THE ELEMENTARY

GEOMETRY OF CONICS

Cambridge
PRINTED BY C. J. CLAY M.A. AND SON
AT THE UNIVERSITY PRESS

THE ELEMENTARY

GEOMETRY OF CONICS

BY

C. TAYLOR D.D.

MASTER OF ST JOHN'S COLLEGE CAMBRIDGE

FOURTH EDITION REVISED AND ENLARGED

CAMBRIDGE
DEIGHTON BELL AND CO.
LONDON GEORGE BELL AND SONS
1883

PREFACE TO THE THIRD EDITION.

THE Geometry of Conics has been recast in the third edition, so as to serve as an introduction to a larger work now approaching completion.

The characteristic feature of the edition is the use of the Eccentric Circle, which contributes to a concise and uniform treatment of the three species of conics.

The Asymptotes of the hyperbola are shewn to be coincident with its self-conjugate diameters, and their properties are deduced from a limiting case of a property of conjugate diameters in general.

The principle that Chord-properties should be proved independently of Tangent-properties is still adhered to, although in the general rearrangement of the text it seemed no longer desirable to confine the two classes of properties to separate chapters.

The work now to some extent resembles my first work on *Geometrical Conics*, published in 1863; but the general chapter has been made more complete than I was then able to make it.

October 1879.

PREFACE TO THE FOURTH EDITION.

A FOURTH edition having been called for the work has been revised and in a measure enlarged by the insertion of some further corollaries (Arts. 40, 54, 59) and a chapter on Curvature and also of a Lemma on points at infinity and a Scholium on the metric properties of diameters which are needful for a right conception of the hyperbola in itself and in its relation to the ellipse.

The collection of Problems has been reconstructed by Mr J. S. Yeo, Fellow of St John's College.

A sketch of the history of this branch of Mathematics from the earliest times will be found in the Prolegomena to the work referred to in the preface to the third edition as approaching completion, and since published under the name of *An Introduction to the Ancient and Modern Geometry of Conics.*

November 1883.

CONTENTS.

THE GEOMETRY OF CONICS.

INTRODUCTION.

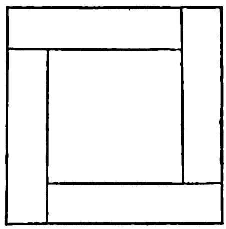
WE shall have occasion in the course of the work to assume the following

LEMMAS.

A. *To prove geometrically that*

$$(a+b)^2-(a\sim b)^2=4ab.$$

If four rectangles whose sides are equal to a and b be fitted symmetrically about the square on $a\sim b$, the whole figure will make up the square on $a+b$.



Therefore $$(a+b)^2=(a\sim b)^2+4ab,$$

or $$(a+b)^2-(a\sim b)^2=4ab.$$

The same is proved in Euclid II. 8, but by an unsymmetrical construction which shews only a gnomon of equal area instead of the four rectangles.

In illustration of the above, let PQ be any straight line bisected in O, and let Y be any point in PQ produced (Figure Art. 13); then

$$PY^2 - QY^2 = (OY + OP)^2 - (OY - OP)^2$$
$$= 4OY \cdot OP = 2OY \cdot PQ^*.$$

B. *The distance of any point in a straight line produced or within it from the middle point of the line is half the sum or difference of its distances from the extremities of the line.*

For let mM be any straight line and L its middle point. And first let a point R be taken in mM produced. Then

$$RL - RM = \tfrac{1}{2} mM = Rm - RL,$$

or

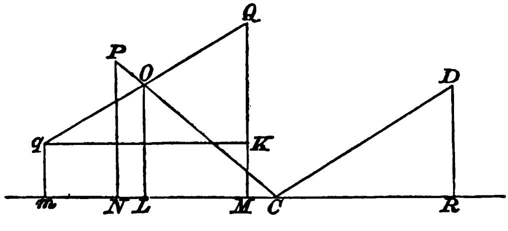
$$RL = \tfrac{1}{2}(RM + Rm).$$

Next let a point N be taken within mM. Then

$$NM - NL = \tfrac{1}{2} mM = Nm + NL,$$

or

$$NL = \tfrac{1}{2}(NM - Nm).$$



C. *The ordinate of the middle point of a straight line is equal to half the sum or difference of the ordinates of its extremities.*

From the extremities of a straight line Qq and from its middle point O let parallels QM, qm, OL be drawn to meet

* Otherwise thus. Since $PY + QY = 2OY$ (Lemma B), and $PY - QY = PQ$; therefore (Euc. II. 5, Cor.) $PY^2 - QY^2 = 2OY \cdot PQ$.

any given straight line or *axis;* and let these parallels be called the *ordinates* of the points from which they are drawn.

Draw qK parallel to mM to meet QM; and first let Q and q lie on the same side of the axis. Then, since qK cuts off from OL a length equal to $\frac{1}{2}QK$, therefore

$$OL - qm = \tfrac{1}{2}(QM - qm),$$

or

$$OL = \tfrac{1}{2}(QM + qm).$$

Next draw a figure in which Q and q lie on opposite sides of the axis. Then it may be shewn in like manner that

$$OL = \tfrac{1}{2}(QM \sim qm).$$

That is to say, the ordinate of O is equal to half the sum or difference of the ordinates of Q and q according as these points lie on the same side or on opposite sides of the axis.

D. *The sum of the squares of the distances of any point from the extremities of any straight line is double of the sum of the squares of its distance from the middle point of the line and of half the line.*

For if SS' be the given straight line, P the given point, PN a perpendicular to SS', and C the middle point of SS',

then $$SP^2 = CS^2 + CP^2 + 2CS \,.\, CN,$$

and $$S'P^2 = CS^2 + CP^2 - 2CS' \,.\, CN;$$

therefore by addition, since CS' is equal to CS,

$$SP^2 + S'P^2 = 2CS^2 + 2CP^2.$$

E. *To divide a given straight line in a given ratio of majority or of minority.*

A ratio is said to be a ratio of equality, majority or minority according as it is equal to unity, or greater or less than unity.

(i) First let SX be the given straight line, which is to be divided in a given ratio of majority. Draw SH in any direction, and produce it to K, so that $SH : HK$ may be equal to the given ratio. Join KX, and draw HA parallel to KX to meet SX in A. Then since

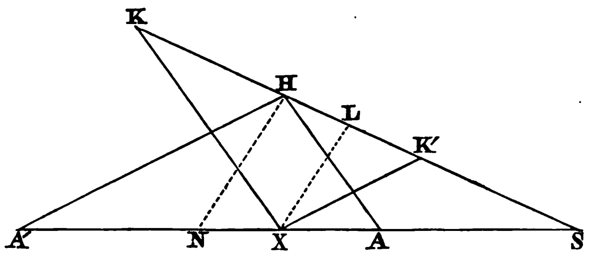
$$SA : AX = SH : HK, \qquad \text{[Euc. VI. 2.}$$

the point A divides SX internally in the given ratio.

Upon HS take HK' equal to HK. Join $K'X$, and draw HA' parallel to it to meet SX produced in A'. Then

$$SA' : A'X = SH : HK' = SH : HK,$$

or the point A' divides SX externally in the given ratio.



In this case the points A and A' will always lie on the *same side* of S, because K and K' lie on the same side of S.

(ii) Next let it be required to divide SX in a given ratio of minority $SH : HK$. Draw a figure in which HK is greater than SH. Then, using the same lines of construction as before, we determine the two required points of division A and A', which must always lie on *opposite sides* of S, because K and K' lie on opposite sides of S.

It is evident that there is only one point at a finite distance which bisects a straight line, or divides it in a ratio of *equality*. But the point ∞ at infinity on any straight line SX likewise divides it in a ratio of equality $S\infty : X\infty$.

F. *To divide a given straight line in a ratio greater or less than a given ratio.*

In the figure given above let it be required to divide SX in a ratio greater than the ratio of *majority* $SA : AX$. This is done by taking the point of division N anywhere between

A and A'. For if XL be drawn as in the figure parallel to NH to meet SH in L, then

$$\begin{aligned} SN : NX = SH : HL &> SH : HK \\ &> SA : AX. \end{aligned}$$

In like manner it may be shewn that every point in AA' produced either way divides SX in a ratio less than $SA : AX$.

Next, if $SA : AX$ be a ratio of *minority*, as in the second case of Lemma E, it may be shewn in like manner that SX is divided in a ratio less than $SA : AX$ by every point in AA', and in a ratio greater than $SA : AX$ by every point in AA' produced.

G. *Harmonic section of a straight line.*

If a straight line SX be divided internally and externally in the same ratio at A and A' so that

$$SA' : A'X = SA : AX;$$

then

$$\begin{aligned} SA' : SA &= A'X : AX \\ &= SA' \sim SX : SX \sim SA, \end{aligned}$$

or SA', SX, SA are in harmonic progression.

Hence SX is said to be divided harmonically at A and A'.

The relation between SA', SX, SA may also be written in the form

$$\frac{1}{SA} + \frac{1}{SA'} = \frac{2}{SX}.$$

Notice, as a limiting case, that SX is divided harmonically by its middle point and its point at infinity, for if A' be taken at infinity SA becomes equal to $\frac{1}{2}SX$.

H. *Opposite points at infinity coincide.*

This Lemma is necessary for the right understanding of the genesis of the hyperbola.

Take an unlimited straight line PP', and let OM be the perpendicular to it from an assumed point O without it.

The line may be regarded as traced by a point P lying on a ray OP which turns continuously about O. For adjacent positions P and P' of the tracing point the angle of rotation POP' is small: conversely we may say that the smallness of this angle is the test of the adjacence of P and P'.

But if P and P' be on opposite sides of and indefinitely remote from M, we may still pass, viz. through infinity, from P to P' by turning OP through an indefinitely small angle. Such points are therefore quasi-adjacent, and the opposite points at infinity on the line are quasi-coincident.

From this it follows that every straight line, or system of parallels, has one point only at infinity, as was assumed in Lemmas E and G.

DEFINITIONS.

[*Further definitions will be given at the beginnings of the chapters, as occasion arises.*]

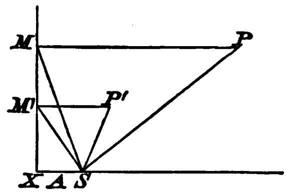
1. A CONIC SECTION*, or briefly a *Conic*, is the curve traced in a plane by a point which moves in such a way that its distance from a given point is in a constant ratio to its distance from a given straight line. The given point is called the *Focus*, the given straight line the *Directrix*, and the constant ratio the *Eccentricity* of the conic.

If S be the focus, P any point on the conic and PM the perpendicular from it to the directrix, the ratio of SP to PM is constant. If P' be any other point on the conic, and $P'M'$ the perpendicular from it to the directrix, then

$$SP : PM = SP' : P'M',$$

or

$$SP : SP' = PM : P'M'.$$



Let us now take a particular case, and suppose the directrix to be at an infinite distance from the focus. In this case $PM : P'M'$ is a ratio of equality, and therefore $SP : SP'$ is

* The conic sections were so called because they are the curves in which a plane can be made to intersect a cone, as will be shewn in the seventh chapter.

a ratio of equality. That is to say, SP is always equal to SP', and the locus of P is a circle. Thus it appears that our definition of a conic is an extension of the definition of a circle.

2. A conic is called a *Parabola*, an *Ellipse*, or a *Hyperbola*, according as its eccentricity is a ratio of equality, of minority, or of majority.

3. The *Axis* is the unlimited straight line through the focus at right angles to the directrix, and the points in which it meets the conic are called the *Vertices*. When one vertex only is spoken of the vertex which lies between the focus and the directrix is signified.

It is evident from Lemma E that the parabola has only one vertex at a finite distance, and that the ellipse and the hyperbola have each two vertices.

4. The middle point of the line joining the vertices is called the *Centre* of the conic. The ellipse and the hyperbola are called *Central Conics*, in contrast with the parabola which has no centre at a finite distance. The straight line through the centre at right angles to the axis is called the *Conjugate Axis.*

5. A *Chord* of a conic is properly the finite straight line joining any two points on the curve; but the term is also used to denote the unlimited straight line joining any two points on the curve. The extremities of a chord are the points in which it meets the conic.

6. The *Latus Rectum* is the focal chord, or chord through the focus, at right angles to the axis.

7. A *Diameter* is the locus of the middle points of a system of parallel chords: it will be proved that the diameters of conics are straight lines. One diameter is said to be *conjugate* to another when it bisects chords parallel thereto.

8. The *Principal Ordinate*, or briefly the *Ordinate*, of any point is the perpendicular drawn from it to the axis. More generally, the ordinate of any point *to any diameter* is

the line drawn from the point to that diameter in the direction parallel to the conjugate diameter.

9. A *Tangent* to a conic is the limiting position of a chord or secant whose two points of intersection with the curve have become coincident. Thus if P and Q be adjacent points on a conic, and if the chord joining them be turned round P, or be moved about in any other way, until its extremity Q coincides with P, the chord in its limiting position becomes the tangent at P. Hence a tangent is said to be a straight line which passes through two *consecutive* or *coincident* points on the curve.

The chord of contact of two tangents is the chord joining their points of contact.

10. The *Normal* at any point of a conic is the perpendicular to the tangent at that point.

11. If about any point in the plane of a conic a circle be described such that the ratio of its radius to the perpendicular distance of its centre from the directrix is equal to the eccentricity, the circle may be called the eccentric circle of the conic with respect to that point, or briefly the *Eccentric Circle of the Point.*

12. The *Order* or *Degree* of a curve is determined by the number of points in which it can be met by a straight line. Thus a curve of the second order or degree is one which a straight line meets generally in two and never in more than two points.

All the points at infinity in any plane constitute a locus of the first degree, which is called the *Straight Line at Infinity*, since by Lemma H every other straight line in the plane passes through one point only at infinity.

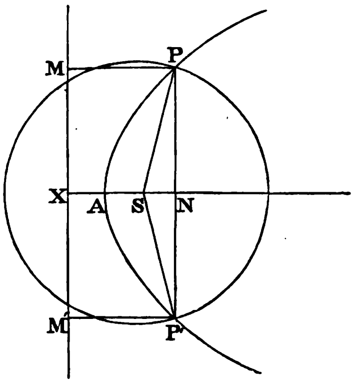
CHAPTER I.

DESCRIPTION OF THE CURVE.

1. *Having given the focus, directrix and eccentricity of a conic, it is required to describe the curve.*

Let S be the focus*, MM' the directrix, and X the point in which the axis meets the directrix. In SX take the point A so that the ratio of SA to AX may be equal to the eccentricity (Lemma E). Then A is the vertex of the conic.

Draw a straight line cutting the axis at right angles in N; and let P and P' be the points in which the line meets



* The planets describe approximately ellipses about the sun in one focus. For this reason the first letter of *Sol* is used, as by Newton, to denote the *Focus*, or as he called it the Umbilicus. We shall use the letters S, A, X as above without further explanation, so that $SA : AX$ will always denote the eccentricity.

the circle described with S as centre and radius SP, such that

$$SP : NX = SA : AX.$$

Draw PM perpendicular to the directrix. Then

$$SP : PM = SP : NX = SA : AX,$$

or P is a point on the conic. In like manner it may be shewn that P' is a point on the conic.

If now we suppose the chord PP' to slide at right angles to the axis, so as to assume all *possible* positions, its extremities will trace out the complete curve.

2. *The three species of conics.*

In order that the line and the circle in the above construction may intersect, the length SN must be less than the radius SP, or

$$SN : NX < SA : AX.$$

In the case of the *Parabola* we must have $SN < NX$. The point N may therefore be taken anywhere in XA produced, and the curve consists of one infinite branch spreading out from the vertex and away from the directrix.

In the *Ellipse*, if A' be the second vertex, the point N may be taken anywhere between A and A' (Lemma F), and the curve consists of one oval branch (Fig. Art. 5) lying on the same side of the directrix with the focus.

In the *Hyperbola*, if A' be the second vertex, the point N may be taken anywhere in AA' produced (Lemma F), and the curve consists of two infinite branches on opposite sides of the directrix.

3. *The symmetry of the curve.*

From the foregoing construction it is evident that the curve is symmetrical with respect to its axis, since its points are always determined in pairs as P and P' in corresponding positions above and below the axis, so that the part of the curve below the axis is the accurate reflexion of the part above the axis.

It follows also that the tangent at A is at right angles to the axis, since when the point N coincides with the vertex, $SP = SA = SP'$; that is to say, the points P and P' coalesce at A, and the chord joining them, which is always at right angles to the axis, becomes the tangent to the conic at its vertex (Def. 9).

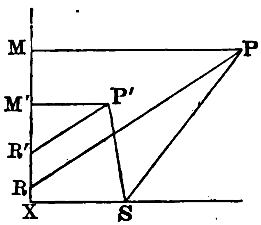
4. *The focal distance of any point on a conic is in a constant ratio to the distance of the point from the directrix measured parallel to any fixed straight line which meets the directrix.*

From any two points P and P' on the conic draw PR and $P'R'$ in any fixed direction to meet the directrix, and draw PM and $P'M'$ perpendicular to the directrix.

Then $\qquad SP : PM = SP' : P'M',$ $\qquad$ [Def. 1.

and $\qquad PM : PR = P'M' : P'R',$

by similar triangles.



Therefore $\qquad SP : PR = SP' : P'R',$

or (since we may consider P' and $P'R'$ to remain fixed whilst P varies) the focal distance SP varies as the distance PR to the directrix measured in any given direction.

Conversely, every point P which satisfies the above relation is a point on the conic.

Notice in particular that if any chord PQ meet the directrix in R,

$$SP : PR = SQ : QR.$$

5. *A conic is a curve of the second order.*

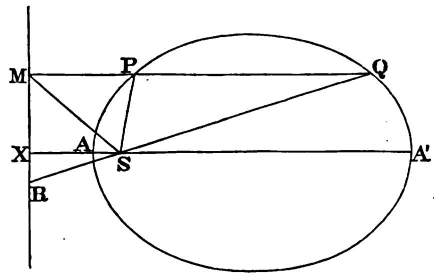
Let a straight line drawn in any direction cut the conic in P and meet the directrix in M. Make the angle MSR equal to MSP, and let the line RS produced meet MP in Q.

Then, by Euclid VI. A or 3, since SM bisects the angle PSR or its supplement,

$$SQ : SP = QM : PM,$$

or

$$SQ : QM = SP : PM;$$



and therefore Q is a point on the curve (Art. 4); and it is evident that no third point of intersection of the line PQ with the conic can be determined.

It follows that a straight line which meets a conic will in general meet it in two points, and never in more than two. A conic is therefore a curve of the second order or degree (Def. 12).

In the *Ellipse* P and Q lie on the same side and in the *Hyperbola* on opposite sides of M.

In the *Parabola* if PM be parallel to the axis and therefore equal to SP,

$$\angle MSR = MSP = SMP = MSX,$$

or SR coincides with SX and the point Q recedes to infinity. Hence every straight line parallel to the axis of a parabola meets the curve in one point only at a finite distance.

6. *To describe a conic of given focus, directrix and eccentricity by means of the eccentric circle of any assumed point.*

Describe the eccentric circle of any point O* in the plane of the conic (Def. 11), and let a straight line through S meet the circle in p and the directrix in R.

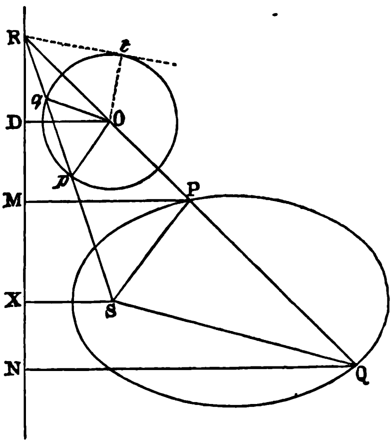
Let the focal radius parallel to pO meet RO in P, and let OD and PM be perpendicular to the directrix.

Then since SP, pO and PM, OD are parallels,

$$SP : Op = PR : OR$$
$$= PM : OD,$$

or

$$SP : PM = Op : OD$$
$$= \text{the eccentricity.}$$



Hence, as p moves round the circle, P traces the conic which was to be described.

Conversely, if any point O be taken on a chord PQ of a conic, the eccentric circle of O will meet SR (drawn to the

* Let SL be the semi-latus rectum, and let the ordinate of O meet the axis in N and XL in K. Then since $KN : OD = KN : NX = SL : SX = SA : AX$, the radius of the circle must be taken equal to KN.

point of concourse R of the chord with the directrix) in points p and q lying upon radii parallel to SP and SQ.

The student should now draw figures in which the eccentric circle touches or cuts the directrix, and trace the corresponding conics, which will be in the one case parabolas and in the other hyperbolas.

CHAPTER II.

THE GENERAL CONIC.

We shall commence by proving some of the principal properties which are common to the parabola, the ellipse and the hyperbola.

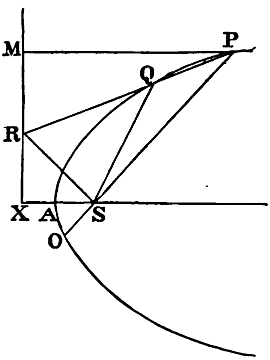
The Tangent.

PROPOSITION I.

7. *Each of the two tangents which can be drawn to a conic from any point on its directrix subtends a right angle at the focus.*

Let P and Q be adjacent points on the curve, and let PQ produced meet the directrix in R. Then it may be shewn (Art. 4) that

$$SP : SQ = PR : QR;$$



and therefore SR bisects the angle which SQ makes with PS produced. [Euc. VI. A.

Let PS meet the curve again in O. Then since the angles RSQ and RSO are always equal, therefore in the limit, when Q coalesces with P, each of them becomes a right angle, and RP, which becomes the tangent at P (Def. 9), subtends a right angle at S. In like manner it may be shewn that the other tangent which can be drawn to the conic from R subtends a right angle at S.

Corollary.

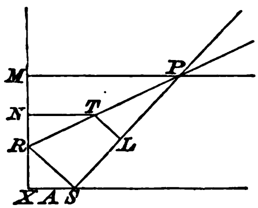
Hence it appears that the tangents at the extremities of any focal chord OP meet at a point R lying on the directrix, and such that SR is at right angles to OP. Conversely, if tangents be drawn to a conic from any point R on the directrix their chord of contact will be the focal chord at right angles to RS. The tangents at the extremities of the latus rectum meet at X.

PROPOSITION II.

8. *If from any point* T *on the tangent at* P *there be drawn perpendiculars* TL *and* TN *to* SP *and the directrix, the ratio of* SL *to* TN *will be constant and equal to the eccentricity.*

For if the tangent at P meet the directrix in R, and if PM be a perpendicular to the directrix, then since SR is at right angles to SP (Prop. I.) and is therefore parallel to TL, it follows that

$$SL : SP = TR : PR$$
$$= TN : PM.$$



Therefore $$SL : TN = SP : PM$$
$$= SA : AX.$$

The line SL is equal to the radius of the eccentric circle of T.

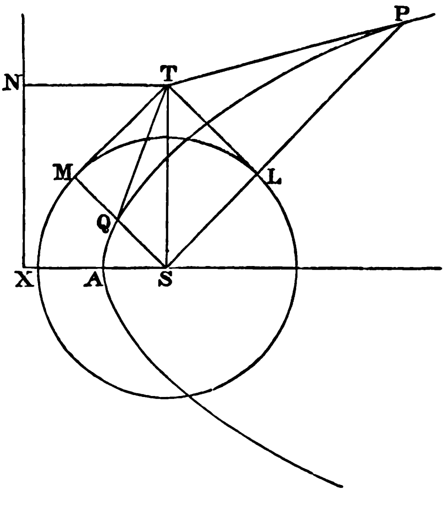
Corollary.

To draw a pair of tangents to a conic from a given external point T, with S as centre describe a circle equal to the eccentric circle of T, and draw the tangents TL and TM to the circle (Fig. Art. 9); then by the converse of the proposition SL and SM will pass through the points of contact P and Q of the required tangents. Draw SR at right angles to SL to meet the directrix in R; then TR is one of the two tangents. Draw SR' at right angles to SM to meet the directrix in R'; then TR' is the second tangent from T.

PROPOSITION III.

9. *The two tangents which can be drawn to a conic from any external point subtend equal or supplementary angles at the focus.*

For if TP and TQ be the two tangents to a conic from



the point T, and TL, TM, TN be perpendiculars to SP, SQ and the directrix, then by Prop. II., since T lies on the tangent at P,

$$SL : TN = SA : AX;$$

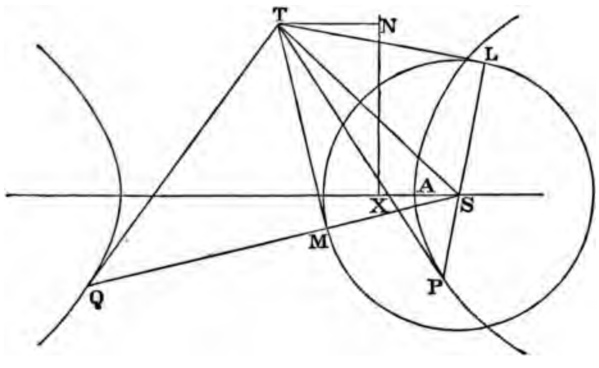
and since T lies on the tangent at Q,

$$SM : TN = SA : AX.$$

Therefore in the right-angled triangles STL and STM the side SL is equal to SM; and the hypotenuse ST is common; and therefore the angle TSL is equal to TSM.

Now (i) if TP and TQ touch the same branch of the conic, the angles which they subtend at S are either *equal* to TSL and TSM (as in the above figure) or *supplementary* thereto. In either case the two tangents subtend EQUAL angles at S.

But (ii) if TP and TQ touch opposite branches of a hyperbola (as in the next figure), so that one and one only of



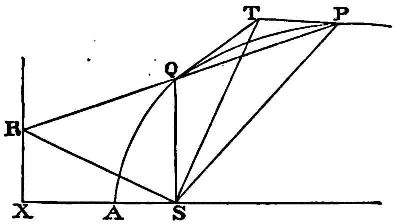
the lines SL and SM has to be produced backwards to P or Q, the two tangents will subtend SUPPLEMENTARY angles at S.

PROPOSITION IV.

10. *The point of concourse of any two tangents to a conic and the point in which their chord of contact meets the*

directrix lie upon a pair of focal radii which include a right angle.

If P and Q be points on the same branch of a conic, and if PQ meet the directrix in R, then, as in Prop. I., SR bisects



the angle which SQ makes with PS produced; that is to say, it bisects the supplement of PSQ.

Also, if the tangents at P and Q meet in T, the line ST bisects the angle PSQ (Prop. III.).

Therefore ST and SR bisect supplementary angles, and are therefore at right angles to one another.

If TP and TQ touch opposite branches of a hyperbola, then (completing the second figure of Art. 9) it may be shewn that in this case also the angle TSR is a right angle.

The Normal.

PROPOSITION V.

11. *The normal at any point of a conic meets the axis at a distance from the focus which is to the focal distance of the point in a constant ratio equal to the eccentricity.*

For if the tangent at P meet the directrix in R, the circle on PR as diameter will pass through S, since PSR is a right angle; and it will likewise pass through M, the projection of P upon the directrix; and the normal at P will touch the circle, since it is at right angles to its diameter PR.

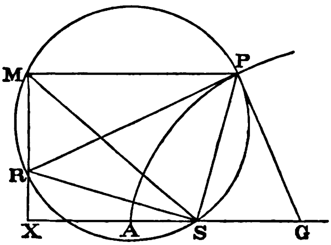
Let the normal meet the axis in G.

Then $\angle\, SPG = SMP,$

in the alternate segment of the circle, and

$$\angle\, PSG = SPM,$$

by parallels.



Therefore the triangles SPG and SPM are similar, and

$$SG : SP = SP : PM$$
$$= SA : AX,$$

or SG varies as SP, as was to be proved.

Conversely, if in AS produced there be taken a point G such that

$$SG : SP = SA : AX,$$

the line PG will be the normal at P.

PROPOSITION VI.

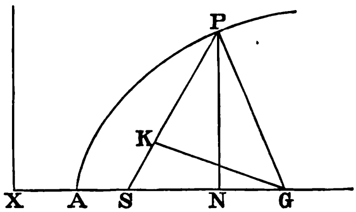
12. *At any point of a conic the projection of the normal (terminated by the axis) upon the focal radius is equal to the semi-latus rectum.*

Let the normal at P meet the axis in G: draw GK perpendicular to SP: and draw PN perpendicular to the axis.

Then by similar right-angled triangles SKG and SNP,

$$SK : SN = SG : SP$$
$$= SP : PM \qquad \text{[Prop. v.}$$
$$= SP : NX.$$

Hence $SP - SK : NX - SN = SP : NX$;



that is to say, PK is to SX in a constant ratio equal to the eccentricity, and is therefore equal to the semi-latus rectum.

Note that when P is taken at an extremity of the latus rectum PK coalesces with the semi-latus rectum.

Diameters.

PROPOSITION VII.

13. *The locus of the middle points of any system of parallel chords of a conic is a straight line which meets the directrix on the straight line through the focus at right angles to the chords.*

Let PQ be any one of a system of parallel chords, V the point in which the focal perpendicular upon them meets the directrix, R and Y the points in which PQ meets the directrix and SV respectively.

Then since $SP : PR = SQ : QR$, [Art. 4.

therefore (supposing for example that SP is greater than SQ)

$$SP^2 - SQ^2 : PR^2 - QR^2 = SP^2 : PR^2;$$

or, subtracting SY^2 from each of the first two squares,

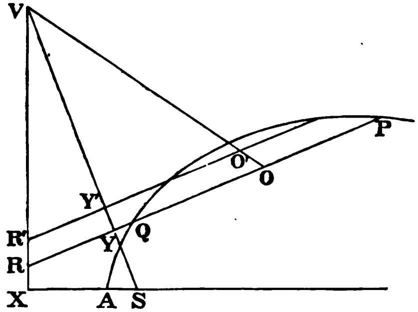
$$PY^2 - QY^2 : PR^2 - QR^2 = SP^2 : PR^2.$$

[Euc. I. 47.

But if O be the middle point of PQ,

$$PY^2 - QY^2 = 2OY.PQ; \qquad \text{[Lemma A.}$$

and similarly, $\quad PR^2 - QR^2 = 2OR.PQ.$



Hence $\quad OY : OR = SP^2 : PR^2,$

which (Art. 4) is a *constant ratio* so long as PQ is drawn in a specified direction.

Hence if any other chord be drawn parallel to PQ, and if O', Y', R' be the new positions of O, Y, R, it follows that

$$O'Y' : O'R' = OY : OR;$$

and hence that the points O, O', V lie in a straight line*.

If now we suppose the point O' to remain fixed whilst PQ moves parallel to itself, the point O will always lie upon a *fixed* straight line $O'V$, as was to be proved.

Corollary 1.

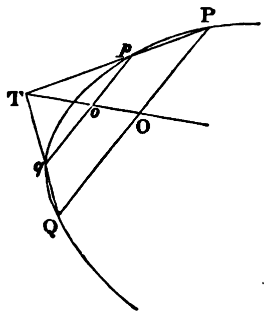
The tangent at either extremity of a diameter is parallel to the chords which the diameter bisects, since any one of the bisected chords may be supposed to move parallel to itself until its segments vanish together and its extremities coalesce, so that it becomes a tangent, viz. at an extremity of the bisecting diameter. If a diameter meets the conic in two

* For if they do lie on a straight line, $O'Y' : OY = O'V : OV = O'R' : OR$. The required converse may be easily deduced by a *reductio ad absurdum*.

points, the tangents at those points are parallel to the *ordinates* (Def. 8) of that diameter and to one another.

Corollary 2.

If PQ and pq be any two of a system of parallel chords, and O and o be their middle points, which will lie on a fixed diameter, it is evident that Pp and Qq will meet at a point T lying on that diameter. Hence, making pq move



parallel to itself until it coalesces with PQ, so that TP and TQ become the tangents at P and Q, we see that *the tangents at the extremities of any chord meet upon the diameter which bisects the chord;* and conversely, that the diameter to any external point bisects the chord of contact of the two tangents from that point.

PROPOSITION VIII.

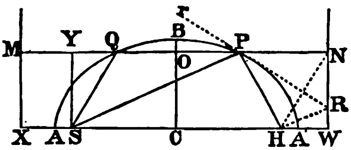
14. *Every central conic is divided symmetrically by its conjugate axis, and has a second focus and directrix.*

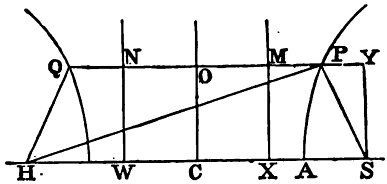
Let AA' be the transverse axis of a central conic, PQ any chord parallel thereto, and M the point in which PQ or its prolongation meets the directrix.

Bisect PQ in O, and draw SY perpendicular to PQ; then it may be shewn, precisely as in Prop. VII., that

$$OY : OM = SP^2 : PM^2;$$

and hence that the locus of O is a straight line at right angles to AA'.





Since this straight line must also bisect AA' (which is a limiting position of the chord PQ), it meets AA' at the centre C of the conic, and coincides with the *conjugate axis* (Def. 4).

It is evident that this line divides the curve into two parts such that each is the exact reflexion of the other; and hence that the curve has a second focus H and directrix NW, the exact counterparts of the original focus and directrix with reference to which the conic was defined.

Corollary 1.

From the symmetry of a central conic with respect to its two axes, it is manifest that every chord through its centre is bisected at that point, and hence that all diameters pass through the centre. It is further evident that any two diameters or focal chords equally inclined to either axis are equal to one another; and that any two tangents to the conic from a point on either axis are likewise equal.

Corollary 2.

In Art. 13 let the diameter parallel to PQ meet the directrix in V'. Then in the triangle CVV', since VS is at right angles to CV' and CS to VV', therefore S is the orthocentre and $V'S$ is at right angles to CV*. Hence the diameter CV' bisects chords parallel to CV, or if one diameter be conjugate to a second, the second is conjugate to the first.

* The three perpendiculars of a triangle meet in a point, which is called its *orthocentre*.

The Segments of Chords.

PROPOSITION IX,

15. *The semi-latus rectum is a harmonic mean between the segments of any focal chord.*

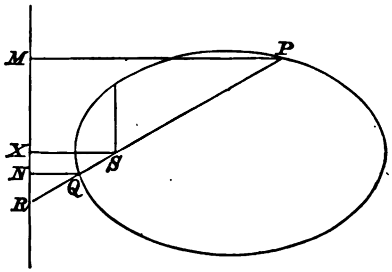
Let a focal chord PQ or its prolongation meet the directrix in R, and let PM, SX, QN be perpendiculars to the directrix.

Then, since

$$SP : SQ = PM : QN = PR : QR,$$

therefore $$PR - SR : SR - QR = PR : QR,$$

or PR, SR, QR are in harmonic progression. [Lemma G.



But by parallels, and from the definition of the curve, if l be used to denote the semi-latus rectum,

$$PR : SR : QR = PM : SX : QN$$
$$= SP : l : SQ.$$

Therefore also SP, l, SQ are in harmonic progression.

Corollary.

It is easy to deduce that

$$l \,.\, PQ = l\,(SP + SQ) = 2SP \,.\, SQ.$$

Hence, if pq be any other focal chord,

$$PQ : pq = SP \,.\, SQ : Sp \,.\, Sq,$$

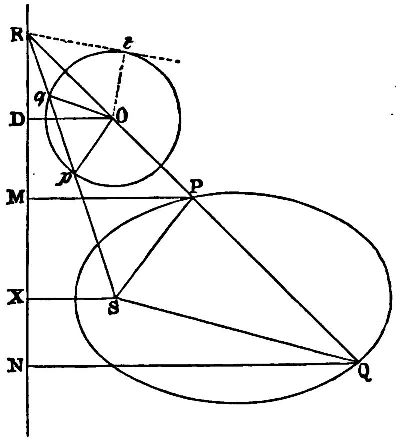
or *focal chords are to one another as the rectangles contained by their segments.*

PROPOSITION X.

16. *A chord of a conic being divided at any point, to determine the rectangle contained by its segments.*

Let PQ be any chord of a conic, and O any point on PQ or PQ produced; it is required to determine the magnitude of the rectangle $OP \,.\, OQ$.

Let the chord meet the directrix in R, and let OD be a perpendicular to the directrix. Draw the eccentric circle of



O, and let it cut SR in p and q. Then it may be shewn that Op is parallel to PS and Oq to QS. [Art. 6.

Therefore $$OP : Sp = OR : Rp,$$

and $$OQ : Sq = OR : Rq.$$

$$\text{Hence } OP \,.\, OQ : Sp \,.\, Sq = OR^2 : Rp \,.\, Rq$$
$$= OR^2 : Rt^2,$$

if Rt be one of the tangents from R to the circle*.

[Euc. III. 36.

In this result it is to be noticed (i) that the magnitude of $Sp \,.\, Sq$ depends only upon the *position* of O, since when O is given its eccentric circle is given, and $Sp \,.\, Sq$ is constant; and (ii) that the ratio $OR^2 : Rt^2$ depends only upon the *direction* of PQ, since when the angle ORD is given, OR varies as OD, and therefore (by the definition of the eccentric circle) as Ot, and therefore OR^2 varies as $OR^2 \sim Ot^2$, or as Rt^2.

Corollary 1.

If through any other point O' there be drawn a chord $P'Q'$ parallel to PQ, and if p' and q' be the corresponding positions of p and q, viz. on the eccentric circle of O', it follows (since the ratio $OR^2 : Rt^2$ depends only upon the *direction* of the chord) that

$$OP \,.\, OQ : Sp \,.\, Sq = O'P' \,.\, O'Q' : Sp' \,.\, Sq',$$

where the consequents depend only upon the *positions* of O and O'. If therefore any second pair of parallel chords be drawn through the same points O and O', we have the general theorem that:

The ratio of the rectangles contained by the segments of any two intersecting chords of a conic is the same as for any other two chords parallel to the former, each to each.

Corollary 2.

This ratio is equal to that of the parallel focal chords (Art. 15, Cor.); and in a central conic to that of the squares of the semi-diameters parallel to the chords; and in the general conic to the ratio of the squares of any pair of tangents

* When P and Q lie on opposite branches of a hyperbola the point R falls within the circle. In this case draw Rt the semichord at right angles to OR in the circle, and apply Euclid III. 35.

parallel to the chords, since a tangent is defined as a chord whose extremities are coincident.

Corollary 3.

If a circle and a conic intersect in four points, their common chords are equally inclined, in opposite pairs, to the axis of the conic. For if POQ and pOq be one of the three pairs of common chords of a circle and a conic, the rectangles $PO \cdot OQ$ and $pO \cdot Oq$ are as the focal chords parallel to PQ and pq (Cor. 2); and the same rectangles are equal to one another by a property of the circle. Therefore the focal chords parallel to PQ and pq are equal, and are therefore equally inclined to the axis. [Art. 14, Cor. 1.

CHAPTER III.

THE PARABOLA.

17. THE annexed further definitions will be required in the present chapter.

The principal *Abscissa* or *Absciss* of any point with respect to a parabola is the finite segment cut off from the axis by the principal ordinate of the point. The abscissa or absciss of a point *to any diameter* is the finite segment cut off from it by the ordinate of the point to that diameter. [Def. 8.

The *Parameter* of any diameter of a parabola is the focal chord which it bisects: thus the latus rectum is the parameter of the axis.

The *Subtangent* at any point of a conic is the intercept upon the axis between the tangent and the ordinate of the point; and the *Subnormal* is the intercept between the normal and the ordinate. The subtangent *to any diameter* is the intercept thereupon between the tangent and the ordinate of its point of contact to that diameter.

18. The eccentricity of the parabola being a ratio of equality, the semi-latus rectum is equal to SX, and therefore to $2SA$. It is to be noted also that in the case of the parabola SG becomes equal to SP (Art. 11), and the eccentric circles of all points (Def. 11) touch the directrix.

The property of diameters in Art. 13 has been proved for all conics without distinction; but we shall also shew that it can be proved with peculiar ease for the special case of the parabola.

Chord-Properties *.

PROPOSITION I.

19. *The principal ordinate of any point on a parabola is a mean proportional to its abscissa and the latus rectum.*

If PN and AN be the principal ordinate and abscissa of any point P on the parabola, we have to shew that

$$PN^2 = 4AS \cdot AN.$$

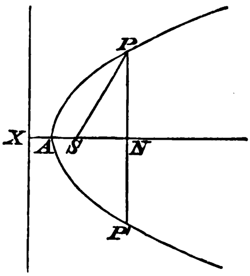
By Euclid I. 47 and from the definition of the parabola,

$$PN^2 + SN^2 = SP^2 = NX^2.$$

Hence (taking for example the case in which AN is greater than AS),

$$PN^2 + (AN - AS)^2 = (AN + AS)^2$$
$$= (AN - AS)^2 + 4AS \cdot AN.$$

[Lemma A.



Therefore PN^2 is equal to $4AS \cdot AN$; or in other words, PN is a mean proportional to the abscissa AN and the latus rectum (Art. 18).

* By chord-properties (as distinguished from tangent-properties) we understand such properties as do not presuppose the definition of a tangent (Def. 9). It is desirable to avoid using tangent-properties to prove chord-properties, the reverse order being the only natural one, so long as we regard a tangent as the limit of a chord.

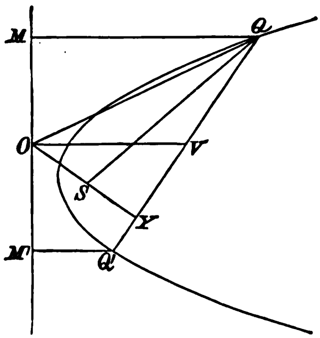
Conversely, if PN and AN be thus related, the locus of P will be a parabola of latus rectum equal to $4AS$.

PROPOSITION II.

20. *The locus of the middle points of any number of parallel chords of a parabola is a straight line parallel to the axis; and the bisecting line meets the directrix on the straight line through the focus at right angles to the common direction of the chords.*

Let QQ' be any one of a system of parallel chords, and let QM and $Q'M'$ be perpendiculars to the directrix.

Let the focal perpendicular upon the chords meet QQ' in Y and the directrix in O; and let the parallel to the axis through O (which is a *fixed* straight line) meet QQ' in V. Then will V be the middle point of QQ'.



For by Euclid I. 47 and from the definition of the parabola (taking the case in which Y lies in OS produced),

$$OM^2 = OQ^2 - QM^2 = OQ^2 - SQ^2$$
$$= OY^2 - SY^2;$$

and OM'^2 may be shewn to have the same value.

And since OM and OM' are thus equal, the line OV parallel to the axis bisects QQ'; that is to say, it bisects *every chord which* is at right angles to OS.

It is hence evident that every straight line parallel to the axis of a parabola is a diameter (Def. 7) of the curve, and that *all diameters of a parabola are parallel to the axis and to one another.*

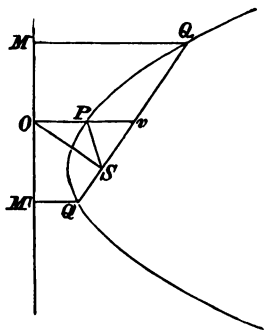
PROPOSITION III.

21. *The parameter of any diameter of a parabola is equal to four times the focal distance of its extremity.*

Let a diameter meet its parameter QSQ' (Def. Art. 17) in v, the curve in P, and the directrix in O, so that vSO is a right angle. [Prop. II.

Let fall perpendiculars QM and $Q'M'$ upon the directrix. Then

$$QQ' = SQ + SQ' = QM + Q'M'$$
$$= 2vO. \qquad \text{[Lemma C.}$$



And because vSO is a right angle, and $SP = PO$, therefore vO is a diameter of the circle round OSv, and P is its centre.

Hence $$QQ' = 2vO = 4SP,$$
or the focal chord QQ' is equal to four times the focal distance of the extremity of the diameter of which it is the parameter. In particular, as we have already seen, the latus rectum or principal parameter is equal to $4AS$.

PROPOSITION IV.

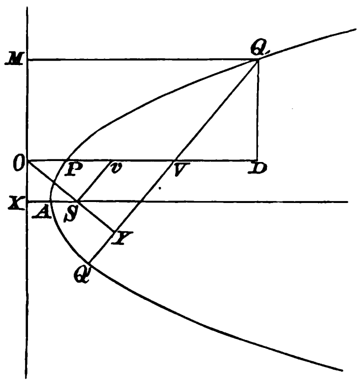
22. *The ordinate of any point on a parabola to any diameter is a mean proportional to the parameter of the diameter and the abscissa of the point.*

Let Q be any point on a parabola, QV and PV its ordinate and abscissa to any diameter: we have to shew that

$$QV^2 = 4SP \cdot PV,$$

the parameter of the diameter through P being equal to $4SP$. [Prop. III.

Let the diameter meet its parameter in v and the directrix in O; and let OS, which is at right angles to Sv and QV (Prop. II.), meet QV in Y.



Then it may be shewn that, if QD and QM be perpendiculars to the diameter and the directrix,

$$QD^2 = OM^2 = OY^2 \sim SY^2. \qquad \text{[Art. 20.}$$

Moreover by parallels,

$$OY : OV = SY : vV; \qquad \text{[Euc. VI. 2.}$$

and therefore (supposing for example that OY is greater than SY)

$$OY^2 - SY^2 : OV^2 - vV^2 = OY^2 : OV^2$$
$$= QD^2 : QV^2,$$

by similar triangles OYV and QDV. And since the antecedents in this proportion are equal, the consequents are equal, so that

$$QV^2 = OV^2 - vV^2.$$

And since P is the centre of the circle round OSv (Art. 21), therefore OV is equal to $PV + SP$, and vV to $PV - SP$; and therefore from above,

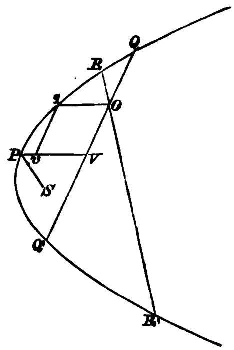
$$QV^2 = (PV + SP)^2 - (PV - SP)^2,$$
$$= 4SP . PV; \qquad \text{[Lemma A.}$$

that is to say, the ordinate QV is a mean proportional to the abscissa PV and the parameter $4SP$.

PROPOSITION V.

23. *The rectangles contained by the segments of any two intersecting chords are as the parameters of the diameters which bisect them.*

Take any two chords QQ' and RR' intersecting in a point O, within or without the parabola, and let the diameter through O meet the parabola in q.



Bisect QQ' in V, let the diameter through V meet the curve in P, and draw the ordinate qv to that diameter. Then, taking the case in which O lies within the parabola,

$$QO \,.\, OQ' = QV^2 - OV^2, \qquad \text{[Euc. II. 5, Cor.}$$
$$= QV^2 - qv^2,$$
$$= 4SP \,.\, PV - 4SP \,.\, Pv; \qquad \text{[Prop. IV.}$$

and therefore, since $PV - Pv$ is equal to vV or qO,

$$QO \,.\, OQ' = 4SP \,.\, qO.$$

Similarly $$RO \,.\, OR' = 4SP' \,.\, qO,$$
if P' be the extremity of the diameter which bisects RR'; and the same may be proved for the case in which O lies without the parabola.

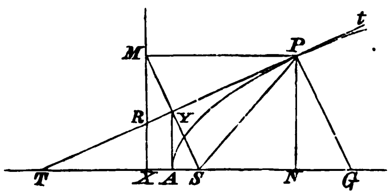
Therefore the rectangles $QO \,.\, OQ'$ and $RO \,.\, OR'$ are as $4SP$ to $4SP'$, or as the parameters of the diameters which bisect QQ' and RR' (Prop. III.); in accordance with what was proved for all conics without distinction in Art. 16.

Tangent-Properties.

PROPOSITION VI.

24. *The tangent to a parabola at any point is the bisector of the angle which the focal distance of the point makes with the diameter produced.*

Let the tangent at P meet the directrix in R, and let the diameter be produced through P without the curve to meet the directrix in M.



Then PSR is a right angle (Art. 7), and $SP = PM$, and PR is common to the right-angled triangles PSR and MPR.

Therefore their angles at P are equal, or the tangent PR is the bisector of the angle SPM.

It is further evident that RP produced makes equal angles with SP and PM; and that if the tangent meet the axis in T, the angles SPT and STP are equal, or the tangent together with SP and the axis determine an isosceles triangle.

Corollary 1.

If PN be the principal ordinate of P, it follows that

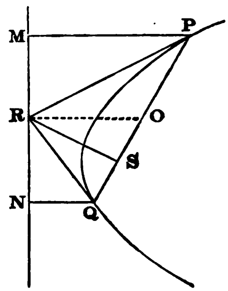
$$ST = SP = NX = AN + AS;$$

and therefore $\quad AN = ST - AS = AT,$

or *the subtangent is double of the abscissa.* [Def. Art. 17.

Corollary 2.

If PQ be any focal chord and MN its projection upon the directrix, and if SR be drawn at right angles to PQ to meet the directrix; it is evident from above that the bisector of the



angle SRM is the tangent at P, and that the bisector of SRN is the tangent at Q. Hence

$$\angle PRQ = \tfrac{1}{2}SRM + \tfrac{1}{2}SRN = \text{a right angle},$$

or the tangents at the extremities of any focal chord meet *at right angles* upon the directrix, and conversely.

PROPOSITION VII.

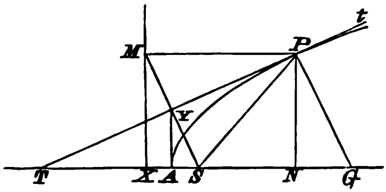
25. *The normal at any point makes equal angles with the focal distance and the axis, and it bisects the interior angle between the focal distance and the diameter to the point.*

If the normal at P meets the axis in G, then $SG = SP$ (Art. 18), and therefore PG makes equal angles with SP and SG.

If the diameter at P be drawn, and any point D be taken upon it *within* the curve, then by parallels and from above,

$$\angle DPG = PGS = GPS,$$

or PG bisects the angle SPD.



Corollary.

If AN be the principal abscissa of P, then since

$$SG = SP = NX,$$

therefore

$$NG = SG - SN = NX - SN = SX,$$

or *the subnormal is equal to semi-latus rectum.* [Def. Art. 17.

PROPOSITION VIII.

26. *The foot of the focal perpendicular to any tangent lies on the tangent at the vertex; and the square of the focal perpendicular to any tangent varies as the focal distance of its point of contact.*

Let the tangent at P meet the axis in T, and let PN be the ordinate of P.

(i) Draw the tangent at the vertex A, and let it meet PT in Y; then will SY be perpendicular to PT.

For since A is the middle point of NT (Art. 24, Cor. 1), therefore by parallels PN and AY (Art. 3), Y is the middle point of PT; and therefore the triangles SPY and STY, having their sides SP and PY equal to ST and TY respectively, and the side SY common, have their angles at Y equal.

Therefore SY is at right angles to PT, and conversely the foot of the focal perpendicular SY upon the tangent at P lies on the tangent at A.

(ii) Moreover, since the tangents YA and YP to the parabola subtend equal angles at S (Art. 9), the right-angled triangles SAY and SYP are similar, so that

$$SA : SY = SY : SP,$$

or

$$SY^2 = SA \cdot SP,$$

and SY^2 varies as SP, as was to be proved.

PROPOSITION IX.

27. *The exterior angle between any two tangents to a parabola is equal to the angle which either of them subtends at the focus.*

Let the tangents at P and Q intersect in T, and let them meet the axis in p and q. Take any point O in pS produced, and produce TS to any point t.

Then by Euclid I. 32, and because SPp is an isosceles triangle (Prop. VI.),

$$\angle PSO = SPp + SpP = 2SpP;$$

and similarly

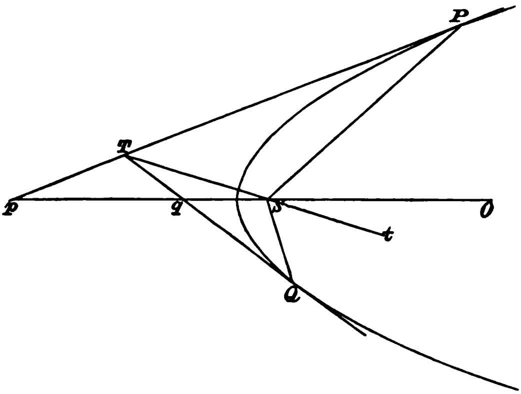
$$\angle QSO = 2SqQ.$$

Hence by addition (taking the case in which P and Q lie on opposite sides of the axis),

$$\angle PSQ = 2(SpP + SqQ) = 2(SpP + pqT),$$
$$= 2\,PTQ.$$

Therefore $\angle PTQ = \frac{1}{2}PSQ = PSt = QSt$; [Art. 9.

and therefore the exterior angle pTq between the tangents is equal to PST or QST, as was to be proved.



PROPOSITION X.

28. *The two tangents from any point to a parabola are the bases of a pair of similar triangles having a common vertex at the focus.*

With the construction of Art. 27 it has been shewn that

$$\angle PTQ = PSt.$$

The former angle is equal to $STP + STQ$, and the latter is equal to $STP + SPT$ (Euc. I. 32). Therefore, taking away STP from each,

$$\angle STQ = SPT.$$

And since the angles at S in the triangles STP and STQ are also equal (Art. 9), the third angle STP is equal to the third angle SQT, and the two triangles are similar, as was to be proved.

Hence $SP : ST = ST : SQ$,

or *the focal distance of the point of concourse of any two tangents to a parabola is a mean proportional to the focal distances of their points of contact.*

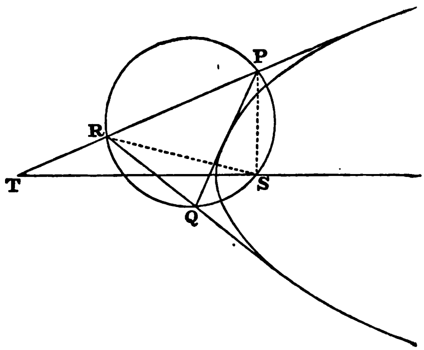
Corollary.

Since the angle STQ is equal to SPT, and therefore to SpP (Prop. VI.), the angle which either tangent makes with the axis (or with the diameter through T) is equal to the angle which the other makes with the focal distance of T.

PROPOSITION XI.

29. *The circumscribed circle of the triangle formed by any three tangents to a parabola passes through the focus.*

Let PQR be any triangle whose three sides touch a parabola, and let any one of its sides, as PR, meet the axis in T.



Then by the last corollary, considering the two tangents which meet in R,

$$\angle SRQ = STP;$$

and next considering the two tangents which meet in P,

$$\angle SPQ = STP.$$

Therefore the angles SPQ and SRQ are equal, and the points $PQRS$ lie on a circle; or in other words, the circle round PQR passes through the focus.

Corollary.

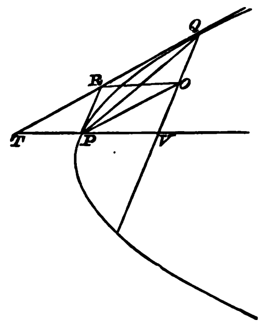
Four tangents to a parabola being given, the circumscribed circles of the four triangles which they determine meet in one point, viz. the focus of the parabola.

PROPOSITION XII.

30. *At any point of a parabola the subtangent to any diameter is double of the abscissa.*

Let Q be any point on a parabola, QV and PV its ordinate and abscissa to any diameter, and let the tangent at Q meet that diameter in T: we have to shew that TV is double of PV. [Def. Art. 17.

Let the tangents at P and Q meet in R, and let PO be drawn parallel to RQ to meet QV in O; then the tangent at P, being a vanishing ordinate to the diameter PV (Art. 13, Cor. 1), is parallel to QV; and therefore $PRQO$ is a parallelogram, and its diagonal RO bisects the diagonal PQ.



But the line drawn from R to bisect the chord of contact of the two tangents from R is a diameter of the parabola (Art. 13, Cor. 2): therefore RO is a diameter, and is parallel to the diameter PV. [Prop. II.

Hence by parallels,

$$PV = RO = PT,$$

or P bisects TV, and the subtangent TV is double of PV.

CHAPTER IV.

THE CENTRAL CONICS.

31. THE central conics are the ellipse and the hyperbola*. It has been shewn (Art. 14) that these are also bifocal, and symmetrical with respect to both axes.

The *Abscissæ* or *Abscisses* of a point to any diameter of a central conic are the segments of the diameter made by the ordinate of the point. The *Central Abscissa*, which is also called absolutely the *Abscissa*, is the distance from the centre to the foot of the ordinate.

The principal axis of a central conic is distinguished as the *Transverse Axis*. The transverse and conjugate (Def. 4) axes of the ellipse are also called its *Major* and *Minor* axes. The term axis is sometimes used to denote the *finite* portion of either of these lines intercepted by the curve; but a special convention has to be made in case of the conjugate axis of the hyperbola, which does not meet the curve in real points. [Art. 33, Cor. 1.

The major and minor auxiliary circles are the circles described upon the transverse and conjugate axes as diameters; but when one *Auxiliary Circle* only is spoken of the circle on the transverse axis is signified. It is easily seen from the next article that this is identical with the eccentric circle of the centre of the conic, the ratio of its radius CA to CX being equal to the eccentricity.

* We shall sometimes give proofs applicable to both curves, but with figures for the ellipse only. In all such cases the student should draw the figures for the hyperbola also.

Supplemental Chords are chords drawn from any point on the curve to opposite extremities of any diameter.

The *Conjugate Parallelogram* is the figure formed by drawing parallels to each of two conjugate diameters through both extremities of the other.

The locus of the point of concourse of a pair of tangents at right angles will be shewn to be a circle (Art. 40), which we shall term the *Orthocycle**.

32. *The segments of the transverse axis.*

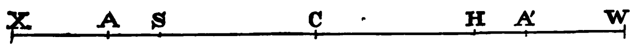
In any central conic the vertices A and A' divide SX so that

$$SA : AX = SA' : A'X;$$

therefore $$SA : AX = SA \sim SA' : AX \sim A'X$$
$$= SA + SA' : AX + A'X,$$

or by Lemma B, if C be the centre of the conic,

$$SA : AX = 2CS : 2CA = 2CA : 2CX.$$



Thus each of the ratios $CS : CA$ and $CA : CX$ is equal to the eccentricity, and

$$CS \,.\, CX = CA^2.$$

The Ordinate.

PROPOSITION I.

33. *The square of the principal ordinate of any point on a central conic is in a constant ratio to the rectangle contained by its abscisses.*

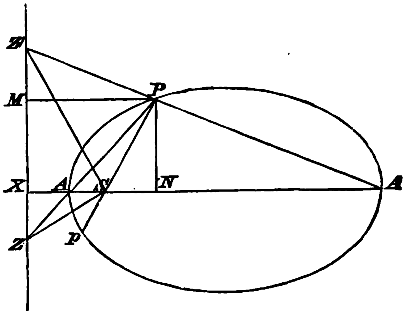
Let A and A' be the vertices, PN the ordinate of any point P on the conic, Z and Z' the points in which AP and $A'P$ or their prolongations meet the directrix.

* It has also been named the *Director Circle*, since in the parabola it *degenerates into the* directrix and the line at infinity.

If PM be a perpendicular to the directrix, and therefore parallel to the axis,

$$SP : SA' = PM : A'X \qquad \text{[Def. 1.}$$
$$= PZ' : A'Z';$$

therefore SZ' bisects the angle PSX, and in like manner (if p be taken in PS produced) SZ bisects the supplementary angle pSX.



Hence the angle ZSZ' is a right angle, and

$$ZX . Z'X = SX^2 = \text{a constant}.$$

Moreover, since PN is parallel to the directrix,

$$PN : AN = ZX : AX,$$

and $$PN : A'N = Z'X : A'X.$$

Hence $$PN^2 : AN . A'N = ZX . Z'X : AX . A'X$$
$$= SX^2 : AX . A'X,$$

or PN^2 is in a constant ratio to $AN . A'N$.

Corollary 1.

Let the length CB be determined by the proportion,

$$CB^2 : CA^2 = SX^2 : AX . A'X.$$

Then in the ellipse

$$PN^2 : AN . A'N = PN^2 : CA^2 - CN^2 = CB^2 : CA^2,$$

where CB is equal to the semi-axis conjugate, with which PN coincides when N is taken at C. In the hyperbola

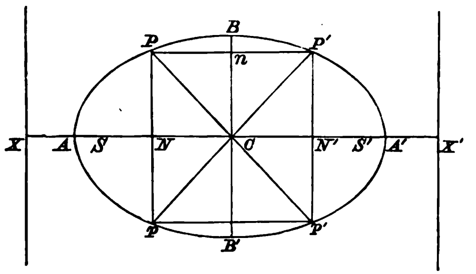
$$PN^2 : AN . A'N = PN^2 : CN^2 - CA^2 = CB^2 : CA^2,$$

where we may agree to call CB *the length of the semi-axis conjugate*, although (Art. 2) this axis does not meet the curve in real points. The results of this corollary may also be written,

$$CB^2 \pm PN^2 : CN^2 = CB^2 : CA^2,$$

the positive sign being taken in the hyperbola and the negative in the ellipse.

If Pn be an ordinate to the conjugate axis, it is easily deduced that in the ellipse,



$$Pn^2 : CB^2 - Cn^2 = CA^2 : CB^2,$$

or

$$CA^2 - Pn^2 : Cn^2 = CA^2 : CB^2;$$

and in the hyperbola,

$$Pn^2 : CB^2 + Cn^2 = CA^2 : CB^2.$$ [Fig. Art. 49.

PROPOSITION II.

34. *The semi-axis conjugate is a mean proportional to the segments of the transverse axis made by either focus; and the latus rectum is a third proportional to the transverse and conjugate axes.*

Since $CS : CA = SA : AX = SA' : A'X,$ [Art. 32.

therefore $CS + CA : CA = SA + AX : AX = SX : AX$

and $\quad CS \sim CA : CA = SA' \sim A'X : A'X = SX : A'X.$

Hence $\quad CS^2 \sim CA^2 : CA^2 = SX^2 : AX \,.\, A'X$

$= CB^2 : CA^2,$ [Art. 33.

or $\quad CS^2 \sim CA^2 = AS \,.\, A'S = CB^2.$

If LSL' be the semi-latus rectum, so that AS and $A'S$ are the abscissæ of L,

$SL^2 : AS \,.\, A'S = CB^2 : CA^2,$ [Prop. I.

and therefore $\quad SL^2 : CB^2 = CB^2 : CA^2.$

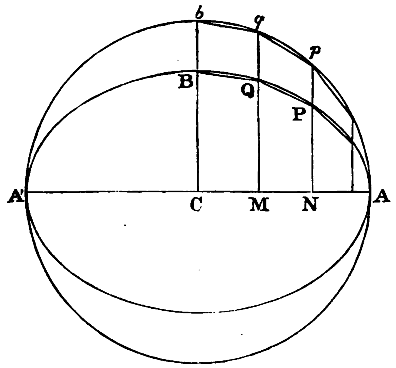
Therefore $SL \,.\, CA = CB^2$, and LL' is a third proportional to the transverse and conjugate axes.

PROPOSITION III.

35. *If the principal ordinates of all points on ellipse be produced outwards in the ratio of the major to the minor axis, the points to which they are produced lie on the circumference of the auxiliary circle, and conversely.*

Let P be any point on an ellipse, and let its principal ordinate PN be produced outwards to p in the ratio of CA to CB, so that

$PN^2 : pN^2 = CB^2 : CA^2 = PN^2 : AN \,.\, A'N.$ [Prop. I.



Then, pN^2 being equal to $AN \,.\, A'N$, the locus of p is the circle on AA' as diameter, as was to be proved.

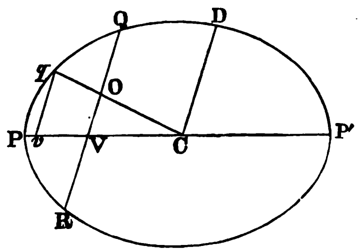
Conversely, if the ordinates pN, qM... in the circle be cut in the ratio of minority $CB : CA$, the points of section P, Q... will all lie on an ellipse whose semi-axes are equal to CA and CB.

In virtue of this relation we are able to deduce a whole class of properties of the ellipse from properties of the circle* by the method of Orthogonal Projection, which will be explained in Chapter VIII.

PROPOSITION IV.

36. *At any point on a central conic the square of the ordinate to any diameter is in a constant ratio to the product of the corresponding abscissæ.*

Let QVR be any double ordinate of a given diameter PP', and PV and $P'V$ the corresponding abscissæ. We have to shew that QV^2 (or RV^2) varies as $PV.VP'$.



This follows at once from Art. 16, Cor. 1, where it is shewn that $QV.VR$ (which is in this case equal to QV^2) is in a constant ratio to $PV.VP'$, so long as the directions of QR and PP' continue unchanged.

Put $CD^2 : CP^2$ equal to this constant ratio, so that

$$QV^2 : PV.VP' = CD^2 : CP^2.$$

* The corresponding chords PQ and pq always meet on the axis. Hence the tangents at P and p meet on the axis, viz. in a point T such that $CN.CT = Cp^2 = CA^2$ (Fig. Art. 45). Cf. Art. 68.

In the ellipse it is evident that CD is equal to the semi-diameter to which QV is parallel, and that

$$PV \, . \, VP' = CP^2 - CV^2,$$

in all cases.

In the hyperbola also we may agree to call CD *the length of the semi-diameter conjugate to* CP, although it will be seen (in the section on conjugate diameters) that one and one only of every two conjugate diameters meets the curve in real points. Supposing CP to meet the hyperbola, then

$$QV^2 : CV^2 - CP^2 = CD^2 : CP^2,$$

or $$QV^2 + CD^2 : CV^2 = CD^2 : CP^2.$$

Hence also $$Qv^2 : Cv^2 + CD^2 = CP^2 : CD^2,$$

if Qv (equal to CV) be an ordinate to the diameter CD, which does *not* meet the curve.

Corollary.

Hence, and by Art. 16, Cor. 2, if FF' be the focal chord parallel to CD, and LL' be the latus rectum,

$$FF' : LL' = CD^2 : CB^2 = CD^2 : \tfrac{1}{2}LL' \, . \, CA. \quad \text{[Art. 34.}$$

Therefore $$FF' \, . \, CA = 2\,CD^2,$$

or any focal chord is a third proportional to the transverse axis and the diameter parallel to the chord.

The Focal Distances.

PROPOSITION V.

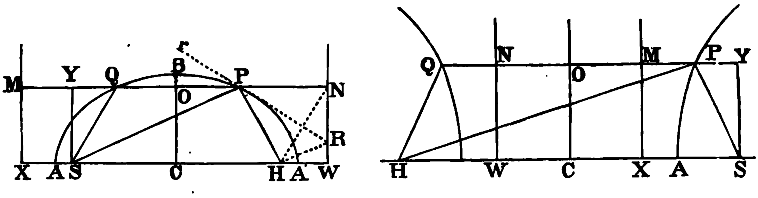
37. *The sum or difference of the focal distances of any point on a central conic is constant and equal to the transverse axis.*

Let H be the second focus (Art. 14) and W the foot of the corresponding directrix. Through any point P on the curve draw a parallel to the axis to meet the directrices in M and N. Then

$$SP : PM = HP : PN.$$

Therefore

$$SP \pm HP : PM \pm PN = SP : PM,$$



where $PM + PN$ is equal to WX in the ellipse, and $PM \sim PN$ is equal to WX in the hyperbola.

Therefore $SP \pm HP$ is constant, the upper sign being taken for the ellipse, and the lower sign for the hyperbola.

By making P coincide with A, it is easily seen that in either case the constant length is equal to the transverse axis, so that

$$SP \pm HP = AA'.$$

Corollary.

In the ellipse, if B be an extremity of the minor axis,

$$SB = HB = CA.$$

The Tangent.

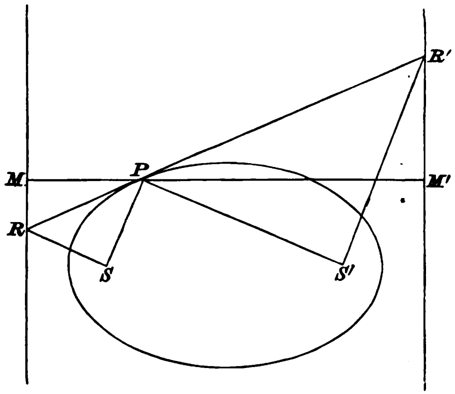
PROPOSITION VI.

38. *The tangent at any point of a central conic makes equal angles with the focal distances of the point.*

Let the tangent at P meet the directrices in R and R', and let the parallel to the axis through P meet the directrices in M and M'. Then, if S and S' be the corresponding foci,

$$SP : S'P = PM : PM' = PR : PR',$$

or the sides about the angles at P in the triangles PSR and $PS'R'$ are proportionals.



Moreover the angles at S and S' in those triangles (being right angles) are equal.

Therefore the triangles are similar and their angles at P are equal (Euc. VI. 7); that is to say, the tangent at P makes equal angles with the focal distances SP and $S'P$, and conversely.

In the ellipse the tangent at P falls without the angle SPS', and bisects the angle which $S'P$ makes with SP produced (Fig. Art. 39). In the hyperbola the tangent bisects the angle SPS' internally.

PROPOSITION VII.

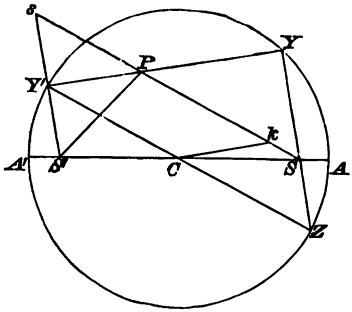
39. *The projections of the two foci upon any tangent to a central conic lie on the auxiliary circle.*

Let S and S' be the foci: Y and Y' the feet of the perpendiculars let fall from them upon the tangent at any point P. Then will Y and Y' lie on the auxiliary circle.

In the case of the ellipse, let SP produced meet $S'Y'$ in

4—2

s; then the tangent PY' bisects the angle $S'Ps$ (Prop. VI.), and $Y's$ is equal to $Y'S'$.



And since Y' has been shewn to be the middle point of $S's$, and the centre C of the conic bisects SS', therefore CY' is parallel to Ss and equal to $\frac{1}{2}Ss$. That is to say,

$$CY' = \tfrac{1}{2}(SP + Ps) = \tfrac{1}{2}(SP + PS')$$
$$= CA, \qquad \text{[Prop. v.}$$

or Y' lies on the auxiliary circle. And in like manner it may be shewn that Y lies on the auxiliary circle.

In the hyperbola it may be shewn in like manner that

$$CY = CY' = \tfrac{1}{2}(SP \sim S'P) = CA,$$

and thus that Y and Y' lie on the auxiliary circle.

Corollary 1.

Conversely, if Y be any point on the auxiliary circle the straight line drawn from Y at right angles to SY will be a tangent to the conic. It is hence evident that the extremities of any focal chord YSZ of the auxiliary circle are the projections of the focus S upon a pair of *parallel tangents* to the conic.

Corollary 2.

If the diameter parallel to the tangent at P meet SP in k, then

$$Pk = CY' = CA.$$

PROPOSITION VIII.

40. *The rectangle contained by the perpendiculars from either focus of a central conic upon any two parallel tangents, or by the perpendiculars from the two foci upon any tangent, is constant and equal to the square of the semi-axis conjugate.*

Draw perpendiculars SY and SZ to any two parallel tangents. Then since Y and Z lie on the auxiliary circle (Prop. VII.) and YSZ is a straight line,

$$SY \,.\, SZ = SA \,.\, SA' = CB^2, \qquad \text{[Prop. II.}$$

as was to be proved.

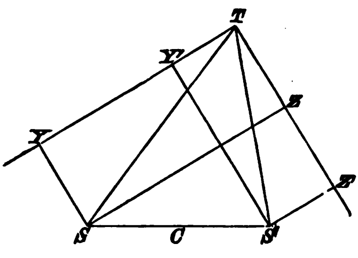
Again, with the construction of Art. 39, since $Y'YZ$ is a right angle $Y'Z$ is a diameter of the circle. Hence the triangles CSZ and $CS'Y'$, having their angles at C equal and their sides CS, CZ equal to CS', CY' each to each, are equal in all respects.

Therefore $\qquad SY \,.\, S'Y' = SY \,.\, SZ = CB^2,$

as was to be proved.

Corollary 1.

If tangents be drawn to an ellipse from any point T, and if Y, Y' and Z, Z' be the projections of the foci upon them, then since $SY \,.\, S'Y' = SZ \,.\, S'Z'$, it is easily seen that the



angle STY must be equal to $S'TZ$; for according as the angle STY is equal to or greater or less than $S'TZ$, the angle $S'TY'$ is equal to or greater or less than STZ', and the ratio of $SY.S'Y'$ to $ST.S'T$ is equal to or greater or less than the ratio of $SZ.S'Z'$ to $ST.S'T$.

Corollary 2.

In the preceding figure let the tangents be supposed to be at right angles. Draw the auxiliary circle, viz. through the points $YY'ZZ'$, and draw TO touching it in O. Then

$$TO^2 = TY.TY' = SZ.S'Z' = CB^2,$$

and $$CT^2 = CO^2 + TO^2 = CA^2 + CB^2.$$

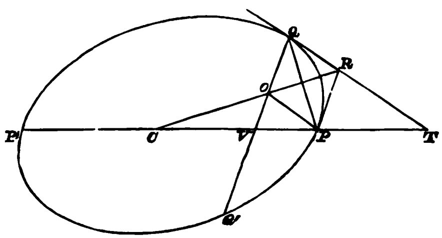
That is to say, *the locus of the point of concourse of a pair of tangents at right angles is a circle*, viz. the *Orthocycle**. In the hyperbola it will be found that the radius of the orthocycle is equal to $\sqrt{(CA^2 - CB^2)}$. Consequently no real tangents at right angles can be drawn to a hyperbola in which CA is less than CB.

PROPOSITION IX.

41. *A variable tangent to a central conic meets any fixed diameter at a distance from the centre which varies inversely as the abscissa of its point of contact to that diameter.*

First Case.

Let PCP' be a given diameter which *meets* the curve in real points P and P'. Let the tangent at any point Q meet



* See also note, page 44.

that diameter in T, and let QV be the ordinate of Q. Then we shall shew that

$$CV \, . \, CT = CP^2.$$

Let the tangent at P, which is parallel to QV, meet QT in R, and let PO be drawn parallel to RQ to meet QV in O. Then $OPRQ$ is a parallelogram, and its diagonal RO bisects PQ.

But RO, since it bisects the chord of contact of the tangents from R, is a diameter and passes through the centre C of the conic. [Art. 13, Cor. 2.

Therefore by parallels,

$$CV : CP = CO : CR = CP : CT,$$

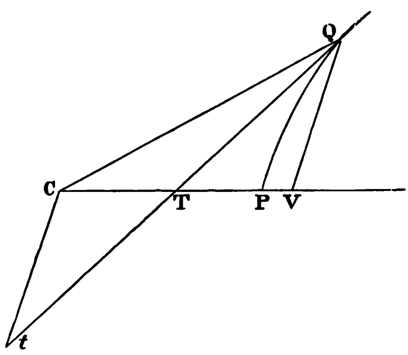
or $CV \, . \, CT$ is equal to CP^2, as was to be proved.

Second Case.

Let CP and CD be given conjugate semi-diameters of a hyperbola, whereof the former meets and the latter does *not meet* the curve. [Art. 36.

Draw QV and QT as in the first case, and produce QT to meet CD in t. Then by parallels,

$$Ct : CT = QV : VT,$$



or
$$QV.Ct : CV.CT = QV^2 : CV.VT$$
$$= QV^2 : CV^2 - CV.CT.$$

Hence by the first case and by Art. 36,
$$QV.Ct : CP^2 = QV^2 : CV^2 - CP^2$$
$$= CD^2 : CP^2,$$
or
$$Cv.Ct = QV.Ct = CD^2,$$
if Cv be the abscissa of Q to the diameter CD.

Corollary.

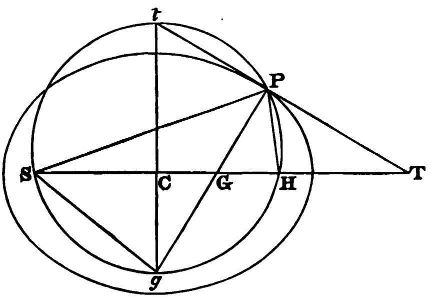
If the tangent at any point P meet the transverse and conjugate axes in T and t, and if PN and Pn be ordinates to those axes, then
$$CN.CT = CA^2, \text{ and } Cn.Ct = CB^2.$$

The Normal.

PROPOSITION X.

42. *The normal at any point bisects the angle between the focal distances of the point, internally in the case of the ellipse and externally in the case of the hyperbola.*

Let S and H be the foci, PG the normal at any point P, and TPt the tangent at that point.



Then in the ellipse, PG being at right angles to the tangent,

$$\angle SPG + SPt = HPG + HPT.$$

And it has been shewn that the angles SPt, HPT are equal and that the tangent falls without the angle SPH (Art. 38).

Therefore $$\angle SPG = HPG,$$

or the normal bisects SPH *internally*.

In the hyperbola it may be shewn in like manner that the normal at P bisects the angle SPH *externally*.

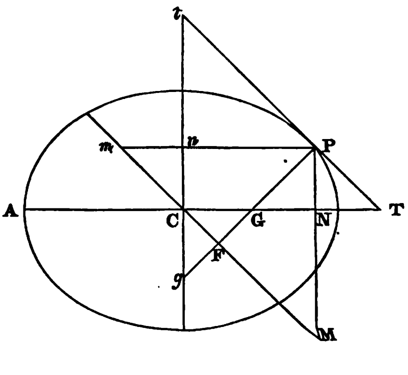
Corollary.

If the circle round SPH meet the conjugate axis in g and t, it is evident that the arcs Sg, Hg and the arcs St, Ht are equal. Hence Pg and Pt are the two bisectors of the angle SPH, and *the circle round SPH passes through the points in which the tangent and normal at P meet the conjugate axis.*

PROPOSITION XI.

43. *At any point of a central conic the normal, terminated by either axis, varies inversely as the central perpendicular upon the tangent.*

Let the tangent at any point P meet the transverse and



conjugate axes in T and t, and let the normal meet them in G and g.

Let PN and Pn be ordinates to those axes, and let PN and Pn or their prolongations meet the diameter parallel to the tangent at P in M and m, and let the normal meet that diameter in F.

Then, the angles at N and F being right angles, a circle goes round $FGNM$, and therefore

$$PG \,.\, PF = PN \,.\, PM = Cn \,.\, Ct$$
$$= CB^2. \qquad \text{[Art. 41, Cor.}$$

In like manner, the angles at n and F being right angles,

$$Pg \,.\, PF = Pn \,.\, Pm = CN \,.\, CT$$
$$= CA^2.$$

Therefore PG and Pg vary inversely as PF, or as the central perpendicular upon the tangent at P.

Corollary.

Hence

$$NG : CN = NG : Pn = PG : Pg = CB^2 : CA^2,$$

or *the subnormal varies as the abscissa.* [Def. Art. 17.

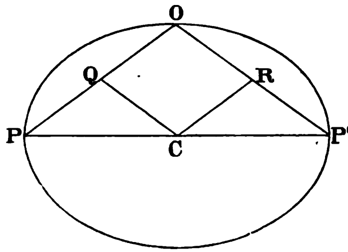
Conjugate Diameters.

PROPOSITION XII.

44. *Supplemental chords are parallel to conjugate diameters, and conversely.*

Let PCP' be any diameter and O any point on the curve. Then will the supplemental chords OP and OP' (Def. Art. 31) be parallel to a pair of conjugate diameters.

For if Q be the middle point of OP, and R the middle point of OP', the line CQ which bisects two sides of the triangle OPP' is parallel to the third side $P'O$, and in like manner CR is parallel to PO.



Therefore the diameters CQ and CR are conjugate, since each bisects one chord (and therefore all chords) parallel to the other. [Art. 13.

Conversely, from any given point O on the curve or from the extremities of any given diameter PP' there can be drawn a pair of supplemental chords OP and OP' parallel to any assumed pair of conjugate diameters.

In the hyperbola it is evident that of every two supplemental chords one lies within and the other without the curve, and hence that *one and one only of every two conjugate diameters meets the hyperbola.* We shall in consequence have occasion to give separate proofs of some of the properties of conjugate diameters for the special case of the hyperbola.

Corollary.

This proposition determines the relation between the directions of any two conjugate diameters. For in Art. 33, where AP and $A'P$ may be parallel to any two such diameters,

$$PN^2 : AN . A'N = CB^2 : CA^2;$$

and therefore if the ratio of PN to AN (or the direction of one of the diameters) be given, the ratio of PN to $A'N$ (or the direction of the conjugate diameter) is known. Conversely any two diameters whose directions are thus related will be conjugate provided that they lie in adjacent quadrants in the case of the ellipse (Fig. Art. 45), or in the same quadrant or opposite quadrants in the case of the hyperbola. If CP and CD be conjugate radii, CN and CR their projections upon the axis, it is easily deduced that

$$PN.DR : CN.CR = CB^2 : CA^2.$$

PROPOSITION XIII.

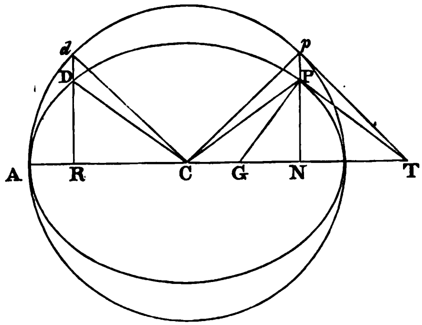
45. *The sum of the squares of any two conjugate diameters is constant in the ellipse, and the difference of their squares is constant in the hyperbola.*

(i) If CP and CD be conjugate semi-diameters of an ellipse, PN and DR the principal ordinates of their extremities, then by the property of ordinates (Art. 33),

$$PN^2 : CA^2 - CN^2 = DR^2 : CA^2 - CR^2 = CB^2 : CA^2;$$

and by a property of conjugate radii,

$$PN.DR : CN.CR = CB^2 : CA^2. \quad \text{[Art. 44, Cor.}$$



In virtue of the former relation, according as CA^2 is equal *to or greater* or less than $CN^2 + CR^2$, the ratios $PN : CR$

and $DR : CN$ are simultaneously equal to or greater or less than $CB : CA$, and the ratio $PN . DR : CN . CR$ is accordingly equal to or greater or less than $CB^2 : CA^2$.

Hence, if CP and CD are conjugate,

$$PN : CR = DR : CN = CB : CA,$$

and $$CN^2 + CR^2 = CA^2;$$

and therefore $$PN^2 + DR^2 = CB^2.$$

By addition, $$CP^2 + CD^2 = CA^2 + CB^2,$$

or *the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes of the ellipse**.

(ii) In the hyperbola, as will be proved in Art. 52†,

$$CP^2 \sim CD^2 = CA^2 \sim CB^2;$$

and, this result being assumed, it may be deduced that

$$PN : CR = DR : CN = CB : CA,$$

as in the ellipse.

Corollary 1.

If P be any point on an ellipse, $(SP + S'P)^2 = 4CA^2$, and $SP^2 + S'P^2 = 2CS^2 + 2CP^2$ (Lemma D). Therefore, subtracting and dividing by two,

$$SP . S'P = 2CA^2 - CS^2 - CP^2 = CA^2 + CB^2 - CP^2$$
$$= CD^2,$$

as may also be proved in like manner for the hyperbola.

Corollary 2.

If the normal at P meet the transverse axis in G, then by similar triangles (CD being at right angles to the normal)

$$PG : CD = PN : CR = CB : CA.$$

* Another proof will be given in Art. 66.

† The reason for treating the ellipse and the hyperbola differently in this proposition is given in Art. 44 and in the Scholium, p. 70.

And in like manner it may be shewn that

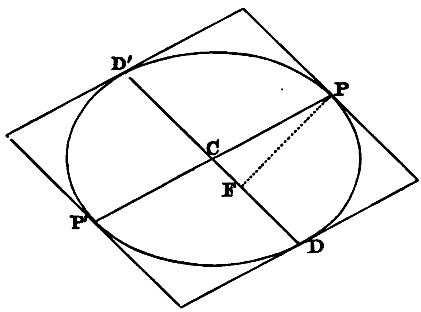
$$Pg : CD = CA : CB,$$

where Pg is the normal terminated by the conjugate axis.

PROPOSITION XIV.

46. *The conjugate parallelogram is of constant area and equal to the rectangle contained by the axes.*

Let PCP' and DCD' be a pair of conjugate diameters, and let a conjugate parallelogram be constructed by drawing parallels to each of them through the extremities of the other. [Def. Art. 31.



Let the normal at P meet DD' in F and the transverse axis in G. Then since

$$PG : CD = CB : CA, \quad \text{[Art. 45, Cor. 2.}$$

therefore

$$PF . PG : PF . CD = CB^2 : CA . CB.$$

Hence, the antecedents being equal by Prop. XI.,

$$PF . CD = CA . CB;$$

and the conjugate parallelogram is equal to $4PF . CD$ or $AA' . BB'$*.

* *A second proof* for the hyperbola alone will be found in Art. 52 (ii).

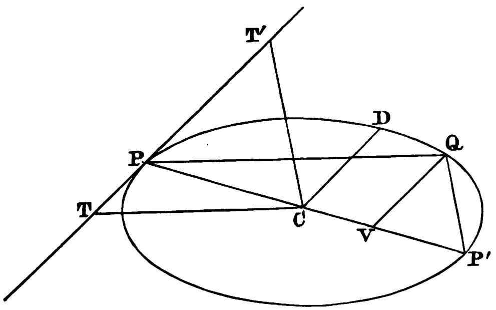
In the ellipse the parallelogram described as above completely envelopes the curve: in the hyperbola two of its sides only touch the curve (Fig. Art. 50).

PROPOSITION XV.

47. *The intercepts on any tangent between the curve and any two conjugate diameters contain a rectangle equal to the square of the semi-diameter parallel to the tangent.*

Draw the tangent at any point P, the diameter PCP' and the conjugate semi-diameter CD.

Let any second pair of conjugate diameters meet the tangent at P in T and T', and draw the supplemental chords QP and $P'Q$ parallel to CT and CT'. [Art. 44.



Then by similar triangles, if QV be an ordinate to PP',

$$PT : CP = QV : PV,$$

and $$PT' : CP = QV : P'V.$$

Hence $$PT \cdot PT' : CP^2 = QV^2 : PV \cdot P'V$$

$$= CD^2 : CP^2, \qquad \text{[Art. 36.}$$

or $PT \cdot PT'$ is equal to CD^2, as was to be proved.

Corollary.

In the hyperbola the points T and t in which any two conjugate diameters meet the tangent at Q lie on the same side of Q, as in the second figure of Art. 41, and the two diameters may be supposed to coalesce. In this case the intercept QT becomes equal to the parallel semi-diameter CD, as will be noticed more particularly in the next chapter.

CHAPTER V.

THE ASYMPTOTES.

48. If a straight line and a curve, being produced, continually approach one another but never actually meet until they are produced to infinity, the straight line is said to be an *Asymptote* of the curve. It will be seen that every hyperbola has two asymptotes.

If CE and CP be two diameters of a hyperbola lying in the same quadrant, and EPN be an ordinate to the transverse axis, the diameters will be conjugate provided that

$$EN . PN : CN^2 = CB^2 : CA^2 *.$$

If CP be made to coalesce with CE, it follows that

$$EN : CN = CB : CA,$$

a relation which determines the position of the *self-conjugate* diameter CE. The curve has also a second self-conjugate diameter CE' making the same angle as CE with the axis.

Since one of every two conjugate diameters meets and the other does not meet the hyperbola (Art. 44), a diameter which is self-conjugate should meet and yet not meet the curve: accordingly it will appear that the self-conjugate diameters coincide with the asymptotes, which meet the curve at infinity and do not meet it at any assignable distance from the centre.

For the sake of uniformity of enunciation we shall at the outset speak of the lines CE and CE' as the asymptotes, and shall then shew conversely that they possess the property from which the term asymptote is derived. [Prop. I. Cor.

* In Art. 44, Cor., let CD meet the ordinate of P in E.

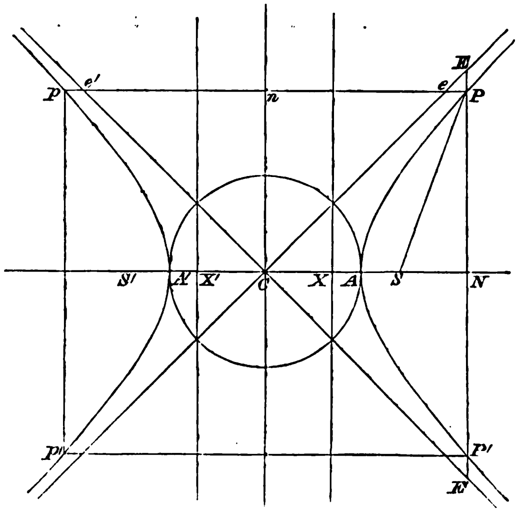
It is to be noticed that the asymptotes are the diagonals of the rectangle formed by drawing parallels to each axis through both extremities of the other. On the tangent at A take a length AL equal to CB, then will CL be equal to CS (Art. 34), and $CL : CA$ to the eccentricity. Also it may be easily proved that the feet of the focal perpendiculars upon the asymptotes lie on the auxiliary circle, and that each perpendicular is equal to CB.

PROPOSITION I.

49. *The rectangle contained by the distances of any point on a hyperbola from its two asymptotes is of constant magnitude.*

(i) If the principal ordinate of any point P on the hyperbola meet the axis in N and the asymptotes in E and E', then

$$PN^2 + CB^2 : CN^2 = CB^2 : CA^2 \qquad \text{[Art. 33.}$$
$$= EN^2 : CN^2; \qquad \text{[Art. 48.}$$



and therefore $PN^2 + CB^2$ is equal to EN^2 and

$$PE . PE' = EN^2 - PN^2 = CB^2.$$

(ii) Next let Pe be drawn in any specified direction to meet CE. Then the triangle PEe is given in species and Pe varies as PE, and therefore $Pe \,.\, PE'$ is constant.

In like manner, if Pe' be drawn in any specified direction* to meet CE' the length Pe' varies as PE', and therefore $Pe \,.\, Pe'$ is constant.

Taking for example the case in which PO the distance of P from one asymptote is measured parallel to the other (Fig. Art. 52), and CO is therefore equal to the distance of P from the latter asymptote measured parallel to the former, we have $PO \,.\, CO$ constant, and it may be shewn by making P coincide with the vertex that

$$PO \,.\, CO = \tfrac{1}{2} CS \,.\, \tfrac{1}{2} CS = \tfrac{1}{4}(CA^2 + CB^2). \qquad \text{[Art. 48.}$$

Corollary.

Since $PE \,.\, PE'$ is constant and PE' continually increases with CN, therefore PE at the same time continually decreases. Hence CE and the curve continually approach one another but never actually meet until produced indefinitely†.

PROPOSITION II.

50. *The intercepts on any tangent to a hyperbola between the curve and its asymptotes are equal to one another and to the parallel semi-diameter; and the opposite intercepts on any chord between the curve and its asymptotes are equal to one another.*

(i) Let the tangent at P be parallel to the semi-diame-

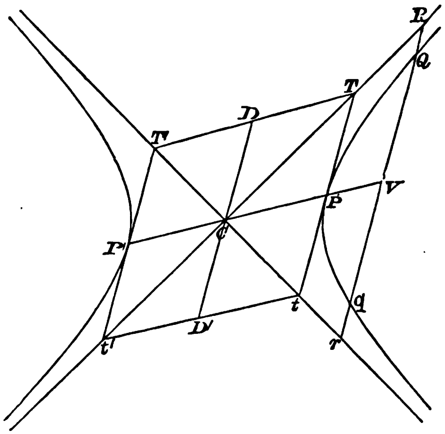
* In this construction Pe and Pe' are not required to be in the same straight line. In the special case in which Pee' is a straight line and parallel to the axis it may be shewn that $Pe \,.\, Pe' = CA^2$. The like may be shewn (Art. 51) if CA be *any* radius, regarded as fixed for the time being.

† The two branches of a hyperbola constitute one complete and continuous curve, which may be regarded as the locus of a point moving as follows. First let the point trace the quadrant AP and recede to infinity in the direction CE: then, travelling in the same direction, it reappears at the opposite extremity of the asymptote (Lemma H), traces the branch $p'A'p$, and finally the quadrant $P'A$. This appears plainly when the hyperbola is traced with the help of the eccentric circle, the point p (Art. 6) being supposed to move continuously round the circle, governing the motion of the point P which *traces the conic.*

ter CD, and let it meet any two conjugate diameters in T and T'', so that

$$PT \, . \, PT'' = CD^2. \qquad \text{[Art. 47.}$$

Let CT'' coalesce with CT, so that CT becomes a self-conjugate diameter or asymptote; then PT^2 is equal to CD^2,



and in like manner, if the same tangent meet the other asymptote in t, Pt^2 is also equal to CD^2. Therefore

$$PT = CD = Pt,$$

as was to be proved.

It readily follows that every two parallel tangents as TPt and $T'P't'$ terminated by the asymptotes are sides of a conjugate parallelogram (Def. Art. 31), and conversely that every conjugate parallelogram has its diagonals coincident with the asymptotes.

(ii) Next let any chord Qq parallel to the tangent TPt be produced to meet the asymptotes in R and r. Then the diameter CP bisects Rr, and it also bisects the chord Qq. [Art. 13, Cor. 1.

Therefore the opposite intercepts QR and qr are equal *and the opposite* intercepts Qr and qR are equal, as was to be proved. And the same may be shewn in like manner if the chord be supposed parallel to the diameter PP' which meets the curve in real points.

PROPOSITION III.

51. *Any chord of the asymptotes is divided at either of the points in which it meets the curve into segments to which the parallel radius is a mean proportional.*

In the figure of Prop. II. let V be the middle point of Qq and Rr. Then by Art. 36 and by parallels (supposing for example that Q and q lie on the same branch of the curve),

$$QV^2 + CD^2 : CV^2 = CD^2 : CP^2 = PT^2 : CP^2$$
$$= RV^2 : CV^2.$$

Therefore, the antecedents being equal,

$$RQ \,.\, Qr = Rq \,.\, qr = RV^2 - QV^2 = CD^2,$$

or CD is a mean proportional to RQ, Qr and to Rq, qr*.

PROPOSITION IV.

52. *The difference of the squares of any two conjugate semi-diameters of a hyperbola is equal to the difference of the squares of the semi-axes; and the triangle contained by the asymptotes and any tangent is equal to the rectangle contained by the semi-axes.*

(i) Let the tangent at P meet the asymptotes in L and M. Draw PY perpendicular to CL, and bisect CL in O. Then, P being the middle point of LM (Prop. II.), OP is parallel to the asymptote CM, and the triangle OYP is given in species and OY varies as OP.

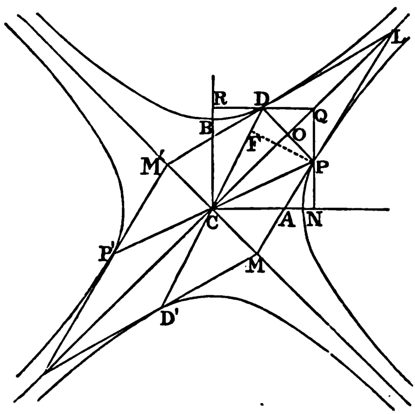
* We may deduce the same from Prop. I. by taking any chord parallel to a fixed diameter, and then supposing it to coalesce with the diameter itself (if it meet the curve), or with the parallel tangent.

Also, by Euc. I. 47 and Lemma A,

$$CP^2 \sim PL^2 = CY^2 \sim LY^2 = 4CO \, . \, OY,$$

where OY varies as OP, and therefore inversely as CO. [Prop. I.

Therefore $CP^2 \sim CD^2$ or $CP^2 \sim PL^2$ has a constant magnitude, which may be shewn by taking P at the vertex to be equal to $CA^2 \sim CB^2$.



(ii) The rectangle $CL \, . \, CM$ is equal to $2CO \, . \, 2PO$ or CS^2 (Art. 49). Therefore the triangle LCM is of constant area, and it may be shewn by taking P at the vertex to be equal to $CA \, . \, CB$.

The *Conjugate Parallelogram*, which is four times the triangle CTt (Art. 50), is therefore equal to $AA' \, . \, BB'$.

Scholium.

Although, in accordance with an ancient convention, we have assigned specific lengths to those diameters of the hyperbola which do not meet the curve (Art. 36), such diameters have not-*withstanding no* real extremities or magnitudes, and it is therefore impossible to treat the ellipse and the hyperbola altogether similarly so long as we restrict ourselves to the conception of real points. It may seem that in the corollary of Art. 33 the two curves have been treated similarly: but this is not the case. In the ellipse, regarded as the locus of a point P whose ordinate and abscissa are connected by the relation,

$$PN^2 : CA^2 - CN^2 = CB^2 : CA^2,$$

we find (1) by making PN vanish that the transverse semi-axis is equal to CA, and (2) by making CN vanish that the conjugate semi-axis is equal to CB. In the hyperbola, from the relation,

$$PN^2 : CN^2 - CA^2 = CB^2 : CA^2,$$

we find (1) by making PN vanish that the transverse semi-axis is equal to CA, and (2) by making CN vanish that the square of the conjugate semi-axis is equal to the negative quantity $-CB^2$. In like manner, from the relation of Art. 36,

$$QV^2 : CV^2 - CP^2 = CD^2 : CP^2,$$

we find by making CV vanish that the square of the semi-diameter conjugate to CP is equal to the negative quantity $-CD^2$. For $-CB^2$ write $C\beta^2$ and for $-CD^2$ write $C\delta^2$; then the hyperbola may be treated as a quasi-ellipse, in which

$$PN^2 : CA^2 - CN^2 = C\beta^2 : CA^2,$$

and

$$QV^2 : CP^2 - CV^2 = C\delta^2 : CP^2.$$

As a property of this "ellipse" we have

$$CP^2 + C\delta^2 = CA^2 + C\beta^2,$$

or

$$CP^2 + (-CD^2) = CA^2 + (-CB^2),$$

which agrees with Art. 52, except that strictly speaking we should say that the *sum* of the squares of any two conjugate diameters of a hyperbola is constant, the square of one of every two such diameters being negative and its length therefore imaginary. In geometrical proofs every step has its explanation upon the figure. But let the student, shutting his eyes to the figure, regard any property of the ellipse (for example) as implicitly contained in the relation between its ordinates and abscissæ, and he will see that from any such property, in so far as it is expressible in terms of CB^2 and CD^2, he may pass at once to a corresponding property of the hyperbola by writing $-CB^2$ for CB^2 and $-CD^2$ for CD^2. Thus, the radius of the orthocycle in an ellipse (Art. 40, Cor. 2) being equal to $\sqrt{(CA^2+CB^2)}$, its radius in the hyperbola is equal to $\sqrt{(CA^2-CB^2)}$. In the ellipse $SY \,.\, S'Y'=CB^2$ (Art. 40); therefore in the hyperbola $SY \,.\, S'Y'=-CB^2$, or $SY \,.\, (-S'Y')=CB^2$, one of the perpendiculars having to be regarded as positive and the other as negative because they are drawn in opposite directions, the tangent passing between the foci. The same principles will be seen to be applicable to other properties: for example, to the theorems of Arts. 46 and 47.

CHAPTER VI.

THE EQUILATERAL HYPERBOLA.

53. THE *Equilateral Hyperbola* is a hyperbola whose latus rectum is equal to its transverse axis. Its two axes being equal (Art. 34), its asymptotes are at right angles, and it is therefore called also the *Rectangular Hyperbola*. Its eccentricity is the ratio of the diagonal to the side of a square, the foot X of the S-directrix bisects CS, and

$$\tfrac{1}{2}CS^2 = CA^2 = 2CX^2 = 2SX^2. \qquad \text{[Art. 32.}$$

Also $$PN^2 = AN \,.\, A'N = CN^2 - CA^2, \qquad \text{[Art. 33.}$$

and $$QV^2 = PV \,.\, P'V = CV^2 - CP^2 *, \qquad \text{[Art. 36.}$$

since by Prop. II. every diameter is equal to its conjugate.

The normal at P terminated by either axis is equal to CP or CD (Art. 45, Cor. 2), and therefore also to the intercept on the tangent between the curve and either asymptote. [Art. 50.

It remains to prove certain of the more distinctive properties of this variety of the hyperbola, which bears the same kind of relation to the general hyperbola that the circle (or equilateral ellipse) bears to the general ellipse.

PROPOSITION I.

54. *The angles between any two diameters of an equilateral hyperbola are equal to the angles between the conjugate diameters.*

Let any two diameters meet the S-directrix in V and V'; and draw SZ at right angles to SV and SZ' at right angles

* This may also be deduced as a corollary from Prop. I.

to SV', so that SZ and SZ' are the directions of the diameters conjugate to CV and CV'. [Art. 13.

Then since X is the middle point of CS, [Art. 53.

$$\angle\, VCV' = VSV' = ZSZ',$$

or the diameters CV and CV' are inclined at the same angles as their conjugates. Thus *any two conjugate diameters* (Fig. Art. 55) make equal angles with the two axes: they therefore *make equal angles with either asymptote, and make complementary angles with either axis.*

Corollary 1.

Let PP' be any diameter, Q and R any two points on the curve; then since supplemental chords are parallel to conjugate diameters (Art. 44) the angle QPR is equal or supplementary to $QP'R$, and therefore *any chord of an equilateral hyperbola subtends equal or supplementary angles at the extremities of any diameter.*

Corollary 2.

In the second figure of Art. 41 (supposing the hyperbola to be equilateral) it may be shewn by similar triangles that

$$QT \,.\, Qt = CQ^2;$$

and likewise that $$CV \,.\, VT = QV^2,$$

or $$CV \,.\, CT = CV^2 - QV^2 = CP^2.$$

Corollary 3.

Let ABC be a given triangle, O the centre of an equilateral hyperbola circumscribed to it, and abc the middle points of its sides. Then since Ob and Oc are the diameters bisecting the chords AB and AC respectively, they contain an angle equal to the given angle BAC. The locus of O is therefore a circle passing through b and c, and likewise through a, since bc evidently subtends the given angle at a. That is to say, *the locus of the centre of an equilateral hyperbola circumscribed to a given triangle is the nine-point circle of the triangle.*

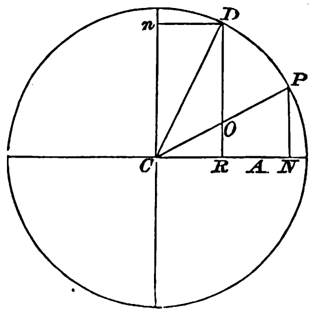
PROPOSITION II.

55. *Conjugate diameters and diameters at right angles are equal in the equilateral hyperbola; and the ordinates and abscissæ of the extremities of any two conjugate semi-diameters are alternately equal to one another.*

(i) The difference of the squares of any two conjugate diameters being equal to the difference of the squares of the axes (Art. 45), and the axes being in this case equal, it follows that every diameter is equal to its conjugate.

(ii) Let CP and CD be conjugate semi-diameters, PN and DR ordinates to the axis; and let PN produced meet the hyperbola again in P', so that $CP' = CP = CD$. Then

$$\angle P'CD = PCN + DCN = \text{a right angle}, \qquad [\text{Prop. I.}$$



or any two equal semi-diameters CP' and CD lying in adjacent quadrants are at right angles, and conversely.

(iii) Since the triangles CPN and CRD are similar (Prop. I.), and $CP = CD$, it readily follows that

$$PN = CR, \text{ and } DR = CN,$$

as was to be proved.

Corollary.

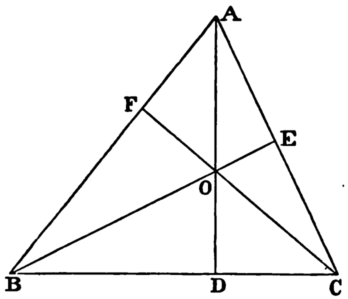
Hence and from the relation $CN^2 - PN^2 = CA^2$ it may be deduced that $\triangle CPD = \frac{1}{2} CA^2$. The conjugate parallelogram is therefore equal to $4 CA^2$.

PROPOSITION III.

56. *If an equilateral hyperbola circumscribes a triangle it passes through its orthocentre, and conversely.*

Let AD, BE, CF be the three perpendiculars of a triangle ABC, and O the point in which they cointersect, which is called the orthocentre of the triangle.

Let a hyperbola be supposed to pass through the four points $ABCO$. Then since $AD.DO = BD.DC$, its diameters parallel to AD and BC are equal (Art. 16, Cor. 2), and in like manner its diameters parallel to BE and AC are equal, as also are those parallel to CF and AB.



Hence the hyperbola through $ABCO$, having more than one pair of equal diameters at right angles, must be equilateral (Prop. II.); and it may be inferred conversely that every equilateral hyperbola circumscribing a triangle ABC must pass through its orthocentre O.

CHAPTER VII.

THE CONE.

57. An unlimited straight line which passes through a fixed point in space and makes a constant angle with a fixed straight line through the point generates a surface which is called a *Cone.* The fixed point is called the *Vertex,* the fixed line the *Axis,* and the variable line in any position is called a *Side* or a *Generating Line* of the cone. The constant angle between any two opposite sides of the cone is called its *Vertical Angle.* The complete cone consists of two infinite sheets situated on opposite sides of the vertex, as in the last figure of this chapter.

This species of cone is more fully described as the *right circular* cone, the sections of it made by planes at right angles to its axis being evidently circles. Any such section may be regarded as the *Base* of the cone.

In the special case in which the vertex is at infinity, and the generating lines are therefore all parallel to the axis and at right angles to the base, the surface is called no longer a cone but a *Cylinder.*

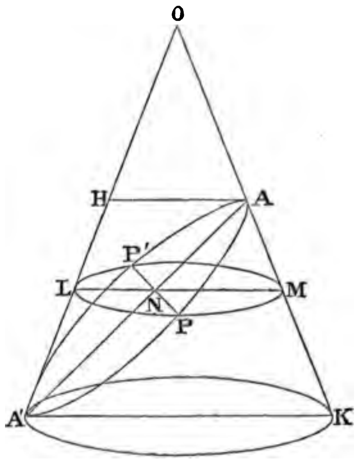
In what follows we shall consider a plane through the axis to be the *Plane of Reference* and the *Sections* to be made by planes at right angles thereto, so that the principal axis of a section will always lie in the plane of reference. It will appear from the following propositions that a plane section of a cone is in general a parabola, an ellipse or a hyperbola.

The Ordinate.

PROPOSITION I.

58. *The square of the principal ordinate in any section varies as the rectangle contained by the corresponding abscissæ.*

Let AA' be the axis of the section and PN the perpendicular upon it from any point P of the section. Draw the



circular sections of the cone through P, A, A', and let their diameters in the plane of reference be LM, AH, $A'K$ respectively.

Then by similar triangles

$$LN : A'N = AH : AA',$$

and $$MN : AN = A'K : AA';$$

and by a property of the circle

$$PN^2 = LN . MN.$$

Therefore $PN^2 : AN . A'N = AH . A'K : AA'^2$,

or the locus of P is an ellipse or a hyperbola, according as the plane of the section cuts all the generating lines of the cone on the same side of the vertex, as above, or cuts both sheets of the cone, as in the last figure of Art. 59. In either case the conjugate axis is a mean proportional to AH and $A'K$, and therefore *the semi-axis conjugate is a mean proportional to the perpendiculars from the vertices of the section to the axis of the cone.*

When the plane of the section is parallel to a side of the cone, it may be shewn in like manner that PN^2 varies as AN and the section is a parabola.

Corollary.

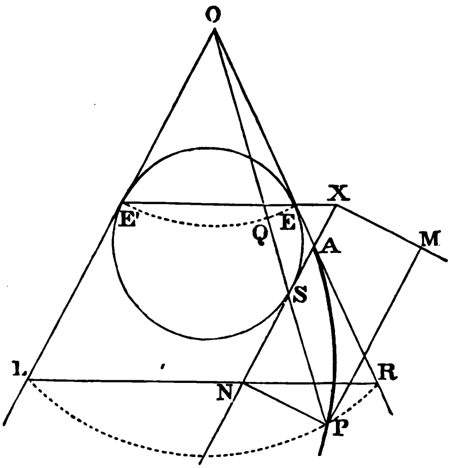
In the cylinder (in which all circular sections are equal) an oblique section is always an ellipse having its minor axis equal to the diameter of the base, whilst its major axis may be of any length greater than that diameter. Conversely *any ellipse may be regarded as a plane section of a right cylinder described on a circular base equal to the minor auxiliary circle of the ellipse.*

The Focal Spheres.

PROPOSITION II.

59. *If a sphere be inscribed in a cone so as to touch the plane of a section, the point of contact of the sphere with the plane will be a focus of the section, and the plane of contact of the sphere with the cone will meet the plane of the section in the corresponding directrix.*

(i) Let a sphere be drawn touching a cone along the circle EQE', and touching the plane of a section at the point S; then will S be a focus of the section, and the



corresponding directrix will be the line MX in which the plane of contact EQE' meets the plane of the section.

For let P be any point on the section, PY* a perpendicular to the plane of contact, PM a perpendicular to MX, and Q the point in which the side of the cone through P meets the plane of contact.

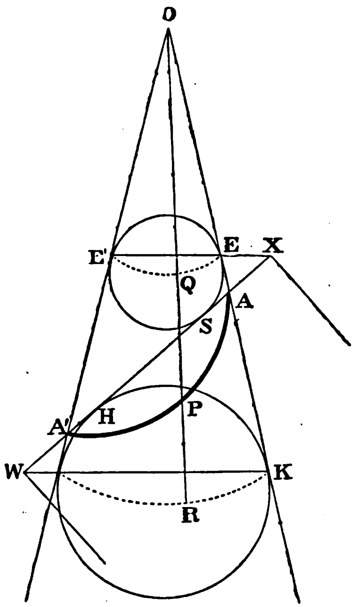
Then since the angle QPY is always equal to half the vertical angle of the cone, and the angle MPY to the angle between the axis of the section and that of the cone, it follows that $PQ : PY$ and $PY : PM$ and therefore also $PQ : PM$ are constant ratios.

Hence, the tangents PS and PQ to the sphere being equal, SP also is in a constant ratio to PM; that is to say,

* *This line is to be supplied in the diagram.*

the locus of P is a conic having the point S and the line MX for a focus and directrix.

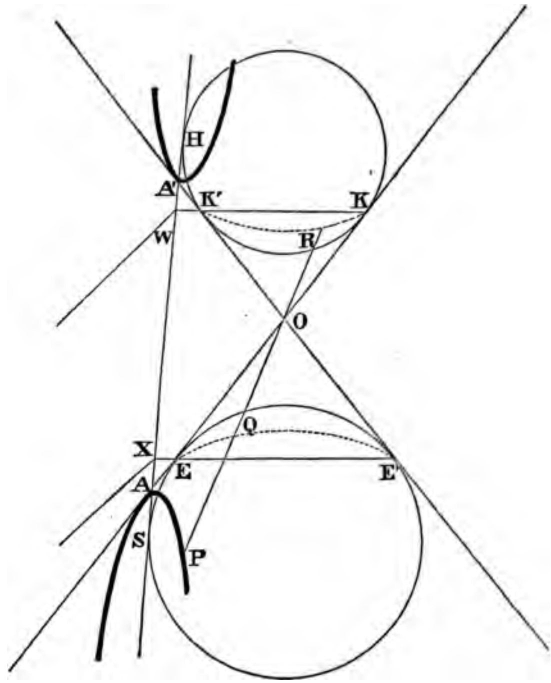
(ii) Every elliptic section has two focal spheres touching its plane on opposite sides, and every hyperbolic section two focal spheres touching its plane on the same side: thus the two foci S, H and their directrices are determined in each case.



Let the side of the cone through any point P of the section touch the focal spheres in Q and R, as in the annexed figures. Then PS and PQ, being tangents to the S-sphere, are equal, and PH and PR, being tangents to the H-sphere are equal.

Hence, in the ellipse,

$$SP + HP = PQ + PR = QR.$$



And in the hyperbola,

$$SP \sim HP = PQ \sim PR = QR.$$

In either case QR is of constant length, and may be shewn, viz. by making P coincide with A, to be equal to

$$AS \pm AH, \text{ or } AA'.$$

Corollary.

Draw the tangent at P to the section and take any point T upon it. Then, PS and PQ, as being tangents to the S-sphere, are equal, and likewise TS and TQ are equal, and PT is common to the two triangles SPT and QPT. Hence their angles at P are equal, or *at any point of a section the tangent makes equal angles with the focal distance and the side of the cone.*

CHAPTER VIII.

ORTHOGONAL PROJECTION.

60. THE *Orthogonal Projection,* or briefly the *Projection* of any point in space upon a plane, is the foot of the perpendicular let fall from the point to the plane. The projection of any line or figure is determined by the projections of its several points. It is evident that the projections of any figure upon parallel planes are equal and similar.

PROPOSITION I.

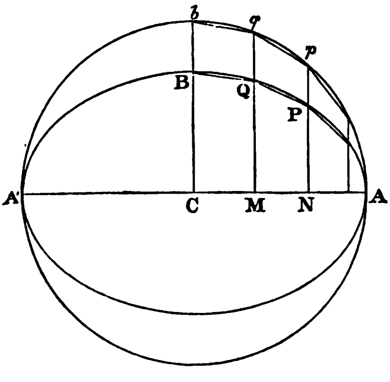
61. *Any ellipse may be projected into a circle equal to its minor auxiliary circle; and a circle may be projected into an ellipse of any eccentricity having its major axis equal to the diameter of the circle.*

(i) Describe a right cylinder upon a circular base equal to the minor auxiliary circle of the given ellipse; then the ellipse may be placed so as to coincide with one of the plane sections of the cylinder. [Art. 58, Cor.

The plane of the section is to be taken at such an inclination to the plane of the base that the projection of the major axis of the section upon the base may be equal to its minor axis.

(ii) All the ordinates to a diameter AA' in a circle are cut in the same ratio of minority $CB : CA$ by an ellipse whose major axis is AA' and whose minor axis is equal to $2CB$. [Art. 35.

Hence if the plane of the circle be turned about AA' through a certain angle*, every point P on the ellipse will



lie vertically under the corresponding point p of the circle, or the ellipse will be the orthogonal projection of the circle. Thus the circle is projected into an ellipse having its major axis equal to AA', whilst its eccentricity increases with the acute angle between its plane and the plane of the circle and may be of any magnitude between zero and unity.

PROPOSITION II.

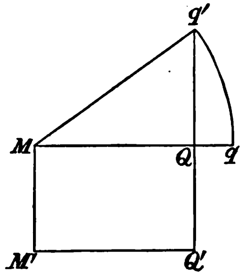
62. *The projections of parallel straight lines are parallel straight lines, and every line or segment in a system of parallels is in the same ratio to its projection.*

(i) The projection of a straight line upon a plane is the common section of that plane with the plane drawn at right angles to it through the line, since their common section evidently contains the projections of all points upon the line and of those only.

Let Mq' be a finite straight line and $M'Q'$ its projection, so that the plane $Mq'Q'M'$ is at right angles to the plane of projection.

* In both cases of the proposition the cosine of the angle between the two planes must be equal to $CB : CA$.

Then if Mq' be regarded as a variable line belonging to a system of *parallels*, the planes by which it is projected



will be parallel, and their common sections $M'Q'$ with the plane of projection will be parallel to one another, as was to be proved.

(ii) Moreover, if MQ be drawn equal and parallel to $M'Q'$, the angle $q'MQ$ will be constant, and the ratio of Mq' to MQ or $M'Q'$ will therefore be invariable, as was to be proved.

In the special case in which the original parallels and their projections are at right angles to the common section of their planes the constant angle $q'MQ$ is equal to *the angle between the planes.*

PROPOSITION III.

63. *Any area in one plane is to its projection upon any other plane in a ratio which depends only upon the angle between the planes.*

In the one plane draw any number of perpendiculars pN, qM,... to the common section of the planes, and let PN, QM,... be their projections upon the other plane, so that

$$PN : pN = QM : qM = CB : CA,$$

where $CB : CA$ is a ratio determined by *the angle between the planes**. [Prop. II.

* Compare the figure of Art. 61, supposing the planes of the two curves to be inclined at an angle whose cosine is equal to CB : CA.

It follows that every rectilinear figure $pNMq$ determined by the pair of perpendiculars pN, qM is to its projection $PNMQ$ as CA to CB; and the aggregate of any number of such figures is in the same ratio to the aggregate of their projections.

But any rectilinear figure in the primitive plane may be divided into elements by means of perpendiculars drawn as above; and any curvilinear figure may be regarded as the limit of a rectilinear figure whose adjacent angular points are indefinitely near to one another. Therefore any area whatever in the one plane is to its projection upon the other plane in the ratio of CA to CB, as was to be proved.

PROPOSITION IV.

64. *The points of concourse of lines and of their projections correspond to one another, and the tangent to a curve at any point corresponds to the tangent at the projection of the point.*

(i) Since the projection of any line is determined by the projections of its several points, if any number of lines straight or curved pass through a point, their projections must all pass through the projection of the point. For example, if a chord of any curve be drawn through a *fixed* point its projection will pass through the corresponding fixed point in the plane of projection.

(ii) If a straight line and a curve intersect in adjacent points P and Q, the projections of the straight line and the curve will intersect at the projections p and q of those points. Hence, the projecting lines Pp and Qq being always parallel, if the points P and Q coalesce their projections must also coalesce; that is to say, the tangent to the original curve at P projects into the tangent at p to the projection of the curve.

We shall conclude by briefly indicating the method of applying these propositions, with reference in the first instance *to the ellipse.*

65. *The Area of the Ellipse.*

It may be shewn by projecting a circle into an ellipse that the area of any ellipse is to that of its auxiliary circle as CB to CA, and hence that the area of the ellipse is equal to $\pi . CA . CB$.

66. *Conjugate Diameters.*

The middle points of any system of parallel chords in a circle may be projected into the middle points of a system of parallel chords in an ellipse. But in the circle parallel chords are bisected by a straight line; therefore in the ellipse also parallel chords have their middle points in a straight line. Hence it appears that diameters at right angles in the circle correspond projectively to conjugate diameters in an ellipse.

Hence a simple construction for drawing conjugate diameters to an ellipse.

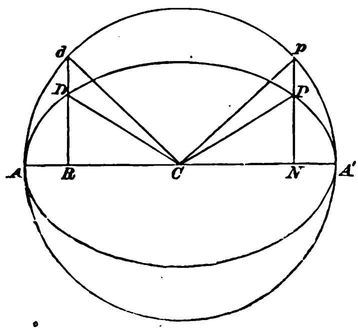
On the circumference of its auxiliary circle take points p and d which subtend a right angle at C, and let their ordinates to the axis, viz. pN and dR, intersect the ellipse in P and D: then will CP and CD be conjugate radii.

It is easily seen that pN is equal CR, and hence that

$$CN^2 + CR^2 = CN^2 + pN^2 = CA^2.$$

It then follows as in Art. 45 (i.) that

$$CP^2 + CD^2 = CA^2 + CB^2.$$



67. *The Segments of Chords.*

Project an ellipse into a circle, or a circle into an ellipse. Let POQ, $P'OQ'$ be any two intersecting chords of the ellipse and CD, CD' the parallel semi-diameters: poq, $p'oq'$ the corresponding chords of the circle and cd, cd' the parallel radii.

Then by Prop. II. and by known properties of the circle,

$$\begin{aligned} OP \,.\, OQ : CD^2 &= op \,.\, oq : cd^2 \\ &= op' \,.\, oq' : cd'^2 \\ &= OP' \,.\, OQ' : CD'^2, \end{aligned}$$

or the rectangles contained by any two intersecting chords of an ellipse are as the squares of the parallel semi-diameters.

68. *The Tangent.*

Any two tangents to a circle meet upon the diameter bisecting their chord of contact, viz. at a point T such that

$$CV \,.\, CT = CP^2,$$

where V is the middle point of the chord, and P an extremity of the diameter CT. It readily follows by orthogonal projection that the same is true in the ellipse; and in like manner the property of Art. 47 may be first proved for the circle and then transferred by projection to the ellipse.

69. *Properties of Polars.*

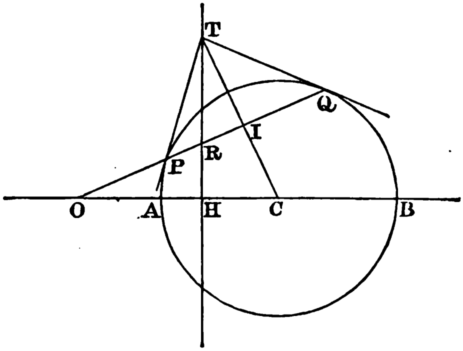
(*a*) *If a chord of a circle passes through a fixed point the tangents at its extremities intersect on a fixed straight line, and conversely.*

In a circle of radius CA draw any chord PQ through a point O, and let TP and TQ be the tangents at its extremities. Draw the perpendicular TH to CO, and let TH meet the chord in R, and let CT meet it in I.

Then since the angles at H and I are right angles the points H, I, T, O are concyclic, and

$$CH \,.\, CO = CI \,.\, CT = CA^2.$$

Hence if O be a fixed point, CH is constant and RH is a fixed straight line, and conversely.



The line RH is called the *Polar* of O, and the point O is called the *Pole* of RH.

If the point O be taken without the circle its polar will be the chord of contact of the tangents from O: if the point be taken within the circle its polar will lie wholly without the curve.

(*b*) *Any chord of a circle is divided harmonically by any point through which it passes and the polar of the point.*

For in the preceding figure the points C, I, R, H are concyclic, so that

$$\begin{aligned} OR \,.\, OI = OH \,.\, OC &= OC^2 \sim CO \,.\, CH \\ &= CO^2 \sim CA^2 \\ &= OP \,.\, OQ. \end{aligned}$$

Hence by Lemma B, since I is the middle point of PQ,

$$2OP \,.\, OQ = 2OR \,.\, OI = OR\,(OP + OQ),$$

and therefore PQ is divided harmonically at O and R.

These properties of polars may be extended to the ellipse by orthogonal projection. We may remark in passing that they are also true of the general conic: thus (to take a special case) the directrix is the polar of the focus. [Art. 7, Cor.

70. *The Equilateral Hyperbola.*

It readily follows from the property of the principal ordinate (Art. 33) that an equilateral hyperbola may be projected into a hyperbola of any eccentricity, and *vice versa.*

In the equilateral hyperbola let the length of any semi-diameter CD which does not meet the curve be *defined* by the condition that it shall be equal to the conjugate semi-diameter CP: let it be granted further that

$$PN^2 \sim DR^2 = CN^2 \sim CR^2 = CA^2, \qquad \text{[Art. 55.}$$

and that the triangle CPD is equal to CA^2.

It may then be deduced by projection that in the general hyperbola the difference of the squares of any two conjugate semi-diameters is equal to $CA^2 \sim CB^2$, and the area of the conjugate parallelogram to $AA'.BB'$.

In like manner the property of the ordinate to any diameter in the general hyperbola (Art. 36) may be deduced from the special case of the equilateral hyperbola.

CHAPTER IX.

CURVATURE.

71. It is evident that a circle and a conic cannot intersect in an odd number of points, and it may be inferred from Art. 16, Cor. 3, that they cannot intersect in more than four points.

Let a circle and a conic intersect in four points $Q'PQP'$, two or more of which may be supposed to become coincident. And first let Q' coalesce with P. Then the circle and the conic have simple contact at P and intersect at Q and P', and their common tangent and their common chord QP' are equally inclined to the axis of the conic.

Next let Q also coalesce with P. Then the circle *both touches and cuts* the conic at P, and their common tangent and their common chord PP' are equally inclined to the axis of the conic.

Lastly let P' also coalesce with P. Then the circle touches the conic without cutting it at P and does not meet it again.

72. The circle which is the limit of a circle described so as to touch a conic at P and cut it at an adjacent point Q which ultimately coalesces with P is called the *Circle of Curvature* of the conic at P. Its centre, radius and diameter are called the *Centre, Radius* and *Diameter of Curvature* at P, and its chord drawn from P in any direction is called the *Chord of Curvature* of the conic at P in that direction. The

circle of curvature in general cuts the conic at their point of contact; but when this point lies on an axis of the conic the circle touches the conic without cutting it at that point and does not meet it again.

PROPOSITION I.

73. *The focal chord of curvature at any point of a conic is equal to the focal chord of the conic parallel to the tangent at that point.*

Let a circle touch a conic at P and cut it at an adjacent point Q.

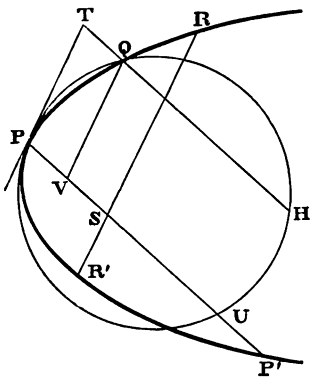
Draw the focal chord PSP' cutting the circle in U, and the focal chord RSR' parallel to the tangent at P; and let the parallel through Q to PP' meet the tangent in T, the circle in H and the conic in K.

Then by Art. 16, Cor. 2,

$$TP^2 : TQ.TK = RR' : PP',$$

and by a property of the circle,

$$TP^2 = TQ . TH;$$



and therefore

$$TH : TK = RR' : PP'.$$

Hence in the limit, when Q coalesces with P and therefore THK with PUP',

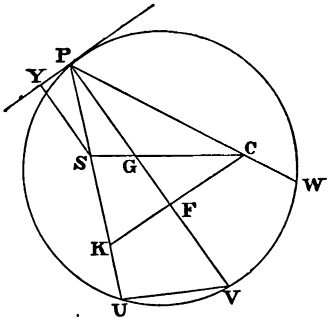
$$PU : PP' = RR' : PP',$$

or the focal chord of curvature PU is equal to the focal chord RR' parallel to the tangent at P.

Corollary 1.

If PV be the chord of curvature at P in any direction, and SY be drawn parallel to it to meet the tangent in Y, it may be shewn that

$$PV = \frac{SP}{SY} . PU = \frac{SP}{SY} . RR',$$



which determines the chord of curvature of a conic at any point in any direction.

Corollary 2.

It may now be deduced that in the Parabola, the chord of curvature at P through the focus, or parallel to the axis, being equal to $4SP$ (Art. 21), the diameter of curvature is equal to

$$\frac{4SP^2}{SY}, \text{ or } \frac{4SY^3}{SA^2},$$

where SY is the focal perpendicular upon the tangent at P; and in the Central Conics that, the focal chords of curvature being equal to $\frac{2CD^2}{CA}$ (Art. 36, Cor.), the central chord of curvature is equal to $\frac{2CD^2}{CP}$ (Art. 39, Cor. 2), and the diameter of curvature to $\frac{2CD^3}{CA \, . \, CB}$, or $\frac{2CD^2}{PF}$ (Art. 46).

Corollary 3.

The focal chord of curvature at P in any conic is equal to $\frac{S'P}{2CA} \, . \, 4SP$ (Art. 45, Cor. 1), and the ratio $\frac{S'P}{2CA}$ is equal to or less or greater than unity according as the conic is a parabola, an ellipse or a hyperbola. Conversely *a conic is a parabola, an ellipse or a hyperbola according as its focal chord of curvature at any point P is equal to or less or greater than* $4SP$*, the point P being supposed to lie on the S-branch in the case of the hyperbola. This may also be proved by shewing that SG is equal to or less or greater than SP according as S, in Art. 75, lies at the middle point of PU, or at a greater or less distance from P.

PROPOSITION II.

74. *At any point of a conic the chord of curvature in any direction is to the chord of the conic in the same direction as the focal chord parallel to the tangent is to the focal chord parallel to the chord of curvature.*

Let a circle meet a conic in three adjacent points QPQ', and let a chord PU of the circle meet QQ' in V and the conic

* The square of the velocity V at any point P of an orbit described under the action of a central force F being measured by

$$2F \times \tfrac{1}{4} \text{ (chord of curvature through centre of force)},$$

a conic so described about S will be a parabola, an ellipse or a hyperbola according as V^2 is equal to or less or greater than $2F \, . \, SP$. See Newton's *Principia Lib.* I. prop. 6 (sect. 2), and prop. 17 (sect. 3).

again in P'. Then it may be shewn by the method of Prop. I. that VU is to VP' as the focal chord parallel to PP' is to the focal chord parallel to QQ'. The required result follows by making the points Q and Q' coalesce with P, in which case the circle evidently becomes the circle of curvature as defined in Art. 72.

Thus the chord of curvature in any direction is determined. For example the central chord of curvature is at once seen to be equal to

$$\frac{CD^2}{CP^2} . 2CP, \text{ or } \frac{2CD^2}{CP}.$$

75. *A construction to determine the centre of curvature at any point of a conic.*

From Art. 73, Cor. 2, it readily follows that at any point of a conic,

$$\text{radius of curvature} = \frac{(\text{normal})^3}{(\frac{1}{2}\text{ lat. rect.})^2}.$$

Hence the following construction for the centre of curvature at P.

In the figure of Art. 12 draw GU at right angles to PG to meet SP, and UO at right angles to SP to meet PG. Then

$$\frac{PO}{PU} \cdot \frac{PU}{PG} = \frac{PG^2}{PK^2},$$

or
$$PO = \frac{PG^3}{PK^2},$$

and therefore PO is the radius and O the centre of curvature at P.

In this construction PU is the semi-chord of curvature through S, and is equal to $\dfrac{PG^2}{PK}$.

PROBLEMS.

The General Conic.

1. The distance of any point outside a conic from the focus is to its perpendicular distance from the directrix in a ratio greater than the eccentricity, and conversely.

2. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the ellipse lies wholly within the parabola, and the parabola wholly within the hyperbola.

3. Conics having the same focus and directrix do not meet.

4. If SL be the semi-latus rectum, and SD be drawn parallel to PR (Art. 4) to meet the directrix, then

$$SP : PR = SL : SD.$$

5. The segments of any focal chord subtend equal (or supplementary) angles at the foot of the directrix.

6. Determine the pole of the latus rectum and the polar of the focus. [Art. 69.

7. If chords PR, QR be produced to meet the directrix in p, q, the angle between the focal radii to p, q will be equal (or supplementary) to half the angle between the focal radii to P, Q.

8. If two tangents TP, TQ meet any third tangent in p, q, the angle between the focal radii to p, q will be equal (or supplementary) to half the angle between the focal radii to P, Q.

9. The lines joining the extremities of any two focal chords of a conic intersect on the directrix, and the focal distances of their intersections are at right angles.

10. If the directrix be removed to an infinite distance from the focus, as in Def. 1, the conic becomes a circle. Find what properties of the circle are contained in the theorem of Art. 8 and in Probs. 7, 8, 9.

11. If the focus and two points of a conic be given, the directrix must pass through one of two fixed points.

12. Having given two points on a conic and its focus and eccentricity, shew how to describe the curve.

13. Having given two points on a conic and its directrix and eccentricity, shew how to describe the curve.

14. Having given the focus and three points of a conic, shew how to describe the curve.

15. If PN be the principal ordinate of any point P of a conic, then

$$SP \pm SL : SN = \text{the eccentricity},$$

where SL is the semi-latus rectum.

16. Determine the condition that a chord of a conic may be greater than, equal to, or less than the diameter of the eccentric circle of its middle point.

17. If the point p (Art. 6) describe a series of circles about the same centre O, the point P will describe a series of conics having the same focus and directrix; and the eccentricities of the conics will be to one another as the radii of the circles.

18. If pm be drawn perpendicular to the directrix (Art. 6), then $pm \,.\, PM = OD \,.\, SX$. Hence shew that every focal chord is divided harmonically by the focus and directrix.

19. At any point of a conic the tangent makes a greater or less angle with the focal distance than with the perpendicular to the directrix according as the eccentricity is a ratio of minority or of majority.

20. If two conics have a common focus, their common chord or chords will pass through the point of concourse of their directrices.

21. If they also touch each other and the tangent at the point of contact be drawn, and if from any point on this common tangent second tangents be drawn to the conics, the line joining their points of contact will pass through the focus.

22. If the tangent at any point of a conic meet the directrix in D, and the latus rectum in L, then

$$SL : SD = SA : AX.$$

23. The tangents at the ends of a focal chord meet the latus rectum at points equidistant from the focus.

24. The focal distance of any point on a conic is equal to the ordinate of the point produced to meet the tangent at an extremity of the latus rectum.

25. If a chord of a conic subtend a constant angle at the focus, its envelope and the locus of its pole are conics having the same focus and directrix, and the eccentricities of the three are proportionals. [Art. 8.

26. The vertex of a circumscribed triangle whose base subtends a constant angle at the focus, lies on a conic having the same focus and directrix. [Prob. 8.

27. If TP be a tangent from T to a conic, and if the ordinate of T meet the curve in Q, the projection of ST upon SP is equal to SQ.

28. Shew that PN and QM in Art. 15 meet on the axis.

29. Prove that the line PR in Art. 8 is a tangent according to Euclid's definition. Prove also that Prop. II. may be extended to chords as follows:—If O be any point on a chord PQ, whose pole is T, and M be the projection of O upon the directrix, and if the perpendicular from O to ST meet SP, or SQ, in L; then

$$SL : OM = SA : AX.$$

30. Given the focus of a conic and a focal chord, the locus of the extremities of the latus rectum is a circle.

31. Given a chord of a conic and the angle which it subtends at the focus, the focal distance of its pole passes through a fixed point.

32. Any two tangents to a conic intercept upon a tangent drawn parallel to their chord of contact a length which is bisected at the curve.

33. The portion of any tangent to a conic intercepted between the tangents at the ends of the parallel focal chord subtends a right angle at the focus, and is divided by its point of contact into segments equal to the distance of that point from the focus.

34. If M be the projection upon the directrix of any point P on a conic, shew that SM meets the tangent at the vertex upon the bisector of the angle SPM.

35. Two sides of a triangle being given in position, if the third subtend a constant angle at a fixed point determine its envelope. [Prob. 26.

36. If a fixed straight line intersect a series of conics which have the same focus and directrix, the envelope of the tangents to the conics at the points of section will be a conic, having the same focus, and touching both the fixed straight line and the directrix of the series of conics.

37. If SY be the focal perpendicular on the tangent at P, and X the foot of the directrix, then

$$SY : YX = SA : AX.$$

Determine the locus of Y, and shew that it is the envelope of the circle on SP as diameter.

38. Shew also that XY meets the latus rectum at the foot of the perpendicular to it from P.

39. If the diameter at a point P on a conic bisect the chord normal at Q, the diameter at Q bisects the chord normal at P.

40. The normal PG (Art. 11) becomes equal to the semi-latus rectum when P coalesces with the vertex.

41. The perpendicular from G to SP varies as the ordinate of P; and the line joining the foot of this perpendicular to the foot of the ordinate of P is parallel to SM.

42. If Q be any point on the normal at P, and if QL be a perpendicular to SP, and QM a perpendicular to the ordinate of P, then

$$QL : PM = SA : AX.$$

43. The perpendicular to any focal chord from the intersection of the normals at its extremities meets the chord at a distance from one extremity equal to the focal distance of the other; and the parallel to the axis from the intersection of the normals bisects the chord.

44. If P be the pole of any normal chord of a conic, and Q the point in which it meets the directrix, the circle SPQ passes through an extremity of the chord.

45. A circle which touches a conic and has its centre upon the axis intercepts a constant length upon the focal radius to either point of contact.

46. If QQ' be the focal chord at right angles to the normal PG, then

$$PG^2 = SQ \,.\, SQ'. \qquad \text{[Prob. 33, Art. 15.}$$

47. Construct a conic of which an arc is given.

48. The parallel diameters of two similarly situated central conics of the same eccentricity bisect the same systems of parallel chords. If the two conics be concentric ellipses or hyperbolas (*or equal parabolas* whose axes are coincident), shew that any

chord of the exterior conic is divided into pairs of equal segments by the interior, and that any chord of the former which touches the latter is bisected at the point of contact.

49. The angle between any two chords of a conic is equal to the angle subtended at the focus by the portion of the directrix intercepted by the diameters which bisect the chords.

50. The polar of any point with respect to a conic meets the directrix on the diameter which bisects the focal chord through that point.

51. Any chord of a conic and the diameter which bisects it meet the axis and the directrix respectively upon a line parallel to the focal distance of the pole of the chord. Hence shew that in a central conic the polar of any point meets the axis at a distance from the centre which varies inversely as the central abscissa of the point.

52. The triangle whose vertices are the focus of a conic and the intersections of the tangent and the diameter at any point with the axis and the directrix respectively has its orthocentre at the point in which the tangent meets the directrix.

53. Given the focus and directrix of a conic, shew that the polar of a given point with respect to it passes through a fixed point.

54. Deduce from Art. 16 that the square of the ordinate at any point of a conic varies either as the distance of the foot of the ordinate from the vertex, or as the rectangle contained by the segments into which it divides the axis.

55. Any focal chord of a conic and the diameter which bisects it meet the directrix (or any fixed straight line perpendicular to the axis) at points whose ordinates contain a constant rectangle. Hence find the locus of the middle point of a focal chord. Determine also the locus of the foot of the perpendicular in Prob. 43.

56. Find the locus of the middle point of a chord of a conic which passes through a fixed point in the axis.

57. Deduce from Art. 10 that a line drawn from any point in the directrix of a conic is cut harmonically by the point, the curve, and the polar of the point.

58. If OTO' touch a conic in T, and if OPQ, $O'P'Q'$ be a pair of parallel chords, then

$$OT^2 : O'T^2 = OP \, . \, OQ : O'P' \, . \, O'Q'.$$

59. Hence shew that a line drawn through *any* point is divided harmonically by the point, the curve, and the polar of the point. [Art. 13, Cor. 2.

60. If a circle be drawn touching a conic at P and cutting it at Q and R, then will QR and the tangent at P be equally inclined to the axis. Shew how to draw a circle touching a conic at any given point (other than a vertex) and also cutting it at the same point.

The Parabola.

61. If the ordinates or focal distances of all points on a parabola be cut in a given ratio the locus of the points of section will in either case be a parabola.

62. The perpendicular from P to a chord AP meets the axis at a distance equal to the latus rectum from the foot of the ordinate of P.

63. Circles whose radii are in arithmetical progression touch a given straight line on the same side at a given point. If to each circle a tangent parallel to the given line be drawn, it will cut the circle next larger in points lying on a parabola.

64. Prove the following construction. Take any ordinate NP, and draw PM equal and parallel to NA. Divide NP into any number of equal parts, and through the points of section draw parallels $p_1, p_2, p_3, \ldots$ to the axis. Divide MP into the same number of equal parts in points 1, 2, 3,... Then the lines $p_1, p_2, p_3 \ldots$ meet $A1$, $A2$, $A3$,... respectively on the parabola.

65. Deduce from Art. 19 that the middle points of all parallel chords of a parabola are at the same distance from the axis.

66. A point on a parabola being given, if the focus also be given the envelope of the directrix will be a circle; or if the directrix be given the locus of the focus will be a circle.

67. The directrix and one point being given, prove that the parabola will touch a fixed parabola to which the given straight line is tangent at the vertex.

68. Given the directrix of a parabola and two points on the curve, two positions of the focus can be determined; or given the *focus and two* points, two positions of the directrix can be determined.

69. If two parabolas have a common focus, their common chord passes through the intersection of their directrices and bisects the angle between them.

70. If two parabolas have a common directrix, their common chord bisects the straight line joining their foci at right angles.

71. Find the locus of the centre of a circle which passes through a given point and touches a given straight line; or which touches a given circle and a given straight line.

72. Find the locus of a point which moves so that its shortest distance from a given circle is equal to its perpendicular distance from a given diameter of that circle.

73. Determine the position of P so that the triangle SPG (Art. 24) may be equilateral.

74. If an equilateral triangle circumscribe a parabola, the focal radii to its vertices pass each through the opposite point of contact. [Art. 27.

75. A focal chord being drawn to meet the tangent at a constant angle, determine the locus of their intersection.

76. The circle on a chord of a parabola as diameter does not meet the directrix unless the chord passes through the focus.

77. The circle described on any focal chord of a parabola as diameter touches the directrix; and the circle on any focal radius touches the tangent at the vertex.

78. Circles being described on the segments of a focal chord as diameters, the straight line joining their centres subtends right angles at the intersections of their common tangents.

79. Prove also that the squares of their common tangents vary as the length of the chord.

80. A point within a parabola is nearer to the focus than to the directrix.

81. If P be any point on a parabola whose focus is S, and PM be perpendicular to the directrix, prove that the line bisecting the angle SPM is the tangent at P, according to Euclid's definition of a tangent.

82. In Art. 20 shew that the angle QOQ' is equal to the angle MYM'. In what case are these angles right angles?

83. Prove the following construction for drawing tangents to a parabola from an external point T. With centre T and radius TS describe a circle cutting the directrix in M and N, then the diameters through these points meet the curve in the points of contact of the tangents.

84. Shew that all parabolas are similar curves.

85. A parabola being given find its axis and focus.

86. Shew how to place in a parabola a focal chord of given length.

87. The perpendicular to a chord of a parabola from its middle point V meets the axis at a distance equal to SX from the foot of the ordinate of V.

88. Shew that the locus of the middle point of a focal chord, or of any chord which meets the axis in a fixed point, is another parabola.

89. If PQ be a focal chord of a parabola, $SA \, . \, PQ = SP \, . \, SQ$.

90. The semi-latus rectum is a mean proportional between the principal ordinates of the ends of a focal chord. And if AM, AM' be the corresponding abscissae, then $AM \, . \, AM' = AS^2$.

91. If PQ be a focal chord, AP, AQ meet the latus rectum at distances from S equal to the ordinates of Q and P.

92. If a leaf of a book be folded so that one corner moves along an opposite side the direction of the crease touches a parabola.

93. The locus of the vertex of a parabola which has a given focus and a given tangent is a circle.

94. A triangle revolves about its vertex in one plane: prove that at any instant the directions of motion of all the points of its base are tangents to a parabola.

95. The diameters through the extremities of any focal chord of a parabola meet the chords joining them to the vertex upon the directrix and intercept upon it a length which subtends a right angle at the focus.

96. In Art. 22 shew that $QD^2 = 4AS \, . \, PV$.

97. From a point O on the directrix of a parabola are drawn two tangents, and through the focus S two straight lines parallel

to these tangents; shew that the part of the directrix intercepted between these parallels is bisected in O.

98. A circle can be described touching any two diameters of a parabola and the focal radii to their extremities.

99. A chord QQ' is cut in O by a diameter which meets the curve in P. Shew that if R be a point on the curve whose abscissa is PO, and PV, PV' be the abscissae of Q, Q', then

$$QV^2 - Q'V'^2 : QV^2 - OR^2 = QV + Q'V' : QV.$$

Deduce that $QV \cdot Q'V' = OR^2$ and $PV \cdot PV' = PO^2$.

100. Any triangle whose base is parallel to the axis of a parabola has its remaining sides in the subduplicate ratio of the parallel focal chords. [Art. 30.

101. If PQ be a chord which subtends a right angle at A, and AN, AM be the principal abscissae of P, Q, then PQ passes through the fixed point K in the axis, where $AK = 4AS$, and

$$AN \cdot AM = PN \cdot QM = 16AS^2.$$

102. A chord PQ of a parabola is a normal at P and subtends a right angle at the vertex: prove that SQ is three times SP.

103. If a circle cut a parabola in points 1, 2, 3 above the axis and in a point 4 below it, the difference of the ordinates of 1, 3 is to the difference of their abscissae as the sum of the ordinates of 2, 4 is to the difference of their abscissae. Deduce that the ordinate of 4 is equal to the sum of the ordinates of 1, 2, 3.

Examine the cases in which (1) 1, 2 coalesce, (2) 1, 2, 3 coalesce.

104. If 1, 2 and 3, 4 lie on opposite sides of the axis the sum of the ordinates of 1, 2 is equal to the sum of the ordinates of 3, 4.

105. If a circle and a parabola touch in one point and intersect in two others, the diameters of the parabola at the latter points will meet the circle again on a parallel to the tangent at the former.

106. If R be a point on a parabola and RS be produced to T so that $ST = SR$ and the tangents TP, TQ be drawn, prove that the circle TPQ touches the curve in R.

107. The tangents at P, Q meet in T, and O is the centre of the circle TPQ: prove that OT subtends a right angle at S and that the circle OPQ passes through S. [Art. 27.

108. Two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

109. Two given parabolas have the same focus and axis, and any line through the focus cuts them in P, Q, P', Q': shew that the tangents at these points form a rectangle one diagonal of which goes through the focus.

110. Shew that the locus of intersection of tangents which are at right angles to two parabolas which have a common focus and axis is a straight line perpendicular to the axis. Examine the case in which the directrices of the two parabolas coincide.

111. If from any point T on a fixed tangent a second tangent TP be drawn, the angle STP will be constant. Hence shew that if two fixed tangents be cut by any third in points p, q, the triangle Spq will have its angles constant.

112. PQR being a circumscribed triangle, the perpendiculars from P, Q, R to SP, SQ, SR are concurrent.

113. If one triangle can be inscribed in a given circle so that its three sides touch a given parabola, any number of triangles can be so described.

114. Deduce from Art. 29 that if the tangents at P, Q meet in T, the circle through P touching QT in T passes through the focus.

115. Two fixed straight lines intersect in O: prove that any circle through O and through another fixed point S meets the two fixed lines again in points such that the chord joining them touches a fixed parabola whose focus is S.

116. The locus of the centre of the circle circumscribing the triangle formed by two fixed tangents to a parabola and any other tangent is a straight line.

117. If two tangents to a parabola be cut by any third, their alternate segments have the same ratio, and this ratio is constant when the two tangents are fixed.

118. If the two tangents from any point on the axis of a parabola be cut by any third tangent, their alternate segments will be equal.

119. If T be the point of concourse of the tangents to a parabola at P and Q, and if p, q be the points in which any third tangent intersects them, then

$$\frac{Tp}{TP} + \frac{Tq}{TQ} = 1.$$

120. Shew that the envelope of a straight line which is cut in a constant ratio by three fixed straight lines is a parabola touching the three fixed lines. [Prob. 111.

121. The side PQ of a circumscribed triangle PQR meets the directrix in D, and RN drawn perpendicular to PQ meets SD in N; prove that N lies on the circle PQR. Deduce that if a parabola be inscribed in a triangle, its directrix passes through the orthocentre.

122. Shew that *one* parabola can in general be described touching four given straight lines.

123. Deduce from properties of the parabola the following geometrical theorems:—

(i) If from any point on the circumscribed circle of a triangle perpendiculars be let fall upon its three sides their feet will be collinear.

(ii) The circumscribed circles of the four triangles formed by any four straight lines meet in a point.

(iii) The orthocentres of the four triangles formed by any four straight lines are collinear.

124. Two fixed tangents are drawn to a parabola: prove that the locus of the centre of the nine points' circle of the triangle formed by these and any other tangent is a straight line. [Prob. 116.

125. If from the focus S of a parabola, SY, SZ be drawn perpendicular to the tangent and normal at any point, YZ will be parallel to the axis.

126. The normals at the ends of a focal chord intersect at right angles upon its diameter, and the locus of their intersection is a parabola.

127. The normal at any point is equal to twice the focal perpendicular upon the tangent, and is also a mean proportional between the focal distance of the point and the latus rectum.

128. Two circles whose centres are on the axis of a parabola touch the parabola and one another. Prove that the difference of their radii is equal to the latus rectum.

129. Two points are taken on a parabola such that the sum of the parts of the normals intercepted between the points and the axis is equal to the part of the axis intercepted between the normals: prove that the difference of the normals is equal to the latus rectum.

130. The perpendicular SY being drawn to any tangent, a straight line is drawn through Y parallel to the axis to meet in Q the straight line through S parallel to the tangent: prove that the locus of Q is a parabola.

131. The normal at any point is equal to the ordinate which bisects the subnormal at that point.

132. The perpendicular to a normal to a parabola at the point in which the normal meets the axis envelopes an equal parabola, and the focal vector of the point at which the normal is drawn meets the envelope at the point in which the perpendicular touches it.

133. The locus of the foot of the focal perpendicular on the normal is a parabola.

134. The squares of the normals at the ends of a focal chord are together equal to the square of twice the normal perpendicular to the chord. [Prob. 127.

135. The diameter through one end of a focal chord bisects the chord normal at the other.

136. A normal chord of a parabola produced to meet the directrix subtends a right angle at the pole of the chord; and the polar of the middle point of the chord meets the focal radius to its point of concourse with the directrix upon the normal at its further extremity.

137. If T be the pole of a chord PQ normal at P, and AN be the abscissa of P, shew that

$$PQ : PT = PN : AN.$$

138. Prove also that the straight line drawn from S at right angles to ST bisects QT.

139. If a parabola be made to roll upon an equal parabola, their vertices being initially coincident, the locus of the focus of the former will be the directrix of the latter.

140. The tangent at any point meets the directrix and the latus rectum in points equidistant from the focus.

141. The vertex of a constant angle whose sides envelope a *parabola* traces a hyperbola having the same focus and directrix. [Art. 8.

142. Tangents being drawn to a parabola from any point T, the diameters through their points of contact meet any secant PQ which passes through T in M and N: shew that

$$TM^2 = TN^2 = TP \,.\, TQ.$$

143. Given two chords of a parabola, find the direction of its axis, and shew that there are two solutions.

144. If a circle and a parabola touch and cut one another at the same point (Prob. 60), their common chord is equal to four times their common tangent at that point, terminated by the axis.

145. If R be any point on the tangent at P to a parabola and if the diameter through R meet the curve in Q, then will RP^2 vary as RQ.

146. A diameter meeting a chord and the tangent at an end of it is cut by the curve in the ratio in which it cuts the chord.

147. Draw a chord which shall be cut in a given ratio by a given diameter.

148. The intercepts on any diameter of a parabola between any two tangents and the ordinates of their points of contact to that diameter are equal; and the triangle contained by the two tangents and the diameter is equal to half the trapezium bounded by their chord of contact, the two ordinates and the diameter.

149. Three tangents to a parabola form a triangle equal to half the triangle determined by their points of contact.

150. The area of a parabolic segment is to a triangle of the same base and altitude as four to three.

The Central Conics.

151. In Art. 33 shew that $Z'Ap$ and ZpA' are straight lines.

152. The sides AD, DC of a rectangle $ABCD$ are divided into the same number of equal parts, and straight lines are drawn from B, A respectively to the points of section. Shew that corresponding lines in the two series meet on an ellipse whose axes are equal to the sides of the rectangle.

153. A parallelogram $ABCD$ has its diagonal AC at right angles to the side AB. If CD be divided into any number of equal parts, and straight lines be drawn from A to the points of

section, and if AC be divided into the same number of equal parts and straight lines be drawn from B to the points of section, then will corresponding lines in the two series meet on a hyperbola.

154. Given one focus of a central conic, a point on the curve, and the length of the axis; find the locus of the further focus, and the locus of the centre.

155. If two ellipses whose major axes are equal have a common focus, they will intersect in two points only; and their common chord will be at right angles to the straight line joining their centres. [Art. 5.

156. What is the locus of the centre of a circle which touches two fixed circles?

157. Given a central conic, find its centre and foci.

158. Shew that the sum (or difference) of the focal distances of any point without the conic is greater than the transverse axis, and conversely.

159. Draw a tangent to a conic parallel to a given line.

160. A conic is drawn touching an ellipse at the extremities A, B of the axes, and passing through the centre C of the ellipse; prove that the tangent at C is parallel to AB.

161. If the perpendicular from the centre on the tangent at P meet the focal distance SP produced in R, the locus of R is a circle whose diameter is equal to the transverse axis.

162. Given a focus of an ellipse, the length of the transverse axis, and that the second focus lies on a straight line, prove that the ellipse will touch two fixed parabolas having the given focus for focus.

163. The circle inscribed in the triangle SPS' touches SP in M, and SS' in N. Prove that $PM = AS$, and $AN = SP$.

164. From a point in the auxiliary circle straight lines are drawn touching the ellipse in P and P'; prove that SP is parallel to $S'P'$.

165. A diameter of an ellipse varies inversely as the perpendicular focal chord of the auxiliary circle. [Art. 46.

166. If SY, SZ be perpendiculars on two tangents which meet in T, the line through T perpendicular to YZ will pass through S'.

167. Given a focus S and two tangents, the locus of the second focus is a straight line.

168. If SY, SZ be drawn perpendicular respectively to the tangent and normal at any point, YZ will pass through the centre.

169. The ordinates to the axes at the points in which a common diameter meets the major and minor auxiliary circles of an ellipse intersect two and two on the ellipse.

170. A given point P in a given straight line AB which slides between two fixed straight lines at right angles traces an ellipse, whose semiaxes are equal to AP and BP. [Art. 35.

171. Deduce the theorems of Art. 43.

172. A circle can be drawn through the foci and the intersection of any tangent with the tangents at the vertices.

173. Any diameter is divided harmonically by a double ordinate and the point of concourse of the tangents at its extremities.

174. The exterior angle between two tangents to an ellipse is an arithmetic mean to the angles which the chord of contact subtends at the two foci. What is the corresponding theorem when the direction of the chord of contact falls between the foci?

175. The focal radii to the two ends of a diameter make equal angles with the tangents thereat.

176. The ordinate PN bisects the angle YNY' (Art. 39), and the points $YNCY'$ are concylic.

177. Also $SY^2 : CB^2 = SP : 2CA \pm SP$.

178. Also, if CD be the radius conjugate to CP,

$$SY : SP = CB : CD.$$

179. The normal at P is a harmonic mean to SY, $S'Y'$ (Art. 39), and is bisected by $S'Y$ and by SY'.

180. Tangents being drawn from any point on a circle through the foci, shew that the bisectors of the angles between them pass through fixed points. [Art. 40, Cor. 1.

181. If the tangent and normal meet either axis in T, G, then $CG \cdot CT = CS^2$.

182. If P be any point on a conic whose foci are S and S', the circles on SP, $S'P$ as diameter touch the auxiliary circle and have for their radical axis the ordinate of P.

183. The pole of the tangent at P with respect to the auxiliary circle lies on the ordinate of P.

184. A circumscribing parallelogram which has two corners on the directrices has the other two on the auxiliary circle. [Art. 7, Cor.

185. Prove that if one rectangle can be inscribed in a given circle so that its sides touch a given conic, any number of rectangles can be so described.

186. If an ellipse inscribed in a triangle has one focus at the orthocentre, the other focus will be at the centre of the circumscribing circle.

187. Prove also that the transverse axis of the ellipse is equal to the radius of the nine-points' circle of the triangle, and that the ellipse has double contact with the circle.

188. If an ellipse slide between two straight lines at right angles the locus of its centre is a circle.

189. The straight line joining the foci subtends at the pole of a chord half the sum or difference of the angles which it subtends at the extremities of the chord.

190. The portion of a normal chord intercepted between the directrices subtends at the pole of the chord half the sum of the angles which the straight line joining the foci subtends at the extremities of the chord. [Prob. 44.

191. The pole of any straight line with respect to a central conic may be found by joining the points in which it meets the directrices to the nearer foci, and drawing perpendiculars through the latter to the joining lines.

192. Every ellipse has one pair of equal conjugate diameters, and they coincide with the diagonals of the rectangle formed by the tangents at the extremities of the axes. Has the hyperbola any corresponding property?

193. If CP and CD be conjugate radii of an ellipse,

$$(SP - CA)^2 + (CA - SD)^2 = CS^2.$$

194. When is the sum or difference of conjugate diameters *greatest, and* when least?

195. The tangent from N to the circle on XX' (Art. 49) as diameter varies as the normal at P, and the tangent to the auxiliary circle varies as PN.

196. If N be a point in AA' produced the circles described about S, S' with radii AN, NA' meet on the hyperbola. What is the corresponding construction for the ellipse?

197. From a fixed point O, OP is drawn to a given circle. Find the envelope of a straight line through P inclined at a constant angle to OP.

198. In a central conic a circle through P and either G or g cuts off from the focal distances lengths whose sum is constant.

199. Given in an ellipse a focus and two points, the other focus describes a hyperbola.

200. TP, TQ are the tangents from T; prove that a circle can be described with T as centre so as to touch SP, HP, SQ, and HQ, or these lines produced. What does this become for the parabola?

201. If P, Q be points on a central conic, a confocal passes through the intersections of SP, $S'Q$ and SQ, $S'P$; and the tangents at these points and at P, Q cointersect.

202. If PP', DD' be conjugate diameters of a hyperbola and Q any point on the curve, shew that $QP^2 + QP'^2$ exceeds $QD^2 + QD'^2$ by a constant quantity.

203. Given two points of a parabola and the direction of its axis, the locus of the focus is a hyperbola.

204. A chord which subtends a right angle at the vertex meets the axis in a fixed point.

205. P being any point on an ellipse, the locus of the centre of the circle inscribed in the triangle SPS' is an ellipse. What is the locus of the centre of the circle touching the transverse axis of an ellipse, SP, and $S'P$ produced?

206. In a hyperbola the locus of the centre of the circle inscribed in the triangle SPS' is a straight line; and the locus of the centre of the circle touching the transverse axis, SP and $S'P$ produced, is a hyperbola.

207. If a hyperbola touches the sides of a quadrilateral inscribed in a circle and if one focus lies on the circle, the other lies on the circle.

208. The triangle whose base is equal to the transverse axis, and its remaining sides to the focal distances of any external point, has its vertical angle equal to the angle between the tangents to the conic from that point and its remaining angles to the angles which either tangent subtends at the foci.

[Arts. 39; 40, Cor. 1.

209. The projection of the normal at any point, terminated by the conjugate axis, upon either focal distance is equal to the semi-axis transverse.

210. The focal distances of g (Art. 42) meet the directrices upon the parallel to the axis through P.

211. If AM and $A'M$ be taken on the axis equal to the focal distances of any point P on an ellipse, then

$$CP^2 = CB^2 + CM^2.$$

Deduce the property of the principal ordinate.

212. If two ellipses having equal axes be placed vertex to vertex, and one of them roll upon the other, either of its foci will describe a circle about a focus of the latter.

213. The common diameters of equal, similar and concentric ellipses are at right angles. [Art. 14, Cor. 1.

214. The diagonals of any parallelogram circumscribing a conic are conjugate diameters; and the sides of any inscribed parallelogram are parallel to conjugate diameters.

215. The sum or difference of the reciprocals of the squares of any two diameters at right angles is constant.

216. The inscribed parallelogram whose diameters are at right angles envelopes a circle, the reciprocal of the square of whose radius is equal to $\frac{1}{CA^2} \pm \frac{1}{CB^2}$.

217. Determine the positions of a chord of an ellipse which subtends right angles at both foci. [Art. 42, Cor.

218. The opposite sides of a quadrilateral described about an *ellipse subtend* supplementary angles at either focus.

219. If a triangle ABC circumscribe a conic the sum of the angles subtended by BC at the foci will exceed the angle A by two right angles.

220. An ellipse touches the sides of a triangle; prove that if one of its foci move along the arc of a circle passing through two of the angular points of the triangle, the other will move along the arc of a circle passing through the same two angular points.

221. A circumscribed quadrilateral whose diagonals meet at the centre of the conic must be a parallelogram.

222. If P and Q be points on a conic, CM and CN their abscissae, and T the point in which PQ meets the axis, then

$$CT\,(PM - QN) = PM\,.\,CN - QN\,.\,CM\,^*.$$

223. If CP, CD and CP', CD' be conjugate radii, and if PN, DR be ordinates to CP', then

$$CN^2 \pm CR^2 = CP'^2\,; \quad PN^2 \pm DR^2 = CD'^2\,;$$

and
$$PN : CR = DR : CN = CD' : CP'.$$

224. The vertices of the conjugate parallelograms of an ellipse lie on a similar ellipse, and their polars envelope a similar ellipse. What are the corresponding properties of the hyperbola?

225. The parallelograms whose diagonals are any two diameters and their conjugates respectively are equal.

226. With the orthocentre of a triangle as centre two ellipses are described, the one touching its sides and the other passing through its vertices: prove that they are similar, and that their homologous axes are at right angles. [Art. 46.

227. If two ellipses having equal major axes be inscribed in a parallelogram, their foci determine an equiangular parallelogram.

228. Any circle through the focus S and the further vertex A' of a hyperbola whose eccentricity is two, meets the curve in three points P, Q, R which determine an equilateral triangle; and conversely the circumscribing circle of any equilateral triangle inscribed in a hyperbola whose eccentricity is two, passes through a focus and the further vertex.

* Equate the areas $(CPT - CQT)$ and $(CPM + PMNQ - QCN)$.

8—2

229. Any one of a series of conterminous circular arcs may be trisected by drawing a pair of hyperbolas whose eccentricity is two, and whose centres and vertices trisect the chord of the arc. How does it appear from this construction that the problem, to trisect a given angle, admits of three solutions?

230. Prove that in Prob. 228 the focal radii SP, SQ, SR meet the curve in three other points which determine an equilateral triangle; and shew that the triangle PQR envelopes a fixed parabola having S and the S-directrix for focus and directrix.

231. Draw a pair of conjugate diameters inclined at a given angle; and thence determine the axes and foci.

232. If two points E and E' be taken in the normal PG to an ellipse such that $PE = PE' = CD$, where CD is the radius conjugate to CP, the loci of E and E' are circles, whose diameters are equal to the sum and difference of the axes of the ellipse.

233. Prove also that the axes bisect the angles between the lines CE, CE'. Deduce a construction for determining the axes of an ellipse when two conjugate diameters are given.

234. For a hyperbola, the loci of E and E' are hyperbolas having their axes equal to the sum and difference of the axes of the given hyperbola.

235. The tangent at P meets any two conjugate diameters in T, t, and TS, tH meet in Q; prove that the triangles SPT, HPt, TQt are similar, and also that the area of the triangle CPT varies inversely as that of CPt.

236. If the tangent at Q (Fig. Art. 41) meet two parallel tangents in R and R', then will the radius parallel to the tangent be a mean proportional to QR and QR'. Shew also that the radius parallel to RP is a mean proportional to PR, $P'R'$. [Art. 47.

237. The common tangents to an ellipse and to a circle through the foci touch the circle in points lying on the tangents at the ends of the minor axis.

238. If any two points P, Q be given on a conic, prove that a third point R may be found so that the angle PRQ is a *maximum* by the following construction.

Draw a tangent parallel to PQ, touching the ellipse in K, and draw KR perpendicular to the major axis, cutting the curve again in R*.

239. The two points on a central conic at which any chord subtends the greatest and least angles are at the ends of a diameter equal to that which bisects the chord.

240. If two chords be drawn from any point of a conic equally inclined to the normal at that point, the tangents at their further extremities will intersect upon the normal †.

241. Supplemental chords of a conic which are equally inclined to the curve at their common point have their poles upon the orthocycle, and their sum or difference is equal to the diameter of the same ‡.

242. A bifocal conic being defined as the locus of a point P the sum or difference of whose distances from two fixed points S, S' is constant, prove, by taking Euclid's definition, that the tangent is the bisector (external or internal) of the angle SPS'; and prove also the property of the directrices and the property of the principal ordinate §.

243. Prove that two confocal conics of the same species do not meet ||.

244. If from any point of a conic tangents are drawn to a confocal conic, these tangents are equally inclined to the normal at the point. [Art. 40, Cor. 1.

245. The bisectors of the angles between the tangents from any point are tangent and normal to the confocals through that point. Prove that confocal conics cut at right angles.

* The chord PQ must subtend *equal* angles at R, and a consecutive point on the curve. Hence R lies on one of the segments of circles described upon the chord so as to touch the conic.

† If PQ, PQ' be two such chords and the normal PP' and the tangent at P meet QQ' in K, T respectively, $Q'KQT$ is a harmonic range. Then see Prob. 59.

‡ Here the normal PP' is an ordinate of QQ', and is parallel to the tangent at Q.

§ If P' be any point on the bisector of the angle SPS' then, for the ellipse, $SP'+S'P'>SP+S'P$, and similarly for the hyperbola. Also if in the ellipse $SP+S'P=2CA$ and CN be the abscissa, it may be shewn that

$$SP^2 \sim S'P^2=4CS \,.\, CN \text{ and } SP \sim S'P=2e \,.\, CN,$$

where e stands for the ratio $CS : CA$. It may also be shewn (Lemma D) that

$$(SP+S'P)^2+(SP\sim S'P)^2=4\,(CS^2+CN^2+PN^2).$$

Similar remarks apply to the hyperbola.

|| Conics which have the same foci are called confocal conics.

246. Draw figures illustrating a system of confocal conics, shewing that special cases are,—a circle, a straight line perpendicular to the major axis, the line joining the foci, and its complement *.

To what does the theorem that confocal conics cut at right angles reduce when the two foci coalesce?

247. Shew that if the sides of a rectangle touch two confocal conics, its vertex lies on a fixed concentric circle.

248. Shew that for any two confocal conics, the difference of the squares of the distances from the centre of parallel tangents is constant.

249. If a circle be drawn through the foci of two confocal conics, cutting them in P, Q, the tangents at P, Q will intersect on the circumference of the circle.

250. If a chord of a central conic be produced to meet the directrices, the parts produced will subtend equal angles at the pole of the chord.

The Asymptotes.

251. Given the asymptotes of a hyperbola and a point on the curve, determine the foci and directrices.

252. The perpendicular from a focus to an asymptote meets it upon the corresponding directrix, and the point of intersection lies on the auxiliary circle. Shew also that the point of contact of the tangent from such a point of intersection lies on a focal radius parallel to the asymptote.

253. The asymptotes of a hyperbola may be regarded as tangents whose points of contact are at infinity. [Art. 7.

254. At any point P of a hyperbola SP is equal to a line drawn parallel to an asymptote to meet the directrix.

255. If the tangent at any point P cut an asymptote in T, and SP cut the same asymptote in Q, then $SQ = QT$.

256. The line joining a pair of adjacent extremities of any two conjugate diameters is parallel to one asymptote and is bisected by the other.

* *The remainder* of an unlimited straight line from which any part has *been taken away* is called the *complement* of that part.

257. The asymptotes and any two conjugate diameters divide any straight line harmonically*.

258. Supposing the axes of a hyperbola to vanish whilst its eccentricity remains unaltered, determine the limiting form of the curve†.

259. Given a pair of conjugate diameters, two hyperbolas can be described: these have the same asymptotes, and every two diameters conjugate in the one are conjugate in the other‡.

260. If two conjugate diameters of a hyperbola be equal, every two conjugate diameters must be equal and the asymptotes must be at right angles.

261. If two hyperbolas have the same asymptotes a chord of one touching the other is bisected at the point of contact.

262. If tangents be drawn to a hyperbola from any point on the conjugate hyperbola, their chord of contact will touch the opposite branch of the latter and be bisected at its point of contact.

263. The tangent to a hyperbola at P meets an asymptote in T and TQ is drawn parallel to the other to meet the curve in Q; prove that if PQ meet the asymptotes in L and M, the line LM will be trisected at P and Q.

264. A hyperbola can be drawn through the ends of any two radii of an ellipse so as to have the conjugate diameters as asymptotes.

265. If through two points R, R' of a hyperbola lines be drawn parallel to the asymptotes forming the parallelogram $RTR'T'$, shew that TT' goes through the centre C, and that $CT \cdot CT' = CP^2$, where P is a point where CTT' cuts the curve.

266. If the abscissae on either asymptote of any number of points on a hyperbola are in arithmetical progression, their ordinates are in harmonical progression.

267. The straight lines joining the points in which any two tangents to a hyperbola meet the asymptotes are parallel; and the intercepts which the tangents make upon the asymptotes are bisected by their chord of contact.

* Prove that they divide the tangent at a vertex harmonically. The theorem is really contained in Prob. 256. See Lemma G.

† Here the foci and directrices coalesce into the centre and the line along which the minor axis was measured.

‡ Two hyperbolas thus related are said to be conjugate.

268. Find the locus of a point which divides the part of any tangent intercepted between the asymptotes in a constant ratio.

269. Find the locus of the centroid of a triangle of constant area contained by one variable and two fixed straight lines.

270. If the ordinate at P to either axis meets the nearer asymptote in E, the perpendicular through E to the asymptote passes through the point in which the normal at P meets that axis.

271. The four normals to a hyperbola and its conjugate at points lying upon a perpendicular to either axis meet one another upon that axis.

272. Any tangent and its normal meet the asymptotes and the axes respectively in four points which lie on a circle passing through the centre of the curve; and the radius of this circle varies inversely as the perpendicular from the centre on the tangent.

273. The intercept on any tangent between the asymptotes subtends at the further focus an angle equal to half the angle between them: it also subtends a constant angle at the intersection of its normal with either axis.

274. The chords of intersection of any circle with the asymptotes are equally inclined to the axis. Shew that this agrees with Prob. 258.

275. The products of the segments of any two intersecting chords of the asymptotes are as the parallel focal chords of the hyperbola.

276. Any circle which touches both branches makes an intercept equal to the transverse axis on either asymptote.

277. The lines joining a variable point on a hyperbola to two fixed points on it intercept a constant length on either asymptote.

278. The axes of the two parabolas which have a common focus and pass through two given points are parallel to the asymptotes of the hyperbola which passes through their focus and has the given points for foci. [Prob. 68.

279. If an ellipse and a confocal hyperbola intersect in P, an asymptote passes through the point on the auxiliary circle of the ellipse which corresponds to P.

280. The area of the hyperbolic sector determined by any two radii CP, CQ is equal to the area cut off from the space between the asymptotes by the parallels from P and Q to either asymptote.

The Equilateral Hyperbola.

281. The subnormal at any point of an equilateral hyperbola is equal to the central abscissa; the tangent from the foot of the ordinate to the auxiliary circle is equal to the ordinate, and the projection of the normal (terminated by either axis) upon either focal vector is equal to the semi-axis.

282. The centre of an equilateral hyperbola circumscribing an equilateral triangle lies on the inscribed circle of the triangle, and the centre of the circle lies on the hyperbola.

283. The locus of the middle point of a straight line which cuts off a constant area from a corner of a square is an arc of a rectangular hyperbola.

284. CY being drawn perpendicular to the tangent at a point P of an equilateral hyperbola, the angle PCY is bisected by the transverse axis, and the triangles PCA, CAY are similar.

285. If CP, CD be conjugate radii of a rectangular hyperbola, then will D be the reflexion of P with respect to one of the asymptotes.

286. Any two conjugate semi-diameters contain equal and similar triangles with the ordinates and abscissae of their extremities to any other diameter.

287. The ends of the equal conjugate diameters of a series of confocal ellipses lie on the confocal equilateral hyperbola.

288. If PQ and $P'Q$ be any pair of supplemental chords of a rectangular hyperbola, the bisectors of the angle PQP' are parallel to the asymptotes; and if the tangent at Q and its ordinate to PP' meet that diameter in T and V, then CP and TP' subtend equal angles at Q, and the circle round CQT touches QV.

289. Diameters at right angles bisect chords at right angles; and any chord subtends equal or supplementary angles at the ends of a perpendicular chord.

290. Of two chords at right angles or conjugate in direction, one and one only is a chord of a single branch.

291. On opposite sides of a chord of a rectangular hyperbola equal segments of circles are described. Shew that the four points in which the completed circles meet the curve again are the vertices of a parallelogram.

292. Tangents to a parabola which include the supplement of half a right angle intersect on an equilateral hyperbola. [Art. 8.

293. Every right-angled triangle inscribed in an equilateral hyperbola has its hypotenuse parallel to the normal at the opposite angle. Hence shew how to draw a tangent at any given point on the curve. [Art. 56.

294. The base of a triangle and the sum or difference of its base angles being given, the locus of its vertex is an equilateral conic. Determine the asymptotes of the hyperbola by supposing the vertex of the triangle to be infinitely distant. [Art. 54.

295. The chords connecting the ends of a fixed diameter of a circle and of any double ordinate of the same intersect upon an equilateral hyperbola.

296. If on an arc AB of a circle whose centre is O there be taken two points P, Q such that arc $AP=2$ arc BQ, then a rectangular hyperbola described on AO as diameter so as to pass through the intersection of OB with the tangent to the circle at A will also pass through the intersection of AP, OQ.

297. Shew also that the hyperbola and the completed circle intersect in three points (other than A) which determine an equilateral triangle; and deduce that the problem, to trisect a given angle, admits of three solutions.

298. The opposite arcs cut off by any two diameters subtend equal angles at any point on the curve.

299. If two concentric rectangular hyperbolas are such that the axes of one are the asymptotes of the other, they cut each other at right angles, and any common tangent subtends a right angle at the centre.

300. A circle through the centre and two points of a rectangular hyperbola passes also through the intersection of the lines drawn from each of the two points parallel to the polar of the other.

301. Ellipses being inscribed in a parallelogram, their foci lie on an equilateral hyperbola. [Prob. 294.

302. A conic through the centres of the four circles which touch the sides of a triangle is a rectangular hyperbola, and its centre is on the circumscribing circle. [Art. 56.

303. A conic through the four common points of two rectangular hyperbolas is itself a rectangular hyperbola.

304. The tangents to an equilateral hyperbola at the vertices of an inscribed triangle meet two and two on the lines joining the feet of the perpendiculars of the triangle. [Prob. 59.

305. If each vertex of a triangle be the pole of the opposite side with respect to an equilateral hyperbola, the circumscribing circle will pass through the centre of the hyperbola*.

306. A circle and an equilateral hyperbola intersect in four points: if one of their common chords is a diameter of the hyperbola, the other is a diameter of the circle, and the tangents to the circle at the ends of this diameter are ordinates of the diameter of the hyperbola.

307. The circles described upon the six common chords of any two equilateral hyperbolas as diameters cut one another orthogonally in opposite pairs.

308. The circles described on parallel chords of an equilateral hyperbola as diameters have a common radical axis.

309. A circle meets an equilateral hyperbola in four points O, P, Q, R and OO', PP', QQ', RR' are diameters of the hyperbola: prove that O' is the orthocentre of the triangle PQR, and similarly for the others.

310. Two circles touch the same branch of an equilateral hyperbola and touch each other in the centre: prove that the chord of the hyperbola joining the points of contact subtends at the centre an angle equal to the angle of an equilateral triangle.

* A triangle each of whose vertices is the pole of the opposite side with respect to a conic is called a *self-conjugate* or *self-polar triangle*.

The Cone.

311. In Art. 59 shew that $AS = A'H$, and $SH = OA \pm OA'$.

312. A conic section may be regarded as the locus of a point the sum or difference of whose distances from a point in its plane and a point without it is constant.

313. Express the eccentricity of a section of a cone in terms of the angles which the axis of the cone makes with its sides and with the axis of the section*.

314. The sections by identical planes of the cones touching two given spheres have their eccentricities in a constant ratio.

315. All sections of a right cone made by planes parallel to tangent planes of the cone are parabolas, and their foci lie on a cone having with the first a common vertex and axis.

316. If two or more plane sections have the same directrix, the corresponding foci lie on a straight line through the vertex of the cone.

317. The sphere having for diameter the line joining the centres of the focal spheres of a section contains its auxiliary circle.

318. The perpendiculars upon any tangent to a section from the centres of its focal spheres are at right angles to one another. Deduce that the product of the focal perpendiculars upon the tangent is constant.

319. If T be any point on the tangent at P to a section, the two tangents to it from T are inclined at the same angle as the tangents TQ, TR [Art. 59 (ii.)] to the spheres. Deduce the property of the orthocycle. [Art. 59. Cor.

320. The conjugate axis of a section is a mean proportional to the diameters of its focal spheres.

321. The latus rectum of any section whose plane touches a sphere about the vertex of the cone as centre is equal to the diameter of the circular sections whose planes touch the sphere.

* One form of the value of the eccentricity is $\cos\alpha \sec\beta$, where α and β are the inclinations of the axis of the cone to the axis of the section, and to a side of the cone respectively.

322. The sections of a cone by parallel planes are similar curves; and the asymptotes of the hyperbolic sections made by parallel planes are parallel to the sides of the cone which lie on the parallel plane through its vertex.

323. Shew how to cut a section of maximum eccentricity from a given cone.

324. A conic section may be regarded as the locus of a point the sum or difference of the tangents from which to two fixed circles is constant.

325. The vertex of a right circular cone which contains a given ellipse lies on a certain hyperbola, and its axis touches the hyperbola.

Projection.

326. Prove by the method of projection that tangents to an ellipse at the extremities of any chord intersect on the diameter which bisects the chord.

327. Deduce from a known property of the circle that the area of the conjugate parallelogram of an ellipse is constant.

328. TP, TQ are tangents to an ellipse, and CP', CQ' are the parallel radii; prove that PQ is parallel to $P'Q'$.

329. Any two similar and coaxal ellipses may be projected into concentric circles. Hence shew that a chord of an ellipse which always touches a similar and coaxal ellipse is bisected at its point of contact, and that it cuts off a constant area from the outer ellipse: and shew that the portions of any chord intercepted between the two curves are equal.

330. Find the locus of the point of intersection of the tangents at the extremities of pairs of conjugate diameters of an ellipse.

331. Find the locus of the middle point of the lines joining the extremities of conjugate diameters.

332. Any two radii of a circle and a pair of radii at right angles thereto determine equal triangles: what is the corresponding property of the ellipse?

333. The orthogonal projection of a parabola is a parabola; and an ellipse or hyperbola may be projected into an ellipse or hyperbola of any eccentricity.

334. If a chord of an ellipse and the tangents at its extremities contain a constant area, the chord cuts off a constant area from the ellipse and touches a similar ellipse, and the tangents at its extremities intersect on another similar ellipse.

335. A polygon described about an ellipse so as to have its sides bisected at their points of contact is of constant area, and the polygon formed by joining every two successive points of contact is of constant area.

336. Any double ordinate to a given diameter of an ellipse being divided into segments whose product is constant, the point of section traces a similar coaxal ellipse.

337. Prove that the greatest triangle which can be inscribed in an ellipse is that which has its sides parallel to the tangents at its angular points and its centroid at the centre of the ellipse.

338. The least triangle circumscribing a given ellipse has its sides bisected at the points of contact.

339. The greatest ellipse which can be inscribed in a given parallelogram is that which bisects its sides.

340. If a triangle be inscribed in an ellipse, the parallels through its vertices to the diameters bisecting the opposite sides meet in a point.

341. Parallel chords drawn to an ellipse through the extremities of conjugate diameters meet the curve again at the extremities of conjugate diameters.

342. Through the centre of an ellipse and the points of concourse and contact of any two tangents a similar and similarly situated ellipse can be drawn.

343. Through a given internal point draw a straight line cutting off a minimum area from a given ellipse.

344. If the tangent at the vertex A of an ellipse meets a similar coaxal ellipse in T and T', any chord of the former drawn from A is equal to half the sum or difference of the parallel chords of the latter through T and T'.

345. The tangents to an ellipse at P and P' are parallel, any two conjugate diameters meet them in D and D', and any third tangent meets them in T and T'; shew that

$$PD : PT = P'T' : P'D'.$$

346. A triangle ABC inscribed in an ellipse has its centroid at the centre of the ellipse; shew that the tangents at the opposite extremities of the diameters through A, B, C form a triangle similar to and four times as great as the triangle ABC.

347. If two conjugate hyperbolas having a pair of conjugate diameters of an ellipse for asymptotes cut the ellipse at points lying on four diameters 1, 2, 3, 4 taken in order: then will 1, 3 and 2, 4 be conjugate in the ellipse, and 1, 4 and 2, 3 in the hyperbolas.

348. The locus of the middle point of a chord of an ellipse drawn through a fixed point is a similar ellipse, having its centre midway between the fixed point and the centre of the given ellipse.

349. Given the directions of two sides of a triangle inscribed in a given ellipse, determine the envelope of its third side.

350. Prove that the perpendiculars from any point on a circle to a fixed chord and to the tangents at its extremities are continued proportionals. What is the corresponding property of the ellipse?

Curvature.

351. The radius of curvature at an extremity of the latus rectum of a parabola is equal to twice the normal.

352. The diameter at either extremity of the latus rectum of a parabola passes through the centre of curvature at its other extremity.

353. Determine the position of the common chord of a parabola and its circle of curvature at an extremity of the latus rectum.

354. The circle of curvature at a point P of a conic cuts off from the diameter through P a portion equal to the parameter of that diameter*.

355. If the tangent at any point P of a parabola meet the axis in T, and if the circle of curvature meet the curve in Q, then $PQ = 4PT$.

* The *parameter* of any diameter of a central conic is defined as a third proportional to that diameter and its conjugate.

356. At any point P of a parabola, if PY be the projection of SP upon the tangent, the chord of curvature through the vertex is a third proportional to AP and $2PY$.

357. If R be the middle point of the radius of curvature at P in a parabola, PR subtends a right angle at S.

358. The radius of curvature at any point of a parabola is double the portion of the normal intercepted between the curve and the directrix.

359. Shew that the centre of curvature may be regarded as the point of ultimate intersection of two consecutive normals to the conic.

360. If Q and Q' are points on a parabola on the same side of the axis and V the middle point of QQ', shew that the ordinate of the point of concourse of the normals at Q and Q' is to the ordinate of V as the product of the ordinates of Q and Q' to the square of the semi-latus rectum. Hence determine the ordinate of the centre of curvature at P and the length of the radius of curvature.

361. The tangent from any point of a parabola to the circle of curvature at its vertex is equal to the abscissa of the point.

362. The envelope of the common chords of a parabola and its circles of curvature is a parabola, and the locus of their middle points is a parabola.

363. If P, P', P'' be points on a parabola, P, P' on one side of the axis, and P'' on the other side, and the normals at P, P', P'' cointersect, prove that the sum of the ordinates of P and P' is equal to the ordinate of P''*.

364. If from the vertex of a parabola chords AR and AR' be drawn equally inclined to the axis, the normals at the extremities of any chord parallel to AR intersect upon the normal at R'; and the centre of curvature at the extremity of the diameter which bisects AR lies upon the normal at R'.

365. A circle through the vertex of a parabola cuts the curve in general in three other points, the normals at which cointersect. Prove also that the centroid of the triangle formed by the three points lies on the axis. [Prob. 103.

* If the normals cointersect at a point whose projection on the axis is Z, we get $PN \cdot ZG = P'N' \cdot ZG' = P''N'' \cdot ZG''$; whence, after some reduction, $PN + P'N' = P''N''$.

366. Prove that at the vertex A of a conic the radius of curvature is equal to $AS(1+e)$, where e is the eccentricity, and at the vertex B to $\frac{CA^2}{CB}$.

367. If the osculating circle at a vertex of an ellipse passes through the further focus, determine the eccentricity*.

368. The circle through the foci of an ellipse and the extremity B of its minor axis will cut the minor axis in the centre of curvature at B.

369. An ellipse, a parabola, and a hyperbola have the same vertex and the same focus: shew that the curvature at the vertex of the parabola is greater than that of the hyperbola and less than that of the ellipse†.

370. If P be a point of an ellipse equidistant from the minor axis and a directrix, the circle of curvature at P will pass through one of the foci.

371. The tangent at P in an ellipse meets the axes in T and t, and CP is produced to meet the circle TCt in L: prove that $2PL$ is equal to the central chord of curvature at P, and that $CL \cdot CP$ is constant.

372. The circle of curvature at an extremity of one of the equal conjugate diameters of an ellipse passes through its other extremity.

373. Find the points on a central conic at which the diameter of curvature is a mean proportional to the axes.

374. At any point P of a rectangular hyperbola, the radius of curvature varies as CP^3.

375. At any point of a rectangular hyperbola the diameter of the curve is equal to the central chord of curvature.

376. At any point P of a rectangular hyperbola if CP be produced to Q, so that $PQ = CP$, and QO be drawn perpendicular to CQ to meet the normal at P in O, then O is the centre of curvature at P.

377. At any point of a rectangular hyperbola the normal chord is equal to the diameter of curvature‡.

* The circle of curvature at a point P on a conic is the circle of closest contact with the conic at P, and is called its *Osculating Circle* at that point.

† *Curvature* is measured by the reciprocal of the radius of curvature.

‡ This may be proved from properties of the centroid, orthocentre and centre of circumscribing circle of a triangle by taking three near points on the curve ultimately coincident and using Art. 56.

378. At any point P of a rectangular hyperbola, if PN be perpendicular to an asymptote, the chord of curvature in the direction PN is equal to $\frac{CP^2}{PN}$.

379. From the point in which the tangent to an ellipse at P meets the major axis a straight line is drawn bisecting one of the focal distances and meeting the other in Q. Prove that PQ is one-fourth of the focal chord of curvature at P.

380. A hyperbola which touches an ellipse, and has a pair of its conjugate diameters for asymptotes has the same curvature as the ellipse at their points of contact.

381. At a point P of an ellipse the chord of curvature in the direction of the ordinate PM is to PM as $2CD^2$ is to BC^2.

382. In a central conic let the diameter CD parallel to the tangent at P meet PQ, the common chord of the ellipse and the circle of curvature at P, in K; then will $PQ.PK$ be equal to $2CD^2$, and the like is true for *any* chord of curvature PQ'.

383. The normal chord which divides an ellipse most unequally is a diameter of curvature, and is inclined at half a right angle to the axis*.

384. Prove that if two straight lines make *supplementary* angles with any third straight line, their projections make supplementary angles with the projection of that third line.

Hence, or otherwise, prove that if the circles of curvature at the extremities of two conjugate radii CP and CD of an ellipse meet the curve again in Q and R, PR is parallel to DQ.

385. Given a point O on a circle, three positions may be found on the curve of a point P such that OP and the tangent at P make supplementary angles with a given diameter, and the three positions of P determine an equilateral triangle.

Deduce by projection that there are three points on an ellipse, lying at the vertices of a maximum inscribed triangle, whose osculating circles cointersect at a given point on the ellipse.

Prove also that the normals at the three points cointersect, and that the four points lie on a circle†. [Prob. 337.

* The normal in two consecutive positions must cut off equal areas, and must be bisected at the centre of curvature.

† If ABC be a maximum inscribed triangle, and the osculating circle at A meet the curve again in A', the tangent at A is parallel to BC, which is *therefore equally* inclined to the axis with AA'.

Miscellaneous.

386. If PQ be a chord of a conic, and if the parallel focal chord F meet the tangent at P in T, then

$$PQ \, . \, ST = F \, . \, SP.$$

387. Given the focus of a conic inscribed in a triangle, find the points of contact.

388. If two chords AB, CD of a conic (not being parallel to one another) make equal angles with the axis, then will AC, BD and likewise AD, BC make equal angles with the axis.

389. If a chord of a conic subtends equal angles at the extremities of another chord, it subtends equal angles at the extremities of any chord parallel to the latter.

390. The tangent to a conic at a given point meets any two parallel tangents in points whose focal distances meet on a fixed circle, having its centre on the normal at the given point.

391. If two focal chords of a hyperbola be conjugate in direction, the lines joining their extremities meet on the asymptotes, and in the equilateral hyperbola pass through fixed points on the asymptotes.

392. One triangle being inscribed and another circumscribed to a parabola, if their sides be parallel each to each they will be in the ratio of four to one.

393. If a parabola be inscribed in a given triangle, each chord of contact passes through a fixed point which lies on the bisector of the corresponding side of the triangle.

394. If TP, TQ be tangents to an ellipse, and CP', CQ' the parallel radii, shew that the triangles TPQ and $CP'Q'$ are together equal to the trapezium $CPTQ$, and likewise to the triangle of Prob. 208.

Prove also that

$$TP \, . \, TQ + CP' \, . \, CQ' = TS \, . \, TH^*.$$

395. Shew from Art. 6 that if the directrix of a conic and the eccentric circle of any point have a point in common, the corresponding point on the conic is at an infinite distance.

Deduce that the hyperbola has two real points at infinity, the ellipse none, and the parabola two coincident points†.

* We have to shew that the triangle of Prob. 208 is equal to $PTQ + P'CQ'$; which follows from Prob. 225, taking into account that $PCQ = \frac{1}{2}(PSQ + PHQ)$.

† Or, in other words, the parabola touches the *line at infinity*, the hyperbola cuts it in real, and the ellipse in imaginary points.

396. Deduce from Art. 37 and Prob. 170 two methods of describing an ellipse mechanically.

397. If a straight rod $S'L$ be moveable in one plane about the end S', and a string LPS, fastened at L and another fixed point S, be stretched in contact with the rod by a pencil P, then the pencil will trace one branch of a hyperbola whose foci are S and S'. How may the other branch be traced? Deduce a method of describing a parabola mechanically.

398. If ABC be a triangle whose sides touch a conic at the points a, b, c, then

$$Ab \,.\, Bc \,.\, Ca = Ac \,.\, Ba \,.\, Cb.$$

What is the corresponding theorem when the conic cuts the sides of the triangle? [Art. 16.

399. A central conic which passes through four given points has a pair of conjugate diameters parallel to the axes of the two parabolas which can be drawn through the same four points*.

400. Prove that in general two parabolas and any number of central conics can be drawn through four given points; and that no two conics can intersect in more than four points.

* Let TP, TQ be tangents to an ellipse, and OAB, OCD chords parallel to them. Determine a diameter of each of the two parabolas through A, B, C, D (Prob. 143); then PQ and the diameter through T in the ellipse are parallel to the diameters of the parabola.

Many of the above Problems are taken from the larger work referred to in the Preface. A few references to it may be found useful.

Prob. 10: see p. 22, Scholium A.

Prob. 59: see p. 32, Art. 18.

Prob. 101 and Prob. 204. These are particular cases of a general theorem, for which see Art. 120, Cor. 2, or Art. 144.

Prob. 150. This is proved by the method of infinitesimals; see Art. 32.

Most of the foot-notes are taken from the same source.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. & SON, AT THE UNIVERSITY PRESS.

September 1883.

A CLASSIFIED LIST

OF

EDUCATIONAL WORKS

PUBLISHED BY

GEORGE BELL & SONS.

Full Catalogues will be sent post free on application.

BIBLIOTHECA CLASSICA.

A Series of Greek and Latin Authors, with English Notes, edited by eminent Scholars. 8vo.

Æschylus. By F. A. Paley, M.A. 18*s.*

Cicero's Orations. By G. Long, M.A. 4 vols. 16*s.*, 14*s.*, 16*s.*, 18*s.*

Demosthenes. By R. Whiston, M.A. 2 vols. 16*s.* each.

Euripides. By F. A. Paley, M.A. 3 vols. 16*s.* each.

Homer. By F. A. Paley, M.A. Vol. I. 12*s.*; Vol. II. 14*s.*

Herodotus. By Rev. J. W. Blakesley, B.D. 2 vols. 32*s.*

Hesiod. By F. A. Paley, M.A. 10*s.* 6*d.*

Horace. By Rev. A. J. Macleane, M.A. 18*s.*

Juvenal and Persius. By Rev. A. J. Macleane, M.A. 12*s.*

Plato. By W. H. Thompson, D.D. 2 vols. 7*s.* 6*d.* each.

Sophocles. Vol. I. By Rev. F. H. Blaydes, M.A. 18*s.*

——— Vol. II. Philoctetes. Electra. Ajax and Trachiniæ. By F. A. Paley, M.A. 12*s.*

Tacitus: The Annals. By the Rev. P. Frost. 15*s.*

Terence. By E. St. J. Parry, M.A. 18*s.*

Virgil. By J. Conington, M.A. 3 vols. 14*s.* each.

An Atlas of Classical Geography; Twenty-four Maps. By W. Hughes and George Long, M.A. New edition, with coloured Outlines. Imperial 8vo. 12*s.* 6*d.*

Uniform with above.

A Complete Latin Grammar. By J. W. Donaldson, D.D. 3rd Edition. 14*s.*

GRAMMAR-SCHOOL CLASSICS.

A Series of Greek and Latin Authors, with English Notes. Fcap. 8vo.

Cæsar: De Bello Gallico. By George Long, M.A. 5*s.* 6*d.*

——— Books I.–III. For Junior Classes. By G. Long, M.A. 2*s.* 6*d.*

——— Books IV. and V. in 1 vol. 1*s.* 6*d.*

Catullus, Tibullus, and Propertius. Selected Poems. With Life. By Rev. A. H. Wratislaw. 3*s.* 6*d.*

Cicero: De Senectute, De Amicitia, and Select Epistles. By George Long, M.A. 4*s*. 6*d*.

Cornelius Nepos. By Rev. J. F. Macmichael. 2*s*. 6*d*.

Homer: Iliad. Books I.–XII. By F. A. Paley, M.A. 6*s*. 6*d*.

Horace. With Life. By A. J. Macleane, M.A. 6*s*. 6*d*. [In 2 parts, 3*s*. 6*d*. each.]

Juvenal: Sixteen Satires. By H. Prior, M.A. 4*s*. 6*d*.

Martial: Select Epigrams. With Life. By F. A. Paley, M.A. 6*s*. 6*d*.

Ovid: the Fasti. By F. A. Paley, M.A. 5*s*.

Sallust: Catilina and Jugurtha. With Life. By G. Long, M.A. 5*s*.

Tacitus: Germania and Agricola. By Rev. P. Frost. 3*s*. 6*d*.

Virgil: Bucolics, Georgics, and Æneid, Books I.–IV. Abridged from Professor Conington's Edition. 5*s*. 6*d*.—Æneid, Books V.–XII. 5*s*. 6*d*. Also in 9 separate Volumes, 1*s*. 6*d*. each.

Xenophon: The Anabasis. With Life. By Rev. J. F. Macmichael. 5*s*. Also in 4 separate volumes, 1*s*. 6*d*. each.

——— The Cyropædia. By G. M. Gorham, M.A. 6*s*.

——— Memorabilia. By Percival Frost, M.A. 4*s*. 6*d*.

A Grammar-School Atlas of Classical Geography, containing Ten selected Maps. Imperial 8vo. 5*s*.

Uniform with the Series.

The New Testament, in Greek. With English Notes, &c. By Rev. J. F. Macmichael. 7*s*. 6*d*.

CAMBRIDGE GREEK AND LATIN TEXTS.

Æschylus. By F. A. Paley, M.A. 3*s*.

Cæsar: De Bello Gallico. By G. Long, M.A. 2*s*.

Cicero: De Senectute et de Amicitia, et Epistolæ Selectæ. By G. Long, M.A. 1*s*. 6*d*.

Ciceronis Orationes. Vol. I. (in Verrem.) By G. Long, M.A. 3*s*. 6*d*.

Euripides. By F. A. Paley, M.A. 3 vols. 3*s*. 6*d*. each.

Herodotus. By J. G. Blakesley, B.D. 2 vols. 7*s*.

Homeri Ilias. I.–XII. By F. A. Paley, M.A. 2*s*. 6*d*.

Horatius. By A. J. Macleane, M.A. 2*s*. 6*d*.

Juvenal et Persius. By A. J. Macleane, M.A. 1*s*. 6*d*.

Lucretius. By H. A. J. Munro, M.A. 2*s*. 6*d*.

Sallusti Crispi Catilina et Jugurtha. By G. Long, M.A. 1*s*. 6*d*.

Sophocles. By F. A. Paley, M.A. 3*s*. 6*d*.

Terenti Comœdiæ. By W. Wagner, Ph.D. 3*s*.

Thucydides. By J. G. Donaldson, D.D. 2 vols. 7*s*.

Virgilius. By J. Conington, M.A. 3*s*. 6*d*.

Xenophontis Expeditio Cyri. By J. F. Macmichael, B.A. 2*s*. 6*d*.

Novum Testamentum Græcum. By F. H. Scrivener, M.A. 4*s*. 6*d*. An edition with wide margin for notes, half bound, 12*s*.

CAMBRIDGE TEXTS WITH NOTES.

A Selection of the most usually read of the Greek and Latin Authors, Annotated for Schools. Fcap. 8vo. 1s. 6d. each, with exceptions.

Euripides. Alcestis.—Medea.—Hippolytus.—Hecuba.—Bacchæ. Ion. 2s.—Orestes.—Phoenissæ.—Troades.—Hercules Furens. By F. A. Paley, M.A.

Æschylus. Prometheus Vinctus.—Septem contra Thebas.—Agamemnon.—Persæ.—Eumenides. By F. A. Paley, M.A.

Sophocles. Œdipus Tyrannus.—Œdipus Coloneus.—Antigone. By F. A. Paley, M.A.

Homer. Iliad. Book I. By F. A. Paley, M.A. 1*s.*

Terence. Andria.—Hauton Timorumenos.—Phormio.—Adelphoe. By Professor Wagner, Ph.D.

Cicero. De Senectute, De Amicitia, and Epistolæ Selectæ. By G. Long, M.A.

Ovid. Selections. By A. J. Macleane, M.A.

Others in preparation.

PUBLIC SCHOOL SERIES.

A Series of Classical Texts, annotated by well-known Scholars. Cr. 8vo.

Aristophanes. The Peace. By F. A. Paley, M.A. 4*s.* 6*d.*
——— The Acharnians. By F. A. Paley, M.A. 4*s.* 6*d.*
——— The Frogs. By F. A. Paley, M.A. 4*s.* 6*d.*

Cicero. The Letters to Atticus. Bk. I. By A. Pretor, M.A. 4*s.* 6*d.*

Demosthenes de Falsa Legatione. By R. Shilleto, M.A. 6*s.*
——— The Law of Leptines. By B. W. Beatson, M.A. 3*s.* 6*d.*

Plato. The Apology of Socrates and Crito. By W. Wagner, Ph.D. 7th Edition. 4*s.* 6*d.*
——— The Phædo. 6th Edition. By W. Wagner, Ph.D. 5*s.* 6*d.*
——— The Protagoras. 4th Edition. By W. Wayte, M.A. 4*s.* 6*d.*
——— The Euthyphro. 2nd Edition. By G. H. Wells, M.A. 3*s.*
——— The Euthydemus. By G. H. Wells, M.A. 4*s.*
——— The Republic. Books I. & II. By G. H. Wells, M.A. 5*s.* 6*d.*

Plautus. The Aulularia. By W. Wagner, Ph.D. 3rd Edition. 4*s.* 6*d.*
——— Trinummus. By W. Wagner, Ph.D. 3rd Edition. 4*s.* 6*d.*
——— The Menaechmei. By W. Wagner, Ph.D. 4*s.* 6*d.*

Sophoclis Trachiniæ. By A. Pretor, M.A. 4*s.* 6*d.*

Terence. By W. Wagner, Ph.D. 10*s.* 6*d.*

Theocritus. By F. A. Paley, M.A. 4*s.* 6*d.*

Thucydides. Book VI. By T. W. Dougan, M.A., Fellow of St. John's College, Cambridge. [*Immediately.*

Others in preparation.

CRITICAL AND ANNOTATED EDITIONS.

Ætna. By H. A. J. Munro, M.A. 3*s.* 6*d.*

Aristophanis Comœdiæ. By H. A. Holden, LL.D. 8vo. 2 vols. 23*s.* 6*d.* Plays sold separately.
——— Pax. By F. A. Paley, M.A. Fcap. 8vo. 4*s.* 6*d.*

Catullus. By H. A. J. Munro, M.A. 7*s.* 6*d.*

Corpus Poetarum Latinorum. Edited by Walker. 1 vol. 8vo. 18*s.*

Horace. Quinti Horatii Flacci Opera. By H. A. J. Munro, M.A. Large 8vo. 1*l.* 1*s.*

Livy. The first five Books. By J. Prendeville. 12mo. roan, 5*s.* Or Books I.-III. 3*s.* 6*d.* IV. and V. 3*s.* 6*d.*

Lucretius. Titi Lucretii Cari de Rerum Natura Libri Sex. With a Translation and Notes. By H. A. J. Munro, M.A. 2 vols. 8vo. Vol. I. Text. (New Edition, Preparing.) Vol. II. Translation. (Sold separately.)

Ovid. P. Ovidii Nasonis Heroides XIV. By A. Palmer, M.A. 8vo. 6*s.*

Propertius. Sex Aurelii Propertii Carmina. By F. A. Paley, M.A. 8vo. Cloth, 9*s.*

Sex. Propertii Elegiarum. Libri IV. By A. Palmer. Fcap. 8vo. 5*s.*

Sophocles. The Ajax. By C. E. Palmer, M.A. 4*s.* 6*d.*

Thucydides. The History of the Peloponnesian War. By Richard Shilleto, M.A. Book I. 8vo. 6*s.* 6*d.* Book II. 8vo. 5*s.* 6*d.*

LATIN AND GREEK CLASS-BOOKS.

Auxilia Latina. A Series of Progressive Latin Exercises. By M. J. B. Baddeley, M.A. Fcap. 8vo. Part I. Accidence. 2nd Edition, revised. 2*s.* Part II. 4th Edition, revised. 2*s.* Key to Part II. 2*s.* 6*d.*

Latin Prose Lessons. By Prof. Church, M.A. 6th Edit. Fcap. 8vo. 2*s.* 6*d.*

Latin Exercises and Grammar Papers. By T. Collins, M.A. 4th Edition. Fcap. 8vo. 2*s.* 6*d.*

Unseen Papers in Latin Prose and Verse. With Examination Questions. By T. Collins, M.A. 2nd Edition. Fcap. 8vo. 2*s.* 6*d.*

——— in Greek Prose and Verse. With Examination Questions. By T. Collins, M.A. Fcap. 8vo. 3*s.*

Analytical Latin Exercises. By C. P. Mason, B.A. 3rd Edit. 3*s.* 6*d.*

Latin Mood Construction, Outlines of. With Exercises. By the Rev. G. E. C. Casey, M.A., F.L.S., F.G.S. Small post 8vo. 1*s.* 6*d.* Latin of the Exercises. 1*s.* 6*d.*

Scala Græca: a Series of Elementary Greek Exercises. By Rev. J. W. Davis, M.A., and R. W. Baddeley, M.A. 3rd Edition. Fcap. 8vo. 2*s.* 6*d.*

Greek Verse Composition. By G. Preston, M.A. Crown 8vo. 4*s.* 6*d.*

Greek Particles and their Combinations according to Attic Usage. A Short Treatise. By F. A. Paley, M.A. 2*s.* 6*d.*

BY THE REV. P. FROST, M.A., ST. JOHN'S COLLEGE, CAMBRIDGE.

Eclogæ Latinæ; or, First Latin Reading-Book, with English Notes and a Dictionary. New Edition. Fcap. 8vo. 2*s.* 6*d.*

Materials for Latin Prose Composition. New Edition. Fcap. 8vo. 2*s.* 6*d.* Key, 4*s.*

A Latin Verse-Book. An Introductory Work on Hexameters and Pentameters. New Edition. Fcap. 8vo. 3*s.* Key, 5*s.*

Analecta Græca Minora, with Introductory Sentences, English Notes, and a Dictionary. New Edition. Fcap. 8vo. 3*s.* 6*d.*

Materials for Greek Prose Composition. New Edit. Fcap. 8vo. 3*s.* 6*d.* Key, 5*s.*

Florilegium Poeticum. Elegiac Extracts from Ovid and Tibullus. New Edition. With Notes. Fcap. 8vo. 3*s.*

BY THE REV. F. E. GRETTON.

A First Cheque-book for Latin Verse-makers. 1*s.* 6*d.*

A Latin Version for Masters. 2*s.* 6*d.*

Reddenda; or Passages with Parallel Hints for Translation into Latin Prose and Verse. Crown 8vo. 4*s.* 6*d.*

Reddenda Reddita (*see below*).

Anthologia Græca. A Selection of Choice Greek Poetry, with Notes. By F. St. John Thackeray. *4th and Cheaper Edition.* 16mo. 4s. 6d.

Anthologia Latina. A Selection of Choice Latin Poetry, from Nævius to Boëthius, with Notes. By Rev. F. St. John Thackeray. Revised and Cheaper Edition. 16mo. 4s. 6d.

By H. A. Holden, LL.D.

Foliorum Silvula. Part I. Passages for Translation into Latin Elegiac and Heroic Verse. 9th Edition. Post 8vo. 7s. 6d.

——— Part II. Select Passages for Translation into Latin Lyric and Comic Iambic Verse. 3rd Edition. Post 8vo. 5s.

——— Part III. Select Passages for Translation into Greek Verse. 3rd Edition. Post 8vo. 8s.

Folia Silvulæ, sive Eclogæ Poetarum Anglicorum in Latinum et Græcum conversæ. 8vo. Vol. I. 10s. 6d. Vol. II. 12s.

Foliorum Centuriæ. Select Passages for Translation into Latin and Greek Prose. 8th Edition. Post 8vo. 8s.

TRANSLATIONS, SELECTIONS, &c.

*** Many of the following books are well adapted for School Prizes.

Æschylus. Translated into English Prose by F. A. Paley, M.A. 2nd Edition. 8vo. 7s. 6d.

——— Translated into English Verse by Anna Swanwick. Post 8vo. 5s.

Horace. The Odes and Carmen Sæculare. In English Verse by J. Conington, M.A. 8th edition. Fcap. 8vo. 5s. 6d.

——— The Satires and Epistles. In English Verse by J. Conington, M.A. 5th edition. 6s. 6d.

——— Illustrated from Antique Gems by C. W. King, M.A. The text revised with Introduction by H. A. J. Munro, M.A. Large 8vo. 1l. 1s.

Horace's Odes. Englished and Imitated by various hands. Edited by C. W. F. Cooper. Crown 8vo. 6s. 6d.

Lusus Intercisi. Verses, Translated and Original, by H. J. Hodgson, M.A., formerly Fellow of Trinity College, Cambridge. 5s.

Propertius. Verse Translations from Book V., with revised Latin Text. By F. A. Paley, M.A. Fcap. 8vo. 3s.

Plato. Gorgias. Translated by E. M. Cope, M.A. 8vo. 7s.

——— Philebus. Translated by F. A. Paley, M.A. Small 8vo. 4s.

——— Theætetus. Translated by F. A. Paley, M.A. Small 8vo. 4s.

——— Analysis and Index of the Dialogues. By Dr. Day. Post 8vo. 5s.

Reddenda Reddita: Passages from English Poetry, with a Latin Verse Translation. By F. E. Gretton. Crown 8vo. 6s.

Sabrinæ Corolla in hortulis Regiæ Scholæ Salopiensis contexuerunt tres viri floribus legendis. Editio tertia. 8vo. 8s. 6d.

Theocritus. In English Verse, by C. S. Calverley, M.A. New Edition, revised. Crown 8vo. 7s. 6d.

Translations into English and Latin. By C. S. Calverley, M.A. Post 8vo. 7s. 6d.

——— into Greek and Latin Verse. By R. C. Jebb. 4to. cloth gilt. 10s. 6d.

Between Whiles. Translations by B. H. Kennedy. 2nd Edition. revised. Crown 8vo. 6s.

REFERENCE VOLUMES.

A Latin Grammar. By Albert Harkness. Post 8vo. 6*s.*

——— By T. H. Key, M.A. 6th Thousand. Post 8vo. 8*s.*

A Short Latin Grammar for Schools. By T. H. Key, M.A., F.R.S. 14th Edition. Post 8vo. 3*s.* 6*d.*

A Guide to the Choice of Classical Books. By J. B. Mayor, M.A. Revised Edition. Crown 8vo. 3*s.*

The Theatre of the Greeks. By J. W. Donaldson, D.D. 8th Edition. Post 8vo. 5*s.*

Keightley's Mythology of Greece and Italy. 4th Edition. 5*s.*

A Dictionary of Latin and Greek Quotations. By H. T. Riley. Post 8vo. 5*s.* With Index Verborum, 6*s.*

A History of Roman Literature. By W. S. Teuffel, Professor at the University of Tübingen. By W. Wagner, Ph.D. 2 vols. Demy 8vo. 21*s.*

Student's Guide to the University of Cambridge. 4th Edition revised. Fcap. 8vo. 6*s.* 6*d.*; or in Parts.—Part 1, 2*s.* 6*d.*; Parts 2 to 9, 1*s.* each.

CLASSICAL TABLES.

Latin Accidence. By the Rev. P. Frost, M.A. 1*s.*

Latin Versification. 1*s.*

Notabilia Quædam; or the Principal Tenses of most of the Irregular Greek Verbs and Elementary Greek, Latin, and French Construction. New Edition. 1*s.*

Richmond Rules for the Ovidian Distich, &c. By J. Tate, M.A. 1*s.*

The Principles of Latin Syntax. 1*s.*

Greek Verbs. A Catalogue of Verbs, Irregular and Defective; their leading formations, tenses, and inflexions, with Paradigms for conjugation, Rules for formation of tenses, &c. &c. By J. S. Baird, T.C.D. 2*s.* 6*d.*

Greek Accents (Notes on). By A. Barry, D.D. New Edition. 1*s.*

Homeric Dialect. Its Leading Forms and Peculiarities. By J. S. Baird, T.C.D. New Edition, by W. G. Rutherford. 1*s.*

Greek Accidence. By the Rev. P. Frost, M.A. New Edition. 1*s.*

CAMBRIDGE MATHEMATICAL SERIES.

Algebra. Choice and Chance. By W. A. Whitworth, M.A. 3rd Edition. 6*s.*

Euclid. Exercises on Euclid and in Modern Geometry. By J. McDowell, M.A. 3rd Edition. 6*s.*

Trigonometry. Plane. By Rev. T. Vyvyan, M.A. 2nd Edit. 3*s.* 6*d.*

Conics. The Geometry of. 3rd Edition. By C. Taylor, D.D. 4*s.* 6*d.*

Solid Geometry. By W. S. Aldis, M.A. 3rd Edition. 6*s.*

Rigid Dynamics. By W. S. Aldis, M.A. 4*s.*

***Elementary* Dynamics.** By W. Garnett, M.A. 3rd Edition. 6*s.*

Heat. An Elementary Treatise. By W. Garnett, M.A. 2nd Edit. 3s. 6d.

Hydromechanics. By W. H. Besant, M.A., F.R.S. 4th Edition. Part I. Hydrostatics. 5s.

Mechanics. Problems in Elementary. By W. Walton, M.A. 6*s.*

CAMBRIDGE SCHOOL AND COLLEGE TEXT-BOOKS.

A Series of Elementary Treatises for the use of Students in the Universities, Schools, and Candidates for the Public Examinations. Fcap. 8*vo.*

Arithmetic. By Rev. C. Elsee, M.A. Fcap. 8vo. 11th Edit. 3*s.* 6*d.*

Algebra. By the Rev. C. Elsee, M.A. 6th Edit. 4*s.*

Arithmetic. By A. Wrigley, M.A. 3*s.* 6*d.*

——— A Progressive Course of Examples. With Answers. By J. Watson, M.A. 5th Edition. 2s. 6d.

Algebra. Progressive Course of Examples. By Rev. W. F. M'Michael, M.A., and R. Prowde Smith, M.A. 2nd Edition. 3s. 6d. With Answers. 4s. 6d.

Plane Astronomy, An Introduction to. By P. T. Main, M.A. 4th Edition. 4s.

Conic Sections treated Geometrically. By W. H. Besant, M.A. 4th Edition. 4s. 6d. Solution to the Examples. 4s.

Elementary Conic Sections treated Geometrically. By W. H. Besant, M.A. [*In the press.*

Statics, Elementary. By Rev. H. Goodwin, D.D. 2nd Edit. 3*s.*

Hydrostatics, Elementary. By W. H. Besant, M.A. 10th Edit. 4*s.*

Mensuration, An Elementary Treatise on. By B. T. Moore, M.A. 6*s.*

Newton's Principia, The First Three Sections of, with an Appendix; and the Ninth and Eleventh Sections. By J. H. Evans, M.A. 5th Edition, by P. T. Main, M.A. 4s.

Trigonometry, Elementary. By T. P. Hudson, M.A. 3*s.* 6*d.*

Optics, Geometrical. With Answers. By W. S. Aldis, M.A. 3*s.* 6*d.*

Analytical Geometry for Schools. By T. G. Vyvyan. 3rd Edit. 4*s.* 6*d.*

Greek Testament, Companion to the. By A. C. Barrett, A.M. 5th Edition, revised. Fcap. 8vo. 5s.

Book of Common Prayer, An Historical and Explanatory Treatise on the. By W. G. Humphry, B.D. 6th Edition. Fcap. 8vo. 4s. 6d.

Music, Text-book of. By H. C. Banister. 11th Edit. revised. 5*s.*

——— Concise History of. By Rev. H. G. Bonavia Hunt, B. Mus. Oxon. 6th Edition revised. 3s. 6d.

ARITHMETIC AND ALGEBRA.

See foregoing Series.

GEOMETRY AND EUCLID.

Euclid. The First Two Books explained to Beginners. By C. P. Mason, B.A. 2nd Edition. Fcap. 8vo. 2*s*. 6*d*.

The Enunciations and Figures to Euclid's Elements. By Rev. J. Brasse, D.D. New Edition. Fcap. 8vo. 1*s*. On Cards, in case, 5*s*. 6*d*. Without the Figures, 6*d*.

Exercises on Euclid and in Modern Geometry. By J. McDowell, B.A. Crown 8vo. 3rd Edition revised. 6*s*.

Geometrical Conic Sections. By W. H. Besant, M.A. 4th Edit. 4*s*. 6*d*. Solution to the Examples. 4*s*.

Elementary Geometrical Conic Sections. By W. H. Besant, M.A. [*In the press.*

Elementary Geometry of Conics. By C. Taylor, D.D. 3rd Edit. 8vo. 4*s*. 6*d*.

An Introduction to Ancient and Modern Geometry of Conics. By C. Taylor, M.A. 8vo. 15*s*.

Solutions of Geometrical Problems, proposed at St. John's College from 1830 to 1846. By T. Gaskin, M.A. 8vo. 12*s*.

TRIGONOMETRY.

Trigonometry, Introduction to Plane. By Rev. T. G. Vyvyan, Charterhouse. Cr. 8vo. 3*s*. 6*d*.

Elementary Trigonometry. By T. P. Hudson, M.A. 3*s*. 6*d*.

An Elementary Treatise on Mensuration. By B. T. Moore, M.A. 5*s*.

ANALYTICAL GEOMETRY AND DIFFERENTIAL CALCULUS.

An Introduction to Analytical Plane Geometry. By W. P. Turnbull, M.A. 8vo. 12*s*.

Problems on the Principles of Plane Co-ordinate Geometry. By W. Walton, M.A. 8vo. 16*s*.

Trilinear Co-ordinates, and Modern Analytical Geometry of Two Dimensions. By W. A. Whitworth, M.A. 8vo. 16*s*.

An Elementary Treatise on Solid Geometry. By W. S. Aldis, M.A. 2nd Edition revised. 8vo. 8*s*.

Elementary Treatise on the Differential Calculus. By M. O'Brien, M.A. 8vo. 10*s*. 6*d*.

Elliptic Functions, Elementary Treatise on. By A. Cayley, M.A. Demy 8vo. 15*s*.

MECHANICS & NATURAL PHILOSOPHY.

Statics, Elementary. By H. Goodwin, D.D. Fcap. 8vo. 2nd Edition. 3*s*.

Dynamics, A Treatise on Elementary. By W. Garnett, M.A. 3rd Edition. Crown 8vo. 6*s*.

Dynamics. Rigid. By W. S. Aldis, M.A. 4*s*.

Elementary Mechanics, Problems in. By W. Walton, M.A. New Edition. Crown 8vo. 6*s*.

Theoretical Mechanics, Problems in. By W. Walton. 2nd Edit. revised and enlarged. Demy 8vo. 16*s*.

Hydrostatics. By W. H. Besant, M.A. Fcap. 8vo. 10th Edition. 4*s*.

Hydromechanics, A Treatise on. By W. H. Besant, M.A., F.R.S. 8vo. 4th Edition, revised. Part I. Hydrostatics. 5*s*.

Dynamics of a Particle, A Treatise on the. By W. H. Besant, M.A. [*Preparing.*

Optics, Geometrical. By W. S. Aldis, M.A. Fcap. 8vo. 3*s*. 6*d*.

Double Refraction, A Chapter on Fresnel's Theory of. By W. S. Aldis, M.A. 8vo. 2*s*.

Heat, An Elementary Treatise on. By W. Garnett, M.A. Crown 8vo. 2nd Edition revised. 3*s*. 6*d*.

Newton's Principia, The First Three Sections of, with an Appendix; and the Ninth and Eleventh Sections. By J. H. Evans, M.A. 5th Edition. Edited by P. T. Main, M.A. 4*s*.

Astronomy, An Introduction to Plane. By P. T. Main, M.A. Fcap. 8vo. cloth. 4*s*.

Astronomy, Practical and Spherical. By R. Main, M.A. 8vo. 14*s*.

Astronomy, Elementary Chapters on, from the 'Astronomie Physique' of Biot. By H. Goodwin, D.D. 8vo. 3*s*. 6*d*.

Pure Mathematics and Natural Philosophy, A Compendium of Facts and Formulæ in. By G. R. Smalley. 2nd Edition, revised by J. McDowell, M.A. Fcap. 8vo. 3*s*. 6*d*.

Elementary Course of Mathematics. By H. Goodwin, D.D. 6th Edition. 8vo. 16*s*.

Problems and Examples, adapted to the 'Elementary Course of Mathematics.' 3rd Edition. 8vo. 5*s*.

Solutions of Goodwin's Collection of Problems and Examples. By W. W. Hutt, M.A. 3rd Edition, revised and enlarged. 8vo. 9*s*.

Mechanics of Construction. With numerous Examples. By S. Fenwick, F.R.A.S. 8vo. 12*s*.

Pure and Applied Calculation, Notes on the Principles of. By Rev. J. Challis, M.A. Demy 8vo. 15*s*.

Physics, The Mathematical Principle of. By Rev. J. Challis, M.A. Demy 8vo. 5*s*.

TECHNOLOGICAL HANDBOOKS.

Edited by H. TRUEMAN WOOD, Secretary of the Society of Arts.

1. **Dyeing and Tissue Printing.** By W. Crookes, F.R.S. 5*s*.
2. **Glass Manufacture.** By Henry Chance, M.A.; H. J. Powell, B.A.; and H. G. Harris. 3*s*. 6*d*.
3. **Cotton Manufacture.** By Richard Marsden, Esq., of Manchester. [*Immediately.*
4. **Telegraphs and Telephones.** By W. H. Preece, F.R.S. [*Preparing.*

HISTORY, TOPOGRAPHY, &c.

Rome and the Campagna. By R. Burn, M.A. With 85 Engravings and 26 Maps and Plans. With Appendix. 4to. 3*l*. 3*s*.

Old Rome. A Handbook for Travellers. By R. Burn, M.A. With Maps and Plans. Demy 8vo. 10*s*. 6*d*.

Modern Europe. By Dr. T. H. Dyer. 2nd Edition, revised and continued. 5 vols. Demy 8vo. 2*l*. 12*s*. 6*d*.

The History of the Kings of Rome. By Dr. T. H. Dyer. 8vo. 16*s*.

The History of Pompeii: its Buildings and Antiquities. By T. H. Dyer. 3rd Edition, brought down to 1874. Post 8vo. 7*s*. 6*d*.

Ancient Athens: its History, Topography, and Remains. By T. H. Dyer. Super-royal 8vo. Cloth. 1*l*. 5*s*.

The Decline of the Roman Republic. By G. Long. 5 vols. 8vo. 14*s*. each.

A History of England during the Early and Middle Ages. By C. H. Pearson, M.A. 2nd Edition revised and enlarged. 8vo. Vol. I. 16*s*. Vol. II. 14*s*.

Historical Maps of England. By C. H. Pearson. Folio. 3rd Edition revised. 31*s*. 6*d*.

History of England, 1800–15. By Harriet Martineau, with new and copious Index. 1 vol. 3*s*. 6*d*.

History of the Thirty Years' Peace, 1815–46. By Harriet Martineau. 4 vols. 3*s*. 6*d*. each.

A Practical Synopsis of English History. By A. Bowes. 4th Edition. 8vo. 2*s*.

Student's Text-Book of English and General History. By D. Beale. Crown 8vo. 2*s*. 6*d*.

Lives of the Queens of England. By A. Strickland. Library Edition, 8 vols. 7*s*. 6*d*. each. Cheaper Edition, 6 vols. 5*s*. each. Abridged Edition, 1 vol. 6*s*. 6*d*.

Eginhard's Life of Karl the Great (Charlemagne). Translated with Notes, by W. Glaister, M.A., B.C.L. Crown 8vo. 4*s*. 6*d*.

Outlines of Indian History. By A. W. Hughes. Small Post 8vo. 3*s*. 6*d*.

The Elements of General History. By Prof. Tytler. New Edition, brought down to 1874. Small Post 8vo. 3*s*. 6*d*.

ATLASES.

An Atlas of Classical Geography. 24 Maps. By W. Hughes and G. Long, M.A. New Edition. Imperial 8vo. 12*s*. 6*d*.

A Grammar-School Atlas of Classical Geography. Ten Maps selected from the above. New Edition. Imperial 8vo. 5*s*.

First Classical Maps. By the Rev. J. Tate, M.A. 3rd Edition. Imperial 8vo. 7*s*. 6*d*.

***Standard Library* Atlas of Classical Geography.** Imp. 8vo. 7*s*. 6*d*.

PHILOLOGY.

WEBSTER'S DICTIONARY OF THE ENGLISH LANGUAGE. With Dr. Mahn's Etymology. 1 vol. 1628 pages, 3000 Illustrations. 21s. With Appendices and 70 additional pages of Illustrations, 1919 pages, 31s. 6d.

'THE BEST PRACTICAL ENGLISH DICTIONARY EXTANT.'—*Quarterly Review*, 1873.

Prospectuses, with specimen pages, post free on application.

New Dictionary of the English Language. Combining Explanation with Etymology, and copiously illustrated by Quotations from the best Authorities. By Dr. Richardson. New Edition, with a Supplement. 2 vols. 4to. 4*l*. 14*s*. 6*d*.; half russia, 5*l*. 15*s*. 6*d*.; russia, 6*l*. 12*s*. Supplement separately. 4to. 12*s*.

An 8vo. Edit. without the Quotations, 15*s*.; half russia, 20*s*.; russia, 24*s*.

Supplementary English Glossary. Containing 12,000 Words and Meanings occurring in English Literature, not found in any other Dictionary. By T. L. O. Davies. Demy 8vo. 16*s*.

Folk-Etymology. A Dictionary of Words perverted in Form or Meaning by False Derivation or Mistaken Analogy. By Rev. A. S. Palmer. Demy 8vo. 21*s*.

Brief History of the English Language. By Prof. James Hadley, LL.D., Yale College. Fcap. 8vo. 1*s*.

The Elements of the English Language. By E. Adams, Ph.D. 19th Edition. Post 8vo. 4*s*. 6*d*.

Philological Essays. By T. H. Key, M.A., F.R.S. 8vo. 10*s*. 6*d*.

Language, its Origin and Development. By T. H. Key, M.A., F.R.S. 8vo. 14*s*.

Synonyms and Antonyms of the English Language. By Archdeacon Smith. 2nd Edition. Post 8vo. 5*s*.

Synonyms Discriminated. By Archdeacon Smith. Demy 8vo. 16*s*.

Bible English. By T. L. O. Davies. 5*s*.

The Queen's English. A Manual of Idiom and Usage. By the late Dean Alford 6th Edition. Fcap. 8vo. 5*s*.

Etymological Glossary of nearly 2500 English Words derived from the Greek. By the Rev. E. J. Boyce. Fcap. 8vo. 3*s*. 6*d*.

A Syriac Grammar. By G. Phillips, D.D. 3rd Edition, enlarged. 8vo. 7*s*. 6*d*.

A Grammar of the Arabic Language. By Rev. W. J. Beamont, M.A. 12mo. 7*s*.

DIVINITY, MORAL PHILOSOPHY, &c.

Novum Testamentum Græcum, Textus Stephanici, 1550. By F. H. Scrivener, A.M., LL.D., D.C.L. New Edition. 16mo. 4*s*. 6*d*. Also on Writing Paper, with Wide Margin. Half-bound. 12*s*.

By the same Author.

Codex Bezæ Cantabrigiensis. 4to. 26*s*.

A Full Collation of the Codex Sinaiticus with the Received Text of the New Testament, with Critical Introduction. 2nd Edition, revised. Fcap. 8vo. 5*s*.

A Plain Introduction to the Criticism of the New Testament. With Forty Facsimiles from Ancient Manuscripts. 3rd Edition. 8vo. 18*s*.

Six Lectures on the Text of the New Testament. For English Readers. Crown 8vo. 6s.

The New Testament for English Readers. By the late H. Alford, D.D. Vol. I. Part I. 3rd Edit. 12s. Vol. I. Part II. 2nd Edit. 10s. 6d. Vol. II. Part I. 2nd Edit. 16s. Vol. II. Part II. 2nd Edit. 16s.

The Greek Testament. By the late H. Alford, D.D. Vol. I. 6th Edit. 1l. 8s. Vol. II. 6th Edit. 1l. 4s. Vol. III. 5th Edit. 18s. Vol. IV. Part I. 4th Edit. 18s. Vol. IV. Part II. 4th Edit. 14s. Vol. IV. 1l. 12s.

Companion to the Greek Testament. By A. C. Barrett, M.A. 5th Edition, revised. Fcap. 8vo. 5s.

The Book of Psalms. A New Translation, with Introductions, &c. By the Very Rev. J. J. Stewart Perowne, D.D. 8vo. Vol. I. 5th Edition, 18s. Vol. II. 5th Edit. 16s.

——— Abridged for Schools. 4th Edition. Crown 8vo. 10s. 6d.

History of the Articles of Religion. By C. H. Hardwick. 3rd Edition. Post 8vo. 5s.

History of the Creeds. By J. R. Lumby, D.D. 2nd Edition. Crown 8vo. 7s. 6d.

Pearson on the Creed. Carefully printed from an early edition. With Analysis and Index by E. Walford, M.A. Post 8vo. 5s.

An Historical and Explanatory Treatise on the Book of Common Prayer. By Rev. W. G. Humphry, B.D. 6th Edition, enlarged. Small Post 8vo. 4s. 6d.

The New Table of Lessons Explained. By Rev. W. G. Humphry, B.D. Fcap. 1s. 6d.

A Commentary on the Gospels for the Sundays and other Holy Days of the Christian Year. By Rev. W. Denton, A.M. New Edition. 3 vols. 8vo. 54s. Sold separately.

Commentary on the Epistles for the Sundays and other Holy Days of the Christian Year. By Rev. W. Denton, A.M. 2 vols. 36s. Sold separately.

Commentary on the Acts. By Rev. W. Denton, A.M. Vol. I. 8vo. 18s. Vol. II. 14s.

Notes on the Catechism. By Rev. Canon Barry, D.D. 6th Edit. Fcap. 2s.

Catechetical Hints and Helps. By Rev. E. J. Boyce, M.A. 4th Edition, revised. Fcap. 2s. 6d.

Examination Papers on Religious Instruction. By Rev. E. J. Boyce. Sewed. 1s. 6d.

Church Teaching for the Church's Children. An Exposition of the Catechism. By the Rev. F. W. Harper. Sq. Fcap. 2s.

The Winton Church Catechist. Questions and Answers on the Teaching of the Church Catechism. By the late Rev. J. S. B. Monsell, LL.D. 3rd Edition. Cloth, 3s.; or in Four Parts, sewed.

The Church Teacher's Manual of Christian Instruction. By Rev. M. F. Sadler. 28th Thousand. 2s. 6d.

Short Explanation of the Epistles and Gospels of the Christian Year, with Questions. Royal 32mo. 2s. 6d.; calf, 4s. 6d.

***Butler's* Analogy of Religion**; with Introduction and Index by *Rev. Dr. Steere.* New Edition. Fcap. 3s. 6d.

Lectures on the History of Moral Philosophy in England. By W. Whewell, D.D. Crown 8vo. 8*s.*

Kent's Commentary on International Law. By J. T. Abdy, LL.D. New and Cheap Edition. Crown 8vo. 10*s.* 6*d.*

A Manual of the Roman Civil Law. By G. Leapingwell, LL.D. 8vo. 12*s.*

FOREIGN CLASSICS.

A series for use in Schools, with English Notes, grammatical and explanatory, and renderings of difficult idiomatic expressions. Fcap. 8vo.

Schiller's Wallenstein. By Dr. A. Buchheim. 3rd Edit. 6*s.* 6*d.* Or the Lager and Piccolomini, 3*s.* 6*d.* Wallenstein's Tod, 3*s.* 6*d.*

——— **Maid of Orleans.** By Dr. W. Wagner. 3*s.* 6*d.*

——— **Maria Stuart.** By V. Kastner. 3*s.*

Goethe's Hermann and Dorothea. By E. Bell, M.A., and E. Wölfel. 2*s.* 6*d.*

German Ballads, from Uhland, Goethe, and Schiller. By C. L. Bielefeld. 3rd Edition. 3*s.* 6*d.*

Charles XII., par Voltaire. By L. Direy. 4th Edition. 3*s.* 6*d.*

Aventures de Télémaque, par Fénélon. By C. J. Delille. 2nd Edition. 4*s.* 6*d.*

Select Fables of La Fontaine. By F. E. A. Gasc. 14th Edition. 3*s.*

Picciola, by X. B. Saintine. By Dr. Dubuc. 11th Thousand. 1*s.* 6*d.*

FRENCH CLASS-BOOKS.

Twenty Lessons in French. With Vocabulary, giving the Pronunciation. By W. Brebner. Post 8vo. 4*s.*

French Grammar for Public Schools. By Rev. A. C. Clapin, M.A. Fcap. 8vo. 9th Edition, revised. 2*s.* 6*d.*

French Primer. By Rev. A. C. Clapin, M.A. Fcap. 8vo. 5th Edit. 1*s.*

Primer of French Philology. By Rev. A. C. Clapin. Fcap. 8vo. 2nd Edit. 1*s.*

Le Nouveau Trésor; or, French Student's Companion. By M. E. S. 16th Edition. Fcap. 8vo. 3*s.* 6*d.*

F. E. A. GASC'S FRENCH COURSE.

First French Book. Fcap. 8vo. 86th Thousand. 1*s.* 6*d.*

Second French Book. 42nd Thousand. Fcap. 8vo. 2*s.* 6*d.*

Key to First and Second French Books. Fcap. 8vo. 3*s.* 6*d.*

French Fables for Beginners, in Prose, with Index. 15th Thousand. 12mo. 2*s.*

Select Fables of La Fontaine. New Edition. Fcap. 8vo. 3*s.*

Histoires Amusantes et Instructives. With Notes. 14th Thousand. Fcap. 8vo. 2*s.* 6*d.*

Practical Guide to Modern French Conversation. 12th Thousand. Fcap. 8vo. 2s. 6d.

French Poetry for the Young. With Notes. 4th Edition. Fcap. 8vo. 2s.

Materials for French Prose Composition; or, Selections from the best English Prose Writers. 16th Thousand. Fcap. 8vo. 4s. 6d. Key, 6s.

Prosateurs Contemporains. With Notes. 8vo. 6th Edition, revised. 5s.

Le Petit Compagnon; a French Talk-Book for Little Children. 11th Thousand. 16mo. 2s. 6d.

An Improved Modern Pocket Dictionary of the French and English Languages. 30th Thousand, with Additions. 16mo. Cloth. 4s. Also in 2 vols. in neat leatherette, 5s.

Modern French-English and English-French Dictionary. 2nd Edition, revised. In 1 vol. 12s. 6d. (formerly 2 vols. 25s.)

GOMBERT'S FRENCH DRAMA.

Being a Selection of the best Tragedies and Comedies of Molière, Racine, Corneille, and Voltaire. With Arguments and Notes by A. Gombert. New Edition, revised by F. E. A. Gasc. Fcap. 8vo. 1s. each; sewed, 6d.

Contents.

Molière:—Le Misanthrope. L'Avare. Le Bourgeois Gentilhomme. Le Tartuffe. Le Malade Imaginaire. Les Femmes Savantes. Les Fourberies de Scapin. Les Précieuses Ridicules. L'Ecole des Femmes. L'Ecole des Maris. Le Médecin malgré Lui.

Racine:—Phédre. Esther. Athalie. Iphigénie. Les Plaideurs. La Thébaïde; ou, Les Frères Ennemis. Andromaque. Britannicus.

P. Corneille:—Le Cid. Horace. Cinna. Polyeucte.

Voltaire:—Zaïre.

GERMAN CLASS-BOOKS.

Materials for German Prose Composition. By Dr Buchheim. 7th Edition Fcap. 4s. 6d. Key, 3s.

Wortfolge, or Rules and Exercises on the Order of Words in German Sentences. By Dr. F. Stock. 1s. 6d.

A German Grammar for Public Schools. By the Rev. A. C. Clapin and F. Holl Müller. 2nd Edition. Fcap. 2s. 6d.

Kotzebue's Der Gefangene. With Notes by Dr. W. Stromberg. 1s.

ENGLISH CLASS-BOOKS.

A Brief History of the English Language. By Prof. Jas. Hadley, LL.D., of Yale College. Fcap. 8vo. 1s.

The Elements of the English Language. By E. Adams, Ph.D. 19th Edition. Post 8vo. 4s. 6d.

The Rudiments of English Grammar and Analysis. By E. Adams, Ph.D. 8th Edition. Fcap. 8vo. 2s.

By C. P. Mason, Fellow of Univ. Coll. London.

First Notions of Grammar for Young Learners. Fcap. 8vo. 10th Thousand. Cloth. 8d.

Word Building in English, reprinted for Elementary Schools from the Shorter English Grammar. 3d.

First Steps in English Grammar for Junior Classes. Demy 18mo. 32nd Thousand. 1s.

Outlines of English Grammar for the use of Junior Classes. 31st Thousand. Crown 8vo. 2s.

English Grammar, including the Principles of Grammatical Analysis. 25th Edition. 86th Thousand. Crown 8vo. 3s. 6d.

A Shorter English Grammar, with copious Exercises. 8th Thousand. Crown 8vo. 3s. 6d.

English Grammar Practice, being the Exercises separately. 1s.

Practical Hints on Teaching. By Rev. J. Menet, M.A. 6th Edit. revised. Crown 8vo. cloth, 2s. 6d.; paper, 2s.

How to Earn the Merit Grant. A Manual of School Management. By H. Major, B.A., B.Sc. Part I. Infant School, 3s. Part II. 4s. Complete, 6s.

Test Lessons in Dictation. 2nd Edition. Paper cover, 1s. 6d.

Questions for Examinations in English Literature. By Rev. W. W. Skeat, Prof. of Anglo-Saxon at Cambridge University. 2s. 6d.

Drawing Copies. By P. H. Delamotte. Oblong 8vo. 12s. Sold also in parts at 1s. each.

Poetry for the School-room. New Edition. Fcap. 8vo. 1s. 6d.

Geographical Text-Book; a Practical Geography. By M. E. S. 12mo. 2s.

The Blank Maps done up separately, 4to. 2s. coloured.

Loudon's (Mrs.) Entertaining Naturalist. New Edition. Revised by W. S. Dallas, F.L.S. 5s.

——— **Handbook of Botany.** New Edition, greatly enlarged by D. Wooster. Fcap. 2s. 6d.

The Botanist's Pocket-Book. With a copious Index. By W. R. Hayward. 3rd Edit. revised. Crown 8vo. cloth limp. 4s. 6d.

Experimental Chemistry, founded on the Work of Dr. Stöckhardt. By C. W. Heaton. Post 8vo. 5s.

Double Entry Elucidated. By B. W. Foster. 12th Edit. 4to. 3s. 6d.

A New Manual of Book-keeping. By P. Crellin, Accountant. Crown 8vo. 3s. 6d.

Picture School-Books. In Simple Language, with numerous Illustrations. Royal 16mo.

School Primer. 6d.—School Reader. By J. Tilleard. 1s.—Poetry Book for Schools. 1s.—The Life of Joseph. 1s.—The Scripture Parables. By the Rev. J. E. Clarke. 1s.—The Scripture Miracles. By the Rev. J. E. Clarke. 1s.—The New Testament History. By the Rev. J. G. Wood, M.A. 1s.—The Old Testament History. By the Rev. J. G. Wood, M.A. 1s.—The Story of Bunyan's Pilgrim's Progress. 1s.—The Life of Christopher Columbus. By Sarah Crompton. 1s.—The Life of Martin Luther. By Sarah Crompton. 1s.

BOOKS FOR YOUNG READERS.

A Series of Reading Books designed to facilitate the acquisition of the power of Reading by very young Children. In 8 vols. limp cloth, 8d. each.

Suitable for Infants.

The Cat and the Hen. Sam and his Dog Redleg. Bob and Tom Lee. A Wreck.

The New-born Lamb. The Rosewood Box. Poor Fan. Sheep Dog.

Suitable for Standards I. & II.

The Story of Three Monkeys.

Story of a Cat. Told by Herself.

The Blind Boy. The Mute Girl. A New Tale of Babes in a Wood.

The Dey and the Knight. The New Bank Note. The Royal Visit. A King's Walk on a Winter's Day.

Queen Bee and Busy Bee.

Gull's Crag.

A First Book of Geography. By the Rev. C. A. Johns. Illustrated. Double size, 1s.

BELL'S READING-BOOKS.

FOR SCHOOLS AND PAROCHIAL LIBRARIES.

The popularity of the 'Books for Young Readers' is a sufficient proof that teachers and pupils alike approve of the use of interesting stories, in place of the dry combination of letters and syllables, of which elementary reading-books generally consist. The Publishers have therefore thought it advisable to extend the application of this principle to books adapted for more advanced readers.

Now Ready. Post 8vo. Strongly bound.

Suitable for Standards II. & III.

Grimm's German Tales. (Selected.) 1*s.*
Andersen's Danish Tales. (Selected.) 1*s.*
Great Englishmen. Short Lives for Young Children. 1*s.*
Great Englishwomen. [*In the press.*
Lamb's Tales from Shakespeare. (Selected.) 1*s.*
Edgeworth's Tales. A Selection. 1*s.*

Standards IV. & V.

Friends in Fur and Feathers. By Gwynfryn. 1*s.*
Parables from Nature. (Selected.) By Mrs. Gatty. 1*s.*
The Light of Truth, and other Parables. By Mrs. Gatty. 1*s.*
Masterman Ready. By Capt. Marryat. (Abgd.) 1*s.* 6*d.*
Settlers in Canada. By Capt. Marryat. (Abdg.) 1*s.* 6*d.*
Gulliver's Travels. (Abridged.) 1*s.*
Robinson Crusoe. 1*s.* 6*d.*

Standard VI.

The Vicar of Wakefield. Abridged. 1*s.*
Marie; or, **Glimpses of Life in France.** By A. R. Ellis. 1*s.*
Poetry for Boys. By D. Munro. 1*s.*
Southey's Life of Nelson. (Abridged.) 1*s.*
Life of the Duke of Wellington, with Maps and Plans. 1*s.*
The Romance of the Coast. By J. Runciman. 1*s.*

London: Printed by STRANGEWAYS & SONS, Tower Street, St. Martin's Lane.