec. causes it to describe a horizontal circle. Find also the time of revolution.

6. If the velocity of the bob of a conical pendulum is v and the length l, prove that the inclination to the vertical is given by

$$\sin^2\theta/\cos\theta = v^2/gl,$$

and shew that this equation has one and only one solution applicable to the question.

122. Train on a Curved Level Line.

If a railway carriage is moving on a curved horizontal track (the sleepers being laid horizontally), there must be a force exerted towards the centre of the curve. This force is the horizontal component of the pressure of the rails on the wheels. Resolve these pressures into their vertical and horizontal components, and let

F_1 = horizontal component acting on all the wheels on one side of the carriage,

F_2 = horizontal component acting on all the wheels on the other side of the carriage,

R_1 = vertical component acting on all the wheels on first side of the carriage.

R_2 = vertical component acting on all the wheels on second side of the carriage,

r = radius of curvature of the curve,

v = velocity of train;

then since there is a horizontal acceleration $\frac{v^2}{r}$

$$F_1 + F_2 = \frac{mv^2}{r},$$

and since there is no vertical acceleration

$$R_1 + R_2 - mg = 0.$$

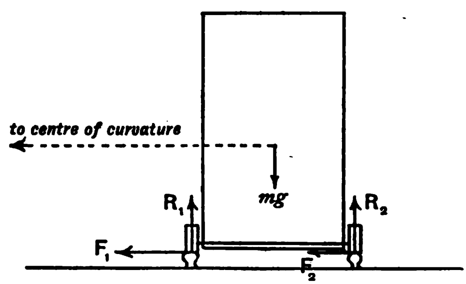


FIG. 107.

The wheels exert pressures on the rails equal and opposite to those shewn. As F_1 and F_2 act in the same line there is no possibility of separating them mathematically, and the amount of the total force $F_1 + F_2$ which acts on each rail depends on the shape of the wheels and inequalities in the track. If the flanges of the wheels are towards one another, that is, on the inner side of each wheel, the lateral pressure is practically borne by the outer rail, that is, $F_1 = 0$ nearly.

These lateral pressures (F_1 and F_2) are very objectionable, as they tend to rapidly destroy the track. To avoid them the sleepers are tilted up so that the plane of the track is normal to the resultant pressure. There will then be no lateral pressure, the centripetal force being supplied by the resolved part of the normal pressure itself.

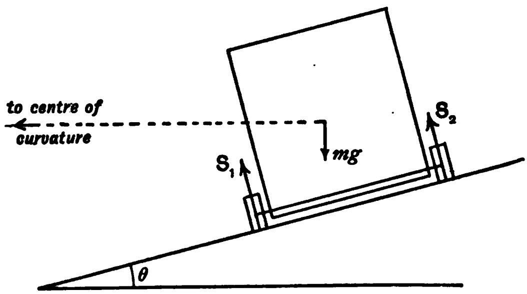


FIG. 108.

If the inclination of the sleepers is θ, and the resultant reactions S_1 and S_2, then resolving horizontally and vertically

$$S_1 \sin\theta + S_2 \sin\theta = \frac{mv^2}{r},$$

$$S_1 \cos\theta + S_2 \cos\theta - mg = 0,$$

or

$$(S_1 + S_2)\sin\theta = \frac{mv^2}{r},$$

$$(S_1 + S_2)\cos\theta = mg;$$

$$\therefore \tan\theta = \frac{v^2}{gr},$$

giving the angle at which the sleepers should be laid for a train travelling at a velocity v. Exactly the same results apply to a banked-up cycle or motor track.

If various trains travel round the curve at different rates, it is impossible to get rid of the lateral pressure except for those of one particular speed, those travelling at a greater speed exerting lateral pressure outwards, and those travelling at a smaller speed exerting lateral pressure inwards (*i.e.* down the slope). But it is best to tilt the sleepers to suit the faster trains, for, as will be seen from the following examples, if the angle of tilt is such as to suit trains at 30 miles an hour, a train at 45 miles an hour will produce a much greater lateral pressure than a train of the same weight at 15 miles an hour. Or, if the sleepers are tilted to suit a velocity of 30 miles an hour, the faster trains should slow down to that speed on approaching the curve, while the slower ones should increase their speed if possible.

Example **1.** Find the lateral pressure when an engine of 50 tons weight travels round an untilted track of 440 yards radius at 30 miles an hour.

2. Find the angle at which the track in Example 1 should be tilted to get rid of the lateral pressure. Find also the amount one rail should be raised above the other if the gauge is 5 ft. 3 ins.

3. If the track is tilted at the angle required to avoid lateral pressure when the velocity is v ft./sec., prove that the lateral pressure outwards when an engine of mass m lbs. travels on the curve at velocity v' ft./sec. is

$$mg\frac{v'^2 - v^2}{\sqrt{v^4 + g^2r^2}} \text{ lbls. or } m\frac{v'^2 - v^2}{r} \text{ nearly.}$$

4. At what angle should a motor racing track be banked if it is 300 yds. in diameter, and the motors are expected to travel at about 60 miles/hr.?

5. Find the lateral pressure when the track in Question 1 is tilted for a velocity of 30 miles an hour and the engine travels

(1) at a speed of 45 miles an hour,

(2) „ „ 15 „ „

6. Find the angle at which a motor track of 110 yards radius should be banked up for motors travelling at 45 miles an hour.

123. When a motor turns a corner the case is again the same as the railway carriage on a level line.

To calculate the values of R_1 and R_2 it is necessary to introduce the idea of moments. A discussion of moments will be found in Part II., Chapter II. Using the first results of that chapter (Arts. 178-179), it is possible to find R_1 and R_2 as follows:

We have, as before,

$$R_1 + R_2 = mg. \quad \text{...(1)}$$

Also, as the forces are equivalent to $\frac{mv^2}{r}$ through the centre of mass, the sum of the moments of the forces about any point is the same as the moment of the $\frac{mv^2}{r}$.

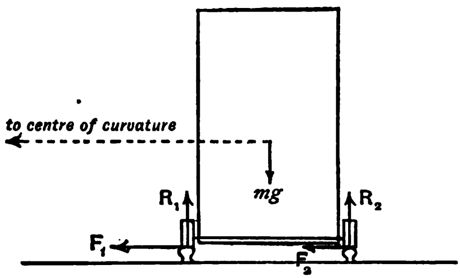


FIG. 109.

If we take moments about the point where the vertical through the centre of mass meets the ground, and if

$2a$ = distance between the wheels,

h = height of centre of mass above the ground.

$$R_2 a - R_1 a = \frac{mv^2}{r} h,$$

$$R_2 - R_1 = \frac{mv^2 h}{ra}. \quad \text{.........................(2)}$$

From (1) and (2),

$$R_1 = \tfrac{1}{2} m \left(g - \frac{v^2 h}{ra}\right),$$

$$R_2 = \tfrac{1}{2} m \left(g + \frac{v^2 h}{ra}\right).$$

Hence, the larger v is, the smaller is R_1, and if $v = \sqrt{\frac{gra}{h}}$,

$R_1 = 0$, or the wheels on the inside of the curve are not pressing on the ground at all. In other words, with a velocity $> \sqrt{\frac{gra}{h}}$ the motor will upset towards the outside of the curve.

A tricycle upsets much more readily, principally on account of the smaller value of a.

124. When a bicyclist rounds a corner he leans over towards the centre of curvature for similar reasons. If we represent the bicycle and rider diagrammatically by AB, G being the centre of mass, and m the total mass of man and machine, AG $= h$, the forces are as in the figure.

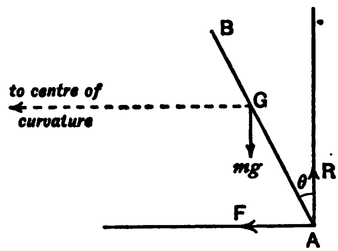


FIG. 110.

In order that the three forces may be equivalent to a single force $\frac{mv^2}{r}$ through G, it is necessary that the resultant of F and R should act through G, that is, along AG, hence

$$\frac{F}{R} = \tan\theta,$$

but

$$F = mv^2/r,$$

$$R = mg;$$

$$\therefore \tan\theta = \frac{v^2}{gr}.$$

Example 1. A motor weighing $1\frac{1}{2}$ tons goes round the corner of a horizontal roadway at 15 miles/hr. describing a circle of 50 ft. radius. What friction is required to prevent side slipping?

2. If the centre of mass of the motor in the last example is at a height 2 ft. 6 ins., and the distance between a pair of wheels 4 ft. 6 ins., find the total vertical pressure borne by the wheels on each side of the car.

3. If $a=2\frac{1}{4}$ ft., $h=2\frac{1}{2}$ ft., $r=40$ ft., $m=1$ ton, find the greatest velocity with which the car can turn the corner with all its wheels on the ground, and find the lateral pressure (friction) exerted at this speed.

4. A motor track of radius r ft. is banked to an angle θ, and a motor of mass m runs round it at a velocity v ft./sec. If the height of the centre of mass of the motor is h ft., and the distance between a pair of wheels $2a$ ft., shew that the total normal pressure on the wheels on the lower side is

$$\frac{m}{2}\left\{\frac{v^2}{r}\left(\sin\theta - \frac{h}{a}\cos\theta\right) + g\left(\cos\theta + \frac{h}{a}\sin\theta\right)\right\}$$

and find the velocity necessary to make the motor upset.

125. Motion of a Particle on a Smooth Curve in a Vertical Plane.

By motion on a smooth curve we may understand that (1) a small bead is threaded on a fine wire and is moving along the wire; or (2) that a particle is moving along inside a fine tube; or (3) it is moving along either the concave or convex side of a strip of metal bent into a curve. In the first and second cases the particle necessarily keeps to one definite path, but in the third it may happen that the particle will leave the curve at some point. In the latter case it will only be necessary here to deal with the motion as long as the particle is in contact with the curve. The mathematical work is the same in all cases, the only difference lying in the fact that in the first and second cases the pressure of the curve on the particle may act at one time towards one side of the curve, and at another towards the other, but in the third case it can only act on the one side.

When the curve is smooth, so that the reaction is perpendicular to the curve, and gravity is the only other force acting, it is easy to get a relation between the velocities at different points by means of the conservation of energy. For

At the highest point

$$\cos\theta = -1,$$

$$\therefore\ v^2 = u^2 - 4ga,$$

hence the particle will reach the highest point if

$$u^2 \geqq 4ga.$$

If $u^2 < 4ga$ the particle will oscillate, the greatest angle reached being given by $v = 0$ or $\cos\theta = (2ga - u^2)/2ga$.

Equation (2) shews that the reaction diminishes as θ increases, and when $\theta = 180°$,

$$R = \frac{mu^2}{a} - 5mg = \frac{m}{a}(u^2 - 5ga),$$

consequently, if $u^2 > 5ga$ the reaction will remain positive all round the circle.

Writing the equations (1) and (2) in the form

$$v^2 = u^2 - 2ga + 2ga\cos\theta,$$

$$\frac{Ra}{m} = u^2 - 2ga + 3ga\cos\theta,$$

we obtain

$$3v^2 - \frac{2Ra}{m} = u^2 - 2ga, \quad \text{.................(3)}$$

consequently, if R vanishes the value of v where $R = 0$ is given by

$$v^2 = \frac{u^2 - 2ga}{3},$$

hence R can never vanish if $u^2 < 2ga$, and if $u^2 > 2ga$, $R = 0$ where $\cos\theta = -\dfrac{u^2 - 2ga}{3ga}$. Since the velocity at the extremity of the horizontal diameter is $\sqrt{u^2 - 2ga}$, if R vanishes, the velocity at the point where it vanishes is less than the velocity at the extremity of the horizontal diameter. Hence R can only vanish on the upper half of the circle.

When $5ga > u^2 > 2ga$, the reaction vanishes somewhere in the upper half of the circle and changes sign, becoming negative, or, in other words, acts along the radius outwards. In other words, the bead is pressing outwards on the circle in the lower part and inwards in the upper part.

Equation (1) shews that the velocity is the same at the same height on the two sides of the vertical, and consequently, when the particle oscillates, the oscillations are symmetrical, and it reaches the same height on each side.

127. Motion on the Inside of a Smooth Vertical Circle.

If the circular wire is replaced by a strip of smooth material on the inside of which the particle is projected the equations are the same, but if R vanishes at any point the particle is then no longer pressing against the circle; in other words, it begins to move in a free path leaving the circle altogether. Its subsequent motion belongs to the chapter on projectiles.

In the same way the equations can be written down for the case when that particle moves from the highest point along the outside of a circular ring.

Example 1. If a particle is placed at the highest point on the outside of a smooth vertical circle (or sphere) and just displaced, it continues to move down. Prove that its velocity is $2\sqrt{ga}\sin\frac{\theta}{2}$ where θ is the angle the radius at a point makes with the vertically upward radius.

Prove also that it will run off the circle when $\cos\theta = \frac{2}{3}$.

2. Prove that if started with any velocity u from the highest point it will leave the circle at once unless $u^2 < ga$.

128. Particle Tied to the End of a Weightless String.

It will be noticed that if a particle is hanging from a fixed point by a light (*i.e.* weightless) string, and is given an initial horizontal velocity u of any magnitude, its motion will be exactly the same as the above case where the particle was caused to move on the inside of a vertical circle. The string can only exert a tension, and if the tension vanishes the string becomes slack and the particle ceases to describe the circle until the string becomes taut again.

Example 1. A bead slides down a smooth wire in the form of a parabola with its axis vertical and vertex upwards. If it starts from rest at the highest point, shew that its velocity at any point is proportional to its distance from the axis.

2. A particle of mass $\frac{1}{4}$ lb. tied to a string of length 8 ft. describes a vertical circle about the other end of the string, which is fixed. Find the velocity at the highest point if the particle just goes round, and find the tension at the lowest point and at the extremity of the horizontal diameter.

3. A particle moving on the inside a vertical circle just reaches the horizontal diameter. Shew that the reaction at any point is proportional to the distance below the horizontal diameter. If it just goes round, shew that the reaction is proportional to the depth below the highest point.

4. An aeroplanist describes a vertical circle of 200 yds. radius. If the velocity at the lowest point of the circle is 120 miles an hour, and the aeroplane is then upside down, what force is required to keep the man in his seat then, if his weight is 150 lbs.?

5. A particle attached to a string 10 ft. long is given a sufficient velocity to just make the complete revolutions. Find the velocity at every 30° of inclination, and deduce approximately the time of a complete revolution.

6. A cyclist loops the loop on a track of 11 ft. radius. Calculate the least speed he must have when upside down at the highest point in order that the cycle may not leave the track. The mass of the machine and rider may be supposed concentrated at a point 3 ft. from the track.

7. A bead slides down a smooth circular wire in a vertical plane, starting from rest at the highest point. Find the horizontal and vertical components of the acceleration at any point, and the resultant. Shew that the vertical component is greatest where $\cos\theta=\frac{1}{3}$ and is there $\frac{4g}{3}$, θ being measured from the highest point. Shew also that the resultant acceleration is greatest at the lowest point.

8. A particle moving on the inside of a smooth vertical circle of radius a is projected from the lowest point with velocity $\sqrt{kga}$, find the horizontal and vertical components of the acceleration in any position, and shew that if the resultant acceleration is f,

$$f=g\sqrt{k^2-4k+5+4(k-2)\cos\theta+3\cos^2\theta}$$

where θ is the angle the radius to the point makes with the vertical downward radius.

Prove that if $k=5, f=g$ when $\theta=180^\circ$;
if $k=2, f=g$ when $\theta=90^\circ$;

and explain the meaning of these results.

If $k=2$, find where (i) the vertical, (ii) the horizontal, (iii) the resultant acceleration, is a maximum or minimum, and find the maximum and minimum values.

129. Relative Rest on a Rotating Wire.

If a bead is placed on a smooth wire or in a fine smooth tube in the form of a plane curve, which is caused to rotate with constant angular velocity ω about a vertical axis in its plane, the bead can in general take one or more definite positions of relative equilibrium, *i.e.* points where it will remain at rest relatively to the wire though describing a circle about the axis of rotation.

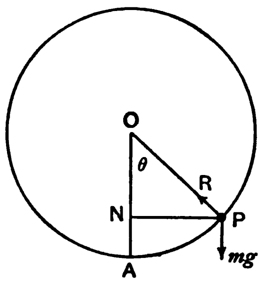


FIG. 113.

If we take the case when the wire is circular and it rotates about a vertical diameter, the equations are the same as for the conical pendulum.

Let a = radius of wire,
r = „ circle described by particle,
θ = angle radius to particle makes with the vertical.

The acceleration of the particle is $\omega^2 r$ along PN in the figure.

∴ resolving vertically

$$R\cos\theta = mg;$$

horizontally $R\sin\theta = m\omega^2 r = m\omega^2 a \sin\theta.$

These equations can be satisfied in two ways:

$$(1)\ \sin\theta = 0, \quad R = mg,$$

$$(2)\ R = m\omega^2 a, \quad \cos\theta = \frac{mg}{R} = \frac{g}{\omega^2 a}.$$

The first solution gives the highest and lowest points on the circle as possible positions of equilibrium, the second gives an inclined position if

$$g/\omega^2 a < 1, \text{ i.e. if } \omega > \sqrt{\frac{g}{a}}.$$

There is, however, a difference between these positions of equilibrium which we will state, though we will not prove the statement. When the inclined position of relative equilibrium exists, it is the only stable position. That is to say, if the particle is slightly displaced from the position it will return to it again and oscillate about the given position. But both the highest and the lowest positions are unstable; that is, if the particle is slightly displaced from one of them it will not return, but move farther from the point.

If, however, $\omega < \sqrt{\frac{g}{a}}$, and therefore the equation $\cos\theta = \frac{g}{\omega^2 a}$ has no solution, the highest and lowest points only are positions of the equilibrium, and of these the highest is unstable and the lowest stable.

Example. A straight smooth tube inclined to the vertical at an angle α rotates with uniform angular velocity ω radians/sec. about a vertical axis intersecting it. Find the distance from the vertical axis of the point in the tube at which a small body can remain in relative equilibrium.

130. Body on the Surface of the Earth. Effect of the Earth's Rotation.

It will be remembered we defined weight as the pressure a body exerts on a horizontal surface. This is, except at the poles, different from the force with which the earth attracts the body, as will be seen from the following considerations.

Take, first, a body of mass m at the equator. It is carried round by the earth in a circle of radius a (the radius of the earth or 3962 miles) with angular velocity $\omega = \frac{2\pi}{86164}$, the period of rotation being not 24 hrs., but the sidereal day of 23 hrs. 56 mins. 4 secs.

Hence there must be a resultant force acting on it $= m\omega^2 a$.

Now, the forces acting on the particle are the pressure of the earth equal to the weight mg acting outwards, and the

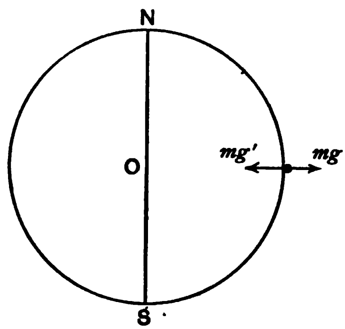


FIG. 114.

attraction of the earth, which we will denote by mg', acting inwards.

Hence,
$$mg' - mg = m\omega^2 a,$$
$$g' - g = \omega^2 a.$$

Now, putting in the values of ω and a, it will be found that
$$\frac{\omega^2 a}{g} = \frac{1}{289} \text{ nearly},$$

hence,
$$g' - g = \frac{g}{289},$$
$$g = \frac{289}{290} g'.$$

If there were no rotation g would be equal to g', or the weight of the body would be equal to the attraction of the earth, but on account of the rotation the weight is diminished from this value by about $\frac{1}{290}$ of its amount.

131. The above is the case at the equator, for other latitudes the effect is a little more complicated. We shall suppose the earth's attraction the same as before, and towards the centre, as if the earth was a sphere. In latitude λ the radius

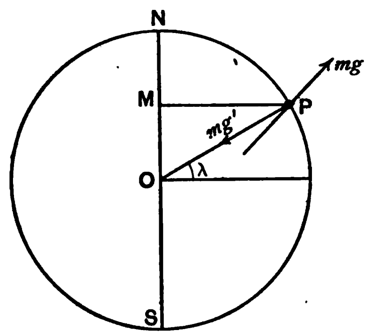


FIG. 115.

of the circle described by the particle is $a \cos \lambda$, and it will be seen at once that in order to have a resultant $m\omega^2 a \cos \lambda$ along PM in the figure, the forces cannot both be along the radius; in fact, the reaction to the weight mg, and consequently the weight, cannot even pass through the centre of the earth.

Suppose mg makes an angle θ with the radius, it is most convenient to resolve the forces along the radius and tangent to the meridian, and we have

along the radius $mg' - mg \cos \theta = m\omega^2 a \cos^2 \lambda, \quad \ldots\ldots\ldots(1)$

" " tangent $mg \sin \theta = m\omega^2 a \sin \lambda \cos \lambda\,; \ldots(2)$

$$\therefore \tan \theta = \frac{\omega^2 a \sin \lambda \cos \lambda}{g' - \omega^2 a \cos^2 \lambda}.$$

Now $$\frac{\omega^2 a}{g'} \text{ is about } \frac{1}{290},$$

hence in the denominator we can neglect the term $\omega^2 a \cos^2 \lambda$ in comparison with g', and we have

$$\tan\theta = \frac{\omega^2 a \sin\lambda \cos\lambda}{g'} \text{ nearly}$$

$$= \frac{1}{290} \sin\lambda \cos\lambda \quad \text{,,}$$

$$= \frac{1}{580} \sin 2\lambda,$$

the greatest value of $\tan\theta$ occurs consequently where

$$\lambda = 45°,$$

$$\tan\theta = \frac{1}{580},$$

and $$\theta_{max} = \frac{1}{580} \text{ nearly}$$

$$= 6' \text{ nearly}.$$

Hence, the greatest angle the weight makes with the direction of the centre of the earth is 6′. The direction in which the weight acts is, of course, the direction of the plumb-line, and is also the direction of the normal to an unruffled liquid surface.

Also, from (1), since

$$\cos\theta = 1 \text{ very nearly}$$

$$mg' - mg = m\omega^2 a \cos^2\lambda \text{ very nearly},$$

and the loss of weight on account of rotation is

$$m\omega^2 a \cos^2\lambda = \frac{mg}{290} \cos^2\lambda.$$

At the poles evidently $g' = g$, and there is no loss of weight.

132. It must also be noticed that on account of the non-spherical shape of the earth the attraction exerted on a body at different parts of the earth's surface varies, being greatest at the pole and least at the equator. Both this effect and the effect of rotation tend therefore to make g diminish as one goes from the poles to the equator, and the value of g in any latitude at the sea-level may be very approximately expressed, in metric units, by

$$g = 978{\cdot}0(1 + {\cdot}0053 \sin^2\lambda),$$

so that the variation from the equator to the pole is a little more than one-half per cent.

133. Effect of Attraction of the Sun on the Weight of a Body. Tide-generating Forces.

It is well known that the tides are due to a tendency of the oceans to heap up, at two extremities of the diameter

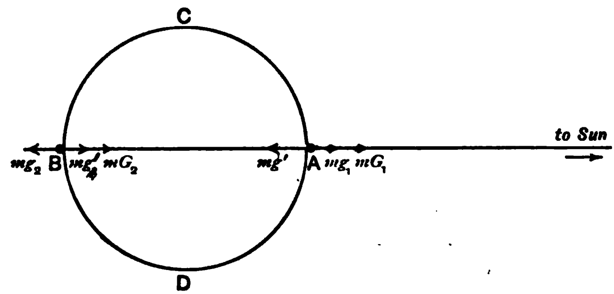


FIG. 116.

of the earth, under the attraction of the moon and sun. We will be able to get an idea of how the effects are produced if we examine the effect of the attraction of these bodies in producing a minute alteration of the weight of a body on the surface of the earth. We will only examine the effects in the simplest possible case, namely, at a point on the equator at a time when the sun is directly overhead. And though the effect of the moon is greater than that of the sun, it will be more convenient to take the case of the latter on account of its much greater distance.

Following, then, the same method as in the case of the rotation of the earth,

let mg_1 be the reaction to the weight,
mg' „ attraction of the earth,
mG_1 „ „ „ sun,
ω „ angular velocity of rotation of earth,
Ω „ „ „ revolution about sun,
V „ velocity of earth in its orbit,
a „ radius of earth,
R „ radius of orbit of earth.

Now the acceleration of the point is the resultant of its acceleration relative to the centre of the earth and the acceleration of the centre relative to the sun, and is therefore $\omega^2 a - \Omega^2 R$ towards the centre of the earth.

Hence, we have

$$mg' - mg_1 - mG_1 = m(\omega^2 a - \Omega^2 R),$$

or

$$g' - g_1 - G_1 = \omega^2 a - \Omega^2 R.$$

Now G_1 is the acceleration due to the sun at the surface of the earth (*i.e.* at distance $R - a$ from the sun), and $\Omega^2 R$ is the acceleration due to the sun at the centre of the earth (at distance R from the sun).

Hence, by the law of gravitation,

$$\frac{G_1}{R^2} = \frac{\Omega^2 R}{(R-a)^2},$$

$$G_1 = \Omega^2 R \frac{R^2}{(R-a)^2}$$

$$= \Omega^2 R \left(1 + \frac{2a}{R}\right) \text{ very nearly,}$$

since $\frac{a}{R}$ is very small;

$$\therefore\ g' - g_1 - \Omega^2 R\left(1 + \frac{2a}{R}\right) = \omega^2 a - \Omega^2 R;$$

$$\therefore\ g' - g_1 = \omega^2 a + 2a\Omega^2. \quad \text{......(1)}$$

Before examining the meaning of this result, let us take a point at the opposite side of the earth, and suppose there

$$G_1 \text{ is changed to } G_2,$$
$$g_1 \quad ,, \quad ,, \quad g_2,$$

the other quantities remaining as they were.

We now have

$$g' + G_2 - g_2 = \omega^2 a + \Omega^2 R,$$

$$\frac{G_2}{R^2} = \frac{\Omega^2 R}{(R+a)^2};$$

$$\therefore\ G_2 = \Omega^2 R\left(1 - \frac{2a}{R}\right);$$

$$\therefore\ g' - g_2 = \omega^2 a + 2a\Omega^2 \ldots\ldots\ldots\ldots(2)$$

as before. So that the effect is the same at both sides of the earth.

The $\omega^2 a$ is the same term as before, shewing the diminution in weight due to the rotation of the earth; and the $2\Omega^2 a$ shews another diminution in weight of the same amount when the sun is on the meridian whether it is midday or midnight.

It can be proved in the same way that at points like C and D in the figure, an *increase* in g of $\Omega^2 a$ is produced, and hence the result is the same as if we had minute forces acting outwards at A and B and acting inwards at C and D. The result of these forces acting on the waters of the ocean is to cause the tendency of the waters to heap up under the sun and moon, and on the opposite sides of the earth.

It does not follow, however, that high water is always under the sun or moon; on account of the rotation of the earth the high tide travels round the earth, and the time of high tide depends on the rate a wave can travel in the oceans of the earth, which again depends on the depth of the ocean and the presence of land.

The acceleration $2a\Omega^2$ will be seen to be very small, having only $2 \times \frac{1}{365} \times \frac{1}{365}$ of the effect due to the rotation of the earth.

As we have seen (Art. 119), $\Omega^2 = \frac{\lambda S}{R^3}$, if λ is the constant of gravitation and S the mass of the sun.

This tide-raising force is therefore $\frac{2\lambda S a}{R^3}$, and consequently varies inversely as the cube of the distance, and directly as the mass, of the tide-raising body. It is on account of the cube of the distance coming into the matter that the effect of the moon is greater than the sun; for, of course, the direct attraction of the sun on the earth is greater than that of the moon.

Example. Compared with the mass of the earth as unity, that of the moon is 0·0123 and the sun's is $3{\cdot}16 \times 10^5$. The distance of the moon is $2{\cdot}38 \times 10^5$ miles and of the sun $9{\cdot}28 \times 10^7$ miles. Compare the tide-raising forces of the moon and sun.

134. Tension in a Rotating Band.

Suppose a circular band, such as a leather belt or a metal wire, to rotate about an axis through its centre and perpendicular to its plane. There will be produced a tension in the band on account of the angular velocity. If we suppose the axis of rotation to be vertical, gravity will not come into the question. The tension will then be the same throughout. We will also suppose the dimensions of the cross section small compared with the radius of the band. We can think of a small arc ACB subtending a small angle θ at O as moving in a circle under the action of the two tensions each T at the ends A and B. If, therefore, we resolve the forces along the normal at the middle points of the arc, since the tangent makes an angle $\frac{\theta}{2}$ with each of the tensions, we have along the normal

$$2T \sin \frac{\theta}{2} = ma\theta \, . \, \omega^2 a,$$

where m is the mass per unit length,

$a\theta$ the length of the element AB,

$\therefore$ $ma\theta$ the mass " " "

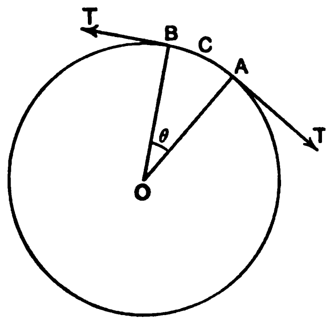


FIG. 117.

Now θ being small $\quad \sin\frac{\theta}{2}=\frac{\theta}{2}$ nearly;

$$\therefore \quad 2T.\frac{\theta}{2}=ma^2\omega^2\theta;$$

$$\therefore \quad T=ma^2\omega^2=mv^2,$$

if v is the linear velocity of the band.

If the angular velocity is great enough the band will break; for, if T_{max} is the maximum tension it can stand without breaking, it is only necessary to have

$$\omega>\sqrt{\frac{T_{max}}{ma^2}},$$

or the maximum angular velocity allowable is

$$\omega_{max}=\sqrt{\frac{T_{max}}{ma^2}}.$$

This equation can conveniently be put into another form. If A is the area of cross section of the belt in square cms., and T_0 the ultimate tensile strength, or the maximum tension in dynes per square centimetre which the material will stand without breaking, and ρ the density of the material,

$$T_{max} = T_0 A,$$
$$m = \rho A;$$
$$\therefore \ \omega_{max} = \sqrt{\frac{T_0}{\rho a^2}}; \quad \therefore \ v_{max} = \sqrt{\frac{T_0}{\rho}},$$

shewing that for a given material the maximum angular velocity is independent of the area of cross section, but depends on the radius of the circle, while the maximum linear velocity which equals $a\omega_{max}$ is independent of everything but the nature of the material.

Example **1.** A cast-iron fly-wheel of mean radius 12 ins. and cross section 4 square inches weighs 450 lbs. per cubic foot. Find

(i) the total tension when it makes 100 revolutions per min.,

(ii) the maximum number of revolutions per minute possible, and the maximum linear velocity if the ultimate tensile strength is 7·5 tons weight per sq. inch.

2. Shew that if the tension in a rotating ring is limited to T lbs.-wt. per sq. ft., the kinetic energy is limited to $\frac{T}{2}$ ft.-lbs. per cubic foot.

EXAMPLES.

1. A plate rotates in its own plane with uniform angular velocity ω about a fixed centre O. A circle of radius r and centre C, distant c from O, is drawn on the plate, and a point P describes this circle with uniform speed v relative to the plate. Find the velocity and acceleration of P relative to O when the angle OCP $= \theta$.

2. Prove that the force acting on a particle of mass m perpendicular to its direction of motion at the instant is $\frac{2mv^2y}{x^2}$ where y is the deflection from that line of motion, when an infinitesimal distance x along it has been described.

3. AB, BC are two rods of lengths a, b jointed together at B and capable of rotating in a plane about the point A. If the angles AB, BC make with a fixed line AX increase uniformly at rates ω, ω', find the velocity and acceleration of C at time t in terms of ω, ω', a, b, where t is the time measured from a moment at which the rods were in a straight line along AX.

a circle having its centre on AA′. Neglecting gravity, shew that the extended length of the string is

$$2l\Big/\left(1-\frac{m\omega^2 l}{2\lambda}\right)$$

where λ is the modulus of the string.

19. A particle C of mass m is attached by strings CA, CB to two fixed points A, B in the same vertical line, A being above B, and the whole rotates about AB. Shew that in order that the strings may be stretched ω^2 must be greater than $g/b \cos A$, and if this holds, find the tensions.

20. A bead C of mass m is capable of sliding on a string AB attached to two points A and B in the same vertical line, A being above B; if the whole length of the string is l, prove that the string is vertical if

$$\omega^2 < 2gl/(l^2-c^2),$$

and that if the string is not vertical $gl/ab\omega^2 = \cos A - \cos B$, and BC is horizontal if $\omega^2 = 2gl^2/c(l^2-c^2)$.

21. A particle C of mass m is attached by strings AC and BC, each of length a, to a fixed point A, and to a ring of mass M sliding without friction on a vertical rod AB. If C rotates about AB, prove that the strings will be inclined to the rod if

$$\omega^2 > \frac{g}{a}\cdot\frac{2M+m}{m}.$$

22. Two particles of masses m, m' are connected by an elastic string of modulus λ and natural length c, and can move on a horizontal table.

Prove that they can revolve in circles about a common centre with uniform angular velocity ω if

$$\omega^2 < \frac{\lambda}{c}\left(\frac{1}{m}+\frac{1}{m'}\right).$$

Explain what happens if ω^2 is greater than this.

23. A block of wood weighing 10 lbs. hangs by a light rope 8 ft. long. A bullet of 2 oz. is fired horizontally into the block, remaining embedded in it, and the rope swings to an angle 60° with the vertical. Find the velocity of the bullet. (Treat the block as a particle.)

24. Equal masses are attached to a weightless string at the corners of a regular polygon formed by it.

If the system rotates in a horizontal plane with uniform angular velocity round the centre of the polygon, prove that the tension of the string is ρv^2 where

$$\rho = \frac{\text{total mass of particles}}{\text{total length of string}},$$

and v is the speed of each particle.

25. If the string of a conical pendulum, whose bob is of mass m, passes up through a smooth ring of mass M, which can slide freely on a smooth vertical rod to which the upper end of the string is attached, prove that the angle of the pendulum is given by

$$\cos\theta = m/(\mathrm{M}+m),$$

and that

$$m\omega^2 l = (\mathrm{M}+m)g,$$

if l is the length of the inclined portion of the string.

26. The bobs of two equal pendulums of length l are connected by a string of length $2a$, and the whole rotates with angular velocity ω about a vertical line bisecting the string. Find the tension of the string, assuming the velocity sufficient to tighten it.

27. A simple pendulum of length l and of mass m is attached to a block of mass M resting on a rough horizontal plane. The bob is projected from the lowest position with energy sufficient to raise it through an arc of 30°. Find the velocity of the bob and the tension when an angle θ has been described, supposing the block not to shift, and shew that it will not shift if the coefficient of friction

$$> m\sqrt{3}/(4\mathrm{M}+3m).$$

CHAPTER VIII.

THE SIMPLE PENDULUM.

135. Simple Harmonic Motion on a Curve.

We have defined simple harmonic motion as a certain kind of motion in a straight line, but motion on a curve may also be spoken of as simple harmonic in the following sense.

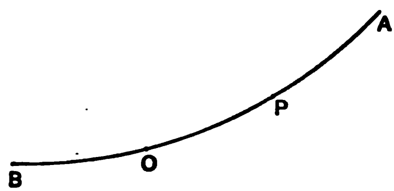


FIG. 118.

If a particle is moving along a curve AOB in such a way that the acceleration at P along the tangent is proportional to the distance, measured along the curve from a fixed point O, and is directed towards O, then the motion will have the same properties as the simple harmonic motion in a straight line, and can be called simple harmonic. For example, the period of oscillation will be independent of the amplitude of the oscillation.

In the pendulum we have a case of motion on a curve which is not exactly simple harmonic, but which becomes more and more nearly simple harmonic the smaller the amplitude of the oscillation.

136. The simple pendulum consists of a heavy particle or bob attached to a fixed point by a weightless string and swinging only in a vertical plane. It is thus a case of motion in a vertical circle which we have partly discussed already. We shall examine the motion a little further in the case when the oscillations are small, that is, when the string always makes a small angle with the vertical, say not more than 5°. The maximum angle made by the string with the vertical will be called the angular amplitude, or, simply, the amplitude.

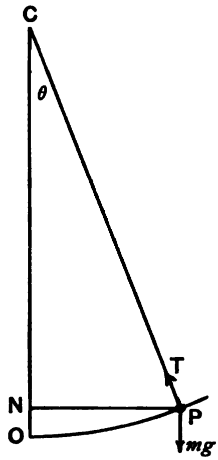


FIG. 119.

Let l = length of string,

$\angle OCP = \theta$ = angle it makes with vertical at any time.

arc $OP = s$,

so that

$$\theta = \frac{s}{l},$$

u = velocity at lowest point A,

v = ,, ,, any ,, P;

then we have

$$v^2 = u^2 - 2gl(1 - \cos\theta)$$

$$= u^2 - 4gl\sin^2\frac{\theta}{2}$$

$$= u^2 - 4gl\frac{\theta^2}{4} \text{ nearly}$$

$$= u^2 - gl\frac{s^2}{l^2}$$

$$= u^2 - \frac{g}{l}s^2 = \frac{g}{l}\left(\frac{lu^2}{g} - s^2\right).$$

If this equation was exact, the motion would be stric simple harmonic motion on the curve from a compar with the equation for S.H.M. giving the velocity in terms of the displacement, namely,

$$v=\sqrt{\mu(a^2-x^2)}.$$

As the equation is not exact, the motion is only approximately simple harmonic, the period being

$$2\pi\sqrt{\frac{l}{g}}.$$

We can arrive at the result, also, easily, by considering the forces acting. These are the tension and the weight. If we resolve along the tangent at P and away from O, we get

$$mf=-mg\sin\theta\,;$$
$$\therefore\ f=-g\sin\theta$$
$$=-g\theta \text{ nearly}$$
$$=-g\frac{s}{l}=-\frac{g}{l}s.$$

Thus the acceleration along the curve is nearly proportional to the displacement from O, and therefore the motion is nearly simple harmonic with period $2\pi\sqrt{\frac{l}{g}}$.

137. This result is, of course, only true in the ideal circumstances stated. In practice there are several points in which the experiments must differ from the theory. In the first place, no actual pendulum can agree with our definition of a simple pendulum, yet the error made when a bullet is attached to a long silk thread is very small. In the next place, the formula only applies to infinitely small amplitudes of swing, an accurate expression for finite amplitudes being

$$2\pi\sqrt{\frac{l}{g}}\left\{1+\frac{1}{4}\sin^2\frac{\alpha}{2}+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\sin^4\frac{\alpha}{2}+\ldots\right\},$$

where α is the amplitude.

Example. With this expression find the percentage increase in the period for amplitude 60° over that for infinitely small amplitude.

Thirdly, in ordinary experiments air resistance comes into play which slightly increases the period, and also causes the amplitude of oscillation to gradually decrease.

138. Seconds' Pendulum.

It will be borne in mind that we have used the term *period* to denote the time of a complete oscillation from the extreme position on one side to the other and back to the starting point. Half this period, or the time from one extreme to the other, is often called the time of swing, and formerly it was more usual to speak of the time of the swing than of the time of a complete oscillation. Hence it is that a pendulum whose time of swing is 1 second, or time of a complete oscillation 2 seconds, is called a seconds' pendulum. Thus the length of the seconds' pendulum is given by

$$2 = 2\pi\sqrt{\frac{l}{g}},$$

or

$$l = \frac{g}{\pi^2} = \frac{980}{9 \cdot 87} = 99 \cdot 3 \text{ cms.}$$

139. Value of g.

The pendulum gives much the easiest and most accurate method of finding the value of g, for, from the formula

$$t = 2\pi\sqrt{\frac{l}{g}},$$

we have

$$g = \frac{4\pi^2 l}{t^2}.$$

In even the roughest experiment with a pendulum about 6 feet long, t can be found to $\frac{1}{100}$ second or, say, to $\frac{1}{300}$ of the period by taking the total time of 100 oscillations, and l, likewise, is easy to measure to the same degree of accuracy, and the resulting value of g will be not more than 1 per cent. wrong.

Newton's Experiments. In consequence of the accuracy of determination of g, the pendulum affords the most convenient way of verifying the fact that g is the same for all bodies at the same place on the earth's surface.

Newton experimented by using pendulums with hollow bobs of the same size in which he could place various materials, and he found that the time of oscillation was the same in all cases for the same length of pendulum, and he consequently deduced that g was the same for all substances. More refined experiments of the same nature confirm Newton's result.

Compound pendulum. Any pendulum which is not a simple pendulum, and therefore any pendulum actually used for experiment, is called a compound pendulum. We will discuss the compound pendulum later.

140. Clock Rate.

A pendulum-clock consists essentially of a pendulum with apparatus for keeping up and counting its oscillations. If at one time the clock is going correctly, and afterwards for any reason the time of oscillation is changed, the clock will gain or lose accordingly.

Thus, if the period when going correctly is t secs., and it is increased to $t+\tau$, the number of oscillations in p secs. is diminished from

$$n=\frac{p}{t} \text{ to } n-\nu=\frac{p}{t+\tau},$$

and the change $$\nu=\frac{p}{t}-\frac{p}{t+\tau}=\frac{p\tau}{t(t+\tau)}.$$

We need only consider the case when $\frac{\tau}{t}$ is very small, as it will be always in practice.

We then have $\nu=\frac{p\tau}{t^2}$ very nearly, and the proportional loss in number of oscillations

$$=\frac{\nu}{n}=\frac{p\tau}{t^2}\Big/\frac{p}{t}=\frac{\tau}{t},$$

and as it loses ν oscillations in n, so it loses ν secs. in n secs.

Thus, in one day the clock loses

$$24\times 60\times 60\times\frac{\tau}{t}=86400\times\frac{\tau}{t}\text{ secs.}$$

141. *Effect of change in l or g.*

If the length l is increased by a quantity λ, for example, by increase in temperature, or if g is increased by γ, as when a clock is brought down towards sea-level from a height, the time of oscillation will be altered in either case. We may take the two effects separately, but it will be seen by taking the two at once that the total effect is the sum of the effects produced by each separately, an example of the general rule called the principle of the superposition of small effects.

We have, then, at first,

$$t=2\pi\sqrt{\frac{l}{g}}$$

afterwards

$$t+\tau=2\pi\sqrt{\frac{l+\lambda}{g+\gamma}}=2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{1+\lambda/l}{1+\gamma/g}}$$

$$=2\pi\sqrt{\frac{l}{g}}\left(\frac{1+\lambda/2l}{1+\gamma/2g}\right)\text{ approximately,}$$

λ/l and γ/g being always small fractions,

$$=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\lambda}{2l}-\frac{\gamma}{2g}\right);$$

$$\therefore\ \tau=2\pi\sqrt{\frac{l}{g}}\left(\frac{\lambda}{2l}-\frac{\gamma}{2g}\right)$$

$$\frac{\tau}{t}=\frac{\lambda}{2l}-\frac{\gamma}{2g}.$$

Thus, if there is only increase in length to consider, the percentage increase in the time of oscillation is half the percentage increase in the length.

When g changes but the length remains unchanged there is a percentage diminution in the time of oscillation equal to half the percentage increase in g.

142. *Change in g due to position.*

We have seen that g is affected by the rotation and shape of the earth, and is consequently different in different latitudes.

Again, if a person climbs a hill of height h, g is diminished to $g-\gamma$ according to the law of gravitation, the attraction of the earth being (nearly) inversely proportional to the square of the distance from the centre; thus, if R is the radius of the earth (in the same units as h),

$$\frac{g}{\frac{1}{R^2}}=\frac{g-\gamma}{\frac{1}{(R+h)^2}},$$

or

$$gR^2=(g-\gamma)(R+h)^2;$$

$$\therefore\ \gamma(R+h)^2=g(2Rh+h^2);$$

$$\therefore\ \frac{\gamma}{g}=\frac{h(2R+h)}{(R+h)^2}.$$

Now, h/R being very small, we can write this

$$\frac{\gamma}{g}=\frac{2h}{R}.$$

On the other hand, if a person descends a mine the value of g again changes. If the earth were of uniform density the attraction at an interior point would be proportional to the distance from the centre (as was proved by Newton), and we should have, if d is the depth,

$$\frac{g}{R}=\frac{g-\gamma}{R-d},$$

$$gR-gd=gR-\gamma R,$$

or

$$\frac{\gamma}{g}=\frac{d}{R}.$$

The density of the earth, however, increases towards the centre, and it has been found that g increases at first in going down a mine. (See Poynting, *Mean Density of the Earth*, p. 32.)

143. Forced Oscillations of a Simple Pendulum.

As a simple example of a forced oscillation, suppose the point of support P of a simple pendulum of length l is caused

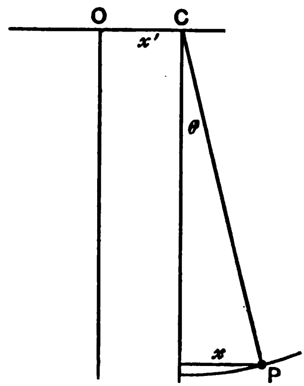


FIG. 120.

(" forced ") to move horizontally with simple harmonic motion of amplitude c and period T′.

Let $T\left(=2\pi\sqrt{\frac{l}{g}}\right)$ = period of the free oscillation. The position of C is given by

$$x' = c\cos\frac{2\pi t}{T'},$$

if we measure times from an instant when C is at its extreme position. The acceleration of C is therefore

$$-\frac{4\pi^2}{T'^2}c\cos\frac{2\pi t}{T'}.$$

The acceleration of P is the resultant of its acceleration relative to C and the acceleration of C.

If we call f the component acceleration perpendicular to CP of P relative to C, the total acceleration of P in this direction will be

$$f - \frac{4\pi^2}{T'^2} c \cos \frac{2\pi t}{T'} \cos \theta.$$

The component force perpendicular to CP is $-mg \sin \theta$;

$$\therefore \ m\left(f - \frac{4\pi^2}{T'^2} c \cos \frac{2\pi t}{T'} \cos \theta\right) = -mg \sin \theta.$$

If the oscillations are small, θ is always small and $\cos \theta = 1$, $\sin \theta = \theta$ nearly;

$$\therefore \ f - \frac{4\pi^2}{T'^2} c \cos \frac{2\pi t}{T'} = -g \sin \theta.$$

If x is the horizontal displacement of P relative to C,

$$\sin \theta = \frac{x}{l};$$

$$\therefore \ f = -\frac{g}{l} x + \frac{4\pi^2}{T'^2} c \cos \frac{2\pi t}{T'}$$

$$= -\frac{4\pi^2}{T^2} x + \frac{4\pi^2}{T'^2} c \cos \frac{2\pi t}{T'}.$$

Hence, if the forced oscillation is given by

$$x = a \cos \frac{2\pi t}{T'},$$

$$f = -\frac{4\pi^2}{T'^2} a \cos \frac{2\pi t}{T'};$$

$$\therefore \ -\frac{4\pi^2}{T'^2} a \cos \frac{2\pi t}{T'} = -\frac{4\pi^2}{T^2} a \cos \frac{2\pi t}{T'} + \frac{4\pi^2}{T'^2} c \cos \frac{2\pi t}{T'};$$

$$\therefore \ \frac{1}{T'^2} a = \frac{1}{T^2} a - \frac{1}{T'^2} c;$$

$$\therefore \ a = \frac{\frac{1}{T'^2}}{\frac{1}{T^2} - \frac{1}{T'^2}} c = \frac{T^2}{T'^2 - T^2} c;$$

$$\therefore \ x = \frac{T^2 c}{T'^2 - T^2} \cos \frac{2\pi t}{T'}.$$

Therefore if $T' > T$, or the oscillation of C is slower than the natural oscillation of the pendulum, the phases of the pendulum oscillation and of C are the same, that is, P and C are at their maximum displacement in the same direction at the same time.

But if the oscillation of C is faster than the natural oscillation of the pendulum P and C are at their maximum displacements in opposite directions at the same time, and will always be moving in opposite directions.

It must be noticed that the free oscillation may exist in addition to the forced one. For example, besides the forced oscillation of period T', the pendulum above may have superposed on this a free oscillation of period T. But it will be found in many cases that the free oscillation tends to die away, on account of friction of some kind, while the forced oscillation keeps up as long as the exciting cause remains.

EXAMPLES.

1. Find to three places of decimals the time of oscillation of a pendulum 6 ft. long if $g = 32 \cdot 18$ ft./sec^2.

2. Find the number of oscillations made in half an hour by a pendulum a metre long if $g = 980$.

3. A pendulum 180 cm. long is found to make 223 oscillations in 10 minutes. Find the value of g.

4. What change would be required in the length of the pendulum in Example 3 to alter the number of oscillations in 10 minutes to 225.

5. A clock keeps correct time at 10° C., what will it lose per day when the average temperature is 25°, supposing the coefficient of expansion of the rod of the pendulum to be 0·000011 per 1° C.?

6. If a clock goes correctly at 0° C., what rise of temperature will cause it to lose at the rate of 10 secs. per day if the coefficient of expansion of the rod is 0·000008?

7. A clock with a seconds pendulum is gaining 2 minutes a day. What alteration should be made in the length of the pendulum to make it go correctly?

8. Find the alteration in g due to going up a mountain a mile high if g is 980 cm./sec^2. at the foot, and shew that a clock going correctly at the foot would lose almost 21·8 secs. per day at the top.

9. A clock which was going correctly is taken to the top of a mountain and, at the same temperature, it is found that the pendulum has to be shortened by a two-thousandth of its length to keep it correct. Find the height of the mountain.

10. The string of a simple pendulum is 1 metre long and will break with a tension of 2·5 kgms. weight. If the bob is 2 kgms., shew that the vertical motion of the bob must not be greater than $\frac{1}{8}$ metre.

11. A simple pendulum makes oscillations of amplitude 15°. Shew that the tension of the string when vertical is 1·07 times the weight of the bob nearly.

12. A block of mass M rests on a smooth horizontal table and contains a smooth spherical cavity in which a particle of mass m rests. Starting from rest the block is kept moving with constant acceleration f by means of a horizontal force applied to it. Shew that the motion of the particle relative to the block is pendulum motion about a radius inclined at an angle α to the vertical where $\tan \alpha = f/g$, and that the ratio of the horizontal forces in the highest and lowest positions is

$$1 + 2mf^2/M(f^2+g^2).$$

CHAPTER IX.

PROJECTILES.

144. If a body is projected from the surface of the earth in any direction it is acted on by two forces, its weight, constant in magnitude and direction (the minute variation in this being negligible), and the air resistance. The latter follows a law which is not known accurately, being found experimentally to obey, apparently, different laws at different velocities, the laws given varying from the first power of the velocity at very low velocities to the sixth power at 1100 feet a second, and again diminishing to lower powers at higher velocities, though, of course, the resistance increases continuously from the lowest to the highest velocities.

The results in the present chapter will be obtained on the supposition that the resistance is negligible. They can consequently only represent the truth approximately when the velocities are small, such as when a stone is thrown by hand. For velocities such as occur in modern bullets and cannon shot, they cannot be trusted even to give an approximation to the truth. Notwithstanding this, the results are of great interest historically and otherwise.

145. Neglecting air resistance then, or, in other words, supposing the body to be projected in a vacuum, we have gravity only acting, and it is most convenient to take horizontal and vertical components throughout. We then base the work as usual on the independence of the two components of the motion. As there is only a vertical force (the weight), there is only a vertical acceleration g downwards.

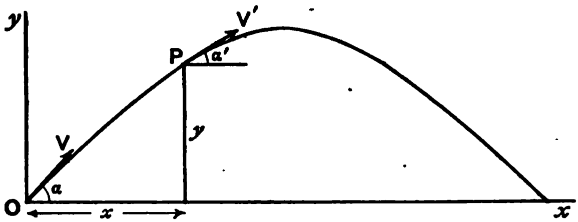


FIG. 121.

We shall use the following notation :

V = initial velocity or velocity of projection,

α = angle V makes with the horizontal or the angle of projection,

V' = velocity at the end of t secs.,

α' = angle V' makes with the horizontal,

x, y = the horizontal and vertical distances travelled in t secs.

Now the initial horizontal velocity $= V \cos \alpha$,
and " " vertical " upwards $= V \sin . \alpha$.

The horizontal acceleration $= 0$,
" vertical " $= -g$.

The motion in the horizontal direction is consequently unaccelerated, while the vertical motion obeys the laws of uniform acceleration.

Hence the component velocities at time t are :

in the horizontal direction $V \cos \alpha$, and
" vertical " $V \sin \alpha - gt$.

Also, the horizontal distance described in t secs. is

$$V \cos \alpha \, . \, t = x, \quad \text{.......(1)}$$

and the vertical $V \sin \alpha \, . \, t - \frac{1}{2}gt^2 = y$ ....(2)

The resultant velocity at time t is given in direction and magnitude by

$$V' \cos \alpha' = V \cos \alpha, \quad \text{.................................(3)}$$

$$V' \sin \alpha' = V \sin \alpha - gt, \quad \text{..........................(4)}$$

squaring and adding,

$$\therefore \; V'^2 = V^2 - 2Vgt \sin \alpha + g^2t^2, \quad \text{.........................(5)}$$

and dividing,

$$\tan \alpha' = \frac{V \sin \alpha - gt}{V \cos \alpha}, \quad \text{.......................(6)}$$

(5) can be written by the help of (2),

$$V'^2 = V^2 - 2gy, \quad \text{.....................................(7)}$$

which is the equation of energy, for the loss in kinetic energy

$$= \tfrac{1}{2}mV^2 - \tfrac{1}{2}mV'^2,$$

and the work done against gravity $= mgy$;

$$\therefore \; \tfrac{1}{2}mV^2 - \tfrac{1}{2}mV'^2 = mgy;$$

$$\therefore \; V^2 - V'^2 = 2gy.$$

146. *Time of Flight and Range.*

If the body is projected from a point in a horizontal plane, it strikes the plane again at a time T given by $y = 0$,

or
$$V \sin \alpha \, . \, T - \tfrac{1}{2}gT^2 = 0,$$

$$T = \frac{2V \sin \alpha}{g}. \quad \text{..........................(8)}$$

This time is consequently called the time of flight.

The distance from the point of projection to the point where it strikes the plane again is the Range, and is given by (1), t being the time of flight;

or
$$\text{Range} = R = \frac{2V \sin \alpha}{g} . V \cos \alpha$$

$$= \frac{2V^2 \sin \alpha \cos \alpha}{g} = \frac{V^2 \sin 2\alpha}{g}. \quad \text{............(9)}$$

If the velocity of projection is given, the range is greatest when

$$\sin 2\alpha = 1,$$

or

$$2\alpha = 90^\circ,$$

$$\alpha = 45^\circ,$$

the maximum range being

$$R_{max} = \frac{V^2}{g}. \qquad \text{.............................(10)}$$

To hit a given object on the horizontal plane with a given velocity of projection, we have R, a given value;

$$\therefore \frac{V^2 \sin 2\alpha}{g} = R,$$

$$\sin 2\alpha = \frac{gR}{V^2}.$$

If $\frac{gR}{V^2} < 1$, in other words, if $R <$ maximum range there are two values for 2α, one acute and one obtuse, and if we call these $2\alpha_1$ and $2\alpha_2$,

$$2\alpha_2 = 180^\circ - 2\alpha_1;$$

$$\therefore \alpha_1 + \alpha_2 = 90^\circ;$$

$\therefore$ α_1 and α_2 are complementary, or putting it in the form

$$\alpha_2 - 45^\circ = 45^\circ - \alpha_1,$$

the two directions of projection make equal angles with the direction giving the maximum range.

147. *Greatest Height Reached for given Velocity and Angle of Projection.*

When the projectile is at the highest point it must be moving, at the instant, horizontally, or the vertical velocity is zero. Hence the time to the highest point is given by

$$V \sin \alpha - gt = 0; \qquad \text{...............................by (4)}$$

$$\therefore t = \frac{V \sin \alpha}{g} = \tfrac{1}{2} \text{ the time of flight.}$$

The maximum height is now given by (2),

$$y_{max} = V\sin\alpha \cdot \frac{V\sin\alpha}{g} - \tfrac{1}{2}g\frac{V^2\sin^2\alpha}{g}$$

$$= \frac{V^2\sin^2\alpha}{2g},$$

and the horizontal distance at the maximum height is

$$\frac{V^2\sin\alpha\cos\alpha}{g}.$$

148. *Form of Path.*

Returning to equations (1) and (2), we have from (1):

$$t = \frac{x}{V\cos\alpha},$$

and if we substitute in (2)

$$y = V\sin\alpha\frac{x}{V\cos\alpha} - \tfrac{1}{2}g\frac{x^2}{V^2\cos^2\alpha}$$

or

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}. \quad \text{.......................(11)}$$

This equation, giving y as a quadratic function of x, shews that the path is a parabola, but we will deduce more about the path in another way.

149. *Symmetry of Path.*

To find the horizontal distance from the point of projection when the particle is at a given height h, we have to solve equation (11),

$$\frac{gx^2}{2V^2\cos^2\alpha} - x\tan\alpha + h = 0,$$

or

$$x^2 - \frac{2V^2\cos^2\alpha\tan\alpha}{g}x + \frac{2V^2h\cos^2\alpha}{g} = 0.$$

This gives two values of x, and if we call them x_1 and x_2, we have from the theory of quadratic equations

$$x_1 + x_2 = \frac{2V^2\cos^2\alpha\tan\alpha}{g} = \frac{2V^2}{g}\sin\alpha\cos\alpha;$$

$$\therefore\ x_1 + x_2 = R$$

= twice horizontal distance of the highest point,

or $$x_2 - \frac{R}{2} = \frac{R}{2} - x_1.$$

Hence the two points at a height h are equally distant horizontally from the highest point, hence the curve is symmetrical about a line through the highest point.

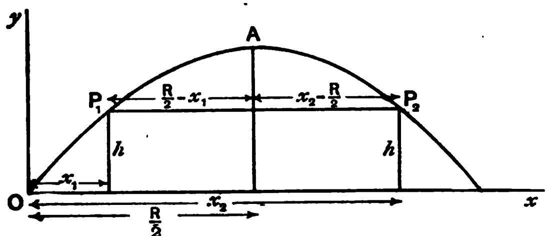


FIG. 122.

Also, since the horizontal velocity is constant, the time from P_1 to A is the same as from A to P_2 at the same height as P_1.

The time to a given height h may be deduced directly from (2) thus:

$$\tfrac{1}{2}gt^2 - V \sin\alpha \cdot t + h = 0$$

or $$t^2 - \frac{2V}{g}\sin\alpha \cdot t + \frac{2h}{g} = 0,$$

and if t_1 and t_2 are the two solutions,

$$t_1 + t_2 = \frac{2V \sin\alpha}{g}$$

= twice the time to highest point.

Shewing again that the time from height h to the highest point is the same as from the highest point to the height h.

These results shew that the curve is symmetrical, and that the time taken in travelling along a portion of the upward path is the same as that in travelling along the similar portion of the downward. In other words, the motion from the highest point to the ground is exactly the reverse of the motion from the ground to the highest point.

150. To examine the nature of the curve more exactly, it is therefore sufficient to start from the highest point. At that point the horizontal velocity is $V \cos \alpha$ and vertical

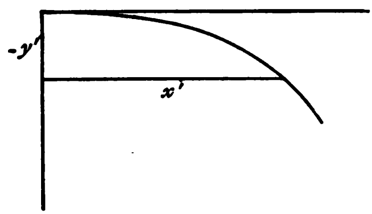


FIG. 123.

velocity 0. t' secs. after the particle has passed the highest point the horizontal and vertical distances x', y' from the highest point are given by

$$x' = V \cos \alpha \,.\, t',$$

$$y' = -\tfrac{1}{2}gt'^2;$$

$$\therefore\ y' = -\frac{gx'^2}{2V^2\cos^2\alpha}$$

$$= -\frac{g}{2V^2\cos^2\alpha}\,.\,x'^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

This is the fundamental property of the parabola—the square of the ordinate is proportional to the abscissa.

151. *Note on Geometrical Properties of the Parabola.*

The parabola is generally defined as the locus of a point P, which moves so that its distance from a fixed point S is equal to its distance from a fixed line MX.

S is called the focus and MX the directrix.

With this definition, in the figure,

$$PS = PM,$$
$$AS = AX,$$
$$PN^2 = PS^2 - SN^2 = PM^2 - SN^2$$
$$= XN^2 - SN^2 = (XN + NS)(XN - NS)$$
$$= 2AN \cdot 2AS,$$

or
$$PN^2 = 4AS \cdot AN. \quad \text{.......................................(13)}$$

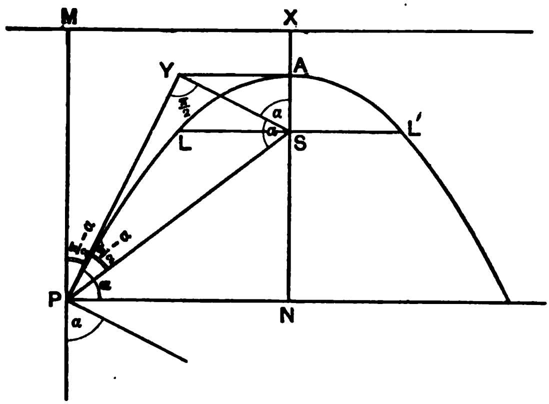


FIG. 124.

If LSL′, parallel to MX, is drawn through S, LL′ is called the latus rectum, and the size of the parabola is determined by the length of LL′.

Now $LL' = 2SL = 2SX = 4SA.$

152. Comparing (13) with (12), we see that the projectile describes a parabola whose latus rectum is

$$\frac{2V^2\cos^2\alpha}{g} \quad \text{.............................(14)}$$

The height of the focus $= SN = AN - AS$

$$= \frac{V^2\sin^2\alpha}{2g} - \frac{V^2\cos^2\alpha}{2g} = -\frac{V^2\cos 2\alpha}{2g},$$

and the focus is below the horizontal plane if $\alpha < 45°$.

The height of the directrix = XN

$$= AN + AX = \frac{V^2 \sin^2\alpha}{2g} + \frac{V^2 \cos^2\alpha}{2g} = \frac{V^2}{2g};$$

hence the height of the directrix is the height from which a body would have to drop vertically to get up a velocity equal to the actual velocity in the path. This is often expressed by saying that the velocity at any point is that due to a fall from the directrix.

153. *Curvature of a Parabola.*

The following results are proved in any book on geometrical conics.

(1) The tangent PY bisects the angle between PS and PM.

(2) If SY is perpendicular to PY, Y is on the tangent at A.

(3) The triangles PSY, YSA are similar.

Assuming these results, it is interesting to deduce from the dynamical results the curvature κ of the parabola at any point.

For, since the acceleration along the normal at P is κV^2, and the normal makes an angle α with the vertical,

$$\kappa V^2 = g \cos\alpha,$$

but

$$V^2 = 2g \,.\, PM = 2g \,.\, PS;$$

$$\therefore \kappa = \frac{\cos\alpha}{2 \,.\, PS}.$$

Now, $\angle SPY = \angle MPY = \frac{\pi}{2} - \alpha$;

$$\therefore \cos\alpha = \sin SPY = \frac{SY}{SP} = \frac{AS}{SY} = \sqrt{\frac{SY \,.\, AS}{SP \,.\, SY}} = \sqrt{\frac{AS}{SP}};$$

$$\therefore \kappa = \frac{AS^{\frac{1}{2}}}{2 \,.\, SP^{\frac{3}{2}}}.$$

154. We may obtain an easy graphical representation of the velocity at any point. Let OA represent the initial velocity in direction and magnitude, and AB, BC, CO ... each represent

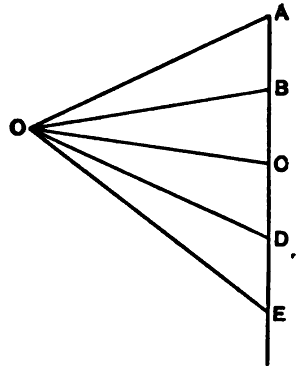


FIG. 125.

g on the same numerical scale as the velocity. Then it will be seen at once that OB, OC, OD ... represent the magnitude and direction of the velocity at the end of 1, 2, 3 ... secs.

ABC ... is, in fact, the hodograph as defined in Art. 117.

155. *Direction of Projection to Hit a given Point.*

To hit a point Q whose horizontal and vertical distances from the point of projection are x, y, we have to solve (11) for α, that is,

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha)$$

$$\tan^2\alpha - \frac{2V^2}{gx}\tan\alpha + 1 + \frac{2V^2y}{gx^2} = 0,$$

giving a quadratic equation for $\tan\alpha$, and two possible directions of projection if the roots are real,

i.e. if $$\frac{4V^4}{g^2x^2} > 4\left(1+\frac{2V^2y}{gx^2}\right),$$

i.e. if $$V^4 > g^2x^2 + 2V^2gy,$$

i.e. if $$(V^2-gy)^2 > g^2(x^2+y^2),$$

i.e. if $$V^2 > g(y+\sqrt{x^2+y^2}).$$

Therefore, if $V^2 < g(y+\sqrt{x^2+y^2})$ it is impossible to project the body to pass through the required point.

Example 1. A stone is projected with a velocity of 100 ft./sec. at an elevation of 30° from a tower 150 ft. high. Find

(i) the time of flight,

(ii) the point where it strikes the horizontal through the foot of the tower,

(iii) the magnitude and direction of the velocity on striking the ground.

Here, if we measure upwards from the point of projection, the vertical distance described when it strikes the ground is -150 ft. The component initial velocities are $100\times\frac{1}{2}$ vertically and $100\times\frac{\sqrt{3}}{2}$ horizontally;

$\therefore$ if t is the time of flight,

$$50t-16t^2=-150;$$

$$\therefore\ t=5 \quad\text{or}\quad -\tfrac{15}{8},$$

the negative solution not being applicable.

The horizontal distance of the point where it strikes the ground from the tower is

$$100\times\frac{\sqrt{3}}{2}\times 5=433 \text{ ft.}$$

The vertical velocity at the ground is

$$50-32\times 5=-110,$$

the horizontal velocity is $100\frac{\sqrt{3}}{2}=86{\cdot}6$.

The resultant velocity is 140 ft./sec., and the angle it makes with the horizontal is

$$\tan^{-1}\frac{110}{50\sqrt{3}}=51°\,47'.$$

2. A body is projected so that the components of the initial velocity are 110 ft./sec. horizontally and 75 ft./sec. vertically.
Find the range, time of flight, and maximum height reached.

3. A body is projected with velocity 90 ft./sec. at an angle of 60° with the horizontal.
Find its position, resultant velocity, and direction of motion at the end of 3·5 secs.

4. If a projectile has a range of 15 miles and remains 30 secs. in the air, find the initial velocity and angle of projection (on the supposition of no air resistance).

5. If the maximum range of a gun is 18 miles, find (on the supposition of no air resistance) the initial velocity and the maximum height reached.

6. What is the greatest range for a particle projected with a velocity of 80 ft./sec., and what would be the angle of projection to give half this range ?

7. What is the least velocity with which a cricket ball can be thrown 100 yds. ?

8. A body is projected with a velocity of 95 ft./sec. at an angle of 34°. Find the range and greatest height reached.

9. A gun is fired horizontally at a height of 6 ft. above the ground. If the shot strikes the ground at a distance of 1200 ft., find its initial velocity.

10. A stone is thrown with velocity 60 ft./sec. from a height of 20 ft. above level ground, and at an elevation of 20°. Find where it strikes the ground, and the magnitude and direction of its velocity then.

11. A stone is projected with velocity V at an elevation α from a point at a height h above the horizontal plane. Prove that the range R on the horizontal plane is given by

$$R^2 - \frac{2V^2}{g} R \sin\alpha\cos\alpha - \frac{2hV^2}{g}\cos^2\alpha = 0,$$

and explain the meaning of the negative solution of the equation.

12. If the range on the level is 250 feet, and the maximum height reached $31\frac{1}{4}$ feet, find the initial velocity and angle of projection.

13. If t is the time from the point of projection to a point P at the height h, and t' is the time from P to the ground again, prove that $h = \frac{1}{2}gtt'$.

14. At what angle should a body be projected with velocity 80 ft./sec. to hit a point on the horizontal at a distance of 120 ft. ?

15. At what angle should a body be projected with velocity

80 ft./sec. to just pass over a wall 12 ft. 6 ins. high at a distance of 100 ft. ?

16. Find the velocity and direction required to throw a stone to just pass horizontally over a wall 8 ft. above the thrower's shoulder and 60 ft. away.

17. Will a stone thrown with velocity 70 ft./sec. at an elevation of 20° pass over a fence at a horizontal distance of 30 yds. and 4 feet higher than the point where it leaves the thrower's hand ?

18. A stone is thrown with a velocity of 80 ft./sec. at an elevation of 30° from the top of a vertical cliff 200 ft. high. Find where it would strike the water.

Where would it strike the water if it was thrown at an angle of depression of 30° ?

Find in each case the direction and magnitude of the velocity on striking the water.

19. If a stone is projected with velocity V from height h to hit a point on the level at a horizontal distance R from the point of projection, shew that the angle of projection is given by

$$R^2\tan^2\alpha-\frac{2V^2}{g}.\,R\tan\alpha+R^2-\frac{2hV^2}{g}=0.$$

Hence deduce that the maximum range on the level for this velocity is

$$\sqrt{V^4/g^2+2hV^2/g},$$

and that if R' is this maximum range and α the angle of projection to give the maximum

$$\tan\alpha=V^2/gR',$$
$$\tan 2\alpha=R'/h.$$

20. If a body is projected with velocity V to hit a point Q at horizontal and vertical distances x, y from the point of projection, and if $V^2=g(y+\sqrt{x^2+y^2})$, prove that there is one value of α only. Shew that in this case if $\frac{y}{x}=\tan i$,

$$\tan\alpha=\sec i+\tan i,$$

and hence that the direction of projection bisects the angle between OQ and the vertical, or $\alpha=\frac{\pi}{4}+\frac{i}{2}$.

When

$$V^2>g(y+\sqrt{x^2+y^2}),$$

so that there are two values α_1 and α_2.

prove that

$$\tan(\alpha_1+\alpha_2)=\tan\left(\frac{\pi}{2}+i\right)$$
$$=\tan 2\alpha.$$

and therefore that the two directions of projection make equal angles with the vertical and OQ.

21. Find the velocity and direction with which a body must be projected to just pass over two parallel walls 20 feet apart, and each 6 feet higher than the point of projection if the nearer is 30 feet from the point of projection. Find also the maximum height to which it will rise.

22. Two walls are 40 ft. apart, and one is 6 ft. and the other 10 ft. higher than the point of projection of particle, projected from a point 20 feet from the lower wall so as to just pass over the two walls. Find the initial velocity and direction of projection.

156. *Errors Produced by a Small Error in the Elevation.*

To hit an object at a horizontal distance x and vertical height y above the point of projection the angle of elevation is determined by the equation

$$y = x \tan \alpha - \frac{gx^2}{2V^2}(1 + \tan^2 \alpha).$$

If α is determined correctly, but a small error θ is made in projecting the body so that the angle of projection is $\alpha + \theta$ where θ is small, so that we can omit θ^2, etc., then at a horizontal distance x the height of the projectile will be $y + k$, say, where k is small (compared with x). We proceed to find k for a given θ. If we suppose θ given in circular measure, we have

$$\tan \theta \approx \theta,$$

where $\approx$ stands for, 'is nearly equal to';

$$\therefore \quad \tan(\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta}$$

$$\approx \frac{\tan \alpha + \theta}{1 - \theta \tan \alpha}$$

$$\approx (\tan \alpha + \theta)(1 + \theta \tan \alpha)$$

$$\approx \tan \alpha + \theta(1 + \tan^2 \alpha)$$

$$\approx \tan \alpha + \theta \sec^2 \alpha;$$

$$\therefore \quad \tan^2(\alpha + \theta) \approx (\tan \alpha + \theta \sec^2 \alpha)^2$$

$$\approx \tan^2 \alpha + 2\theta \tan \alpha \sec^2 \alpha.$$

Now, $y+k=x\tan(\alpha+\theta)-\frac{gx^2}{2V^2}\{1+\tan^2(\alpha+\theta)\}$

$$\approx x(\tan\alpha+\theta\sec^2\alpha)$$

$$-\frac{gx^2}{2V^2}(1+\tan^2\alpha+2\theta\tan\alpha\sec^2\alpha);$$

$$\therefore\ k\approx\theta\left(x\sec^2\alpha-\frac{gx^2}{2V^2}.2\tan\alpha\sec^2\alpha\right)$$

$$\approx\theta\sec^2\alpha\left(x-\frac{gx^2}{V^2}\tan\alpha\right).$$

If θ is measured in degrees,

$$k\approx\frac{\pi\theta}{180}\sec^2\alpha\left(x-\frac{gx^2}{V^2}\tan\alpha\right).$$

If the point aimed at is on the horizontal plane through the point of projection, x is the range,

$$\therefore\ x=\frac{2V^2}{g}\sin\alpha\cos\alpha;$$

$$\therefore\ k\approx\frac{\pi\theta}{180}x\sec^2\alpha\left(1-\frac{gx}{V^2}\tan\alpha\right)$$

$$\approx\frac{\pi\theta}{180}x\sec^2\alpha(1-\tan\alpha.2\sin\alpha\cos\alpha)$$

$$\approx\frac{\pi\theta}{180}x\sec^2\alpha(1-2\sin^2\alpha)$$

$$\approx\frac{\pi\theta}{180}x\sec^2\alpha(\cos^2\alpha-\sin^2\alpha)$$

$$\approx\frac{\pi\theta}{180}x(1-\tan^2\alpha).$$

This expression gives the height at which the projectile will pass over the object aimed at. It is positive if $\alpha<45°$, but if $\alpha>45°$ (or the higher angle of projection to hit the point is chosen), the error causes the projectile to fall short of the object.

Hence, substituting in (11), x is given by

$$x \tan i = x \tan \alpha - \frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha};$$

$$\therefore\ x = \frac{2V^2\cos^2\alpha}{g}(\tan\alpha - \tan i);$$

$$\therefore\ \text{the range} = R = \frac{2V^2\cos^2\alpha}{g}(\tan\alpha - \tan i)\sec i$$

$$= \frac{2V^2\cos^2\alpha}{g}\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin i}{\cos i}\right)\frac{1}{\cos i}$$

$$= \frac{2V^2}{g}\frac{\cos\alpha}{\cos^2 i}(\sin\alpha\cos i - \cos\alpha\sin i)$$

$$= \frac{2V^2}{g}\frac{\cos\alpha}{\cos^2 i}\sin(\alpha - i).$$

To find the maximum range for a given velocity but different angles of projection, it is convenient to put this in the form

$$R = \frac{V^2}{g}\frac{1}{\cos^2 i}\{\sin(2\alpha - i) - \sin i\}, \quad \text{.............(13)}$$

and the first term in the bracket being the only one that varies, R is greatest when

$$2\alpha - i = \frac{\pi}{2},$$

and

$$R_{max} = \frac{V^2}{g\cos^2 i}(1 - \sin i)$$

$$= \frac{V^2(1 - \sin i)}{g(1 - \sin^2 i)} = \frac{V^2}{g(1 + \sin i)}.$$

The value of α, which gives the maximum range, is given by

$$2\alpha - i = \frac{\pi}{2},$$

or

$$\alpha + (\alpha - i) = \frac{\pi}{2},$$

shewing that the direction of projection makes an angle $\alpha - i$ with the vertical, and therefore bisects the angle between the vertical and the inclined plane.

To hit a given point on the plane, in other words, to get a given range R, we have to solve the equation (13), or

$$\sin(2\alpha - i) - \sin i = \frac{Rg\cos^2\alpha}{V^2};$$

$$\therefore \sin(2\alpha - i) = \sin i + \frac{gR\cos^2\alpha}{V^2},$$

giving two values of $2\alpha - i$ (if R is less than R_{max}), calling the two values of α, α_1 and α_2.

$$2\alpha_1 - i + 2\alpha_2 - i = \pi,$$

$$\alpha_1 + \alpha_2 = \frac{\pi}{2} + i,$$

or

$$\alpha_1 - i = \frac{\pi}{2} - \alpha_2,$$

which expresses the fact that the one direction is inclined to the plane at the same angle as the other is to the vertical.

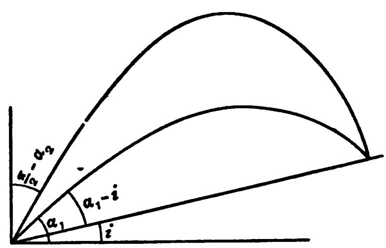


FIG. 128.

In the same way the range of an inclined plane not passing through the origin may be found.

The maximum range of any inclined plane may also be found conveniently by finding the intersection of the plane with the envelope of the paths.

Example. Find the maximum range on a plane of inclination i which meets the horizontal through the point of projection at distance a from the point of projection.

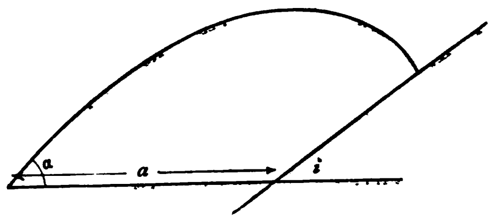


FIG. 129.

159. The motion of a projectile relative to an inclined plane may often be conveniently dealt with by resolving the velocity and acceleration into components along and perpendicular to the incline.

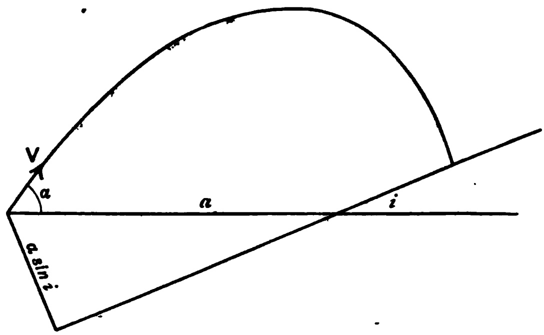


FIG. 130.

Thus the initial velocity may be resolved into components :

$V \cos(\alpha - i)$ parallel to the plane upwards,

$V \sin(\alpha - i)$ perpendicular to the plane away from the plane.

The components of acceleration are :

$-g \sin i$ parallel to the plane,

$-g \cos i$ perpendicular to the plane.

We have now uniform acceleration in each of the two directions, and the component velocities at time t are

$$V\cos(\alpha-i)-g\sin i\,.\,t, \text{ parallel to the plane;}$$

and $$V\sin(\alpha-i)-g\cos i\,.\,t, \text{ perpendicular to the plane;}$$

and the distances described are

$$V\cos(\alpha-i)\,.\,t-\tfrac{1}{2}g\sin i\,.\,t^2, \text{ parallel to the plane;}$$

$$V\sin(\alpha-i)\,.\,t-\tfrac{1}{2}g\cos i\,.\,t^2, \text{ perpendicular to the plane.}$$

From which any results can be deduced in the method first used in this chapter.

Illustrations of the foregoing theory.

1. A body is projected with a velocity V from the foot of an inclined plane of inclination i. Find the direction of projection that it may strike the plane perpendicularly.

With the notation of the last article, if t is the time until it strikes the plane, the velocity parallel to the plane is then zero, and the distance described perpendicular to the plane also zero.

$$\therefore\ V\cos(\alpha-i)-g\sin i\,.\,t=0,$$

and $$V\sin(\alpha-i)t-\tfrac{1}{2}g\cos i\,.\,t^2=0;$$

$$\therefore\ \tan(\alpha-i)=\tfrac{1}{2}\cot i,$$

giving $\alpha-i$.

Notice that the result does not depend on the initial velocity.

2. A particle is projected from a point on a plane of inclination i at an angle θ with the plane to hit an object on the plane. In artillery θ is called the tangent elevation, $\theta+i$ is the quadrant elevation. The range (as we have seen in Art. 158) is

$$R=\frac{2V^2}{g}\frac{\cos\alpha}{\cos i}\sin(\alpha-i)$$

$$=\frac{2V^2}{g}\frac{\cos(\theta+i)\sin\theta}{\cos^2 i}.$$

Now the larger V is, the smaller is θ, and for a small value of θ,

$$R=\frac{2V^2}{g}\frac{\cos i\,.\,\theta}{\cos^2 i},$$

if θ is in circular measure;

$$\therefore\ \theta=\frac{gR\cos i}{2V^2}.$$

Now if i is only a few degrees, $\cos i$ is nearly unity, and θ is practically the same whether the line of sight is horizontal ($i=0$) or inclined at a small angle. For example, if $i=10^\circ$ the tangent elevation is only diminished by $1\frac{1}{2}$ per cent. of its value for the same horizontal range.

3. A person in a boat throws a stone at elevation α and with velocity V relative to the boat, which is moving with velocity v in the direction of the object aimed at. Find the range.

If the ratio v/V is small, find the alteration in the elevation necessary on account of the motion of the boat to hit a point at a given horizontal distance.

The horizontal and vertical velocities of the stone are

$$V\cos\alpha + v, \quad V\sin\alpha;$$

$$\therefore \text{ the range is } \quad \frac{2V\sin\alpha(V\cos\alpha+v)}{g}.$$

To hit an object at horizontal range R, if α is the elevation for a fixed point of projection, and $\alpha-\theta$ the elevation for the case of the moving boat,

$$R=\frac{2V^2\sin\alpha\cos\alpha}{g}=\frac{2V\sin(\alpha-\theta)\{V\cos(\alpha-\theta)+v\}}{g};$$

$$\therefore \quad V\sin\alpha\cos\alpha = V\sin(\alpha-\theta)\cos(\alpha-\theta)+v\sin(\alpha-\theta);$$

$$\therefore \quad \sin 2\alpha - \sin 2(\alpha-\theta)=\frac{2v}{V}\sin(\alpha-\theta),$$

$$2\sin\theta\cos(2\alpha-\theta)=\frac{2v}{V}\sin(\alpha-\theta).$$

If $\frac{v}{V}$ is small, θ is small, and

$$\theta\cos 2\alpha=\frac{v}{V}\sin\alpha,$$

θ being in circular measure;

$$\therefore \quad \theta=\frac{v\sin\alpha}{V\cos 2\alpha}.$$

4. Stones are projected with a maximum velocity U ft./sec. over a wall of height k at a horizontal distance h from the point of projection. Prove that those that fall nearest to the wall are projected at the highest angle and with the greatest velocity, and find the breadth of the zone of safety on the ground behind the wall.

If a stone projected with velocity V at an angle just grazes the wall,

$$k = h\tan\alpha - \frac{gh^2}{2V^2\cos^2\alpha}\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

also the range
$$R = \frac{2V^2}{g}\sin\alpha\cos\alpha\,;$$

$$\therefore\quad k = h\tan\alpha - \frac{h^2}{\cos^2\alpha}\,\frac{\sin\alpha\cos\alpha}{R}$$

$$= h\tan\alpha - \frac{h^2}{R}\tan\alpha\,;$$

$$\therefore\quad R = \frac{h^2\tan\alpha}{h\tan\alpha - k}.$$

The distance behind the wall of the point where it strikes the ground is

$$R - h = \frac{hk}{h\tan\alpha - k}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and this (the breadth of the zone of safety) diminishes as α increases, or the smallest value is given by making α as large as possible.

Putting the breadth of the zone of safety

$$= R' = R - h\,;$$

$$\therefore\quad h\tan\alpha - k = \frac{hk}{R'}\,;$$

$$\therefore\quad \tan\alpha = k\left(\frac{1}{h} + \frac{1}{R'}\right);$$

and substituting in (1), we have

$$k = hk\left(\frac{1}{h} + \frac{1}{R'}\right) - \frac{gh^2}{2V^2}\left\{1 + k^2\left(\frac{1}{h} + \frac{1}{R'}\right)^2\right\};$$

$$\therefore\quad \frac{hk}{R'} = \frac{gh^2}{2V^2}\left\{1 + k^2\left(\frac{1}{h^2} + \frac{2}{hR'} + \frac{1}{R'^2}\right)\right\};$$

$$\therefore\quad R'^2\left(1 + \frac{k^2}{h^2}\right) + 2R'\left\{\frac{k^2}{h} - \frac{V^2k}{gh}\right\} + k^2 = 0,$$

giving
$$R' = \frac{hk}{g(h^2+k^2)}\{V^2 - gk \pm \sqrt{(V^2-gk)^2 - g^2(h^2+k^2)}\}. \quad \ldots\ldots\ldots(3)$$

This gives the two values of R' corresponding to the two values of the angle of projection to graze the wall, and the smaller value, (with the negative sign), corresponding to the higher elevation, has to be taken.

We can write this

$$R' = \frac{hk}{g(h^2+k^2)} \cdot \frac{g^2(h^2+k^2)}{V^2 - gk + \sqrt{(V^2-gk)^2 - g(h^2+k^2)}}$$

$$= \frac{ghk}{V^2 - gk + \sqrt{(V^2-gk)^2 - g^2(h^2+k^2)}}, \quad \text{..................(4)}$$

and since V^2 is necessarily greater than gk, R′ diminishes as V increases; hence, if the maximum velocity is U, the breadth of the zone of safety is found by putting U for V in (3) or (4).

5. Stones are projected with a maximum velocity of 60 ft./sec. at a boy who shelters himself behind a wall 10 ft. high and 60 ft. from the point of projection. Find the safe distance from the wall at the ground and at 5 ft. above the ground.

Here, substituting in the equation,

$$y = x \tan \alpha - \frac{gx^2}{2V^2}(1 + \tan^2 \alpha).$$

We find $\tan \alpha = \frac{13}{4}$ or $\frac{1}{2}$.

Taking the higher inclination the range is 63·2 ft. nearly, and the safe distance at the ground is 3·2 ft. To find the safe distance at a height of 5 ft. put $y = 5$, and we have

$$5 = \tfrac{13}{4}x - \tfrac{185}{3600}x^2,$$

giving $x = 61{\cdot}6$ nearly,

or the safe distance $= 1{\cdot}6$ ft.

6. A particle is projected with velocity 90 ft./sec. at an angle 20° with a plane of inclination of 10°.

Find its range on the plane when the projection is

(1) up the plane,
(2) down the plane.

7. A body is projected from the foot of an inclined plane of elevation 30° with velocity 80 ft./sec. at an angle 60° with the horizontal. Find

(1) the time of flight,
(2) its greatest distance from the plane,
(3) its range on the plane,
(4) the velocity and direction of motion when it strikes the plane.

8. If the maximum distance a person can throw a cricket ball on the level is 100 yds., what is the maximum distance on an upward slope of 1 in 5?

9. A body is projected with velocity V at the elevation required for maximum range on a plane of inclination i. Find the direction of motion when it strikes the plane, and prove that it is independent of V.

10. Shew that the greatest height which a projectile with initial velocity V can reach on a vertical wall at a distance a from the point of projection is

$$V^2/2g - ga^2/2V^2.$$

EXAMPLES.

1. Two bodies projected at the same time with the same velocity from a point O in different directions strike the same point A in the horizontal plane at the end of t, t' secs. respectively. Prove that $OA = \frac{1}{2}gtt'$.

2. The path of an unresisted projectile fired from O with velocity V at elevation α intersects a straight line inclined at an angle i to the horizontal in points P and Q. Shew that the middle point of PQ is at a horizontal distance $V^2 \cos^2 \alpha (\tan \alpha - \tan i)/g$ from O, and find the distance from O of the parallel straight line which is grazed by the projectile.

3. Of two bodies projected from O with the same velocity, one strikes the top of a pillar of height h in t secs. and the other the foot A (on the same horizontal as O) in t' secs.; prove that

$$OA^2 = \{g^2t^2t'^2 - (2h + gt^2)^2\}t'^2/\{4(t'^2 - t^2)\}.$$

4. Material is to be projected over a horizontal ledge a direct distance l from A and a vertical depth h below it.

Shew that the work required is at least $\frac{l-h}{2}$ ft.-lbs. per lb. of material.

5. Two particles are projected from the same point O at an interval of time T with the same velocity in the same direction. The first is above the level of O when the second is projected. Shew that their shortest distance apart occurs when they are at the same level, and $= u$T where u is the horizontal component of the velocity of projection. Find, also, the time at which this occurs.

6. A railway carriage is travelling at 60 km./hr. on a straight track. A small heavy body is thrown out horizontally with a velocity relative to the carriage of 10 m./sec. at right angles to its length. Neglecting the resistance of the air, find [illegible] body reaches the level ground 3 m. below the poin[illegible]

7. Find the possible angles of projection for a stone to be thrown with velocity 80 ft./sec. from the top of a tower 100 ft. high to hit a point on the ground 200 ft. from the foot of the tower.

8. A projectile is to have a range R on the horizontal, and to reach a maximum height h; find the time of flight, the initial velocity and elevation.

9. A particle is projected with velocity v to hit a point at a horizontal distance x and vertical distance y from the point of projection. Obtain an equation to find the time of flight, and deduce the condition that the point can be hit with the given initial velocity.

10. From a gun placed on a horizontal plane which can fire a shell with velocity $\sqrt{2gH}$ it is required to throw a shell over a wall of height h, and the elevation of the gun cannot exceed α where $\alpha < 45°$. Shew that this will be possible only if $h < H \sin^2\alpha$, and that if this condition is satisfied the gun must be fired from within a strip of the plane whose breadth is $4 \cos\alpha \sqrt{H(H \sin^2\alpha - h)}$.

11. A shot fired at a mark in a horizontal plane goes a feet beyond it. When a screen of thickness t is placed at the muzzle of the gun, and perpendicular to the length of the gun, the shot falls b feet short. Prove that the shot will hit the mark if the thickness of the screen is reduced to $\frac{at}{a+b}$. (Assume the velocity to be uniformly retarded as it passes through the screen.)

12. A gun fires a shot with velocity 800 ft./sec. to hit an object 900 ft. above the point of projection, and at a horizontal distance $9000\sqrt{3}$ feet from it. At what elevation should the gun be pointed?

If the gun is pointed a quarter of a degree above the lower of the two possible elevations, at what height above the object will it pass, to the nearest foot?

13. A ball is projected from a given point with velocity V so as to strike a vertical wall above a height h. Prove that the points on the wall towards which the ball can be directly projected lie within a circle of radius $\sqrt{\frac{V^4}{g^2} - \frac{hV^2}{g} - a^2}$, if a is the distance of the wall from the point of projection.

14. A straight pipe of length a can be supported in any direction, one end being attached to a given tap on the surface of the ground. Water issues from the free end with a constant velocity $\sqrt{gh}$. Prove that when the pipe is inclined at angle α to the horizontal the range is given by

$$R^2 - 2R(a \cos\alpha + h \sin\alpha \cos\alpha) + a^2 \cos^2\alpha = 0.$$

Deduce that the range is greatest when

$$h \cos 2\alpha = 2a \sin^3\alpha, \quad \text{.....................................(i)}$$

and that the greatest range is $a \cos\alpha/\cos 2\alpha$, α being given by (i).

15. The portion of a vertical wall that can be covered by a jet from a fire engine at a distance c from it is a parabola of height $(4h^2 - c^2)/4h$ and breadth $2\sqrt{4h^2 - c^2}$ if the velocity of the jet is $\sqrt{2gh}$.

16. A projectile is to pass through a point x, y and be there travelling in a direction making an angle β with the horizontal; find the velocity and direction of projection.

17. A projectile fired from a given point grazes a wall of height h at a distance a, and reaches the ground at a distance R. Shew that the elevation is given by $\tan\alpha = \frac{Rh}{a(R-a)}$, and find the velocity of projection.

18. A small horizontal target is to be hit by a projectile fired from a point at a distance h vertically below and x horizontally from the target. Shew that the target can be hit either on the upper or the lower side if

$$V^2 > 2gh\left(1 + \frac{x^2}{4h^2}\right),$$

and on the upper side only if

$$V^2 < 2gh\left(1 + \frac{x^2}{4h^2}\right),$$

and

$$> g\{h + \sqrt{h^2 + x^2}\},$$

while it cannot be hit at all if

$$V^2 < g\{h + \sqrt{h^2 + x^2}\}.$$

19. A smooth sphere is fixed on a horizontal table, and a particle runs down it having been just displaced from rest at the highest point. Find where it hits the table.

20. Oil is flying off a horizontal axle of radius r, which is rotating with angular velocity ω. Shew that no drop reaches a height above the centre line of the axle greater than

$$\omega^2 r^2/2g + g/2\omega^2.$$

21. AB is the horizontal line of intersection of two inclines of elevations θ, ϕ. A ball is to be thrown with given velocity V from the former so as to land as high as possible on the other. Shew that the projection must be from a point in AB unless $2\theta + \phi > \frac{\pi}{2}$. If this inequality is satisfied, shew that the distance from AB that can be reached is $V^2/2g \cos\theta \sin(\theta + \phi)$, and find the point of projection.

22. A particle projected from the foot of an inclined plane strikes the plane at right angles at a distance d from the point of projection. Shew that its greatest distance from the plane during the flight is $\frac{1}{4}d \cot \alpha$, if α is the inclination of the plane to the horizon.

23. A particle is projected with velocity u from the top of a tower of height h on an inclined plane. Shew that the greatest distance from the foot of the tower that the particle can fall is

$$\frac{u^2 \tan \alpha \sec \alpha}{g} + \sec^2 \alpha \sqrt{\frac{u^4}{g^2} + \frac{2u^2h}{g} \cos^2 \alpha}$$

if α is the inclination of the plane.

24. A particle is projected from a point whose perpendicular distance from an incline of elevation 60° is h. Prove that it cannot strike the plane at right angles if the square of the velocity of projection

$$< \tfrac{1}{2} gh(\sqrt{13} - 1).$$

25. If β is the inclination of an inclined plane to the vertical, and α the angle between the direction of projection and the incline, shew that the range when $\alpha \neq \frac{\beta}{2}$ is less than the range when $\alpha = \frac{\beta}{2}$ by $2V^2 \sin^2\left(\alpha - \frac{\beta}{2}\right) \Big/ g \sin^2 \beta$.

26. A particle moving on the inside of a smooth vertical circle of radius a has a velocity of $\sqrt{\frac{7}{2} ga}$ at the lowest point. Find where it will leave the circle, and find the position of the highest point reached afterwards.

CHAPTER X.

COLLISION.

160. We have already given some of the simplest results relating to collision, and in the present chapter we wish to examine more fully into the actions which take place between the colliding bodies.

When two bodies collide actions go on between them which, in general, involve forces which are large in comparison with the forces, such as gravity, to which the bodies are usually subject. This follows from the fact that the bodies are in contact for a very short time, usually a small fraction of a second, which interval, however, is sufficient to produce an appreciable, it may be large, change of momentum in each body. Hence the rate of change of momentum or the force acting must be large.

On colliding the bodies undergo a deformation near the point of contact, and, in consequence, remain for a short time in contact with one another over a small area, and the bodies separate again as the deformation disappears.

The deformation is shewn on a large scale when a wet rubber ball strikes a wooden floor. The ball leaves a mark of considerable area, shewing that it was deformed by the collision.

We may imagine the time of contact to be divided into two parts, during the first of which the deformation increases gradually to a maximum, and during the second diminishes again till the bodies separate completely. The pressure between the two bodies alters correspondingly, increasing gradually, but rapidly, to a maximum, which occurs when the deformation is a maximum, and then diminishing to zero as the bodies separate.

The physical theory of collision was worked out by Hertz in 1882 for bodies colliding with small relative velocity, and he obtained the following results for two equal spheres of the same material and radius r colliding directly with relative velocity v:

The time during which the spheres are in contact is $ar/v^{\frac{1}{5}}$.

The radius of the circular areas that come into contact $=brv^{\frac{2}{5}}$.

The total pressure at maximum deformation $=cr^2v^{\frac{6}{5}}$.

The maximum pressure intensity $=dv^{\frac{2}{5}}$.

Where a, b, c, d are constants expressible in terms of the elastic properties of the material.

Hertz gives the following results for two steel spheres of radii 2·5 cm. and relative velocity 1 cm./sec.:

Time of impact $=0·00038$ sec.

Radius of circular area of contact $=0·013$ cm.

Total pressure at maximum deformation $=2470$ gms. wt.

Maximum pressure intensity (pressure per square centimetre) $=7300$ kgm./cm.^{2}.

Experiments have since been made on the time of contact, and have confirmed Hertz's results.

161. Direct Impact of Two Spheres.

Let us now examine into the relations between the velocities before and after the collision, taking first the case when the spheres are moving in the same line, and let us represent the velocities thus:

	(m_1)	(m_2)
before collision	$\rightarrow u_1$	$\rightarrow u_2$
after collision	$\rightarrow u_1'$	$\rightarrow u_2'$

FIG. 131.

Now if we consider the distance between the centres, at the time of the greatest compression this distance is a minimum, and therefore its rate of change is then zero, or the relative velocity of the centres is zero. Thus, at the instant of greatest compression the velocities of the two balls must be the same. Let this velocity be U.

Now in whatever way the pressure may vary during the impact, it still holds as a result of the law of action and reaction that at any instant the pressure on m_1 is equal and opposite to the pressure on m_2, and consequently the impulse in any time on m_1 is equal and opposite to that on m_2, and hence again the change of momentum in any time on m_1 is equal and opposite to that on m_2. Hence also the total momentum is unchanged.

Putting these statements in symbols we have

$$m_1(u_1 - u_1') = -m_2(u_2 - u_2'),$$

or

$$m_1u_1 + m_2u_2 = m_1u_1' + m_2u_2'. \quad \text{.................(1)}$$

The common velocity U at time of greatest compression is given by

$$m_1U + m_2U = m_1u_1 + m_2u_2,$$

$$U = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1u_1' + m_2u_2'}{m_1 + m_2}. \quad \text{..............(2)}$$

162. To get a further relation between the velocities we have to return to experiments, and the great importance of the law of action and reaction in Newton's theory led in his time to many investigations on its truth, in which experiments on impact formed an important part. For just as we have deduced the equality of the total momentum before and after impact from the equality of action and reaction, so we may deduce the latter from the former, measuring the momentum experimentally.

Newton himself made the most accurate and complete series of experiments at the time by a method which is briefly as follows:

Two balls of the same or different materials and weights are hung up, each by two threads, so that they touch at the height of their centres when at rest, and can move in the same vertical plane.

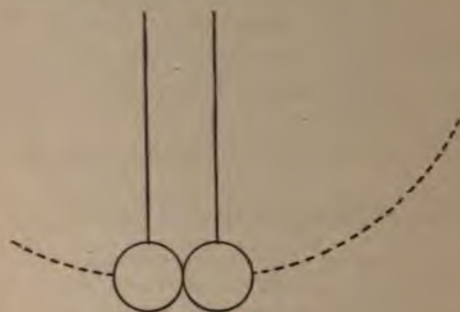


FIG. 132.

When drawn aside and released they move as pendulums, and if released at the same instant they reach the lowest point together, and rebound. The velocities before and after impact can be deduced at once from the height from which they fall and the height to which they rise afterwards.

Let a, a' be the heights above the position of rest from which they fall,

„ b, b' be the heights above the position of rest to which they rise afterwards,

and suppose the velocities of both were reversed by the collision; then the velocities just before the collision were

$\sqrt{2ga}$, $\sqrt{2ga'}$ towards one another,

and just after $\sqrt{2gb}$, $\sqrt{2gb'}$ away from one another.

By varying a, a' and the masses, Newton deduced his experimental law of impact that

The relative velocity after impact bears a ratio to the relative velocity before impact which is independent of the masses and velocities before impact, and depends only on the nature of the balls.

This may be expressed thus:

$$u_2' - u_1' = -e(u_2 - u_1) \qquad (3)$$

and the factor e may be called the *coefficient of restitution* or *coefficient of impact.*

The equation (3) can only be taken as an approximate experimental result. In other words, the equation is satisfied approximately for two given balls colliding with the small relative velocities occurring in the experiments. At higher velocities different coefficients might be required to represent the facts.

It will be noted carefully that the relative velocity after impact is always in the opposite direction to the relative velocity before; hence the minus sign in the above equation.

The student should always subtract the velocities in the same order, and insert the minus sign.

The value of e differs for different materials, and as examples of its magnitude may be quoted from Hodgkinson's Report to the British Association, 1834, the following:

For two balls of the same material

Glass	0·94
Ivory	0·81
Cast iron	0·66
Cork	0·65
Lead	0·20

The coefficient of restitution can never be greater than unity.

The two equations (1) and (3) are sufficient to find the velocities after impact when the velocities before, the masses, and coefficient of restitution are given.

Example. Two bodies of masses 6 and 4 lbs., and coefficient of restitution 0·5, are approaching one another with velocities 10 and 8 feet per second, find the velocities after collision.

Masses	6 lbs.	4 lbs.
Velocities before	$\rightarrow 10$	$\leftarrow 8$
,, after	$\rightarrow v$	$\rightarrow v'$.

FIG. 183.

By momentum $6v+4v'=6\times 10+4\times(-8)=28.$

By Newton's experimental law,

$$v-v'=-\tfrac{1}{2}\{10-(-8)\}=-9,$$

whence $v=-0{\cdot}8, \quad v'=8{\cdot}2.$

163. Impulse and Kinetic Energy.

To help to get an idea of the actions going on during the time of contact, we will calculate the impulse and change in kinetic energy during each of the two parts of the interval, namely, from beginning up to the time of greatest compression, and secondly, from the time of greatest compression to the end.

Let $\mathrm{I}_1=$ the impulse in the first part

and $\mathrm{I}_2=$ the impulse in the second,

then

$$\begin{aligned}
\mathrm{I}_1 &= m_1u_1-m_1\mathrm{U}\\
&= m_1\left\{u_1-\frac{m_1u_1+m_2u_2}{m_1+m_2}\right\}\\
&= \frac{m_1m_2}{m_1+m_2}(u_1-u_2)\\
\mathrm{I}_2 &= m_1\mathrm{U}-m_1u_1'\\
&= m_1\left\{\frac{m_1u_1'+m_2u_2'}{m_1+m_2}-u_1'\right\}\\
&= \frac{m_1m_2}{m_1+m_2}(u_2'-u_1')\\
&= \frac{m_1m_2}{m_1+m_2}e(u_1-u_2)\\
&= e\mathrm{I}_1,
\end{aligned}$$

and the total impulse

$$=I_1(1+e)=\frac{m_1m_2}{m_1+m_2}(1+e)(u_1-u_2).$$

Let $\quad E_1=$ the loss of kinetic energy in the first part

and $\quad E_2=$ the loss in the second part.

$$E_1=\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2-\tfrac{1}{2}m_1U^2-\tfrac{1}{2}m_2U^2$$

$$=\tfrac{1}{2}\{m_1u_1^2+m_2u_2^2-(m_1+m_2)U^2\}$$

$$=\tfrac{1}{2}\left\{m_1u_1^2+m_2u_2^2-(m_1+m_2)\left(\frac{m_1u_1+m_2u_2}{m_1+m_2}\right)^2\right\}$$

$$=\tfrac{1}{2}\left\{m_1u_1^2+m_2u_2^2-\frac{m_1^2u_1^2+2m_1m_2u_1u_2+m_2^2u_2^2}{m_1+m_2}\right\}$$

$$=\tfrac{1}{2}m_1m_2\frac{u_1^2+u_2^2-2u_1u_2}{m_1+m_2}$$

$$=\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2$$

and

$$E_2=\tfrac{1}{2}m_1U^2+\tfrac{1}{2}m_2U^2-\tfrac{1}{2}m_1u_1'^2-\tfrac{1}{2}m_2u_2'^2$$

$$=\tfrac{1}{2}\left\{(m_1+m_2)\left(\frac{m_1u_1'+m_2u_2'}{m_1+m_2}\right)^2-m_1u_1'^2-m_2u_2'^2\right\}$$

$$=-\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1'-u_2')^2$$

$$=-\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}e^2(u_1-u_2)^2$$

$$=-e^2E_1.$$

Thus in the first part kinetic energy is lost, being changed into energy of deformation and molecular energy, and is partly restored during the second part, but with a loss on the whole of

$$(1-e^2)E_1 \quad \text{or} \quad \tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

164. This result may also be obtained in the following way, illustrating former equations.

The change of kinetic energy due to an impulse I acting on a body whose velocity changes from u to v is

$$I\frac{u+v}{2}. \quad \text{(Art. 42.)}$$

Now the total impulse

$$=I=\frac{m_1m_2}{m_1+m_2}(1+e)(u_1-u_2)$$

and acts in the direction shewn.

	(m_1)	(m_2)
Impulse	$I \leftarrow$	$\rightarrow I$
Velocity before impact	$u_1 \rightarrow$	$\rightarrow u_2$
" after "	$u_1' \rightarrow$	$\rightarrow u_2'$

FIG. 134.

Thus m_1 loses kinetic energy $=I\dfrac{u_1+u_1'}{2}$

and m_2 gains $I\dfrac{u_2+u_2'}{2}$;

$\therefore$ the total loss is

$$\frac{I}{2}\{u_1+u_1'-u_2-u_2'\}$$

$$=\frac{I}{2}\{u_1-u_2-e(u_1-u_2)\}$$

$$=\tfrac{1}{2}I(1-e)(u_1-u_2)$$

$$=\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

Considering this expression for the loss of energy, we see that each factor is necessarily positive for $e<1$, and $(u_1-u_2)^2$ is positive whether u_1-u_2 is positive or negative. If in any case e were >1 kinetic energy would be gained at the collision, which seems impossible. Hence energy is always lost except in the two special cases

(1) where $u_1=u_2$ and there is no true collision;
(2) when $e=1$.

The latter case does not occur in any actual body, though e approaches unity in the case of glass spheres, but in the kinetic theory of gases, the molecules of a gas are usually regarded as spheres for which $e=1$. Such spheres are frequently called perfectly elastic.

An interesting case occurs when two equal and perfectly elastic spheres collide. Here $e=1$ and $m_1=m_2$; hence the equation of momentum is

$$m_1u_1+m_1u_2=m_1u_1'+m_1u_2'$$

$$\therefore\ u_1+u_2=u_1'+u_2' \quad\text{.......(1)}$$

and Newton's experimental law gives

$$u_1'-u_2'=-(u_1-u_2)$$

$$=-u_1+u_2 \quad\text{.......(2)}$$

and from (1) and (2)

$$\left.\begin{aligned}u'&=u_2\\ u_2'&=u_1\end{aligned}\right\}$$

or the two spheres interchange velocities.

Example 1. Two spheres, A and B, of masses 2 and 3 lbs., moving in the same direction, in the same straight line, with velocities 6 and 3 ft./sec. respectively, collide. After the impact the velocity of B is 5 ft./sec.; find that of A, the coefficient of impact, and the loss of kinetic energy.

2. Two spheres, A and B, of masses 4 and 8 lbs., moving with velocities 9 and 3 ft./sec. in opposite directions, collide. If A rebounds with velocity 1 ft./sec., find the velocity of B after the impact, the coefficient of impact, and the loss of kinetic energy.

3. If two balls of equal mass and moving with velocities u_1, u_2 in the same straight line collide, prove that the velocities after collision are

$$\tfrac{1}{2}(1-e)u_1+\tfrac{1}{2}(1+e)u_2 \quad\text{and}\quad \tfrac{1}{2}(1+e)u_1+\tfrac{1}{2}(1-e)u_2.$$

4. A and B are masses of 9 and 6 lbs. moving in the same direction with velocities 8 and 4 ft./sec. respectively. Prove that after the collision the velocity of B cannot be greater than 8·8, nor that of A smaller than 4·8 ft./sec.

5. B is a mass of 10 lbs. moving with velocity 12 ft./sec., A is moving in the opposite direction with velocity 8 ft./sec. Shew

that, if the mass of A is greater than 40 lbs., it cannot possibly have its velocity reversed in direction, after the collision.

Find the velocities after collision if

(i) A is 20 lbs. and $e=0{\cdot}6$,

(ii) A is 40 lbs. and $e=0{\cdot}8$.

6. A sphere of mass m overtakes and impinges directly on a sphere of mass m', the coefficient of impact being e. Shew that the former cannot have its velocity reversed if $m>em'$.

7. A is a sphere of mass m moving with velocity u, and overtakes B, whose mass is em where e is the coefficient of impact. Shew that the velocity of B after collision is u, and that of A is

$$(1-e)u+eu'.$$

8. A, B, C are three balls of the same material in a straight line. A is projected towards B with velocity u, and B afterwards collides with C. Find the velocity of B after impact with C, and the final velocity of C. If the masses of A and C are given, shew that the greatest velocity will be given to C when the mass of B is a geometric mean between those of A and C.

9. A, B, C are three balls of the same material at rest in a straight line. A is projected towards B. Shew that if the masses are equal, no second collision between A and B can take place.

10. A series of n equal balls are in a row with a coefficient of impact e between each pair. The first is started with velocity u towards the next. Shew that the velocity of the rth ball after the collisions is $(1-e)(1+e)^{r-1}u/2^r$, except the last, whose velocity is $(1+e)^{n-1}u/2^{n-1}$.

11. A ball is projected vertically upwards with velocity 80 ft./sec., and when it is at the highest point a similar ball is projected vertically upwards from the same point with the same velocity. If the coefficient of impact is $0{\cdot}6$, find when each reaches the ground again.

12. Two equal balls impinge horizontally at a height h above the ground with relative velocity $2v$. Shew that they reach the ground at a distance $2ev\sqrt{2h/g}$ apart.

165. Oblique Impact of Two Smooth Spheres.

Let us next consider the case where the colliding spheres are not moving in the line joining the centres. We will use the following notation:

Let u_1, v_1 be the components of velocity of m, along and perpendicular to the line of centres just before impact.

Let u_2, v_2 be corresponding components for m_2.
,, u_1', v_1' ,, components for m_1 just after impact.
,, u_2', v_2' ,, corresponding components for m_2.
,, V_1, V_2 ,, resultant velocities before impact.
,, V_1', V_2' ,, ,, ,, after ,,
,, α_1, α_2 ,, angles the resultant velocities make with the line of centres before impact.
,, α_1', α_2' ,, angles the resultant velocities make with the line of centres after impact.

These can be conveniently represented on a diagram thus:

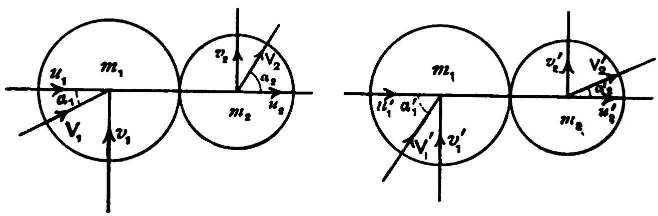


FIG. 135.

Component and resultant velocities before impact. Component and resultant velocities after impact.

We will at first suppose the spheres are frictionless, that is, that the impulse between them is entirely along the line joining the centres.

We then have

(1) v_1, v_2 are unaltered by the collision, for there is no impulse perpendicular to the line of centres

or $$v_1' = v, \quad v_2' = v_2.$$

(2) The total momentum along the line of centres is unchanged

or $$m_1u_1' + m_2u_2' = m_1u_1 + m_2u_2.$$

(3) Newton's experimental law holds for the velocities along the line of impact

or $$u_1' - u_2' = -e(u_1 - u_2).$$

These equations may, of course, be expressed in terms of V_1, V_2, V_1', V_2', α_1, α_2, α_1', α_2', but if the resultant velocities are given in direction and magnitude, they must always be resolved in the two directions as above to write down the equations.

The total loss of kinetic energy

$$= \tfrac{1}{2} m_1(u_1^2 + v_1^2) + \tfrac{1}{2} m_2(u_2^2 + v_2^2)$$
$$- \tfrac{1}{2} m_1(u_1'^2 + v_1'^2) - \tfrac{1}{2} m_2(u_2'^2 + v_2'^2)$$
$$= \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 - \tfrac{1}{2} m_1 u_1'^2 - \tfrac{1}{2} m_2 u_2'^2$$
$$= \tfrac{1}{2} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_1')^2$$

by the same working as before.

166. Oblique Impact of Rough Spheres.

When two spheres in general collide, there is a frictional impulse perpendicular to the line of centres, and this impulse is usually assumed to be μ times the normal impulse, where μ is the ordinary coefficient of friction between the two bodies. Assuming this, the equations will be formed in the following way:

(i) The momentum in each direction is unchanged;

$$\therefore\ m_1u_1' + m_2u_2' = m_1u_1 + m_2u_2 \quad \text{......(1)}$$

and

$$m_1v_1' + m_2v_2' = m_1v_1 + m_2v_2 \quad \text{......(2)}$$

(ii) Newton's experimental law holds for the velocities in the line of centres;

$$\therefore\ u_1' - u_2' = -e(u_1 - u_2) \quad \text{......(3)}$$

(iii) If I is the impulse on either ball in the line of centres

$$I = m_1(u_1 - u_1')$$

and the impulse at right angles to this is μI or $\mu m_1(u_1 - u_1')$;

$$\therefore\ m_1(v_1 - v_1') = \mu m_1(u_1 - u_1') \quad \text{......(4)}$$

These four equations are sufficient to find the component velocities after impact, when the velocities before impact, the masses, the coefficient of restitution, and coefficient of friction are given.

167. Impact against a Fixed Plane.

In the cases considered above, two spheres were colliding. If we suppose one of these spheres to increase indefinitely in size and mass, we reach ultimately the case of a sphere impinging against a fixed plane. In this case the equation of momentum will drop out, but Newton's experimental equation will still hold for the velocity perpendicular to the plane. We need not, of course, deduce our equations from the former case, but will form them afresh in the same way.

Let u, v be the components of velocity perpendicular and parallel to the plane before impact, and u', v' the components after impact.

As u is necessarily towards the wall, and u' away from it, we will measure these components as positive when in opposite directions, so that we may represent the velocities thus:

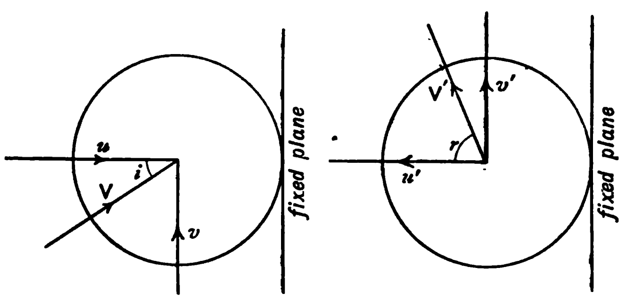


FIG. 136.

i. Component and resultant velocities before impact. ii. After impact.

If there is no frictional impulse, v is unchanged by the impact, but u is reversed in direction and diminished in magnitude so that

$$v' = v$$

$$u' = eu.$$

The relation between the angles of incidence (i) and of reflection (r) becomes therefore

$$\cot r = \frac{u'}{v'} = \frac{eu}{v} = e \cot i.$$

If there is friction, we must calculate the normal impulse by the help of the experimental law, and deduce the tangential impulse, which is μ times the normal.

Thus with the same notation

$$u' = eu \qquad \text{.......(1)}$$

$\therefore$ normal impulse $= m(u+u') = mu(1+e)$;

$\therefore$ tangential impulse $= \mu mu(1+e)$;

$\therefore$ $m(v-v') = \mu mu(1+e)$;

$$\therefore\ v' = v - \mu u(1+e) \qquad \text{.......(2)}$$

and (1) and (2) give the solution of the problem.

Example **1.** If a sphere drops from a height h on to a horizontal plane, and rebounds to a height nh where n is a proper fraction, prove that the coefficient of impact is $\sqrt{n}$.

2. A sphere drops from a height h on to a horizontal plane, and the coefficient of impact is e; if it is allowed to continue rebounding, prove the following results:

(1) after the first impact it rises to a height he^2,

(2) between the nth and $(n+1)$th impacts it rises to a height he^{2n},

(3) the interval of time between the nth and $n+1$th impact is $2e^n\sqrt{\frac{2h}{g}}$.

Hence shew that though it makes an infinite number of rebounds they occupy a total time from when it was first released of

$$\sqrt{\frac{2h}{g}} \cdot \frac{1+e}{1-e},$$

and that it travels a total distance (up and down) of

$$h\frac{1+e^2}{1-e^2}.$$

Find the whole time occupied to the end of the rebounding, and the whole distance travelled if the ball is dropped from a height of 16 ft. and $e = 0{\cdot}6$.

3. An elastic ball is let drop on to a horizontal plane. At a height of 9 ft. above the plane the velocity is 32 ft./sec. in falling and 18 ft./sec. on rising. Find the coefficient of impact, and prove that after the next impact it will not reach the height of 9 ft.

4. A particle is projected from a point on a smooth horizontal plane with velocity V at an elevation α, and continues to rebound. Shew that the range between the nth and $(n+1)$th impacts is $V^2e^n \sin 2\alpha/g$, and hence that the total horizontal distance travelled before rebounding ceases is $V^2 \sin 2\alpha/g(1-e)$, and the time during which it is rebounding is $2V \sin \alpha/g(1-e)$.

5. A particle after falling from rest through a distance h strikes a smooth plane inclined at an angle i to the horizontal. If the coefficient of impact is e, prove that the range between the first and second impacts is $4he(1+e) \sin i$.

6. A particle after falling from rest through h feet strikes a smooth plane of inclination 30° and rebounds, striking the plane again h feet lower down; prove that $e=0{\cdot}37$ nearly.

7. A ball moving at angle of 45° with the normal to a rough plane impinges on the plane and rebounds. If $e=\frac{1}{2}$ and $\mu=\frac{1}{3}$, shew that after the impact the direction of motion again makes the angle 45° with the normal and the velocity is half of what it was before the collision.

8. Shew that if a body rebounds from a rough plane, making the angle of reflexion equal to the angle of incidence, these angles are each

$$\tan^{-1}\left(\mu\frac{1+e}{1-e}\right).$$

9. A ball is projected with velocity 48 ft./sec. at elevation 15°, and, after striking a smooth vertical wall at a distance of 12 feet, returns to the point of projection. Prove that $e=0{\cdot}5$.

10. A ball is projected with velocity V at an elevation α, and, after striking a smooth vertical wall at a distance a, returns to the point of projection. Prove that

$$ga(1+e)=eV^2 \sin 2\alpha.$$

Shew that it will also return to the starting point if projected at the complementary elevation.

11. A ball strikes a smooth plane. If the angles of incidence and reflection are 40° and 60°, find e.

EXAMPLES.

1. A body A impinges directly on another B at rest. If the mass of B, the kinetic energy of A, and the coefficient of restitution are given, shew that the velocity of B after impact is greatest when the mass of A is equal to the mass of B.

2. A cylinder of mass M fits a hole bored centrally in another cylinder of mass M′ so that there is a constant resistance F to their relative motion. A mass m impinges axially on the end of M with velocity v. Assuming the initial motion of M is the same as if M′ were not present, and taking a coefficient of restitution e, find the subsequent displacement of M relative to M′ (which it is not supposed to leave).

3. A mass m_1 impinges directly on a mass m_2 at rest, the coefficient of restitution being e. Shew that after impact

(i) m_1 continues to move in the same direction if $m_1 > em_2$,

(ii) the momentum of m_1 is the greater if $m_1 > m_2(1+2e)$,

(iii) the kinetic energy of m_1 is the greater if

$$2m_1 > m_2\{1+4e+e^2+(1+e)\sqrt{1+6e+e^2}\},$$

or

$$2m_1 < m_2\{1+4e+e^2-(1+e)\sqrt{1+6e+e^2}\}.$$

4. A ball is projected along a smooth horizontal table with velocity u normal to a wall, and impinges directly on a similar ball at rest between it and the wall.

If the coefficient of restitution between the two balls, and also between the ball and the walls, are each e, prove that if the velocity of the first ball is never reversed, its velocities before successive impacts are given by

$$2u_{n+2}-(1-e)^2u_{n+1}+2e^2u_n=0.$$

Hence prove that the velocity will not be reversed if

$$1-6e+e^2>0.$$

5. Three equal and similar balls A, B, C lie on a smooth table with their centres in a straight line. A is projected directly towards B. The final velocity of C is $\frac{9}{16}$ of the initial velocity of A. Shew that the coefficient of restitution is $\frac{1}{2}$.

6. If two particles of masses m, m' moving with velocities v, v' impinge directly, prove that the condition that each loses the same amount of kinetic energy is

$$(3+e)(mv+m'v')+(1-e)(mv'+m'v)=0.$$

7. A small ball hangs suspended by a string of length l and is in contact with a vertical wall. The ball is pulled back so that the string makes an angle θ with the vertical in a plane normal to the wall and let go. Given that the coefficient of restitution is e, shew that the angular amplitude of the rebound after n impacts is $2\sin^{-1}\left(e^n \sin\frac{\theta}{2}\right)$.

8. Two balls of masses 1 and 3 lbs. are connected by an elastic string of modulus 10 lbs.-wt., and held apart so that the length of the string is 1 ft. If the natural length of the string is 6 inches, find the velocities they possess when they collide, and, if the coefficient of restitution is 0·8, find their velocities after impact, and the maximum length of string after the first impact.

9. A small elastic sphere is projected with velocity V from the foot of a vertical wall, and strikes a second parallel wall at a distance a, and after rebounding strikes the first wall at P. Shew that the greatest height of P above the point of projection is

$$\frac{1}{2g}\left\{V^2 - \frac{(1+e)^2 g^2 a^2}{e^2 V^2}\right\}.$$

10. A particle moving on a smooth horizontal table strikes alternately two smooth vertical walls inclined at an angle 45°. If the coefficient of restitution is e, and at the fourth impact the particle is moving perpendicularly to the wall then struck, shew that $e < \sqrt{2} - 1$.

11. A particle is projected from a point in an inclined plane, and at the rth impact strikes the plane perpendicularly, and at the nth is at the point of projection.

Prove that $$e^n - 2e^r + 1 = 0.$$

12. A sphere of mass m' on a horizontal table is tied to a fixed point by a stretched inelastic string. Another sphere of mass m impinges directly on it with velocity u in a direction making an angle α with the direction of the string (produced), and is reduced to rest by the collision. Prove that if v, w are the velocities of m' along the perpendicular to the line of impact after the collision,

$$w = u\sqrt{\frac{e(m - em')}{m'}},$$

and shew that the kinetic energy after impact is equal to e times the kinetic energy before. (Assume Newton's experimental law to hold still.)

13. Two smooth spheres of masses m_1, m_2 moving with velocities u_1, u_2 at right angles to one another collide.

Prove that their directions of motion after impact are also at right angles if the coefficient of restitution

$$= \frac{m_1^2 u_1 + m_2^2 u_2 \tan \alpha}{m_1 m_2 (u_1 + u_2 \tan \alpha)}$$

where α is the angle u_1 makes with the line of centres.

14. Two balls each of mass 1 lb. are placed on a smooth horizontal plane and connected by a light elastic string of natural length 6 ft. and modulus 12 lbs. weight. The balls are drawn apart to a distance of 8 ft. and released. Find the velocity of the balls when the string becomes slack, the velocity after impact, and the subsequent maximum elongation, the coefficient of impact being 0·5.

PART II.

DYNAMICS OF RIGID BODIES.

CHAPTER I.

KINEMATICS.

In all the previous work we have been dealing with the motion of bodies which were particles or could be treated as particles. That is to say, rotations of the body about any point or axis were not existent. We now proceed to examine some simple cases where such rotations do exist. We will confine ourselves to the case where the motion is in one plane, and the forces act in, or parallel to, this plane.

168. The term **rigid body** is intended to denote a body which remains unchanged in shape and size however much it may move about. Thus, any two given points in the body always remain at the same distance from one another.

169. A body is said to be **moving in two dimensions** or **to be moving in one plane** when a plane fixed in the body moves in a plane fixed in space. Different points of the body then move in parallel planes. For example, if a cube slides about on a horizontal table with the same face always in contact with the table it is said to be moving in one plane. In this case all points of the cube move in horizontal planes. In such a case all points in the body in a straight line perpendicular to the planes of motion are moving in the same way at the same instant, and therefore the motion of the whole body is determined by the motion of the points in any one of these planes.

If a body is only capable of motion in one plane, and two points of the body which are moving in the same plane are placed in fixed positions, the whole body is fixed; if only one was placed in a fixed position the body would still be capable of rotation about an axis through that point. Hence if the displacements of two such points in the body are given it should be possible to deduce the displacement of every point. We shall shew immediately that this can be done graphically.

170. Translation and Rotation.

A translation is a displacement of a body in which every point moves through equal and parallel distances.

Any displacement in one plane of a rigid body other than a translation can be produced by the rotation of the body about some point.

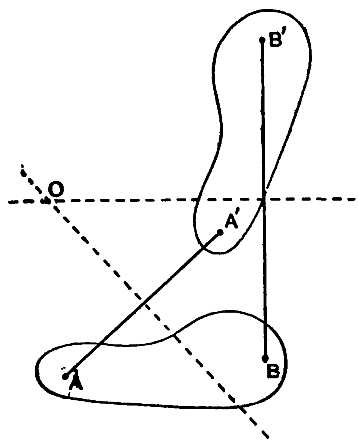


FIG. 137.

To prove this let A, B be two points in the body in its first position, and A′, B′ the positions after the displacements.

We have seen that the position of the body is fixed completely by these two points.

Bisect AA′ and BB′ at right angles by lines meeting in O. If O is at a finite distance the triangles OAB, OA′B′ are congruent, and hence O is the same point relative to the body in the two positions; that is to say, if O is in the body, it is the same point of the body in each position; if O is outside the body, the body may be conceived as indefinitely extended, or O may be thought of as connected to the body by two wires, and when the body moves from one position to the other O remains fixed, and the body would be brought from the old position to the new by a rotation about O. Since the angles OAB, OA′B′ are equal, so also are the angles OAA′, OBB′, or in such a rotation any two points such as A, B undergo the same angular displacement.

If AA′ and BB′ are parallel, the perpendicular bisectors are either parallel or coincident, and two cases thus arise which may be represented thus (remembering that AB = A′B′):

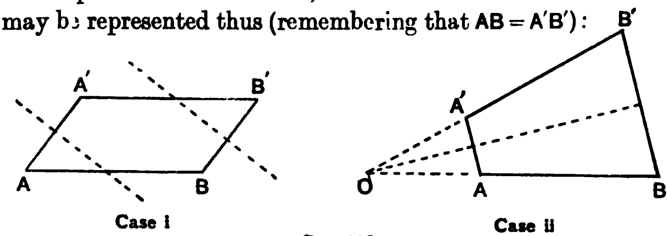


FIG. 138.

In the first case O is at infinity and the displacements AA′, BB′ are themselves equal and parallel, and every other point will also move the same distance in the same direction. This is a motion of translation only.

In the second case AA′, BB′ are parallel but not equal, and the intersection of AB and A′B′ is the point satisfying the condition that a rotation of the body about it will bring AB into the position A′B′, and therefore the whole body into its new position.

171. Combination of Translation and Rotation.

The displacement of a rigid body may also be represented by a rotation about any assumed point together with a translation of the body as a whole.

For in Fig. 139, with the former letters, let C be a third point, and suppose it is required to represent the displacement by a translation of the whole together with the rotation about C, and let C′ be the new position of C.

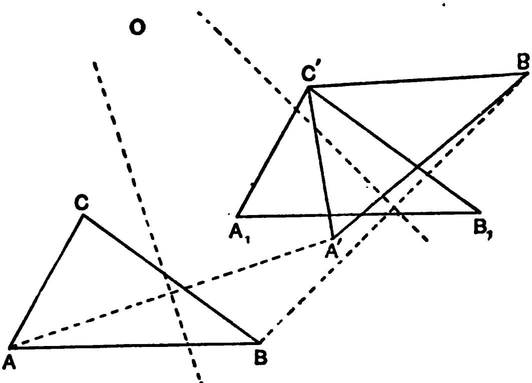


FIG. 139.

By a translation of the whole CAB comes to $C'A_1B_1$, then since $CA' = CA_1$ by a rotation about C′, A_1 can be brought to A′, and consequently the two points C and A having come into their positions, the whole body is now in its new position.

Further, the angle of rotation about C′ in this case is the same as the rotation about O in the previous representation.

For in the rotation about O the lines OC, CA have come into the position OC′, C′A′, hence the rotation about O is measured by the angle COC′, and since

$$\angle OCA = OC'A'$$

$$\angle COC' = \text{angle between } CA \text{ and } C'A'.$$

In the rotation about C′, $C'A_1$ comes to C′A′; therefore the angle of rotation is AC′A′, which is equal to the angle between CA and C′A′. Hence the angle of rotation is the same in both cases, and this angle may also be expressed as the angle through which any line in the body turns as the body passes from one position to the other.

Consequently, it does not matter what point like C (or *base-point* as it is frequently termed), is selected as the centre of the rotation which combined with a translation gives the complete displacement, the angular rotation is the same for all such points. The magnitude and direction of the translation depend, on the other hand, on the base point selected.

172. Instantaneous Centre.

If we consider the angles described and the displacements effected in the above manner in a short time, we arrive at the result that any motion at any instant, except a motion of translation, may be thought of as an angular velocity about an instantaneous centre of rotation, or as an angular velocity about any assumed point combined with a velocity of translation as a whole, and that, whatever the assumed point, the angular velocity in all such representations of the motion is the same, and may be called the angular velocity of the body.

It is generally convenient in dynamical discussions, when treating the motion as a combination of translational and angular velocities, to take the centre of mass as the assumed point. The reason will be seen when we come to the consideration of the forces acting.

173. When a body moves in a plane the instantaneous centre in general will also move in the plane, and also in the body, that is, fresh points of the body may become instantaneous centres from time to time. If the instantaneous centre remains fixed both in the body and space, the body is said to be rotating about a fixed point, as when a rod is hung up by the end and oscillates like a pendulum in a vertical plane.

But when a body does not continue to rotate about a fixed point, the position of the instantaneous centre changes both in the plane and in the body.

Thus, let us take as example the motion of a wheel of radius a rolling along a plane with uniform velocity v. The motion

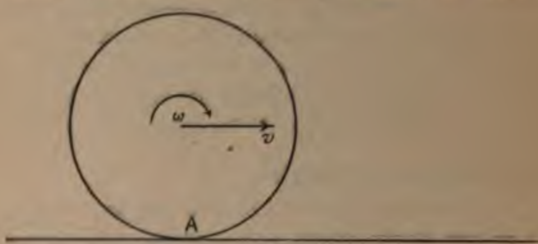


FIG. 140.

may be regarded as a combination of a velocity v of the whole with an angular velocity ω about the centre. Now the wheel goes a distance $2\pi a$ in making a complete revolution, and hence takes $\frac{2\pi a}{v}$ secs. to make the revolution, hence the angular velocity about the centre is

$$\frac{2\pi}{\frac{2\pi a}{v}}=\frac{v}{a},$$

hence $v=a\omega$.

If we consider the point A in contact with the ground its velocity is compounded of v (due to the translation and the same as for any other point in the body), towards the right and $a\omega$ towards the left due to the angular velocity about C, and therefore the point has resultant velocity $=v-a\omega=0$.

Thus the point A is instantaneously at rest, or is the instantaneous centre, and the whole wheel can be regarded as g about A at the instant. This is otherwise clear, for if A had a component velocity in the horizontal direction it would be sliding along the road, and if it had a component vertical velocity it would leave the road.

It is interesting, as illustrating the idea of the instantaneous centre, to compare the motion of the circle with that of a polygon of a large number of sides rolling on the plane.

In the case of the wheel it will be seen that as the wheel rolls along the instantaneous centre takes all positions on the circumference of the wheel. This locus of the instantaneous centre in the body is called the *Body Centrode.* In the same way the instantaneous centre has a locus in space which is called the *Space Centrode*; in the above example the space centrode is the line on which the wheel rolls.

174. Determination of the Instantaneous Centre.

The position of the instantaneous centre at any moment can in general be determined if the directions of the velocities

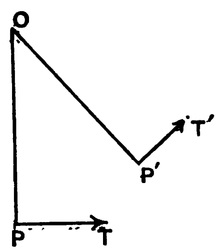


FIG. 141.

of two separate points are known. For if O is the instantaneous centre, any point P must be moving at the instant perpendicular to OP. Conversely, as P is moving in the direction PT, the instantaneous centre must be on a line through P perpendicular to PT. If a second point P′ is moving in the direction P′T′, the instantaneous centre must be on a line through P′ perpendicular to P′T′. The intersection of these two lines will give the instantaneous centre.

If the lines PO, P′O′ in Fig. 142 are parallel, the instantaneous centre is at infinity, and the motion at the instant is one of pure translation.

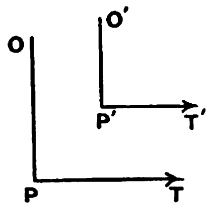


FIG. 142.

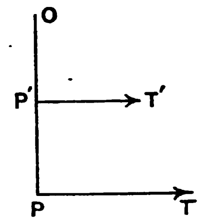


FIG. 143.

If the lines PO, P′O coincide as in Fig. 143, the instantaneous centre cannot be deduced from the directions alone, but the direction of the motion of another point not on the line *PP′* will be required to determine the instantaneous centre; or it can be determined from the velocities of P and P′ alone *if* the magnitude of these velocities are given, for if v, v' are these velocities, and ω is the angular velocity about the instantaneous centre O in PP′

$$v = \mathrm{OP} \,.\, \omega$$

$$v' = \mathrm{OP}'\omega$$

$$\frac{\mathrm{OP}}{\mathrm{OP}'} = \frac{v}{v'}, \text{ giving the position of O.}$$

As an example, suppose a rod AB moves with its ends always constrained to slide along two straight lines at right angles.

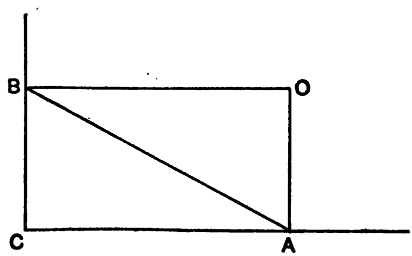


FIG. 144.

The instantaneous centre O is at the intersection of the perpendiculars at A and B to the straight lines.

175. As we have seen, the motion at any instant may be thought of as an angular velocity about any assumed base point combined with a velocity of translation of the body as a whole. Usually the centre of mass is chosen as the base point.

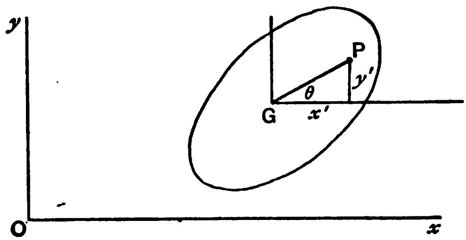


FIG. 145.

Let u, v be the component velocities parallel to fixed axes Ox and Oy of G (the centre of mass),

ω be the angular velocity,

r be the distance of any other point from G,

then the velocity of P relative to G is $r\omega$ perpendicular to GP, and its components parallel to Ox and Oy are

$$-r\omega \sin\theta, \quad r\omega\cos\theta;$$

$\therefore$ the components of the velocity of P relative to O are

$$u - r\omega\sin\theta,$$
$$v + r\omega\cos\theta.$$

If x', y' are the coordinates of P relative to G, the component velocities can be written

$$u - \omega y', \quad v + \omega x'.$$

The coordinates relative to G of the instantaneous centre can therefore be found by putting these two velocities zero, or

$$\left.\begin{aligned} u - \omega y' &= 0 \\ v + \omega x' &= 0 \end{aligned}\right\}$$

giving

$$x' = -v/\omega$$
$$y' = u/\omega.$$

Notice that the equations shew that there is one and only one point which is instantaneously at rest, the case when ω is zero being excepted.

Example. A rod of length $2a$ slides with its ends on two straight lines at right angles, find the velocity of any point on the rod.

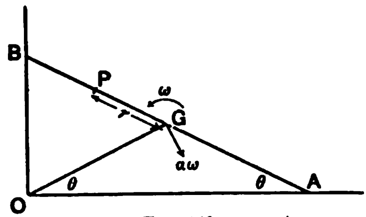


FIG. 146.

Here the centre of mass describes a circle, and since the angles GAO, GOA are equal, if ω is the angular velocity of the rod, ω is also the angular velocity (in the opposite sense) of OG, and the velocity of G is $a\omega$ perpendicular to OG, hence its components along OA and OB are $a\omega \sin\theta$, $-a\omega\cos\theta$.

$\therefore$ the component velocities of P are

$$a\omega \sin\theta - r\omega\sin\theta = (a-r)\omega\sin\theta,$$

and

$$-a\omega\cos\theta - r\omega\cos\theta = -(a+r)\omega\cos\theta.$$

176. Acceleration of any Point in a Rigid Body.

If a body is rotating about a fixed axis the acceleration of any point can readily be deduced. For since all points have

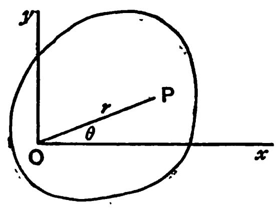


FIG. 147.

the same angular velocity at a given instant, therefore the rate of change of angular velocity is the same for all at the same instant, or the angular acceleration of all points about the axis is the same, and may be called the angular acceleration of the body.

Let ω = angular velocity at the instant,

A = " acceleration "

A particle at distance r from the axis is describing a circle, and its accelerations are

$$\omega^2 r \text{ along PO}$$

and $\mathsf{A}r$ perpendicular to PO. (See Art. 118.)

If the body is moving freely in the plane it will be best to think of its velocity as made of a velocity of translation equal to that of the centre of mass, and a rotation about the centre of mass. The acceleration of any particle will be the resultant of the acceleration of the centre of mass, and the acceleration relative to the centre of mass.

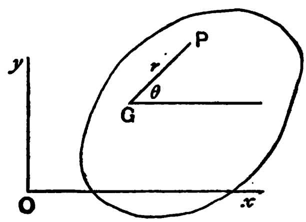


FIG. 148.

If Ox, Oy are fixed axes and G the centre of mass of the body, a point P has, relative to the centre of mass, accelerations

$$\omega^2 r \text{ along PG,}$$

and $\mathsf{A}r$ perpendicular to PG;

these are equivalent to

$$-\omega^2 r\cos\theta - \mathsf{A}r\sin\theta \text{ along O}x,$$

and $-\omega^2 r\sin\theta + \mathsf{A}r\cos\theta$ along Oy;

hence, if f, g are the accelerations parallel to Ox and Oy of the centre of gravity itself, the total acceleration of the point is equivalent to $f - \omega^2 r \cos\theta - \text{A}r\sin\theta$ along Ox,

and $\quad g - \omega^2 r\sin\theta + \text{A}r\cos\theta$ along Oy.

These may be written

$$f - \omega^2 x' - \text{A}y',$$

and

$$g - \omega^2 y' + \text{A}x',$$

if x', y' are the coordinates of the point relative to the centre of mass.

EXAMPLES.

1. A circular disk of radius a is rolling along a straight line, and at a given instant the velocity of the centre is u and its acceleration f. Prove that there is a point in the disk whose acceleration is zero, and that if r is its distance from the centre and θ the angle, the radius to it makes with the perpendicular to the straight line,

$$\tan\theta = u^2/af,$$

$$r = a^2 f/\sqrt{u^4 + a^2 f^2} = a\cos\theta.$$

Shew that the acceleration of any other point is proportional to its distance from this one.

2. A circular disk rolls with uniform velocity on the inside of a circle of centre O of twice the radius of the disk. If ω is the angular velocity of the centre of the disk about O, shew that the angular velocity of the disk about its centre is also ω, and shew that any point on the circumference of the disk describes a straight line.

3. A rod always touches a fixed circle, and one end moves along a fixed tangent to the circle, prove that the centrodes are both parabolas.

4. A uniform stick is thrown into the air, and its centre of mass moves in a vertical line, while it rotates about the centre in a vertical plane with constant angular velocity ω. If the initial velocity of the centre of mass is u, and the stick is initially vertical, find the velocity and acceleration at time t of the two points at distance r from the centre.

5. Prove that there is one point in the body (Art. 176) whose acceleration is zero, and find its coordinates.

CHAPTER II.

ROTATION ABOUT A FIXED AXIS; MOMENTS OF INERTIA.

177. We now pass on to consider the effect of forces on the motion of a body rotating about a fixed axis, and we will suppose the forces to act in planes perpendicular to the axis.

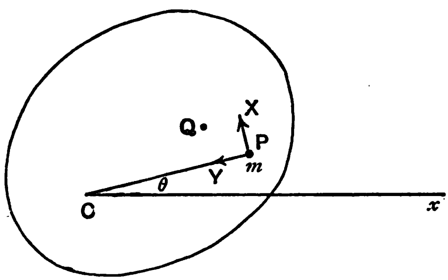


FIG. 149.

Let the body of any shape be capable of rotating about a fixed axis through C perpendicular to the plane of the paper, and let the forces acting be either in the plane of the paper or in parallel planes.

To determine the motion it will be necessary in the first place to consider the motion of each particle of the body, treating it as obeying the laws of motion in the ordinary way.

In discussing the effects of forces on a rigid body it is necessary to use the term *moment* frequently; we shall therefore define *moment* first and give one theorem relating to moments.

178. The *Moment* or *Torque* of a force about a point X is the product P . XN of the force into the perpendicular from the point to the line of action of the force. Geometrically,

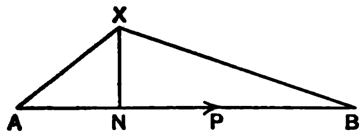


FIG. 150.

if AB represents the force in magnitude, the moment about X is represented by twice the area of the triangle XAB, since this triangle equals

$$\tfrac{1}{2}XN \,.\, AB = \tfrac{1}{2}P \,.\, XN.$$

We will see that the moment measures the tendency of a force to produce rotation in a rigid body about the point.

179. The following is the fundamental theorem about moments:

The sum of the moments of two forces (*in a plane*) *about any point* (*in the same plane*) *is equal to the moment of their resultant about the point.*

Let P and Q be two forces meeting at O, and X the point about which the moments are taken.

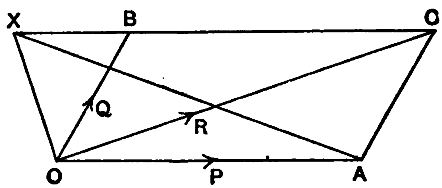


FIG. 151.

Through X draw a line XB parallel to P, meeting the line of action of Q in B, and take OB to represent Q. On the same scale let OA represent P, and complete the parallelogram OACB so that OC represents the resultant R of P and Q.

Now, the moments of P, Q and R about X are respectively twice the triangles XOA, XOB, XOC.

Also, $$\triangle XOA = \triangle COB = \triangle BOC;$$

$$\therefore \triangle XOA + \triangle XOB = \triangle XOB + \triangle BOC = \triangle XOC,$$

which proves the result for forces meeting in a point.

The same result applies if the forces are parallel, but as we have not so far discussed parallel forces we will refer to any book on statics for the proof.

Since this result is true for any two forces, by compounding forces continually we have the more general result:

The sum of the moments about a point of any number of forces is equal to the moment of the resultant (if they have one) about the point.

In any case the sum of the moments is equal to the sum of the moments of any other system of forces obtained from the first by composition according to the parallelogram law or law for parallel forces.

180. Returning now to the rotating body (Fig. 149),

let ω = angular velocity of the body at time t,

A = ,, acceleration ,, ,, ,,

r = distance of a particle P of mass m from the axis,

the acceleration of P will be rA perpendicular to CP and $r\omega^2$ along PC, and the forces acting on the particle at P are equivalent to mrA perpendicular to CP and $mr\omega^2$ along PC.

Since these expressions are the exact equivalent of the forces acting on the particle, if X and Y are the components of the forces in the same directions it follows that the moment X$r = mr^2$A, and consequently if we add up for all the particles of the body, $$\Sigma mr^2 A = \Sigma Xr.$$

Now, the forces on the particle having been replaced by X and Y in the given directions, Y has no moment about C, and, consequently, Xr is the total moment about C of all the forces acting on m, and ΣXr is the total moment about C of all the forces on all the particles of the body.

These forces include all the internal reactions between the different particles of the body, but if we consider the reactions between two particles P and Q, the action on P is equal, opposite, and in the same straight line as the reaction on Q. Hence, when we add the moments of these about C, the moments cancel one another, and we are left with the moments of the external forces only. The reaction of the axis, also, will not come into the equation as its moment about the axis is zero.

Hence, if we denote by L the sum of the moments of the external forces, we have the equation :

$$\Sigma mr^2 A = L$$

or

$$A\Sigma mr^2 = L,$$

since A is the same for all particles ;

or

$$IA = L,$$

where I is written in place of Σmr^2.

In this and the following equations, in using British units I may be said to be in lb.-ft.^{2} units, A in radian/sec.^{2}, and L, which is the product of a force into a distance, is in lbl.-ft. units.

181. Kinetic Energy of the Body.

We can get the kinetic energy of the rotating body immediately from the fact that the K.E. of the particle m at P is

$$\tfrac{1}{2}mr^2\omega^2.$$

Since kinetic energy is a scalar quantity, the kinetic energies of the different particles simply add together, and the total kinetic energy of the body is

$$\begin{aligned} &\Sigma\tfrac{1}{2}mr^2\omega^2 \\ &= \tfrac{1}{2}\omega^2\Sigma mr^2 \\ &= \tfrac{1}{2}I\omega^2. \end{aligned}$$

182. The quantity I or Σmr^2, which thus comes into all questions concerning rotating bodies, is called the **Moment of Inertia** of the body. The Moment of Inertia will be different for different axes of rotation, and, consequently, in talking about the M.I. of the body, the axis about which it is taken has to be specified.

183. The equations,

$$\text{moment of forces} = \text{I}A \qquad (1)$$

$$\text{kinetic energy} = \tfrac{1}{2}\text{I}\omega^2 \qquad (2)$$

are analogous to the equations for a single particle.

$$\text{Force} = mf,$$

$$\text{kinetic energy} = \tfrac{1}{2}mv^2,$$

moment of inertia replacing mass, moment of a force or torque replacing force, and angular velocity and acceleration replacing linear velocity and acceleration.

In the same way other equations in the motion of a particle have their analogue in the rotating body; thus the equation

$$\text{impulse} = m(v' - v),$$

is replaced by

$$\text{moment of impulses} = \text{I}(\omega' - \omega), \qquad (3)$$

and $\text{I}\omega$ is called the angular momentum of the body just as mv is called the linear momentum of the particle.

It will be seen that

angular momentum = sum of moments of linear momentum of the separate particles,

for this sum of the moments of momentum

$$= \Sigma mr\omega \,.\, r = \Sigma mr^2\omega$$

$$= \omega\Sigma mr^2 = \text{I}\omega,$$

and the equation (3) follows in the same way as (1); for since the impulse on a particle

$$= m(v' - v) = m(r\omega' - r\omega)$$

$$= mr(\omega' - \omega);$$

$\therefore$ the sum of the moments of the impulses

$$= \Sigma mr(\omega' - \omega)r$$
$$= (\omega' - \omega)\Sigma mr^2 = \mathrm{I}(\omega' - \omega).$$

If two bodies are rotating about the same axis, and collide or become suddenly united in any way, there is an impulse on each and an impulse at the axis. The moment of the impulse on either body is equal to its change of angular momentum. Thus, let

I_1, I_2 be the moments of inertia of the two bodies about the axis;

ω_1, ω_2 their angular velocities before collision;

ω_1', ω_2' the angular velocities after collision;

Q the moment of the impulse;

then
$$\mathrm{Q} = \mathrm{I}_1(\omega_1' - \omega_1),$$
$$-\mathrm{Q} = \mathrm{I}_2(\omega_2' - \omega_2);$$
$$\therefore\ \mathrm{I}_1(\omega_1' - \omega_1) + \mathrm{I}_2(\omega_2' - \omega_2) = 0;$$
$$\therefore\ \mathrm{I}_1\omega' + \mathrm{I}_2\omega_2' = \mathrm{I}_1\omega_1 + \mathrm{I}_2\omega_2,$$

or the total angular momentum is unchanged.

This equation corresponds to the equation of momentum for two particles colliding, and is a special case of a more general principle called the conservation of angular momentum.

184. Work done by a Torque.

The expression for the work done by a force in a displacement likewise has its analogue in the expression for the work done in a rotation.

For if a force acting on the rotating body be resolved into components R and T along and perpendicular to the radius to the point of application, the work done in a small rotation θ is
$$\mathrm{T}\,.\,r\theta = \mathrm{T}r\,.\,\theta = \mathrm{L}\theta,$$

if L is the moment of the force about C.

For any displacement the total work done by the force is $\Sigma L\theta$, corresponding to the total work ΣPs done by a varying force in the displacement of a particle.

If the moment of the forces remains constant, this $= L\theta$ where θ is the total angle turned through.

185. Having got the fundamental equations giving the angular acceleration in terms of the torque acting, the angular velocity and angle described have to be found by the same methods as in the case of the linear acceleration, velocity and distance.

Thus the following equations hold for the case of *uniform* angular acceleration :

$$\omega' = \omega + At,$$

$$\theta = \omega t + \tfrac{1}{2}At^2,$$

$$\omega'^2 = \omega^2 + 2A\theta,$$

where $\omega =$ the initial angular velocity,

and $\omega' =$,, angular velocity at time t,

$\theta =$,, angle described in t secs.

$A =$,, constant angular acceleration.

We have supposed the forces to act in directions perpendicular to the axis. The moment of a force in this case is obtained thus. Let AB be the axis and P a force acting on

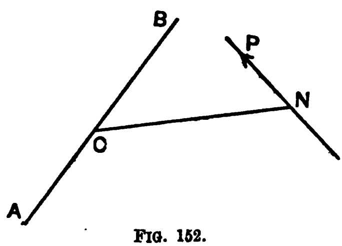


FIG. 152.

the body in a plane perpendicular to the axis, the plane cutting the axis at C, and let CN be the perpendicular from C on the force. The moment of the force about the axis is P . CN, and the moments of different forces simply add together, paying attention to algebraic sign.

186. Moments of Inertia.

The most striking fact about the equations for rotating bodies is the appearance of the moment of inertia.

This, as we have remarked, depends on the axis to which it refers, and we will consequently have to give some theorems on the relations between moments of inertia about different axes, and also some calculations of moments of inertia in special cases.

187. Theorem of Parallel Axes.

If the moment of inertia about an axis through the centre of mass is I_G, *the moment of inertia about a parallel axis at a distance* p *from this is* $I_G + Mp^2$ *if* M *is the total mass.*

First take the case of a disk or lamina, and suppose that the axes about which the moments are taken lie in the plane

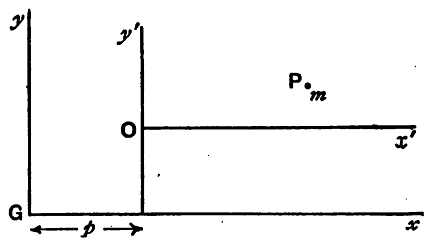


FIG. 153.

of the disk. Take the axis through the centre of mass G as the axis of y, and a line at right angles through G as axis of x. If we take another axis Oy' parallel to Gy and at a distance p from it, and I_O is the moment of inertia about Oy', we have to prove

[illegible]

Now, if x is the distance of a particle of mass m from Gy, its distance from Oy' is $x-p$;

$$\therefore\ I_O = \Sigma m(x-p)^2$$
$$= \Sigma mx^2 + \Sigma mp^2 - 2\Sigma mxp$$
$$= \Sigma mx^2 + p^2\Sigma m - 2p\Sigma mx$$
$$= \Sigma mx^2 + p^2\Sigma m$$
$$= I_G + Mp^2,$$

since by definition of the centre of mass, $\frac{\Sigma mx}{\Sigma m}$ is the x coordinate of the centre of mass, and is therefore zero since the C.M. is the origin;

$$\therefore\ \frac{\Sigma mx}{\Sigma m} = 0;$$
$$\therefore\ \Sigma mx = 0.$$

If the axes about which the moments of inertia are to be compared are perpendicular to the disk, and the new point O is at distances p, q from Gy, Gx,

$$I_O = \Sigma m\text{OP}^2 = \Sigma m\{(x-p)^2 + (y-q)^2\}$$
$$= \Sigma m(x^2+y^2) + \Sigma m(p^2+q^2) - 2\Sigma mxp - 2\Sigma myq$$
$$= \Sigma m(x^2+y^2) + (p^2+q^2)\Sigma m - 2p\Sigma mx - 2p\Sigma my$$
$$= \Sigma m\,.\,\text{GP}^2 + M(p^2+q^2)$$
$$= I_G + M(p^2+q^2) = I_G + M\cdot\text{OG}^2.$$

Since as before $\Sigma mx = 0 = \Sigma my$.

It is evident that the result applies to the case when the body is a cylinder with the axis parallel to the length of the cylinder. For the cylinder may be thought of as divided into a large number of equal thin laminae whose moments of inertia about the axis are equal and add together.

In fact, the proof shews that the result holds for any body whatsoever, for if the axis runs through the centre of mass, the equations,

$$\Sigma mx = 0,\quad \Sigma my = 0$$

still hold, and, consequently, also the whole proof given.

188. ***The moment of inertia of a disk about a perpendicular to its plane and meeting it in a point O is the sum of the moments of inertia of the disk about any two axes at right angles in the disk and intersecting one another at O.***

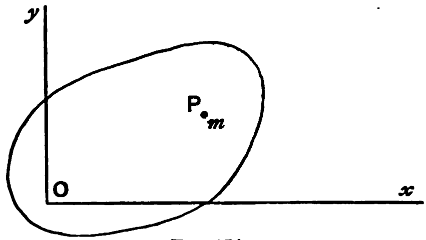


FIG. 154.

For if Ox, Oy are two axes at right angles in the plane of the disk, the moment of inertia about the axis through O perpendicular to the plane

$$= \Sigma m OP^2 = \Sigma m(x^2 + y^2) = \Sigma mx^2 + \Sigma my^2$$

and $\Sigma mx^2 = \text{M.I. about } Oy$

$$\Sigma my^2 = \text{M.I. about } Ox.$$

189. *To find the relation between the M.I.'s of a disk about different axes in the plane of the disk and drawn through the same point.*

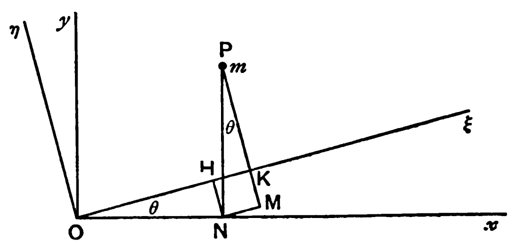


FIG. 155.

Let Ox, Oy be two axes at right angles, $O\xi$ another straight line making an angle θ with Ox. The perpendicular η from

P on $O\xi$ is in the figure

$$=PM-NH$$
$$=PN\cos\theta-ON\sin\theta$$
$$=y\cos\theta-x\sin\theta;$$

$\therefore$ moment of inertia about $O\xi$

$$=\Sigma m(y\cos\theta-x\sin\theta)^2$$
$$=\Sigma m(y^2\cos^2\theta+x^2\sin^2\theta-2xy\sin\theta\cos\theta)$$
$$=\cos^2\theta\Sigma my^2+\sin^2\theta\Sigma mx^2-2\sin\theta\cos\theta\Sigma mxy.$$

This is generally written

$$A\cos^2\theta+B\sin^2\theta-2F\sin\theta\cos\theta,$$

where $A=\Sigma my^2=$ M.I. about Ox,

$B=\Sigma mx^2=$ M.I. about Oy,

$F=\Sigma mxy$,

and F is called the product of inertia with respect to Ox, Oy.

Hence, if the M.I.'s about two lines Ox, Oy, at right angles, are known, and also the product of inertia with respect to O and Oy, we can write down the M.I. about any line through O in the plane.

If the disk is symmetrical with respect to either Ox or Oy, the product of inertia will evidently vanish; for, if it is symmetrical about Ox, corresponding to any term $m_1x_1y_1$ in Σmxy, there will be another $m_1x_1(-y_1)$, and these two will cancel one another.

190. If the disk is not symmetrical about either axes, it is still possible to find a pair of rectangular axes for which the product of inertia vanishes, for the product of inertia with respect to $O\xi$, $O\eta$ is $\Sigma m\xi\eta$.

Now $\xi=OK=OH+HK$

$$=x\cos\theta+y\sin\theta;$$

$$\therefore\ \Sigma m\xi\eta = \Sigma m(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)$$
$$= \cos\theta\sin\theta(\Sigma my^2 - \Sigma mx^2) + (\cos^2\theta - \sin^2\theta)\Sigma mxy$$
$$= (\mathrm{A} - \mathrm{B})\sin\theta\cos\theta + \mathrm{F}(\cos^2\theta - \sin^2\theta)$$
$$= \tfrac{1}{2}\{(\mathrm{A} - \mathrm{B})\sin 2\theta + 2\mathrm{F}\cos 2\theta\},$$

and this is zero if

$$\tan 2\theta = 2\mathrm{F}/(\mathrm{B} - \mathrm{A}).$$

Now, whatever the values of A, B, F, it is always possible to find one and only one value of 2θ less than 180° which will satisfy this equation, and consequently make $\Sigma m\xi\eta = 0$.

The axes for which the product of inertia vanishes are called *principal axes* of the disk.

Hence, at any point in the disk a pair of rectangular axes exist which are principal axes. If now we take these axes as the axes of x and y and call the moments about them A and B, supposing $\mathrm{A} > \mathrm{B}$, then the M.I. about any other line in the disk through O is

$$\mathrm{A}\cos^2\theta + \mathrm{B}\sin^2\theta.$$

Now $\qquad \mathrm{A}\cos^2\theta + \mathrm{B}\sin^2\theta = \mathrm{A} - (\mathrm{A} - \mathrm{B})\sin^2\theta,$

and since $\qquad \mathrm{A} > \mathrm{B},$

this expression $< \mathrm{A}$.

Also, $\qquad \mathrm{A}\cos^2\theta + \mathrm{B}\sin^2\theta = (\mathrm{A} - \mathrm{B})\cos^2\theta + \mathrm{B},$

and is therefore $> \mathrm{B}$;

$\therefore$ A and B are the greatest and least moments of inertia about axes through O in the disk. They will be called the *principal moments of inertia* at A.

From the above propositions it will be seen that, in the case of a disk, if the mass of the disk, the directions of the principal axes at the centre of mass G are known, and also the magnitudes of the principal moments of inertia, it is possible first to find the moment of inertia about any other axis through G in the plane, and then to find the moment of inertia about any parallel axis in the plane, and also to deduce the moment of inertia about any axis perpendicular to the disk.

191. Calculation of Moments of Inertia.

The calculation of moments of inertia has usually to be performed by the integral calculus. We will give the calculus proofs for a few cases, and also give alternative proofs without the calculus and deduce other standard cases.

192. A Thin Uniform Rod.

To find the moment of inertia about an axis through the centre and perpendicular to the length.

(A) By calculus.

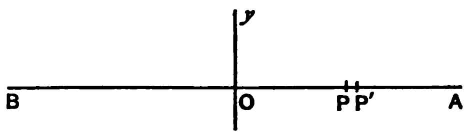


FIG. 156.

Let AB be the rod, and let

m = mass of rod per unit length,

$2l$ = whole length,

M = total mass = $m \cdot 2l$,

PP′ be an element of the rod,

OP = x,

PP′ = dx;

$\therefore$ mass of PP′ = mdx.

M.I. of PP′ about Oy = $mdx \cdot x^2$;

$\therefore$ total moment

$$= \Sigma mdx \cdot x^2$$

$$= \int_{-l}^{l} mx^2\,dx = m\left[\frac{x^3}{3}\right]_{-l}^{l} = \frac{2ml^3}{3}$$

$$= \text{M}\,\frac{l^2}{3}.$$

(B) By algebraic methods.

Let the rod be divided into a large number $2n$ of equal bits each of mass $\frac{M}{2n}$; the nearer end of the r^{th} from the centre is at a distance $(r-1)\frac{l}{n}$ from the centre, and, consequently, its moment of inertia about the centre is $\frac{M}{2n}\left\{(r-1)\frac{l}{n}\right\}^2$ nearly, taking the distance from the centre as the same as the distance of the nearer end from the centre.

∴ M.I. of all of one-half of the rod

$$=\frac{Ml^2}{2n^3}(1^2+2^2+\ldots+\overline{n-1}^2) \text{ nearly}$$

$$=\frac{Ml^2}{2n^3}\frac{(n-1)n(2n-1)}{6}$$

$$=\frac{Ml^2}{12}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right).$$

By making n infinite we get the true moment of inertia of half the rod; this gives $\frac{Ml^2}{6}$.

The other half of the rod contributes the same moment of inertia,

and the total M.I. $=\frac{Ml^2}{3}$.

193. Other Moments of Inertia deduced from the above.

(1) The M.I. of the rod about an axis through one end perpendicular to the rod is (by Art. 187)

$$M\frac{l^2}{3}+Ml^2=M\frac{4l^2}{3}.$$

(2) M.I. of a **rectangular lamina** about an axis through the centre perpendicular to a pair of sides.

If the sides of the lamina are $2a$, $2b$, and we want the moment

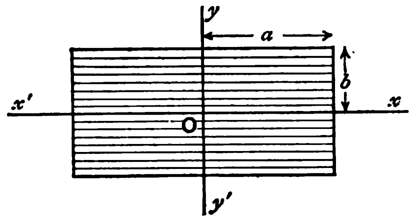


FIG. 157.

of inertia about the axis yOy' in the figure, we can divide the rectangle into strips parallel to xOx', and the M.I. of each strip

$$=\text{mass of strip}\times\frac{a^2}{3};$$

$\therefore$ total moment of inertia

$$=\text{total mass}\times\frac{a^2}{3}$$

$$=M\frac{a^2}{3}.$$

Similarly, the M.I. about xOx' is $M.\frac{b^2}{3}$, and by Art. 188 the M.I. about an axis through O perpendicular to the plane is

$$M\frac{a^2+b^2}{3}.$$

The axes Ox, Oy are axes of symmetry, and therefore for them $F=0$;

$\therefore$ the moment of inertia about an axis through O lying in the plane of the disk, and making an angle θ with Ox, is

$$\frac{M}{3}(b^2\cos^2\theta+a^2\sin^2\theta).$$

(3) M.I. of a **rectangular parallelopiped** (or **cuboid**) about an axis through its centre perpendicular to a pair of faces.

If the lengths of the sides are $2a$, $2b$, $2c$, and if the axis is perpendicular to the faces whose edges are $2b$, $2c$ (Ox in the

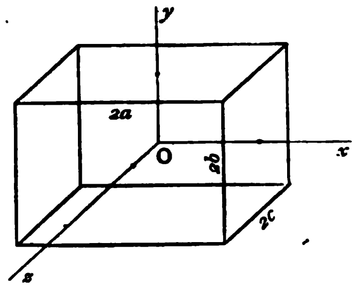


FIG. 158.

figure), we can divide the parallelopiped into laminae by drawing planes parallel to these faces, and the M.I. of each of these laminae is its mass $\times \frac{b^2+c^2}{3}$;

$\therefore$ M.I. of whole parallelopiped

$$= \text{total mass} \times \frac{b^2+c^2}{3}$$

$$= M\frac{b^2+c^2}{3}.$$

Example **1.** Prove that the M.I. of a rod about an axis through the centre and inclined at an angle θ to the rod is $\frac{Ml^2}{3}\sin^2\theta$, and deduce the M.I.'s of a parallelogram of sides $2a$ and $2b$, and angle θ about its sides.

2. Shew that the moments of inertia about all lines through the centre of a uniform square lamina and in its plane are equal.

3. Shew that for lines through a corner of a square in its plane the M.I. is least about the diagonal through the corner, and greatest about a line parallel to the other diagonal. Find these moments of inertia, and deduce the product of inertia relative to the two sides.

4. The moments of inertia of a rectangular lamina about its edges are $M\frac{4b^2}{3}$ and $M\frac{4a^2}{3}$.

5. Find the moment of inertia of a rectangular lamina about a diagonal.

6. Find the M.I. of a cuboid about an axis through the centre of one face and parallel to an edge of that face.

7. Find the M.I. about an edge of the cuboid.

194. Moment of Inertia of a **fine circular wire** (or **hoop**) about an axis through its centre perpendicular to its plane, (or the axis of the circle).

As all points are equally distant from the centre, the M.I. $= Ma^2$ if a is the radius.

Also, since the M.I.'s about any two diameters are equal, and the sum of the moments of inertia about two diameters at right angles is equal to the M.I. about a perpendicular to the plane of the wire through the point of intersection (Art. 188), it follows that the M.I. about a diameter is $M\frac{a^2}{2}$.

195. M.I. of **circular lamina** about its axis.

(i) Calculus proof.

Let $a =$ the radius,

$m =$ mass per unit area,

$M =$ total mass

$= m \cdot \pi a^2$.

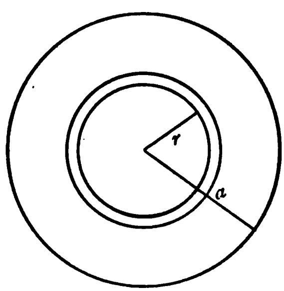


FIG. 159.

Divide the disk into rings by concentric circles, and let r be the radius and dr the breadth of one of the rings.

$\therefore$ the area of the ring is $2\pi r dr$;

$\therefore$ its mass is $m \,.\, 2\pi r dr$;

$\therefore$ by Art. 194 its M.I. about its axis is $2\pi m r dr \,.\, r^2$;

$\therefore$ total M.I. of lamina

$$= \Sigma 2\pi m r dr \,.\, r^2$$

$$= 2\pi m \int_0^a r^3 dr$$

$$= 2\pi m \frac{a^4}{4} = m\frac{\pi a^4}{2}$$

$$= \mathrm{M}\frac{a^2}{2}.$$

(ii) Algebraical proof of same theorem.

If we divide the disk into n rings of equal breadth the area of the r^{th} from the centre is

$$\pi\left\{\left(\frac{ra}{n}\right)^2 - \left(\frac{\overline{r-1}a}{n}\right)^2\right\}$$

$$= \frac{\pi a^2}{n^2}(2r-1),$$

and its M.I. about the axis is

$$m\frac{\pi a^2}{n^2}(2r-1)\,.\left(\frac{ra}{n}\right)^2 \text{ nearly};$$

$\therefore$ the total M.I.

$$= \lim_{n\to\infty} \frac{m\pi a^4}{n^4}\sum_{r=1}^{r=n}(2r^3 - r^2).$$

Now $\displaystyle \lim_{n\to\infty}\frac{\Sigma r^3}{n^4} = \text{Lim}\,\frac{1}{4}\,\frac{n^2(n+1)^2}{n^4}$

$$= \text{Lim}\,\frac{1}{4}\,\frac{(n+1)^2}{n^2} = \text{Lim}\,\frac{1}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$= \frac{1}{4}.$$

Also $\lim_{n\to\infty} \frac{\Sigma r^2}{n^4} = \text{Lim}\,\frac{1}{6}\,\frac{n(n+1)(2n+1)}{n^4}$

$$= \text{Lim}\,\frac{1}{6}\,\frac{1}{n}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = 0;$$

∴ total moment of inertia

$$= 2\,\frac{m\pi a^4}{4} = \frac{Ma^2}{2}.$$

The M.I. about a diameter is consequently $\frac{Ma^2}{4}$ by the same argument as for the circular wire.

196. Moment of inertia of a **circular cylinder** about its **axis.**

Since the cylinder can be divided into circular disks, the moment of inertia of each of which about the axis is

$$\text{its mass} \times \frac{a^2}{2};$$

∴ the total moment of inertia will be $M\frac{a^2}{2}$ if M is the total mass.

197. Moment of inertia of a **circular cylinder** about a **straight line through the centre perpendicular to the axis.**

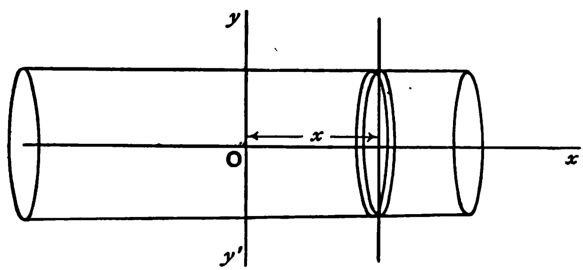


FIG. 160.

Divide the cylinder into laminae by planes perpendicular to the axis.

Let ength of cylinder,
nass of a lamina,
ts distance from the axis Oy about which the M.I. is required.

The lamina about a diameter $= m\frac{a^2}{4}$;

$\therefore$ the out Oy

$$= m\frac{a^2}{\cdot} + mx^2 \text{ by Art. 187};$$

$\therefore$ the total M.I. of

$$= \Sigma m\frac{a^2}{4} +$$

now $\Sigma\frac{ma^2}{4} = M\frac{a^2}{4}$ if $M =$ total mass.

$\Sigma mx^2 =$ moment of inertia of a uniform rod whose mass is the mass of the cylinder about an axis through the centre perpendicular to its length;

$$\therefore \Sigma mx^2 = M\frac{l^2}{3};$$

$\therefore$ total moment of inertia

$$= M\left(\frac{a^2}{4} + \frac{l^2}{3}\right).$$

Example **1.** Find maximum and minimum moments of inertia for a circular disk about lines in its plane through a given point in the plane.

2. Find the M.I. about its axis of a flat circular ring of radii a and b.

3. Find the M.I. about the axis of a circular cylinder of internal and external radii 10 and 12 inches and weighing 100 lbs.

4. Find the M.I. of a circular disk about a line perpendicular to its plane and passing through the circumference of the disk.

5. Find the M.I. of a hollow cylinder about a line through its centre and perpendicular to its axis.

198. Moment of inertia of a **sphere** about a diameter. Calculus proof. Divide the sphere into laminae by planes perpendicular to the diameter. A plane at a distance x from the centre cuts the sphere in a circle of radius $\sqrt{a^2-x^2}$.

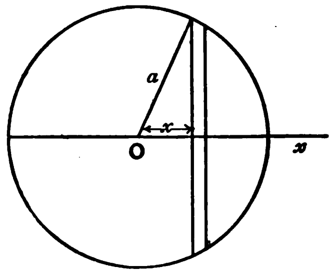


FIG. 161.

The mass of the lamina is, therefore,

$$\rho\pi(a^2-x^2)dx,$$

where ρ is the mass per unit volume, or density.

The M.I. of the lamina about Ox (the axis of the lamina) is, by Art. 195,

$$\rho\pi(a^2-x^2)dx\frac{a^2-x^2}{2};$$

$\therefore$ total M.I.

$$=\int_{-a}^{a}\rho\frac{\pi}{2}(a^4-2a^2x^2+x^4)\,dx$$

$$=\rho\frac{\pi}{2}\left[a^4x-\frac{2a^2x^3}{3}+\frac{x^5}{5}\right]_{-a}^{a}$$

$$=\rho\frac{\pi}{2}\left(2a^5-\frac{4a^5}{3}+\frac{2a^5}{5}\right)$$

$$=\rho\frac{\pi}{2}\frac{16a^5}{15}$$

$$=\frac{8}{15}\pi\rho a^5$$

$$=\mathrm{M}\times\frac{2a^2}{5}$$

since the mass of a sphere is $\frac{4}{3}\pi\rho a^3$.

Example 1. Find the M.I. of a sphere about a tangent line.

2. Find the M.I. about a diameter of a hollow sphere of mass M and internal and external radii a and b.

199. *Results proved.*

Collecting the principal results proved above, we have the following table of moments of inertia :

Thin rod of length $2a$ about an axis through its centre perpendicular to its length $= M\frac{a^2}{3}$.

Rectangular lamina about an axis through its centre perpendicular to the sides of length $2a = M\frac{a^2}{3}$.

Rectangular lamina of sides $2a$, $2b$ about an axis through its centre perpendicular to its plane $= M\frac{a^2+b^2}{3}$.

Cuboid of edges $2a$, $2b$, $2c$, about an axis through its centre parallel to the edges $2a = M\frac{b^2+c^2}{3}$.

Circular wire of radius a about its axis Ma^2.

" " " " a diameter $M\frac{a^2}{2}$.

" disk " " its axis $\frac{Ma^2}{2}$.

" " " " a diameter $M\frac{a^2}{4}$.

" cylinder " " the axis $\frac{Ma^2}{2}$.

Circular cylinder of radius a and length $2l$ about axis through centre of mass perpendicular to its length $M\left(\frac{a^2}{4}+\frac{l^2}{3}\right)$.

Sphere of radius a about a diameter $M.\frac{2a^2}{5}$.

EXAMPLES.

1. Find the moment of inertia, angular momentum, and kinetic energy of a circular disk of diameter 2 ft. and thickness 2 ins. made of iron weighing 477 lbs. per cubic foot and rotating about its axis 150 times a minute.

2. Find the kinetic energy and angular momentum of a sphere of mass 250 lbs. and diameter 1 ft. rotating about a diameter 50 times a minute.

3. How much kinetic energy does a flywheel give up while the number of rotations per minute drops from 150 to 90, if it consists of a uniform circular disk of radius 18 inches and mass 400 lbs.

4. In a flywheel its mass of 120 lbs. may be regarded as concentrated round its rim of circumference 66 inches. It is set spinning at 2,400 revolutions a minute, and by suitable gearing it is made to drive machinery that absorbs one horse power. For how long a period will the machinery be driven before the speed is reduced by one-half, assuming that no energy is lost in transmission.

5. Shew that the M.I. of a uniform rectangular plate about any line in its plane is the same as that of four equal particles each of mass $\frac{1}{12}$ of the whole at the angles, and a fifth of $\frac{2}{3}$ of the whole mass at the centre.

Shew, also, that the M.I. about any line perpendicular to the plane is the same for the disk and for these particles.

6. Shew that in order that two disks may have equal moments of inertia about all axes in their plane it is sufficient that

(1) their masses are equal,
(2) their centres of mass coincide,
(3) their principal axes at the centre of mass coincide,
(4) their moments of inertia about these principal axes are respectively the same.

7. Find the masses of a set of five particles, four at the middle points of the sides of a rectangular disk and the fifth at the centre, which will have the same moment of inertia about all lines in the plane as the disk.

CHAPTER III.

ROTATION ABOUT A FIXED AXIS; EXAMPLES.

200. We proved the following dynamical equations for a body rotating about a fixed axis:

$$IA = L = \text{moment of force},$$

$$I(\omega' - \omega) = Q = \text{moment of impulses},$$

$$\tfrac{1}{2}I(\omega'^2 - \omega^2) = L\theta = \text{work done} = \text{change in kinetic energy}.$$

We also have the following kinematical results for uniform angular acceleration:

$$\left.\begin{aligned} \omega' - \omega &= At \\ \theta &= \omega t + \tfrac{1}{2}At^2 \\ \omega'^2 &= \omega^2 + 2A\theta \end{aligned}\right\}.$$

We will now take some examples of motion of this kind.

201. A body is capable of rotating about a horizontal axis through its centre of mass, and is acted on by the force due to a weight attached to a string wound round the axle and attached at one end to the axle.

A useful arrangement of this kind for experiments on moments of inertia is shewn in the diagram, where a framework carries heavy weights which can be moved to different distances along the arms.

If these weights are symmetrically placed on the arms, the centre of mass is on the axis, and the only force other than friction acting on the body and having a moment about the axis is the tension T of the string. Supposing friction non-existent or negligible, let

m = the mass of the hanging weight,

I = M.I. of rotating system,

r = radius of axle on which the string is wound,

f = acceleration of hanging weight,

and other letters as before.

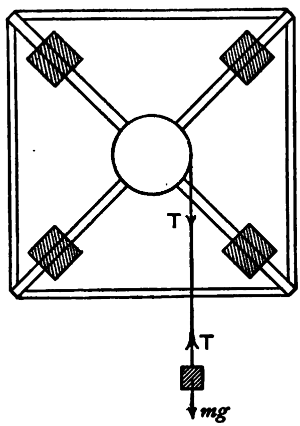


FIG. 162.

Then, fo the hanging weight

$$mg - \mathrm{T} = mf, \quad \ldots\ldots(1)$$

and for the rotating body,

$$\mathrm{T}r = \mathrm{IA}, \quad \ldots\ldots(2)$$

but

$$f = r\mathrm{A}\ ; \quad \ldots\ldots(3)$$

for the velocity and acceleration of the weight are the same as the velocity and tangential acceleration of a point on the circumference of the axle, and the latter must $= r\mathrm{A}$.

From (1) and (2), eliminating T,

$$mgr = IA + mrf$$
$$= IA + mr^2A,$$
$$A = \frac{mgr}{I + mr^2}.$$

The above result was arrived at on the supposition that there was no friction, but if friction exists it produces a torque about the axis which is practically constant whatever the velocity of the apparatus. This torque may first be balanced by a weight μ hung on the string, and the expression for the angular acceleration produced by the extra mass m will be

$$\frac{mgr}{I + (m+\mu)r^2};$$

$m + \mu$ being now the total weight, for the equations would be

$$(m+\mu)g - T = (m+\mu)f,$$
$$Tr - Fr' = IA,$$
$$\mu gr = Fr',$$

where F is the friction acting at a distance r' from the axis, r' being the radius of axle on which the apparatus turns and which is usually less than r; whence

$$mgr = \{I + (m+\mu)r^2\} A.$$

If the extra weight m is caught off after a time by an arrangement as in Atwood's machine the body will continue to rotate with uniform angular velocity, which may be readily measured by observing the time required for a number of revolutions.

We can thus verify the kinematical equations, which will in this case take the form (initial velocity = 0):

$$\omega = At,$$
$$\theta = \tfrac{1}{2} At,$$
$$\omega^2 = 2A\theta.$$

We can also vary the moment of inertia by shifting the weights along the arms, and the nearer the weights are to the extremities the greater the moment of inertia, and, consequently, the smaller the angle described in any time.

If the same angle θ is described under the action of the same weight m, in experiments with different moments of inertia, we have

$$\theta = \frac{1}{2}\frac{mgt^2}{I + (m+\mu)r^2} = \frac{1}{2}\frac{mgt'^2}{I' + (m+\mu)r^2},$$

where I' is the moment of inertia in the second experiment and t' the time taken;

$$\therefore \ \theta = \frac{1}{2}\frac{mg(t^2 - t'^2)}{I - I'}$$

by subtracting numerators and denominators of the equal fractions;

$$\therefore \ t^2 - t'^2 = \frac{2\theta}{mg}(I - I'); \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

now the change $I - I'$ in the moment of inertia depending only on the shift of the weights can be readily calculated from the results of the last chapter and the equation (1) verified.

202. Effect of Inertia of the Wheel in Atwood's Machine.

Suppose a string passing over a pulley, as in Atwood's machine. If the pulley turns with the string so that no slipping takes place, there is friction between string and pulley, and the tensions of the two portions of the string will not be the same.

If the acceleration of each weight is f, the angular acceleration of the pulley is $\frac{f}{r}$, and the equations of motion are

for m $\qquad mg - T = mf, \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$

„ m' $\qquad T' - m'g = m'f, \ \ldots\ldots\ldots\ldots\ldots\ldots (2)$

„ pulley $\qquad (T - T')r = I\frac{f}{r}; \ \ldots\ldots\ldots\ldots\ldots\ldots (3)$

multiply (1) and (2) by r, and add the three equations ·

$$(m-m')gr=\left\{(m+m')r+\frac{I}{r}\right\}f,$$

$$f=\frac{(m-m')g}{m+m'+\frac{I}{r^2}},$$

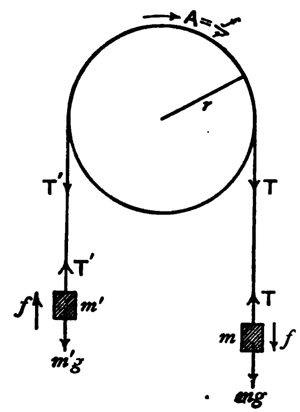


FIG. 163.

shewing that the inertia of the wheel can be allowed for by a correction to the denominator of the expression $\frac{m-m'}{m+m}g$, which would be the acceleration if the inertia of the wheel is neglected.

If the pulley is a uniform disk of mass M,

$$\frac{I}{r^2}=\frac{M}{2}.$$

If it is a wheel with its mass practically concentrated along the circumference,

$$\frac{I}{r^2}=M.$$

Usually $\frac{I}{r^2}$ is small compared with m or m'.

Example. Prove that

$$\frac{T}{T'}=\frac{mm'+\frac{I}{2r^2}m}{mm'+\frac{I}{2r^2}m'}.$$

203. *Second Method of Treatment of Atwood's Machine,—by Energy.*

Supposing the apparatus to start from rest, after the weight m has fallen a distance s let its velocity be v, the angular velocity of the wheel is then $\frac{v}{r}$, and the total kinetic energy

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}m'v^2+\tfrac{1}{2}I\frac{v^2}{r^2}$$

$$=\tfrac{1}{2}\left(m+m'+\frac{I}{r^2}\right)v^2,$$

the loss of potential energy is

$$mgs-m'gs=(m-m')gs;$$

$$\therefore\ \tfrac{1}{2}\left(m+m'+\frac{I}{r^2}\right)v^2=(m-m')gs,$$

shewing that we get a relation of the form holding for uniform acceleration

$$\tfrac{1}{2}v^2=fs,$$

and

$$f=\frac{(m-m')g}{m+m'+\frac{I}{r^2}}.$$

204. The Torsion Pendulum.

If a bar AB hangs horizontally, supported by a wire at its middle point, it can oscillate in the horizontal plane if displaced from the position of equilibrium. The apparatus is consequently called a torsion pendulum. Weights C, D, can be moved along the bar to any position in order to alter the moment of inertia.

In a position of equilibrium there is no torsion (twist) in the wire, but if the bar is rotated in the horizontal plane, the wire is twisted and a couple is brought into action on

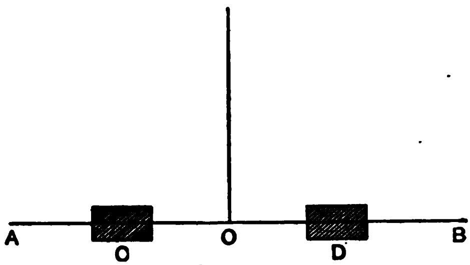


FIG. 164.

account of this twist which tends to restore the bar to its equilibrium position. This couple is found experimentally to be proportional to the angular displacement θ; we can therefore represent the couple by $\tau\theta$, where τ is a constant depending on the nature, length, and diameter of the wire.

The equation of motion is therefore

$$\mathrm{IA} = -\tau\theta,$$

where I = moment of inertia about the axis of the wire;

$$\therefore \quad \mathrm{A} = -\frac{\tau}{\mathrm{I}}\theta.$$

We consequently have the angular acceleration proportional to the angular displacement and in the opposite direction, hence the equation corresponds exactly to the equation for simple harmonic motion, and we can say at once the solution is of exactly the same form, namely,

$$\theta = c\cos\left(\sqrt{\frac{\tau}{\mathrm{I}}}t + \gamma\right),$$

and the oscillations have a period

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{\tau}},$$

being strictly isochronous.

If we make two experiments, in the second of which the weights have been shifted a carefully observed amount so as to alter the moment of inertia to I′ and the period to T′,

$$T = 2\pi\sqrt{\frac{I}{\tau}},$$

$$T' = 2\pi\sqrt{\frac{I'}{\tau}},$$

hence
$$\frac{T'^2 - T^2}{T^2} = \frac{I' - I}{I},$$

or
$$I = (I' - I)\frac{T^2}{T'^2 - T^2},$$

and this gives a method of determining experimentally the moment of inertia of such a vibrating system, for the difference $I' - I$ is readily calculable from the previous results, if the movable weights are accurately made hollow cylinders.

For example, if the moment of inertia of an irregular disk about an axis through its centre of mass and perpendicular to its plane is required, we could attach it horizontally with its centre of mass at the middle point of the rod, and determine in the above way the total moment of inertia of the system; we can also determine in the same way the moment of inertia with the disk removed, and the difference of the two will give the moment of inertia of the disk itself.

205. Fly-wheels.

The action of a fly-wheel may be explained as depending on the equation

$$\tfrac{1}{2}I(\omega'^2 - \omega^2) = \text{change in energy.}$$

In the steam engine we have reciprocating motion of the piston transformed by means of the crank and connecting rod into a circular motion of the axle, and then by belts energy may be transmitted to the machines in which it is made use of. In the running of the engines there will always be irregularity in the rate at which energy is supplied and used. For example, if the supply of energy increased without a corresponding increase in the demand, the increase in the supply (if there were no fly-wheel) would cause all the moving parts to move much more rapidly, and so the machinery

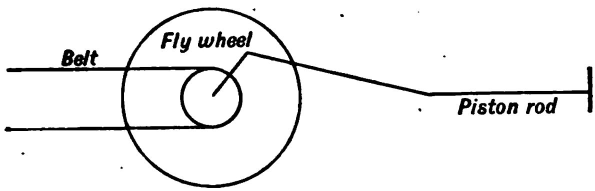


FIG. 165.

would run very unevenly. But if there is a big fly-wheel a large amount of the extra supply of energy will be taken up in making it rotate more rapidly, and if the moment of inertia is very big the change $\omega'^2 - \omega^2$ will be correspondingly small, and the smaller, the bigger is the moment of inertia. Hence a fly-wheel can be looked on as a part of the engine which acts as a reservoir of energy, storing it up when more than the average quantity is being supplied, and giving it out again when less is being supplied or more being used up.

Another way of putting it is that sudden changes in the driving force or resistances are equivalent to impulses acting, which produce sudden changes in the velocity, including, of course, the angular velocity of the rotating parts. Now the moment of the impulses $= I(\omega' - \omega)$ where $\omega' - \omega$ is the change in angular velocity. Hence for given impulses the change in angular velocity will be the smaller the bigger the moment of inertia of the fly-wheel, or the bigger the moment of inertia the more evenly will the machinery work.

It will be remembered that energy is used up in the first place by the fly-wheel in getting up speed, but when once a uniform velocity is attained, no further energy is spent on the fly-wheel except what is required to overcome the friction at the bearings of the axle.

206. The Compound Pendulum.

Any ordinary pendulum is called a compound pendulum in distinction from the simple pendulum, which, we have seen, is a purely ideal conception. If we take any rigid body and support it so that it can oscillate about a horizontal axis it

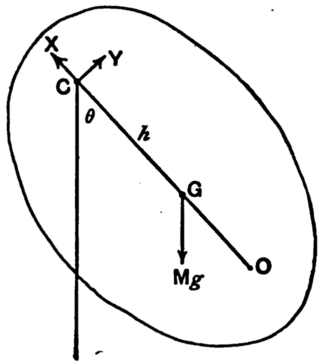


FIG. 166.

will be called a compound pendulum. Let the figure represent any compound pendulum, the axis passing through C and being perpendicular to the plane of the paper. Let G be the centre of mass, and let the plane of the paper contain G.

The forces acting on the body are the weight Mg through G, and a reaction at C which we can represent if required by components X, Y along and perpendicular to CG.

Let $CG = h$,

θ = angle CG makes with the vertical at any time,

A = angular acceleration.

The only moment about C is the moment of the weight, which $= Mgh \sin\theta$;

$$\therefore \ IA = -Mgh \sin\theta.$$

If θ is kept small always,

$$IA = -Mgh\theta,$$

or

$$A = -\frac{Mgh}{I}\theta,$$

and the angular acceleration is proportional to the angular displacement, and consequently the motion is simple harmonic with a period

$$T = 2\pi\sqrt{\frac{I}{Mgh}}.$$

Of course, the approximation to simple harmonic motion is exactly the same as in the simple pendulum.

Now let the moment of inertia about the axis through G perpendicular to the plane of the figure be represented by Mk^2;

$$\therefore\ I = M(k^2 + h^2)$$

$$T = 2\pi\sqrt{\frac{k^2 + h^2}{gh}}.$$

$\frac{k^2 + h^2}{h}$ is called the length of the equivalent simple pendulum, for a simple pendulum of this length will have the same time of oscillation as the compound pendulum.

Put then

$$\frac{k^2 + h^2}{h} = l,$$

$$k^2 = h(l - h),$$

and suppose

$$CO = l;$$

$$\therefore\ GO = l - h;$$

$$\therefore\ k^2 = CG \cdot GO.$$

If the pendulum was now dismounted and made to swing about an axis through O, the length of the equivalent simple pendulum would be

$$\frac{k^2 + (l-h)^2}{l-h} = \frac{h(l-h) + (l-h)^2}{l-h}$$

$$= h + (l - h) = l,$$

or the same as before. Thus the times about the axes through C and O [illegible].

Conversely, if we can find parallel axes such that the times of oscillation about them are the same, that the centre of mass lies in the plane of the two axes between them and at unequal distances from them, then the distance between the parallel axes is the length of the equivalent simple pendulum for this time of oscillation.

It is necessary to notice that the centre of mass must not be midway between the axes, for then the times of oscillation about them would be the same whatever distance the axes were apart.

This fact (proved by Huygens) is made use of in Kater's pendulum for the accurate determination of g.

A pendulum is made with two knife edges C and O on opposite sides of the centre of mass, a weight W can be moved

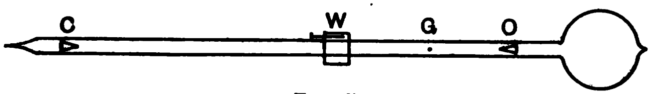


FIG. 167.

along the bar of the pendulum by a screw, and the experiment consists in adjusting the weight until the observed times of oscillation about C and O are the same. The distance CO is then the length of the equivalent simple pendulum for the observed time of oscillation. Hence, having found the length l and the time T, we have

$$T = 2\pi\sqrt{\frac{l}{g}},$$

$$g = \frac{4\pi^2 l}{T^2}.$$

We need not discuss the details of the experiments, for which we will refer to Poynting and Thomson's *Properties of Matter*, p. 12.

It is necessary to remark that when an ordinary pendulum is set swinging, the principal reason for its coming to rest gradually is the air resistance, and this has to be allowed or the experiment has to be performed in vacuo, as it often is. There may be a minute frictional torque at the supports which also tends to stop it, but in a good pendulum with steel knife edges resting on agate planes this is evanescent.

Example 1. Find the time of oscillation and length of equivalent simple pendulum for a thin rod 1 metre long swinging about a horizontal axis through one end.

2. Find the same if the rod is a metre long and the section is a square whose side is 2 cms.

3. Find the time of oscillation and the length of the equivalent simple pendulum for a pendulum consisting of a heavy sphere of radius 4 cms. supported by a string of negligible weight, the centre of the sphere being 1 metre below the point of suspension.

207. *Reaction at the Axis.*

If the angular velocity at any moment is ω, and angular acceleration A, the accelerations of the centre of mass are $\omega^2 h$ and Ah along and perpendicular to GC (see Fig. 166). Hence we have the following:

$$\text{along GC,} \qquad M\omega^2 h = X - Mg\cos\theta;$$

$$\text{perpendicular to GC,} \qquad MAh = Y - Mg\sin\theta;$$

giving X and Y when the angular velocity and acceleration are known.

If the pendulum is started from rest with CG at any given angle α with the vertical, the angular velocity in any position can be determined by the principle of energy. For the loss of P.E. is $Mgh(\cos\theta - \cos\alpha)$, and the gain in K.E. $= \frac{1}{2}I\omega^2$,

$$\tfrac{1}{2}I\omega^2 = Mgh(\cos\theta - \cos\alpha),$$

$$\omega^2 = \frac{2Mgh}{I}(\cos\theta - \cos\alpha),$$

also we found that $\qquad A = -\frac{Mgh}{I}\sin\theta.$

208. *Centre of Percussion.*

If a pendulum is hanging at rest, and is struck a horizontal blow whose impulse is P at a distance x from the axis of suspension, there will be in general an impulsive reaction at the axis.

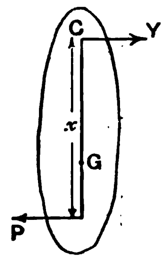


FIG. 168.

If the pendulum begins to move with angular velocity ω, and the impulse on the axis is Y (which must be horizontal), we have

$$\left.\begin{aligned} &\text{resultant impulse} &&= \mathsf{P} - \mathsf{Y} = \mathsf{M}h\omega, \\ &\text{moment of impulse} &&= \mathsf{P}x = \mathrm{I}\omega, \end{aligned}\right\}$$

$$\frac{\mathsf{P} - \mathsf{Y}}{\mathsf{P}x} = \frac{\mathsf{M}h}{\mathrm{I}},$$

$$\mathsf{Y} = \mathsf{P}\left(1 - \frac{\mathsf{M}hx}{\mathrm{I}}\right);$$

hence the reaction at the axis is zero, if

$$x = \mathrm{I}/\mathsf{M}h.$$

Now $$\frac{\mathrm{I}}{\mathsf{M}h} = \frac{\mathsf{M}(k^2 + h^2)}{\mathsf{M}h} = \frac{k^2 + h^2}{h} = l$$

= the length of the equivalent simple pendulum.

Hence, if the line of action of the blow passes through the centre of oscillation of the pendulum there is no jar on the axis.

The centre of oscillation has, therefore, also got the name of the centre of percussion.

209. The Ballistic Pendulum.

An important use has been made of a pendulum with a massive bob and considerable time of swing to determine the velocity of rifle bullets. The bob is sometimes made in the shape of a hollow iron cylinder closed at one end, and with a block of wood inserted in the hollow. Let a bullet of mass m moving with velocity v strike the pendulum in a horizontal line through the centre of oscillation and remain embedded in it. The angular momentum remains unchanged; hence, if $CO = l$,

$$mvl = I\omega.$$

If the pendulum now swings through an angle α, we have, since the loss in K.E. is equal to the gain of potential,

$$\tfrac{1}{2}I\omega^2 = Mgh(1 - \cos\alpha).$$

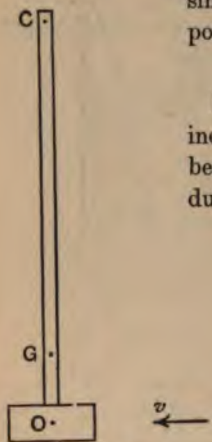


FIG. 169.

In these equations I is the moment of inertia of the pendulum including the embedded bullet, and M is the mass of pendulum and bullet:

$$mvl = I\omega,$$

$$\tfrac{1}{2}I\omega^2 = Mgh(1 - \cos\alpha)$$

$$= 2Mgh\sin^2\frac{\alpha}{2},$$

$$\omega = 2\sqrt{\frac{Mgh}{I}}\sin\frac{\alpha}{2},$$

$$mvl = 2\sqrt{IMgh}\sin\frac{\alpha}{2};$$

also, if the time T of oscillation is determined,

$$T = 2\pi\sqrt{\frac{I}{Mgh}},$$

$$I = \frac{MghT^2}{4\pi^2};$$

$$\therefore\ mvl = \frac{T}{\pi}Mgh\sin\frac{\alpha}{2},$$

$$v = \frac{TMgh}{\pi ml}\sin\frac{\alpha}{2},$$

giving the velocity of the bullet.

The angle α is usually measured by a piece of tape attached to the bottom of the pendulum, and wound on a reel so that the amount unwound can be measured and the amplitude of the oscillation determined.

In the above, M, I, and T may be taken as the quantities relating to the pendulum itself before the bullet is embedded in it, the errors so made being small.

The ballistic pendulum method of determining the velocity of a bullet is no longer used, being replaced by chronograph methods giving the instants at which a bullet is at measured distances from the gun.

EXAMPLES.

1. A door 3 feet wide weighs 100 lbs. What constant force would have to be applied 30 inches from the line of hinges, and always perpendicular to the door, to turn it through a right angle in 2 secs. from rest? Find, also, the angular velocity and kinetic energy at the end of the 2 secs. (neglect friction).

2. A door 3 ft. wide weighs 120 lbs. What impulsive torque would start it moving with an angular velocity of 1 rad./sec.? If, after being started with this velocity, a constant frictional torque brings it to rest in 3 secs., find the magnitude of this frictional torque and the angle described in the 3 secs.

3. A wheel and axle have a moment of inertia of 20 lb.-ft^2. A weight of 2 lbs. is tied to a string coiled round the axle which is horizontal and of diameter 6 ins. Find the angular acceleration of the wheel and the time taken to make 10 revolutions from rest, neglecting friction.

4. A weight of 2 lbs. is attached to a string coiled round an axle which is horizontal and of diameter 6 ins. A constant frictional torque acts at the bearings. If the axle describes 5 revolutions in 10 secs. from rest, and the weight then reaches the ground, and the axle turns through one more revolution before coming to rest, find the frictional torque.

5. A wheel and axle is capable of turning about a horizontal axis, and a mass m_1 is attached to a string wound round the axle. If the radius of the axle is a, and the weight is found to fall s_1 ft. from rest in t secs., and when m_1 is replaced by m_2 a distance s_2 is fallen from rest in t secs., find the M.I. of the wheel and axle, supposing a constant frictional torque to act.

6. An Atwood's machine consists of a wheel of radius 3 ins., whose weight of 6 oz. may be supposed concentrated in its circumference. If weights of 12 and 16 oz. are attached to the string, find the acceleration of the weights

(1) when friction at the axle is neglected,

(2) if there is a constant frictional torque of $\frac{1}{8}$ oz. wt.-ft. units.

7. A mass M is attached to one end of a rod of length l and mass m per unit length, which rotates with uniform angular velocity ω on a smooth horizontal plane about the other end, which is fixed. Prove that the tension of the rod at a distance r from the fixed end is

$$M\omega^2 l + \tfrac{1}{2} m\omega^2 (l^2 - r^2).$$

8. Two particles of masses 3 and 1 lb. are fixed at the ends A, B of a weightless rigid rod 3 feet long capable of turning freely about a point O between A and B. If AO=2 ft. and OB=1 ft., and the rod is just disturbed from its position of unstable equilibrium, find the velocities of A and B when the rod is passing through the position of stable equilibrium.

9. A uniform circular disk of mass M and radius r is loaded at one point of its circumference with a mass m, and can rotate about a frictionless horizontal axis through its centre. If displaced slightly from the position of equilibrium, find the time of a small oscillation.

10. Find the time of oscillation and the length of the E.S. pendulum of a pendulum consisting of a weightless string 5 ft. long to which is attached an iron sphere 6 ins. diameter. (Take $g=32{\cdot}18$.)

11. Find the time of oscillation of a pendulum consisting of a uniform bar of length 100 cms. and cross section a square of side 2 cms., the axis of suspension being through the centre of a section and parallel to an edge of the section, and at a distance of 10 cms. from the end.

What would the calculated time be if the thickness of the rod were neglected? (Take $g=980$.)

12. A heavy thin uniform rod of length 3 ft. and weight 10 lbs. can rotate about a horizontal axis through one end. If it is struck a blow through the centre of oscillation which causes it to just make complete revolutions, find the initial angular velocity and the impulse of the blow.

13. A pendulum consists of a thin uniform bar 4 ft. long and weighing 6 lbs., the axis of rotation being at one end. Find the horizontal impulse which must be given to it at the centre of percussion when at rest to make it just describe complete circles, neglecting friction. Find, also, the direction and magnitude of the reaction at the axis when it is

(1) vertically downwards,
(2) vertically upwards,
(3) horizontal.

14. The pendulum of the last question is struck a horizontal blow through the centre of mass to give it the same angular velocity as in that example. Find the magnitude of the impulse and the impulsive reaction at the axis of suspension.

15. The motion of a door is resisted by a constant friction couple, and it is found that a constant force of 4 lbs.-wt., perpendicular to the door, acting $2\frac{1}{2}$ feet from the line of hinges, causes it to describe a right angle in 2 secs., while, if applied at a distance of 3 ft., it causes it to describe a right angle in 1·8 secs. Find the moment of inertia of the door and the magnitude of the frictional torque.

16. The mass of a wheel weighing 40 lbs. may be considered as concentrated in a circle of 18 ins. in diameter. What is its kinetic energy when rotating at 1 revolution per sec., and what impulsive torque would produce this velocity from rest?

With what pressure should a brake press on the axle whose radius is 2 ins. in order to stop the wheel in (1) 3 secs., (2) 3 revolutions, if the coefficient of friction is 0·3?

17. A ballistic pendulum has a mass of 80 lbs. and moment of inertia 1000 lbs.-ft^2. about its axis of suspension. Its time of oscillation is 2·2 secs. A bullet of mass 2 oz. is fired horizontally into it in a line through the centre of oscillation, and the pendulum swings through 16°. Find

(1) the length of the equivalent simple pendulum,
(2) the distance of the centre of mass from the axis of suspension,
(3) the initial angular velocity of the pendulum,
(4) the velocity of the bullet.

18. A ballistic pendulum weighs 100 lbs., and the length of its equivalent simple pendulum is 7·2 feet. A force of 50 lbs. applied horizontally at the centre of oscillation deflects it through an angle of 31° (tan 31° = 0·6). A shot of weight 2 oz., moving at 2400 ft./sec., strikes it at the centre of oscillation and remains embedded in it. Find through what angle the pendulum will swing.

19. A rod of weight 2 kgms. and length 60 cms., whose thickness may be neglected, is suspended by a wire through its centre, and can oscillate in a horizontal plane. If a torque of $\frac{1}{10}$ of a kgm.-wt.-metre unit is required to turn it through 90°, find the time of a small oscillation.

20. A thin uniform rod of length $2l$ and mass M can turn freely about a vertical axis through its centre and perpendicular to its length. To the ends are attached two horizontal strings which pass in opposite directions over pulleys, and to which equal weights are attached. If the rod is displaced slightly from its position of equilibrium, find the time of oscillation. (Assume the pulleys so far off that the strings are always in the same direction and the tension remains equal to the tension in the equilibrium position.)

Obtain a numerical result when

$$M = 6 \text{ lbs.}, \quad m = 2 \text{ lbs.}, \quad l = 4 \text{ ft.}$$

We have proved that the accelerations are

$$f - r\omega^2 \cos\theta - rA \sin\theta,$$
$$g - r\omega^2 \sin\theta + rA \cos\theta;$$
$$\therefore\ X = m(f - r\omega^2 \cos\theta - rA \sin\theta), \quad \text{............(1)}$$
$$Y = m(g - r\omega^2 \sin\theta r + A \cos\theta). \quad \text{............(2)}$$

The moment about O of the forces on m is

$$L = Y \,.\, ON - X \,.\, PN = Y(\bar{x} + r\cos\theta) - X(\bar{y} + r\sin\theta)$$
$$= -m(\bar{y} + r\sin\theta)(f - r\omega^2\cos\theta - rA\sin\theta)$$
$$+ m(\bar{x} + r\cos\theta)(g - r\omega^2\sin\theta + rA\cos\theta). \quad \text{............(3)}$$

Now, if we add up for the whole body,

$$\Sigma mr\cos\theta = \Sigma mx = 0,$$

since $r\cos\theta$ or x is the coordinate of P relative to G;

also, $\Sigma mr\sin\theta = \Sigma my = 0$;

hence, from (1), $\Sigma X = \Sigma mf = f\Sigma m = Mf$,

,, (2), $\Sigma Y = \Sigma mg = Mg$.

Note that in all the following summations $\bar{x}$, $\bar{y}$, f, g, which refer to the centre of mass, can be taken in front of the summation sign.

From (3), $\Sigma L = -\Sigma m(\bar{y} + r\sin\theta)(f - r\omega^2\cos\theta - rA\sin\theta)$

$$+ \Sigma m(\bar{x} + r\cos\theta)(g - r\omega^2\sin\theta + rA\cos\theta)$$
$$= \Sigma m(\bar{x}g - r^2\omega^2\cos\theta\sin\theta + r^2A\cos^2\theta)$$
$$- \Sigma m(\bar{y}f - r^2\omega^2\cos\theta\sin\theta - r^2A\sin^2\theta),$$

the remaining terms vanishing.

$$\therefore\ \Sigma L = \Sigma m(\bar{x}g - \bar{y}f) + \Sigma mr^2A$$
$$= (\bar{x}g - \bar{y}f)\Sigma m + A\Sigma mr^2$$
$$= M(\bar{x}g - \bar{y}f) + IA,$$

where I is the M.I. about an axis through the C.M. perpendicular to the plane.

Thus the equations are:

$$\left.\begin{aligned} Mf &= \Sigma X, \\ Mg &= \Sigma Y, \\ M(\bar{x}g - \bar{y}f) + IA &= \Sigma L, \end{aligned}\right\} \quad \begin{aligned} &\text{..........................(4)} \\ &\text{..........................(5)} \\ &\text{..........................(6)} \end{aligned}$$

as before (Art. 180) in ΣX, ΣY, ΣL only the external forces appear, the internal reactions cancelling one another.

211. If we take O at the instantaneous position of the centre of mass, $\bar{x}$, $\bar{y}$ are zero, and

$$IA = \Sigma L ; \quad \ldots\ldots\ldots\ldots(7)$$

thus the equation for the angular acceleration about the centre of mass is independent of the motion of the centre of mass; in other words, the equation is the same as if the C.M. were a fixed point.

The equations are now in their simplest form

$$Mf = \Sigma X, \qquad Mg = \Sigma Y, \qquad IA = \Sigma L.$$

Of these, (4) and (5) had been obtained before in Art. 81, and shew that in any case the centre of mass moves as if all the mass were collected there, and the forces were transferred to act at the centre of mass parallel to their original directions.

212. Energy of a Rigid Body Moving in a Plane.

If u, v are the components of the velocity of the centre of mass, the velocities of P are

$u - r\omega \sin\theta$ parallel to Ox, $\quad v + r\omega\cos\theta$ parallel to Oy;

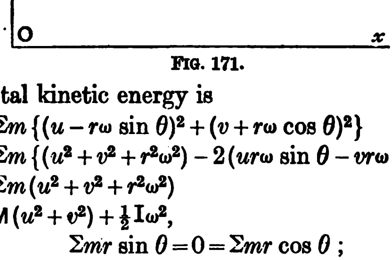


FIG. 171.

∴ the total kinetic energy is

$$\tfrac{1}{2}\Sigma m\{(u - r\omega\sin\theta)^2 + (v + r\omega\cos\theta)^2\}$$
$$= \tfrac{1}{2}\Sigma m\{(u^2 + v^2 + r^2\omega^2) - 2(ur\omega\sin\theta - vr\omega\cos\theta)\}$$
$$= \tfrac{1}{2}\Sigma m(u^2 + v^2 + r^2\omega^2)$$
$$= \tfrac{1}{2}M(u^2 + v^2) + \tfrac{1}{2}I\omega^2,$$

since $\quad \Sigma mr\sin\theta = 0 = \Sigma mr\cos\theta$;

hence the kinetic energy may be regarded as composed of two parts:

(i) $\frac{1}{2}M(u^2+v^2)$, which is the K.E. the body would have if collected at its centre of mass and moving with the velocity of the centre of mass.

(ii) $\frac{1}{2}I\omega^2$, which is the K.E. which it would have if rotating about the C.M. as a fixed point.

These two parts of the kinetic energy are spoken of as the kinetic energy of translation and rotation respectively.

Examples of Motion in a Plane.

213. *A reel has a weightless thread wound round it, and the thread being held fixed at one end, the reel falls in a vertical plane, its axis remaining horizontal, find the motion.*

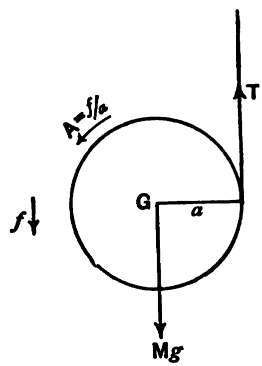


FIG. 172.

The extremity of the horizontal diameter is the instantaneous centre, and if the velocity of the C.M. is v at any moment, the angular velocity of the reel is $\frac{v}{a}$; hence, also, if f is the acceleration of the C.M.,

$$\frac{f}{a} \text{ is the angular acceleration.}$$

The equations are:

vertically, $\qquad Mf = Mg - T;$

moments about G, $\qquad I\frac{f}{a} = Ta;$

$$\therefore\ Mfa + I\frac{f}{a} = Mga;$$

$$f = \frac{Ma}{Ma + \frac{I}{a}}g = \frac{Ma^2}{Ma^2 + I}g.$$

If the reel is a uniform cylinder,

$$I = Ma^2/2,$$
$$f = 2g/3,$$
$$T = M(g - f) = Mg/3.$$

214. *Find the acceleration of a circular disk (or cylinder) rolling down an inclined plane without sliding.*

Let α = inclination of plane to horizon,

f = acceleration down plane,

$\frac{f}{a}$ = angular acceleration about the C.M.,

R = the normal reaction,

F = the friction.

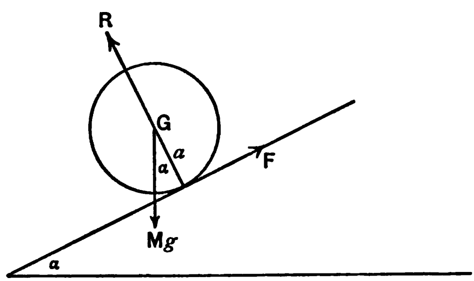


FIG. 173.

The equations are :

resolving along the plane, $Mf = Mg \sin\alpha - F$;...............(1)

moments about G, $I\frac{f}{a} = Fa$;..........................(2)

$$\therefore\ Mfa + I\frac{f}{a} = Mga\sin\alpha;$$

$$\therefore\ f = \frac{Ma}{Ma + \frac{I}{a}}g\sin\alpha$$

$$= \frac{Ma^2}{Ma^2 + I}g\sin\alpha \quad \text{..............(3)}$$

For a circular disk or cylinder,

$$I = Ma^2/2;$$

$$\therefore\ f = \tfrac{2}{3}g\sin\alpha.$$

The friction and normal reaction can now be found.

From (1), $F = -Mf + Mg\sin\alpha = \frac{1}{3}Mg\sin\alpha.$

Resolving perpendicular to the plane,

$$R = Mg\cos\alpha;$$

$$\therefore\ \frac{F}{R} = \tfrac{1}{3}\tan\alpha.$$

215. The body in the last example consequently rolls down if the coefficient of friction is as great as $\frac{1}{3}\tan\alpha$. If the coefficient of friction were less than this, the body would slide as well as rotate.

If the body instead of being a cylinder is a sphere, the equations (1), (2), (3) hold, but

$$I = M2a^2/5,$$

$$f = \tfrac{5}{7}g\sin\alpha.$$

If, in the case of the cylinder, the coefficient of friction is less than $\frac{1}{3}\tan\alpha$, the body will slide and rotate. There is now no connection between the angular and linear velocity, but instead we have the relation between the friction and normal reaction; thus, if the coefficient of friction is μ, we have, forming the equations in the same way as before,

$$\left.\begin{aligned} Mf &= Mg\sin\alpha - \mu R, \\ 0 &= Mg\cos\alpha - R, \\ IA &= \mu Ra; \end{aligned}\right\}$$

$$\therefore\ f = g(\sin\alpha - \mu\cos\alpha),$$

$$A = \frac{\mu Mga\cos\alpha}{I},$$

and for the disk or cylinder,

$$A = \frac{2\mu g}{a}\cos\alpha.$$

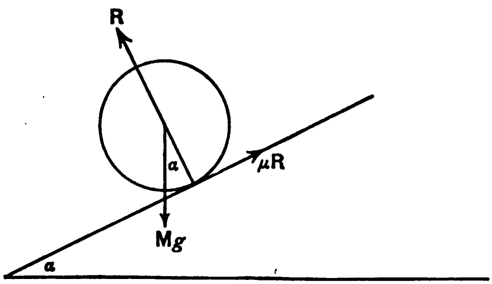


FIG. 174.

Example 1. Find the kinetic energy of (*a*) a uniform cylinder or disk, (*b*) a sphere, rolling along a horizontal plane, and determine the fraction of this which is rotational.

2. The ends of a uniform rod move along two straight lines at right angles. If the angular velocity of the rod at a given instant is ω, find the velocity of the centre of mass and the kinetic energy of the rod.

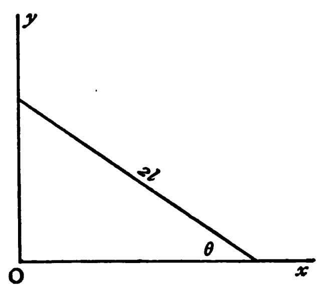


FIG. 175.

3. If in Example 2 the line Oy is vertical, and there is no friction (a rod sliding down with its ends on smooth horizontal and vertical

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about the C.M. (There can be no component velocity along AB, as there is no impulse in that direction.)

Hence
$$\mathrm{P} = \mathrm{M}v,$$
$$\mathrm{P}l = \mathrm{I}\omega.$$
giving v and ω.

Now the rod may be thought of as moving initially about some instantaneous centre which must be in the line AB, since G is moving perpendicularly to AB. If the distance of this point from the centre towards B is x, then

$$v - x\omega = 0 \text{ (being the velocity of the point)};$$
$$\therefore \frac{\mathrm{P}}{\mathrm{M}} = \frac{\mathrm{P}l}{\mathrm{I}}x,$$
$$x = \frac{\mathrm{I}}{\mathrm{M}l} = \mathrm{M}\frac{l^2}{3}\Big/\mathrm{M}l = \frac{l}{3}.$$

Or the instantaneous centre is the centre of oscillation corresponding to a centre of suspension at the end struck.

218. *A rod of length 2l has small rings attached to its two ends which slide on two smooth wires at right angles. If it makes an angle θ with one of them and is struck a blow, at its centre of impulse* P, *parallel to this wire, find the initial velocity.*

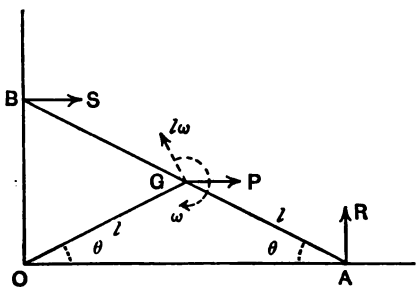

FIG. 177.

In this case, as the position is determined by a single co-ordinate, so the motion is determined by the angular velocity alone.

The centre of mass G describes a circle about O, and if ω is the angular velocity of the rod (which is also the rate of change of θ), ω will also be the angular velocity of G about O, and the linear velocity of G is $l\omega$ perpendicular to OG. In the figure the linear and angular velocities are marked on the supposition that θ is increasing. Impulsive reactions R, S act at A, B perpendicular to Ox, Oy.

The equations are:

resolving along Ox, $\quad P+S=-Ml\omega\sin\theta$, ...(1)

" $\quad Oy$, $\quad R=Ml\omega\cos\theta$, ...(2)

moments about G, $\quad Rl\cos\theta-Sl\sin\theta=-I\omega$; ....(3)

hence eliminating R and S,

$$Ml^2\omega\cos^2\theta+(Pl\sin\theta+Ml^2\omega\sin^2\theta)=-I\omega,$$

$$Pl\sin\theta=-I\omega-Ml^2\omega,$$

$$\omega=-\frac{Pl\sin\theta}{I+Ml^2}=-\frac{Pl\sin\theta}{M\frac{l^2}{3}+Ml^2}$$

$$=-\frac{3}{4}\frac{P}{Ml}\sin\theta.$$

The minus sign shewing that the rod begins to move in the opposite direction to that marked.

Whence also $\quad R=-\frac{3}{4}P\sin\theta\cos\theta,$

$$S=+\tfrac{3}{4}P\sin^2\theta-P=-P(1-\tfrac{3}{4}\sin^2\theta).$$

219. AB, BC *are two equal uniform rods smoothly jointed together and lying in the same straight line on a smooth horizontal table. The end* A *is struck by a blow, of impulse* P, *perpendicular to* AB, *find the initial motions.*

In this case there will be an impulse produced at the joint which we may express by two components in the directions marked. To shew the reactions clearly the rods are drawn with the ends at B separated, but it is to be remembered that these ends remain in contact. It is easy to see that the component X vanishes.

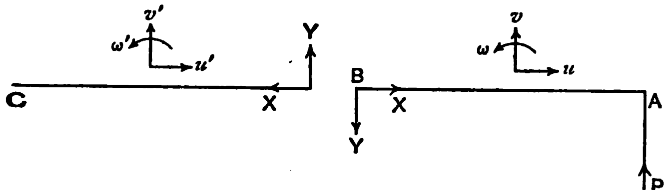

FIG. 178.

However, we will form the equations as if X existed, and then the solution of these will shew that

$$X=0.$$

Let $u, v, \omega, u', v', \omega'$, be the linear and angular velocities of the two rods.

Then we have

for AB along AB, $X = Mu$; ......(1)

perpendicular to AB, $P - Y = Mv$; ......(2)

moments about C.M. of AB,

$$Pl + Yl = I\omega = M\frac{l^2}{3}\omega,$$

or $P + Y = M\frac{l}{3}\omega$; ......(3)

similarly, for BC, $X = -Mu'$, ...(4)

$Y = Mv'$, ...(5)

$$Yl = I\omega' = M\frac{l^2}{3}\omega',$$

or $Y = M\frac{l}{3}\omega'$. ......(6)

So far we have six equations with six unknown velocities and two reactions, so that to solve them two more equations are wanted. These two are geometrical, expressing the fact that the ends of the rods at B remain together, and therefore have equal component velocities in the two directions.

projectile; and also discovered (his earliest discovery in 1583) the isochronism of the pendulum.

Although Kepler's (1571-1630) name appears in elementary dynamics in connection with motion in a circle, the laws he formulated were entirely empirical, and had no effect on the development of mechanics until in Newton's hands they helped to establish the law of gravitation.

More important work in dynamics was done by Newton's great contemporary Huygens (1629-1695), who was the first to prove (in 1673) the formula v^2/r for the acceleration in uniform circular motion. He also, at the same time as Wren and Wallis in England, made important experiments on collision, and he further gave the theory of the compound pendulum, and proved the convertibility of the centres of suspension and oscillation.

The parallelogram of forces had been stated for statical cases by various earlier writers.

To Newton (1642-1727) is due an immense proportion of the elementary dynamics. He obtained wonderfully clear and definite ideas on the fundamental notions of dynamics, and expressed these in the laws of motion, which have almost universally served ever since with slight modification, as the most convenient basis for building up the science of dynamics. In the statement of these laws there are two great ideas which may be regarded as the leading ideas in Newton's work, and specially due to him. They are the idea of mass and its connection with force, and the law of action and reaction.

He carried out many experimental investigations, the experiments on collision alone shewing that he was a master in experimental work as well as in theoretical. Another very important series of experiments were those on the time of oscillation of pendulums of different materials, by which he shewed that g was the same for all bodies. The law of gravitation was one of his great discoveries, and the theory of simple harmonic motion was due to him.

The foundations of rigid dynamics, though deducible from Newton's work, were not laid down till 1743 by d'Alembert (1717-1783), and the name "Moment of Inertia" was first used by Euler (1707-1783), who studied a number of cases of motion of a rigid body, and gave the equations of motion in the most general form.

The laws of friction were obtained by Coulomb (1736-1806) in 1781.

It is only necessary to add that the idea of the instantaneous centre is due to Chasles (1793-1880), and that the vector methods, of which the merest beginnings have been given in this book, are due to various mathematicians of the nineteenth century, the notation used being that due to Gibbs.

ANSWERS.

CHAPTER I.

p. 6. 1. $-9,\ -99,\ 11,\ 101.$ 2. $2,\ \infty;\ 1{\cdot}990,\ 2{\cdot}010,\ 101,\ -99.$

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p. 19. $v = 4t$, $s = 72$ ft.

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5. -3 ft./sec^2., 9 ft. 6. 2 ft./sec^2., $8{\cdot}72$ ft./sec.

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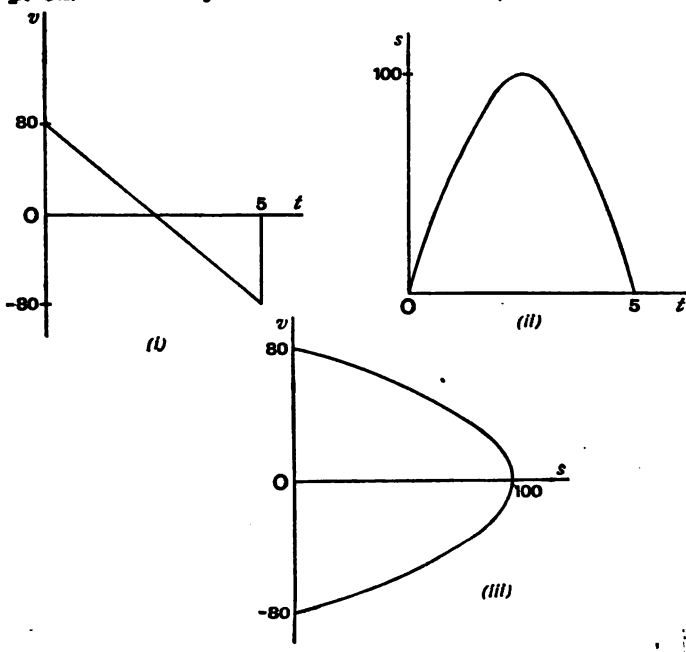
8. $-\frac{1}{3}$ ft./sec^2., $8\frac{1}{6}$ ft., 7 secs. 9. $22{\cdot}4$ miles/hr.

10. $\sqrt{\{(u^2+v^2)/2\}}$. 11. Yes, $2{\cdot}45$ ft.

p. 32. 1. $u^2 > 2gs$. 2. 100 ft., 5 secs.



p. 32. **2.** 100 ft., 80 ft./sec. **3.** 31·3 m./sec., 3·19 secs.
4. 27·7 ft./sec. **5.** 7·03 secs., No.

p. 33. **6.** 32 ft./sec., 240 ft. **8.** 0·56, 4·44, 0·36 secs.
9. $\sqrt{8h(1-k)/g}$. **11.** 230 ft.

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p. 36. **9.** 40 secs., 400 ft. from O.

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7. 8·96 tons-wt. **8.** 2738 ft., 186·7 secs.
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3. 53·1 ft./sec., 372 secs., 5952 yds.
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5. 3·2 ft./sec^2. upwards.
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7. 64 ft./sec., 64 ft. **8.** 253·5, 684 ft.-lbs.
9. 2·45, 7·55 kgm.-metres.
10. 1·4 ft./sec., $1{\cdot}715\times10^6$, 1715 ft.-lbs.
12. $Mu/(M+m)$, $mu/(M+m)$. **13.** 312.

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17. 3013 lbs.-wt., 241·1, 94·2 miles/hr.

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2. 1·14 ft./sec., 4·340 ft.-lbs.

4. 19·4 ft./sec., 6·06 ft.-lbs., 145·4 lbs.-wt., 0·0043 sec.

5. 6·20 ft./sec., 2·75 tons-wt.

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3. 4 ft./sec²., 7·875 lbs.-wt. 4. $\sqrt{(m-m')g/2k}$, 10 ft./sec.

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4. 292 H.P., 52·2 miles/hr.

5. $m\left\{\frac{308}{3}\frac{v}{t}+n\right\}$ lbs.-wt.; $\frac{mv}{375}\left\{\frac{308}{3}\frac{v}{t}+n\right\}$ H.P.

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p. 97. 1. 1 ft., 3·2 ft./sec., 0·6 lb.-wt.-sec.

2. 3·16 sec., 1·69 ft./sec., 0·53 ft., 0·105, 0·422 lb.-wt.-sec.

3. 2, 3·765 lbl.-sec., 3·765 ft./sec.

p. 98. 1. 0·0928g, 0·311g, 0·249g, 0·155g.

2. 2·91, 8·73 ft./sec²., 1·455 lbs.-wt.

p. 99. 3. $\frac{2g}{13}$, $\frac{g}{13}$; 2·31 lbs.-wt.

4. 0·2g, 0·067g, 0·067g, 160 gms.-wt., 0·078g, 138·5, 184·6 gms.-wt.

5. (1) 0·125g, 0·25g, 225, 112·5 gms.-wt.
(2) 0·152g, 0·303g, 230·3, 104·5 gms.-wt.

p. 102. (1) $4{\cdot}214 \times 10^5$. (2) 745·8. (3) 0·00134.

p. 102. 1. 187 yds., 254 lbs.-wt.

2. 0·586 ft./sec².; 1680, 1120, 560 lbs.-wt.

3. 28·4 ft./sec²., 10·6 ft./sec. 4. 0·291.

p. 103. 5. 0·081. 6. $mu^2/2g$, $mu^2/17600t$.

7. 1061. 8. 288, 180 tons.

9. $\frac{M}{448}\left(1+\frac{v^2}{1936}\right)$, 124·1 ft./sec., 1011.

10. $\frac{550H}{M}$ ft./sec., $550H/v'$ lbs.-wt., $\left(\frac{v}{v'}-1\right)g$.

p. 103. 11. $75H/14Mv - (a+bv^2)/70$. 13. 12·8, 0·0011.

p. 104. 14. 396 lbs. 15. 0·106.

16. 1·47 ft./sec., 15·1 ft./sec². 21. 21·1 lbs.

p. 105. 22. 36·3 kgms. 24. $\frac{1}{2}\frac{Mm}{M-m}(v-V)^2$.

25. 2036, 964 ft./sec. 26. 3·53.

27. $\sqrt{2MEg/m(M+m)}$, $\sqrt{2mEg/M(M+m)}$. 28. 2560 ft./sec.

30. $V \pm \sqrt{2m_2E/m_1(m_1+m_2)}$, $V \mp \sqrt{2m_1E/m_2(m_1+m_2)}$.

p. 106. 31. $m(V-v)/M$; $2h\Big/\left\{V+v-\frac{m}{M}(V-v)\right\}$.

34. 0·0037. 35. 51, 56 kgms.-wt.

36. 1·56 lbs.-wt., 0·16; 3·2 ft. 37. 14·1 ft./sec., 0·39, 0·31 sec.

p. 107. 39. $(u+v)^2/2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)F$; $(m_1u-m_2v)/(m_1+m_2)$.

40. 0·29, 32·9 ft./sec². 41. Equal. 43. 1, $\frac{1}{3}$ lbs.-wt., $g/3$.

44. $l+\left(\frac{m}{m+m'}\right)^2\left(\frac{v^2}{2g}-l\right)$; $\left(\frac{m}{m+m'}\right)^2\left(\frac{v^2}{2g}-l\right)$.

45. (i) $\{(4m-2M)g-Mf\}/(M+4m)$
$\{(2m-M)g+2mf\}/(M+4m)$; $Mm(3g+f)/(M+4m)$.
(ii) f, $m(g-f)$, $(M+m)f+(M-m)g$.

p. 108. 46. $(2m'-m+M)g/(4m'+m+M)$,
$2(2m'-m+M)g/(4m'+m+M)$,
$2m(3m'+M)g/(4m'+m+M)$,
$m'(3m-M)g/(4m'+m+M)$.

47. Velocities $m'v/(m+m')$, $mv/(m+m')$.
Accelerations $m'f/(m+m')$, $mf/(m+m')$,
Tension $mm'f/(m+m')$.

48. 7·1 ft./sec².; 6·2, 4·9 lbs.-wt.; 2·2 ft.

49. 4·4 ft./sec².; 6·9, 4·6 lbs.-wt.; 1·0 ft.

50. $Mm(1+n)g/(M+n^2m)$, $(M-nm)g/(M+n^2m)$,
$n(M-nm)g/(M+n^2m)$; $2nv/g$; $nmMv/(M+n^2m)$,
$mn^2v/(M+n^2m)$.

51. $(mn-M)g/(M+n^2m)$, $n(mn-M)g/(M+n^2m)$,
$Mm(1+n)g/(M+n^2m)$,
afterwards same with M replaced by 2M.
Velocities just after
$(mn-M)gt/(2M+mn^2)$, $n(mn-M)gt/(2M+mn^2)$.
Impulsive tension
$nmM(mn-M)gt/(M+mn^2)(2M+mn^2)$.

52. 4·44, 3·39 ft./sec.; 5·22, 13·57 lbl.-sec.; 3·06 ft.

p. 109. 53. 151 cms.; 5·55 secs.

54. $\frac{17}{2}\left(\frac{8}{9}\right)^n$ secs., $17\times\left(\frac{8}{9}\right)^{2n}$ ft.; 70 secs., 68 ft.

55. $\frac{2}{g}\left(\frac{Q}{P+Q}\right)^{n-1}v$; $\frac{2}{g}\frac{Q}{P-Q}\left(\frac{Q}{P+Q}\right)^{n-1}v$,
where $v^2=2(P-Q)ga/(P+Q)$.
Impulse $=P\left(\frac{Q}{P+Q}\right)^n v$.
Total time $=3\sqrt{2a(P+Q)/(P-Q)g}$.

57. 4·27 ft./sec²., 4·33 lbs.-wt., 19·26 ft.

p. 110. 58. $\frac{(M'+m-M)(M+M')}{(M'+m+M)(M-M')}h$ below, and $\left(\frac{M+M'}{M+m+M'}\right)^2 h$ above, where the weight is caught off.

60. Between n^{th} and $n+1^{th}$, $\tau\sqrt{2gh}+\frac{1}{2}(2n-1)g\tau^2$.

CHAPTER IV.

p. 116. 3. 7·5 m./hr.

p. 126. 4. 8 ft. 5. 1·94 sec., 12·39 ft./sec.

6. 24·6 ft./sec., 214·9 secs.

7. 8·33 ft., 11·55 ft./sec.; 0·833, 1·44 secs.

8. 24·4 ft., 12·9 ft./sec.

10. $\sqrt{4a/g\sin 2\theta}$, if θ is the angle made with the vertical and a = distance of the starting point from the line.

11. 880 ft. 12. 14·63, 17·63 tons-wt.

p. 127. 13. 51·03 ft./sec., 119·3 secs., 7637 ft. 14. 251·4 yds.

15. 434. 16. 25·9 ft./sec. 17. 1 : 9.

19. 320 H.P.; 12 lbs.-wt. per ton.

20. 6133 lbs.-wt., 490·7 H.P. 21. 10·5 miles/hr.

22. $m\left(2240\sin\alpha+n+\frac{308}{3}\frac{v}{t}\right)$lbs.-wt.,
$\frac{mv}{375}\left(2240\sin\alpha+n+\frac{308}{3}\frac{v}{t}\right)$.

p. 128. 23. 1606 yds., 13·9 miles/hr.; 164, 473 secs.

24. 7·06 ft./sec². 26. 0·97 ft./sec². 27. 0·375.

28. 4·334 ft., 0·10 sec. after the string breaks, 1·33 secs.

29. $0·246g$. 30. 3·12 ft./sec².

31. 10·67 ft./sec²., 9·33 lbs.-wt., 2·99 ft./sec².

p. 129. 32. $(m\sin\alpha - n$ [illegible] $\beta)g/(m+n)$.

p. 131. 35. Wedge 4·[illegible]/sec²., particle relative to wedge 22·9 ft./sec².

p. 135. 1. 36960 f[illegible] 0·16 H.P. 3. 184 kgm.-metres.

p. 141. 2. $(m-m')^2 g/(m+m')^2$.

3. 5 ft./sec., making an angle 53° 8′ with the first.

4. 12·17 ft./sec., making 34° 43′ with Ox and meeting Ox 3 ft. from O.

p. 141. 1. 6·21, 16·97 miles/hr.

4. 43·8 ft. from starting point; velocity 29·2 ft./sec. at 34° 43′ with force 3.

6. $2mv \sin \frac{\pi}{n}$ towards the centre.

p. 142. 7. 83·1 ft./sec. perpendicular to incline.

8. $\cos^{-1} \frac{au+bv}{bu+av}$; $\cos^{-1} \frac{a}{b}$.

9. 2475 lbs.-wt. 11. 66 miles/hr. 12. $\sqrt{(2240-na)/nb}$.

14. $u^2/2g(\sin\alpha+\mu\cos\alpha)$,

$$\frac{u}{g}\left\{\frac{1}{\sin\alpha+\mu\cos\alpha}+\frac{1}{\sqrt{\sin^2\alpha-\mu^2\cos^2\alpha}}\right\}.$$

p. 143. 16. $\mathrm{M}u\cos\alpha/(\mathrm{M}+m)$, $u\sin\alpha$, $mu\cos\alpha/(\mathrm{M}+m)$, $\frac{1}{2}mu^2(\mathrm{M}+m\sin^2\alpha)/(\mathrm{M}+m)$.

19. Wedge $\frac{1}{2}\cdot\frac{mgt^2\cos\alpha(\sin\alpha-\mu\cos\alpha)}{\mathrm{M}+m\sin\alpha(\sin\alpha-\mu\cos\alpha)}$,

particle relative to wedge

$$\frac{1}{2}\cdot\frac{gt^2(\mathrm{M}+m)(\sin\alpha-\mu\cos\alpha)}{\mathrm{M}+m\sin\alpha(\sin\alpha-\mu\cos\alpha)}.$$

21. 15° 33′, 0·380g; block 0·278g, weight relative to block 0·658g.

p. 144. 23. $\frac{(m-\mathrm{M})(\sin\alpha\pm\mu\cos\alpha)g}{\mathrm{M}\sin\alpha(\sin\alpha\pm\mu\cos\alpha)+m}$.

24. Acceleration $g\sin(\alpha+\beta)$ at angle $\alpha-\beta$ with horizontal.

25. 17·9 ft./sec². at 63° 26′ with horizontal, 6·71 lb.-wt.-sec.

CHAPTER VI.

p. 163. 2. 5 ins., 0·262 sec., 10 ft./sec. 4. 12, 5 ft./sec.

5. 0·287, 0·122, 0·097, 0·084, 0·076, 0·071, 0·068, 0·066, 0·064, 0·064.

6. 0·12, 0·37, 0·60, 0·82, 1·02, 1·19, 1·34, 1·45, 1·53, 1·56.

7. 0·833, 0·416.

8. Centre 2 ft. from first point, amplitude 3·6 ft., max. vel. 3·6 ft./sec.

9. 4 ins. from first point, amplitude 2·69 ft., 2·09 secs., 8·06 ft./sec.

p. 164. 10. 1·97, 3·22 ft. 11. 135·8, 32·2 lbs.-wt.

13. 0·76 ft., 0·37 secs. 15. 0·0314 cm./sec., 98·7 cm./sec².

p. 170. 3. $2\pi\sqrt{ma/\mathrm{T}g}$, $\sqrt{\{(l'-l)^2+mau^2/\mathrm{T}g\}}=c$ say, $\mathrm{T}gc/a$, $c\sqrt{\mathrm{T}g/ma}$.
4. 0·351 sec., 4·47 ins. 5. $(l'-l)\sqrt{2\lambda/ml}$, $2\pi\sqrt{ml/2\lambda}$.

p. 171. 6. 0·196 secs., 16 ft./sec. 7. $2\pi\sqrt{ml/\lambda}+4\sqrt{ml^3/\lambda(l'-l)^2}$.
8. Distances from fixed point $\frac{1}{15}$, $\frac{13}{15}$, $\frac{3}{15}$ ft., etc.; final position $\frac{7}{15}$ ft.
9. 0·454 sec., 11·55 ft./sec.

p. 192. 3. $\mathrm{M}\left\{g-\frac{4\pi^2}{\mathrm{T}^2}x\right\}$, where x is the height above the mean position.
4. 1·36 sec., 4·62 ft./sec. 5. 0·248, 0·351 secs.
6. $2mgl(\sin\alpha-\mu\cos\alpha)/\lambda$; moves up again if $\tan\alpha>3\mu$; distance 0·462 ft; does not return.

p. 193. 10. 6 ft./sec., $12\sin 4t$ ft./sec^2.
12. 1·46 ft./sec., 0·373 ft., 0·544 sec.

p. 194. 13. $\{\mathrm{M}(v+2\mathrm{V})-mv\}/(\mathrm{M}+m)$, $\{m(\mathrm{V}+2v)-\mathrm{MV}\}/(\mathrm{M}+m)$ from one another; $\pi\sqrt{m\mathrm{M}/(m+\mathrm{M})\mathrm{T}}$.
17. $\sqrt{2gb-\mathrm{T}b^2g/\mathrm{M}a}$; $\sqrt{\mathrm{M}a/\mathrm{T}g}\cos^{-1}(\mathrm{M}a-\mathrm{T}b)/\mathrm{M}a$; $2\pi\sqrt{\mathrm{M}a/\mathrm{T}g}$.

CHAPTER VII.

p. 199. 1. 7·85. 2. 23·47. 3. $7{\cdot}29\times10^{-6}$. 4. $1{\cdot}99\times10^{-7}$.

p. 203. 1. 688·4 ft./sec^2. 2. $0{\cdot}111\cos\lambda$.
3. Angle with radius 4° 33′, 6° 48′, 13° 25′, 90°; magnitude 150·5, 100·7, 51·4, 11·93 ft./sec^2.

p. 204. 4. 0·0194 ft./sec^2. = 0·593 cm./sec^2.
5. $\frac{1}{112}$ = 0·00890 ft./sec^2 = $g/3600$ nearly.

p. 205. 1. $1{\cdot}98\times10^{33}$ gm. 2. $6{\cdot}76\times10^{-8}$.

p. 206. 3. $5{\cdot}97\times10^{27}$ gm. 4. 84·5 mins.
6. 64·5 ft.; radial 40, 20 ft./sec^2.; tangential 3·2, 3·2 ft./sec^2.; 0·078, 0·040 lb.-wt., at angles 4° 34′, 9° 4′ with the radius.

p. 207. 7. 66·6 lbs.-wt. 8. 64·5 kgms.-wt.
9. 3, 1·43 lbs.-wt. 10. 35·8 ft./sec.
11. $\frac{1}{2\pi}\sqrt{\mathrm{M}g/ml}$. 12. 8·05, 6·04 kgms.-wt.
13. 221·4 cm./sec., 0·705. 14. 12·4 cm.
15. 1·27. 16. $2mn^2\pi^2l$.

p. 208. 17. $m\omega^2ab\cos\mathrm{B}/c$, $m\omega^2ab\cos\mathrm{A}/c$.

p. 211. 2. 3·75 lb.-wt.-sec., 3·14 secs.
3. 25·9 ft./sec., 9·17 ft. 5. 60°, 1·36 secs.

p. 214. 1. 2·29 tons-wt. 2. 2° 37′, 2·89 ins.
4. 28° 16′. 5. 2·86, 1·72 tons-wt. 6. 22° 25′.

p. 216. 1. 1016 lbs.-wt. 2. 1115, 2245 lbs.-wt. 3. 35·8 ft./sec.

p. 217. 4. $\sqrt{gr(h\sin\theta + a\cos\theta)/(h\cos\theta - a\sin\theta)}$.

p. 222. 2. 16 ft./sec.; 3, 1·5 lbs.-wt. 4. 392 lbs.-wt.
5. 2·26 secs. 6. 16 ft./sec.
7. $g(2\sin\theta - 3\sin\theta\cos\theta)$, $g(1+2\cos\theta - 3\cos^2\theta)$,
$g\sqrt{(5 - 8\cos\theta + 3\cos^2\theta)}$.
8. Horizontal $g\{(k-2)\sin\theta + 3\sin\theta\cos\theta\}$,
vertical $g\{(k-2)\cos\theta + 2\cos^2\theta - \sin^2\theta\}$;
when $k=2$, horizontal max. when $\theta=45°$,
min. when $\theta=0°$ or $90°$;
vertical max. when $\theta=0°$,
min. when $\theta=90°$;
resultant max. when $\theta=0°$,
min. when $\theta=90°$;
max. and min. values respectively $3g/2$, 0, $2g$, g, $2g$, g.

p. 224. $g\cot\alpha/\omega^2$.

p. 231. 2·31.

p. 233. 1. 42·8 lbs.-wt., 3960, 414·7 ft./sec.

p. 233. 1. $\sqrt{(v^2 + \omega^2c^2 - 2v\omega c\sin\theta)}$
at angle $\tan^{-1}(\omega c - v\cos\theta)/v\sin\theta$ with OC.
$\sqrt{(v^4/r^2 + \omega^4c^2 - 2v^2\omega^2c\cos\theta/r)}$
at angle $\tan^{-1}v^2\sin\theta/(v^2\cos\theta - \omega^2cr)$ with OC.
3. $\sqrt{\{a^2\omega^2 + b^2\omega'^2 + 2ab\omega\omega'\cos(\omega-\omega')t\}}$,
$\sqrt{\{a^2\omega^4 + b^2\omega'^4 + 2ab\omega^2\omega'^2\cos(\omega-\omega')t\}}$.

p. 234. 4. $6{\cdot}0\times10^{27}$ gms., $6{\cdot}7\times10^{-8}$, $2{\cdot}1\times10^{33}$ gms., $3{\cdot}7\times10^{27}$ dynes.
7. $mg\sin\theta$, $mgv\sin\theta$; $mg\cos\theta + \frac{mv^2}{r}$. 12. 61·5 ft., 0·16.

p. 235. 13. $\sqrt{\lambda r(2a-r)/ma}$, $\lambda r(3r-5a)/2a^2$, where $AP=r$.
15. $T_1 = m\omega^2(l + a\sin B)$, $T_2 = T_1\cos A/\sin C$,
$T_3 = T_1\cos B/\sin C$.
16. 148·5 cm./sec.; 125, 917 gms.-wt. 17. $gl/(g - 4\pi^2n^2a)$.

p. 236. 19. $m\sin A(\omega^2b\cos A - g)/\sin C$,
$m(\omega^2b\sin A\cos B + g\sin B)/\sin C$.
23. 1296 ft./sec. 26. $m\omega^2a - mga/\sqrt{l^2 - a^2}$.

CHAPTER VIII.

p. 240. 7 per cent.

p. 247. 1. 2·713 secs. 2. 896·8. 3. 981·6.
4. 3·23 cm. 5. 7·1 secs. 6. 28°·9 C.
7. 0·28 cm. 8. 0·495. 9. 5221 ft.

CHAPTER IX.

p. 260. 2. 515·6 ft., 4·7 sec., 87·9 ft.
3. 157·5 ft. horizontally, 76·8 ft. vertically; resultant vel. 56·4 ft./sec. at 37° 8′ with horizontal downwards.
4. 2683 ft./sec., 10° 18′. 5. 1744 ft./sec., 4·5 miles.
6. 200 ft., 15° or 75°. 7. 98 ft./sec.
8. 261·5, 44·1 ft. 9. 1960 ft./sec.
10. Horizontal distance 108·8 ft., 69·9 ft./sec. at 36° 11′.
12. 100 ft./sec., 26° 34′. 14. 18° 26′ or 71° 34′.
15. 22° 44′ or 74° 24′. 16. 87·8 ft./sec., 14° 56′. 17. No.

p. 261. 18. 346·4, 173·2 ft. horizontally. Velocity 138·5 ft./sec. in each case at 60° with horizontal.

p. 262. 21. 66·4 ft./sec., 17° 45′, 6·4 ft. 22. 73·8 ft./sec., 20° 8′.

p. 264. 2. 1·6 ft., 6·1 ft.

p. 268. $-a \sec i - \frac{V^2}{g} \tan i \sec i + \frac{V^2}{g} \sec^2 i \sqrt{1 + ga \sin 2i / V^2}$.

p. 272. 6. 154·6, 175·8 ft.
7. 2·89 sec., 28·9 ft., 133·3 ft., 46·2 ft./sec. at 60° with plane.
8. 250 ft. 9. $45° + \frac{i}{2}$ with the plane.

p. 273. 2. $V^2 \cos^2 a \cos i (\tan a - \tan i)^2 / 2g$.
5. $v/g - T/2$ after second starts, where v is the vertical velocity.
6. Opposite the window at horizontal distance 7·82 m. from it, 13·04 m. in advance of the point where it was when projected.

p. 274. 7. 0° or 63° 26′. 8. $2\sqrt{2h/g}$, $\sqrt{g(2h + R^2/8h)}$, $\tan^{-1} 4h/R$.
12. 30° or 63° 18′, 49·9 ft.

p. 275. 16. $\tan a = \frac{2y}{x} - \tan\beta$, $V^2 = 2g\left\{y + \frac{1}{4} x \sec^2\beta \Big/ \left(\frac{y}{x} - \tan\beta\right)\right\}$.
17. $V^2 = g\{R^2h^2 + a^2(R-a)^2\}/2ah(R-a)$.
19. Distance from point of contact of sphere $= 1·46a$ if a is the radius.

p. 276. 26. Distances from centre $3\sqrt{3}a/8$ horizontal, $11a/16$ vertical.

CHAPTER X.

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2. 2 ft./sec., 0·25, 5·625 ft.-lbs.
5. (i) $-2·67$, 9·33; (ii) 0·8, 16·8.

p. 286. 8. $A(B-Ce)(1+e)u/(A+B)(B+C)$, $AB(1+e)^2u/(A+B)(B+C)$.
11. 1·54, 3·04 secs. after the collision.

p. 290. 2. 4 secs., 34 ft. 3. 0·75. 11. 0·48.

p. 292. 2. $\frac{1}{2}\frac{MM'}{M+M'}\cdot\frac{m^2(1+e)^2}{(M+m)^2}\cdot\frac{v^2}{F}$.

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PART II.

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horizontal $\pm r\omega\cos\omega t$.
Acceleration vertical $-g\mp\omega^2r\cos\omega t$,
horizontal $\pm\omega^2r\sin\omega t$.
5. Coordinates relative to centre of mass
$(\omega^2f-Ag)/(\omega^4+A^2)$, $(Af+\omega^2g)/(\omega^4+A^2)$.

CHAPTER II.

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p. 323. 5. $M\frac{2a^2b^2}{3(a^2+b^2)}$. 6. $M\frac{b^2+4c^2}{3}$. 7. $M\frac{4(b^2+c^2)}{3}$.

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13. 2·60 lb.-wt.-secs. ;
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In OB, $\frac{m'ab\text{F}\sin\alpha\cos\text{B}}{(ma^2+m'b^2)\sin\text{B}} - m'\omega^2 b$.

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