AN ELEMENTARY TREATISE ON GRAPHS

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BY

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PREFACE.

My object in the preparation of this text-book has been to present the subject of graphs in a connected form, simple enough in the early stages for the mere beginner while including in the ultimate development such of its more important applications as come within the range of elementary mathematics. The present tendency of mathematical teaching is perhaps to overestimate the value of graphical methods and to depreciate unduly those of analysis; but in spite of the evils attendant upon the reaction from the neglect of graphical methods, these possess, when judiciously used, a high educational value and are of essential importance to all engaged in experimental work.

From the educational point of view a graph has the great merit of representing in a simple manner the fundamental notion of functional dependence. The beginner's conceptions of a variable are usually very crude, and it is necessary that they should be clear and definite if he is to understand mathematical principles and processes; as an aid to the right comprehension of a variable, the graph renders very great service. But the graphical method may also be badly used; one of these bad uses is, in my judgment, the too common practice of plotting a graph from an insufficient number of points. The behaviour of a function, for example, in the neighbourhood of its turning values cannot be adequately understood by the beginner unless he tests it in typical cases by calculating the values of the function for a succession of values of the argument at small intervals. The process known as "cramming" is quite possible in graphical work and is less excusable there than in other departments of mathematics.

I have included, as opportunity arose, many applications of a practical kind, and I am deeply indebted to my colleagues Professors Longbottom, Maclean and Watkinson for the use of their Laboratory Note-books, on which I have drawn heavily for examples. In the text and among the Exercises examples occur which have been manufactured simply to illustrate certain processes, but examples in which the data are stated to be experimental are of course taken directly from the record of the experiments. The answers given are such as can be obtained by the methods illustrated in the text; they have been worked out by my friends Mr. John Dougall and Mr. John Miller and will be found, it is hoped, to be as accurate as the data warrant.

The Tables at the end of the book are sufficient for the calculations required in the examples; in questions on gradients however there would in some cases be an advantage in using seven-figure Tables.

Besides the gentlemen already named, my friends Dr. J. S. Mackay, Dr. A. Morgan, Mr. P. Bennett, Mr. W. A. Lindsay and Mr. P. Pinkerton have been kind enough to take an interest in the preparation of the book, and for their help in proof reading I tender them my hearty thanks. I owe a special debt of gratitude to Professor R. A. Gregory and Mr. A. T. Simmons for their advice in all matters bearing on the passage of the book through the press. The work of proof reading has however been made comparatively simple by the excellence of the printing, and I gratefully acknowledge my debt to the printing staff of Messrs. MacLehose.

GEORGE A. GIBSON.

GLASGOW, August, 1904.

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CHAPTER I.

STEPS. COORDINATES. PLOTTING OF POINTS.

1. Positive and Negative Numbers. In ordinary arithmetic, numbers are not distinguished as positive and negative; the signs + and − are used simply to indicate the operations of addition and subtraction, and the number to be subtracted must not be greater than that from which it is to be taken away. The introduction of negative numbers in algebra removes this restriction on the number to be subtracted, and there is no confusion caused by using the signs + and −, not only to indicate the operations of addition and subtraction, but also to distinguish positive and negative numbers. The interpretation of positive and negative numbers as representing credit and debit, gain and loss, and similar notions, will be familiar to the student; we will consider a certain geometrical interpretation which is of special importance in graphical work.

2. Steps. Let A and B be two points on an unlimited straight line $X'X$ (Fig. 1), and let the segment AB be thought of as traced out by a point moving along $X'X$ from A to B. In this motion the point moves a definite distance in a definite direction and the segment AB, when considered as a straight line having a definite length and drawn in a definite direction, is called a **directed segment** or, more shortly, a **step**. In naming the step, the point from which the motion begins, the *initial* point of the step, is written first; the other end of the step may be called the *final* point. Thus, AB denotes the step traced out by a point moving from A to B, while BA denotes the step traced out by a point moving from B to A; the *step* BA therefore is not the same as the *step* AB.

Two steps AB and CD are defined to be equal when, and only when, they agree in the following three respects:

(1) they have the same length,

(2) they lie on the same straight line or on parallel straight lines, and

(3) D is on the same side of C as B is of A.

The student must particularly note that equality of steps means not merely equality in length but also sameness in

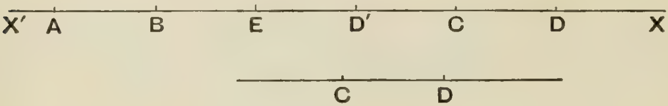


Fig. 1.

direction. Thus, if D' is at the same distance from C as D is but on the opposite side (Fig. 1), the *steps* AB and CD' are not equal; they are different steps because, though they have the same length, the direction from C to D' is not the same as that from A to B. In tracing AB the point moves to the right while in tracing CD' it moves to the left; AB may therefore be called a **right** step and CD' a **left** step. The right steps AB and $D'C$ are equal; the left step CD' is equal to the left step BA.

3. Positive and Negative Steps. Whatever be the relative positions of the three points A, B, C on a straight line (Fig. 2 shows all the possible cases) a point which has moved along the line from A to B and then from B to C will be at the same distance from A and on the same side of A as if it had moved directly from A to C. The single step AC is therefore called the **sum** of the two steps AB and BC, and the operation of adding steps is expressed by the equation

$$AB+BC=AC. \qquad\qquad (1)$$

To find the sum of the steps AB and CD when, as in Fig. 1, the final point B of the first step does not coincide with the initial point C of the second step, mark off the step BE equal to the step CD; the sum of AB and BE, that is AE, is the sum of AB and CD. Of course, not only must BE be of the same length as CD, but E must be on the same side of B that D is of C.

If C coincides with A the step AC becomes the step AA; the step AA since it has no length is called the zero step, and is denoted by 0. Equation (1) becomes in this case

$$AB+BA=0. \qquad \text{.......................(2)}$$

The form of this equation at once suggests that we should write

$$BA=-AB. \qquad \text{.........................(3)}$$

Now if AB is a right step BA is a left step and equation (3) states that a left step is equal to the right step of the same length taken with the negative sign. We are thus led to consider steps as **algebraic** quantities, the **sign** of the step being interpreted as indicating the **direction** in which the step is traced out. If we agree to call a right step positive then a left step will be negative; if the left step be called positive then the right step will be negative. It does not matter which is considered positive but usually it is the right step that we shall consider positive; if $X'X$ is vertical the upward step will usually be considered positive.

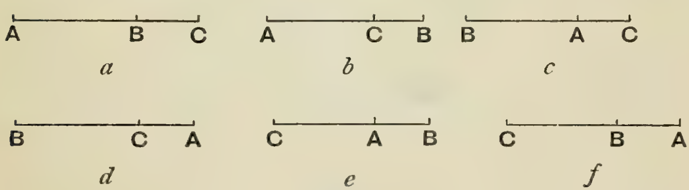


Fig. 2.

It will be an easy and instructive exercise to test by inspection of the different cases of Fig. 2 that the rule for adding steps is exactly the same as that for algebraic addition, right and left steps corresponding to positive and negative numbers.

Thus, in (a) the sum of the two right steps AB and BC is the right step AC; in (f) the sum of the two left steps AB and BC is the left step AC; in (e) the sum of the right step AB and the left step BC (the length of the step BC being greater than that of AB) is the left step AC. These correspond exactly to the formulae

$$(+3)+(+2)=(+5); \quad (-3)+(-2)=(-5);$$
$$(+3)+(-5)=(-2).$$

Again, to see what is meant by subtracting a step write equation (1) in the form

$$BC=AC-AB. \qquad (4)$$

By the meaning of the *sum* of BA and AC we have

$$BC=BA+AC,$$

that is, by interchanging the terms BA and AC,

$$BC=AC+BA; \qquad (5)$$

and now, by comparing equations (4) and (5), we see that the *subtraction* of the step AB is equivalent to the addition of the *opposite* or *reversed* step BA; exactly as in algebra, the subtraction of a number is equivalent to the addition of the number with its sign changed.

Example. A, B, C, D are four points on a straight line; find the position of the point P when

(i) $AP=AB+CD$, (ii) when $AP=AB-CD$.

Consider the cases in which neither C nor D lies between A and B and in which one of them lies between A and B. Take definite lengths, say AB two inches and CD three inches, or AB two inches and DC three inches, and compare with algebraical results; note for example that when CD is a right step of 3 inches DC is a left step of 3 inches.

4. Geometrical Representation of Numbers. Let $X'X$ (Fig. 3) be an unlimited straight line, O a fixed point on it; let U be another fixed point on it, say to the right of O. Take A, B to the right of O and A', B' to the left of O, making the length of OA and of OA' twice that of OU and the length of OB and of OB' thrice that of OU.

Considering OU, OA, OA'... as *steps* we have

$$OA = 2OU, \quad OA' = -OA = -2OU;$$
$$OB = 3OU, \quad OB' = -OB = -3OU.$$

If OU is taken as the *unit step*, that is the step of unit length in the positive direction (for example, a right step of one inch), it may be denoted by the number 1. The numbers 2 and -2 will then denote the steps OA and OA'

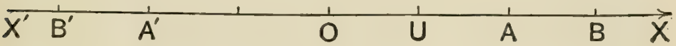


Fig. 3.

respectively, and the steps may be taken as representing the numbers. Similarly the numbers 3 and -3 will denote the steps OB and OB' and the steps will represent the numbers.

Quite generally, if $OP = aOU$, the number a will denote the step OP and OP will represent the number a; if a is positive P will be to the right of O but if a is negative P will be to the left of O. Since OU is the unit step, we may write simply $OP = a$; the numerical value of a gives the length of OP, the sign of a gives the direction of OP.

It is this method of representing numbers that is employed in defining coordinates (§ 5).

5. Coordinates. Let $X'OX$, $Y'OY$ (Fig. 4) be two unlimited straight lines at right angles to each other. Take a point P in the plane of the diagram and draw PM, PN perpendicular to $X'X$, $Y'Y$ respectively. For this point P the steps OM, ON are definitely fixed; and conversely, when the steps OM, ON are given, P is definitely determined as the point of intersection of the perpendiculars MP, NP.

Let OU be the unit step for the direction $X'X$ and OV the unit step for the direction $Y'Y$; we will for the present suppose these steps to be of the same length, say one inch (1″), but there is no necessity that they should be of the same length (see §§ 11, 24).

The step OM, or its equal the step NP, will be positive when P is to the right of $Y'Y$ but negative when P is to the left of $Y'Y$; the step ON or its equal the step MP will be positive when P is above $X'X$ but negative when P is below $X'X$.

Suppose now that

$$OM = xOU; \quad ON = yOV.$$

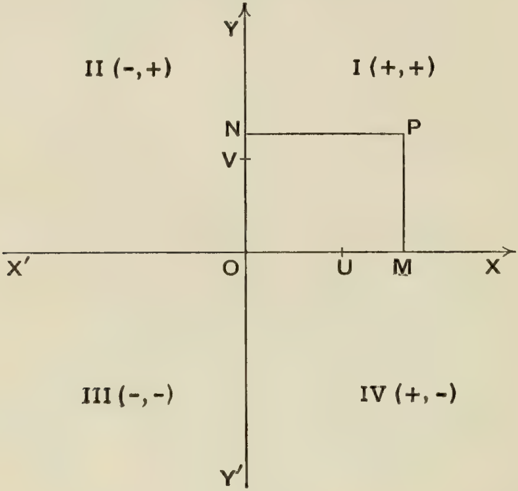


Fig. 4.

The numbers x, y are called the **coordinates** of P with respect to the **coordinate axes** $X'X$, $Y'Y$; x is the **abscissa**, y is the **ordinate** and P is described shortly as "the point (x, y)." In thus describing the point the first coordinate is understood to be the abscissa and the second the ordinate. The axes will be always assumed to be at right angles to each other. O is called the **origin of coordinates**; it is the point $(0, 0)$.

The axes $X'X$ and $Y'Y$ are often called the **x-axis** and the **y-axis** respectively; similarly the abscissa is often called **the x of a point** and the ordinate **the y of the point.**

The axes divide the plane into four compartments or

quadrants; the first quadrant (I) is bounded by OX and OY, the second (II) by OY and OX', the third (III) by OX' and OY', and the fourth (IV) by OY' and OX. The **signs** of the coordinates show at once the **quadrant** in which a point lies: in I the signs (the first being that of the abscissa) are $+, +$; in II, $-, +$; in III, $-, -$; and in IV, $+, -$.

When a point is specified by its coordinates, that is when the values of x and y are given, the process of marking its position on the diagram is called **plotting the point**. This process is made very easy by using "squared paper" or "section paper," that is, paper ruled twice over with two sets of equidistant parallel lines, the lines of one set being perpendicular to those of the other. In most papers every tenth line, sometimes every fifth, is rather heavier than the rest or is coloured differently.

To indicate the position of a point, a small cross is used or a small circle is drawn round the point; a mere dot should never be used to indicate the position of the point. All lines should be drawn with a sharp, hard pencil. The best results are obtained by using two pencils: one with a needle-point for marking points on the diagram, the other with a sharp chisel-edge for drawing fine lines.

The following example shows how to proceed:

Example. Plot the points $A(13, 12)$, $B(-8, 12)$, $C(-8, -6)$, $D(13, -6)$; find the lengths of the sides and the area of the quadrilateral $ABCD$ (Fig. 5).

Let the unit of length be one division of the paper. To serve as a guide in plotting the points, the number 10 is placed at the point where the 10th line to the right of O crosses $X'X$ and also at the point where the 10th line above O crosses $Y'Y$. Other leading points are shown by the number -10 placed 10 units to the left of O and 10 units below O.

Now to plot A move *to the right* 13 units, then *up* 12; to plot B move *to the left* 8 units, then *up* 12; to plot C move *to the left* 8 units, then *down* 6; finally to plot D move *to the right* 13 units, then *down* 6.

The beginner is advised to read the sign of a coordinate as "to the right" or "to the left," "up" or "down."

$ABCD$ is clearly a rectangle. BA, CD are each 21 units and DA, CB are each 18 units.

The rectangle is divided by the horizontal lines into 18 strips, and each strip contains 21 small squares; the area of $ABCD$ is therefore 18×21, that is 378, times the area of a small square.

In the diagram the side OE of a large square is one inch and therefore one division of the paper is one-tenth of an inch. Since one division represents the number 1 the **scale** of the figure is stated by saying that "one-tenth of an inch represents unity" or "$\frac{1}{10}$ inch $=1$" or thus "$1''=10$.

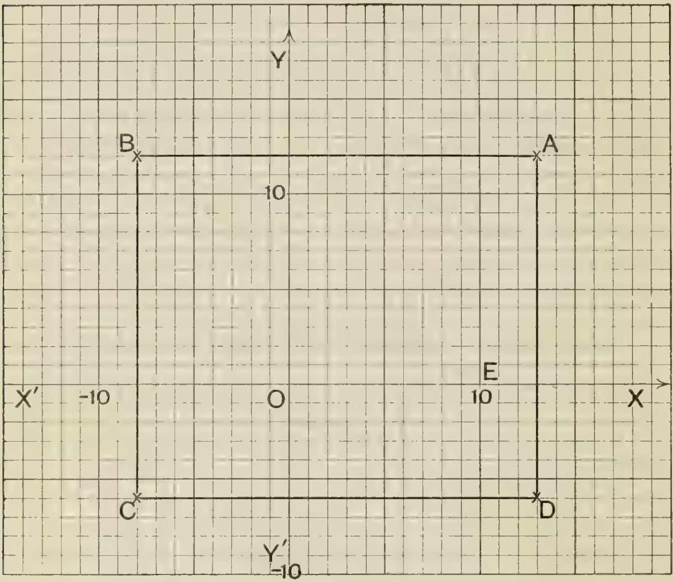


Fig. 5.

The number 21, which gives the length of BA and CD, represents 21 tenths of an inch; BA, CD are therefore $2{\cdot}1''$. Similarly DA, CB are $1{\cdot}8''$.

The area of a small square is one-hundredth of a square inch; the area of $ABCD$ is therefore 378 hundredths of a square inch, that is 3·78 square inches.

EXERCISES. I.

In this set of Exercises let the unit of length be one division of the paper. Assuming that one division is one-tenth of an inch, state lengths and areas thus (taking as an example the problem just worked):

$$BA=21\ (2{\cdot}1\text{ in.});\quad ABCD=378\ (3{\cdot}78\text{ sq. in.}).$$

Plot the points in examples 1–20:

1. (10, 10).	**2.** (5, 5).	**3.** (7, 7).	**4.** (16, 16).
5. $(-10, -10)$.	**6.** $(-5, -5)$.	**7.** $(-7, -7)$.	**8.** $(-16, -16)$.
9. (8, 12).	**10.** $(-8, 12)$.	**11.** $(-8, -12)$.	**12.** $(8, -12)$.
13. (7, 17).	**14.** (17, 7).	**15.** $(-13, 6)$.	**16.** $(13, -6)$.
17. (14, 0).	**18.** (0, 14).	**19.** $(-14, 0)$.	**20.** $(0, -14)$.

Plot the four points in each of the examples 21–25; show that in each case the four points are the vertices of a rectangle and find the sides and the area of each rectangle:

21. (4, 2), (20, 2), (20, 14), (4, 14). **22.** (7, 0), (23, 0), (23, 23), (7, 23).

23. (8, 12), $(-7, 12)$, $(-7, -6)$, $(8, -6)$.

24. $(-2, 6)$, $(-14, 6)$, $(-14, -16)$, $(-2, -16)$.

25. $(-13, 0)$, $(-13, -15)$, $(15, -15)$, (15, 0).

Plot the three points in each of the examples 26–33 and find in each case the area of the triangle of which the three points are the vertices:

26. (0, 0), (20, 0), (20, 20). **27.** (4, 6), (22, 6), (22, 22).

28. $(-8, -4)$, $(-8, 7)$, (12, 7). **29.** (16, 8), $(-13, 8)$, $(-13, -5)$.

30. $(-15, -15)$, $(15, -15)$, (0, 10). **31.** (10, 20), $(-10, 20)$, $(5, -10)$.

32. (16, 12), $(-10, 0)$, $(16, -12)$. **33.** (12, 14), $(-14, 4)$, $(12, -8)$.

6. Plotting of Points. Additional Examples. Areas.

Example 1. Plot the points $A(2·5, 1)$, $B(-1, 1·5)$, $C(-1·5, -1·5)$, $D(1, -2)$. Join AB, BC, CD, DA and give the coordinates of the points where these lines cross the axes.

In this example take a larger scale than in § 5; let the unit steps OU, OV (Fig. 6) be each one inch.* In this case the distance between any two consecutive lines is one-tenth of the unit and therefore represents 0·1. The point midway between O and U is 0·5 of the unit to the right of O and at this point the number 0·5 is placed. Similarly 0·5 is placed at the point midway between O and V. The point on $X'X$ marked -1 is 1 unit to the left of O; the point on $Y'Y$ marked -2 is 2 units below O and so on.

To plot A move to the right 2·5 units, then up 1; to plot B move to the left 1 unit, then up 1·5 and so on.

AB crosses $Y'Y$ at E, and E lies, as far as we can judge, midway between the 3rd and 4th lines above the point marked 1. OE is thus greater than 1·3 by half of 0·1, that is OE is equal to $1·3+0·05$ or 1·35; the sign is + since OE is a positive step. The coordinates of E are therefore (0, 1·35). (See the remarks on the estimation of distance at the end of example 3.)

BC crosses $X'X$ at F, midway between the 2nd and 3rd lines to the left of the point marked -1; hence OF is $-1·25$, the sign being negative since OF is a left step. F is thus the point $(-1·25, 0)$.

* The diagram from which Fig. 6 is reproduced was drawn to this scale.

Similarly, G is the point $(0, -1{\cdot}8)$ and H the point $(2, 0)$.

OV is 1 inch and $OE = 1{\cdot}35\ OV$; the second figure after the decimal point therefore represents hundredths of an inch. It requires careful drawing and thin lines to secure accuracy in this second decimal; besides, in many of the cheaper papers, the errors due to irregular spacing of the lines amount to more than a unit in the second decimal.

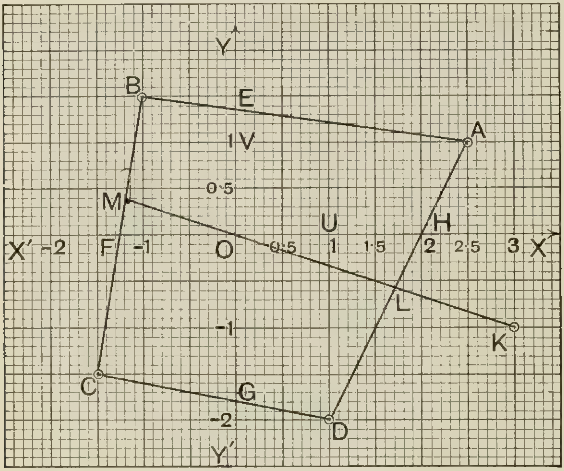


Fig. 6.

Example 2. On Fig. 6 plot the point $K(3, -1)$; let KO cut AD at L and let KO produced cut BC at M. State the coordinates of L and M.

The x of the point L is rather greater than $1{\cdot}7$, say $x = 1{\cdot}71$; the y of L is negative and is numerically less than $0{\cdot}6$, say $y = -0{\cdot}57$. L is therefore the point $(1{\cdot}71, -0{\cdot}57)$.

M is the point $(-1{\cdot}18, 0{\cdot}39)$.

Example 3. At what point does the horizontal line through V (Fig. 6) cut BC, and at what point does the vertical through $(1{\cdot}3, 0)$ cut OK?

The point on BC is $(-1{\cdot}08, 1)$; the point on OK is $(1{\cdot}3, -0{\cdot}43)$.

Facility in reading off distances can only be gained by practice; gross errors, such as the misplacing of the decimal point or the omission of the negative sign, are easily avoided by making a rough estimate and then comparing this estimate with the results obtained from the more careful inspection of the figure.

Another matter requires notice, namely :—**the numbers that are estimated for the lengths of lines should not suggest a degree of accuracy above that which the scale of the drawing admits.** Thus in examples

1–3 one division of the paper is one-tenth of an inch and represents 0·1; on this scale a length which is judged to be say two-thirds of a division should not be stated as 0·06 but as 0·07, which is the nearest two-place decimal approximation to $\frac{2}{3}$ of 0·1. This approximation implies that distances may be estimated to hundredths of an inch but not to thousandths; this standard of approximation is the one we shall assume.

Similarly, on the same scale, $3\frac{2}{7}$ would be plotted as 3·29; $\sqrt{3}$ as 1·73; $\dfrac{1}{\sqrt{3}}$ as 0·58 and so on.

The beginner must be particularly careful not to state results to a number of figures beyond what the scale admits.

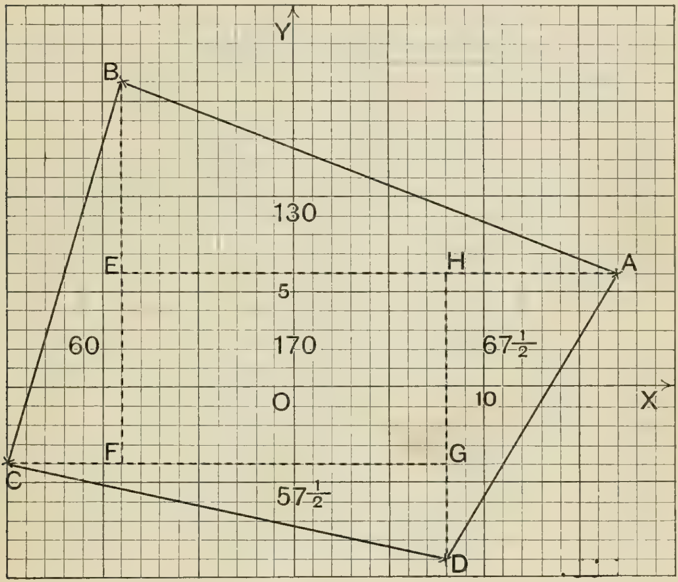


Fig. 7.

It may be noted that, when in example 1 it is stated that OH is 2, all that is meant is that, if OH does differ from 2, the difference is less than one-hundredth; properly stated, OH is 2·00, though in such cases it seems customary to omit the zeros.

Before reading the following examples the beginner should try some of the Exercises II., 1–18.

Example 4. Plot the points $A(17, 6)$, $B(-9, 16)$, $C(-15, -4)$, $D(8, -9)$ and find the area of the quadrilateral $ABCD$ (Fig. 7).

Take one division as unit of length ; 10 divisions = 1 inch.

The dotted lines divide $ABCD$ into four right-angled triangles and a rectangle, the lines being drawn parallel to the axes.

The triangle ABE is half the rectangle whose adjacent sides are EA and EB. The side EA contains 26 units and the side EB 10, so that the rectangle contains 260 and the triangle 130 small squares. In the same way the areas of the other triangles are found.

Again, EH contains 17 and FE 10 units, so that the rectangle $EFGH$ contains 170 small squares. Hence

$$\begin{aligned} ABCD &= EFGH + ABE + BCF + CDG + DAH \\ &= \ \ 170 \ + 130 + \ 60 \ + 57\tfrac{1}{2} + 67\tfrac{1}{2} \\ &= 485. \end{aligned}$$

Since one division represents one-tenth of an inch, one small square represents one-hundredth of a square inch and the area of $ABCD$ is 4·85 square inches.

By a similar process the quadrilateral $ABCD$ in Fig. 6 is found to contain 950 small squares ; its area is therefore $9\frac{1}{2}$ times the square of side OU.

When the figure is bounded wholly or partially by curved lines the area can be found to a fair approximation by counting squares. When only a part of a square lies within the area the usual rule is *to count* 1 *when the part looks greater than half a complete square, but to count* 0 *when the part looks less than half a complete square* ; a part that appears to be exactly a half may be counted as $\frac{1}{2}$.

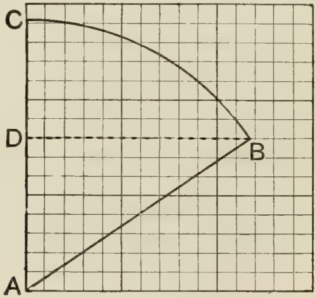


Fig. 8.

In Fig. 8 the area ABC contains about 98 small squares. The triangle ABD is $\frac{1}{2}AD \, . \, DB$; $AD = 8$, $DB = 11·7$ so that ABD is 46·8.

Example 5. Show by measurement that the sides of the quadrilateral in Fig. 6 are

$$AB = 3·54, \quad BC = 3·04, \quad CD = 2·55, \quad DA = 3·35.$$

7. Trigonometric Ratios. Good practice in reading off distances is furnished by the trigonometric ratios. The three principal ratios are defined as follows.

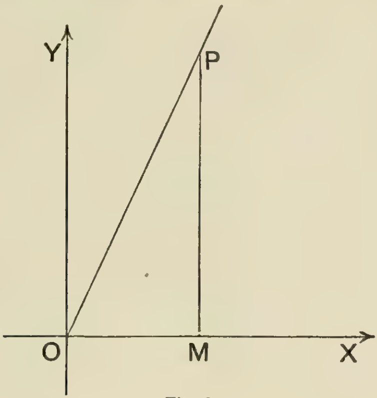


Fig. 9.

Let one arm of an angle A coincide with OX, the positive direction of the x-axis. On the other arm take any point P and draw PM perpendicular to OX.

When A is an acute angle, P will lie in the first quadrant and its coordinates OM, MP will be positive numbers. When A is an obtuse angle, P will lie in the second quadrant; the abscissa of P will then be *negative* but the ordinate will be positive. The line OP, which is the hypotenuse of the right-angled triangle OMP, is *always* to be considered positive. The three fractions or ratios

$$\frac{MP}{OP}, \quad \frac{OM}{OP}, \quad \frac{MP}{OM}$$

are called respectively

the sine, the cosine, the tangent

of the angle A or XOP. The phrase "sine of the angle A" is usually contracted to "$\sin A$"; similarly "$\cos A$" and "$\tan A$" mean "cosine of the angle A" and "tangent of the angle A" respectively. Hence

$$\sin A = \frac{MP}{OP}, \quad \cos A = \frac{OM}{OP}, \quad \tan A = \frac{MP}{OM}.$$

Note that MP is the **ordinate** and OM the **abscissa** of the point P; or, again, MP is the side **opposite** to the angle A and OM the side **adjacent** to the angle A in the **right-angled triangle** OMP. When the angle A is greater than a right angle the words "opposite" and "adjacent" are not very appropriate.

In calculating these ratios from measurements OP should be not less than two inches.

EXERCISES. II.

In examples 1–15 let one inch represent unity.

Plot the points in examples 1–15 :

1. $(2{\cdot}5,\ 1{\cdot}5)$.	**2.** $(1{\cdot}5,\ 2{\cdot}5)$.	**3.** $(2{\cdot}7,\ 1{\cdot}8)$.
4. $(-2{\cdot}3,\ 1{\cdot}4)$.	**5.** $(-3{\cdot}2,\ -1{\cdot}3)$.	**6.** $(2{\cdot}1,\ -1{\cdot}6)$.
7. $(1{\cdot}54,\ 1{\cdot}63)$.	**8.** $(2{\cdot}60,\ 1{\cdot}72)$.	**9.** $(0{\cdot}37,\ 1{\cdot}49)$.
10. $(-2{\cdot}76,\ -1{\cdot}23)$.	**11.** $(-1{\cdot}98,\ 0{\cdot}81)$.	**12.** $(0{\cdot}88,\ -0{\cdot}71)$.
13. $(1\frac{1}{3},\ 2\frac{2}{3})$.	**14.** $(1\frac{3}{7},\ 1\frac{4}{7})$.	**15.** $(\sqrt{2},\ \sqrt{3})$.

Plot the points in examples 16–18, taking one inch to represent 10 :

16. $(6\frac{1}{3},\ 7\frac{2}{3})$.	**17.** $(8\frac{3}{7},\ 9\frac{4}{7})$.	**18.** $(10\sqrt{2},\ 10\sqrt{3})$.

Plot the four points in each of the examples 19–24 and find the sides and the area of each of the quadrilaterals having the four points as vertices. Scale $1''=1$.

19. $(3{\cdot}5,\ 2)$, $(1{\cdot}5,\ 2)$, $(1{\cdot}5,\ -1)$, $(3{\cdot}5,\ -1)$.

20. $(2{\cdot}7,\ 3)$, $(0{\cdot}4,\ 3)$, $(0{\cdot}4,\ -1{\cdot}2)$, $(2{\cdot}7,\ -1{\cdot}2)$.

21. $(1{\cdot}8,\ 1{\cdot}3)$, $(-2{\cdot}4,\ 1{\cdot}3)$, $(-2{\cdot}4,\ -0{\cdot}7)$, $(1{\cdot}8,\ -0{\cdot}7)$.

22. $(2\frac{3}{4},\ 1\frac{1}{2})$, $(-3\frac{1}{4},\ 1\frac{1}{2})$, $(-3\frac{1}{4},\ -2\frac{1}{2})$, $(2\frac{3}{4},\ -2\frac{1}{2})$.

23. $(1{\cdot}24,\ 2{\cdot}62)$, $(0,\ 2{\cdot}62)$, $(0,\ 0)$, $(1{\cdot}24,\ 0)$.

24. $(1{\cdot}86,\ 2{\cdot}27)$, $(-2{\cdot}14,\ 2{\cdot}27)$, $(-2{\cdot}14,\ -1{\cdot}45)$, $(1{\cdot}86,\ -1{\cdot}45)$.

Find the coordinates of the point of intersection of the straight lines AC, BD and the area of the quadrilateral $ABCD$ in each of the examples 25–28 :*

25. $A(2,\ 1)$, $B(-2,\ 2)$, $C(-1,\ -1)$, $D(3,\ -1)$.

26. $A(1{\cdot}7,\ 2{\cdot}3)$, $B(-1{\cdot}8,\ 1{\cdot}3)$, $C(-1{\cdot}6,\ -0{\cdot}5)$, $D(2{\cdot}1,\ 0{\cdot}3)$.

27. $A(2\frac{1}{2},\ 1\frac{1}{3})$, $B(2,\ -\frac{3}{5})$, $C(-1\frac{1}{4},\ -1\frac{2}{3})$, $D(-1,\ 1\frac{3}{7})$.

28. $A(3{\cdot}8,\ 2{\cdot}3)$, $B(0{\cdot}4,\ 1{\cdot}6)$, $C(-1{\cdot}3,\ -2{\cdot}2)$, $D(2{\cdot}4,\ -1{\cdot}7)$.

* In some cases it may be convenient to draw through A, B, C, D parallels to the axes outside the quadrilateral, forming a circumscribed rectangle. $ABCD$ will then be the rectangle diminished by four triangles.

Find the area of the triangles whose vertices are the points in examples 29–34 :

29. (0, 0), (2·4, 0·5), (2·4, 2·1).

30. (0, 0), (−2·3, 0·8), (−2·3, −1·4).

31. (0, 0), (1·5, 2), (0·6, 3).

32. (0·6, 0·4), (2·8, 1·3), (1·3, 2·4).

33. (1·6, 1·2), (−1, 2·3), (−0·4, −1).

34. (2·4, −1·8), (−2·6, 2·3), (−1, −1·4).

Draw, using a protractor, the angles in examples 35–46 and calculate from measurements their three trigonometric ratios :

35. 25°. **36.** 30°. **37.** 35°. **38.** 55°. **39.** 60°. **40.** 65°.
41. 115°. **42.** 120°. **43.** 125°. **44.** 145°. **45.** 150°. **46.** 155°.

8. Distance between two points. Let P (Fig. 10) be the point (a, b) and Q the point (c, d); draw PM, QN perpendicular to $X'X$ and PR parallel to $X'X$, PR meeting NQ or NQ produced at R.

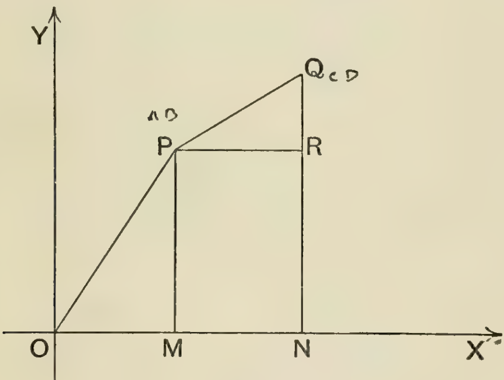


Fig. 10.

The *steps* PR and MN are equal; but

$$MN = MO + ON = -OM + ON = ON - OM = c - a, \ldots(1)$$

and therefore $PR = c - a$. In the same way we find

$$RQ = NQ - NR = NQ - MP = d - b. \ldots\ldots\ldots\ldots(2)$$

These expressions for the steps MN (or PR) and RQ are true whatever be the positions of P and Q. If PR be called the x-component and RQ the y-component of the step PQ (*from* P *to* Q) the results (1) and (2) may be stated thus:

$$x\text{-component of step } PQ = (x \text{ of } Q) - (x \text{ of } P), \quad \ldots\ldots(1')$$

$$y\text{-component of step } PQ = (y \text{ of } Q) - (y \text{ of } P), \quad \ldots\ldots(2')$$

The numerical value of $c-a$ gives the length of the step PR or MN while the sign of $c-a$ tells whether the step is right or left.

Now, $$PQ^2 = PR^2 + RQ^2,$$

and therefore $$PQ^2 = (c-a)^2 + (d-b)^2, \quad \ldots\ldots(3)$$

and the length of PQ is given by

$$PQ = \sqrt{\{(c-a)^2 + (d-b)^2\}}. \quad \ldots\ldots(4)$$

The length of OP is given by

$$OP = \sqrt{(OM^2 + MP^2)} = \sqrt{(a^2 + b^2)}. \quad \ldots\ldots(5)$$

Equation (5) is clearly that case of (4) in which Q coincides with O and therefore $c=0$, $d=0$.

To gain familiarity with and confidence in the results (1′), (2′) the beginner should take several positions of P and Q, for example

$$P(-2, 3),\ Q(1, 2);\quad P(3, 2),\ Q(-1, 1);$$
$$P(-2, -3),\ Q(3, -2).$$

Example. Calculate the distance between the points $A(2{\cdot}5, 1)$, $B(-1, 1{\cdot}5)$ shown in Fig. 6, p. 10.

$$\begin{aligned} AB^2 &= (x \text{ of } B - x \text{ of } A)^2 + (y \text{ of } B - y \text{ of } A)^2 \\ &= (-1 - 2{\cdot}5)^2 + (1{\cdot}5 - 1)^2 \\ &= 12{\cdot}25 + 0{\cdot}25 \\ &= 12{\cdot}50, \end{aligned}$$

$$AB = \sqrt{12{\cdot}50} = 3{\cdot}535\ldots.$$

By measurement we found $AB = 3{\cdot}54$ (example 5, p. 12).

The following definitions will save explanations at a later stage.

Definitions. Two points A and B are said to be **symmetric with respect to a straight line** when the line bisects AB and is perpendicular to AB.

Two points A and B are said to be **symmetric with respect to a point** O when O is the middle point of AB.

EXERCISES. III.

Calculate the distance between the pairs of points in examples 1 – 6

1. $(0, 0), (3{\cdot}2, -2{\cdot}3)$. **2.** $(0, 0), (-3{\cdot}2, 2{\cdot}3)$.

3. $(1{\cdot}6, 2{\cdot}3), (2{\cdot}3, 1{\cdot}6)$. **4.** $(-1{\cdot}3, 2{\cdot}1), (2{\cdot}1, 1{\cdot}3)$.

5. $(-2{\cdot}5, -1{\cdot}2), (2{\cdot}5, -3{\cdot}2)$. **6.** $(4{\cdot}3, -2{\cdot}4), (-3{\cdot}4, -2{\cdot}4)$.

7. Show that the following points lie on a circle whose centre is the origin and whose radius is 5.

$$(5, 0),\ (4, 3),\ (3, 4),\ (0, 5),\ (-3, 4),\ (-4, -3),\ (3, -4).$$

8. Show that the following points lie on a circle whose centre is the point (6, 7) and whose radius is 5.

$$(11, 7),\ (10, 10),\ (9, 11),\ (3, 11),\ (2, 4),\ (6, 2).$$

9. Calculate the sides and diagonals of the quadrilaterals in Exercises II. 25, 26 and test your results by measurement.

10. Show from the diagram of § 7 that

(i) $\sin^2 A + \cos^2 A = 1$; (ii) $1 + \tan^2 A = \dfrac{1}{\cos^2 A}$; (iii) $\tan A = \dfrac{\sin A}{\cos A}$.

[$\sin^2 A$ means "the square of $\sin A$," etc.].

11. Verify the formulae (i), (ii), (iii) of example 10 for the ratios found in Exercises II. 36, 38, 46.

12. Find the coordinates of the points symmetric to the following points with respect to the x-axis.

(i) (3, 2); (ii) (–1, 3); (iii) (–2, –1); (iv) (2, 3).

13. Find the coordinates of the points symmetric to the points in example 12 with respect to the y-axis.

14. Find the coordinates of the points symmetric to the points in example 12 with respect to the origin.

CHAPTER II.

EQUATION OF THE STRAIGHT LINE.

9. Coordinates connected by an Equation. We shall now plot some points whose coordinates, x and y, are connected by an equation.

Example 1. In the equation $y=2x+3$ give to x in succession the values

$$-6,\ -3,\ -1,\ 0,\ 1,\ 3,\ 4\,;$$

associate with each value of x the corresponding value of y deduced

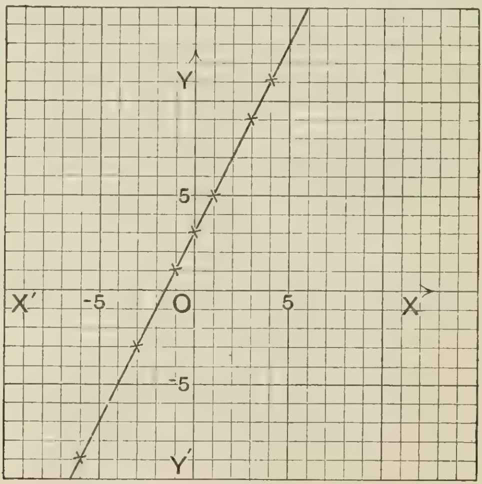


Fig. 11.

from the equation, take each pair of corresponding values of x and y as the coordinates of a point and plot the seven points thus obtained.

When $x=-6$, $y=-9$; when $x=-3$, $y=-3$ and so on. The values may be tabulated as follows:

x	-6	-3	-1	0	1	3	4
y	-9	-3	1	3	5	9	11

Now plot the points $(-6, -9)$, $(-3, -3)$ ... $(4, 11)$. When he has plotted the points the student will probably notice that they seem to lie in a straight line; the observation, if tested by a ruler, will be found correct. Draw the straight line, producing it both ways indefinitely (Fig. 11).

The coordinates of the points $(\frac{1}{2}, 4)$, $(-1\frac{1}{2}, 0)$, $(2\frac{1}{2}, 8)$ satisfy the equation $y=2x+3$; do these points lie on the line? If the points we started with are correctly plotted, the answer is, "Yes."

What is the y of the point on the line for which x is

(i) 5, (ii) $3\frac{1}{2}$, (iii) -2, (iv) -12?

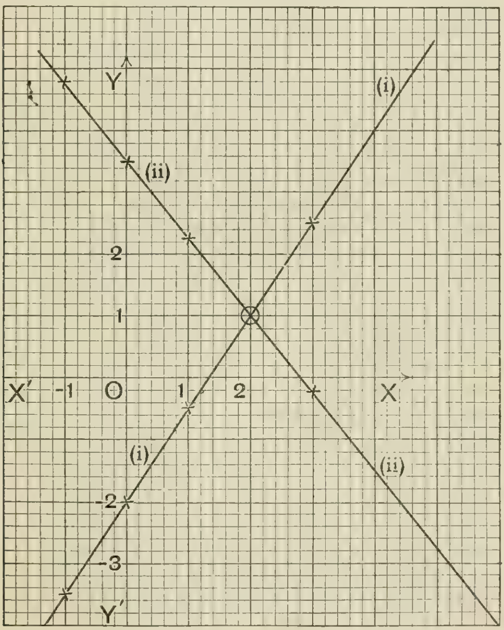


Fig 12.

Do the corresponding values of x and y satisfy the equation $y=2x+3$? For example when $x=5$ the diagram makes $y=13$; do the values $x=5$, $y=13$ satisfy the equation? Obviously they do satisfy it.

Example 2. In the equation $3x-2y=4$ give to x in succession the values $-1, 0, 1, 3$, find the corresponding values of y from the equation and plot the points as in example 1.

The points are $(-1, -3\frac{1}{2})$, $(0, -2)$, $(1, -\frac{1}{2})$, $(3, 2\frac{1}{2})$; these are in a straight line. Draw the line and produce it (Fig. 12 (i)).

From the equation $5x+4y=14$ find the values of y corresponding to the values $-1, 0, 1, 3$ of x and plot the points, using the same axes and scale as before (Fig. 12 (ii)).

The points are $(-1, 4\frac{3}{4})$, $(0, 3\frac{1}{2})$, $(1, 2\frac{1}{4})$, $(3, -\frac{1}{4})$; these again lie in a straight line. Draw the line.

At what point do the lines intersect? Do the coordinates of this point satisfy either or both of the equations?

The point is $(2, 1)$ and the coordinates satisfy both equations.

In examples 1 and 2 the points have been obtained by first choosing values for x and calculating the values of y from the equations. Of course we might have first chosen values for y and calculated the corresponding values of x from the equations. The student may, for example, give to y in example 1 the values $-\frac{1}{2}$, $\frac{1}{2}$, $1\frac{1}{2}$, calculate the corresponding values of x and test whether the points lie on the straight line.

EXERCISES. IV.

In each of the examples 1–14 plot the six points obtained by giving to x the values $-5, -2, 0, 1, 2, 6$ and show by applying a ruler that each set of six lies on a straight line.

Find, by giving to x (or y) other values, other points whose coordinates satisfy one of the equations and test whether the points lie on the straight line constructed from that equation. Do this for examples 1, 8, 13.

Take on each straight line the points whose abscissae are $5, 4, -1, -4$, read off the diagram the corresponding ordinates and then test whether the coordinates of the points satisfy the equation used in constructing the line.

1. $y=x$. **2.** $y=x+2$. **3.** $y=x-2$. **4.** $y=-x$.
5. $y=-x+3$. **6.** $y=-x-3$. **7.** $y=2x$. **8.** $y=2x+4$.
9. $y=2x-4$. **10.** $y=-2x$. **11.** $y=-2x+3$. **12.** $y=-2x-3$.
13. $2x+3y=4$. **14.** $3x-2y+4=0$.

15. Having proved that the points given by equation 1 lie in a straight line how could you show, without calculating the coordinates of each point, that the points given by equations 2 and 3 are in each case in a straight line? Consider in the same way the relation of 5 and 6 to 4, of 8 and 9 to 7, and of 11 and 12 to 10.

16. A point P moves in a plane in such a way that its abscissa with reference to chosen axes is always 2; what is the locus of P, that is what path does P describe?

What is the locus of P if it moves so that its ordinate is always 2?

17. What is the locus of a point in the following cases:
(i) when its x is always -3; (ii) when its y is always -3;
(iii) when its x is always 0; (iv) when its y is always 0;
(v) when its x is always a fixed positive or negative number, $+a$ or $-a$;
(vi) when its y is always a fixed positive or negative number, $+a$ or $-a$?

18. Find any two points, A and B say, whose coordinates satisfy the equation $3x+4y=7$ and any two points, C and D, whose coordinates satisfy the equation $4x-3y=1$. Plot A, B, C, D on the same diagram and read off the coordinates of the point in which the straight lines AB and CD intersect. Test whether the coordinates of this point satisfy both equations.

Try whether other pairs of points, found in the same way as A, B, C, D, give the same straight lines.

19. The same problem as in example 18 for the equations

$$3x-2y=6, \quad 2x+3y=2.$$

20. The same problem as in example 18 for the equations

$$4x-2y+5=0, \quad 5x+8y-15=0.$$

10. Equation of a Straight Line. When pairs of numbers are chosen at random and the points plotted which have these numbers as coordinates, there will usually be no orderly arrangement among the points; they will be scattered all over the diagram. The case is altered however when the coordinates satisfy an equation. The student who has carefully worked through the examples of § 9 and the exercises on pp. 20, 21 must have observed

(i) that not merely the few points whose coordinates were first calculated, but *all* the points he tried whose coordinates satisfied an equation lay on the (unlimited) straight line corresponding to that equation;

(ii) that the coordinates of every point he took on the line satisfied the corresponding equation.

In these examples the equation connecting the coordinates x and y is of the first degree in x and y; in other words each equation is of the *form*

$$ax+by+c=0, \qquad \text{.........................(1)}$$

where a, b, c are numbers. Thus, in example 1, § 9, $a=2$, $b=-1$, $c=3$, for the equation may be written in the form

$$2x-y+3=0.$$

The inference that all points whose coordinates satisfy an equation of the form (1) will lie in a straight line is almost inevitable, after the numerous cases that have been tested; a formal proof that the inference is correct is given in § 14. Meanwhile, assuming the truth of the inference, we see that we have obtained a geometrical meaning for an algebraic equation; namely, whatever be the values of a, b, c the points whose coordinates satisfy equation (1) lie in a straight line, each set of values of a, b, c giving rise to a different line.

It is usual to express this fact by saying that every equation of the first degree in the coordinates, that is, every equation of the form (1) **represents** a straight line; and conversely, that a straight line is **represented** or **given** by an equation of the first degree. The equation is called, with respect to the line, **the equation of the line**; the line is often called **the graph of the equation.**

An equation of the first degree in x and y, since it is the equation of a straight line, is frequently called a **linear equation.**

Test or condition that a given point should lie on the graph of a given equation. How can we tell, without drawing the graph, that a given point (that is, a point whose coordinates are given) lies on the graph of a given equation? The answer is, by testing whether the coordinates satisfy the equation.

For example, does the point $(-4, -4)$ lie on the graph of

$$3x-2y+4=0\,?$$

Yes; because $\quad 3\times(-4)-2\times(-4)+4=0,$

that is, the equation is true when $x=-4$ and $y=-4$.

Does the point (4, 3) lie on the same line? No; because

$$3\times 4-2\times 3+4=10,$$

that is, the equation is not true when $x=4$ and $y=3$.

It is very important that the beginner should thoroughly grasp the fact that **a point does or does not lie on a graph according as its coordinates do or do not satisfy the equation of the graph.**

To draw a straight line, only two points on it are needed; these should be as far apart as possible so that any slight inaccuracy in plotting them may not cause a serious displacement of the line. It is easiest to find the points where the line crosses the axes, but these are seldom the best points to choose.

For example, to draw the graph of

$$3x-2y+4=0$$

we may proceed as follows: The x of all points on the y-axis is zero; but when $x=0$ the equation gives $y=2$, so that the line crosses the y-axis at the point (0, 2). The y of all points on the x-axis is zero; but when $y=0$ the equation gives $x=-1\frac{1}{3}$, so that the line crosses the x-axis at the point $(-1\frac{1}{3}, 0)$. It would be better, however, to find another point than $(-1\frac{1}{3}, 0)$; for example, the point (2, 5) or the point (4, 8).

It is often useful to plot *three* points as a test of accuracy.

It is perhaps worth noting specially that **the equation of the y-axis is $x=0$, and that of the x-axis is $y=0$.** The equation $x=a$, where a is a definite number, represents a line perpendicular to the x-axis, while the equation $y=a$ represents a line parallel to the x-axis. (See examples 16, 17, pp. 20, 21.)

11. Scale Units. Points have often to be plotted whose coordinates differ considerably in magnitude; such points, for example, as (1, 16), (2, 32), (3, 48). In such cases the choice of equal unit steps OU, OV (§ 5) requires either a very small unit length or a very large diagram. We are, however, quite at liberty to choose these unit steps of different lengths; such a choice is quite consistent with the definition of coordinates. Thus, in Fig. 4, $OM=xOU$, $MP=yOV$ and the point P is definitely fixed whether OU and OV have the same length or not.

In many of the most important applications of the method of coordinates the numbers x and y refer to quantities of different kinds, and there is no necessity that the segment which represents a unit of the one quantity should have the same length as that which represents a unit of the other; the scales of representation of the two quantities may, and usually must, be chosen quite independently. As a matter of fact, the student will find as he proceeds that it is in most cases the *relative* and not the *absolute* length of the ordinates that is of importance; if in the same diagram the same unit is used for the ordinates throughout, it does not matter whether it is of the same length as the unit used for the abscissae or not. (See also § 24.)

A proper choice of scales contributes greatly to the usefulness of a diagram; before making his choice the student should find out as far as possible the greatest numbers that have to be represented.

We will now work some examples and show how the graphs may be used to solve equations.

12. Examples on the Straight Line. Solution of Equations.

Example 1. Draw the straight lines given by the equations

(i) $y=10x$, (ii) $y=10x+12$, (iii) $y=10x-12$.

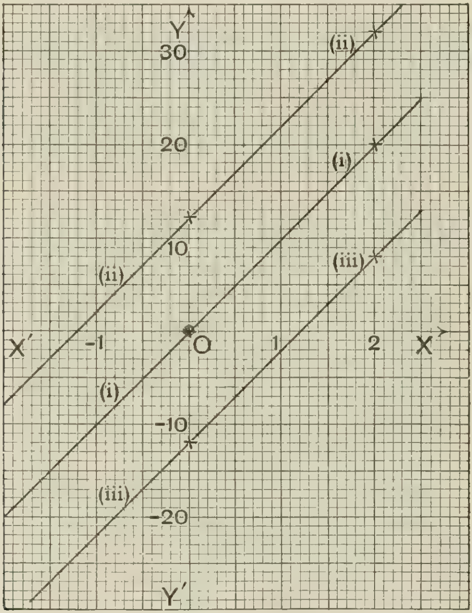


Fig. 13. Scale reduced to one-half.

Equal horizontal and vertical units would give an inconvenient representation. Let 1 inch along OX be the x-unit but let 1 inch along OY count 10 y-units, that is, take the vertical unit line to be $\frac{1}{10}$th of the horizontal unit line.

The origin (0, 0) is a point on (i); to get another point let $x=2$ and we get the point (2, 20). To plot the point (2, 20), move 2 *horizontal* *units* to the right along OX, then 20 *vertical units* upwards; that is, move 2 inches to the right, then 2 inches upwards.

For (ii) and (iii) put 0 and 2 for x; we thus get the points (0, 12), (2, 32) on line (ii) and the points (0, -12), (2, 8) on line (iii).

Fig. 13 shows the lines. They *seem* to be parallel and it is easy to prove that they are so. The line (ii) is simply the line (i) moved 12 units up the diagram; for if we take *any* two points, one on each line, *having the same abscissa*, the ordinate given by (ii) is greater by 12 than that given by (i). Similarly line (iii) is simply line (i) moved 12 units down the diagram.

The student will have no difficulty in seeing that the line given by $y=ax+b$, where a and b are any two numbers, is parallel to that given by $y=ax$; the latter passes through the origin and the former lies b units above it when b is positive, but below it when b is negative.

Example 2. Draw on the same diagram and with the same scales* the straight lines given by the equations

$$\text{(i) } y=4x+10, \quad \text{(ii) } 7x+2y=50$$

and state the coordinates of their point of intersection.

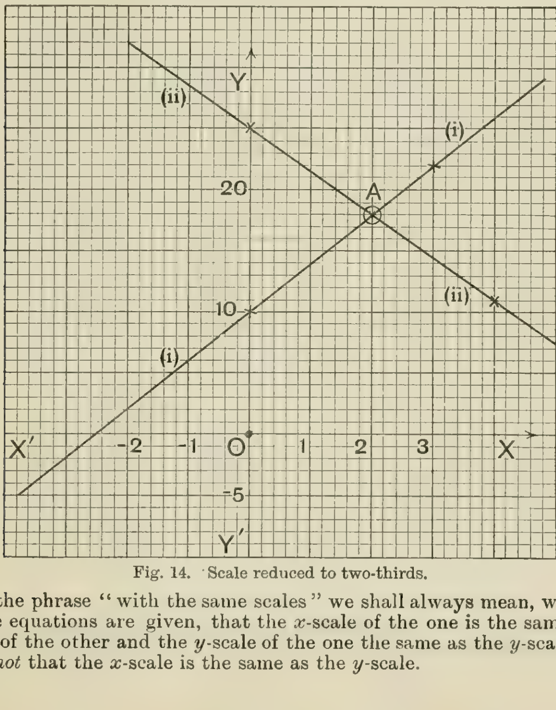


Fig. 14. Scale reduced to two-thirds.

*By the phrase "with the same scales" we shall always mean, when two or more equations are given, that the x-scale of the one is the same as the x-scale of the other and the y-scale of the one the same as the y-scale of the other, *not* that the x-scale is the same as the y-scale.

Two points on line (i) are (0, 10), (3, 22); two points on line (ii) are (0, 25), (4, 11).

For scales, let 1 inch represent the value 2 of x and the value 10 of y.

The lines are shown in Fig. 14. The point of intersection A is (2, 18); so far as we can see from the diagram the x is exactly 2 and the y exactly 18.

Since A lies on *both* lines its coordinates must satisfy *both* equations (§ 10); trial shows that both equations are true when $x=2$, $y=18$. The roots of the simultaneous equations (i) and (ii) are therefore $x=2$, $y=18$.

It is evident that we have now a graphical method of solving two simultaneous equations of the first degree; all that we have to do is to **draw the lines given by the equations and read off the coordinates of their point of intersection.** In applying this method it is essential that *the same scales* should be used for the two equations.

Conversely, to find the point of intersection of two straight lines whose equations are given, we must solve the equations, treating them as simultaneous equations.

The solution of the equation $4x+10=0$ is equivalent to the solution of the simultaneous equations

$$\text{(i) } y=4x+10, \quad \text{(ii) } y=0;$$

we draw the line given by (i) and find where it crosses the line given by (ii), that is, find where it crosses the x-axis, whose equation is $y=0$. The value of x for that point is the root required.

For an equation of the first degree in one unknown the method is of little importance but, as we shall see, it is of great value for equations of higher degrees.

Example 3. Find the equation of the straight line that passes through the points (2, 3), $(-4, 1)$.

Whatever may be the values of a, b, c, the equation

$$ax+by+c=0 \text{.................................(i)}$$

represents a straight line. We must therefore choose the numbers a, b, c so that the equation may be true both when $x=2$ and $y=3$ and also when $x=-4$ and $y=1$. Hence we have to solve the two simultaneous equations

$$2a+3b+c=0, \quad -4a+b+c=0.$$

Since there are only two equations we solve for two of the numbers a, b, c in terms of the third; we get $a=\frac{1}{7}c$, $b=-\frac{3}{7}c$. Substitute these values in (i); c will now occur in every term and may therefore be divided out. Clearing of fractions we find for the required equation

$$x-3y+7=0$$

and it is easy to verify that the given coordinates satisfy the equation.

In later work the equation of the straight line will usually be taken of the form

$$y=ax+b, \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

which is really equivalent to (i), although it contains only two numbers a, b while (i) contains three a, b, c. For, after division by b and transposition of terms, (i) becomes

$$y=-\frac{a}{b}x-\frac{c}{b}, \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

and the *form* is now that of (ii). We may represent the fractional forms $-\frac{a}{b}$, $-\frac{c}{b}$ by single letters, since each letter may represent any number, positive or negative, integral or fractional; we take a, b as standard letters, but the a, b of (ii) are of course not the same as the a, b of (i).

The only exception is the case in which b of equation (i) is zero; that equation is then $ax+c=0$ and represents a straight line perpendicular to the x-axis. If the two given points happen to be in a line perpendicular to the x-axis, the form (ii) would give *two inconsistent equations* for finding a, b.

Thus, if the points are (1, 1), (1, 3), equation (ii) gives

$$1=a+b, \quad 3=a+b$$

and these are inconsistent. Equation (i) however gives

$$a+b+c=0, \quad a+3b+c=0$$

and now $b=0$, $c=-a$ and the equation of the line is

$$ax-a=0, \quad \text{or } x=1.$$

If form (ii) gives inconsistent equations, then form (i) may be taken; but with a very little practice the student will notice at once whether the points are in a line perpendicular to the x-axis, and will be able to write down the equation without calculation.

It should be noticed that the *two* numbers a, b of (ii) and the *two* fractions of (iii) correspond to the property that *two* points determine a straight line.

EXERCISES. V.

1. Find, without drawing the line, which, if any, of the points

$$(3, 2),\ (4, 3),\ (-2, -2),\ (8, 6),\ (5, 4),$$

lie on the line given by $4x-5y=2$.

Solve equations 2–15 graphically and verify your solutions by testing whether the coordinates satisfy *both* equations.

2. $3x-2y=0,\ x-y+1=0.$	**3.** $x-2y+11=0,\ 2x-3y+18=0.$	**4.** $4x-7y=13,\ x-8y=22.$
5. $4x+y=10,\ 3x-4y=17.$	**6.** $2x+4y=15,\ 4x+2y=15.$	**7.** $2x+y+1=0,\ 8x+6y=3$
8. $3x+9y+14=0,\ 9x+12y+2=0.$	**9.** $3x-2y=2,\ 20x-25y+24=0.$	**10.** $y=25x+13,\ y=50x-62.$
11. $4y=75x-124,\ 5y=36x+76.$	**12.** $5x+36y=160,\ 8x+45y=130.$	**13.** $x+16y=112,\ 3x+13y=161.$
14. $2{\cdot}63x+3{\cdot}12y=12,\ 2{\cdot}14x-2{\cdot}36y=5.$	**15.** $23{\cdot}5x+34{\cdot}5y=810,\ 18{\cdot}4x-46{\cdot}6y=857.$	

16. Solutions of the equation $3x+4=a$ are wanted for several values of a; how may the solutions be obtained graphically?

If solutions of $3x+4=bx+c$ are wanted for various values of b and c how may they be obtained graphically?

17. Find the equations of the straight lines through the following pairs of points:

(i) $(5, 6), (-5, -3)$; (ii) $(-7, 8), (7, -8)$; (iii) $(6, -4), (-7, -3)$; (iv) $(6, 7), (-3, 7)$; (v) $(2, -3), (2, 4)$.

18. Find the coordinates of the vertices of the triangle whose sides are given by the equations:

$$x-2y+4=0,\quad x+y+1=0,\quad 5x-y=7.$$

19. Show by solution of equations that the three straight lines whose equations are

$$4x=3y,\quad y=5x-11,\quad 5y=x+17$$

all pass through one point. Verify by drawing the lines.

20. Show that the three points $(3, -1)$, $(-2, 4)$, $(5, -3)$ are in a straight line, and find the equation of the line.

21. Find the equations of the straight lines AC, BD in examples 25–28, Exercises II. (p. 14), and determine the coordinates of the point of intersection of the lines by solving their equations as simultaneous equations.

CHAPTER III.

NOTION OF A FUNCTION. PRACTICAL APPLICATIONS OF GRAPHS.

13. Variable. Constant. Function. As a point moves along the straight line given by the equation $y=6x+5$, the x of the point goes through, or *takes*, a succession of values; the y of the point also goes through a succession of values, but the values that y takes can be calculated from the equation when those of x are known. Or, again, we may say that if we give to x a series of values, y is restricted by the equation to another series of values, and the two series determine a point which moves along the straight line as x goes through its values.

In other words, x is a **variable**; so is y, but since the equation fixes the value of y as soon as a definite value is given to x the variable y is said to be **dependent** on x. Since the values of x are supposed to be first given, x is called the **independent** variable of the equation. We might, of course, first assign values to y and then calculate those of x; y would now be the independent, and x the dependent variable. It is usually a mere matter of convenience which is taken as independent; that variable whose values are the objects of inquiry or calculation is the dependent one.

Another method of stating the connection between two variables, one of which is dependent on the other, is to say that the dependent variable is a **function** of the other variable, which is then often called the **argument** of the function.

The graph of an equation shows very clearly how the function varies as the argument changes. The abscissa is usually taken as the argument or independent variable, and the ordinate then represents the function; the graph is therefore often called **the graph of the function.** Thus, Fig. 13 shows the graphs of the three functions

$$10x, \; 10x+12, \; 10x-12;$$

the two expressions—"the graph of the function $10x$" and "the graph of the equation $y=10x$"—mean the same thing.

Since the graph of the function $ax+b$ is a straight line this function is often called a **linear function** of x.

In the expression $ax+b$ there are three letters, but only one of these is a variable in the sense now explained. The letters a, b denote definite numbers; they fix the particular line we are dealing with. For each set of values of a and b we get one line, and x and y vary from point to point as we go along the line; a change in a or b would give rise to a new line and to a new case of the linear function. Letters such as a, b that retain the same value all through any one investigation are called **constants.**

It is customary to denote constants by the earlier letters of the alphabet $a, b, c \ldots$, and variables by the later letters $z, y, x \ldots$; but when there is any advantage in denoting a variable by a or a constant by z there is of course no reason against doing so.

Example 1. The variables x and y are connected by the equation

$$2xy-3x-5y+7=0;$$

express y explicitly as a function of x.

The equation clearly makes y dependent on x, for if we give to x any value we can calculate the value of y; in mathematical language, the equation is said to *define* y as a function of x. To see more plainly how y depends upon x, solve the equation for y in terms of x; we find

$$(2x-5)y=3x-7$$

and therefore $$y=\frac{3x-7}{2x-5}.$$

y is now said to be expressed *explicitly* as a function of x while, so long as the equation is not solved for y, it is only *implicitly* expressed as a function of x; in the unsolved form of the equation y is an *implicit function* of x while in the solved form it is an *explicit function* of x.

The equation also defines x as a function of y, namely

$$x=\frac{5y-7}{2y-3},$$

as may be seen by solving the equation for x. Both functions are *fractional* functions of their arguments.

Example 2. A stone is thrown vertically upwards with a velocity of V feet per second; express the distance travelled in a given time as a function of the time.

Suppose that in t seconds the stone has risen s feet above the point of projection; then it is shown in books on mechanics that, when the resistance of the air is left out of account,

$$s=Vt-\tfrac{1}{2}gt^2,$$

where g is a constant, equal to 32·2 approximately. The distance travelled is therefore a function of the time; since the time t enters into the expression of the function in the second and no higher degree, the distance s is a **quadratic function** of the time t.

The velocity v at time t is a linear function of the time because

$$v=V-gt.$$

The graph of the velocity v is a straight line; the graph of the distance s is a curved line called a parabola (§ 29).

In this example s, v, t are variables; V, g are constants.

Example 3. A point moves in a circle of radius 5, and centre O, the origin of coordinates; express the ordinate of the point as a function of its abscissa.

Let x, y be the coordinates of P in any one of its positions; then (§ 8)

$$OP^2=x^2+y^2$$

and therefore

$$x^2+y^2=25, \quad\text{.........(i)}$$

so that

$$y=\sqrt{(25-x^2)} \quad\text{.........(ii)}$$

To express y fully we must remember that the root may be either positive or negative; the symbol $\sqrt{(25-x^2)}$ is *two-valued*, namely is either $+\sqrt{(25-x^2)}$ or $-\sqrt{(25-x^2)}$. The $+$ sign goes with points above the x-axis, the $-$ sign with points below that axis.

Equation of a circle. We have here found the equation of a circle. It is easy to find the equation of *any* circle. Let its centre be the point A (a, b) and let its radius be c; then if P (x, y) is any point on it we have (§ 8)

$$(x-a)^2+(y-b)^2=AP^2=c^2 \quad\text{.........(c)}$$

which is the required equation.

The student should verify the equation for different positions of the centre and different values of the radius.

EXERCISES. VI.

1. The base of a triangle is b inches, its height h inches and its area A square inches; write down the equation that connects b, h and A. If h is constant and b, A variable what kind of function is A of b? Represent graphically the relation between b and A when h is constant.

2. The radius of a circle is r, its circumference is c and its area A. What kind of function is (i) c of r, (ii) A of r? Represent graphically the relation between r and c.

3. When a quantity of gas expands at constant temperature, the product of its pressure, p lb. per sq. in., and its volume, v cub. in., is constant, equal to C say. Express p as a function of v.

4. If the effort, E lb., required to raise a load, W lb., is a linear function of the load write down the general expression for E as a function of W.

5. y is given as a function of x by the equation

$$axy + bx + cy + d = 0;$$

express y explicitly as a function of x.

6. Draw (with compasses) the circle whose centre is the origin and whose radius is 5, and find the coordinates of the points in which it is cut by the straight line whose equation is

$$5y = 3x + 10.$$

[In this case the unit length must be the same for the y-scale as for the x-scale.]

7. Draw the circle, centre (2, 3) and radius 3, and find the coordinates of the points in which it is cut by the straight line

$$y = 2x + 3.$$

Of what two simultaneous equations are these coordinates the roots?

8. What are the coordinates of the point or points in which the circle of example 7 cuts (i) the x-axis, (ii) the y-axis? What are the equations that the values of x in case (i) and the values of y in case (ii) satisfy?

9. Find the equations of the following circles:

(i) centre $(-2, 3)$, radius $= 5$. (ii) centre $(2, -3)$, radius $= 5$.
(iii) centre $(-1\frac{1}{2}, -2\frac{1}{2})$, radius $= 6$. (iv) centre $(2{\cdot}4, -2{\cdot}4)$, radius $= 2{\cdot}4$.

10. Show that the equation

$$x^2 + y^2 - 4x + 6y + 7 = 0$$

represents a circle and find its centre and radius.

[The equation may be written

$$(x-2)^2 + (y+3)^2 = 6,$$

that is $$(x-2)^2 + \{y - (-3)\}^2 = (\sqrt{6})^2.$$

By comparing with equation (c), p. 31, we see that this equation represents a circle, centre $(2, -3)$ and radius $\sqrt{6}$ or $2{\cdot}449$.]

11. Show that the following equations represent circles and find their centres and radii:

(i) $x^2+y^2+2x-4y+1=0$. (ii) $x^2+y^2+6x+4y+4=0$.
(iii) $x^2+y^2+8x-12y=12$. (iv) $2x^2+2y^2-6x-2y=3$.

12. Show that the equation (where a, b, c are constants)

$$x^2+y^2+ax+by+c=0$$

represents a circle and find its centre and radius.

[The equation may be written

$$(x+\tfrac{1}{2}a)^2+(y+\tfrac{1}{2}b)^2=+\tfrac{1}{4}a^2+\tfrac{1}{4}b^2-c=\left\{\frac{\sqrt{(a^2+b^2-4c)}}{2}\right\}^2.$$

The centre is $(-\frac{1}{2}a, -\frac{1}{2}b)$; the radius is $\frac{1}{2}\sqrt{(a^2+b^2-4c)}$.]

13. Find the equation of the circle through (2, 0), (0, 1), (3, 4) and give its centre and radius.

[The equation must be of the form given in example 12; determine a, b, c so that that equation may be true when the coordinates of each point are substituted in it. We get three equations, namely

$$4+2a+c=0,\quad 1+b+c=0,\quad 25+3a+4b+c=0,$$

whence $$a=-\tfrac{11}{3},\quad b=-\tfrac{13}{3},\quad c=\tfrac{10}{3}.$$

Hence the required equation is

$$x^2+y^2-\tfrac{11}{3}x-\tfrac{13}{3}y+\tfrac{10}{3}=0$$

and the centre is $(\frac{11}{6}, \frac{13}{6})$ and the radius $\frac{1}{6}\sqrt{170}=2{\cdot}17$. (Compare § 12, example 3.)]

14. Find the centre and radius of the circle through the three points in examples (i)–(iii):

(i) (0, 0), (−5, 0), (0, 6). (ii) (1, 1), (−1, −1), (1, −1).
(iii) (2, 3), (−4, 0), (0, −5).

14. Gradient of a Straight Line. We shall now prove that the equation $y=ax$ represents a straight line; the general case

$$y=ax+b \quad \text{or} \quad ax+by+c=0$$

can then be inferred as in § 12, examples 1 and 3.

First, let a be positive; for definiteness, suppose $a=2$.

In Fig. 15 let $OU=1$; draw UA perpendicular to OU and equal to 2 units of the y-scale. On the unlimited straight line through O and A take any two points P and Q and draw PM and QN perpendicular to $X'X$.

The coordinates of P are both positive, those of Q are both negative, and therefore in both cases the quotient of ordinate by abscissa is positive.

Again, the triangles OMP, ONQ are both equiangular to the triangle OUA; hence

$$\frac{MP}{OM}=\frac{UA}{OU}=2,\quad \frac{NQ}{ON}=\frac{UA}{OU}=2$$

and therefore $MP=2OM$, $NQ=2ON$.

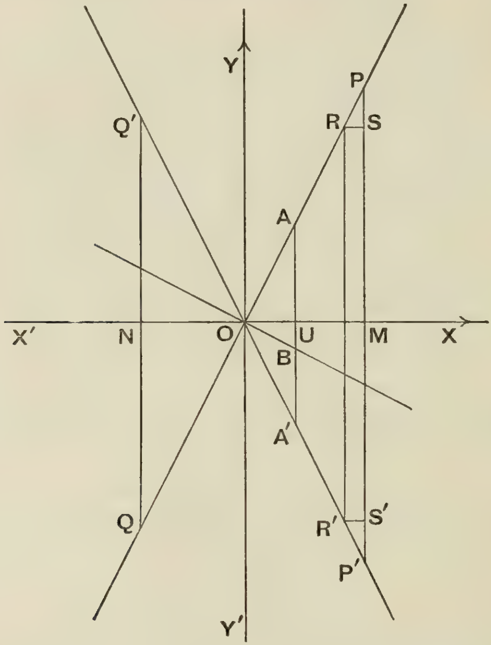


Fig. 15.

If therefore x and y are the coordinates of any point on the line, such as P or Q, we have $y=2x$. In other words, the coordinates of *every* point on the line satisfy the equation $y=2x$. It is easy to prove that if a point is not on the line its coordinates will not satisfy the equation.

Second, let a be negative, say $a=-2$.

Draw UA' downwards perpendicular to OU and let the length of UA' be 2 units of the y-scale: complete the construction as in Fig. 15.

The coordinates of P' are of opposite signs; so are those of Q', and therefore in both cases the quotient of ordinate by abscissa is *negative.* Exactly as in the first case it will now be seen that the coordinates of every point on the line $P'Q'$ satisfy the equation $y = -2x$.

The proof for other values of a is similar to that now given. Obviously when $a=0$ the equation is $y=0$ and represents the x-axis. In all cases therefore the equation $y=ax$ represents a straight line through the origin O; the equation $y=ax+b$ represents a straight line parallel to that given by $y=ax$.

Definition. The coefficient of x in the equation $y=ax+b$ is called the **gradient** (sometimes the **slope**) of the straight line represented by the equation.

The following ways of interpreting the gradient are important:

Geometrically, the x-axis being supposed horizontal and the y-axis vertical, the gradient measures **the rate at which the line rises or falls.** When a is positive the line has a right-hand upward slope; a point rises as it moves towards the right along the line. When a is negative the line has a right-hand downward slope; a point falls as it moves towards the right along the line. When $a=0$ the line is horizontal; the greater a is (numerically) the greater is the angle the line makes with the horizontal. When a is very large the angle is nearly a right angle; when the angle is 90° the gradient will be said to be **infinite.**

The gradient may of course be obtained by considering any portion of the line, long or short. Thus, the gradient of the portion RP (Fig. 15) is the vertical rise SP divided by the horizontal advance RS and this quotient, since the triangles RSP, OUA are equiangular, is equal to UA divided by OU, that is, is equal to 2. Similarly, the gradient of $R'P'$ is the vertical *fall* $S'P'$ divided by the horizontal advance $R'S'$ and this quotient is equal to -2.

Trigonometrically, the gradient a is **the tangent of the angle which the line makes with the x-axis.** When the line has a right-hand downward slope, the angle may be taken to be the *negative* angle XOP' or the *obtuse* angle XOQ'; tan XOP' and tan XOQ' are both negative.

Algebraically, the gradient a measures **the rate at which the function $ax+b$ varies as x varies.** When x increases by any amount, y or $ax+b$ increases by a times as much. If a is negative, y will decrease as x increases; a decrease is to be considered as a negative increase.

For example, let $y=2x+5$. As x increases from 1 to 4, y increases from 7 to 13; that is when x increases by 3, y increases by 6 or twice as much.

Again, let $y=-2x+5$. As x increases from 1 to 4, y changes from 3 to -3; that is when x increases by 3, y decreases by 6 (because $-3=3-6$) which is twice as much as the increase in x.

Since the increase of $ax+b$, when x increases by any amount, is always a times the increase of x, the linear function $ax+b$ is called a **uniformly varying** function of x. The **rate** at which the function varies is **constant** and equal to a; or again, the increase of $ax+b$ is always **in simple proportion** to the increase of x.

Example 1. What is the gradient of the line given by the equation

$$7x+2y=50\,?$$

The equation may be written $y=-\frac{7}{2}x+25$. Hence the gradient is $-\frac{7}{2}$; the line has a right-hand downward slope and falls 7 units for every 2 units of horizontal advance or at the rate 7 in 2.

Example 2. Find the equation of the straight line with gradient $\frac{2}{5}$ passing through the point (3, 5).

Let $y=ax+b$ be the required equation. Then $a=\frac{2}{5}$, and the equation becomes $y=\frac{2}{5}x+b$.

Since the line goes through (3, 5) we have

$$5=\tfrac{2}{5}\times 3+b \quad \text{or} \quad b=\tfrac{19}{5},$$

and the required equation is

$$y=\tfrac{2}{5}x+\tfrac{19}{5} \quad \text{or} \quad 2x-5y+19=0.$$

Similarly it may be shown that the equation of the line with gradient g passing through the point (h, k) is

$$y=gx+k-gh,$$

or, in a form that is more easily remembered,

$$y-k=g(x-h).$$

Example 3. Show that the gradient of the line drawn through any point at right angles to the line $y=ax+b$ is $-\frac{1}{a}$.

The gradient of the line through the origin O perpendicular to the line $y=ax$ will clearly be that required. Draw OB perpendicular to OA (Fig. 15) and let OB cut UA' at B; then, taking OA as the line with gradient a, we have $UA=a$.

Now the triangles BUO, OUA are equiangular, so that

$$\frac{BU}{OU}=\frac{OU}{UA} \text{ and } BU=\frac{1}{UA}=\frac{1}{a}.$$

The *length* of the ordinate UB is $1/a$ but the *sign* of the ordinate UB is opposite to that of UA. Hence both in size and in sign

$$UB=-\frac{1}{a}.$$

But the gradient of OB is UB, since OU is unity.

In this proof it is assumed that the unit line for the ordinates is of the same length as OU, the unit line for the abscissae; if these units are of different lengths, the triangles BUO, OUA will be distorted and will not be similar. The student should note examples 17, 18 in Exercises VII.; if the lines are correctly drawn they will not seem to the eye to be perpendicular to each other.

EXERCISES. VII.

Find the equations of the straight lines through the points in examples 1–4 and state the gradient of each:

1. $(2, 4), (-3, 1)$. **2.** $(-4, 6), (4, -6)$.

3. $(-7, -11), (4, 0)$. **4.** $(3, 7), (-7, 3)$.

Find the equations of the straight lines passing through the point and sloping at the gradient given in examples 5–10:

5. $(0, 0)$, $1{\cdot}5$. **6.** $(3, 2)$, -5. **7.** $(-5, -4)$, $\frac{5}{3}$.

8. $(-3, 6)$, $-2{\cdot}5$. **9.** $(4, -8)$, $-\frac{1}{2}$. **10.** $(6, -3)$, $\frac{1}{5}$.

Find the equations of the straight lines through the point (3, 4) perpendicular to the lines in examples 11–16:

11. $y=3x+7$, **12.** $y=-3x+7$. **13.** $4x-2y=5$.

14. $4x+2y=5$. **15.** $5x+6y+4=0$. **16.** $6x-5y=12$.

17. Taking the unit for the x-scale to be 1 inch and that for the y-scale to be $\frac{1}{10}$th of an inch, draw the straight line $y=10x$ and the straight line through the origin perpendicular to $y=10x$.

18. The same problem as in example 17 for the straight lines

$$y=-10x, \quad y=20x, \quad y=-15x.$$

19. y and z are each linear functions of x, but y increases twice as fast as z; when $x=0$, $y=2$, $z=6$; when $x=12$, $y=z$. At what rate does z increase?

20. y and z are each linear functions of x, but y decreases three times as fast as z increases; when $x=0$, $y=9$, $z=-3$; when $x=1$, $y=1$. At what rate does z increase?

15. Applications of Graphs. We shall now give some illustrations of the way in which graphs may be applied.

The student will probably have noticed that a straight line, referred to coordinate axes, can be used as a kind of multiplication table or of combined addition and multiplication table. Thus the ordinate of line (i), Fig. 14, gives the value of $4x+10$ for every value of x within the range of the diagram; when x is, for example, 1·6 the value of $4x+10$ is at once found from the diagram to be 16·4, because 16·4 is the value of the ordinate when x is 1·6. Similarly the ordinate of line (ii) in the same figure shows that $25-3{\cdot}5x$ is 19·4 for the value 1·6 of x.

When no great accuracy is required a graph may usefully replace a table or serve as a "ready-reckoner," as in the following simple examples:

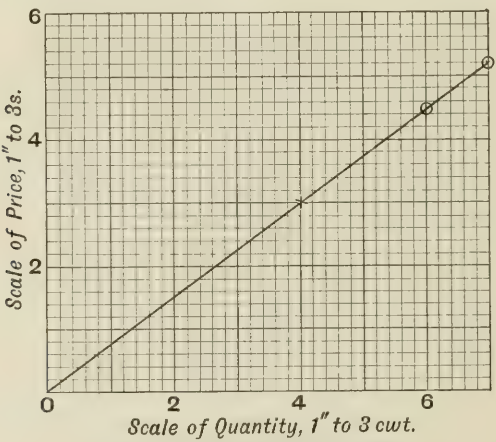


Fig. 16. Scale reduced to two-thirds.

Example 1. Construct a graphical ready-reckoner to show the price of coal at 9d. per cwt.

Let distances measured along OX (Fig. 16) represent the number of cwt., the scale being say 1″ to 2 cwt. and let distances measured along OY represent the cost, the scale being 1″ to 2 shillings.

If x cwt. cost y shillings then $y=\frac{3}{4}x$; the relation between x and y, since this equation is of the first degree, can be represented by a straight line. When $x=0$, $y=0$ and when $x=4$, $y=3$. The line through O and the point (4, 3) will serve as a ready-reckoner.

Thus, when $x=6$, $y=4\frac{1}{2}$; that is, 6 cwt. cost 4s. 6d. Again, when $y=5$, $x=6\frac{2}{3}$; thus for 5s. one can buy $6\frac{2}{3}$ cwt.

It is obvious that with a large sheet of paper it would be possible to obtain from it a considerable range of quantities and prices with fair accuracy.

Example 2. Represent graphically the relation between the Fahrenheit and Centigrade scales of temperature.

Let F and C indicate the readings on the two scales corresponding to the same temperature ; then

$$F=32+\frac{180}{100}C\,; \quad C=\frac{5}{9}(F-32).$$

To indicate with fair accuracy temperatures from, say, 0°C to 100°C a large sheet is necessary, but if a much smaller range is all that is required, a range from 20° C to 50° C for example, we may proceed as follows :

Take the values of F as abscissae, the scale being 1″ to 20° F, and the values of C as ordinates, the scale being 1″ to 10° C. The least value

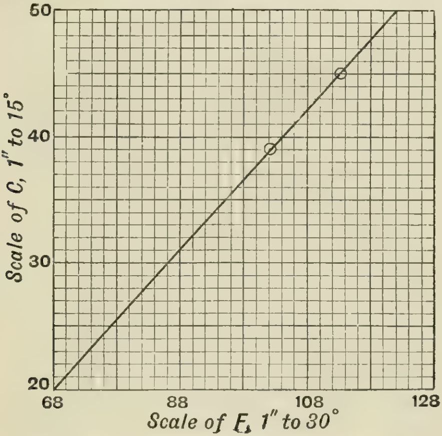


Fig. 17. Scale reduced to two-thirds.

of F that has to be shown is 68 because $F=68$ when $C=20$; since no point *to the left of or below the point* (68, 20) is required, it is convenient to measure the coordinates along lines drawn through this point parallel to the coordinate axes. This device is often useful ; it might be referred to as **a change of axes to parallel axes through the point** (68, 20). (Fig. 17).

The equation between F and C is of the first degree and therefore the relation between F and C will be represented by a straight line; to draw the line take the points (68, 20), (122, 50). It is easy now to read off the diagram corresponding values of F and C; for example

$$100^\circ \text{ F} = 37^\circ{\cdot}8 \text{ C}, \quad 45^\circ \text{ C} = 113^\circ \text{ F}.$$

Determination of a Graph by a limited number of Points. When the relation between two quantities can be expressed by an equation of the first degree the graph that represents that relation, being a straight line, can be drawn after plotting two points representing two pairs of corresponding values of the quantities. When the relation can be expressed by an equation that is not of the first degree it is still possible to draw the graph that represents that relation, as will be shown in subsequent chapters. But in many cases the quantities considered are not given as satisfying an equation; only a *limited number* of corresponding values is given and therefore only a limited number of points can be plotted. To draw the graph that represents the *general* relation between the two quantities (as the straight line for example represents the general relation between the Fahrenheit and Centigrade scales) is in such cases apparently a problem that does not admit of a definite solution; because through a *limited number* of points we can obviously draw as many curves as we please.

The problem however is not so indefinite as it appears to be. In experimental work like that of a physical or chemical laboratory it may usually be assumed that some definite relation or law connects the two quantities considered; when corresponding values of these quantities are taken as abscissa and ordinate and the points plotted, the *simplest curve* that passes evenly among the points may be taken as the graphical representation of that relation or law. When the curve has been drawn it may sometimes be possible to find its equation and thus to obtain an algebraic expression for the relation.

In the case of *statistical results,* on the other hand, it is probably best for the beginner to join successive points by straight lines; when the graph consists of a succession of straight lines each of which makes an angle with the two lines adjacent to it, the graph is called a *broken line* to distinguish it from a *continuous curve* like a circle or a parabola. Problems on prices may also be represented graphically by broken lines.

When used with proper precautions this graphical representation is of the utmost value, but it is only by experience that the student will understand the justification of the assumptions made as well as the limitations inherent in the method.

16. Statistics. Prices. Problems.

Example 1. The following table from Mulhall's *Dictionary of Statistics,* p. 442, gives for the years named the population (in millions) of the United Kingdom, France and Germany:

	1800	1830	1860	1880	1890
United Kingdom, - -	16·2	24·4	29·1	35·3	38·2
France, - - - -	27·35	32·5	36·7	37·6	38·8
Germany, - - -	23·18	29·7	38·1	45·2	48·6

Take the abscissae to represent the time to a scale of 1″ to 30 years, and the ordinates to represent the number of millions in the population to a scale of 1″ to 10 millions; measure these numbers along lines through the point (1800, 16) parallel to the coordinate axes. (Compare § 15, example 2.)

Plot the points for each country and join consecutive points for the respective countries by a straight line; mark the diagram as shown (Fig. 18).

The diagram shows very clearly the comparative rate of growth of population both of the same country at different periods and of different countries at the same period.

Assuming that the growth of population in each period is uniform for that period, we can find the population at any date between 1800 and 1890; to take the straight line as representing the relation between the population and the year during any interval is equivalent to the assumption that the population grows at a uniform rate during that interval, and the gradient of the line measures the rate of growth (§ 14).

For 1845, for example, the ordinates are 26·7, 34·6 and 33·9 respectively and the population is therefore given by these numbers (in millions). Values obtained in this way from a diagram are said to be **interpolated.**

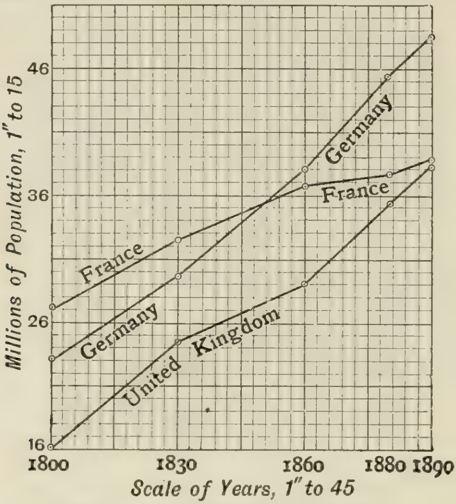


Fig. 18. Scale reduced to two-thirds.

Example 2. In a certain price list the cost (P pence) of saucepans of capacity C pints is given as follows:

C	2	3	4	8	12
P	16	19	22	30	39

What is the probable cost of saucepans of capacity 6 pints and 10 pints respectively?

Plotting as shown in Fig. 19 and joining consecutive points by straight lines, we see that when $C=6$, $P=26$ and when $C=10$, $P=34\frac{1}{2}$; the cost therefore is in one case 2s. 2d. and in the other 2s. $10\frac{1}{2}$d. As a matter of fact, the listed prices are 2s. 2d. and 2s. 9d.; probably the 12-pint saucepan is too dear.

Example 3. If 100 tickets are taken for an excursion the cost of a ticket will be 7s. 6d. but if 150 are taken the cost will be only 6s.; what will be the probable cost of a ticket if 120 are taken?

The receipts from 100 tickets would be 750 shillings and from 150 tickets 900 shillings. Take the number of tickets as abscissae and the number of shillings in the receipts as ordinates and plot as shown in Fig. 20.

When the abscissa is 120 the ordinate of the straight line is 810; the receipts from 120 tickets would therefore be 810 shillings and each ticket would cost 6s. 9d.

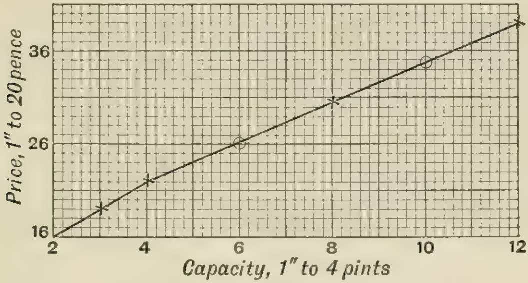


Fig. 19. Scale reduced to one-half.

Another method of solution in this case is the following, which however is merely the algebraic interpretation of the graphical solution:

Let the receipts from x tickets be y shillings. If the receipts are in simple proportion to the number of tickets, then $y=ax$ where a is a

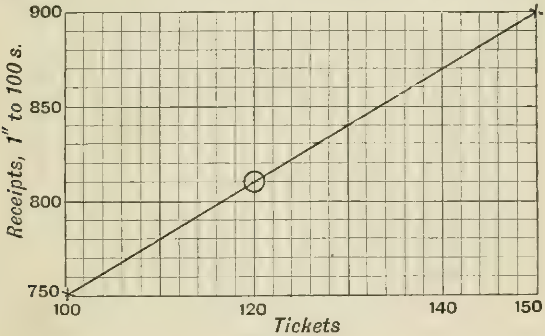


Fig. 20.

constant; but the receipts are not in simple proportion to the number of tickets because the fractions

$$\frac{750}{100} \text{ and } \frac{900}{150}$$

are not equal. Try now the equation $y=ax+b$ where a and b are constants; with this relation between x and y the *rate* at which the receipts increase is constant and equal to a.

To determine a and b we have two pairs of corresponding values of x and y, giving

$$750=100a+b, \quad 900=150a+b,$$

whence $a=3$, $b=450$, and therefore

$$y=3x+450.$$

From this equation we find as before that $y=810$ when $x=120$, so that the cost of one ticket is 6s. 9d.

The beginner should always bear in mind that *a straight line graph implies that, as one quantity changes, the other quantity changes at a constant rate.*

Example 4. At what time between 2 and 3 o'clock are the two hands of a watch (i) together, (ii) 5 minute-spaces apart?

Let abscissae denote the time in minutes after 2 o'clock at which the hands are in any particular position and let ordinates denote the

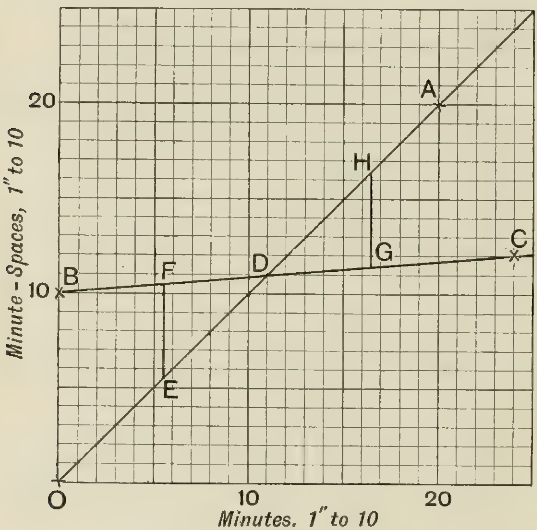


Fig. 21.

number of minute-spaces past 12 o'clock. For abscissae, 1″ may represent 10 minutes and for ordinates 1″ may represent 10 minute-spaces.

The long hand moves at the constant rate of 1 minute-space per minute; the graph that represents its motion is therefore a straight line. This line goes through the origin and the point (10, 10); the point (20, 20) will perhaps give a more accurately placed line than (10, 10). The line is OA (Fig. 21).

The short hand moves at the constant rate of 1 minute-space per 12 minutes. At two o'clock, that is when the abscissa of the point that represents its position is zero, the short hand is 10 minute-spaces in advance of 12 o'clock ; the point that represents its position at 2 o'clock is therefore $B(0, 10)$. Another convenient point is $C(24, 12)$ because in 24 minutes it has advanced 2 minute-spaces ; draw the line BC.

The point D where BC cuts OA corresponds to the position in which the two hands are together ; the abscissa of D is 10·9 and the hands are therefore together at 10·9 minutes past two (approximately).

The hands will be 5 minute-spaces apart at the time represented by the abscissa of a point on the ordinate through which the two lines OA, BC intercept a length of 5 units. By sliding a graduated ruler, keeping its edge parallel to the axis of ordinates, we find there are two ordinates on which the intercepts EF and GH are 5 units ; the corresponding abscissae are 5·5 and 16·4. The required times are therefore 5·5 and 16·4 minutes past 2 ; these numbers are of course approximate.

Data for statistical examples will be found in Mulhall's book, quoted in example 1, in Whitaker's *Almanack,* the *Daily Mail Year Book* and similar compilations. A few examples are given in the following Exercises, but the pupil should be encouraged to obtain the data for himself and to interpret the meaning of the graphs ; the plotting of graphs can be made a most valuable adjunct to the lessons in geography and history.

EXERCISES. VIII.

1. Express graphically the relation (i) between the inch and the centimetre, (ii) between the pound and the kilogramme, given

$$1 \text{ in.} = 2{\cdot}54 \text{ cm.}, \quad 1 \text{ lb.} = 0{\cdot}454 \text{ kg.}$$

From your diagrams find the number of inches in 3·6 centimetres and the number of pounds in 3·2 kilogrammes.

2. Given 1 litre = 1·760 pints find by a graph the number of litres in $3\frac{1}{2}$ pints.

3. Find by a graph the temperature which is expressed by the same number on the Fahrenheit and Centigrade scales.

4. The highest marks obtained in an examination are 132 and the marks are to be reduced so that the highest marks may be 100. Show how to do this graphically and state what marks will be assigned to papers which obtained (i) 100, (ii) 70 marks, giving the marks to the nearest integer.

5. The highest and lowest marks obtained in an examination are 283 and 110 respectively; the marks are to be reduced so that 283 shall become 100 and 110 shall become 50. Show how to do this graphically and state what marks will be assigned to papers which obtained (i) 248, (ii) 124.

6. The tonnage, T thousands of tons, of vessels launched (i) on the Clyde, (ii) from all Scottish yards during the month of June in each of the ten years from 1894 to 1903 is given in the table:

Year, -	1894	1895	1896	1897	1898	1899	1900	1901	1902	1903
(i) T, -	39·7	42·1	27·7	29·2	47·9	36·1	52·0	44·9	39·2	28·4
(ii) T, -	40·7	45·5	28·4	32·0	51·0	38·6	52·5	47·0	47·9	29·9

Illustrate graphically.

7. The number of thousands (N) of people who emigrated from Ireland between 1876 and 1885 is given in the table:

Year, -	1876	1877	1878	1879	1880	1881	1882	1883	1884	1885
N, -	37·5	38·5	41·1	47·0	95·5	78·4	89·1	108·7	75·8	62·0

Illustrate graphically.

8. The number of millions of acres under crops in Ireland during the years 1877 to 1886 is given in the table, where T denotes the total area under crops, M the area under meadow and clover, C the area under cereals and G the area under green crops.*

Year, -	1877	1878	1879	1880	1881	1882	1883	1884	1885	1886
T, -	5·26	5·20	5·12	5·08	5·19	5·08	4·93	4·87	4·95	5·03
M, -	1·92	1·94	1·93	1·90	2·00	1·96	1·93	1·96	2·03	2·09
C, -	1·86	1·83	1·76	1·76	1·77	1·75	1·67	1·59	1·59	1·59
G, -	1·35	1·31	1·29	1·24	1·27	1·24	1·23	1·22	1·21	1·22

Illustrate graphically, putting all the data on one sheet.

* Examples 7, 8 are taken from an interesting little book *Facts about Ireland: A curve-history of recent years* by Alex. B. MacDowall, M.A. (London: Edward Stanford, 1888.)

9. The average annual premiums (£P) for whole life assurance of £100 for the age at entry (A years) is given in Whitaker's *Almanack*, from which the following table is extracted :

A	21	25	30	35	40	45	50
P	1·68	1·83	2·08	2·39	2·80	3·33	4·03

What is the premium for ages 27 and 38 ?

10. The number of years E that a male aged A years may be expected to live (that is, "the expectation of life" as it is called) is given in Whitaker as follows :

A	0	4	8	12	16	20	24	28	32	36
E	41·35	51·01	49·10	45·96	42·58	39·40	36·41	33·52	30·71	27·96

What is the expectation of life of males aged 7, 14, 21, 35 ?

11. The number of years' purchase N of an annuity payable for x years, compound interest at 5 per cent. per annum being allowed, is given in Whitaker as follows :

x	5	9	13	17	21	25	29
N	4·33	7·11	9·39	11·27	12·82	14·09	15·14

What is the number of years' purchase of an annuity payable for 10, 20, 27 years respectively ?

12. A man aged 36, in the receipt of a pension of £100 a year, wishes to commute it for a present payment, interest being reckoned at 5 per cent. How much will he receive ?

(NOTE. The number of years' purchase of an annuity is the ratio of the purchase price to the annual payment.)

13. The cost of fuel, C, per week of 54 hours, for an engine of brake horse-power, P, is given in a certain price list as follows :

P	10	20	50	80	100
C	4s. 11d.	9s. 3d.	21s. 9d.	31s. 8d.	39s. 6d.

What is the probable cost for an engine of 30, 70, 90 horse-power ?

14. The price, p shillings, of carriage cases of length l inches is given in a certain price list as follows :

l	18	20	24	26
p	9	10	12	13

What is the probable price for a case 22 inches long ?

15. A contractor's weekly outlay for wages and incidental expenses was found on the average of several years to be £37 for 20 men, £54 for 30 and £68 for 40. What will be the outlay for 25 and for 35 men?

16. The price, £P, of certain engines of brake horse-power H is given as follows:

H	3	$6\frac{1}{2}$	10	$14\frac{1}{2}$
P	105	160	208	255

What is the probable price of engines of 4 and of 12 horse-power?

17. For a dinner at which there are 60 guests a restaurant keeper charges 10s. 6d. per head but if there are 100 guests the charge is 8s. 6d. per head. What will be the probable charge per head for 75 guests?

18. A cyclist sets out at 9 a.m. from a town A and rides two hours at a speed of 10 miles an hour; he rests half an hour and then returns at a speed of 8 miles an hour. A second cyclist leaves A at 9·30 a.m. and rides at a speed of 7 miles an hour; when and where will the cyclists meet?

19. Two cyclists A and B set out at the same time. A rides for 2 hours at a speed of 9 miles per hour, rests 15 minutes and then continues at 6 miles per hour. B rides without stopping at a speed of 7 miles per hour. When will B overtake A?

20. From the same spot on a circular course one mile in circumference, two boys A and B start at the same moment to walk round it, travelling in the same direction; A walks at 4 and B at 3 miles an hour. How often and at what times will they meet if they walk for an hour and a half?

21. If the boys of example 20 walk in opposite directions round the course how often and at what times will they meet?

22. At what times between 4 and 5 o'clock are the two hands of a watch (i) together, (ii) 15 minute-spaces apart?

23. At what time between 3 and 4 o'clock is the long hand of a watch as far behind the short hand as 10 minutes later it is in front of it?

24. A can do a piece of work in 3 days and B can do it in 5 days; in how many days can they do it when working together?

25. A cistern can be filled by a pipe A in 20 minutes and by a pipe B in 15 minutes while it can be emptied by a pipe C in 12 minutes; if all three pipes are set running when the cistern is empty in what time will it be filled?

26. If in example 25 the pipe C is not opened till A and B have been running for 5 minutes in what time will the cistern be filled?

27. In what proportion must tea at 2s. 6d. per lb. be mixed with tea at 4s. per lb. so that the mixture may be sold at 3s. 6d. per lb.?

28. How many lb. of tea at 2s. 6d. per lb. must be mixed with 6 lb. of tea at 4s. per lb. so that the mixture may be sold at 3s. 6d. per lb.?

17. Continuous Graphs. Physical Applications. We shall now discuss some examples in which the plotted points are to be connected by a smooth curve.

Example 1. Draw a curve to illustrate the variation of temperature in the course of a day from the following data, the temperature being in degrees Fahrenheit.

Time, -	8 a.m.	9 a.m.	10 a.m.	11 a.m.	12 noon.	1 p.m.	2 p.m.
Temp., -	52·2	53·4	61·0	69·8	75·7	77·8	78·1

Time, -	3 p.m.	4 p.m.	5 p.m.	6 p.m.	7 p.m.	8 p.m.
Temp., -	76·9	72·5	67·8	66·8	60·0	51·1

Let times be represented by abscissae to the scale of 1″ to 2 hours and temperatures by ordinates to the scale of 1″ to 10 degrees; measure along lines through the point (8, 50) parallel to the coordinate axes (Fig. 22).

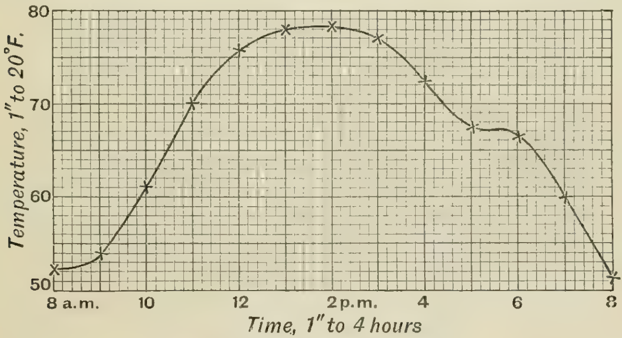


Fig. 22. Scale reduced to one-half.

Join the plotted points by a smooth curve as shown.

By interpolation the temperature at any time during the day can be found; thus at 10.30 it is 65°·5, at 6.15 it is 65°·8.

In the same way a curve representing the variation in the height of the barometer may be drawn. Frequently however the temperature for a week or a month is given by stating the maximum and minimum temperature for each day of the week or month. In such cases the data may be considered statistical and the representative graph is perhaps better shown as a broken line after the manner of statistical graphs.

Example 2. In a test of a Pelton wheel with a constant head of water the brake horse-power (B.H.P.) at N revolutions per minute was found to be as follows :

N	1180	1375	1560	1750	1950	2120	2320	2500	2700	2875
B.H.P.	0·640	0·671	0·669	0·660	0·650	0·600	0·560	0·480	0·380	0·270

Draw a curve to represent the relation between the number of revolutions and the brake horse-power.

Take the values of N as abscissae to a scale of 1″ to 500 and the values of the B.H.P. as ordinates to a scale of 1″ to 0·1 (Fig. 23). On the scale chosen for the ordinates each digit in the values of the ordinate can be represented ; the side of a small square represents 0·01 and by estimation of the divisions of the side of a small square the effect of the third digit after the decimal point can be determined with fair accuracy.

When the points have been plotted a fair curve is drawn free hand to pass through or very near them ; usually some of the points will not fit in to the curve but no one point should be at a relatively great distance from it.

The next example is one of a type that occurs frequently in laboratory work. The plotted points lie approximately in a straight line and it is often essential to obtain the equation of the line. Before proceeding to this example the student should try Exercises IX. 10 and 11. The points will be found to lie on or near a straight line. Since the equation of a straight line is of the form $y=ax+b$ all we have to do to obtain its equation is to select two convenient points on the line, read their coordinates off the diagram and then, by substitution in the equation $y=ax+b$, determine the values of a and b. (Compare § 12, example 3.)

When the graph is not a straight line we are not yet in a position to find its equation; some simple practical cases will be given in later chapters.

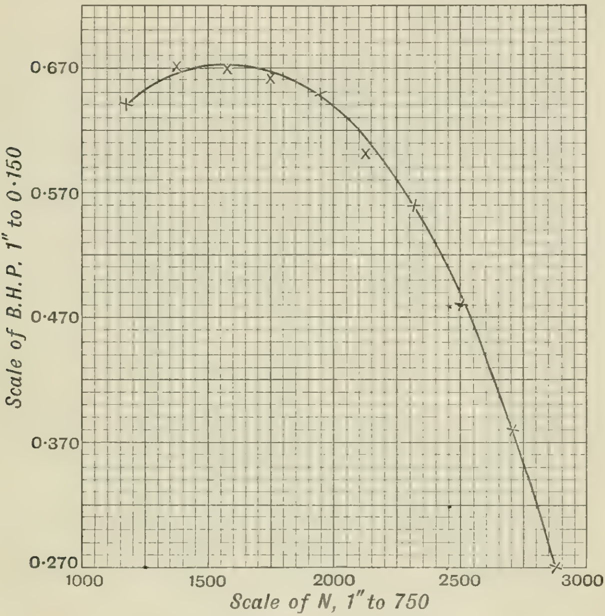


Fig. 23. Scale reduced to two-thirds.

Example 3. In an experiment with a Weston Differential Pulley Block the effort, E lb., required to raise a load, W lb., was found to be as follows:

W	10	20	30	40	50	60	70	80	90	100
E	$3\frac{1}{4}$	$4\frac{7}{8}$	$6\frac{1}{4}$	$7\frac{1}{2}$	9	$10\frac{1}{2}$	$12\frac{1}{4}$	$13\frac{3}{4}$	15	$16\frac{1}{2}$

Plot the loads as abscissae to a scale of 1″ to 10 lb. and the efforts as ordinates to a scale of 1″ to 2 lb. (Fig. 24).

The points lie nearly in a straight line, which is therefore the simplest curve that passes evenly among them. To find the line that best fits the points, stretch a thread on the paper and shift it about till the plotted points are either covered by the thread or about equally distributed on opposite sides of it. It is very unlikely that all the points will be on the straight line, because experimental work is always subject to error, but of course we are only entitled to conclude that the straight line is the proper graph if no points are at relatively great distances from it.

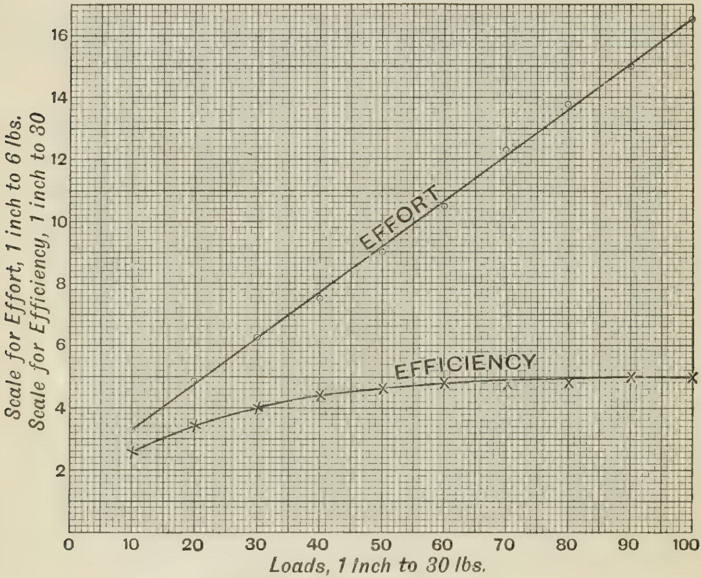


Fig. 24. Scale reduced to one-third.

Since the graph is a straight line, the effort is a linear function of the load ; therefore

$$E = aW + b, \qquad\qquad (1)$$

where a, b are constants. To find the values of a and b, select any two convenient points on the line ; it might happen that the line did not go through any of the plotted points, but in this case it goes through $(30, 6\frac{1}{4})$ and $(100, 16\frac{1}{2})$. Substituting these coordinates in equation (1) we get

$$6\tfrac{1}{4} = 30a + b, \quad 16\tfrac{1}{2} = 100a + b.$$

These equations give $a = 0{\cdot}146\ldots$, $b = 1{\cdot}857\ldots$. We might take $0{\cdot}15$ for a and $1{\cdot}86$ for b ; but if we substitute these values in (1) and then calculate the values of E for W equal to 10, 20 ... it will be found that the calculated values do not agree so closely with the given values as when we take 0·146 for a and 1·86 for b. We take therefore for the relation between E and W, or **the law of the machine** as it is usually called,

$$E = 0{\cdot}146\,W + 1{\cdot}86. \quad\text{.............................(2)}$$

It is always advisable to test the law by calculating E from the equation found and comparing with the given values.

It is shown in books on mechanics that, if r is the velocity ratio of the machine, the work lost through friction and otherwise is proportional, for a given rise of the load, to $rE - W$. The force $rE - W$ is often taken as measuring the friction of the machine; we may denote it by F.

In the case in hand r was 24. From the equation

$$F = 24E - W$$

calculate the values of F, using the given values of E and W, and then plot the points for W and F as has been done for W and E. The points will be found to lie nearly in a straight line and the equation of the line can be found as before. That equation might be got by means of (2); for

$$F = 24E - W = 2{\cdot}504\,W + 44{\cdot}64.$$

This equation should be compared with that obtained from the plotted points.

The efficiency e of the machine, expressed as a percentage, is

$$e = \frac{W}{rE} \times 100 = \frac{100\,W}{24E} = \frac{100\,W}{3{\cdot}504\,W + 44{\cdot}64}, \quad\text{.....................(3)}$$

where the last fraction is obtained by using (2).

Corresponding values of W and e are given by:

W	10	20	30	40	50	60	70	80	90	100
e	12·8	17·1	20·0	22·2	23·1	23·8	23·8	24·2	25·0	25·3

the values of e being calculated from the given values of E and W.

Keeping the scale of W as before plot e as ordinate, to a scale of 1″ to 10. The points obtained are not in this case in a straight line; we therefore draw with a free hand, as in examples 1 and 2, a curved line passing through or near them. Had e been calculated from the last fraction in equation (3) the points would have been distributed a little more regularly than those actually plotted, but the curve obtained would be practically the same as that shown in Fig. 24.

In Exercises IX. several examples are given of quantities connected by a linear law; the method of obtaining the algebraic equation between the quantities is always the same as has been illustrated in this example. The student should note examples 29–31 of the next set. These show how in certain cases the equation of a curved line may be found; similar devices are sometimes useful in other cases (see for example § 34) but except in very simple examples the problem of finding the equation of a curve in this manner is too difficult to be discussed in an elementary book. Fortunately the curves amenable to elementary treatment are of considerable practical importance.

18. General Remarks. The student may have a difficulty in deciding which is the simplest curve that passes evenly among the points. As he proceeds in his study of the graphical representation of equations he will find that all ordinary equations are represented by *smooth* curves, that is, by curves without angular points like the teeth of a saw; the curve bends gradually, there is no abrupt change of direction in passing along it. It is only in very special cases that such abrupt change takes place; the rule is that the curve is well rounded.

Hence when the graph is to represent some physical process, or some relation deduced from observation or experiment, the curve should not, as a rule, possess sharp angles; the bending should be gradual. It may be of use to study the traces of the self-registering instruments so common now for recording the temperature of the atmosphere and the height of the barometer; it is the exception for these graphs to show sharp angles.

In dealing with statistics on the other hand it is perhaps best to follow the method of § 16; problems on prices also may be treated as in that section.

In deducing conclusions from the study of a graph one must not go beyond the range fixed by the data; thus we may find from the graph of example 3, § 17, or the equivalent equation (2), the effort required to raise any weight between 10 and 100 pounds but we are not justified in using it to find the effort to raise 200 pounds. In many cases the law seems to be different for different ranges of the variables: or it may be that the law which holds for a wide range of the variables is somewhat complicated but may be represented approximately for smaller ranges by expressions or graphs that are comparatively simple but that differ for different ranges.

EXERCISES. IX.

1. Draw a curve to represent the variation of temperature given by the following data, the temperature being in degrees Fahrenheit :

Time, -	2 a.m.	4 a.m.	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.
Temp., -	42·2	40·8	38·8	40·8	43·8	42·2	48·7

Time, -	4 p.m.	6 p.m.	8 p.m.	10 p.m.	12 midnight
Temp., -	46·9	42·6	41·3	38·0	34·4

2. Draw a smooth curve to represent the variations in the height of the barometer, H inches :

Time	3 a.m.	6 a.m.	9 a.m.	12 noon	3 p.m.	6 p.m.	9 p.m.	12 night
H	29·87	29·90	30·01	29·96	29·91	29·94	29·98	29·94

3. The maximum and minimum shade temperature, in degrees Fahr., and the height, H inches, of the barometer as recorded at the Observatory Glasgow for June 1–7, 1903, are as follows :

Day, - - - -	1	2	3	4	5	6	7
Max. Temp., - -	59	59	66	68	70	75	69
Min. Temp., - -	49	43	43	47	52	52	53
H, - - - -	29·88	30·12	30·40	30·45	30·39	30·43	30·43

Illustrate these results graphically, putting the two curves of temperature on the same sheet.*

* Numerous exercises like 1–3 can be constructed from the data in the daily newspapers. See also Whitaker's *Almanack* for the several months.

4. The rainfall in inches, and the dust fall, measured by the weight of dust, in grains, falling on a dish of 75 sq. in. area, at Edinburgh during the year 1902 are given as follows :

Month, - -	Jan.	Feb.	Mar.	Apr.	May	June
Rainfall, - -	0·955	0·895	0·805	1·190	2·190	2·145
Dustfall, - -	33	25	36½	160*	49	29

Month, - -	July	Aug.	Sept.	Oct.	Nov.	Dec.
Rainfall, - -	2·835	1·385	1·290	0·795	0·408	1·334
Dustfall, - -	26	80	60	120*	109*	140*

The * indicates that in these months there was sand in the dish.

Illustrate these results graphically.

5. A beaker is filled with water at a temperature of 15° C.; heat is then applied to the beaker and the temperature, T degrees Cent., at the end of t minutes is found to be as follows :

t	0	5	10	15	20	25	30	35	40
T	15	20	24·4	28·4	32	35·2	38·2	41	43·3

Draw the time-temperature curve.

6. In a test the pressure, P lb. per sq. in., corresponding to a delivery of C cub. ft. of water per min. is given by the table :

P	250	400	500	600	750	800	900	1000
C	0·64	0·80	0·91	0·99	1·12	1·15	1·22	1·28

Draw the curve representing the relation between P and C.

Draw the curves representing the relation between the number of revolutions per min. (N) and the brake horse-power (B.H.P.) in examples 7, 8, the data for which were obtained from tests on a Pelton wheel.

7.

N	1150	1450	1770	2100	2400	2720	3040	3340	3675	3975
B.H.P.	0·99	1·10	1·20	1·21	1·15	1·03	0·87	0·53	0·35	0·00

8.

N	1750	2050	2350	2625	2900	3150	3380	3575
B.H.P.	2·38	2·56	2·70	2·77	2·79	2·70	2·57	2·40

N	3850	4040	4270	4475	4650	4825	5000
B.H.P.	2·20	1·93	1·63	1·29	0·89	0·46	0·00

9. Draw a curve representing the efficiency E, in the case of example 7, N being as before the number of revolutions per min.

N	1150	1450	1770	2100	2400	2720	3040	3340	3675	3975
E	38·6	44·6	46·0	46·2	43·8	39·3	33·2	20·2	13·4	0

10. Plot the points given by the table :

x	1	2	3	4	5
y	3·71	3·28	2·86	2·44	2·10

and find the equation of the line on which they lie.

11. Find the equation of the straight line that best fits the following points :

x	0·5	1	1·5	2	2·5	3
y	0·31	0·82	1·29	1·85	2·51	3·02

12. The linear extension, l inches, of a copper wire stretched by a load, W lb., is given by the table :

W	10	20	30	40	50	60
l	0·06	0·11	0·17	0·22	0·275	0·32

Show that the extension is proportional to the load for loads up to 60 lb.

13. In an experiment on the stretching of an iron rod the linear extension, l inches, for a load of W lb. was found to be as follows :

W	600	1100	1600	2100	2600	3100	3600	4100	4600	5100
l	0·004	0·009	0·013	0·018	0·022	0·027	0·032	0·037	0·043	0·050

Show that for loads under 3000 lb. the extension is proportional to the load.

14. A lath of yellow pine, 1″ broad and 0·55″ deep, is supported at points 24″ apart and loaded at the point midway between the points of support. The deflection, d inches, for a load of W lb. is as follows :

W	0	8·6	18·6	28·6	38·6	48·6	58·6	63·6	68·6	69·6	70·6
d	0	0·15	0·36	0·57	0·78	1·00	1·23	1·36	1·70	1·78	1·86

Show that for loads under a certain amount the deflection is proportional to the load and find what the limit of load is.

15. When the points of support of the lath of the preceding example were 12″ apart the results were as follows :

W	0	8·6	28·6	48·6	68·6	88·6	98·6	108·6	118·6	123·6	128·6
d	0	0·02	0·07	0·12	0·17	0·22	0·25	0·29	0·32	0·34	0·37

For what range of load is the deflection proportional to the load ?

In examples 16–18 find the law of the machine and the friction ; plot also the efficiency curve. The notation is that adopted in § 17.

16.

W	10	20	30	40	50	60	70	80	90	100
E	1	$1\frac{5}{8}$	$2\frac{1}{8}$	$2\frac{5}{8}$	$3\frac{1}{4}$	$3\frac{3}{4}$	$4\frac{1}{4}$	5	$5\frac{1}{2}$	6

Velocity ratio = 89.

17.

W	6	11	16	21	26	31	36	41	46	51
E	0·53	0·875	1·22	1·60	1·94	2·31	2·625	3·125	3·31	3·75

Velocity ratio = 51·5.

18.

W	24	44	64	84	104	124	144
E	0·55	0·87	1·10	1·44	1·65	1·95	2·20

Velocity ratio = 85.

19. In an experiment to determine the friction of brass on iron (rubbing surface about 5 square inches) the friction F lb. for a load of W lb. was found to be:

(i) for dry surfaces

W	2	4	6	8	10	13	16
F	0·38	0·88	1·25	1·75	2·25	2·88	3·63

(ii) for lubricated surfaces

W	3	13	23	33	43
F	$\frac{3}{8}$	1	$1\frac{5}{8}$	$2\frac{1}{8}$	$2\frac{5}{8}$

Find the relation connecting F and W in each case.

20. The angle of twist, D degrees, produced by a couple or torque, T pound-inches, in a wire was found to be as follows:

T	1·4	2·75	5·5	8·25	11	13·75	16·5
D	1·5	3	6	9	12·5	15·5	18

Show that the twist is approximately proportional to the torque.

21. The angle of twist, D degrees, produced by the same torque in a wire of length l inches is as follows:

l	4	6	8	10	13	16	20
D	17	26	34·5	43	56	69	86

Show that the twist is approximately proportional to the length.

22. In a comparison of two voltmeters corresponding readings C and K were found to be as follows:

C	3·8	5·5	7·55	9·6	11·5	13·55	15·75
K	11·5	16·5	22·5	28·0	33·5	39·5	45·5

What is the relation between C and K?

23. The battery resistance, b ohms, for a current of C amperes was found in a certain test to be as follows :

b	4·2	4·8	5·0	5·8	7·6	8·5	11·0
C	0·21	0·16	0·14	0·10	0·066	0·06	0·04

Illustrate these results graphically.

24. The temperature, $T°$ C., at the depth D metres below the surface of the ground, as determined by borings at Paruschowitz, Silesia (*Brit. Ass. Report*, 1901), is as follows :

D	6	37	68	99	130	161	192	223	254	285
T	12·1	13·1	14·3	14·6	15·6	16·0	16·5	17·3	18·1	18·9

Plot the points. Show that (roughly) the gradient is about 1° C. in 42 metres ; for the depth from 192 to 285 metres the gradient is more nearly 1° C. in 40 metres.

25. At the greatest depths reached in the borings referred to in example 24 the observations were :

D	1680	1711	1742	1773	1804	1835	1866	1897	1928	1959
T	60·3	61·4	62·1	63·6	64·8	65·5	65·5	66·9	67·5	69·3

Show that the gradient for this range is about 1° C. in 33 metres.

26. A test-tube containing some water, initially at a temperature of 29° C., is plunged into a freezing mixture, and the temperature of the water is read every minute ; readings are taken for several minutes after the water has all frozen. The following table gives the readings, M denoting the number of minutes after starting and T the temperature in degrees Centigrade.

M	0	1	2	3	4 to 12	13	14	15	16	17	18
T	29·0	5·2	0·5	0·2	0·0	−0·6	−2·0	−4·3	−7·0	−9·1	−10

Draw a curve to show the variation of temperature with time.

27. A test-tube containing some ice, initially at a temperature of −10° C., was held in a current of hot air and the temperature of the contents of the test-tube was read every minute (the bulb of the thermometer was imbedded in the ice) ; readings were taken for several minutes after all the ice had melted. Draw a curve to show the variation of temperature with time from the following readings; M denotes the number of minutes after starting and T the temperature in degrees Centigrade.

M	0	1	2	3	4 to 19	20	21	22	23
T	−10·0	−6·5	−3·2	−0·4	0·0	0·5	2·1	4·5	9·0

28. A mass of liquid wax contained in a test-tube was allowed to cool in air. The temperature of the wax was read every two minutes, readings being taken for some time after the wax had solidified. Draw a curve to show the variation of temperature with time from the following readings; T denotes the temperature in degrees Centigrade, M minutes after starting.

M	0	2	4	6	8	10	12	14	16	18
T	75·8	65·9	57·6	51·0	49·3	49·0	49·0	48·9	48·8	48·6

M	20	22	24	26	28	30	32	34	36	38
T	48·2	47·9	47·4	46·8	46·1	45·2	44·1	42·9	41·2	39·5

M	40	42	44	46	48	50
T	37·4	35·2	33·4	31·9	30·6	29·5

29. Plot the points given by the scheme:

x	1·0	1·7	1·9	2·3	3·0	4·3	6·0
y	0·8	1·2	1·3	1·5	1·8	2·1	2·4

and draw a smooth curve passing through or near them.

Put $u=1/x$, $v=1/y$ and calculate the values of u and v corresponding to the values of x and y: thus $u=1$ when $x=1$, and $v=1·25$ when $y=0·8$; $u=0·59$ when $x=1·7$ and $v=0·83$ when $y=1·2$ and so on. Show that the points (u, v) lie on a straight line and therefore that u and v satisfy an equation of the form

$$au+bv+c=0.$$

The equation of the curve on which the points (x, y) lie is therefore

$$a\cdot\frac{1}{x}+b\cdot\frac{1}{y}+c=0, \quad \text{or} \quad ay+bx+cxy=0.$$

30. Find as in example 29 the equation of the curve on which the following points lie :

x	0·84	1·24	2·00	3·34	5·00	6·67
y	10·92	3·64	2·38	1·96	1·82	1·68

31. Find the equation of the curve on which the following points lie :

x	1·3	2·4	3·6	4·9	6·7	8·5
y	14·1	18·8	21·2	22·7	24·0	24·8

CHAPTER IV.

QUADRATIC FUNCTIONS.

19. Plotting of Curves from Equations. When an equation is given that contains x and y, but that is not of the first degree in these variables, it is still possible, by giving a series of values to x, to calculate a corresponding series of values of y and then to plot the points as in § 9. It will be found however that the points do not now lie on a straight line; but, when the difference between successive values of x is small, the points will be arranged in such a way as to suggest a definite curve on which they all lie. If we draw a curve freehand through all the plotted points, adapting the curve to the general trend of the points, it will be seen by trial that the curved line so drawn possesses (within the limits of accuracy prescribed by the diagram) the two properties noted in § 10 as characteristic of the straight line in relation to its equation, namely:

(i) all points whose coordinates satisfy the equation lie on the curve;

(ii) the coordinates of every point on the curve satisfy the equation.

The process thus described is called "plotting the curve from its equation." As in the case of the straight line, the curve* is said **to be represented by** or **to be given by** or **to be the graph of** the equation; in reference to the curve the equation is called **the equation of the curve or graph.**

* It may be well to warn the beginner that the word *curve* is often used to include *straight line* as well as *curved line.*

The equation will define y as a function of x (example 1, p. 30) and the ordinate y will represent the function. Hence the curve is often called **the graph of the function.** Thus the curve represented by an equation such as

$$y = 3x^2 - 2x + 1$$

is often called the graph of the function $3x^2 - 2x + 1$. The properties of a function—its greatest and least values, the way in which it increases or decreases as x changes, etc.,—are usually understood most readily by studying the graphical representation of it.

We shall now plot some simple curves; but we first remind the student of what was said in § 10 about the condition that a point should lie on a curve whose equation is given. For curved as well as straight lines, the sole test is that a point lies on the curve **if and only if** its coordinates satisfy the equation of the curve.

20. Graph of $y = x^2$. For the moment let us confine ourselves to values of x from $x = -2$ to $x = +2$, and let us take the horizontal and vertical unit lines of the same length, say one inch.

To obtain a convincing proof of the form of the graph, we must take the difference between consecutive values of x fairly small; we must plot the curve, so to speak, *point by point*. The imagination of experience will enable the student to reduce the number of points whose co-ordinates must be calculated, but his knowledge of curves and of functions will rest on no sound basis unless, to begin with, he plots points enough to assure himself that he has obtained the proper bending of the curve.

Let the successive values of x differ by 0·1, that is let x increase or decrease by 0·1; the successive increments of y will therefore be also fairly small, as the calculations show. Tabulate as follows:

x	0	0·1	0·2	 1	1·1	1·2	 2
y	0	0·01	0·04	 1	1·21	1·44	 4

x	$-0{\cdot}1$	$-0{\cdot}2$	...... -1	$-1{\cdot}1$	$-1{\cdot}2$	...... -2
y	0·01	0·04	 1	1·21	1·44	 4

The student can fill up the gaps; it is advisable in view of graphical work that he should draw up for himself tables showing the values of x^2, x^3, x^4 for values of x from $x=0$ to $x=2$, at intervals of 0·1 (as above); and from $x=2$ to $x=10$ at intervals of 0·5, that is, for $x=2{\cdot}5$, 3, 3·5.... Only positive values of x need be taken.

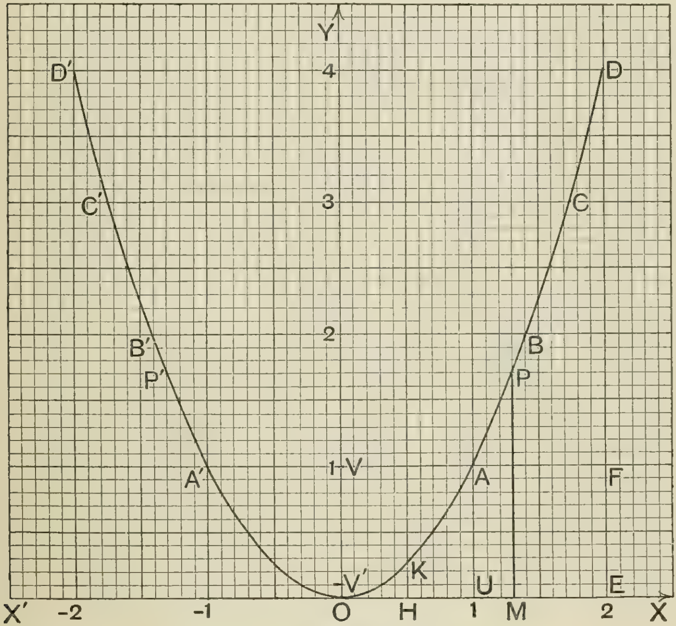


Fig. 25.

Now plot the points

$$(0, 0),\ (0{\cdot}1,\ 0{\cdot}01)\ldots,\ (-0{\cdot}1,\ 0{\cdot}01),\ (-0{\cdot}2,\ 0{\cdot}04)\ldots,$$

and draw a curve through them (not merely *near them*); the result is shown in Fig. 25.

The x-axis is a tangent to the curve at the point O.

21. The Symmetry of the Curve. It is obvious that in this case half the calculations might have been avoided, since any two values of x that differ only in sign give the same value of y; thus $y=1{\cdot}96$ both when $x=1{\cdot}4$ and when $x=-1{\cdot}4$. Again, the points $(1{\cdot}4, 1{\cdot}96)$ and $(-1{\cdot}4, 1{\cdot}96)$ are **symmetric** (§ 8, p. 16) with respect to the y-axis; and, in general, to any point P on the curve with a positive abscissa there is a symmetric point P' lying at the same distance to the left of the y-axis as P does to the right. The curve is therefore said to be **symmetrical about the y-axis.**

Hence, to plot this particular curve it is sufficient to calculate y for positive values of x; the points A', B',... on the left of OY are symmetric to the points A, B,... on the right and can be plotted as soon as A, B,... are laid down. In fact, the part OAD will coincide with the part $OA'D'$ if it is turned over and A laid on A' and D on D'; or, again, it may be said that the part $OA'D'$ is the **image** or **reflection** in the y-axis (considered as a mirror) of the part OAD.

As a rule a curve is not symmetrical about either axis, but the student should be on the watch for symmetry because its presence saves labour.

22. Turning Points. Maximum and Minimum Values. As a point moves along the curve (Fig. 25) from any position on the left of OY to any position on the right, the ordinate of the point decreases till the point reaches O and then increases. The point O is therefore called a **turning point** of the graph; and, by analogy, the value of the ordinate (or function) at O—in this case, zero—is called a **turning value** of the ordinate (or function).

In general, those points on a graph at which the ordinate either ceases to decrease and begins to increase, or else ceases to increase and begins to decrease, are called turning points of the graph, and the values of the ordinate (or function) at the turning points are called turning values. The value of the ordinate (or function) at that turning point where it ceases to decrease and begins to increase is a **minimum value**; at a turning point where it ceases to increase and begins to decrease, the ordinate (or function) has a **maximum value.**

The meaning now given of the words maximum and minimum is that generally understood in mathematics and should be particularly noted. A maximum ordinate is one that is greater than any other ordinate of the curve *near it and on either side of it*; it is not necessarily, though it sometimes is, the greatest ordinate of the curve. Similarly, a minimum ordinate is merely one that is less than any other ordinate of the curve near it and on either side of it. A minimum ordinate may even be greater than a maximum one.

For example, on a contour road map the trace of an undulating road has several turning points, but the lowest point of a hollow (at which the height of the road above the datum line is a minimum) may well be at a greater height above the datum line than one of the crests of the road.

Again, let the student note how slowly the length of the ordinate changes near the turning point O in Fig. 25; this property of **slow change near a turning point** is characteristic of turning points on all ordinary graphs and should be verified in all graphs the student draws.

The manner in which the length of the ordinate (which measures the value of the function x^2) changes at different parts of the curve should also be studied. Thus, as x increases from 0 to $\frac{1}{2}$, the ordinate (or function x^2) increases very slowly; as x increases from $\frac{1}{2}$ to 1, the ordinate increases more rapidly; and as x increases from 1 to 2, the ordinate increases still more rapidly.

It will be readily seen that as x increases beyond 2, the ordinate grows very rapidly and, with the units chosen for the diagram, could not be shown on a sheet of moderate size even for such a small value of x as 5 not to say 10. For such cases the vertical unit step must be taken smaller than the horizontal one; in special cases it may be necessary to draw more than one graph, with different scales, so as to get a complete knowledge of the curve. See also § 24.

EXERCISES. X.

1. Draw, with the scales and values of x given in § 20, from $x=-2$ to $x=2$ the graphs of

(i) $y=x^2+1$, (ii) $y=x^2-1$, (iii) $y=-x^2+1$, (iv) $y=-x^2-1$.

State the turning points of the graphs and the turning values of the functions.

2. Draw the graph of $y=10x^2$ from $x=-2$ to $x=2$, taking the values of x in § 20 but making the y-scale one-tenth of the x-scale; say, 1″ representing the value 1 of x and the value 10 of y. Compare the graph with Fig. 25.

3. With the scales and values stated in example 2 draw the graphs of (i) $y=10x^2+10$, (ii) $y=10x^2-10$, (iii) $y=-10x^2+10$, (iv) $y=-10x^2-10$.

State the turning points and turning values.

4. Draw the graph of $y=\frac{1}{10}x^2$ from $x=-2$ to $x=2$ taking the y-scale 10 times the x-scale. Compare with Fig. 25.

5. With the scales of example 4 draw the graphs of

(i) $y=\frac{1}{10}x^2+\frac{1}{10}$, (ii) $y=\frac{1}{10}x^2-\frac{1}{10}$, (iii) $y=-\frac{1}{10}x^2+\frac{1}{10}$, (iv) $y=-\frac{1}{10}x^2-\frac{1}{10}$.

State the turning points and turning values.

6. Draw the graph of $y=x^2$ from $x=0$ to $x=10$, taking the values of x suggested in § 20; for scales let 1″ represent the value 2 of x and the value 20 of y.

How is the graph of $y=-x^2$ related to that of $y=x^2$?

7. On the same axes and with the same scales (§ 12) draw the graphs of $4y=x^2$ and $6y=2x+3$ from $x=-1$ to $x=3$.

State the abscissae of the points of intersection of the two graphs and write down the equation of which these abscissae are the roots.

8. The same problem as in example 7 for the equations

$$y=10-10x^2,\quad 4y=24-11x.$$

9. Plot the points given by the table:

x	0	0·3	0·7	1·2	1·5	1·8	2·4
y	0	0·3	1·6	4·6	7·2	10·4	18·5

and show, by finding the value of a, that they lie on the graph of an equation of the form $y=ax^2$.

10. Plot the points given by the table:

x	0·25	0·37	0·84	1·27	1·65
y	9·5	10·1	14·6	21·9	30·8

and show, by finding the values of a and b, that they lie on the graph of an equation of the form $y=ax^2+b$.

11. State which, if any, of the points

$$(1, 2),\ (-1, 3),\ (-2, 5),\ (2{\cdot}4, 6{\cdot}57),\ (-3, 9),$$

lie on the graph of the equation $4y=3x^2+9$.

12. Find the gradient of the line joining the two points on the graph of $y=x^2$ whose abscissae are

(i) 0 and 1; (ii) 1 and 2; (iii) 2 and 3;
(iv) 1 and 1·5; (v) 1 and 1·1; (vi) 1 and 1·01.

13. Find the gradient of the line joining the two points on the graph of $y=x^2$ whose abscissae are

(i) 1 and $1+h$; (ii) a and $a+h$.

What would you suppose the gradient of the tangent to the graph at the points whose abscissae are 1 and a to be?

23. Graph of $y=ax^2$. For any given value of a, say 2 or 10 or -5, we can plot the graph as in § 20, namely by calculating the values of y for chosen values of x; it will be instructive however to indicate another process.

First, let a be positive, say $a=2$. Denote by y any ordinate of the graph of $2x^2$ and by Y the ordinate of the graph of x^2 *for the same value of* x. Then whatever value x may have, y is twice Y: thus, when $x=\frac{1}{2}$, $y=\frac{1}{2}$, $Y=\frac{1}{4}$; when $x=1$, $y=2$, $Y=1$ and so on. Hence, having first drawn the graph of x^2, we can construct the graph of $2x^2$ by simply doubling each ordinate of the graph of x^2.

In the same way we can construct the graph of $3x^2$ by trebling and the graph of $\frac{1}{2}x^2$ by halving, each ordinate of the graph of x^2; and so on.

The curves above the x-axis in Fig. 26 are the graphs of x^2, $2x^2$ and $\frac{1}{2}x^2$; the diagram is not large enough to show the whole of the graph of x^2 and of $2x^2$ from $x=-2$ to $x=2$.

Secondly, let a be negative. If $a=-1$, the equation is $y=-x^2$ and the graph is clearly symmetrical to that of

$y=x^2$ with respect to the x-axis; because the value of y given by $y=-x^2$, for any chosen value of x, differs *only in sign* from that given by $y=x^2$ for the same value of x.

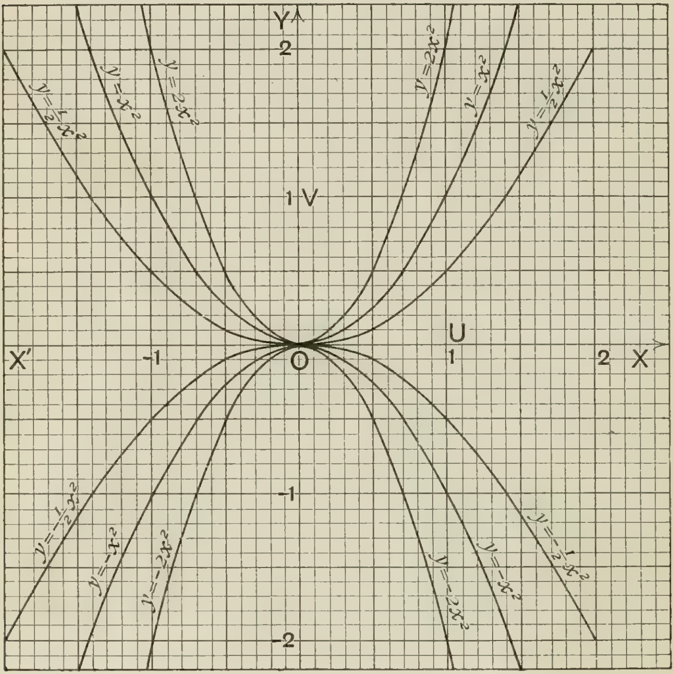


Fig. 26.

The graph of $-2x^2(a=-2)$ may be obtained by doubling the ordinates of that of $-x^2$; or it may be got by taking the image in the x-axis of the graph of $2x^2$. Similarly the graphs of $-\frac{1}{2}x^2$, $-3x^2$ ... may be constructed.

The curves for negative values of a lie below the x-axis in Fig. 26.

The equation $by=cx^2$ may be written $y=\frac{c}{b}x^2$ and is therefore of the form just discussed.

In practice it is usually best to draw the graphs by plotting points but the process just considered shows that the graph of ax^2, for different positive values of a, is of the same general character as that of x^2 and that the graph of ax^2, for different negative values of a, is of the same general character as that of $-x^2$. The greater a is the more rapidly does the graph recede from the x-axis.

If b is positive, the graph of ax^2+b is simply that of ax^2 moved b units up the diagram, for it may be obtained from that of ax^2 by increasing each ordinate by b. Similarly the graph of ax^2-b is that of ax^2 moved b units downwards.

The origin is a turning point on the graph of ax^2; but, if a is negative, the ordinate at the origin, namely zero, is a *maximum*, when considered *algebraically*; because every ordinate except that at the origin is negative and zero is algebraically greater than any negative number.

The curve given by the equation $y=ax^2+b$ is called a **parabola** (§ 29); this equation is a particular case of that of § 29.

24. Change of Scale. There is another method of considering the graph of ax^2 depending on the scales used in plotting it. The graph of $y=x^2$ (Fig. 25) will, if the vertical unit line be properly chosen, represent the graph of $y=ax^2$ *for any positive value of* a.

For example, let $a=10$. When $x=1$, the equation $y=10x^2$ gives $y=10$; let therefore the segment OV which in § 20 represents 1 now represent 10. In other words let the new vertical unit segment OV' be $\frac{1}{10}$th of the former unit segment OV. Every vertical step therefore will now represent a number 10 times as large as it represented on the first scale. ED for example is $4OV$, that is, $40OV'$; when OV is the unit the ordinate of D is 4, but when OV' is the unit the ordinate of D is 40.

Now, every ordinate of the graph of $y=10x^2$ is 10 times the ordinate of the graph of $y=x^2$ for the same value of x; but on the new scale every vertical step represents a number that is 10 times as great as the number it represented on the first scale. Therefore the graph of $y=10x^2$ is simply that of $y=x^2$ with OV', instead of OV, representing unity.

Similarly the graph of $y=x^2$, constructed with OV as unit, will be the graph of $y=ax^2$ (a being positive) provided the scale is changed so that OV shall represent, not 1 but, a. Thus it will be the graph of $2x^2$ if $OV=2$, of $\frac{1}{2}x^2$ if $OV=\frac{1}{2}$ and so on.

The graph of $y=-x^2$ stands in the same relation to that of $y=ax^2$ when a is negative as the graph of $y=x^2$ does to that of $y=ax^2$ when a is positive. Thus the graph of $y=-x^2$ will represent that of $y=-10x^2$ provided $OV=10$ (Fig. 26).

These considerations also show that a change of scale like that just treated is equivalent to a stretching or contracting of all lines in the paper parallel to the y-axis.

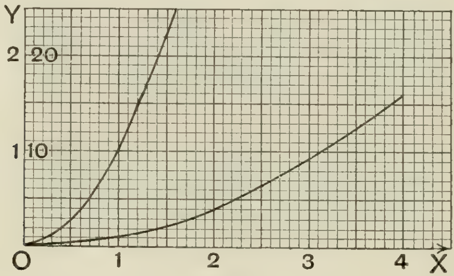


Fig. 27.

In studying the purely geometrical properties of curves it is desirable that the two unit steps OU, OV should be of the same length; but such a choice is often impracticable. The more advanced student will readily see that a change in the length of the steps OU, OV, so long as the lengths are kept equal, merely changes the size and not the shape of the figure because all lines are altered in the same proportion. When OU and OV are of different lengths the curve is distorted and its geometrical properties are often much disguised; for example, a circle would be flattened and appear to be an ellipse.

Fig. 27 shows two curves both of which represent $y=x^2$. In both the x-scale is 1″ to 2, but in the upper curve the y-scale is 1″ to 2 while in the lower curve it is 1″ to 20.

In interpreting a graph it is essential that the scales be known.

From what has been stated in this article and in § 23 the student should now have no difficulty in picturing to himself the graph of $y=ax^2+b$; in employing the graph for the solution of problems very much depends on a proper choice of scales. It will not now be necessary to choose the values of x so near to each other; a few points, to act as guide points, will generally be sufficient. The proper rounding at a turning point should be specially attended to.

Before proceeding to § 25 the student should work several of the examples in Exercises XI. 1–10.

25. Applications of the Graph of ax^2. We shall take two illustrations of the way in which the graph may be usefully applied.

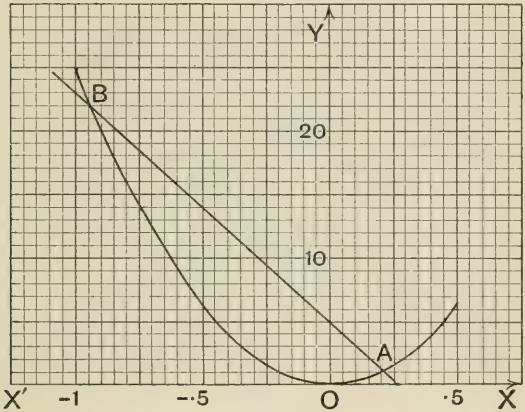


Fig. 28.

Example 1. Solve graphically the equation

$$25x^2+18x-5=0. \qquad \text{(i)}$$

Write the equation in the form

$$25x^2=-18x+5, \qquad \text{(ii)}$$

then draw the graphs of

$$y=25x^2 \qquad \text{(iii)} \quad \text{and} \quad y=-18x+5. \qquad \text{(iv)}$$

These graphs intersect in two points A and B (Fig. 28). The coordinates of A satisfy *both* of the equations (iii) and (iv), because A

is on both graphs. At A therefore the y of (iii) is the same as the y of (iv), and the x of (iii) the same as the x of (iv). Hence the x of the point A is such that

$$25x^2 = -18x + 5;$$

in other words the x of the point A satisfies (ii) which is equivalent to (i).

Similarly we see that the x of B satisfies (i).

Thus, to solve equation (i), **plot the graphs of equations (iii) and (iv) and read off the abscissae of the points of intersection.** These abscissae are the roots of the equation.

A preliminary rough sketch of the graphs will show that they intersect a little to the right of O and a little to the right of the point for which $x = -1$; we only require therefore to plot the graphs carefully near these points.

The roots are approximately 0·21 and −0·93; on the scale to which the figure was originally drawn the roots were read as 0·214 and −0·934. The roots, when the equation is solved algebraically, are 0·2141... and −0·9341....

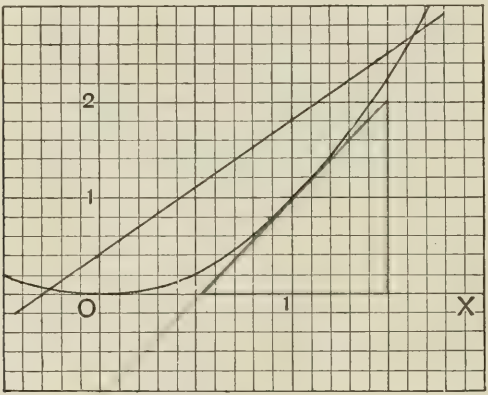


Fig. 29.

In general, the roots of $ax^2 + bx + c = 0$ may be found as the abscissae of the points of intersection of the graphs of

$$y = ax^2 \quad \text{and} \quad y = -bx - c.$$

Sometimes it may be more convenient to take the graphs of

$$y = ax^2 + c \quad \text{and} \quad y = -bx.$$

In many cases however it is preferable to use the method shown in the next example.

Example 2. Solve the equation $523x^2-726x-213=0$.

Divide by the coefficient of x^2, express the fractions as two-place decimals and write the equation in the form $x^2=1{\cdot}39x+0{\cdot}41$.

To draw the linear graph take the points (1, 1·80) and (−1, −0·98); when the line is drawn note, as a test of accuracy, whether it crosses the y-axis at the distance 0·41 above the origin.

A rough sketch of the graph of x^2 shows that the two abscissae are 1·6... and −0·2... ; the roots are then easily found to be 1·64 and −0·25 (Fig. 29).

When the coefficients are large this method should be taken; indeed, it is usually the best method. If many equations have to be solved it is useful to have a well-drawn graph of x^2. The straight line need not be actually drawn ; a ruler placed in the position for drawing the line will enable the roots to be read.

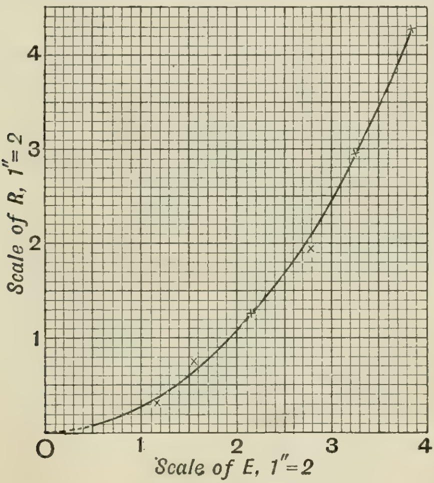


Fig. 30.

Example 3. Corresponding values of two quantities E and R are given by the table :

E	0·50	1·12	1·53	2·16	2·74	3·25	3·83
R	0·06	0·33	0·72	1·26	1·92	2·94	4·22

the values being subject to small errors ; find some simple relation between E and R.

When the points (E, R) are plotted (Fig. 30) the curve suggests that R is proportional to E^2 ; try therefore if the equation $R=aE^2$ will suit the table. To find a take the point $(2, 1{\cdot}09)$ which is on the graph; this point gives

$$1{\cdot}09 = 4a; \quad a = 0{\cdot}2725.$$

Try another point, say $(3, 2{\cdot}46)$; this gives

$$2{\cdot}46 = 9a; \quad a = 0{\cdot}273\ldots.$$

We might therefore take $a = 0{\cdot}273$, which gives the relation

$$R = 0{\cdot}273E^2.$$

When the values of R are calculated from this equation, for the different values of E, the results are found to agree pretty well with the given values; the above relation is therefore the one sought.

When the curve suggests the equation $R = aE^2 + b$, two points must be taken to determine the two numbers a, b, exactly as in the case of the linear graph (§ 17). In this case it is sometimes easier to plot, not the points (E, R) but the points (E^2, R). That is, when the graph suggests the equation $R = aE^2 + b$, begin over again; calculate the values of E^2, take these values as abscissae and the corresponding values of R as ordinates. If E^2 be denoted by F, say, and if it is found that the points (F, R) lie on a straight line, then F and R satisfy the linear equation $R = aF + b$, so that E and R satisfy the quadratic equation $R = aE^2 + b$. Naturally, this method involves a good deal of calculation but it is sometimes very useful.

A better method of determining a when $R = aE^2$ is the following. Calculate the quotient R/E^2 for each pair of corresponding values; for the above set these quotients are, in order,

$$0{\cdot}240,\ 0{\cdot}263,\ 0{\cdot}307,\ 0{\cdot}270,\ 0{\cdot}256,\ 0{\cdot}278,\ 0{\cdot}288.$$

These quotients are not equal but, allowance being made for the errors of observation, they may be considered as equal. Hence R/E^2 is constant, so that $R = aE^2$.

The value to be taken for a is the *mean* of the quotients, that is, the sum of the quotients divided by the number of them, in this case 7. We find

$$\text{sum of quotients} = 1{\cdot}902; \quad \text{mean} = \frac{1{\cdot}902}{7} = 0{\cdot}272;$$

so that $R = 0{\cdot}272E^2$. The value of a suggested by the points taken on the graph was $0{\cdot}273$; one value can hardly be considered much better than the other.

EXERCISES XI.

1. Graph the equations $y = 100x^2$ and $y = 100x^2 - 164$ from $x = 0$ to $x = 5$.

2. Graph the equation $y = 250 - 16x^2$ for positive values of y.

3. Graph the equation $22x^2 + 5y = 80$ for positive values of y.

4. Draw to a large scale the graph of $y=x^2$ from $x=6$ to $x=7$; from the graph find, as accurately as your scales allow, $\sqrt{45}$. (The origin of coordinates should be outside the sheet.)

5. Draw the graph of $y^2=x$. How is this graph related to that of $y=x^2$?

More generally, how is the graph of $x=ay^2$ related to that of $y=ax^2$?

6. On the same axes and with the same scales draw the graphs of $x^2=y$ and $y^2=8x$, carrying the curves sufficiently far to make sure that you have got all their points of intersection. State the abscissae of the points of intersection and write down the equation of which these abscissae are the roots.

7. The same problem as in example 6 for the equations

$$x^2=5y, \quad y^2=12x.$$

8. The same problem as in example 6 for the equations

$$x^2=-5y, \quad y^2=12x.$$

9. The same problem as in example 6 for the equations

$$x^2=y+10, \quad y^2=x+4.$$

10. The same problem as in example 6 for the equations

$$9x^2+4y=50, \quad y^2+25=17x.$$

Solve the equations in examples 11–16:

11. $9x^2-5x-2=0$.

12. $25x^2-13x-60=0$.

13. $3{\cdot}2x^2+1{\cdot}3x-2=0$.

14. $332x^2-576x-428=0$.

15. $1{\cdot}8x^2-9{\cdot}36x+8{\cdot}72=0$.

16. $2{\cdot}15x^2-1{\cdot}87x-8{\cdot}53=0$.

17. Find the greater positive root of the equation

$$3{\cdot}2x^2-53x+112=0.$$

Find the relation between x and y in examples 18–20.

18.

x	0·5	0·8	1·0	1·4	1·8	2·5	3
y	2·8	3·9	5·0	7·9	11·7	20·8	29·0

19.

x	1·0	1·5	2·0	2·5	3·0	3·5
y	16·10	36·21	64·38	100·6	144·9	197·2

20.

x	1	2	3	4	5	6	8
y	6·1	19·2	41·2	71·9	111·5	160	283·2

21. A particle moves in a straight line and its distance, s feet, from a fixed point in its line of motion t seconds after starting is given by the table:

t	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
s	11	$14\frac{1}{2}$	20	$27\frac{1}{2}$	$37\frac{1}{2}$	$49\frac{1}{2}$

Find an equation between s and t.

22. A point is moving in a plane and its horizontal and vertical coordinates, x feet and y feet respectively, t seconds after starting are given by the equations

$$x=100t, \quad y=144-16t^2.$$

Plot the path of the point and find when and at what distance from the origin it reaches the horizontal through the origin.

23. $A, B, C, D, E, \ldots$ are n points in a plane. The straight line AB is horizontal; BC slopes upwards (to the right) at the gradient 0·1; CD slopes upwards at the gradient 0·2; DE slopes upwards at the gradient 0·3 and so on. The projection on the horizontal of each of the lines $BC, CD, DE, \ldots$ is equal to AB which has the length 1. Taking the middle point of AB as origin and axes along and perpendicular to AB as axes of coordinates, show that all the points lie on a curve given by an equation of the form $y=ax^2+b$ and find the values of a and b.

24. Given the table of corresponding values:

V	8·23	11·63	18·40	26·02	82·28
D	1	2	5	10	100

find a relation between V and D.

25. In Kelvin's *Mathematical and Physical Papers*, vol. i., p. 448, corresponding values of two quantities V and T are given as follows:

V	46·9	51·5	68·1	72·7	78·7	84·8	104·5	130·2	133·2	145·4
T	27·5	32	46·5	57·5	67·5	74	91	151	172	191

Verify that, approximately, $T=0·01026\,V^2$.

26. If V and T are given by the table:

V	7·08	15·36	23·04	30·71
T	2·5	13·5	36·5	48

show that, approximately, $T=0·0567\,V^2$.

26. Graph of $y = ax^2 + bx + c$. We will draw the graph for two typical cases, (i) for a a positive number, (ii) for a a negative number.

(i) Draw the graph of $y = 4x^2 - 8x - 7$ from $x = -3$ to $x = 5$.

Calculate first the values of y for the integral values of x; we thus obtain the table:

x	-3	-2	-1	0	1	2	3	4	5
y	53	25	5	-7	-11	-7	5	25	53

The greatest value of y within the range is 53; y also takes negative values up to -11. We may now choose the scales, taking the vertical unit line, say $\frac{1}{10}$th the horizontal one, and then plot the above points.

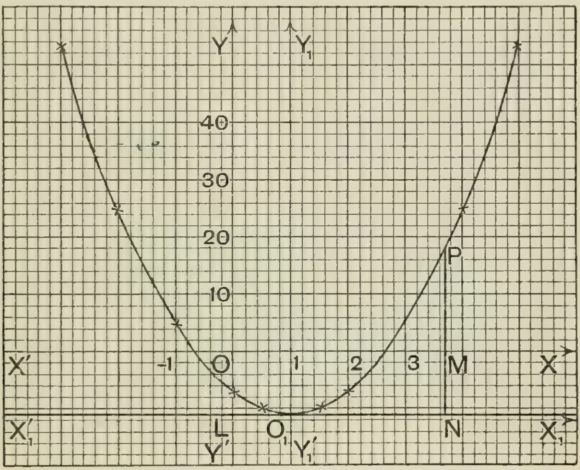


Fig. 31.

It is obvious that the graph will have a turning point at or near the point $(1, -11)$; we should therefore find one or two points near this one and on each side of it. Make, then, the supplementary table:

x	0·5	0·7	0·8	0·9	1·1	1·2	1·3	1·5
y	-10	$-10·64$	$-10·84$	$-10·96$	$-10·96$	$-10·84$	$-10·64$	-10

This table is much fuller than there is usually any need for, but it has been given to show how slowly the ordinate changes near the turning point $(1, -11)$.

The graph may now be drawn freehand. (Fig. 31.)

(ii) Draw the graph of $y=7+8x-4x^2$ from $x=-3$ to $x=5$.

The value of y in this equation differs *only in sign* from that of y in (i) for the same value of x we therefore plot the points $(-3, -53)$, $(-2, -25)\ldots$, $(5, -53)$. This graph is the image of the first one in the x-axis. (Fig. 32.)

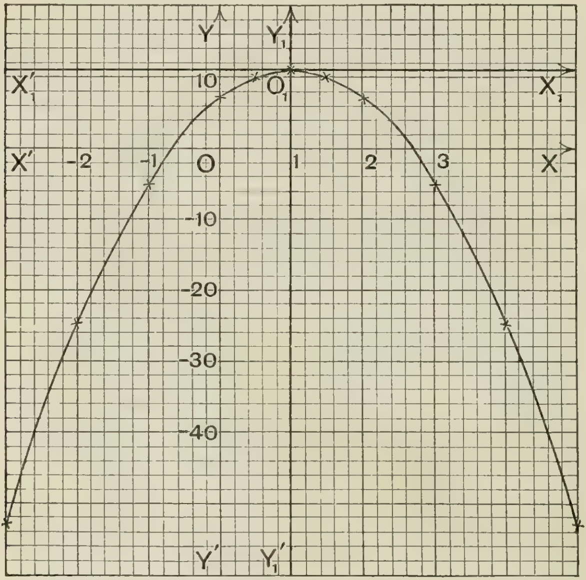


Fig. 32.

The two equations just discussed are of the form

$$y=ax^2+bx+c.$$

As will be seen in § 29 the value of a determines the *shape* of the curve; the values of b and c determine its position with respect to the coordinate axes. When a is positive, the curve is *concave upwards* (Fig. 31); when a is negative, the curve is *convex upwards* (Fig. 32). The curve is called **a parabola** (§ 29).

Another method of drawing the graph is to plot with the same scales the graphs of ax^2 and $bx+c$ and then to add the ordinates. This method is of great importance for more complicated curves and will be illustrated in drawing the graph of a cubic function (§§ 37, 38).

27. Application to Quadratic Equations and Quadratic Relations. We shall discuss two applications of the graph of ax^2+bx+c.

Example 1. Solve the equation $4x^2-8x-7=0$.

The roots of this equation are the values of x that satisfy the simultaneous equations

$$y=4x^2-8x-7 \text{ ......(i)}, \quad y=0 \text{ ....(ii)};$$

in other words, **they are the abscissae of the points where the graph of equation (i) crosses the x-axis.**

From Fig. 31 we see that the roots are 2·66 and −0·66.

Similarly we see that the roots of

$$4x^2-8x-7=10 \text{ ....}(\alpha)$$

are the abscissae of the points where the graph of (i) is cut by the straight line $y=10$. From Fig. 31 the roots are seen to be 3·29 and −1·29.

When a graph is to be used merely for the purpose of solving an equation it need not be traced except for points on it near the x-axis (or other line) and there it should be traced as accurately as possible. To find the *neighbourhood* of the points where it crosses the x-axis, observe that the value of y given by a value of x a little less than the root is *of opposite sign* to that given by a value of x a little greater than the root.

For example, take $y=4x^2-8x-7$. When $x=2$, $y=-7$ and when $x=3$, $y=5$; the curve therefore must cross the x-axis at some point between $x=2$ and $x=3$. Similarly, when $x=0$, $y=-7$, and when $x=-1$, $y=5$; the curve therefore must cross between $x=0$ and $x=-1$. The neighbourhoods of the two roots being thus found, a few values of y will give the shape of the curve near these points and thus the roots themselves.

In the same way to solve equation (α) find values of x, not differing much from each other, that make y a little less and a little greater than 10.

As examples the student may try to solve some of the equations 11–16, p. 77.

Example 2. Find a relation between x and y that will satisfy the following system of values:

x	0	0·5	1	1·5	2	2·5	3
y	5·4	6·3	6·6	6·1	5·0	3·2	0·6

When the points are plotted and a smooth curve drawn to fit them (Fig. 33) the curve suggests that x and y satisfy a relation of the form

$$y = ax^2 + bx + c.$$

To determine whether the suggestion is correct, take three points on the curve so as to obtain three equations for finding the numbers a, b, c. Take the three points for which x has the values 0, 1, 2 respectively. These give $\quad 5{\cdot}4 = c\,;\ \ 6{\cdot}6 = a + b + c\,;\ \ 5 = 4a + 2b + c,$

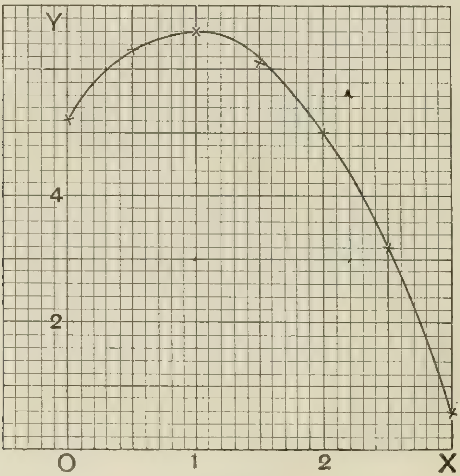


Fig. 33.

from which we obtain

$$a = -1{\cdot}4,\quad b = 2{\cdot}6,\quad c = 5{\cdot}4.$$

The relation between x and y becomes

$$y = 5{\cdot}4 + 2{\cdot}6x - 1{\cdot}4x^2.$$

The values of y calculated from this equation agree well with the given values.

This example is specially simple; it is quite obvious that if the given numbers were large the calculations would be very laborious. It is not however difficult in any case to plot the points and to obtain from the curve a suggestion as to the algebraic relation between the quantities; but more powerful mathematical methods than are employed in this book are often required for the practical evaluation of the coefficients. In Mr. Bashforth's works on the Resistance of the Air to the Motion of Projectiles excellent examples will be found of the more difficult type.*

EXERCISES. XII.

Draw the graphs of equations 1–6 for values of x from $x=-5$ to $x=5$. State the turning points and say whether the value of y at the turning point is a maximum or a minimum.

1. $y=2x+x^2$. **2.** $y=2x-x^2$. **3.** $y=4x+x^2$.

4. $y=4x-x^2$. **5.** $y=10x+4x^2$. **6.** $y=10x-4x^2$.

7. Graph the function $13+30x-9x^2$; extend the graph far enough to obtain the roots of the equations

(i) $9x^2-30x-13=0$. (ii) $9x^2-30x-24=0$.

8. Graph the function $10+3{\cdot}4x-0{\cdot}6x^2$. Find its maximum value and the values of x for which it vanishes.

Find as accurately as you can by means of a graph the maximum or the minimum value of each of the functions 9–11 and state the value of x for which the function has its turning value.

9. $(x-1)(x-3)$. **10.** $(2x+3)(x-\frac{1}{2})$. **11.** $x(12-x)$.

12. Show by a graph the relation between the area and one side of a rectangle the perimeter of which is 72 inches. What is the greatest area the rectangle can have?

13. x and y are two numbers such that $3x+4y=48$; what are the values of x and y when the product xy has its greatest value?

14. A point P moves along the straight line given by the equation

$$x+5y=60,$$

and M, N are the projections of P on the coordinate axes OX, OY. What is the greatest value of the rectangle $OMPN$, the coordinates of P being positive?

15. Corresponding values of u and v are given as follows:

u	1	2	3	4	5	6	7
v	25	41	55	67	77	85	91

* A Mathematical Treatise on the Motion of Projectiles. By Francis Bashforth. (London: Asher & Co., 1873.)

Show that u and v are connected by an equation of the form

$$v = au^2 + bu + c$$

and find the values of a, b, c.

16. Corresponding values of t and R are given as follows :

t	1	1·5	2	2·5	3	4
R	11	14	15·5	16·5	16	13

Test whether R is a quadratic function of t.

17. The resistance, R ohms, of a wire at t deg. Cent. is given by the table :

t	0	5	10	15	20	25	30	35	40
R	25	25·49	25·98	26·48	26·99	27·51	28·03	28·55	29·08

Show that $R = 25(1 + at + bt^2)$ and find the values of a and b. What is the value of R when $t = 12$ and when $t = 33$?

18. The following values are taken from a table of experimental results :

t	11·94	15·09	19·20	24·64	31·88	36·42
e	272	279	286	297	310	315

Show that the relation between t and e may be represented very approximately by an equation of the form

$$e = a + bt + ct^2$$

and find the most probable values of a, b, c.

19. Solve graphically the simultaneous equations

$$y + 20 = x^2, \quad 2y = 56 + 13x - 35x^2.$$

20. Graph the equation $x = 4y^2 - 8y - 7$. What is the maximum or minimum value of x ?

21. Graph the equations

$$\text{(i)}\ x = 14 - 24y + 9y^2\ ; \qquad \text{(ii)}\ 5x = 25 + 12y - 5y^2.$$

22. Solve graphically the simultaneous equations

$$y = 2 + 2x - x^2, \quad x = 14 - 24y + 9y^2.$$

23. A point is moving in a plane and at time t seconds from a chosen instant its distances from two rectangular axes OY, OX in the plane are x, y, feet respectively where

$$x = 400t, \quad y = 100t - 16t^2.$$

What path does the point describe? For what value of t is y a maximum and what are then the values of y and x? For what values of t is y zero?

24. If $x=5-6t$, $y=5+6t-t^2$, where x, y, t have the same meanings as in the preceding example, trace the path of the point and answer the same questions as in example 23.

28. Change of Origin. If the graph of $y=4x^2$ is plotted with the same scales as are taken for the graph of (i) § 26 it will be found that the two graphs can be made to coincide, by superposition; in other words, they are the same curves but they occupy different positions with respect to the coordinate axes. The student should make the test for himself; it is easily done by using tracing paper.

In general, the graph of ax^2+bx+c can be made to coincide, by superposition, with that of ax^2 if both graphs are drawn with the same scales. The proof of the general proposition depends on changing the origin of coordinates; we will indicate the method fully for the equation

$$y=4x^2-8x-7. \qquad \text{(i)}$$

By the method of "completing the square" equation (i) may be written

$$y+11=4(x-1)^2. \qquad \text{(ii)}$$

Now let $$x-1=X, \quad y+11=Y, \qquad \text{(iii)}$$
and equation (ii) becomes

$$Y=4X^2. \qquad \text{(iv)}$$

The graph of (iv), with X, Y as coordinates, is obviously the same graph as that of $y=4x^2$, with x, y as coordinates, provided the scales are the same. To see the meaning of the coordinates X, Y notice that, by equations (iii),

$$X=0 \text{ gives } x=1; \quad Y=0 \text{ gives } y=-11.$$

Let O_1 (Fig 31) be the point $(1, -11)$ and draw $X_1'X_1$, $Y_1'Y_1$ horizontally and vertically through O_1; *X, Y are the coordinates, referred to the axes $X_1'X_1$, $Y_1'Y_1$ of the point whose coordinates referred to the axes $X'X$, $Y'Y$ are x, y.* For, if $X_1'X_1$ cut $Y'Y$ at L and if the perpendicular from the point P (x, y) cut $X'X$ at M and $X_1'X_1$ at N we have

$$x=OM, \quad y=MP, \quad X=O_1N, \quad Y=NP.$$

Also the step $LO_1=1$ and the step $LO=11$; *OL is the step* -11.

Now $x=LO_1+O_1N=1+X$; $x-1=X$.

$y=NP-NM=NP-LO=Y-11$; $y+11=Y$.

This proves that the change from x and y to X and Y is simply equivalent to choosing the point $O_1(1, -11)$ as a new origin and measuring the coordinates X, Y along the axes through O_1 parallel to the old axes.

The transformation given by equations (iii) is called **change of the origin, the new axes being parallel to the old axes.**

It is a very simple problem to show, in general, that if the coordinates of the new origin are a and b and if the coordinates of any point P are x and y when referred to the old axes, and are X and Y when referred to the new axes

$$x=a+X, \quad y=b+Y; \quad x-a=X, \quad y-b=Y. \text{......(A)}$$

Notice that the coordinates of the new origin are obtained by putting $X=0$ and $Y=0$.

Take now the general case $y=ax^2+bx+c$. This may be written, by the method of completing the square,

$$y+\frac{b^2-4ac}{4a}=a\left(x+\frac{b}{2a}\right)^2.$$

Let $$x+\frac{b}{2a}=X, \quad y+\frac{b^2-4ac}{4a}=Y, \text{...............(B)}$$

and the equation becomes $Y=aX^2$, the graph of which is clearly the same as that of $y=ax^2$.

The new origin is the point given by the equations

$$x=-\frac{b}{2a}, \quad y=-\frac{b^2-4ac}{4a}, \text{.................(C)}$$

these values being obtained by putting $X=0$, $Y=0$ in equations (B). The point given by (C) is the turning point of the graph; the line through this point parallel to the x-axis is a tangent to the graph.

29. The Parabola. The curve given by the equation

$$y=ax^2+bx+c \text{........................(1)}$$

is called **a parabola**; from the discussion in the last article it is plain that its shape depends only on a.

The straight line about which the curve is symmetrical (OY in Fig. 25; O_1Y_1 in Figs. 31, 32) is called **the axis** of the parabola. The point in which the axis meets the curve (O or O_1) is called **the vertex** of the parabola. The number $1/a$ is sometimes called **the parameter** of the parabola.

The parabola is not a closed curve like the circle; it extends to infinity on both sides of its axis, because the equation $y=ax^2$ gives a real value of y for every real value of x and when x becomes very large so does y.

The vertex of the parabola given by equation (1) is always either the *highest* or the *lowest* point of the curve; it is the highest when a is negative, the lowest when a is positive. The knowledge of the position of the vertex is of great assistance in tracing the curve, not only because it is the highest or the lowest point on the curve but because the curve is symmetrical about the vertical line through it.

30. Average Gradient. The gradient of a straight line is the vertical rise from *any* point P on it to *any other* point Q on it divided by the horizontal advance from P to Q; the same quotient is obtained whatever two points are taken on the line. The quotient obtained by taking two points on a curved line however will clearly depend on the positions of *both points*; in Fig. 25, for example, the quotients for the three portions OK, OA, AD of the curve are

$$\frac{HK}{OH}=\tfrac{1}{2},\quad \frac{UA}{OU}=1,\quad \frac{FD}{AF}=3.$$

When a point is moving along a curve, the direction in which it is moving when it has reached the point P is that of the tangent to the curve at P; the gradient of the tangent line is therefore taken as the gradient of the curve at the point P. We are not yet in a position to calculate this gradient, though we can calculate approximations to it by finding the gradient of the chord PQ, where Q is a point on the curve near P. The gradient of the chord, or secant, PQ is called **the average gradient of the arc** PQ; this number, when multiplied by the horizontal advance from P to Q, will give the actual rise or fall in passing along the curve from P to Q. When Q is very close to P the gradient of the chord will clearly differ very little from that of the tangent.

The gradient of a straight line measures the rate of increase of the ordinate or of the function which it represents. Similarly, the average gradient of a portion PQ of a graph measures **the average rate of increase** of the ordinate, or of the function which it represents, as the abscissa or argument increases from its value at P to its value at Q. When the argument is denoted by x we speak of the average x-gradient of the function; when by t, of the average t-gradient and so on, but if no ambiguity is to be feared the x and the t may be omitted.

In calculating gradients we always suppose the abscissa to increase algebraically; the amount by which the abscissa increases, that is the horizontal advance from P to Q, may be called the **increment of the abscissa.** The vertical rise or fall from P to Q may be called the **increment of the ordinate**; this increment will be positive if the ordinate of Q is algebraically greater than that of P, but negative if less than that of P.

Hence in all cases

$$\text{average gradient of arc } PQ = \frac{(\text{ord. of } Q) - (\text{ord. of } P)}{(\text{absc. of } Q) - (\text{absc. of } P)}$$

$$= \frac{\text{increment of ord. of } P}{\text{increment of absc. of } P}.$$

Example 1. Find the average gradient of the graph of $y = x^2$ as x increases (i) from 0 to 1, (ii) from 1 to 2, (iii) from 2 to 3, (iv) from -2 to -1, (v) from -1 to 0.

(i) When $x=0$, $y=0$ and when $x=1$, $y=1$; the increment of x is 1 and the increment of y is also 1 so that

$$\text{av. grad.} = \frac{1-0}{1-0} = 1.$$

(ii) When x increases from 1 to 2, y increases from 1 to 4, so that the increment of x is 1 and the increment of y is 3 and therefore

$$\text{av. grad.} = \frac{4-1}{2-1} = \frac{3}{1} = 3.$$

(iii) When x increases from 2 to 3 we find in the same way

$$\text{av. grad.} = \frac{9-4}{3-2} = \frac{5}{1} = 5.$$

(iv) When $x=-2$, $y=4$ and when $x=-1$, $y=1$; the increment of x is 1 and the increment of y is -3. Note that y changes *from* 4 *to* 1 and that the increment is obtained by subtracting the value *from* which it has changed from the value *to* which it has changed. The increment of y is in this case negative and the arc has a right-hand downward slope.

$$\text{av. grad.} = \frac{1-4}{-1-(-2)} = \frac{-3}{1} = -3.$$

(v) In this case

$$\text{av. grad.} = \frac{0-1}{0-(-1)} = \frac{-1}{1} = -1.$$

These gradients give a rough idea of the steepness of the graph along different portions of it; thus in case (iii) the average steepness is 5 times as great as in case (i). From the point of view of rates the average rate at which the function x^2 increases as x increases from 2 to 3 is 5 times as great as when x increases from 0 to 1.

Example 2. Find the average gradient of the graph of $y=x^2$ as x increases (i) from 2 to 2·5, (ii) from 2 to 2·1, (iii) from 2 to 2·01, (iv) from 2 to $2+h$.

$$\text{(i)}\quad \text{av. grad.} = \frac{(2{\cdot}5)^2-2^2}{2{\cdot}5-2} = 4{\cdot}5.$$

$$\text{(ii)}\quad \text{av. grad.} = \frac{(2{\cdot}1)^2-2^2}{2{\cdot}1-2} = 4{\cdot}1.$$

$$\text{(iii)}\quad \text{av. grad.} = \frac{(2{\cdot}01)^2-2^2}{2{\cdot}01-2} = 4{\cdot}01.$$

For case (iv) observe that when $x=2+h$, $y=(2+h)^2$; hence

$$\text{(iv)}\quad \text{av. grad.} = \frac{(2+h)^2-2^2}{(2+h)-2} = 4+h.$$

It will be noticed that (iv) includes (i), (ii), (iii); to obtain (i) from (iv) put $h=0{\cdot}5$, to obtain (ii) put $h=0{\cdot}1$, and to obtain (iii) put $h=0{\cdot}01$.

When h is very small, say $h=0{\cdot}01$ or $0{\cdot}001$, the direction of the chord PQ will be very nearly the same as the direction of the tangent to the graph at P. The student may try to give a sound (not merely a plausible) reason for the conclusion that the gradient of the tangent at P is exactly 4; test the conclusion by drawing the tangent.

Example 3. When a stone falls freely from rest under gravity the distance it falls in t seconds is $16t^2$ feet approximately. What is the average velocity of the stone during (i) one second, (ii) half a second, (iii) one-tenth of a second, (iv) the fraction h of a second, each of these intervals of time being reckoned from the instant given by $t=2$, that is, just after the stone has been falling for 2 seconds?

Let s denote the number of feet the stone falls in t seconds; then

$$s=16t^2. \qquad \text{.....................................(1)}$$

(i) To find the distance the stone falls in case (i) we subtract the distance it falls from rest in 2 seconds from the distance it falls from rest in 3 seconds; these distances are obtained by putting t equal to 2 and 3 respectively in equation (1). Hence the number of feet the stone falls in case (i) is $16\times3^2-16\times2^2=80$.

Now **the average velocity** with which the stone falls during any interval of time is obtained by dividing the number of feet in the distance it falls during the interval by the number of seconds in the interval. In this case the number of feet is 80 and the number of seconds 1, so that the quotient is 80. The average velocity is therefore said to be *80 feet per second.*

It is clear that if the stone fell for 1 second with the *uniform* velocity of 80 feet per second, the distance it would fall would be 80 feet; the average velocity is thus equal to that uniform velocity with which in the same time the stone would fall through the distance it actually travels.

(ii) The number of feet the stone falls in this case is

$$16\times(2\tfrac{1}{2})^2-16\times2^2=36,$$

and the time during which it falls is $\frac{1}{2}$ second, so that, dividing 36 by $\frac{1}{2}$ we find the average velocity to be *72 feet per second.*

(iii) In this case the number of feet per second in the average velocity is

$$\frac{16\times(2\cdot1)^2-16\times2^2}{0\cdot1}=65\cdot6.$$

(iv) The distance the stone falls in $(2+h)$ seconds is $16\,(2+h)^2$ feet, so that the distance it falls in the fraction h of a second is, in feet,

$$16\,(2+h)^2-16\times2^2=64h+16h^2.$$

The average velocity during the fraction h of a second is therefore

$$\frac{64h+16h^2}{h}, \text{ that is, } 64+16h \text{ feet per second.}$$

We shall now state these results in a general form. In t_1 seconds let the stone fall s_1 feet; in (t_1+h) seconds let it fall s_2 feet. Then the distance, in feet, that it falls during the interval of h seconds is s_2-s_1, and we have $\quad s_1=16t_1^2, \quad s_2=16\,(t_1+h)^2$

so that $\quad s_2-s_1=16\,(t_1+h)^2-16t_1^2=32t_1h+16h^2.$

The average velocity during the interval, h seconds, that succeeds the first t_1 seconds of its fall, is

$$\frac{s_2-s_1}{h} \text{ feet per second,}$$

that is, $\quad 32t_1+16h$ feet per second.

Let the graph of $s=16t^2$ be drawn, with t as abscissa ; then, clearly, if P is the point on it whose abscissa is t_1 and Q the point whose abscissa is t_1+h, **the average velocity during the interval h seconds is simply the average gradient of the arc PQ.**

The velocity at time t_1 seconds is the gradient of the tangent to the graph at P.

Again, since the average rate at which s increases, as t increases from t_1 to t_1+h, is the quotient of the increment s_2-s_1 of s by the increment h of t, we see that the average velocity during the interval h seconds is **the average rate at which the function s or $16t^2$ increases as t increases from t_1 to t_1+h.**

All cases of average velocity are treated as in these examples. As soon as the relation between the distance, s feet say, travelled in time, t seconds, is known we can calculate the distance, s_2-s_1 feet, travelled during any interval, h seconds ; the quotient $(s_2-s_1)/h$ is the average velocity, in feet per second, during the h seconds. The student should note how, as in cases (i), (ii), (iii), the quotient comes nearer and nearer to a fixed number as the interval is made smaller and smaller ; case (iv) shows that, however small h may be, the quotient will never be quite 64 but may be brought as near to 64 as we please by sufficiently diminishing h.

What property will the number 64 measure (a) with respect to the graph of $s=16t^2$, (b) with respect to the motion of the stone?

EXERCISES. XIII.

Find the coordinates of the vertex, the equation of the axis and the equation of the tangent at the vertex of each of the parabolas in examples 1–4, and write each of the four equations in the form $Y=aX^2$. Sketch the parabolas.

1. $y=3x^2-12x+8$. **2.** $y=9+30x-25x^2$.

3. $3y=5x^2-7x-4$. **4.** $5y=8-11x-4x^2$.

Write each of the equations 5–8 in the form $X=aY^2$. Hence show that each equation represents a parabola ; find the coordinates of the vertex, the equation of the axis and the equation of the tangent at the vertex. Sketch the parabolas.

5. $x=2y^2-12y+21$. **6.** $x=4+12y-3y^2$.

7. $5x=4y^2-24y+21$. **8.** $7x=5+24y-9y^2$.

9. If $y=x^2+2x+3$ calculate the value of y for each of the following values of x : (i) 3, (ii) 3·1, (iii) $3+h$, (iv) a, (v) $a+h$.

What is the increment of y when x increases (α) from 3 to 3·1, (β) from 3 to $3+h$, (γ) from a to $a+h$?

10. If $y=15+20x-4x^2$ what is the increment of y as x increases (i) from 2 to 2·5, (ii) from 2 to $2+h$, (iii) from 5 to 6, (iv) from 5 to 5·5, (v) from 5 to $5+h$?

Find the average gradient of the arc PQ of the graphs of equations 11–19. In each case several values of the abscissa of Q are stated for one value of that of P; several gradients have therefore to be calculated and the student should note how these gradients change as the difference between the abscissae of P and Q becomes less and less. The probable value of the gradient of the tangent to the graph at the point P should be stated.

11. $y=x^2+3$; x of $P=3$; x of $Q=4$, $3·5$, $3·1$, $3·01$, $3+h$.

12. $y=5x-x^2$; x of $P=3$; x of $Q=4$, $3·5$, $3·1$, $3·01$, $3+h$.

13. $y=10+3x-2x^2$; x of $P=0$; x of $Q=1$, $0·5$, $0·1$, $0·01$, h.

14. $y=12-6x+x^2$; x of $P=-2$; x of $Q=-1$, $-1·5$, $-1·9$, $-1·99$, $-2+h$.

15. $y=x^2-8x+6$; x of $P=4$; x of $Q=5$, $4·5$, $4·1$, $4·01$, $4+h$.

16. $y=10+9x-x^2$; x of $P=4$; x of $Q=5$, $4·5$, $4·1$, $4·01$, $4+h$.

17. $y=5+7x-3x^2$; x of $P=2$; x of $Q=3$, $2·5$, $2·1$, $2·01$, $2+h$.

18. $y=6+4x-x^2$; x of $P=a$; x of $Q=a+h$.

19. $y=ax^2+bx+c$; x of $P=u$; x of $Q=u+h$.

20. A point is moving in a straight line, and at time t seconds from a chosen instant its distance from a fixed point on the line is s feet, where
$$s=100t-16t^2.$$

Find the average velocity of the point as t increases (i) from 4 to 5, (ii) from 4 to $4·5$, (iii) from 4 to $4·1$, (iv) from 4 to $4·01$, (v) from 4 to $4+h$. With what velocity is the point moving when $t=4$?

21. Find the average velocity of the point whose motion is specified in example 20, as t increases from t_1 to t_1+h. With what velocity is the point moving when $t=t_1$?

22. If the relation between s and t is given by the equation
$$s=Vt-\tfrac{1}{2}gt^2$$
find the average velocity of the moving point as t increases from t_1 to t_1+h. What is the velocity of the point when $t=t_1$?

23. If $x=400t$, $y=100t-16t^2$, what is the average rate at which x and y increase as t increases from t_1 to t_1+h? At what rates are x and y increasing when $t=t_1$?

24. A point is moving in a straight line with a velocity of v feet per second, and at time t seconds from a chosen instant the relation between v and t is given by the equation
$$v=50+36t-9t^2.$$

What is the average rate at which the velocity changes as t increases from t_1 to t_1+h?

CHAPTER V.

FRACTIONAL FUNCTIONS. CUBIC AND BIQUADRATIC FUNCTIONS.

31. Infinity. The quotient of a by x is defined to be that number which, when multiplied by x, gives a; but if x is zero the definition fails: the symbol $a/0$ is *not defined.* It is possible however to assign a meaning to this symbol, and in the next section we shall see the graphical interpretation of it.

For simplicity suppose $a=1$. By giving to x smaller and smaller values, say 0·1, 0·01, 0·001 ... we see that $1/x$ takes larger and larger values, namely 10, 100, 1000 .... Further, we can give to x a value small enough to make $1/x$ larger than any assigned number, no matter how large that number may be: for example, to make $1/x$ larger than 10 million we may take x equal to the fraction one divided by 10 million and one. The symbol $1/0$ is therefore taken as representing an infinitely large number or "infinity." The usual symbol for infinity is ∞.

Similarly, *if a is not zero,* $a/0$ also represents an infinitely large number. When the quotient a/x is positive, $a/0$ is said to be positively infinite ($+\infty$); when a/x is negative, $a/0$ is said to be negatively infinite ($-\infty$).

When x is very large, a/x is very small; when x is infinite, a/x is zero.

It must be specially noted that infinity is not a number in the same sense that 2 is a number; for example, it does not follow that ∞/∞ is equal to 1. We are only concerned at present with the *limiting* case of a fraction like a/x; we say nothing about other operations in which the symbol for infinity may appear. Further, $a/0$ *is not necessarily infinite if* $a=0$; the symbol $0/0$ has no meaning of any kind as yet.

32. Fractional Functions, $\frac{a}{x}, \frac{a}{x^2}$. The simplest case is that given by $y=1/x$.

Take first the values of y for positive values of x; they are easily calculated and the curve can be plotted, say from $x=0{\cdot}4$ to $x=3$ (Fig. 34). For smaller values of x however the values of y become very large; a point on the graph as

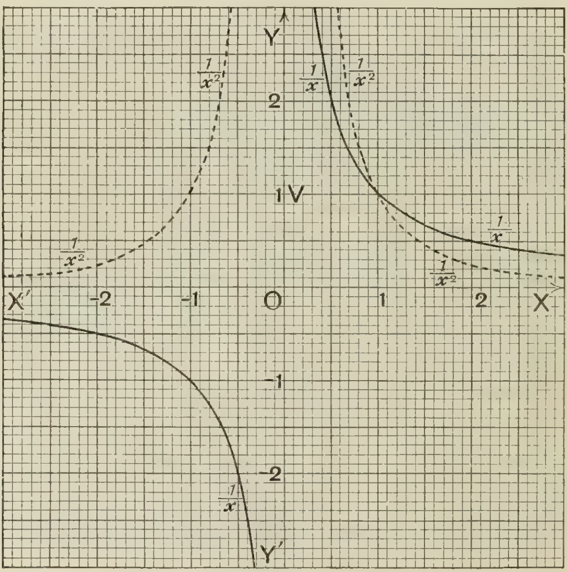


Fig. 34.

it gets near to the y-axis rises to a great distance above the x-axis. So long as y is finite, no matter how large it may be, x is also finite though small and the graph has not reached the y-axis; when the graph reaches the y-axis, x

has become zero and y has become infinite. The graph is in this case said to approach the y-axis **asymptotically**, or, to have the y-axis as an **asymptote**; as a point moves upwards along the graph it gets nearer and nearer to the y-axis, but it does not reach the axis till it has moved off to an infinite distance.

In the same way it may be seen that the x-axis is an asymptote of the graph.

When x is negative, y is also negative, and the graph approaches the negative ends of the two axes asymptotically. The complete curve consists of two branches lying one in the first and the other in the third quadrant; it is called a **hyperbola** (§ 33).

Definition. In general, when a curve has a branch extending to infinity, the branch is said to approach a straight line **asymptotically**, or to have the straight line for an **asymptote**, if, as a point moves off to infinity along the branch, the distance from the point to the straight line **tends towards zero as a limit**—that is, if, as the point moves off to infinity, the distance becomes and remains less than any given length, however small that length may be.

There is a kind of symmetry, called **central symmetry**, about the graph of $1/x$. For let a be any number; then the points $(a, 1/a)$ and $(-a, -1/a)$ are both on the graph because their coordinates satisfy the equation $y = 1/x$. But these points are symmetrical with respect to the origin; therefore to every point on the curve there corresponds another point symmetrical to it with respect to the origin and also on the curve. The curve is in this case said to have the origin as a **centre of symmetry**. The use that may be made of central symmetry in plotting the graph is obvious.

The graph of $1/x$ will be the graph of a/x, when a is positive, provided OV is taken to represent not 1 but a (§ 24).

The graph of $-1/x$ (and therefore of $-a/x$ when a is positive) lies in the second and fourth quadrants. If the axes in Fig. 34 be interchanged so that OY' becomes the new OX and OX becomes the new OY, the graph of $1/x$ will become that of $-1/x$; the number -1 on OY' will become the number 1 on the new OX, and the number 1 on the OX of the diagram will become the number 1 on the new OY.

The graph of $1/x^2$, for positive values of x, resembles that of $1/x$; it lies above that of $1/x$ when x is less than 1, but below it when x is greater than 1. Both the x-axis and the y-axis are asymptotes. The curve is symmetrical about the y-axis and consists of two branches lying in the first and second quadrants. It is represented by the dotted curve in Fig. 34.

The graphs of $1/x^3$, $1/x^4$, ... for positive values of x resemble that of $1/x$, but they approach the x-axis more rapidly when x is greater than 1, and ascend more rapidly when x is less than 1.

33. Rectangular Hyperbola. The function $1/x$ is the simplest case of the fractional function given by the equation

$$y=\frac{ax+b}{cx+d}, \quad \text{.........................(1)}$$

in which both numerator and denominator are linear functions of x. To see the general nature of the graph of (1) consider the equation

$$y=\frac{4x-7}{2x-5}. \quad \text{.........................(2)}$$

This equation may be written

$$y=2+\frac{3}{2x-5} \quad \text{or} \quad y-2=\frac{1\cdot5}{x-2\cdot5}. \quad \text{...........(2')}$$

Now put X for $x-2\cdot5$ and Y for $y-2$, that is, shift the origin (§ 28) to the point O_1 $(2\cdot5, 2)$ and the equation becomes

$$Y=\frac{1\cdot5}{X}. \quad \text{.........................(3)}$$

If therefore we take as new axes the lines $X_1'O_1X_1$, $Y_1'O_1Y_1$, drawn through O_1 parallel to $X'OX$, $Y'OY$ respectively, the graph will be of the same shape as that of $y=1\cdot5/x$; the asymptotes are the lines $X_1'X_1$, $Y_1'Y_1$. The graph is shown in Fig. 35; for negative values of X comparatively little is shown.

For other values of a, b, c, d equation (1) can also be reduced to the form of equation (2′) because

$$\frac{ax+b}{cx+d}=\frac{a}{c}+\frac{(bc-ad)/c^2}{x+d/c}=f+\frac{g}{x+h} \text{ say.}$$

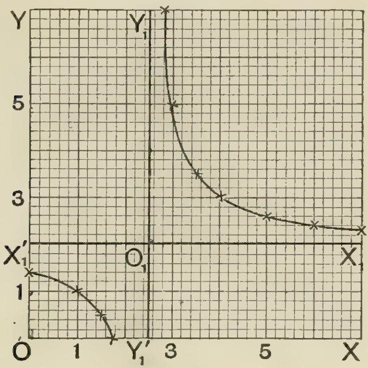


Fig. 35.

If therefore we put X for $x+h$ and Y for $y-f$, equation (1) becomes $Y=g/X$. ....(1′)

In all cases then the graph of (1) resembles that of $y=1/x$, but the asymptotes are not usually the coordinate axes; they are in general parallel to the axes.

To draw the graph of equation (2) it is perhaps best to begin by drawing the asymptotes. The asymptote $Y_1'Y_1$ is given by the value of x that makes y infinite, and is therefore obtained by equating to zero the denominator of the fraction, namely $2x-5$; $Y_1'Y_1$ is the line given by $2x-5=0$ or $x=2{\cdot}5$. In the same way the asymptote $X_1'X_1$ is given by the value of y that makes x infinite; to find it, solve the equation for x in terms of y and then equate the denominator to zero; *or* divide the given fraction by its denominator and equate y to the integral part of the quotient. The equation of $X_1'X_1$ is $y=2$.

When the asymptotes have been drawn the calculation of a few ordinates will readily give the curve.

A case of equation (1) that is of considerable importance is that for which $b=0$. This case has been met with in § 17, example 3. Equation (3) of that example is

$$e=\frac{100\,W}{3{\cdot}504\,W+44{\cdot}64},$$

and the graph is the curved line of Fig. 24. The asymptote parallel to the axis of W is given by

$$e=\frac{100}{3{\cdot}504}=28{\cdot}54,$$

and the curve approaches this asymptote from below.

The graph of equation (1) is called a **rectangular hyperbola.** The word "rectangular" is used because the asymptotes are at right angles to each other; as a rule, the asymptotes of a hyperbola are not at right angles to each other.

34. Applications of the Hyperbola. The graphs just discussed are sometimes useful in suggesting a relation between variables of which a few corresponding values are known; we give some illustrations.

Example 1. The pressure p, measured in centimetres of mercury, corresponding to the volume, v cubic centimetres, of a quantity of air kept at constant temperature was determined experimentally, and the following pairs of corresponding values were obtained:

v	20·7	22·1	23·6	25·4	27·3
p	130·3	121·5	114·1	105·6	98·4

Find an equation that will represent approximately the relation between v and p.

We notice that as v increases p decreases, and when the points (v, p) are plotted the curve through them resembles one of the curves of Fig. 34. The simplest of these curves would give an equation of the form

$$p=a/v \text{ or } pv=a \text{ ....(i)}$$

where a is a constant.

To test whether this relation suits, we form the product of each pair of corresponding values; the products, taken in order, are

2697, 2685, 2693, 2682, 2686.

These numbers are as nearly equal as can be expected, so that the required relation is of the form (i). The best value for the constant a is the mean of the products, that is, their sum divided by 5, the number of them. Hence

$$pv=\frac{13443}{5}=2689. \text{ ....(ii)}$$

The rectangular hyperbola is therefore an **isothermal curve**, because it represents the relation between pressure and volume when the temperature is constant. The equation

$$pv = \text{constant}$$

expresses **Boyle's Law**.

The equation $$pv^n = a, \qquad \text{(iii)}$$

of which the one just treated is a particular case, will be discussed in the next chapter; but we may here note a method by which the determination of the constants n, a in (iii) may be reduced to a problem on the straight line.

Take the logarithm of each member of equation (iii); then

$$\log p + n \log v = \log a.$$

Now put $x = \log v$, $y = \log p$ and we get the linear equation

$$y + nx = \log a. \qquad \text{(iv)}$$

Hence when v, p satisfy equation (iii), x, y satisfy equation (iv). If therefore the points (v, p) seem to lie on a curve with an equation of the form (iii) a good method of testing is to plot the points (x, y) and see whether they lie on a straight line. The values of n and $\log a$ are obtained from the linear graph as in § 17, example 3. The best method, however, of finding a is to calculate the values of pv^n (the value of n being taken from the graph) and then to take the mean of these values; in any case the products pv^n should be tested so as to verify the value of n.

Example 2. Find a simple relation connecting x and y, pairs of corresponding values of these quantities being as in the table.

x	1	2	3	4	5	6	7	8	9
y	2·05	3·23	3·95	4·49	4·87	5·20	5·40	5·60	5·77

Fig. 36 shows the graph, which is of the hyperbolic type. It is evident however that the product xy is not constant, so that we may try equation (1) of § 33.

The curve seems as if, when produced, it would go through the origin. Now, when the hyperbola represented by that equation goes through the origin the term b is zero, and when $b = 0$ the determination of the constants can be reduced in various ways to a problem on the straight line.

Putting $b=0$ in equation (1) § 33 we obtain

$$cxy=ax-dy. \qquad \ldots\ldots(\alpha)$$

Dividing both sides of (α) first by x, next by y and lastly by xy, we derive the three forms

$$cy=a-d\frac{y}{x}\ldots\ldots(\beta); \quad cx=a\frac{x}{y}-d\ldots\ldots(\gamma); \quad c=a\frac{1}{y}-d\frac{1}{x}\ldots\ldots(\delta).$$

Now in (β) put u for y/x, in (γ) put v for x/y and in (δ) put X for $1/x$ and Y for $1/y$; these equations then take the forms

$$cy=a-du\ldots\ldots(\beta'); \quad cx=av-d\ldots\ldots(\gamma'); \quad c=aY-dX\ldots\ldots(\delta').$$

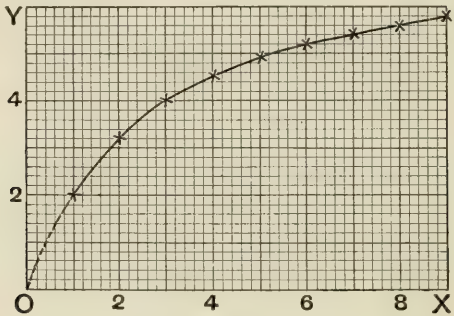


Fig. 36.

Equation (β') represents a straight line when y and u are taken as coordinates; so does equation (γ') when x and v are taken and equation (δ') when X and Y are taken.

To test then whether a graph can be represented by an equation of the form (α) we may use any of the equations (β'), (γ'), (δ'); naturally, we take the equation that gives us the most manageable coordinates.

For the example in hand take (γ'); we therefore form the table, after calculating the values of v by dividing each value of x by the corresponding value of y.

x	1	2	3	4	5	6	7	8	9
$v=\frac{x}{y}$	0·488	0·619	0·760	0·891	1·027	1·154	1·296	1·429	1·560

Plotting these values on a sheet that will allow for v a scale of 1″ to 0·1 (count ordinates from 0·45) we see that the points are very approximately on a straight line. Hence there is a linear relation between

x and v; taking the points for which $x=4$ and $x=8$ we get the equation

$$xy = 7{\cdot}44x - 2{\cdot}62y.$$

It will be found on trial that this equation is satisfied very approximately by the given values of x and y.

When the term b in equation (1) § 33 is not zero these transformations are not applicable. That equation really contains only three independent constants, for it may be written in the form

$$y = \frac{Ax+B}{x+D}.$$

To test this equation we must select three points on the graph which will give three equations to determine A, B, D.

It need hardly be added that similar transformations to those of the present example may easily be devised for special cases. Thus, to test the equation

$$y = a/x^2 + d$$

we may put u for $1/x^2$ and test whether the points (u, y) lie on a straight line. No general rule however can be given; the plotting of the logarithms of the variables, as suggested in example 1 and as will be shown more fully at a later stage, is even more useful than the method just treated.

EXERCISES. XIV.

1. Draw the graph of $y=25/4x$ for positive values of x, and find graphically the roots of the simultaneous equations

$$4xy=25, \quad y+3x=10.$$

2. Graph the equations

$$\text{(i) } xy=10, \quad \text{(ii) } x^2y=10, \quad \text{(iii) } x^3y=10.$$

Find the abscissae of the points in which each of the graphs cuts the straight line given by

$$y+10x=25$$

and write down the equations of which these abscissae are the roots. Will it be necessary to plot each graph for negative values of x in order to find the roots?

3. If p is the pressure in pounds per square inch and v the volume in cubic feet of one pound of air at the temperature 32° F., then $pv=182$. Represent graphically the relation between v and p.

4. Draw to the same axes and with the same scales the curves given by the following equations:

(i) $u=\frac{3}{2}-\frac{1}{2}x^2$ from $x=0$ to $x=1$, $u=\frac{1}{x}$ for $x>1$;

(ii) $v=-x$ from $x=0$ to $x=1$, $v=\frac{1}{x^2}$ for $x>1$;

(iii) $w=-1$ from $x=0$ to $x=1$, $w=\frac{2}{x^3}$ for $x>1$.

These graphs are of importance in the Theory of the Potential (*E.C.*, pp. 154, 155).*

5. Graph the following equations:

(i) $y=10-\frac{1}{x}$; (ii) $y=10+\frac{1}{x}$;

(iii) $y=\frac{x-3}{x-4}$; (iv) $y=\frac{x-4}{x-3}$.

6. Graph the equation

$$xy-3x+2y-4=0$$

and find the abscissae of the points in which it is cut by the straight line $x+y=3$. Of what equation are these abscissae the roots?

7. Graph the equation $y+4=\frac{10}{(x-2)^2}$.

8. The deflection d of a galvanometer for a total resistance R ohms was found to be as follows:

R	6080	5485	4996	4419	3774
d	60	66·5	73	82·5	96·5

Find a relation between R and d.

9. Four yellow-pine laths of the same length 24″ and of the same depth 0·525″ but of variable breadth b inches give, for the same load, a deflection x inches; corresponding values of b and x were found to be as follows:

b	0·54	0·79	1·02	1·26
x	1·08	0·75	0·60	0·46

Show that, roughly, x varies inversely as b.

* The reference is to the author's *Elementary Treatise on the Calculus.* (London: Macmillan.)

10. Boyle's "Table of the Condensation of the Air" by which he verified the law that bears his name is as follows, p representing the pressure in inches of mercury and v being proportional to the volume.

v	48	46	44	42	40	38	36	34	32
p	$29\frac{2}{16}$	$30\frac{9}{16}$	$31\frac{15}{16}$	$33\frac{8}{16}$	$35\frac{5}{16}$	37	$39\frac{4}{16}$	$41\frac{10}{16}$	$44\frac{3}{16}$

v	30	28	26	24	23	22	21	20
p	$47\frac{1}{16}$	$50\frac{5}{16}$	$54\frac{5}{16}$	$58\frac{13}{16}$	$61\frac{5}{16}$	$64\frac{1}{16}$	$67\frac{1}{16}$	$70\frac{11}{16}$

v	19	18	17	16	15	14	13	12
p	$74\frac{2}{16}$	$77\frac{14}{16}$	$82\frac{12}{16}$	$87\frac{14}{16}$	$93\frac{1}{16}$	$100\frac{7}{16}$	$107\frac{13}{16}$	$117\frac{9}{16}$

Verify the law from these data.

11. Determine a relation between x and y from the following data:

x	1·4	1·7	2·3	2·8	3·3
y	2·04	1·38	0·76	0·51	0·37

[Plot either the points ($\log x$, $\log y$) or the points ($1/x^2$, y).]

Apply to examples 12–14 the method of § 34, example 2.

12.

x	1	2	3	4	5	6	7	8
y	2·09	2·90	3·34	3·61	3·79	3·92	4·02	4·10

13.

x	4	8	12	16	20	24	28	32
y	3·50	4·65	5·60	5·90	6·20	6·45	6·65	6·80

14.

x	3·6	4·4	5·2	5·8	6·6	7·2	8·0	8·6
y	30	20·3	16·9	15·1	14·0	13·1	12·4	12·0

15. The numbers in the following table are supposed to be connected by an equation of the form

$$xy = ax + by + c;$$

test the supposition.

x	4·0	6·3	8·7	10·0	12·4	14·0
y	33·8	30·8	28·1	26·7	24·5	23·2

16. F and d are given by the table

d	0·5	1	1·5	2	2·5	3	3·5	4
F	86·5	31·7	21·4	18·0	16·4	15·3	14·9	14·5

Plot the points $(F, 1/d^2)$ and find a relation between F and d.

17. Find a formula that will express the relation between the numbers T, K given by the scheme

T	12	15	20	25	30	38	50	75	100	150
K	536	627	719	773	810	848	883	919	937	956

18. Graph the function $x + 16/x$ from $x = 0·5$ to $x = 10$, and find the values of x and y at the turning point.

19. Illustrate by a graph the relation between the perimeter $2s$ and one side x of a rectangle whose area is 16 square inches. For what value of x is the perimeter least, and what is the least perimeter?

20. Graph the function $x + 32/x^2$ for positive values of x, and find the values of x and y at the turning point.

21. u and v are two positive numbers such that u^2v is equal to 108; what is the least value of $u + v$?

22. The volume of a cylinder is three-eighths of the volume of a sphere of radius 6 inches; for what value of the radius of the cylinder is the sum of the radius and the height of the cylinder a minimum, and what is that minimum sum?

35. Graphs of x^3 and x^4. The graphs are easily traced; the calculations are a little laborious but they need only be made for positive values of x.

The origin is a **centre of symmetry** (§ 32) for the graph of x^3. The curve touches the x-axis at O; but to the right of O the curve is above the axis while to the left of O it is below the axis; the curve *crosses* the axis at the point where it touches it (Fig. 37).

A point, such as O, where a curve crosses its tangent and bends away from it in opposite directions on opposite sides of the point is called a **Point of Inflexion**; the tangent at the point is called an **Inflectional Tangent**.

The graph of x^4 is symmetrical about the y-axis.

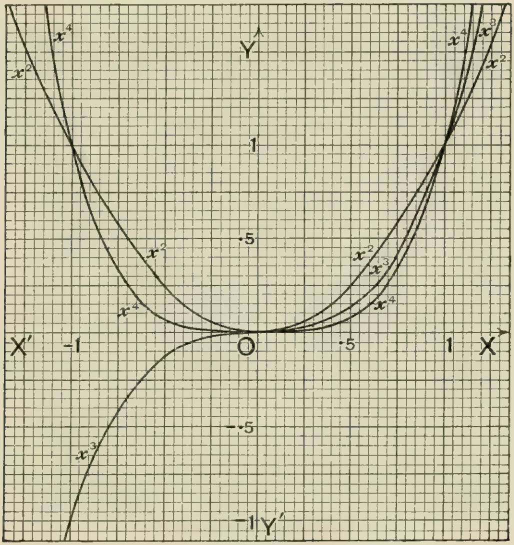


Fig. 37.

In Fig. 37 the graphs of x^2, x^3 and x^4 are shown from $x = -1$ to $x = 1$; they are extended a little to the left and a little to the right, but when x becomes greater than 1 the increase of x^3 and x^4 is so rapid that their graphs cannot be shown on the somewhat large scale of the diagram. The student will do well to draw the graphs say from $x = 0$ to $x = 4$, taking a small vertical unit.

The graphs of ax^3 and ax^4 need no further discussion after the explanations of §§ 23, 24.

36. Cubic Equations. First suppose the term in x^2 to be absent; the equation is therefore of the form

$$ax^3 + bx + c = 0 \quad \text{.........................} (a)$$

As in § 25 we see that the roots are the abscissae of the points of intersection of the curves given by

$$y = ax^3 \text{ and } y = -bx - c.$$

For example take the equation

$$2x^3 - 7x + 3 = 0.$$

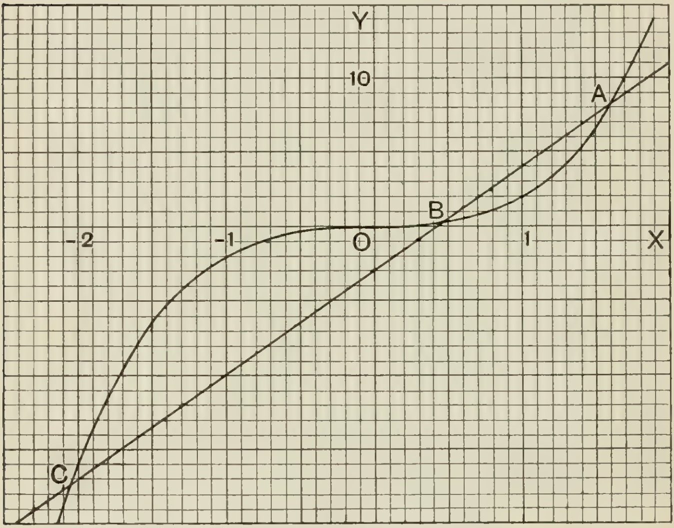


Fig. 38.

In Fig. 38 the curve $ABOC$ is the graph of $2x^3$ and the straight line ABC the graph of $7x-3$. A, B, C are the points of intersection of the graphs and the abscissae of these points are respectively 1·60, 0·46, – 2·06. The equation therefore has three roots, given by these numbers.

It will often be more convenient to divide first by the coefficient of x^3 and to take the graphs of the equations

$$y = x^3 \text{ and } y = -\frac{b}{a}x - \frac{c}{a}.$$

Next, suppose the cubic equation to be complete, that is, of the form

$$ax^3 + bx^2 + cx + d = 0. \qquad \text{.....................}(b)$$

In this case we may take the graphs of

$$y = ax^3 \quad \text{and } y = -bx^2 - cx - d,$$

or of

$$y = x^3 \quad \text{and } y = -\frac{b}{a}x^2 - \frac{c}{a}x - \frac{d}{a},$$

or of

$$y = ax^3 + d \text{ and } y = -bx^2 - cx,$$

but any method involves a good deal of labour (see also §39).

Again, it is easily seen that the roots of (b) are the abscissae of the points of intersection of the parabola and the hyperbola given by the equations

$$y = x^2 \text{ and } (ax+b)y + cx + d = 0$$

(compare Exercises XIV. 1, 2).

Similar methods apply to equations of higher degrees.

Thus, the equation $ax^4 + bx + c = 0$ can be solved by taking the graphs of ax^4 and $-bx-c$.

37. Graph of Cubic Function. To obtain a satisfactory curve by plotting points demands of the beginner a considerable amount of calculation. We shall indicate two methods, taking in both cases the equation

$$y = 2x^3 - 7x + 3.$$

First Method. Take a series of integral values of x, so as to obtain suggestions as to the points where the curve crosses the x-axis and also as to turning points. Form the table

x	-3	-2	-1	0	1	2	3
y	-30	1	8	3	-2	5	36

y has opposite signs when $x = -3$ and when $x = -2$; also the value for $x = -2$ is, numerically, much smaller than that for $x = -3$. Hence the curve must cross the x-axis a little to the left of $x = -2$, and it crosses *from below*.

Similarly we see that the curve crosses the x-axis *from above* between $x = 0$ and $x = 1$; and again, *from below*, between $x = 1$ and $x = 2$.

There will be a turning point (maximum) between $x = -2$ and $x = 0$, and another (minimum) between $x = 0$ and $x = 2$.

A few more values should now be calculated so as to obtain more exactly the points where the curve crosses the x-axis and where it turns. The following table will be sufficient:

x	$-2{\cdot}3$	$-1{\cdot}9$	$-1{\cdot}1$	$-0{\cdot}9$	$0{\cdot}4$	$0{\cdot}5$	$0{\cdot}9$	$1{\cdot}1$	$1{\cdot}5$	$1{\cdot}7$
y	$-5{\cdot}23$	$2{\cdot}58$	$8{\cdot}04$	$7{\cdot}84$	$0{\cdot}33$	$-0{\cdot}25$	$-1{\cdot}84$	$-2{\cdot}04$	$-0{\cdot}75$	$0{\cdot}93$

When x is numerically greater than 3, the term $2x^3$ grows very rapidly (numerically); the curve therefore rises rapidly towards the right and falls rapidly towards the left.

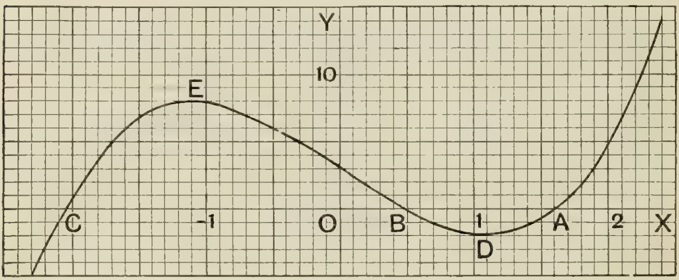


Fig. 39.

The curve is shown in Fig. 39.

The abscissae of the points A, B, C (Fig. 39) are the roots of the equation which was solved in § 36. At the turning point D, $x = 1{\cdot}08$ and at the turning point E, $x = -1{\cdot}08$ (approximately).

Second Method. In this method we make use of the graphs drawn in § 36.

Let $\quad y_1 = 2x^3, \quad y_2 = 7x - 3, \quad y = 2x^3 - 7x + 3;$

then
$$y = y_1 - y_2.$$

In Fig. 40, $y_1 = MP$, $y_2 = MQ$, so that $y = MP - MQ$.

By the rule for subtracting steps (§ 3) we have

$$MP - MQ = MP + QM = QM + MP = QP$$

where it must be remembered that MP, MQ, QP are *steps*, and therefore that their *direction* is as important as their length.

Hence $y = QP$ and, if we mark off the step MR equal to the step QP (not PQ), R will be a point on the required graph. It is easy now to plot points and to obtain a satisfactory curve. The curve is RRR, Fig. 40.

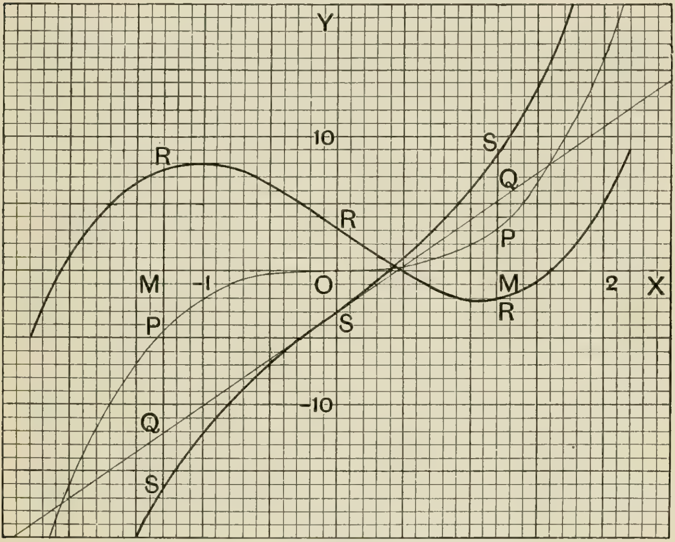


Fig. 40.

Consider now the graph of

$$y = 2x^3 + 7x - 3.$$

In this case $y = y_1 + y_2$. To find the point, S say, such that MS is the *sum* of MP and MQ, mark off from the point P the step PS equal to the step MQ and S will be the required point. The graph is the curve SSS, Fig. 40.

When x is large, y_1 is much larger than y_2; even for $x = 5$ we have $y_1 = 250$, $y_2 = 32$. Hence at points at a moderately great distance to the right or to the left of the y-axis the curves whose ordinates are $y_1 - y_2$ and $y_1 + y_2$ will differ very little from that whose ordinate is y_1. The student should plot on the same diagram the graphs of y_1, $y_1 - y_2$ and $y_1 + y_2$ from $x=5$ to $x=10$ taking the y-scale small, say $1''$ to 250; integral values of x will be sufficient.

The fact that, for large values of x, the term of highest degree determines the behaviour of the graph is of considerable importance in higher work.

38. Building up of a Graph. The method just given of plotting the graphs of one or more terms of the function and then adding, by the rule for the addition of steps, corresponding ordinates of the component graphs is of very

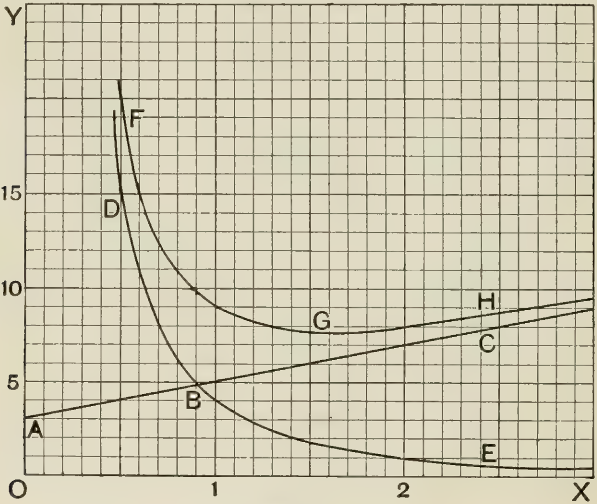


Fig. 41.

great importance and should be carefully studied. When the component graphs are of a well-known shape the resultant graph can be obtained with much less labour, and with more certainty, than by plotting points. In this way the graph of an equation such as

$$y = \frac{2x^3 + 3x^2 + 4}{x^2}$$

can be easily drawn. The equation may be written

$$y = 2x + 3 + \frac{4}{x^2}$$

and the graphs of $2x+3$ and $4/x^2$ can be readily laid down.

In Fig. 41 ABC is the graph of $2x+3$, DBE that of $4/x^2$ and FGH that of $2x+3+4/x^2$; the curves are only drawn for positive values of x. G is the turning point; at G $x=1{\cdot}6$ and $y=7{\cdot}7$ approximately.

When x becomes moderately large the ordinate of the curve differs very little from that of the straight line; clearly the straight line is an asymptote to the curve. On the other hand, when x is a small fraction the ordinate of the curve differs very little from that of the graph of $4/x^2$; the difference, no doubt, is always greater than 3, but 3 is very small compared with $4/x^2$ when x is a small fraction.

39. Solution of Equations. Method of Trial and Error.

When rough approximations to the roots of an equation have been obtained, closer approximations may be got by a process that may be called the method of trial and error.

Take for example the equation

$$3x^3 + 4x^2 - 8x - 7 = 0.$$

A rough sketch of the graphs of $3x^3$ and $7+8x-4x^2$ (Fig. 32) will show that the equation has three roots, equal approximately to $1{\cdot}5$, $-0{\cdot}8$ and $-2{\cdot}1$. To obtain a closer approximation to the first of these roots, notice that when $x=1{\cdot}5$, $y=0{\cdot}125$. The point $(1{\cdot}5, 0{\cdot}125)$ is above the x-axis; when x is greater than $1{\cdot}5$, y is positive so that the root is less than $1{\cdot}5$.

Now try $x=1{\cdot}49$; this gives $y=-0{\cdot}116$ and the point $(1{\cdot}49, -0{\cdot}116)$ is below the x-axis. We therefore try a value of x between $1{\cdot}49$ and $1{\cdot}5$; since $0{\cdot}125$ and $0{\cdot}116$ are nearly equal we try $x=1{\cdot}495$, that is half the sum of $1{\cdot}49$ and $1{\cdot}5$. This gives $y=0{\cdot}0042$.

A still better approximation is $x=1{\cdot}4948$; for this value of x we find $y=-0{\cdot}0006$.

In the same way better approximations to the other two roots are found to be $-0{\cdot}752$ and $-2{\cdot}076$.

In applying this method the graph is only needed to suggest first approximations, though by plotting the portion of the graph near the x-axis on a very large scale we can get the closer approximations in the usual way.

It may be noticed that $1{\cdot}495$ differs from the true value of the root by less than $0{\cdot}07$ per cent. of that value, as may be seen thus. The

root is greater than 1·494 but less than 1·495 and therefore differs from either by less than 0·001. The fractional error is therefore less than
$$\frac{0{\cdot}001}{1{\cdot}494}$$
and the percentage error is less than this fraction multiplied by 100. But
$$\frac{0{\cdot}001}{1{\cdot}494}\times 100 = 0{\cdot}06\ldots < 0{\cdot}07.$$

The methods that have been given of solving an equation are all laborious if more than a moderate approximation to the roots is desired; for more powerful processes see any book on the Theory of Equations or the author's *Calculus*, Chap. XII.

Note on the Cubic Function. The graph of a quadratic function is always a parabola, with its vertex at the highest or at the lowest point of the curve. The following discussion shows that the graph of a cubic function has two distinct forms, one in which there is no turning point and a second in which there are two turning points. The discussion also leads easily to the tests for the nature of the roots of a cubic equation.

In the equation $$y = ax^3 + bx^2 + cx + d \text{ ....(1)}$$
put $X+h$ for x, that is, shift the origin to the point $(h, 0)$; the equation becomes, when arranged in descending powers of X,
$$y = aX^3 + (3ah+b)X^2 + (3ah^2+2bh+c)X + ah^3 + bh^2 + ch + d. \text{ ....(2)}$$

Now choose h so that the coefficient of X^2 shall be zero; therefore $h = -b/3a$. When this value of h is substituted in (2), that equation becomes
$$y = aX^3 + \frac{3ac-b^2}{3a}X + \frac{2b^3 - 9abc + 27a^2d}{27a^2}. \text{ ....(3)}$$

Let us now put $Y' + (2b^3 - 9abc + 27a^2d)/27a^2$ for y and we obtain from (3)
$$Y' = aX^3 + \frac{3ac-b^2}{3a}X. \text{ ....(4)}$$

Finally, for Y' put aY and we get
$$Y = X^3 + \frac{3ac-b^2}{3a^2}X. \text{ ....(5)}$$

It will be noticed that (4) is deduced from (1) by a change of origin to the point (h, k) where
$$h = -\frac{b}{3a},\quad k = \frac{2b^3 - 9abc + 27a^2d}{27a^2}. \text{ ....(6)}$$

Equation (5) is derived from (4) by a change of scale; if a is negative, the change of scale is accompanied by reflection in the X-axis.

The origin is a point of inflexion on the graph of (5); it is also a centre of symmetry, and therefore, in considering the graph of (5), we may restrict ourselves to positive values of X.

If $b^2 = 3ac$, equation (5) becomes $Y = X^3$, the graph of which has no turning point (Fig. 37). We must take now the cases for which (i) $b^2 < 3ac$, and (ii) $b^2 > 3ac$.

(i) Let $(3ac - b^2)/3a^2 = 3m^2$, a positive quantity. (The form $3m^2$ is chosen for the sake of symmetry of notation; in case (ii) the value $-3n^2$ makes the calculations simpler). Equation (5) is for this case

$$Y = X^3 + 3m^2 X. \quad \text{.................................(7)}$$

As X increases from 0 to ∞, Y steadily increases from 0 to ∞, and therefore the graph has no turning point. The graph resembles *SSS* (Fig. 40), the origin for (7) being the point $(0, -3)$ in Fig. 40.

The equation $X^3 + 3m^2 X = 0$ has only one real root, and so also has the equation

$$X^3 + 3m^2 X + l = 0 \quad \text{.................................(8)}$$

where l is any constant; because the graph of $X^3 + 3m^2 X + l$ is simply that of $X^3 + 3m^2 X$, shifted parallel to the Y-axis.

When l has the value k/a, where k is given by (6), equation (8) is equivalent to the equation

$$ax^3 + bx^2 + cx + d = 0. \quad \text{.............................(1')}$$

Hence, when $b^2 < 3ac$ equation (1′) has one, and only one, real root.

(ii) Let $(3ac - b^2)/3a^2 = -3n^2$, a negative quantity. In this case equation (5) takes the form

$$Y = X^3 - 3n^2 X, \quad \text{.................................(9)}$$

which may be written, as an easy calculation shows,

$$Y = (X - n)^2 (X + 2n) - 2n^3. \quad \text{.........................(9')}$$

We may, without loss of generality, assume n as well as X to be positive; equation (9′) then shows that Y is always greater than $-2n^3$, *except when* $X = n$. Hence Y is a minimum, $-2n^3$, when $X = n$; from symmetry we infer that Y is a maximum, $2n^3$, when $X = -n$. The points $(n, -2n^3)$ and $(-n, 2n^3)$ are the turning points of the graph of (9); the graph resembles *RRR* (Fig. 40), the origin for (9) being the point $(0, 3)$ in Fig. 40.

The equation $X^3 - 3n^2 X = 0$ has three real roots, namely 0, $n\sqrt{3}$ and $-n\sqrt{3}$; it is easy from graphical considerations to determine the nature of the roots of the equation

$$X^3 - 3n^2 X + p = 0 \quad \text{..............................(10)}$$

where p is any constant.

The roots of (10) are the abscissae of the points of intersection of the graph of (9) and the straight line $Y = -p$. If the straight line has the turning points of the graph of (9) on opposite sides of it, then it will cut that graph in three points; equation (10) will therefore have three unequal roots. If the line touches the graph at either turning point, equation (10) will have two equal roots and a third root distinct from the equal roots. Lastly, if the line falls above the maximum turning point or below the minimum turning point, it will cut the graph of (9) only once, and therefore equation (10) will have only one root.

Equation (10) therefore will have three, unequal, real roots if $p^2 < 4n^6$; three real roots, two of which are equal, if $p^2 = 4n^6$; and only one real root if $p^2 > 4n^6$.

If we put for n^2 its value $(b^2 - 3ac)/9a^2$, and for p the value k/a, we find, after an easy calculation,

$$27a^4(p^2 - 4n^6) = 4b^3d - b^2c^2 - 18abcd + 4ac^3 + 27a^2d^2. \quad \text{.........(11)}$$

With this value of p, equation (10) is equivalent to equation (1′). Hence equation (1′) has two equal roots when $p^2 = 4n^6$, that is, when the right-hand member of (11) is zero.

The right-hand member of (11) is called the **discriminant** of the cubic equation (1′). (See Exercises XV, 34.)

This note is substantially taken from a paper by Mr. P. Pinkerton in the *Proceedings of the Edinburgh Mathematical Society*, Vol. XXII. (June, 1904).

EXERCISES. XV.

1. From the graph of x^3 find the cube roots of 1·25, 3·75, 6·5.

2. Graph equations of the form $y = ax^3 + b$; for example

$$y = \frac{x^3}{100}, \quad y = \frac{x^3}{100} + 20, \quad y = \frac{x^3}{100} - 20,$$

$$y = -\frac{x^3}{100}, \quad y = -\frac{x^3}{100} + 20, \quad y = -\frac{x^3}{100} - 20,$$

$$y = 100x^3, \quad y = 100x^3 + 80, \quad y = -100x^3 + 80.$$

3. The equation $4x^3 + 3x - 16 = 0$ has one real root; find it to two decimals.

4. Solve $x^3 - 5x - 16 = 0$ [one real root].

5. Solve $8x^3 + 15x - 30 = 0$ [one real root].

Solve equations 6–11.

6. $x^3 - x^2 - 1 = 0$.

7. $8x^3 - 7x^2 + 10 = 0$.

8. $x^3 - 6x^2 + 3x + 5 = 0$.

9. $3x^3 - 4x^2 - 4x + 2 = 0$.

10. $5x^4 - 27x - 10 = 0$.

11. $x^4 - 2x^3 + 7x - 3 = 0$.

12. Graph functions of the form $ax^3 + bx$ and find their maximum and minimum values; for example

(i) $x^3 + x$; (ii) $x^3 - x$; (iii) $x^3 + 16x$; (iv) $16x - x^3$.

What kind of symmetry do the graphs possess?

13. How may the graph of the function $ax^3 + bx + c$ be deduced from that of $ax^3 + bx$? Plot the functions represented by the left side of equations 3, 4, 5 above; give the turning values of each function.

14. Graph functions of the form ax^3+bx^2 and find their turning values; for example

(i) x^3+x^2, (ii) x^3-x^2, (iii) x^2-x^3, (iv) $2x^3-5x^2$.

Deduce the graphs of functions of the form ax^3+bx^2+c.

15. If x is positive find the maximum value of $(1+x)(1-x^2)$.

What is the maximum value of $(R+x)(R^2-x^2)$ when x is positive?

16. A cone is inscribed in a sphere of radius R; if the distance of the base of the cone from the centre of the sphere is x, show that its volume is $\frac{1}{3}\pi(R+x)(R^2-x^2)$. Apply example 15 to find the maximum cone that can be inscribed in the sphere.

17. Graph the equation $y=x^2+16/x$ for positive values of x, and find the minimum value of y.

18. An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water; show that the expense of lining the tank with lead will be least if the depth is half the width.

[If a side of the base is x feet the surface is x^2+4V/x square feet where V is the volume of the tank in cubic feet; since the expense is proportional to the surface the expense will be least when this function is a minimum (take $V=32$).]

19. Graph the equation $y=10(x-1)(x-2)(x-3)$ and find the turning values of y.

20. Graph equations of the form $y=(ax^2+bx+c)/x$, and find the turning values of y; for example

(i) $y=\dfrac{x^2+4}{x}$, (ii) $y=\dfrac{x^2-4}{x}$, (iii) $y=\dfrac{2x^2-x+8}{x}$, (iv) $y=\dfrac{2x^2+3x-2}{x}$.

21. Graph equations of the form $y=(ax^3+bx^2+c)/x^2$; for example (x positive)

(i) $y=\dfrac{x^3+4}{x^2}$, (ii) $y=\dfrac{x^3-4}{x^2}$, (iii) $y=\dfrac{2x^3-x^2+8}{x^2}$.

22. Graph the equations

(i) $y=\dfrac{3x-4}{(x-1)(x-2)}$; (ii) $y=\dfrac{x^3-x^2+x+3}{x-1}$.

23. Graph functions of the form ax^4+bx^2+c and find their turning values; for example

(i) x^4+x^2, (ii) x^2-x^4, (iii) x^4-2x^2-10.

24. Graph the equation $y=5x^4-6x-10$ and find the values of x for which y is zero.

Find the average gradient of the arc PQ of the graphs of equations 25–32; state also the value you would deduce for the gradient of the tangent at P. (Compare Exercises XIII, 11–19.)

25. $y=x^3$; x of $P=1$; x of $Q=2$, 1·5, 1·1, 1·01, $1+h$.

26. $y=x^3$; x of $P=-1$; x of $Q=0$, $-0·5$, $-0·9$, $-0·99$, $-1+h$.

27. $y=x^3$; x of $P=2$; x of $Q=3$, 2·5, 2·1, 2·01, $2+h$.

28. $y=16x-x^3$; x of $P=0$; x of $Q=1$, 0·5, 0·1, 0·01, h.

29. $y=16x-x^3$; x of $P=4$; x of $Q=5$, 4·5, 4·1, 4·01, $4+h$.

30. $y=x^4$; x of $P=1$; x of $Q=2$, 1·5, 1·1, 1·01, $1+h$.

31. $y=\frac{1}{x}$; x of $P=1$; x of $Q=2$, 1·5, 1·1, 1·01, $1+h$.

32. $y=\frac{1}{x^2}$; x of $P=1$; x of $Q=2$, 1·5, 1·1, 1·01, $1+h$.

33. If $V=\frac{1}{x}$ find the average rate at which V changes as x increases from a to $a+h$. At what rate is V changing when $x=a$?

34. If D denote the discriminant of the cubic equation

$$ax^3+bx^2+cx+d=0$$

show that

$$27a^2D=(2b^3-9abc+27a^2d)^2+4(3ac-b^2)^3.$$

By using this expression for D, and applying the results stated on page 113 for equation (8) and on page 114 for equation (10), show that the cubic equation has three, unequal, real roots when D is negative; three real roots, two of which are equal, when D is zero; and one, and only one, real root when D is positive.

35. From the fact that the abscissae of the turning points of the graph of (9), page 113, are the roots of the equation $X^2-n^2=0$ show, by replacing X by its value $x+b/3a$ and n^2 by its value $(b^2-3ac)/9a^2$, that the abscissae of the turning points of the graph of (1), page 112, are the roots of the equation

$$3ax^2+2bx+c=0.$$

36. Apply the result stated in example 35 to the determination of the turning values of the functions in examples 12–16.

CHAPTER VI.

LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

40. Graphs of $\log x$ and 10^x. We go on to consider examples that require logarithms and we begin with the graph of $\log x$ to the base 10; we shall generally use four-figure logarithms.

The argument x of $\log x$ must be positive; when x is a proper fraction $\log x$ is negative, and the beginner may be cautioned to write the value properly. Thus,

$$\log 0{\cdot}2 = \bar{1}{\cdot}301 = 0{\cdot}301 - 1 = -0{\cdot}699;$$

and when x is $0{\cdot}2$, y or $\log x$ is $-0{\cdot}699$, equal to $-0{\cdot}7$ say.

The graph of $\log x$ is ABC in Fig. 42; OY' is an asymptote.

By the definition of a logarithm, $x = 10^y$ when $y = \log x$; that is, x is the antilogarithm of y or the number whose logarithm is y. If y is taken as the argument and x or 10^y as the function, the curve ABC is the graph of the function 10^y.

It is more convenient however to have the graph of 10^x, the argument being measured as usual along the horizontal line. In § 41 it is shown how the graph of 10^x may, without further calculation, be derived from that of 10^y, but it is easy to take out the values of 10^x from the table of antilogarithms. Thus,

$$10^{1{\cdot}5} = \text{antilog. of } 1{\cdot}5 \quad = 31{\cdot}62,$$

$$10^{-0{\cdot}5} = \text{antilog. of } -0{\cdot}5 = \text{antilog. of } \bar{1}{\cdot}5 = 0{\cdot}3162,$$

and so on.

The graph of 10^x is the curve $A'B'C'$ in Fig. 42; OX' is an asymptote.

The graph of 10^{-x} is symmetrical to that of 10^x with respect to the y-axis; because, whatever be the value of a, the value of 10^{-x} when $x=-a$ is equal to that of 10^x when $x=a$.

The curve $A''B''C''$ (Fig. 42) represents $y=10^{-x}$; it approaches the positive end of the x-axis asymptotically.

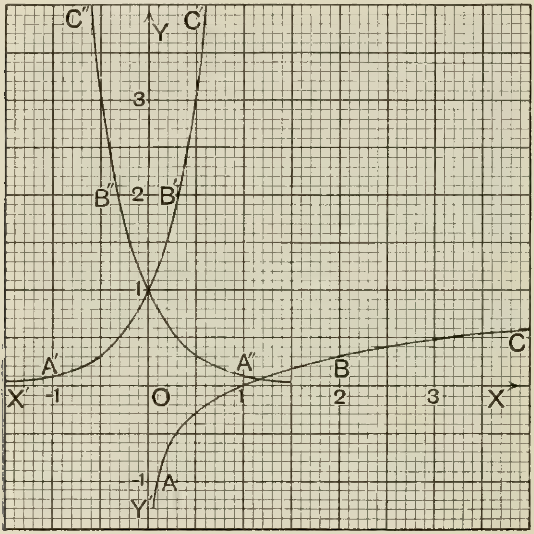


Fig. 42.

Example. Solve the equation $10^{\frac{1}{2}x-1}=6x-8$.

The roots are the abscissae of the points of intersection of the graphs of $y=10^{\frac{1}{2}x-1}$......(i) and $y=6x-8$......(ii)

To plot the graph of (i) take the following values:

x	0	0·5	1	1·5	2	2·5	3	3·5	4	4·5	5
y	0·10	0·18	0·32	0·56	1	1·78	3·16	5·62	10	17·78	31·62

The effect of the second decimal in the values of y will not be clearly seen unless the unit for ordinates is about an inch; for solving the equation however it is more important to have the unit for abscissae fairly large, say 1″ to 1.

To plot the straight line, take the points (2, 4), (4, 16).

Fig. 43 shows the graphs; in the diagram from which this figure is reproduced the roots are read as 1·42 and 4·58.

41. Inverse Functions. The equation $y = \log x$ not only defines y as a function of x but also defines x as a function of y (example 1, p. 30). Two functions defined by the same equation are said to be **inverse** to each other.

The function 10^y, since y occurs in it as an exponent, is called an **exponential function** of y. (See also § 46.) Thus, the logarithmic and the exponential functions are inverse

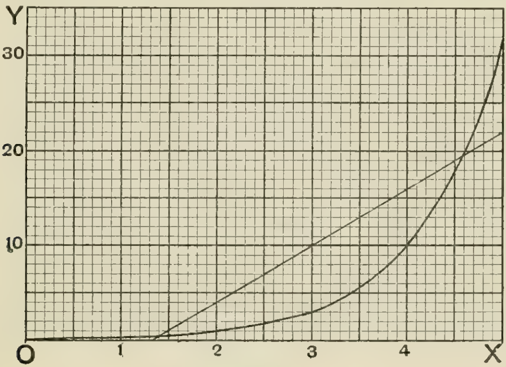


Fig. 43.

to each other. The exponential function is the antilogarithmic function.

In the same way the equation $y = x^3$, when solved for x, gives $x = \sqrt[3]{y}$ and thus defines two functions which are inverse to each other, namely the cube and the cube root.

A function and its inverse, for example $\log x$ and 10^y, are both represented by the same graph; but when one graph is taken as representative of both functions, the argument of one of them is measured along the vertical axis and not, as in the usual graphic representation, along the horizontal. We can get the graph of 10^y into the standard position as follows.

Lift the sheet on which the curve ABC, the graph of $y = \log x$, is drawn; then *turn it over* and place it so that OY is horizontal with Y to the right of O and OX vertical with X above O. If we hold the sheet in this position and look through it against the light we shall see that ABC has come into the position occupied by $A'B'C'$ in Fig. 42. If ABC shows through the sheet when it is laid on another sheet we can prick a few points and get $A'B'C'$; a copy of ABC on tracing paper would be useful. When the graph has been got into the standard position we may write x for y and y for x. Thus, given the graph of $\log x$, we have constructed the graph of 10^x.

Similarly, from the graph of $y=x^2$ we get that of $y^2=x$; that is, from the graph of x^2 we construct that of $\sqrt{x}$, and so on.

EXERCISES. XVI.

1. Graph the three functions

$$\text{(i) } \log(1+x), \quad \text{(ii) } \log(1-x), \quad \text{(iii) } \log\frac{1+x}{1-x}$$

from $x=-0{\cdot}9$ to $x=0{\cdot}9$.

2. Graph the function $10\log(5x+2)$ from $x=0$ to $x=5$ and solve the equation

$$10\log(5x+2)=24-2{\cdot}7x.$$

3. Graph the function $3\log(2{\cdot}4x+3{\cdot}6)$, and solve the equation

$$(2{\cdot}4x+3{\cdot}6)^3=10^{8-1{\cdot}3x}.$$

4. Solve the equation $\quad 10^x=20x.$

5. Graph the function $x\log(1+x)$ from $x=0$ to $x=10$ and solve the equations $\quad$ (i) $(1+x)^x=387{\cdot}4$, $\quad$ (ii) $(1+x)^x=3874$.

6. Draw to the same axes and with the same scales the graphs of the equations $\quad$ (i) $y=x-1$, $\quad$ (ii) $y=2{\cdot}3\log x$, $\quad$ (iii) $y=1-\dfrac{1}{x}$.

Let the values of x range, say, from $0{\cdot}5$ to 5.

Show from the graphs that, except when $x=1$,

$$x-1>2{\cdot}3\log x>1-\frac{1}{x}.$$

7. Draw the graphs of the equations

$$\text{(i) } 100y=\tfrac{1}{2}(10^x-10^{-x}), \quad \text{(ii) } 100y=\tfrac{1}{2}(10^x+10^{-x})$$

from $x=-3$ to $x=3$.

8. Solve the equation $\quad 10^{\frac{3}{4}x-1}=31-5{\cdot}8x.$

9. Solve the equation $\quad 10^{\frac{2}{3}x}=16+4x-x^2.$

10. Graph the equation $y=100x10^{-x}$, and find the maximum value of y, and the value of x for which y is a maximum.

11. Graph the function $x\log x$ from $x=0{\cdot}1$ to $x=5$, and find its turning value, and the value of x for which it turns.

12. Find the average gradient of the arc PQ of the graph of $\log x$, the abscissa of P being $3{\cdot}6$ and the abscissa of Q being successively $4{\cdot}6$, $4{\cdot}1$, $3{\cdot}8$, $3{\cdot}7$.

13. Find the average gradient of the arc PQ of the graph of 10^x, the abscissa of P being 0 and the abscissa of Q being successively 1, 0·5, 0·1, 0·01.

14. The same as example 13, the abscissa of P being 1 and the abscissa of Q being successively 2, 1·5, 1·1, 1·01.

42. Graphs of x^n and $1/x^n$, n fractional. These functions are of considerable importance in mechanics and in physics generally; we restrict ourselves, as a rule, to positive values of x, since it is for positive values alone that the functions are usually defined. If the complete representation of the function is required the student has only to consider whether x or y, or both, can take both positive and negative values.

For example, the equation $y^2=x^3$ gives $y=x^{\frac{3}{2}}$. Here x cannot be negative but the complete value of y is given by $y=+x^{\frac{3}{2}}$ and $y=-x^{\frac{3}{2}}$; the graph corresponding to $-x^{\frac{3}{2}}$ is symmetrical to that of $+x^{\frac{3}{2}}$ and the complete graph consists of these two portions.

Again, $y^3=x^5$ gives $y=x^{\frac{5}{3}}$. Here both x and y may be negative; the complete graph lies in the first and third quadrants like that of x^3.

The remarks in the next three paragraphs apply to the shape of the graph in the first quadrant.

When n is positive and greater than 1, the graph of x^n is like that of x^2 or x^3 in general appearance. Thus, $\frac{5}{2}$ lies between 2 and 3; the graph of $x^{\frac{5}{2}}$ therefore lies between those of x^2 and x^3. These graphs touch the x-axis at the origin.

When n is positive and less than 1, the graph of x^n touches the y-axis at the origin. Thus, if $y=x^{\frac{1}{2}}$ we have $x=y^2$, and the graph is simply the parabola of § 20 placed so that its axis is horizontal and lies along OX instead of, as in Fig. 25, along OY. The graph of $y=x^{\frac{1}{3}}$ is related in a similar way to that of $y=x^3$.

When n is positive, the graph of $1/x^n$ resembles that of $1/x$ or $1/x^2$ and has both OX and OY as asymptotes. For example, the graph of $1/x^{\frac{3}{2}}$ lies between those of $1/x$ and $1/x^2$.

We again remind the beginner that, when the index n is fractional, the function x^n is usually not defined for negative values of x; positive values alone are to be given to x in all practical applications of the function, when n is fractional.

The calculations will as a rule require logarithms.

Example. Graph the equations

$y_1 = 6x^{2·35}$, ...(i) $y_2 = 18 - 4·3x^{1·43}$, ...(ii)

and solve the equation $6x^{2·35} + 4·3x^{1·43} - 18 = 0$. ....(iii)

We have by the rules of logarithms

$$\log(6x^{2·35}) = \log 6 + 2·35 \log x = 0·7782 + 2·35 \log x,$$
$$\log(4·3x^{1·43}) = \log 4·3 + 1·43 \log x = 0·6335 + 1·43 \log x.$$

The value of $4·3x^{1·43}$ must, of course, be first obtained and the result subtracted from 18 to find y_2.

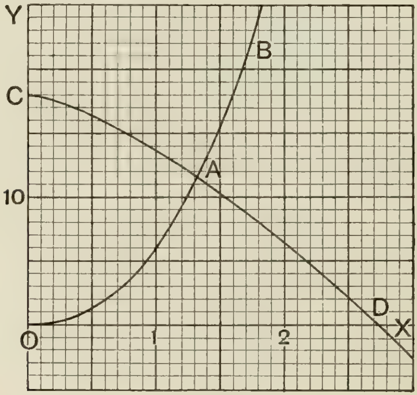


Fig. 44.

In the following table the values are given as found from the four-figure tables, though it will not usually be possible to show the effect of all the decimals on the graph.

x	0	0·5	1	1·5	2	2·5	3
$6x^{2·35}$	0	1·177	6	15·56	30·59	51·68	79·32
$4·3x^{1·43}$	0	1·596	4·3	7·679	11·58	15·94	20·69
y_2	18	16·404	13·7	10·321	6·42	2·06	$-2·69$

In Fig. 44, OAB is the graph of (i), CAD that of (ii).

The root of equation (iii) is the abscissa of A, the point of intersection of the two graphs; its value is 1·32.

The beginner should compare these graphs with those of

$$y=6x^3 \text{ and } y=18-4{\cdot}3x^2;$$

he will see that the remarks as to the resemblance between graphs of functions with fractional indices and those of functions with integral indices are borne out.

43. Adiabatic Curves. To illustrate the case of $1/x^n$ we shall take an adiabatic curve. A given mass of gas is said to expand adiabatically when it expands in such a way that heat neither enters nor leaves it. In an adiabatic expansion the equation connecting the pressure, p lb. per sq. in.

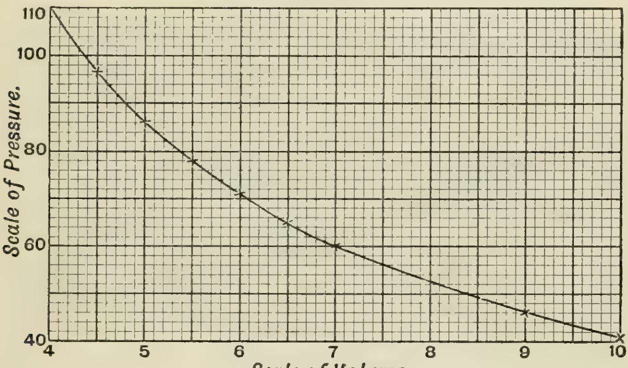


Fig. 45.

say, with the volume of the mass, v cub. ft., is of the form

$$pv^\gamma=\text{constant}.$$

As a definite case, let v be the volume in cub. ft. of one pound of saturated steam and p the pressure in lb. per sq. in. corresponding to the volume v; then approximately

$$pv^{\frac{17}{16}}=480.$$

To calculate p we use the equation

$$\log p=\log 480-\tfrac{17}{16}\log v=2{\cdot}6812-\tfrac{17}{16}\log v.$$

We may take the following set of values:

v	4	4·5	5	5·5	6	6·5	7	8	9	10
p	110	97·1	86·8	78·4	71·5	65·7	60·7	52·7	46·5	41·6

The values of p are given to the nearest three-figure approximation.

The graph is shown in Fig. 45; to get a convenient scale the point (4, 40) is taken as temporary origin.

In general appearance the graph resembles those of Fig. 34. The apparent steepness of the curve depends greatly on the scales; unless attention is paid to the scales one is apt to draw erroneous conclusions from the graph in respect to the value of the index n or γ.

44. Applications. We shall give two examples of the determination of approximate formulae from experimental data, in which the index of one variable is not an integer.

Example 1. The time, t seconds, that it took for water to flow through a triangular (or V) notch, under a pressure head of h feet, till the same quantity was in each case discharged, was determined by experiment to be as follows:

h	0·043	0·057	0·077	0·094	0·100
t	1260	540	275	170	135

Find a formula connecting h and t.

If the points (h, t) are plotted and a smooth curve drawn through them, the curve thus obtained suggests the equation $th^n = a$. The best way of testing the suggestion is that indicated in § 34, namely, to plot the logarithms of t and h. From the equation $th^n = a$ we find

$$\log t + n \log h = \log a, \quad \text{or} \quad y + nx = \log a, \qquad \text{(i)}$$

where $x = \log h$ and $y = \log t$.

We therefore form the table

$x = \log h$	$-1·367$	$-1·244$	$-1·114$	$-1·027$	$-1·000$
$y = \log t$	3·100	2·732	2·439	2·230	2·130

The points (x, y), if carefully plotted, will be found to be distributed very evenly about a straight line whose gradient is, approximately,

$-2{\cdot}5$. Equation (i) is therefore verified and the value of n is $2{\cdot}5$, because the gradient of the line given by equation (i) is $-n$. Hence we have the relation $$th^{2\cdot5}=\text{constant}=a.$$

The value of a obtained from the graph of the straight line is about 0·44, but this value is unimportant; it is rather the relation between h and the quantity discharged per second that is ultimately wanted. In this experiment, the quantity discharged in t seconds was, in each of the five cases, 1800 cubic inches. The discharge Q, in cubic feet per second, was therefore

$$Q=\frac{1800}{1728t}=\frac{1800}{1728a}h^{2\cdot5}.$$

The best value for the coefficient of $h^{2\cdot5}$ is obtained by writing

$$\frac{Q}{h^{2\cdot5}}=\frac{1800}{1728a}=\frac{1800}{1728th^{2\cdot5}}$$

and then calculating the quotient for each of the five pairs of values of h and t. The average of these quotients is 2·34, so that finally we have

$$Q=2{\cdot}34h^{2\cdot5}.$$

Example 2. In a gas-engine test corresponding values of the pressure, p lb. per sq. in., and the volume, v cub. ft., were obtained as shown in the table:

v	3·54	4·13	4·73	5·35	5·94	6·55	7·14	7·73	8·04
p	141·3	115	95	81·4	71·2	63·5	54·6	50·7	45

Find a relation between v and p.
Let $x=\log v$, $y=\log p$ and form the table:

x	0·549	0·616	0·675	0·728	0·774	0·816	0·854	0·888	0·905
y	2·150	2·061	1·978	1·911	1·852	1·803	1·737	1·705	1·653

The points (x, y), when plotted, will be found to be very nearly in a straight line whose gradient is $-1{\cdot}32$.

Hence the relation between v and p is of the form

$$pv^{1\cdot32}=\text{constant}.$$

The value of the constant is about 750.

EXERCISES. XVII.

Graph equations 1–10 for positive values of x and y.

1. $y=x^{\frac{3}{2}}$. 2. $y=x^{\frac{2}{3}}$. 3. $y=x^{\frac{5}{3}}$. 4. $y=x^{\frac{3}{5}}$. 5. $y=x^{2\cdot7}$.
6. $y=x^{0\cdot43}$. 7. $y=\frac{1}{\sqrt{x}}$. 8. $y=\frac{1}{x^{\frac{3}{2}}}$. 9. $y=\frac{1}{x^{2\cdot4}}$. 10. $y=\frac{1}{x^{3\cdot4}}$.

11. Graph the equation

$$y = 3x^{2.5} - 4x^{1.2} - 5$$

and find the value of x for which y is zero.

12. Solve the equation $17x^{2.63} = 43x^{1.42} + 68$.

13. Graph the equation

$$y = 2x + 5 + \frac{4}{x^{2.3}}.$$

For what value of x is the ordinate a minimum, and what is the minimum value?

14. Draw a curve to suit the following values of v and p:

v	3·84	4·85	6·20	8·03	9·20	10·56
p	115·1	89·9	69·2	52·5	45·5	39·2

Find an equation connecting v and p.

15. Find an equation connecting v and p from the following values:

v	3	3·4	4	5·2	6	7·3	8·5	10
p	107·3	89·8	71·5	49·5	40·5	30·8	24·9	19 8

16. The quantity of water, Q lb., discharged per second from a circular orifice in a tank, under a pressure head of h feet, was found by experiment to be as follows:

h	0·583	0 667	0·750	0·834	0·876	0·958	1·000
Q	7·00	7·60	7·94	8·42	8·68	9·04	9·34

Test the formula $Q = ah^n$; the value of n alone need be given.

17. The average velocity v of the efflux of water from a tank, when the pressure head is h, is in inverse proportion to the time t, where h and t are given by the table:

h	30	24	18	12
t	81	90	103	128

Find whether an expression of the form $v = ah^n$ will suit these values; the value of n alone is required.

18. The same problem as in example 17 for the data :

h	30	24	18	12
t	262	290	338	410

19. When the notch in the experiment of § 44, example 1, was rectangular, the following values were obtained :

h	0·028	0·036	0·049	0·069	0·088
t	400	300	180	110	75

Find the equation between h and t.

20. Find a relation between v and p from the following observed data :

v	3·54	4·13	4·73	5·35	5·94	6·55	7·14	7·73
p	45	38	33·3	30	26·6	24	22	19·8

21. Determine a relation between h and v from the following data :

h	10·20	23·80	41·50	46·00	69·24	102·74
v	24·74	37·90	51·67	54·60	65·97	81·43

22. In the following table, V represents a velocity in feet per second and l a length in feet :

l	19·9	45·1	67·5	94·4	109	126
V	10·1	15·2	18·6	22·0	23·6	25·4

Find the relation between l and V.

23. Find the relation between S and T from the following data :

S	240	178	117	71
T	215	178	147	104

24. The following values of x and y are taken from a table :

x	17·0	19·2	20·8	23·6	25·2	26·8	29·6
y	154	221	281	411	500	602	810

Find the relation between x and y.

25. Given the following table of values:

x	17·0	19·2	20·8	23·6	25·2	26·8	29·6
y	81·6	85·0	87·3	91·0	93·1	95·0	98·2

find the relation between x and y.

45. Napierian Logarithms. In many investigations the base of the logarithms is not 10, but a number, usually denoted by e and equal approximately to 2·71828. Logarithms to the base e are called *Napierian*, or *hyperbolic*, or *natural* logarithms, so as to distinguish them from logarithms to the base 10, which are called *common* or *Briggian* logarithms.

Let $y=\log_{10}x$ and $z=\log_e x$; then, by the definition of a logarithm, x is equal to 10^y and also to e^z. Hence

$$10^y=e^z. \qquad\qquad (1)$$

Take the common logarithm of each member of equation (1); therefore

$$y=z\log_{10}e, \text{ that is, } \log_{10}x=\log_e x\times\log_{10}e. \qquad (2)$$

Again, take the Napierian logarithm of each member of equation (1); therefore

$$z=y\log_e 10, \text{ that is, } \log_e x=\log_{10}x\times\log_e 10. \qquad (3)$$

In (2) put 10 for x, or in (3) put e for x; we find in both cases

$$\log_e 10\times\log_{10}e=1. \qquad\qquad (4)$$

Equations (2) and (3) give the rules for changing from one base to the other. The values of $\log_{10}e$ and $\log_e 10$ are

$$\log_{10}e=0·43429,\quad \log_e 10=2·30259.$$

Hence, to convert Napierian to common logarithms, multiply by 0·43429; to convert common to Napierian logarithms, multiply by 2·30259.

For the present, the symbol "log" will mean the common logarithm; when Napierian logarithms are meant, the symbol "$\log_e$" will be used.

46. The Exponential Function. The function e^x is usually called *the* exponential function of x; the choice of e, instead of 10, as the base simplifies to a considerable extent many of the fundamental formulae of higher mathematics.

At the end of the book will be found a table (Table XII.) of values of e^x and e^{-x}.

The graph of e^x resembles that of 10^x. The graph of 10^x is the graph of $e^{2 \cdot 3x}$, because $\log_e 10 = 2 \cdot 3$ approximately, and therefore $10 = e^{2 \cdot 3}$, $10^x = e^{2 \cdot 3x}$, $10^{-x} = e^{-2 \cdot 3x}$.

Thus, the graphs of 10^x and 10^{-x} are also those of $e^{2 \cdot 3x}$ and $e^{-2 \cdot 3x}$.

It should be noted that a mere change of the x-scale turns the graph of e^x into that of e^{ax}. For example, let $a=2$; then, if the step on the x-axis that represents 2 for the graph of e^x be chosen to represent 1 the graph will, with the new scale, represent e^{2x}.

Similarly, the graph of e^x will represent $e^{\frac{1}{2}x}$, provided the step on the x-axis that represents $\frac{1}{2}$ for the graph of e^x be chosen to represent unity.

The graph of 10^x, that is $e^{2 \cdot 3x}$, will represent e^x, provided the step on the x-axis that represents 1 for the graph of 10^x be chosen to represent $2 \cdot 3$.

The proofs of these statements should offer no difficulty at this stage.

EXERCISES. XVIII.

1. Plot to the same axes the graphs of

$$\text{(i) } 10e^{-x}, \quad \text{(ii) } 10(1-e^{-x})$$

from $x=0$ to $x=5$.

2. Graph the equations

$$\text{(i) } y=\tfrac{1}{2}(e^x+e^{-x}), \quad \text{(ii) } y=\tfrac{1}{2}(e^x-e^{-x})$$

from $x=-4$ to $x=4$.

3. Graph the function xe^{-x}; find its maximum value, and the value of x for which it is a maximum.

4. Graph the function e^{-x^2} from $x=-3$ to $x=3$. What kind of symmetry does the graph possess?

5. The pressure of the atmosphere, p lb. per sq. in., at the height x feet above sea level, is given by the equation

$$p=Pe^{-\frac{x}{H}},$$

where P is the pressure at sea level, and H feet the height of the homogeneous atmosphere. Represent graphically the relation between p and x, taking $P=15$, $H=26000$.

6. Solve the equations

(i) $e^x = 2x + 3$; (ii) $4{\cdot}5e^{2{\cdot}5x} = 68x + 47$;

(iii) $12e^{-\frac{1}{2}x} = 5 + 4x - x^2$; (iv) $3{\cdot}6e^{2{\cdot}7x} + 12{\cdot}7e^{1{\cdot}2x} = 65{\cdot}4$.

7. The two equations

$$i = \frac{Q}{T}e^{-\frac{t}{T}}, \quad q = Q(1 - e^{-\frac{t}{T}})$$

where $Q = EC$, $T = RC$ give the current, i amperes, flowing into a condenser, and the charge, q coulombs, in the condenser of capacity C farads, t seconds after being connected with a source of constant potential, E volts, by a circuit containing in series a resistance of R ohms. Q is the final charge and T is the time-constant of the circuit. Represent graphically the current and the charge when

(i) $E = 100$, $R = 400$, $C = 0{\cdot}000\,001$;

(ii) $E = 500$, $R = 1000$, $C = 0{\cdot}000\,004$.

8. What is the value of q (example 7) when $t = T$? State the physical interpretation of T.

9. If q, in example 7, is taken as a function of C, plot the curve from $C = 0$ to $C = 5/10^6$ in the cases

(i) $E = 100$, $R = 200$, $t = 0{\cdot}0001$;

(ii) $E = 100$, $R = 200$, $t = 0{\cdot}0005$.

10. Find a relation between t and v to suit the following values:

t	4·2	4·8	5·0	5·6	5·8
v	2·1	1·6	1·4	1·1	1·0

CHAPTER VII.

TRIGONOMETRIC FUNCTIONS.

47. Trigonometric Functions. Before tracing the graphs of trigonometric functions we remind the student of certain important properties.

It follows at once from the definition of the functions that

$$\sin(x \pm n\,.\,360°) = \sin x\,;\ \cos(x \pm n\,.\,360°) = \cos x\,;$$
$$\tan(x \pm n\,.\,180°) = \tan x,$$

where n is any integer. In other words, when the angle x is increased or diminished by any multiple of 360° the sine and cosine do not change their value. $\operatorname{Sin} x$ and $\cos x$ are therefore called **periodic** functions of x; the angle 360° (or 2π radians, if the angle is measured in radians) is called **the period** of $\sin x$ and $\cos x$. The function $\tan x$ is also periodic, but its period is 180° (or π radians); $\tan x$ is of course unaltered when x is increased or diminished by any multiple of 360° but, since it is unaltered when x is increased or diminished by any multiple of 180°, the period is 180° and not 360°.

In general, a function of x is said to be periodic if the function does not change in value when x is increased or diminished by any multiple of a number a, and a is called the period of the function. It is to be understood that a is the smallest number that will secure this repetition of values.

Their periodicity is one of the most important of the properties of the trigonometric functions. In what follows we restrict ourselves almost entirely to the sine, cosine and tangent.

The following relations are fundamental

(i*a*) $\sin(180° - x) = \sin x$, $\sin(x + 180°) = -\sin x$, $\sin(360° - x) = -\sin x$.

(i*b*) $\cos(180° - x) = -\cos x$, $\cos(x + 180°) = -\cos x$, $\cos(360° - x) = \cos x$.

(i*c*) $\tan(180° - x) = -\tan x$, $\tan(x + 180°) = \tan x$, $\tan(360° - x) = -\tan x$.

(ii*a*) $\cos x = \sin(90° + x)$, (ii*b*) $\cos x = \sin(90° - x)$.

(iii) $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$.

The relations (i) give the usual rules for taking out of the tables the sine, cosine and tangent of an angle greater than 90° ; the student should have these rules thoroughly at command.

Either of the relations (ii) reduces the cosine graph to the sine graph.

The relations (iii) show that $\sin x$ and $\tan x$ are **odd functions** of x ; that is, when x changes its sign but not its numerical value, $\sin x$ and $\tan x$ also change their sign but not their numerical value. On the other hand, $\cos x$ is an **even function** of x ; that is, when x changes its sign but not its numerical value, $\cos x$ does not change either in sign or in numerical value. So far as change of sign is concerned, $\sin x$ and $\tan x$ behave like odd powers of x (x^3, x^5,...) while $\cos x$ behaves like even powers of x (x^2, x^4,...).

Again, if x is the number of degrees and t the number of radians in the same angle, we have the relation

$$\text{(iv)} \qquad t = \frac{\pi x}{180}.$$

In changing from one unit to the other we simply replace x by t or t by x when the angle is the argument of a trigonometric function ; thus, $\sin x$ becomes $\sin t$, the unit of angle being understood. But when the angle is not the argument of a trigonometric function, we must replace x by $180t/\pi$ and t by $\pi x/180$; thus

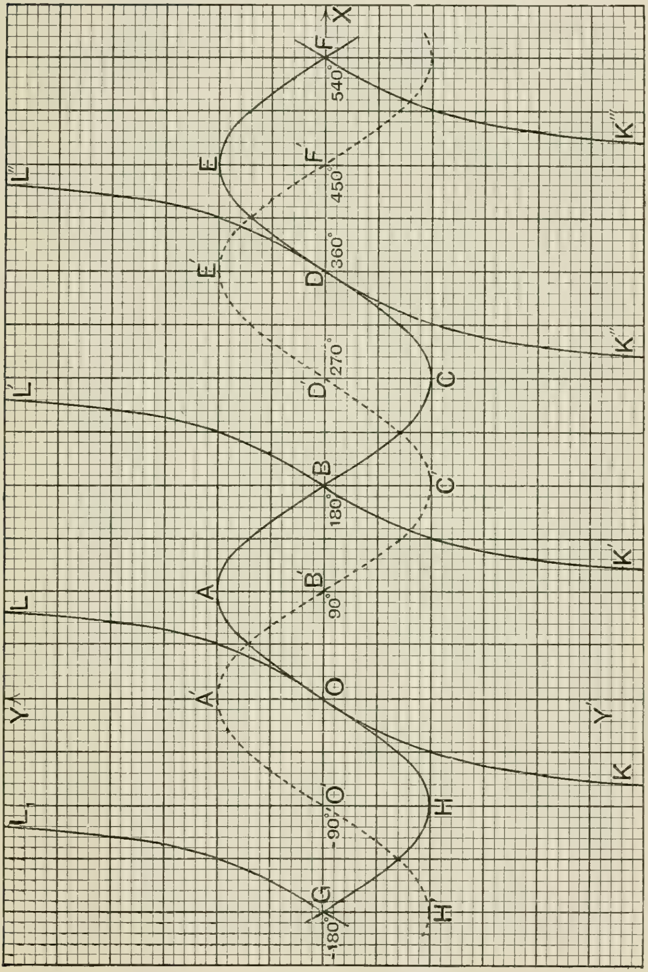
$$t \sin t = \frac{\pi x}{180} \sin x ; \quad 5t \sin\left(2t - \frac{\pi}{3}\right) = \frac{\pi x}{36} \sin(2x - 60°).$$

The graphs of $\sin t$ and $t \sin t$ will be identical with the graphs of $\sin x$ and $\frac{\pi x}{180} \sin x$ respectively ; provided the segment that represents 180 when the degree is the unit of angle is the same as that which represents π when the radian is the unit, the vertical unit of course being the same in both cases.

48. Graphs of the Circular Functions. With the help of the tables the graphs are easily constructed ; or, the values of the functions may be obtained from a circle of unit radius, the circumference being divided by trial, or with the aid of a protractor, into a sufficient number of equal parts. The latter method, when carefully carried out, gives excellent graphs.

In Fig. 46, $OABCD$ is the graph of $\sin x$ from $x = 0°$ to $x = 360°$; DEF continues it on the right to $x = 540°$ and OHG continues it on the left to $x = -180°$. The complete graph of $\sin x$ consists of $OABCD$ and its repetition infinitely often to the right of D and to the left of O.

The dotted curve (Fig. 46) is the graph of $\cos x$; $A'B'C'D'E'$ is the graph of $\cos x$ from $x = 0°$ to $x = 360°$ and



X
Y
Y'
O
O'
A
A'
B
B'
C
C'
D
D'
E
E'
F
F'
G
H
H'
K
K'
K''
K'''
L
L'
L''
L₁
-180°
-90°
90°
180°
270°
360°
450°
540°

Fig. 46.

is simply $ABCDE$, the graph of $\sin x$ from $x=90°$ to $x=450°$, shifted $90°$ to the left (§ 47, ii$a$).

Both of these graphs lie wholly between two straight lines parallel to the x-axis at unit distance above and below that axis; neither $\sin x$ nor $\cos x$ can be numerically greater than unity.

The curve KOL and its repetitions $K'BL'$, $K''DL''$, etc., represent $\tan x$. The function $\tan x$ can take every value between $-\infty$ and $+\infty$; the verticals through B', D' etc., are asymptotes.

The graphs of $\operatorname{cosec} x$, $\sec x$, $\cot x$ are of less importance. Like $\tan x$, $\cot x$ can take every value between $-\infty$ and $+\infty$; neither $\operatorname{cosec} x$ nor $\sec x$ can take any value that is numerically less than unity.

Inverse Circular Functions. The equation $y=\sin x$ not only defines y as a function of x but also defines x as a function of y (compare § 41); x is an angle whose sine is y. Clearly, for any value of y (not greater numerically than 1) there is an infinite number of values of x; for definiteness, we shall represent by the symbol $\sin^{-1}y$ the angle lying between $-90°$ and $90°$ or between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ radians (the extreme angles $-90°$ and $90°$ included) whose sine is y.

Thus,

$$\sin^{-1}\tfrac{1}{2}=30°, \quad \sin^{-1}(-\tfrac{1}{2})=-30°,$$

$$\sin^{-1}1=90°, \quad \sin^{-1}(-1)=-90°.$$

The equation $x=\sin^{-1}y$ is represented by the portion HOA of the sine-curve (Fig. 46).

The same range of angles is represented by the symbol $\tan^{-1}y$; that is, $\tan^{-1}y$ means the angle lying between $-90°$ and $90°$ whose tangent is y. Thus,

$$\tan^{-1}1=45°, \quad \tan^{-1}(-1)=-45°,$$

$$\tan^{-1}(\infty)=90°, \quad \tan^{-1}(-\infty)=-90°.$$

The equation $x=\tan^{-1}y$ is represented by the branch KOL of the tangent-curve (Fig. 46).

When the angle is given by its cosine the range is chosen differently; by the symbol $\cos^{-1}y$ is meant the angle between 0° and 180°, or between 0 and π radians, whose cosine is y. Thus,

$$\cos^{-1}\tfrac{1}{2}=60°, \quad \cos^{-1}(-\tfrac{1}{2})=120°,$$
$$\cos^{-1}1=0°, \quad \cos^{-1}(-1)=180°.$$

The equation $x=\cos^{-1}y$ is represented by the portion $A'B'C'$ of the cosine-curve (Fig. 46).

The graphs of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ can be obtained from those of $\sin^{-1}y$, $\cos^{-1}y$, $\tan^{-1}y$ by the method explained in § 41.

The restrictions on the range of the angle must be remembered in all applications; the student will readily see that, with the above restrictions, the angle is the *smallest* (positive or negative) angle with the given sine, cosine or tangent.

Example. Show that

$$\text{(i)}\ \sin^{-1}x+\cos^{-1}x=90°, \quad \text{(ii)}\ \tan^{-1}x+\cot^{-1}x=90°,$$

where $\cot^{-1}x$ means the angle between 0° and 180° whose cotangent is x.

49. Simple Harmonic Motion. When a point is moving in a straight line in such a way that, at time t, its distance x from a fixed point O on the line is given by the equation

$$x=a\cos(nt+\alpha), \quad \text{or} \quad x=a\sin(nt+\beta) \ldots\ldots\ldots\ldots(1)$$

the point is said to describe a **simple harmonic motion.**

The motion is obviously vibratory, or to and fro; the point moves first in one direction to the distance a from O, then back through O to a distance a on the other side, then returns towards O, and so on. The greatest distance from O that the point reaches, namely a, is called the **amplitude** of the motion.

As t increases from 0 to $2\pi/n$ (or from t_1 to $t_1+2\pi/n$ where t_1 is any value of t) the point makes one complete to and fro motion; $2\pi/n$ is therefore called the **period** of the motion. The reciprocal of the period, namely $n/2\pi$, is sometimes called the **frequency** of the motion. If T is the period and p the frequency, then

$$T=\frac{2\pi}{n}; \ p=\frac{1}{T}=\frac{n}{2\pi}; \ n=\frac{2\pi}{T}=2\pi p.$$

The function $a\cos(nt+\alpha)$, or $a\sin(nt+\beta)$, is frequently called a **simple harmonic function** of t; its graph, that is the cosine curve or the sine curve, is called a simple harmonic curve. The function is of great importance in all branches of physics.

The function of t given by the equation (k positive)

$$x=ae^{-kt}\cos(nt+\alpha) \quad \text{or} \quad x=ae^{-kt}\sin(nt+\beta) \quad \text{.......(2)}$$

is sometimes called a simple harmonic function with decreasing amplitude; the coefficient ae^{-kt} of the cosine or sine is a function of t which decreases as t increases. Physically, the equation represents what is termed *a damped vibration.*

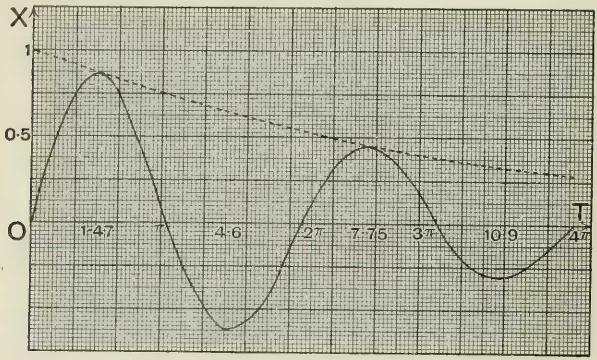


Fig. 47.

Fig. 47 is the graph of

$$x=e^{-\frac{t}{10}}\sin t \quad \text{..........................(3)}$$

and gives some idea of the nature of the function; two waves are shown, but after a few periods of $\sin t$ the height becomes very small. Thus, when $t=10\pi+\frac{\pi}{2}$ we find

$$x=e^{-3\cdot30}\sin\frac{\pi}{2}=0\cdot037.$$

The dotted curve is the graph of $e^{-t/10}$ which touches the other graph near the crests of the waves; at the first crest

$t=1{\cdot}47$, at the second crest $t=7{\cdot}75$. The hollows (the minimum values of x) are given by $t=4{\cdot}6$ and $t=10{\cdot}9$.

The amplitude of the function (2), when t has any value t_1, is ae^{-kt_1}; when t has increased by $\frac{1}{2}T$ (where T is the period $2\pi/n$ of the circular function) the amplitude has decreased to $ae^{-k(t_1+\frac{1}{2}T)}$. The ratio of the first to the second of these amplitudes is

$$ae^{-kt_1} : ae^{-k(t_1+\frac{1}{2}T)} \quad \text{or} \quad e^{\frac{1}{2}kT};$$

the Napierian logarithm of this ratio, namely $\frac{1}{2}kT$, is called the **logarithmic decrement** of the amplitude.

50. Composition of Harmonic Curves. Functions of the form

$$y=a_1\sin(x+\alpha_1)+a_2\sin(2x+\alpha_2)+a_3\sin(3x+\alpha_3)+\ldots(1)$$

occur frequently. Each term is a simple harmonic function. The period of the 2nd term is one half, that of the 3rd term is one third of the period of the first (or fundamental) term; the frequencies are therefore respectively twice and thrice the frequency of the first. Those harmonics in which the coefficient of x is an odd number are called **odd harmonics**; those in which the coefficient is even are called **even harmonics**.

If the angle in the fundamental harmonic is $nx+\alpha_1$, then the angles in the odd harmonics will be $nx+\alpha_1$, $3nx+\alpha_3\ldots$ and in the even harmonics $2nx+\alpha_2$, $4nx+\alpha_4$. ...

To obtain the graph of (1), plot to the same axes the components $a_1\sin(x+\alpha_1)$, $a_2\sin(2x+\alpha_2)$, ... and then add corresponding ordinates (§ 38). The period of y is clearly 360°; the complete graph will therefore consist of repetitions of the portion between $x=0^\circ$ and $x=360^\circ$.

Fig. 48 shows the graph of

$$y=100\sin x+50\sin(3x-40^\circ)\ldots\ldots\ldots\ldots\ldots(2)$$

from $x=0^\circ$ to $x=360^\circ$; the component curves are dotted. The graph of $100\sin x$ is one complete wave; that of $50\sin(3x-40^\circ)$, which is the *third* harmonic, consists of *three* complete waves. The complete representation of y consists of $ABC\ldots K$ and its repetitions.

The function in (2) contains only odd harmonics and the graph possesses, in virtue of this fact, a special kind of symmetry. For, if A is any angle,

$$\sin(x+180^\circ+A) = -\sin(x+A),$$
$$\sin\{3(x+180^\circ)+A\} = -\sin(3x+A),$$
$$\sin\{5(x+180^\circ)+A\} = -\sin(5x+A), \text{ etc.}$$

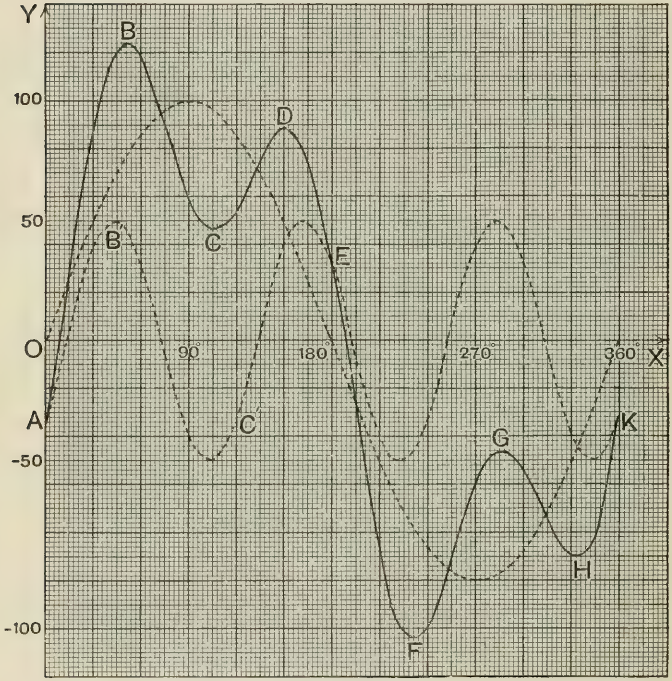


Fig. 48.

Hence the value of y in (2) for $x = x_1 + 180^\circ$ is *simply the negative* of the value for $x = x_1$, where x_1 is any value of x; for example, the value of y for $x = 240^\circ$ is the negative of that for $x = 60^\circ$. The portion of the graph from $x = 180^\circ$ to $x = 360^\circ$, namely $EFGHK$, will therefore, if it be shifted to the left (each point moving parallel to the x-axis) till E comes to the y-axis, be the image of $ABCDE$ in the x-axis.

E will become the image of A, F of B, G of C, H of D and K of E.

The same kind of symmetry will obviously be present whenever y contains only odd harmonics; such cases are of special interest in the theory of Alternate Currents.

If equation (2) contains an absolute term, for example, if the equation is

$$y = 150 + 100 \sin x + 50 \sin(3x - 40°) \quad \text{............(3)}$$

the graph may be obtained by simply shifting $AB \ldots K$ vertically upwards 150 units. The line with respect to which $EFGHK$ (when moved to the left) is symmetrical to $ABCDE$ is no longer the x-axis but is the line parallel to the x-axis at the distance 150 units above it.

Before proceeding to § 51 the student should work several of the earlier examples in Exercises XIX.

51. Decomposition of a Curve into Harmonic Components. There is a remarkable theorem, called **Fourier's Theorem**, which shows that any periodic function of x can be represented by a series of the form

$$y = a_0 + a_1 \sin(x + \alpha_1) + a_2 \sin(2x + \alpha_2)$$
$$+ a_3 \sin(3x + \alpha_3) + a_4 \sin(4x + \alpha_4) + \ldots\ldots \quad \text{(1)}$$

the period of the function being 360° or 2π radians; if the period is $360/n$ degrees or $2\pi/n$ radians, then x is replaced by nx. It is impossible to discuss this theorem here, but there are some simple cases of great practical importance that can be treated graphically. The series (1) is an infinite series but, in the cases referred to, the function y can with sufficient approximation be represented by the sum of two or three harmonic terms.

The problem, then, is :—given a curve, find the harmonic curves which will, when compounded as shown in § 50, produce the given curve. The test of the solution is, of course, that the harmonics found will actually yield the given curve, with sufficient approximation.

We require the following theorem, proved in any text-book of trigonometry :—The sum of n terms of the series

$$\sin A + \sin(A + B) + \sin(A + 2B) + \sin(A + 3B) + \ldots\ldots \quad \text{(2)}$$

where the angles are in arithmetical progression is, unless B is 360° or a multiple of 360°,

$$\frac{\sin\frac{1}{2}nB}{\sin\frac{1}{2}B}\times\sin\{A+\tfrac{1}{2}(n-1)B\};$$

when B is 360° or a multiple of 360° the sum is $n\sin A$, because in these cases each term is equal to $\sin A$.

Note that *the sum is zero* when $\sin\frac{1}{2}nB$, but not $\sin\frac{1}{2}B$, is zero, that is, when nB, but not B, is 360° or a multiple of 360°; for example, when $n=3$ and $B=120°$ the sum is zero, but when $n=3$ and $B=360°$ the sum is $3\sin A$.

If the curve to be analysed has the kind of symmetry noted at the end of § 50 there can be no *even* harmonics in it; we will state the rule however for the general curve given by equation (1), as the method is the same in all cases. *For the present, the term a_0 is supposed to be zero.* (See end of this Article.)

To test whether any harmonic, say the *third*, occurs we have the rule:—divide the period (360° in this case) into *three* equal parts; slide horizontally the two parts of the curve lying between $x=120°$ and $x=240°$, and between $x=240°$ and $x=360°$, till they lie between $x=0°$ and $x=120°$; then add corresponding ordinates of the three parts thus superposed, and divide each resultant ordinate by 3. The equation of the curve so obtained will be

$$y=a_3\sin(3x+\alpha_3)+a_6\sin(6x+\alpha_6)+\ldots\ldots\ldots\ldots(3)$$

that is, it will contain the third harmonic and its multiples, if any of these occur in the given curve, but will not contain any other harmonics.

The proof of the rule is very simple. Let x_1 be any value of x between 0° and 120°; the x of the second part which *after* superposition is x_1 was, *before* superposition, $x_1+120°$; and similarly the x of the third part which after superposition is x_1 was, before superposition, $x_1+240°$. From the term $a_1\sin(x+\alpha_1)$ we therefore get the sum

$$a_1\sin(x+\alpha_1)+a_1\sin(x+120°+\alpha_1)+a_1\sin(x+240°+\alpha_1).$$

In (2) put $A=x+\alpha_1$, $B=120°$, $n=3$; the sum is therefore zero since $\sin\frac{1}{2}nB=\sin 180°=0$ and $\sin\frac{1}{2}B=\sin 60°$, which is not zero.

Similarly, the term $a_2\sin(2x+\alpha_2)$ yields a zero sum. On the other hand, the term $a_3\sin(3x+\alpha_3)$ gives the sum

$$a_3\sin(3x+\alpha_3)+a_3\sin(3x+360^\circ+\alpha_3)+a_3\sin(3x+720^\circ+\alpha_3),$$

which is equal to $3a_3\sin(3x+\alpha_3)$.

In the same way it may be seen that every term, except those containing $3x$, $6x$, $9x$, ... will give a zero sum, while those containing $3x$ and its multiples will give three times the corresponding terms.

Different possibilities for the resultant curve will now be considered.

I. Resultant is a simple sine curve. If the resultant curve is exactly, or with sufficient approximation, a simple sine curve, equation (3) will have only one term on the right-hand side. In the case of Fig. 48, § 50, the resultant curve is simply $AB'C'$; its equation is

$$y=a_3\sin(3x+\alpha_3)=50\sin(3x-40^\circ).$$

The values $a_3=50$, $\alpha_3=-40^\circ$ are obtained from the graph. (The maximum ordinate is 50, which is therefore the value of a_3; the ordinate is zero when $x=13\frac{1}{3}^\circ$ so that $3\times13\frac{1}{3}^\circ+\alpha_3=0$ or $\alpha_3=-40^\circ$. The accuracy of the numbers obtained for a_3 and α_3 is of course conditioned by the scale of the diagram.)

It may happen that the third harmonic is absent and the sixth (but no other) present; the resultant curve given by (3) will, in this case, consist of a simple sine curve with *two* complete waves between $x=0^\circ$ and $x=120^\circ$. If (3) contains only the 9th harmonic then the resultant curve will be a simple sine curve with *three* complete waves between $x=0^\circ$ and $x=120^\circ$, and so on.

II. Resultant is a composite curve. If, however, the resultant curve is not a simple sine curve, proceed as before. Thus, to test if the sixth harmonic is present in the original curve, note that it is the *second* harmonic of the curve given by (3). The period of y in (3) is 120°; therefore divide this period into two equal parts, superpose, add ordinates and divide by 2. The curve so obtained, the second resultant, will be given by

$$y=a_6\sin(6x+\alpha_6)+a_{12}\sin(12x+\alpha_{12})+\dots$$

where $6x$ and its multiples may occur. If this resultant is a simple sine curve of one complete wave it will have for its equation $y = a_6 \sin(6x + \alpha_6)$,

and the values of a_6 and α_6 will be obtained from the graph. The third harmonic of the original curve may now be obtained by subtracting the ordinates of the second resultant from the corresponding ordinates of the first resultant.

The method just explained for finding the third harmonic and its multiples is applicable in all cases. Of course, there is no necessity for the actual superposition of the curves; it will often be more convenient to read corresponding ordinates from the diagram (for example, the ordinates for x, $x + 120°$, $x + 240°$), and then to add them, due regard being paid to sign. The resultant curve would be plotted from these values.

General Rule. To sum up, on the supposition that the first five harmonics may occur; the rule is easily extended if there should happen to be more. The absolute term a_0 is supposed to be zero.

(i) Find the even harmonics by halving the period. (If the first resultant is the x-axis, then no even harmonics are present.) Repeat the operation to find the 4th harmonic, read its constants a_4 and α_4 off this resultant, and then find the 2nd harmonic by subtracting the ordinates of the second resultant from the corresponding ordinates of the first resultant.

(ii) Find the 3rd harmonic, starting from the original curve.

(iii) Find the 5th harmonic, starting from the original curve.

(iv) The first harmonic alone remains to be found. The two constants a_1 and α_1 may be calculated by taking two values of x, say $x = 0°$ and $x = 90°$; the ordinates corresponding to these may be read off the given curve and the other constants are known. Other methods of obtaining a_1, α_1 will readily suggest themselves.

If a_0 is not zero it will appear in every resultant; its value may be determined at the same time as the first resultant simple sine curve from the equation

$$y = a_0 + a_4 \sin(4x + \alpha_4).$$

The x-axis will not in this case be the axis of symmetry of the simple sine curve as it is when a_0 is zero (see § 50, end); the axis of symmetry can be readily found from the resultant curve and its distance above or below the x-axis is the value of a_0. The occurrence of a constant term is therefore tested by the position of the axis of symmetry of the first resultant simple sine curve.

This method of analysing a curve involves a considerable amount of labour, but it is of importance in practice. The more advanced student will be able to diminish the labour by combining analytical and graphical methods. In the exercises will be found a few simple examples for practice.

52. Solution of Equations. Equations in which trigonometric functions occur may often be solved by aid of the graphs of the functions.

An equation of some importance in higher work is

$$\tan x = mx.$$

It is evident that the graph of mx, which is a straight line, will intersect the graph of $\tan x$ infinitely often; the equation has therefore an infinite number of roots. Rough approximations may be obtained from the graph; a full discussion for the case $m=1$ is given in the author's *Calculus*, § 107.

EXERCISES. XIX.

1. Graph the following functions from $x=0^\circ$ to $x=360^\circ$:

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\sin 3x$, (iv) $\cos 3x$,

(v) $\sin 4x$, (vi) $\cos 4x$, (vii) $\sin 5x$, (viii) $\cos 5x$.

State the period of each function.

2. From the graph of $\sin x$ find, merely by changing the origin of coordinates, that of (i) $\sin(x+75^\circ)$, (ii) $\sin(x-75^\circ)$.

How may the graphs of (i) $\sin(nx+A)$, (ii) $\sin(nx-A)$ be obtained from that of $\sin nx$?

3. By what change of scale can the graph of $\sin x$ be interpreted as the graph of (i) $\sin 2x$, (ii) $\sin 3x$, (iii) $\sin\frac{1}{2}x$, (iv) $\sin\frac{1}{3}x$, (v) $\sin nx$?

4. Draw to the same axes the graphs of

(i) $\sin(x+27^\circ)$, (ii) $\cos(x+54^\circ)$, (iii) $\sin(x+27^\circ)+\cos(x+54^\circ)$.

5. Graph the equation

$$y = 10 \sin(x - 36^\circ) + 5 \cos(x + 63^\circ)$$

from $x = 0^\circ$ to $x = 360^\circ$.

What are the turning values of y and what are then the values of x?

Take the same problem as in example 5 for equations 6–11.

6. $y = 100 \sin x - 50 \cos x$. **7.** $y = 50 \sin(x + 18^\circ) + 10 \cos 2x$.

8. $y = 46 \cos(x + 36^\circ) + 30 \cos(3x - 72^\circ)$.

9. $y = 20 \sin x + 10 \sin 3x + 5 \sin 5x$.

10. $y = \sin x + \sin 4x$. **11.** $y = 10 \sin x + 5 \sin(3x - 45^\circ) + 2 \sin 7x$.

12. Graph the following functions from $x = 0^\circ$ to $x = 180^\circ$:

(i) $\dfrac{1}{5 + 3 \cos x}$; (ii) $\dfrac{1}{5 + 3 \sin x}$; (iii) $\dfrac{1}{7 + 5 \cos x + 3 \sin x}$.

13. Graph the following functions for a range of one period:

(i) $\sin 2x \cos x$; (ii) $\cos x \cos 2x$; (iii) $\sin^2 x$; (iv) $\sin^3 x$.

[Use the transformations, $\sin 2x \cos x = \frac{1}{2}(\sin 3x + \sin x)$, etc.]

14. Draw the graphs of

(i) $y = \log \sin x$; (ii) $y = \log \cos x$; (iii) $y = \log \tan x$.

Graph equations 15–18, from $t = 0$ to $t = 1$, the angle being measured in radians.

15. $y = 50 \sin 2\pi t + 10 \sin(4\pi t - 0{\cdot}873)$.

16. $y = 50 \sin 2\pi t + 10 \sin(6\pi t - 0{\cdot}873)$.

17. $y = 100 \sin 2\pi t + 20 \sin(10\pi t - 4{\cdot}189)$.

18. $y = 100 \sin 2\pi t + 60 \sin(6\pi t - 1{\cdot}571) + 10 \sin(10\pi t - 3{\cdot}142)$.

19. Graph the equations

(i) $y = x - \sin x$, from $x = -\pi$ to $x = \pi$.

(ii) $y = x \sin x$, from $x = 0$ to $x = 2\pi$.

(iii) $y = x \cos x$, from $x = 0$ to $x = 2\pi$.

(iv) $y = x \sin^2 x$, from $x = 0$ to $x = \pi$.

20. Graph, from $x = 0$ to $x = \pi$,

$$y = \sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \tfrac{1}{7} \sin 7x.$$

21. Graph, from $x = 0$ to $x = \pi$,

$$y = \sin x - \tfrac{1}{2} \sin 2x + \tfrac{1}{3} \sin 3x - \tfrac{1}{4} \sin 4x.$$

22. Graph, from $x = 0$ to $x = \pi$,

$$y = \sin x - \tfrac{1}{9} \sin 3x + \tfrac{1}{25} \sin 5x - \tfrac{1}{49} \sin 7x.$$

23. Graph the equations

(i) $x = e^{-\frac{t}{20}} \sin(t + 0{\cdot}78)$; (ii) $x = e^{-\frac{t}{20}} \cos(t + 0{\cdot}78)$;

(iii) $x = e^{-10t} \sin(200\pi t - 0{\cdot}5)$; (iv) $x = e^{-10t} \cos(200\pi t - 0{\cdot}5)$.

24. The values of a periodic function y (period 360°) for values of x at intervals of 10°, namely 0°, 10°, 20° ... up to 180° are

$-51{\cdot}96$,	$-12{\cdot}64$,	$34{\cdot}20$,	$80{\cdot}00$,	$116{\cdot}24$,	$136{\cdot}60$,	$138{\cdot}56$,
$123{\cdot}97$,	$98{\cdot}48$,	$70{\cdot}00$,	$46{\cdot}52$,	$33{\cdot}97$,	$34{\cdot}64$,	$46{\cdot}60$,
$64{\cdot}28$,	$80{\cdot}00$,	$86{\cdot}16$,	$77{\cdot}36$,	$51{\cdot}96$.		

The graph has the symmetry noted in § 50. Analyse y into its harmonic components.

25. The same problem as in example 24 for the values

$-19{\cdot}15$,	$-15{\cdot}94$,	$4{\cdot}60$,	$33{\cdot}55$,	$55{\cdot}63$,	$59{\cdot}95$,	$47{\cdot}64$,
$30{\cdot}91$,	$24{\cdot}24$,	$33{\cdot}93$,	$53{\cdot}58$,	$68{\cdot}63$,	$66{\cdot}79$,	$46{\cdot}85$,
$19{\cdot}64$,	$0{\cdot}38$,	$-2{\cdot}05$,	$8{\cdot}68$,	$19{\cdot}15$.		

26. In the following example the intervals are the same as in examples 24, 25, but the value of y for $360^\circ - x$ is the negative of that for x; analyse y into its harmonic components.

0,	$51{\cdot}13$,	$95{\cdot}21$,	$126{\cdot}63$,	$142{\cdot}39$,	$142{\cdot}51$,	$129{\cdot}90$,
$109{\cdot}44$,	$86{\cdot}71$,	$66{\cdot}67$,	$52{\cdot}51$,	$45{\cdot}16$,	$43{\cdot}30$,	$44{\cdot}03$,
$43{\cdot}91$,	$40{\cdot}03$,	$30{\cdot}93$,	$16{\cdot}93$,	0.		

27. Find the two smallest positive roots of the equations

(i) $36 \sin(x+36^\circ) = 55 \sin(3x-56^\circ)$.

(ii) $5 \tan x = 9 \sin(x-45^\circ)$.

In examples 28, 29 the angles are measured in radians.

28. Find the two smallest positive (not zero) roots of each of the equations

(i) $\tan x = x$; (ii) $\tan x = 2x$.

29. Solve the equations

(i) $x = 3 \sin x$; (ii) $x = \cos x$.

30. The chord AB of a circle, centre C, bisects the sector ACB; if the angle ACB is x radians, show that $x = 2 \sin x$ and find x.

31. Find the average rate at which $\sin x$ increases as x increases from 30 to $30+h$ for the values 5, 2, 1, 0·5, 0·1 of h, the angles being measured in degrees.

32. The same problem as in example 31 as x increases from 45 to $45+h$.

The same problem as in example 31 for

33. $\cos x$. **34.** $\tan x$. **35.** $\sin 2x$.

CHAPTER VIII.

CONIC SECTIONS.

53. The Ellipse. In this chapter the equations of the curves called conic sections will be discussed very briefly.

Definition. The locus of a point P which moves so that the **sum** of its distances from two fixed points, S and S', is constant is called an **ellipse**, of which the fixed points S and S' are called the **foci.**

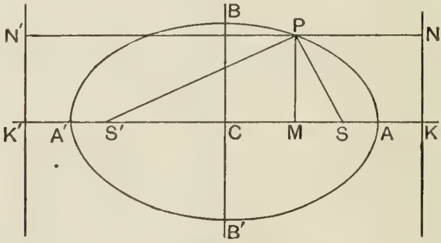


Fig. 49.

Let the constant be $2a$. Bisect $S'S$ (Fig. 49) at C and on $S'S$, produced both ways, take A and A' so that CA and $A'C$ are each equal to a. A and A' are clearly points on the ellipse; $A'A$ is called the **major axis** of the ellipse.

Let $CS = ea$; then e is less than unity. Take $A'A$ as the x-axis and the perpendicular to it through C as the y-axis. Let the coordinates of P be $x = CM$, $y = MP$. Then

$$S'P^2 = S'M^2 + MP^2 = (ea + x)^2 + y^2 = x^2 + y^2 + e^2a^2 + 2eax,$$
$$SP^2 = SM^2 + MP^2 = (ea - x)^2 + y^2 = x^2 + y^2 + e^2a^2 - 2eax.$$

For brevity, let $x^2+y^2+e^2a^2=d$; then

$$S'P=\sqrt{(d+2eax)},\quad SP=\sqrt{(d-2eax)}\ldots\ldots\ldots\ldots(1)$$

and
$$\sqrt{(d+2eax)}+\sqrt{(d-2eax)}=2a.\ldots\ldots\ldots\ldots(2)$$

Square, rearrange and divide by 2; therefore

$$\sqrt{(d^2-4e^2a^2x^2)}=2a^2-d.$$

Square again and reduce, dividing by $4a^2$; therefore

$$-e^2x^2=a^2-d.\ldots\ldots\ldots\ldots(3)$$

Replacing d by its value and rearranging we get

$$(1-e^2)x^2+y^2=(1-e^2)a^2.\ldots\ldots\ldots\ldots(4)$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{(1-e^2)a^2}=1.\ldots\ldots\ldots\ldots(5)$$

Lastly, let $(1-e^2)a^2=b^2$ and we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ldots\ldots\ldots\ldots(E)$$

which is the equation of the ellipse.

When $x=0$, $y=\pm b$. The ellipse therefore cuts the y-axis at B and B' where CB and CB' have each the length b or $a\sqrt{(1-e^2)}$. BB' is called the **minor axis** of the ellipse. C is called the **centre** of the ellipse.

The curve is perhaps most simply constructed by taking points, such as M, between S and S' and describing arcs with S and S' as centres and AM and $A'M$ as radii. The one point M will clearly give 4 points of the curve, two to the left of C and two to the right. Other methods will suggest themselves.

54. The Hyperbola. Definition. The locus of a point P which moves so that the **difference** of its distances from two fixed points, S and S', is constant is called a **hyperbola**, of which the fixed points S and S' are called the **foci**.

Take the same notation as in § 53. In this case A and A' will lie between S and S' (Fig. 50), so that if $CS=ea$ the number e will be greater than unity. Instead of the plus sign in equation (2) we now have the minus sign, but the process of squaring gives the same equations (3), (4), (5) as before. We write (5), however, in the form

$$\frac{x^2}{a^2}-\frac{y^2}{(e^2-1)a^2}=1$$

and put $b^2=(e^2-1)a^2$, which is positive since e is greater than 1. The equation of the hyperbola is thus

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad\qquad (H)$$

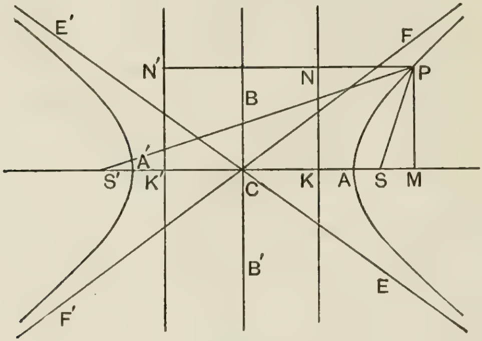


Fig. 50.

From (H) we get

$$y=\pm\frac{b}{a}\sqrt{(x^2-a^2)},$$

so that y is imaginary when x is numerically less than a. No part of the curve therefore lies between the two perpendiculars through A and A' to the major (or **transverse**) axis $A'A$; the curve consists of two branches, one extending to infinity on the right of A and the other to infinity on the left of A'. The segment $B'B$ on the y-axis, where CB and CB' are each of length b, is called the **conjugate axis**; C is the **centre** of the hyperbola.

55. Expression for Focal Distance. Equation (3) § 53 may be written

$$d=a^2+e^2x^2.$$

First adding $2eax$ to each side, next subtracting $2eax$ from each side we find, after taking the square root,

$$\sqrt{(d+2eax)}=a+ex; \quad \sqrt{(d-2eax)}=a-ex.$$

Therefore by § 53 (1) we get for the focal distances $S'P$, SP of the point on the ellipse whose abscissa is x

$$S'P = a + ex, \quad SP = a - ex.$$

(Note that SP is $a - ex$, not $ex - a$, because ex is less than a and the distances SP, $S'P$ are positive.)

For the hyperbola we have

$$S'P = ex + a, \quad SP = ex - a$$

when P is on the right-hand branch; when P is on the left-hand branch the proper expressions are, since x is negative,

$$S'P = -(ex + a), \quad SP = -(ex - a).$$

56. Directrix. Eccentricity. On CA produced in Fig. 49, and on CA between C and A in Fig. 50, take the point K such that $CK = a/e$; draw KN perpendicular to $A'A$ and PN perpendicular to KN. Then for the ellipse

$$PN = MK = CK - CM = \frac{a}{e} - x = \frac{a - ex}{e} = \frac{SP}{e},$$

and for the hyperbola

$$NP = KM = CM - CK = x - \frac{a}{e} = \frac{ex - a}{e} = \frac{SP}{e},$$

so that $$SP : PN = e : 1.$$

Therefore in both cases the ratio of the focal distance SP to the perpendicular distance PN of P from the line KN is equal to the constant e. The line KN is called the **directrix** for the focus S, and the constant e is called the **eccentricity**.

Similarly it may be proved that there is a second directrix $K'N'$ related to the focus S' in the same way as KN is to S; it lies at the distance a/e to the left of C and

$$S'P : PN' = e : 1.$$

57. Conic Sections. The property proved in § 56 is that usually taken as the definition of a conic section, namely :—

Definition. A conic section (or, more briefly, a conic) is the locus of a point P which moves so that its distance from a fixed point S (the focus) is in a constant ratio e (the eccentricity) to its distance from a fixed straight line KN

(the directrix). The conic is an ellipse if e is less than unity, a hyperbola if e is greater than unity, a parabola if e is equal to unity.

That the curve we have called a parabola possesses this property is easily proved. Let Fig. 51 be the graph of the equation

$$py = x^2 \qquad \text{.....................................(1)}$$

and let K, S be points on the y-axis such that $KO = OS = \frac{1}{4}p$. Draw KN perpendicular to KS, and let the perpendicular PN, drawn to KN from the point P on the graph, cut the x-axis at M; also, draw SQ perpendicular to NP.

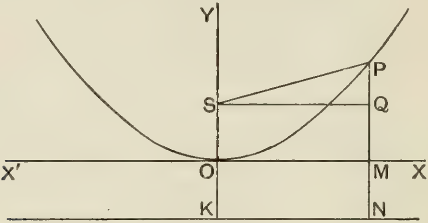


Fig. 51.

If P is the point (x, y) then, since $x = OM$, $y = MP$, $p = 4OS$, equation (1) gives

$$4OS \,.\, MP = OM^2.$$

Now $\qquad SQ = OM,\ QP = MP - OS,\ NP = MP + OS;$

hence $\qquad SP^2 = OM^2 + (MP - OS)^2 = 4OS \,.\, MP + (MP - OS)^2.$

But $\qquad 4OS \,.\, MP + (MP - OS)^2 = (MP + OS)^2 = NP^2,$

and therefore $SP = NP$, so that the curve is a parabola of which S is the focus and KN the directrix.

The circle is the particular case of the ellipse in which $b = a$. But when $b = a$ we must have $e = 0$, because $b^2 = (1 - e^2)a^2$. The circle therefore is a conic of which the eccentricity is zero.

The ellipse (which includes the circle) and the hyperbola are called **central conics**; every chord through the centre C (Figs. 49, 50) is bisected at C. The parabola has no centre.

The points A, A' (Figs. 49, 50) are called the **vertices** of the central conics. The circle on AA' as diameter is called the **auxiliary circle**. (See Exercises XXI., 2, 12, 13, 14.)

58. Equal Roots of a Quadratic Equation. In the next set of Exercises the student will have occasion to apply the tests that the roots of a quadratic equation should be real, and also that they should be equal. The roots of the equation

$$ax^2+bx+c=0$$

are $$x_1=\frac{-b+\sqrt{(b^2-4ac)}}{2a}, \quad x_2=\frac{-b-\sqrt{(b^2-4ac)}}{2a}.$$

x_1 and x_2 are real and different if b^2 is greater than $4ac$; they are real and equal if $b^2=4ac$; they are imaginary if b^2 is less than $4ac$.

Example 1. Find the equation of the tangent at the point (2, 4) on the parabola $y=x^2$.

The equation of every straight line through the point (2, 4) is of the form

$$y-4=m(x-2). \quad \text{(i)}$$

To find the points in which this straight line meets the parabola, we must solve (i) and the equation

$$y=x^2 \quad \text{(ii)}$$

as simultaneous equations. The equation for the abscissae of the points of intersection is

$$x^2=m(x-2)+4, \text{ or } x^2-mx+2m-4=0. \quad \text{(iii)}$$

Now, we know that $x=2$ is one root of (iii); therefore $x-2$ must be a factor of the left-hand side of (iii). In fact, equation (iii) may be written

$$(x-2)(x-m+2)=0.$$

The second value of x is therefore $m-2$. This will be the same as the first value 2 if $m-2=2$, that is, if $m=4$. Therefore the straight line given by the equation

$$y-4=4(x-2) \text{ or } y=4x-4$$

is the tangent.

We may also find the equation as follows: The line given by (i) will meet the parabola only once if the two roots of equation (iii) are equal. But these roots are equal if

$$m^2=4(2m-4) \text{ or } m^2-8m+16=0,$$

that is, if $m=4$.

The equation of the normal to the parabola at (2, 4) is

$$y-4=-\tfrac{1}{4}(x-2) \text{ or } x+4y=18.$$

Definition. The normal at a point P on a curve is the straight line through P perpendicular to the tangent to the curve at P.

Example 2. In how many points does the straight line whose equation is $x=c$ cut the curve whose equation is

$$x^2+xy+y^2=3\,?$$

To find the points of intersection we solve the equations as simultaneous equations. Hence the y of the points of intersection is given by the equation

$$y^2+cy+c^2-3=0.$$

The roots of this equation are

$$y_1=-\tfrac{1}{2}c+\tfrac{1}{2}\sqrt{(12-3c^2)},\quad y_2=-\tfrac{1}{2}c-\tfrac{1}{2}\sqrt{(12-3c^2)}.$$

If $3c^2<12$, that is, if $c^2<4$ the roots are real and unequal, and therefore for these values of c there are two points of intersection.

If $c^2>4$, the roots are imaginary, and therefore if $c^2>4$ the line does not intersect the curve.

If $c^2=4$, the two values y_1, y_2 are equal; therefore the lines whose equations are $x=2$, $x=-2$ meet the curve each in only one point, that is, they are tangents to the curve.

In the same way it may be seen that the lines given by $y=2$, $y=-2$ are tangents.

The curve is an ellipse inscribed in the square whose sides are given by the equations

$$x=2,\quad x=-2,\quad y=2,\quad y=-2\,;$$

and the points of contact are

$$(2,\ -1),\ (-2,\ 1),\ (-1,\ 2),\ (1,\ -2).$$

A second set of Exercises is appended in which many of the simpler and more important properties of the conic sections are stated. The proofs should offer no difficulty, and the theorems may be useful to students who cannot afford the time for a fuller study. The notations of this chapter are adhered to in the Exercises.

EXERCISES. XX.

1. Draw (i) an ellipse, (ii) a hyperbola whose axes are 8 and 6 respectively.

2. Plot the curves given by the following equations, and state the eccentricity of each :—

$$\text{(i)}\ 16x^2+25y^2=400\,;\quad \text{(ii)}\ 16x^2-25y^2=400.$$

3. Plot the curves

$$\text{(i)}\ x^2+4y^2=6x\,;\quad \text{(ii)}\ x^2-4y^2=6x.$$

Show that (i) is an ellipse, (ii) a hyperbola, and find the axes, the eccentricity and the coordinates of the centre of each.

4. Plot the curves

$$\text{(i)}\ y^2=36x-9x^2\,;\quad \text{(ii)}\ y^2=36x+9x^2.$$

Show that (i) is an ellipse whose major axis is vertical; find the axes, the eccentricity and the coordinates of the centre of each.

5. Show that the equations

$$\text{(i)}\ y^2=2Ax-Bx^2\ ;\quad \text{(ii)}\ y^2=2Ax+Bx^2,$$

where B is positive, represent (i) an ellipse, and (ii) a hyperbola, respectively.

6. Plot the graph of the equation $x^2-2xy+3y^2=4$.

[Solve for y: $\qquad y=\frac{1}{3}x\pm\frac{1}{3}\sqrt{(12-2x^2)}$.

$2x^2$ therefore cannot be greater than 12, so that the curve lies between two straight lines perpendicular to the x-axis given by $x=+\sqrt{6}$, $x=-\sqrt{6}$. These lines are tangents to the curve.

Similarly, solving for x we find that y^2 cannot be greater than 2, and the curve lies between two lines parallel to the x-axis given by $y=\sqrt{2}$, $y=-\sqrt{2}$. These lines also are tangents.

The curve crosses the x-axis ($y=0$) where $x=2$ and -2; it crosses the y-axis ($x=0$) where $y=\frac{1}{3}\sqrt{12}$ and $-\frac{1}{3}\sqrt{12}$.

Other values of y can be obtained most readily from the solved equation, each value of x giving two values of y.

The curve is an ellipse.]

7. Plot the equations

$$\text{(i)}\ 2x^2-2xy+y^2=9\ ;\quad \text{(ii)}\ 3x^2+2xy-y^2=9.$$

Write down the equations of the tangents parallel to the coordinate axes.

8. Plot the equations

$$\text{(i)}\ (2x+y)^2=y-2x\ ;\quad \text{(ii)}\ (y-x+1)^2=4(x+y).$$

The curves are parabolas.

9. Show that $3x+8y=25$ is a tangent to the ellipse $x^2+4y^2=25$ and that $5x-4y=9$ is a tangent to the hyperbola $x^2-y^2=9$. Find the coordinates of the point of contact of each tangent and write down the equation of each normal.

10. Find the points of intersection of

$$x^2+5y^2=45 \quad\text{and}\quad x=my+7,$$

and determine m so that the straight line may be a tangent.

11. Determine the value of c in terms of m so that the straight line $y=mx+c$ may be a tangent to the conics

$$\text{(i)}\ 9x^2+16y^2=144\ ;\quad \text{(ii)}\ 9x^2-16y^2=144\ ;$$
$$\text{(iii)}\ b^2x^2+a^2y^2=a^2b^2\ ;\quad \text{(iv)}\ b^2x^2-a^2y^2=a^2b^2.$$

12. The same problem as in example 11 for the curves

$$\text{(i)}\ 4y=x^2\ ;\quad \text{(ii)}\ y=x^2+2x+3\ ;\quad \text{(iii)}\ y^2=4ax.$$

EXERCISES. XXI.

1. The double ordinate through the focus of a central conic is called the **latus rectum** or the **parameter** of the conic; show that it is equal to $2b^2/a$.

For the parabola sketched in Fig. 51 the parameter is the double abscissa through the focus; show that when the parabola is given by $py=x^2$ the latus rectum or parameter is p. (Compare § 29.)

2. On AA' (Fig. 49) as diameter a circle is described; if MP is produced to meet the circle at Q show that

$$MP : MQ = b : a = \text{constant ratio.}$$

[For, $$MP^2 = \frac{b^2}{a^2}(a^2-x^2)\,;\; MQ^2 = a^2 - x^2.$$

This circle is called the auxiliary circle of the ellipse (§ 57); the points P and Q may be called corresponding points.]

3. Deduce from example 2 the following method of constructing an ellipse:—Let M be any point on a fixed diameter AA' of a circle of radius a, MQ the half chord perpendicular to AA' and P a point in MQ such that $MP : MQ = b : a$; the locus of P for all positions of MQ is an ellipse whose axes are $2a$, $2b$.

What is the locus of P when P is taken in MQ produced outside the circle so that $MP : MQ = b : a$?

4. The angle ACQ in example 2 is called the **eccentric angle** of the point $P(x, y)$; if $\angle ACQ = \theta$ show that

$$x = a\cos\theta, \quad y = b\sin\theta.$$

5. On the edge RQ of a straight ruler a fixed point P is taken; the point R is placed on a straight line $Y'Y$ and the point Q on a straight line $X'X$ perpendicular to $Y'Y$, and the ruler is moved about so that R and Q always remain on $Y'Y$ and $X'X$ respectively. Show that P will describe the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $RP = a$, $QP = b$ and x, y are the coordinates of P to the axes $X'X$, $Y'Y$.

Deduce a method of constructing an ellipse.

6. Show from example 2 that an ellipse is the projection of a circle.

7. If P, Q and P', Q' are two pairs of corresponding points on an ellipse and its auxiliary circle show that the chords PP' and QQ' intersect the major axis at the same point, T' say. (Lines to be produced.)

8. If the secant $QQ'T'$ in example 7 is turned till it becomes the tangent to the circle at Q, and if this tangent cut the major axis at T show that PT is the tangent to the ellipse at P.

9. Deduce from example 8 that $CM \,.\, CT = CA^2$. If m is the projection of P on the minor axis, and if PT meet the minor axis at t show that $Cm \,.\, Ct = CB^2$.

10. Show that a point Q is outside or inside an ellipse according as the sum of its focal distances SQ, $S'Q$ is greater than or less than the major axis.

For the hyperbola, show that a point Q lies between the two branches or inside one of the branches according as the difference of its focal distances SQ, $S'Q$ is less than or greater than the transverse axis.

11. Show by example 10 that every point on the bisector of the exterior angle between the focal distances SP, $S'P$ of the point P on an ellipse (except the point P itself) is outside the ellipse, and thus prove that this bisector is the tangent to the ellipse at P.

Show that for the hyperbola the bisector of the angle SPS' is the tangent at P.

[For the ellipse, let the perpendicular from S on the bisector meet $S'P$ produced at P', and let Q be any point, except P, on the bisector.

Then $\qquad SP=P'P,\ SQ=P'Q,\ S'Q+SQ=S'Q+P'Q.$

But $S'Q+P'Q$ is greater than $S'P'$ which is equal to $S'P+SP$, that is, equal to the major axis. Q is therefore outside the ellipse.

The proof for the hyperbola is similar.]

12. If the perpendiculars SZ, $S'Z'$ from the foci of a central conic on the tangent at P meet the tangent at Z, Z' respectively show that $CZ=CA=CZ'$; that is, show that Z, Z' are on the auxiliary circle of the conic.

13. If, in example 12, ZS and $Z'C$ are produced to meet at W prove $CW=CZ'=CA$, $S'Z'=SW$. Then prove $SZ.S'Z'=CB^2$.

[W is on the auxiliary circle and therefore $SZ.SW$, which is equal to $SZ.S'Z'$, is equal to CA^2-CS^2 for the ellipse and to CS^2-CA^2 for the hyperbola. Then compare values of b^2, a^2, a^2e^2 for ellipse and hyperbola.]

14. Deduce from example 13 the following construction for drawing a tangent to a central conic from an external point P:—on SP as diameter describe a circle cutting the auxiliary circle at Q and R; PQ and PR, produced if necessary, are the two tangents from P.

15. If the normal and tangent at P to a central conic meet the major axis at G and T respectively, show that

$$CG.CT=CS^2;\ CG=e^2x=e^2CM.$$

[PG, PT are the bisectors of the angle SPS' and therefore G, T divide SS' internally and externally in the same ratio, from which it follows that $CG.CT=CS^2$. Again, using the values of SP, $S'P$ in § 55, we have

$$S'G:GS=S'P:SP=a+ex:a-ex,$$

whence $\qquad S'G:S'S=a+ex:2a,$

and therefore $\qquad S'G=e(a+ex),\ CG=e^2x.$]

16. From example 15 prove the first theorem of example 9 and then deduce the second theorem.

$$[CM.CT:CG.CT=CM:CG=1:e^2.$$

But $\qquad CG.CT=CS^2=e^2a^2$ and therefore $CM.CT=a^2=CA^2$.

This proof holds for the hyperbola as well as for the ellipse.]

17. Show that $SP.S'P=a^2-e^2x^2$ for the ellipse, but $e^2x^2-a^2$ for the hyperbola.

18. With the notation of example 15 prove that

$$PG^2=(1-e^2)(a^2-e^2x^2).$$

[For $$PG^2=GM^2+MP^2=(1-e^2)^2x^2+y^2\,;$$
then use the value of y^2 in § 53 (4).]

19. If θ is the eccentric angle of a point P on an ellipse show from example 9 that $$CT=a/\cos\theta,\quad Ct=b/\sin\theta,$$
and prove that the equations of the tangent and normal at P are respectively

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1\,;\ \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

20. Find the coordinates of the points in which the line through C parallel to the tangent at P meets the ellipse.

[The line is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=0$; combining with the equation of the ellipse we get two points $D(-a\sin\theta,\ b\cos\theta)$, $D'(a\sin\theta,\ -b\cos\theta)$.

The two semi-diameters CP, CD are said to be **conjugate**; each is parallel to the tangent at the end of the other. The eccentric angle of D is $90^\circ+\theta$, and of D' is $\theta-90^\circ$ or $\theta+270^\circ$.]

21. Show from example 20 that $CP^2+CD^2=CA^2+CB^2$, that is that the sum of the squares of two conjugate semi-diameters is constant.

22. Show from Examples 17 and 20 that $CD^2=SP\,.\,S'P$.

$$[CD^2=a^2\sin^2\theta+b^2\cos^2\theta=a^2-(a^2-b^2)\cos^2\theta=a^2-e^2x^2.]$$

23. From C a perpendicular CF is drawn to the tangent at P; show that the coordinates of F are

$$x=\frac{ab^2\cos\theta}{a^2\sin^2\theta+b^2\cos^2\theta},\quad y=\frac{a^2b\sin\theta}{a^2\sin^2\theta+b^2\cos^2\theta}$$

and that $$CF=\sqrt{(x^2+y^2)}=\frac{ab}{CD}.$$

24. Show from example 23 that the area of the parallelogram formed by the tangents at the ends of two conjugate diameters PCP', DCD' is constant, and equal to $4ab$ or $AA'\,.\,BB'$, the rectangle contained by the axes.

[A quarter of the area is clearly $CF\,.\,CD$ which is equal to ab.]

25. Show that the equations of the tangent and normal at the point (x_1, y_1) on the hyperbola $x^2/a^2-y^2/b^2=1$ are respectively

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1,\quad \frac{a^2}{x_1}x+\frac{b^2}{y_1}y=a^2+b^2.$$

26. Show that the straight lines $y=bx/a$, $y=-bx/a$ are asymptotes of the hyperbola.

[Let $$y_1=\frac{bx}{a},\quad y=\frac{b}{a}\sqrt{(x^2-a^2)}\,;$$

then $$y_1-y=\frac{b}{a}\left\{x-\sqrt{(x^2-a^2)}\right\}=\frac{b}{a}\cdot\frac{a^2}{x+\sqrt{(x^2-a^2)}},$$

and therefore when x becomes very large the difference between y_1, the ordinate of the straight line, and y, the ordinate of the hyperbola, becomes very small.

When $b=a$ the asymptotes are at right angles to each other; the hyperbola, when $b=a$, is called **rectangular.**]

27. From any point $P(x, y)$ on the *rectangular* hyperbola $x^2-y^2=a^2$ PL is drawn perpendicular to the asymptote $E'CE$ (Fig. 50); if $CL=x'$, $LP=y'$ show that

$$x=\frac{x'+y'}{\sqrt{2}},\quad y=\frac{y'-x'}{\sqrt{2}},$$

and therefore that $x^2-y^2=a^2$ becomes $x'y'=\frac{1}{2}a^2$.

[The values of x, y are proved at once by projection. The result shows that when referred to its asymptotes as axes the equation of the rectangular hyperbola is $xy=\frac{1}{2}a^2$. (Compare § 33).]

28. Show that for a parabola the point P is outside or inside the curve according as the distance SP of P from the focus is greater than or less than its distance PN from the directrix.

29. Deduce from example 28 that the bisector of the angle SPN is the tangent at P to the parabola. Show that the normal at P bisects the angle between NP produced and SP.

30. A is the vertex of a parabola; the tangent and normal at P cut the axis of the parabola at T and G respectively; H is the projection of P on the axis, and Z the projection of S on the tangent at P. Prove

$$ST=SP=SG;\quad SP=AS+AH;\quad TA=AH;\quad HG=2AS;$$
$$\angle ASZ=\angle PSZ;\quad SZ^2=AS\,.\,SP.$$

Show also that Z lies on the tangent at the vertex A.

31. Prove from example 30 the following method of drawing a tangent to a parabola from an external point P:—On SP as diameter describe a circle cutting the tangent at the vertex in Q and R; PQ and PR are the two tangents from P.

TABLES.

TABLE I.

SQUARES OF NUMBERS FROM 10 TO 99.

	0	1	2	3	4	5	6	7	8	9
1	100	121	144	169	196	225	256	289	324	361
2	400	441	484	529	576	625	676	729	784	841
3	900	961	1024	1089	1156	1225	1296	1369	1444	1521
4	1600	1681	1764	1849	1936	2025	2116	2209	2304	2401
5	**2500**	**2601**	**2704**	**2809**	**2916**	**3025**	**3136**	**3249**	**3364**	**3481**
6	3600	3721	3844	3969	4096	4225	4356	4489	4624	4761
7	4900	5041	5184	5329	5476	5625	5776	5929	6084	6241
8	6400	6561	6724	6889	7056	7225	7396	7569	7744	7921
9	8100	8281	8464	8649	8836	9025	9216	9409	9604	9801

TABLE II.

SQUARE ROOTS OF NUMBERS FROM 1 TO 9·9.

	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	0·000	0·316	0·447	0·548	0·632	0·707	0·775	0·837	0·894	0·949
1	1·000	1·049	1·095	1·140	1·183	1·225	1·265	1·304	1·342	1·378
2	1·414	1·449	1·483	1·517	1·549	1·581	1·612	1·643	1·673	1·703
3	1·732	1·761	1·789	1·817	1·844	1·871	1·897	1·924	1·949	1·975
4	2·000	2·025	2·049	2·074	2·098	2·121	2·145	2·168	2·191	2·214
5	**2·236**	**2·258**	**2·280**	**2·302**	**2·324**	**2·345**	**2·366**	**2·387**	**2·408**	**2·429**
6	2·449	2·470	2·490	2·510	2·530	2·550	2·569	2·588	2·608	2·627
7	2·646	2·665	2·683	2·702	2·720	2·739	2·757	2·775	2·793	2·811
8	2·828	2·846	2·864	2·881	2·898	2·915	2·933	2·950	2·966	2·983
9	3·000	3·017	3·033	3·050	3·066	3·082	3·098	3·114	3·130	3·146

TABLE III.

SQUARE ROOTS OF NUMBERS FROM 10 TO 99.

	0	1	2	3	4	5	6	7	8	9
1	3·162	3·317	3·464	3·606	3·742	3·873	4·000	4·123	4·243	4·359
2	4·472	4·583	4·690	4·796	4·899	5·000	5·099	5·196	5·292	5·385
3	5·477	5·568	5·657	5·745	5·831	5·916	6·000	6·083	6·164	6·245
4	6·325	6·403	6·481	6·557	6·633	6·708	6·782	6·856	6·928	7·000
5	**7·071**	**7·141**	**7·211**	**7·280**	**7·348**	**7·416**	**7·483**	**7·550**	**7·616**	**7·681**
6	7·746	7·810	7·874	7·937	8·000	8·062	8·124	8·185	8·246	8·307
7	8·367	8·426	8·485	8·544	8·602	8·660	8·718	8·775	8·832	8·888
8	8·944	9·000	9·055	9·110	9·165	9·220	9·274	9·327	9·381	9·434
9	9·487	9·539	9·592	9·644	9·695	9·747	9·798	9·849	9·899	9·950

TABLE IV.

CUBES OF NUMBERS FROM 1 TO 9·9.

	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
1	1·00	1·33	1·73	2·20	2·74	3·37	4·10	4·91	5·83	6·86
2	8·00	9·26	10·65	12·17	13·82	15·62	17·58	19·68	21·95	24·39
3	27·00	29·79	32·77	35·94	39·30	42·87	46·66	50·65	54·87	59·32
4	64·0	68·9	74·1	79·5	85·2	91·1	97·3	103·8	110·6	117·6
5	**125·0**	**132·7**	**140·6**	**148·9**	**157·5**	**166·4**	**175·6**	**185·2**	**195·1**	**205·4**
6	216·0	227·0	238·3	250·0	262·1	274·6	287·5	300·8	314·4	328·5
7	343·0	357·9	373·2	389·0	405·2	421·9	439·0	456·5	474·6	493·0
8	512·0	531·4	551·4	571·8	592·7	614·1	636·1	658·5	681·5	705·0
9	729·0	753·6	778·7	804·4	830·6	857·4	884·7	912·7	941·2	970·3

TABLE V.

RECIPROCALS OF NUMBERS FROM 1 TO 9·9.

	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
1	1·000	0·909	0·833	0·769	0·714	0·667	0·625	0·588	0·556	0·526
2	0·500	0·476	0·455	0·435	0·417	0·400	0·385	0·370	0·357	0·345
3	0·333	0·323	0·313	0·303	0·294	0·286	0·278	0·270	0·263	0·256
4	0·250	0·244	0·238	0·233	0·227	0·222	0·217	0·213	0·208	0·204
5	**0·200**	**0·196**	**0·192**	**0·189**	**0·185**	**0·182**	**0·179**	**0·175**	**0·172**	**0·169**
6	0·167	0·164	0·161	0·159	0·156	0·154	0·152	0·149	0·147	0·145
7	0·143	0·141	0·139	0·137	0·135	0·133	0·132	0·130	0·128	0·127
8	0·125	0·123	0·122	0·120	0·119	0·118	0·116	0·115	0·114	0·112
9	0·111	0·110	0·109	0·108	0·106	0·105	0·104	0·103	0·102	0·101

TABLE VI. LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	**0000**	**0043**	**0086**	**0128**	**0170**	**0212**	**0253**	**0294**	**0334**	**0374**	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	**1761**	**1790**	**1818**	**1847**	**1875**	**1903**	**1931**	**1959**	**1987**	**2014**	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	**3010**	**3032**	**3054**	**3075**	**3096**	**3118**	**3139**	**3160**	**3181**	**3201**	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	**3979**	**3997**	**4014**	**4031**	**4048**	**4065**	**4082**	**4099**	**4116**	**4133**	2	4	5	7	9	10	12	14	16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	**4771**	**4786**	**4800**	**4814**	**4829**	**4843**	**4857**	**4871**	**4886**	**4900**	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
35	**5441**	**5453**	**5465**	**5478**	**5490**	**5502**	**5514**	**5527**	**5539**	**5551**	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	**6021**	**6031**	**6042**	**6053**	**6064**	**6075**	**6085**	**6096**	**6107**	**6117**	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	**6532**	**6542**	**6551**	**6561**	**6571**	**6580**	**6590**	**6599**	**6609**	**6618**	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	**6990**	**6998**	**7007**	**7016**	**7024**	**7033**	**7042**	**7050**	**7059**	**7067**	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

TABLE VI. LOGARITHMS.—*Continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	**7404**	**7412**	**7419**	**7427**	**7435**	**7443**	**7451**	**7459**	**7466**	**7474**	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	**7782**	**7789**	**7796**	**7803**	**7810**	**7818**	**7825**	**7832**	**7839**	**7846**	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	**8129**	**8136**	**8142**	**8149**	**8156**	**8162**	**8169**	**8176**	**8182**	**8189**	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	**8451**	**8457**	**8463**	**8470**	**8476**	**8482**	**8488**	**8494**	**8500**	**8506**	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	**8751**	**8756**	**8762**	**8768**	**8774**	**8779**	**8785**	**8791**	**8797**	**8802**	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	**9031**	**9036**	**9042**	**9047**	**9053**	**9058**	**9063**	**9069**	**9074**	**9079**	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	**9294**	**9299**	**9304**	**9309**	**9315**	**9320**	**9325**	**9330**	**9335**	**9340**	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	**9542**	**9547**	**9552**	**9557**	**9562**	**9566**	**9571**	**9576**	**9581**	**9586**	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	**9777**	**9782**	**9786**	**9791**	**9795**	**9800**	**9805**	**9809**	**9814**	**9818**	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE VII. ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	2	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	1	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	3
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	3	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	5	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	6
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	2	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	6	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	3	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE VII. ANTILOGARITHMS—*Continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	1	2	3	4	4	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	1	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	2	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	3	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	7	8	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	5	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	3	5	6	7	8	9	10
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	7	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	11	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	7	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	14	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	4	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	6	7	9	11	13	15	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	9	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	6	9	11	13	15	17	19
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	21

TABLE VIII. NATURAL SINES.

Deg.	0′ =0·0	6′ =0·1	12′ =0·2	18′ =0·3	24′ =0·4	30′ =0·5	36′ =0·6	42′ =0·7	48′ =0·8	54′ =0·9	1	2	3	4	5
0	**0000**	**0017**	**0035**	**0052**	**0070**	**0087**	**0105**	**0122**	**0140**	**0157**	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	**0872**	**0889**	**0906**	**0924**	**0941**	**0958**	**0976**	**0993**	**1011**	**1028**	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	**1736**	**1754**	**1771**	**1788**	**1805**	**1822**	**1840**	**1857**	**1874**	**1891**	3	6	9	11	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	9	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	**2588**	**2605**	**2622**	**2639**	**2656**	**2672**	**2689**	**2706**	**2723**	**2740**	3	6	8	11	14
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	**3420**	**3437**	**3453**	**3469**	**3486**	**3502**	**3518**	**3535**	**3551**	**3567**	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	**4226**	**4242**	**4258**	**4274**	**4289**	**4305**	**4321**	**4337**	**4352**	**4368**	3	5	8	10	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	**5000**	**5015**	**5030**	**5045**	**5060**	**5075**	**5090**	**5105**	**5120**	**5135**	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	3	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	**5736**	**5750**	**5764**	**5779**	**5793**	**5807**	**5821**	**5835**	**5850**	**5864**	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	10	12
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	5	7	9	11
40	**6428**	**6441**	**6455**	**6468**	**6481**	**6494**	**6508**	**6521**	**6534**	**6547**	2	4	7	9	11
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	7	9	11
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

TABLE VIII. NATURAL SINES—*Continued.*

Deg.	0′ =0·0	6′ =0·1	12′ =0·2	18′ =0·3	24′ =0·4	30′ =0·5	36′ =0·6	42′ =0·7	48′ =0·8	54′ =0·9	1	2	3	4	5
45	**7071**	**7083**	**7096**	**7108**	**7120**	**7133**	**7145**	**7157**	**7169**	**7181**	2	4	6	8	10
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	**7660**	**7672**	**7683**	**7694**	**7705**	**7716**	**7727**	**7738**	**7749**	**7760**	2	4	6	8	9
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	3	5	7	9
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	9
55	**8192**	**8202**	**8211**	**8221**	**8231**	**8241**	**8251**	**8261**	**8271**	**8281**	2	3	5	7	8
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	**8660**	**8669**	**8678**	**8686**	**8695**	**8704**	**8712**	**8721**	**8729**	**8738**	1	3	4	6	7
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	**9063**	**9070**	**9078**	**9085**	**9092**	**9100**	**9107**	**9114**	**9121**	**9128**	1	3	4	5	6
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	4	5	6
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	5	6
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	**9397**	**9403**	**9409**	**9415**	**9421**	**9426**	**9432**	**9438**	**9444**	**9449**	1	2	3	4	5
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	3	3	4
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	**9659**	**9664**	**9668**	**9673**	**9677**	**9681**	**9686**	**9690**	**9694**	**9699**	1	2	2	3	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0	1	2	2	3
80	**9848**	**9851**	**9854**	**9857**	**9860**	**9863**	**9866**	**9869**	**9871**	**9874**	0	1	1	2	2
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	**9962**	**9963**	**9965**	**9966**	**9968**	**9969**	**9971**	**9972**	**9973**	**9974**	0	0	1	1	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	9998	9999	9999	9999	9999	1·0000	1·0000	1·0000	1·0000	1·0000	0	0	0	0	0
							to 4 decimals.								

TABLE IX. NATURAL COSINES.

Deg.	0′ =0·0	6′ =0·1	12′ =0·2	18′ =0·3	24′ =0·4	30′ =0·5	36′ =0·6	42′ =0·7	48′ =0·8	54′ =0·9	1	2	3	4	5
	to 4 decimals.														
0	**1·0000**	**1·0000**	**1·0000**	**1·0000**	**1·0000**	**1·0000**	**9999**	**9999**	**9999**	**9999**	0	0	0	0	0
1	9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	**9962**	**9960**	**9959**	**9957**	**9956**	**9954**	**9952**	**9951**	**9949**	**9947**	0	1	1	1	1
6	9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	**9848**	**9845**	**9842**	**9839**	**9836**	**9833**	**9829**	**9826**	**9823**	**9820**	1	1	2	2	3
11	9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	**9659**	**9655**	**9650**	**9646**	**9641**	**9636**	**9632**	**9627**	**9622**	**9617**	1	2	2	3	4
16	9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	2	3	4
18	9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	**9397**	**9391**	**9385**	**9379**	**9373**	**9367**	**9361**	**9354**	**9348**	**9342**	1	2	3	4	5
21	9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	4	5	6
24	9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	**9063**	**9056**	**9048**	**9041**	**9033**	**9026**	**9018**	**9011**	**9003**	**8996**	1	3	4	5	6
26	8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	**8660**	**8652**	**8643**	**8634**	**8625**	**8616**	**8607**	**8599**	**8590**	**8581**	1	3	4	6	7
31	8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	**8192**	**8181**	**8171**	**8161**	**8151**	**8141**	**8131**	**8121**	**8111**	**8100**	2	3	5	7	8
36	8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	**7660**	**7649**	**7638**	**7627**	**7615**	**7604**	**7593**	**7581**	**7570**	**7559**	2	4	6	8	9
41	7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

TABLE IX. NATURAL COSINES—*Continued.*

Deg.	0′ =0·0	6′ =0·1	12′ =0·2	18′ =0·3	24′ =0·4	30′ =0·5	36′ =0·6	42′ =0·7	48′ =0·8	54′ =0·9	1	2	3	4	5
45	**7071**	**7059**	**7046**	**7034**	**7022**	**7009**	**6997**	**6984**	**6972**	**6959**	2	4	6	8	10
46	6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	**6428**	**6414**	**6401**	**6388**	**6374**	**6361**	**6347**	**6334**	**6320**	**6307**	2	4	7	9	11
51	6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	10	12
55	**5736**	**5721**	**5707**	**5693**	**5678**	**5664**	**5650**	**5635**	**5621**	**5606**	2	5	7	10	12
56	5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	**5000**	**4985**	**4970**	**4955**	**4939**	**4924**	**4909**	**4894**	**4879**	**4863**	3	5	8	10	13
61	4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	**4226**	**4210**	**4195**	**4179**	**4163**	**4147**	**4131**	**4115**	**4099**	**4083**	3	5	8	11	13
66	4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	**3420**	**3404**	**3387**	**3371**	**3355**	**3338**	**3322**	**3305**	**3289**	**3272**	3	5	8	11	14
71	3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	**2588**	**2571**	**2554**	**2538**	**2521**	**2504**	**2487**	**2470**	**2453**	**2436**	3	6	8	11	14
76	2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	8	11	14
78	2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	**1736**	**1719**	**1702**	**1685**	**1668**	**1650**	**1633**	**1616**	**1599**	**1582**	3	6	9	12	14
81	1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	**0872**	**0854**	**0837**	**0819**	**0802**	**0785**	**0767**	**0750**	**0732**	**0715**	3	6	9	12	15
86	0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

TABLE X. NATURAL TANGENTS.

Deg.	0′ =0·0	6′ =0·1	12′ =0·2	18′ =0·3	24′ =0·4	30′ =0·5	36′ =0·6	42′ =0·7	48′ =0·8	54′ =0·9	1	2	3	4	5
0	**0·0000**	**0017**	**0035**	**0052**	**0070**	**0087**	**0105**	**0122**	**0140**	**0157**	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	**0·0875**	**0892**	**0910**	**0928**	**0945**	**0963**	**0981**	**0998**	**1016**	**1033**	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	**0·1763**	**1781**	**1799**	**1817**	**1835**	**1853**	**1871**	**1890**	**1908**	**1926**	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	**0·2679**	**2698**	**2717**	**2736**	**2754**	**2773**	**2792**	**2811**	**2830**	**2849**	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	10	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	6	10	13	16
20	**0·3640**	**3659**	**3679**	**3699**	**3719**	**3739**	**3759**	**3779**	**3799**	**3819**	3	7	10	13	16
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	**0·4663**	**4684**	**4706**	**4727**	**4748**	**4770**	**4791**	**4813**	**4834**	**4856**	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	7	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	11	15	19
30	**0·5774**	**5797**	**5820**	**5844**	**5867**	**5890**	**5914**	**5938**	**5961**	**5985**	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	8	13	17	21
35	**0·7002**	**7028**	**7054**	**7080**	**7107**	**7133**	**7159**	**7186**	**7212**	**7239**	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	4	9	13	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	**0·8391**	**8421**	**8451**	**8481**	**8511**	**8541**	**8571**	**8601**	**8632**	**8662**	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	5	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

TABLE X. NATURAL TANGENTS—*Continued.*

Deg.	0′ =0·0	6′ =0·1	12′ =0·2	18′ =0·3	24′ =0·4	30′ =0·5	36′ =0·6	42′ =0·7	48′ =0·8	54′ =0·9	1	2	3	4	5
45	**1·0000**	**0035**	**0070**	**0105**	**0141**	**0176**	**0212**	**0247**	**0283**	**0319**	6	12	18	24	30
46	·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	24	31
47	·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	26	32
48	·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	**1·1918**	**1960**	**2002**	**2045**	**2088**	**2131**	**2174**	**2218**	**2261**	**2305**	7	14	21	29	36
51	·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	32	39
53	·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	17	25	33	41
54	·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	35	43
55	**1·4281**	**4335**	**4388**	**4442**	**4496**	**4550**	**4605**	**4659**	**4715**	**4770**	9	18	27	36	45
56	·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	9	19	29	38	48
57	·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	22	34	45	56
60	**1·7321**	**7391**	**7461**	**7532**	**7603**	**7675**	**7747**	**7820**	**7893**	**7966**					
61	·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728					
62	·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542					
63	·9626	9711	9797	9883	9970	$\overline{0057}$	$\overline{0145}$	$\overline{0233}$	$\overline{0323}$	$\overline{0413}$					
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348					
65	**2·1445**	**1543**	**1642**	**1742**	**1842**	**1943**	**2045**	**2148**	**2251**	**2355**					
66	·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445					
67	·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627					
68	·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916					
69	·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326					
70	**2·7475**	**7625**	**7776**	**7929**	**8083**	**8239**	**8397**	**8556**	**8716**	**8878**					
71	·9042	9208	9375	9544	9714	9887	$\overline{0061}$	$\overline{0237}$	$\overline{0415}$	$\overline{0595}$					
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506					
73	·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646					
74	·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062					
75	**3·7321**	**7583**	**7848**	**8118**	**8391**	**8667**	**8947**	**9232**	**9520**	**9812**					
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	·7046	7453	7867	8288	8716	9152	9594	$\overline{0045}$	$\overline{0504}$	$\overline{0970}$					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	**5·6713**	**7297**	**7894**	**8502**	**9124**	**9758**	$\overline{\mathbf{0405}}$	$\overline{\mathbf{1066}}$	$\overline{\mathbf{1742}}$	$\overline{\mathbf{2432}}$					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	$\overline{0264}$					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	$\overline{0285}$					
83	8·1443	2636	3863	5126	6427	7769	9152	$\overline{0579}$	$\overline{2052}$	$\overline{3572}$					
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	**11·43**	**11·66**	**11·91**	**12·16**	**12·43**	**12·71**	**13·00**	**13·30**	**13·62**	**13·95**					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

Differences should be calculated from 60° up to about 80° by Proportional Parts.

For minutes not given here a Seven-figure Table should be consulted.

TABLE XI.

RADIAN MEASURE OF ANGLES.

DEG.	0′	10′	20′	30′	40′	50′
0	**0·0000**	**0·0029**	**0·0058**	**0·0087**	**0·0116**	**0·0145**
1	0175	0204	0233	0262	0291	0320
2	0349	0378	0407	0436	0465	0495
3	0524	0553	0582	0611	0640	0669
4	0698	0727	0756	0785	0814	0844
5	**0·0873**	**0·0902**	**0·0931**	**0·0960**	**0·0989**	**0·1018**
6	1047	1076	1105	1134	1164	1193
7	1222	1251	1280	1309	1338	1367
8	1396	1425	1454	1484	1513	1542
9	1571	1600	1629	1658	1687	1716
10	**0·1745**	**0·1774**	**0·1804**	**0·1833**	**0·1862**	**0·1891**
11	1920	1949	1978	2007	2036	2065
12	2094	2123	2153	2182	2211	2240
13	2269	2298	2327	2356	2385	2414
14	2443	2473	2502	2531	2560	2589
15	**0·2618**	**0·2647**	**0·2676**	**0·2705**	**0·2734**	**0·2763**
16	2793	2822	2851	2880	2909	2938
17	2967	2996	3025	3054	3083	3113
18	3142	3171	3200	3229	3258	3287
19	3316	3345	3374	3403	3432	3462
20	**0·3491**	**0·3520**	**0·3549**	**0·3578**	**0·3607**	**0·3636**
21	3665	3694	3723	3752	3782	3811
22	3840	3869	3898	3927	3956	3985
23	4014	4043	4072	4102	4131	4160
24	4189	4218	4247	4276	4305	4334
25	**0·4363**	**0·4392**	**0·4422**	**0·4451**	**0·4480**	**0·4509**
26	4538	4567	4596	4625	4654	4683
27	4712	4741	4771	4800	4829	4858
28	4887	4916	4945	4974	5003	5032
29	5061	5091	5120	5149	5178	5207
30	**0·5236**	**0·5265**	**0·5294**	**0·5323**	**0·5352**	**0·5381**
31	5411	5440	5469	5498	5527	5556
32	5585	5614	5643	5672	5701	5730
33	5760	5789	5818	5847	5876	5905
34	5934	5963	5992	6021	6050	6080
35	**0·6109**	**0·6138**	**0·6167**	**0·6196**	**0·6225**	**0·6254**
36	6283	6312	6341	6370	6400	6429
37	6458	6487	6516	6545	6574	6603
38	6632	6661	6690	6720	6749	6778
39	6807	6836	6865	6894	6923	6952
40	**0·6981**	**0·7010**	**0·7039**	**0·7069**	**0·7098**	**0·7127**
41	7156	7185	7214	7243	7272	7301
42	7330	7359	7389	7418	7447	7476
43	7505	7534	7563	7592	7621	7650
44	7679	7709	7738	7767	7796	7825

Difference

for	is
1′	3
2′	6
3′	9
4′	12
5′	15
6′	17
7′	20
8′	23
9′	26

TABLE XI.

RADIAN MEASURE OF ANGLES—*Continued.*

Deg.	0′	10′	20′	30′	40′	50′
45	**0·7854**	**0·7883**	**0·7912**	**0·7941**	**0·7970**	**0·7999**
46	8029	8058	8087	8116	8145	8174
47	8203	8232	8261	8290	8319	8348
48	8378	8407	8436	8465	8494	8523
49	8552	8581	8610	8639	8668	8698
50	**0·8727**	**0·8756**	**0·8785**	**0·8814**	**0·8843**	**0·8872**
51	8901	8930	8959	8988	9018	9047
52	9076	9105	9134	9163	9192	9221
53	9250	9279	9308	9338	9367	9396
54	9425	9454	9483	9512	9541	9570
55	**0·9599**	**0·9628**	**0·9657**	**0·9687**	**0·9716**	**0·9745**
56	9774	9803	9832	9861	9890	9919
57	9948	9977	1·0007	1·0036	1·0065	1·0094
58	1·0123	1·0152	0181	0210	0239	0268
59	0297	0327	0356	0385	0414	0443
60	**1·0472**	**1·0501**	**1·0530**	**1·0559**	**1·0588**	**1·0617**
61	0647	0676	0705	0734	0763	0792
62	0821	0850	0879	0908	0937	0966
63	0996	1025	1054	1083	1112	1141
64	1170	1199	1228	1257	1286	1316
65	**1·1345**	**1·1374**	**1·1403**	**1·1432**	**1·1461**	**1·1490**
66	1519	1548	1577	1606	1636	1665
67	1694	1723	1752	1781	1810	1839
68	1868	1897	1926	1956	1985	2014
69	2043	2072	2101	2130	2159	2188
70	**1·2217**	**1·2246**	**1·2275**	**1·2305**	**1·2334**	**1·2363**
71	2392	2421	2450	2479	2508	2537
72	2566	2595	2625	2654	2683	2712
73	2741	2770	2799	2828	2857	2886
74	2915	2945	2974	3003	3032	3061
75	**1·3090**	**1·3119**	**1·3148**	**1·3177**	**1·3206**	**1·3235**
76	3265	3294	3323	3352	3381	3410
77	3439	3468	3497	3526	3555	3584
78	3614	3643	3672	3701	3730	3759
79	3788	3817	3846	3875	3904	3934
80	**1·3963**	**1·3992**	**1·4021**	**1·4050**	**1·4079**	**1·4108**
81	4137	4166	4195	4224	4254	4283
82	4312	4341	4370	4399	4428	4457
83	4486	4515	4544	4573	4603	4632
84	4661	4690	4719	4748	4777	4806
85	**1·4835**	**1·4864**	**1·4893**	**1·4923**	**1·4952**	**1·4981**
86	5010	5039	5068	5097	5126	5155
87	5184	5213	5243	5272	5301	5330
88	5359	5388	5417	5446	5475	5504
89	5533	5563	5592	5621	5650	5679

Difference

for	is
1′	3
2′	6
3′	9
4′	12
5′	15
6′	17
7′	20
8′	23
9′	26

TABLE XII.

THE EXPONENTIAL FUNCTION.

x	e^x	e^{-x}	x	e^x	e^{-x}	x	e^x	e^{-x}
0·0	**1·000**	**1·000**	**1·5**	**4·482**	**0·223**	**3·0**	**20·09**	**0·050**
0·1	1·105	0·905	1·6	4·953	0·202	3·5	33·12	0·030
0·2	1·221	0·819	1·7	5·474	0·183	4·0	54·60	0·018
0·3	1·350	0·741	1·8	6·050	0·165	4·5	90·02	0·011
0·4	1·492	0·670	1·9	6·686	0·150	5·0	148·4	0·007
0·5	**1·649**	**0·607**	**2·0**	**7·389**	**0·135**	**5·5**	**244·7**	**0·004**
0·6	1·822	0·549	2·1	8·166	0·122	6·0	403·4	0·002
0·7	2·014	0·497	2·2	9·025	0·111			
0·8	2·226	0·449	2·3	9·974	0·100			
0·9	2·460	0·407	2·4	11·023	0·091			
1·0	**2·718**	**0·368**	**2·5**	**12·18**	**0·082**			
1·1	3·004	0·333	2·6	13·46	0·074			
1·2	3·320	0·301	2·7	14·88	0·067			
1·3	3·669	0·273	2·8	16·44	0·061			
1·4	4·055	0·247	2·9	18·17	0·055			

TABLE XIII.

NUMBERS OFTEN USED IN CALCULATIONS.

π = Ratio of the circumference of a circle to its diameter.
e = Base of the Napierian Logarithms.

Number.	Logarithm.
$\pi = 3·14159$	$0·49715$
$1/\pi = 0·31831$	$\bar{1}·50285$
$\pi^2 = 9·86960$	$0·99430$
$1/\pi^2 = 0·10132$	$\bar{1}·00570$
$\sqrt{\pi} = 1·77245$	$0·24857$
$1/\sqrt{\pi} = 0·56419$	$\bar{1}·75143$
$e = 2·71828$	$0·43429$

To convert **Common into Napierian** Logarithms, multiply by 2·30259.
To convert **Napierian into Common** Logarithms, multiply by 0·43429.

1 radian = 57·29578 degrees.
1 centimetre = 0·3937 inch.
1 inch = 2·5400 centimetres.
1 square centimetre = 0·1550 square inch.
1 cubic centimetre = 0·0610 cubic inch.
1 kilogramme = 2·2046 pound.
1 pound = 453·6 grammes.
1 litre = 1·7598 pints.
= 61·0253 cubic inches.

ANSWERS.

Exercises. I. Page 8.

21. $AB=16$ (1·6 in.); $BC=12$ (1·2 in.); $ABCD=192$ (1·92 sq. in.).
22. $AB=16$ (1·6 in.); $BC=23$ (2·3 in.); $ABCD=368$ (3·68 sq. in.).
23. $AB=15$ (1·5 in.); $BC=18$ (1·8 in.); $ABCD=270$ (2·7 sq. in.).
24. $AB=12$ (1·2 in.); $BC=22$ (2·2 in.); $ABCD=264$ (2·64 sq. in.).
25. $AB=15$ (1·5 in.); $BC=28$ (2·8 in.); $ABCD=420$ (4·2 sq. in.).
26. $AB=20$ (2 in.); $BC=20$ (2 in.); $ABC=200$ (2 sq. in.).
27. $AB=18$ (1·8 in.); $BC=16$ (1·6 in.); $ABC=144$ (1·44 sq. in.).
28. $AB=11$ (1·1 in.); $BC=20$ (2 in.); $ABC=110$ (1·1 sq. in.).
29. $AB=29$ (2·9 in.); $BC=13$ (1·3 in.); $ABC=188·5$ (1·885 sq. in.).
30. $AB=30$ (3 in.); height$=25$ (2·5 in.); $ABC=375$ (3·75 sq. in.).
31. $AB=20$ (2 in.); height$=30$ (3 in.); $ABC=300$ (3 sq. in.).
32. $CA=24$ (2·4 in.); height$=26$ (2·6 in.); $ABC=312$ (3·12 sq. in.).
33. $CA=22$ (2·2 in.); height$=26$ (2·6 in.); $ABC=286$ (2·86 sq. in.).

Exercises. II. Page 14.

19. 2″, 3″, 6 sq. in.
20. 2·3″, 4·2″, 9·66 sq. in.
21. 4·2″, 2″, 8·4 sq. in.
22. 6″, 4″, 24 sq. in.
23. 1·24″, 2·62″, 3·25 sq. in.
24. 4″, 3·72″, 14·88 sq. in.
25. (0·9, 0·26); 9·5.
26. (−0·02, 0·84); 7·11.
27. (0·96, 0·10); 8·30.
28. (1·31, 0·10); 12·92.
29. 1·92. **30.** 2·53. **31.** 1·65. **32.** 1·88. **33.** 3·96. **34.** 5·97.

	Sine.	Cosine.	Tangent.
35.	0·423	0·906	0·466.
36.	0·500	0·866	0·577.
37.	0·574	0·819	0·700.
38.	0·819	0·574	1·43.
39.	0·866	0·500	1·73.
40.	0·906	0·423	2·14.
41.	0·906	−0·423	−2·14.
42.	0·866	−0·500	−1·73.
43.	0·819	−0·574	−1·43.
44.	0·574	−0·819	−0·700.
45.	0·500	−0·866	−0·577.
46.	0·423	−0·906	−0·466.

Exercises. III. Page 17.

1. 3·94. **2.** 3·94. **3.** 0·99. **4.** 3·49. **5.** 5·39. **6.** 7·7.

9. $AB=4{\cdot}12$; $BC=3{\cdot}16$; $CD=4$; $DA=2{\cdot}24$; $AC=3{\cdot}61$; $BD=5{\cdot}83$.
$AB=3{\cdot}64$; $BC=1{\cdot}81$; $CD=3{\cdot}79$; $DA=2{\cdot}04$; $AC=4{\cdot}33$; $BD=4{\cdot}03$.

12. (i) $(3, -2)$; (ii) $(-1, -3)$; (iii) $(-2, 1)$; (iv) $(2, -3)$.

13. (i) $(-3, 2)$; (ii) $(1, 3)$; (iii) $(2, -1)$; (iv) $(-2, 3)$.

14. (i) $(-3, -2)$; (ii) $(1, -3)$; (iii) $(2, 1)$; (iv) $(-2, -3)$.

Exercises. IV. Page 20.

16. A straight line parallel to the y-axis.
A straight line parallel to the x-axis.

17. In all cases the locus is a straight line; in (i), (v) parallel to the y-axis, in (iii) the y-axis itself; in (ii), (vi) parallel to the x-axis, in (iv) the x-axis itself.

Exercises. V. Page 28.

1. $(3, 2)$, $(-2, -2)$, $(8, 6)$.

2. $x=2$; $y=3$.

3. $x=-3$; $y=4$.

4. $x=-2$; $y=-3$.

5. $x=3$; $y=-2$.

6. $x=y=2{\cdot}5$.

7. $x=-2{\cdot}25$; $y=3{\cdot}5$.

8. $x=3{\cdot}33$; $y=-2{\cdot}67$.

9. $x=2{\cdot}8$; $y=3{\cdot}2$.

10. $x=3$; $y=88$.

11. $x=4$; $y=44$.

12. $x=-40$; $y=10$.

13. $x=32$; $y=5$.

14. $x=3{\cdot}41$; $y=0{\cdot}97$.

15. $x=38{\cdot}9$; $y=-3{\cdot}03$.

17. (i) $9x-10y+15=0$. (ii) $8x+7y=0$. (iii) $x+13y+46=0$.
(iv) $y=7$. (v) $x=2$.

18. $(-2, 1)$; $(1, -2)$; $(2, 3)$.

20. $x+y=2$.

21. (i) AC, $2x-3y=1$; BD, $3x+5y=4$; $(17/19, 5/19)$.
(ii) AC, $2{\cdot}8x-3{\cdot}3y+2{\cdot}83=0$; BD, $x+3{\cdot}9y=3{\cdot}27$; $(-0{\cdot}02, 0{\cdot}84)$.
(iii) AC, $12x-15y=10$; BD, $71x+105y=79$; $(0{\cdot}96, 0{\cdot}10)$.
(iv) AC, $1{\cdot}5x-1{\cdot}7y=1{\cdot}79$; BD, $3{\cdot}3x+2y=4{\cdot}52$; $(1{\cdot}31, 0{\cdot}10)$.

Exercises. VI. Page 32.

1. $A=\frac{1}{2}bh$.

3. $p=c/v$.

4. $E=aW+b$.

5. $y=-(bx+d)/(ax+c)$.

6. $(3{\cdot}15, 3{\cdot}89)$; $(-4{\cdot}91, -0{\cdot}95)$.

7. $(1{\cdot}48, 5{\cdot}95)$; $(-0{\cdot}68, 1{\cdot}65)$; $x^2+y^2-4x-6y+4=0$; $y=2x+3$.

8. (i) $(2, 0)$; $x^2-4x+4=0$. (ii) $(0, 0{\cdot}76)$; $(0, 5{\cdot}24)$; $y^2-6y+4=0$.

9. (i) $x^2+y^2+4x-6y=12$. (ii) $x^2+y^2-4x+6y=12$.
(iii) $2x^2+2y^2+6x+10y=55$. (iv) $x^2+y^2-4{\cdot}8x+4{\cdot}8y+5{\cdot}76=0$.

11. (i) $(-1, 2)$; 2. (ii) $(-3, -2)$; 3. (iii) $(-4, 6)$; 8. (iv) $(1{\cdot}5, 0{\cdot}5)$; 2.
14. (i) $(-2{\cdot}5, 3)$; $3{\cdot}91$. (ii) $(0, 0)$; $1{\cdot}414$ $(=\sqrt{2})$. (iii) $\left(\frac{1}{7}, -\frac{11}{14}\right)$; $4{\cdot}22$.

Exercises. VII. Page 37.

1. $3x-5y+14=0$; $\frac{3}{5}$.
2. $3x+2y=0$; $-\frac{3}{2}$.
3. $y=x-4$; 1.
4. $2x-5y+29=0$; $\frac{2}{5}$.
5. $2y=3x$.
6. $y+5x=17$.
7. $5x-3y+13=0$.
8. $5x+2y+3=0$.
9. $x+2y+12=0$.
10. $x-5y=21$.
11. $x+3y=15$.
12. $x-3y+9=0$.
13. $x+2y=11$.
14. $x-2y+5=0$.
15. $6x-5y+2=0$.
16. $5x+6y=39$.
19. $\frac{1}{3}$.
20. $\frac{8}{3}$.

Exercises. VIII. Page 45.

1. $1{\cdot}42''$; $7{\cdot}05$ lb.
2. $1{\cdot}99$.
3. $-40°$.
4. (i) 76; (ii) 53.
5. (i) 90; (ii) 54.
9. £$1{\cdot}93$; £$2{\cdot}64$.
10. $49{\cdot}58$; $44{\cdot}27$; $38{\cdot}65$; $28{\cdot}65$.
11. $7{\cdot}68$; $12{\cdot}43$; $14{\cdot}62$.
12. £1487.
13. 13*s.* 5*d.*; 28*s.* 4*d.*; 35*s.* 7*d.*
14. 11*s.*
15. £45. 10*s.*; £61.
16. £121; £229.
17. 9*s.* 6*d.*
18. 11.54 a.m.; $16{\cdot}8$ miles from A.
19. $4\frac{1}{2}$ hrs.
20. Once; after an hour.
21. Ten times; after $8{\cdot}6$, $17{\cdot}1$, $25{\cdot}7$, $34{\cdot}3$, $42{\cdot}9$, $51{\cdot}4$, 60, $68{\cdot}6$, $77{\cdot}1$, $85{\cdot}7$ minutes.
22. (i) $21{\cdot}8$ min. after 4. (ii) $5{\cdot}5$ and $38{\cdot}2$ min. after 4.
23. $11{\cdot}4$ min. after 3.
24. $1{\cdot}88$ days.
25. 30 min.
26. $17{\cdot}5$ min.
27. 1 lb. at 2*s.* 6*d.* to 2 lb. at 4*s.*
28. 3.

Exercises. IX. Page 55.

10. $y=4{\cdot}10-0{\cdot}41x$.
11. $y=1{\cdot}10x-0{\cdot}28$.
14. About 50. $d=0{\cdot}02W$.
15. About 95 lb.
16. $E=0{\cdot}056W+0{\cdot}46$; $F=3{\cdot}98W+40{\cdot}9$.
17. $E=0{\cdot}072W+0{\cdot}092$; $F=2{\cdot}71W+4{\cdot}74$.
18. $E=0{\cdot}0136W+0{\cdot}24$; $F=0{\cdot}156W+17{\cdot}9$.
19. (i) $F=0{\cdot}226W-0{\cdot}06$; (ii) $F=0{\cdot}056W+0{\cdot}27$.
20. $D=1{\cdot}091T$.
21. $D=4{\cdot}3l$.
22. $K=2{\cdot}833C+0{\cdot}92$.
29. $4{\cdot}17x-4{\cdot}17y=xy$.
30. $30y+65x=42xy$.
31. $576x-27y=20xy$.

Exercises. X. Page 68.

1. (i) $x=0,\ y=1$; (ii) $x=0,\ y=-1$;
(iii) $x=0,\ y=1$; (iv) $x=0,\ y=-1$.

3. (i) $x=0,\ y=10$; (ii) $x=0,\ y=-10$;
(iii) $x=0,\ y=10$; (iv) $x=0,\ y=-10$.

5. (i) $x=0,\ y=1/10$; (ii) $x=0,\ y=-1/10$;
(iii) $x=0,\ y=1/10$; (iv) $x=0,\ y=-1/10$.

7. $-0{\cdot}9$; $2{\cdot}23$; $3x^2-4x-6=0$.

8. $-0{\cdot}51$; $0{\cdot}78$; $40x^2-11x-16=0$.

9. $a=3{\cdot}23$.

10. $y=8x^2+9$.

11. $(-1,\ 3)$, $(2{\cdot}4,\ 6{\cdot}57)$, $(-3,\ 9)$.

12. (i) 1; (ii) 3; (iii) 5; (iv) 2·5; (v) 2·1; (vi) 2·01.

13. (i) $2+h$; (ii) $2a+h$; 2 and $2a$.

Exercises. XI. Page 76.

4. 6·71.

6. 0; 2; $x^4=8x$.

7. 0; 6·69; $x^4=300x$.

8. 0; 6·69; $x^4=300x$.

9. $-3{\cdot}29$; $-3{\cdot}00$; $2{\cdot}72$; $3{\cdot}57$; $x^4-20x^2-x+96=0$.

10. $2{\cdot}04$; $2{\cdot}76$; $81x^4-900x^2-272x+2900=0$.

11. $x=-0{\cdot}27$ or $0{\cdot}82$.

12. $x=-1{\cdot}31$ or $1{\cdot}83$.

13. $x=-1{\cdot}02$ or $0{\cdot}61$.

14. $x=-0{\cdot}56$ or $2{\cdot}30$.

15. $x=1{\cdot}22$ or $3{\cdot}98$.

16. $x=-1{\cdot}60$ or $2{\cdot}47$.

17. 14·1.

18. $y=3x^2+2$.

19. $y=16{\cdot}1x^2$.

20. $y=4{\cdot}4x^2+1{\cdot}6$.

21. $s=4{\cdot}4t^2+10$.

22. $t=3$; $x=300$.

23. $y=x^2/20-1/80$.

24. $V^2=67{\cdot}69\ D$.

Exercises. XII. Page 83.

1. $x=-1$; $y=-1$, min.

2. $x=1$; $y=1$, max.

3. $x=-2$; $y=-4$, min.

4. $x=2$; $y=4$, max.

5. $x=-1{\cdot}25$; $y=-6{\cdot}25$, min.

6. $x=1{\cdot}25$; $y=6{\cdot}25$, max.

7. (i) $-0{\cdot}39$, $3{\cdot}72$; (ii) $-0{\cdot}67$, $4{\cdot}00$.

8. 14·82 when $x=2{\cdot}83$; $-2{\cdot}14$, $7{\cdot}80$.

9. Min. -1 when $x=2$.

10. Min. -2 when $x=-0{\cdot}5$.

11. Max. 36 when $x=6$.

12. 324 sq. in.

13. $x=8$, $y=6$.

14. 180.

15. $v=-u^2+19u+7$.

16. $R=2{\cdot}5+10{\cdot}5t-2t^2$.

[A better result is $R=2{\cdot}68+10{\cdot}3t-1{\cdot}93t^2$, which is however obtained by a method that does not make use of the graph. The student will find that more than one equation can be obtained, in many cases, and that each will give results that agree fairly well with the data. It is not easy to decide which is the best.]

17. $R=25(1+0{\cdot}00388t+0{\cdot}000005t^2)$; $t=12$, $R=26{\cdot}18$; $t=33$, $R=28{\cdot}34$.

18. $e=240(1+0{\cdot}0124t-0{\cdot}000106t^2)$.

21. $x=-1{\cdot}445,\ y=-17{\cdot}91$; $x=1{\cdot}7960,\ y=-16{\cdot}77$. **20.** Min. $=-11$.
22. $x=2, y=2$; $x=-0{\cdot}443, y=0{\cdot}92$; $x=-0{\cdot}099,\ y=1{\cdot}79$; $x=2{\cdot}54, y=0{\cdot}63$.
23. A parabola. $t=3{\cdot}125,\ y=156{\cdot}25,\ x=1250$. $t=0$ and $6{\cdot}25$.
24. $t=3,\ x=-13,\ y=14,\ t=6{\cdot}74$ and $-0{\cdot}74$.

Exercises. XIII. Page 91.

1. $(2, -4)$; $x=2$; $y=-4$; $y+4=3(x-2)^2$.
2. $(0{\cdot}6, 18)$; $x=0{\cdot}6$; $y=18$; $y-18=-25(x-0{\cdot}6)^2$.
3. $(0{\cdot}7, -2{\cdot}15)$; $x=0{\cdot}7$; $y=-2{\cdot}15$; $y+2{\cdot}15=\frac{5}{3}(x-0{\cdot}7)^2$.
4. $\left(-\frac{11}{8}, \frac{249}{80}\right)$; $x=-\frac{11}{8}$; $y=\frac{249}{80}$; $y-\frac{249}{80}=-\frac{4}{5}\left(x+\frac{11}{8}\right)^2$.
5. $x-3=2(y-3)^2$; $(3, 3)$; $y=3$; $x=3$.
6. $x-16=-3(y-2)^2$; $(16, 2)$; $y=2$; $x=16$.
7. $x+3=0{\cdot}8(y-3)^2$; $(-3, 3)$; $y=3$; $x=-3$.
8. $x-3=-\frac{9}{7}\left(y-\frac{4}{3}\right)^2$; $\left(3, \frac{4}{3}\right)$; $y=\frac{4}{3}$; $x=3$.
9. (i) 18, (ii) 18·81, (iii) $18+8h+h^2$, (iv) a^2+2a+3;
(v) $a^2+2a+3+2(a+1)h+h^2$; (α) 0·81, (β) $8h+h^2$, (γ) $2(a+1)h+h^2$.
10. (i) 1, (ii) $4h-h^2$, (iii) -24, (iv) -11, (v) $-20h-4h^2$.
11. 7, 6·5, 6·1, 6·01, $6+h$; 6.
12. $-2,\ -1{\cdot}5,\ -1{\cdot}1,\ -1{\cdot}01,\ -(1+h)$; -1.
13. 1, 2, 2·8, 2·98, $3-2h$; 3.
14. $-9,\ -9{\cdot}5,\ -9{\cdot}9,\ -9{\cdot}99,\ -10+h$; -10.
15. 1, 0·5, 0·1, 0·01, h; 0. **16.** 0, 0·5, 0·9, 0·99, $1-h$; 1.
17. $-8,\ -6{\cdot}5,\ -5{\cdot}3,\ -5{\cdot}03,\ -(5+3h)$; -5.
18. $4-2a-h$; $4-2a$. **19.** $2au+b+ah$; $2au+b$.
20. $-44,\ -36,\ -29{\cdot}6,\ -28{\cdot}16,\ -28-16h$; -28 feet per second.
21. $100-32t_1-16h$; $100-32t_1$ feet per second.
22. $V-gt_1-\frac{1}{2}gh$; $V-gt_1$ feet per sec.
23. 400 and $(100-32t_1-16h)$ feet per sec.; 400 and $(100-32t_1)$ feet per sec.
24. $(36-18t_1-9h)$ feet per sec. per sec.

Exercises. XIV. Page 101.

1. $(2{\cdot}5, 2{\cdot}5)$; $(0{\cdot}83, 7{\cdot}5)$.
2. (i) 0·5, 2; $2x^2-5x+2=0$; (ii) 2·31, 0·76, $-0{\cdot}57$; $2x^3-5x^2+2=0$;
(iii) 0·85, 2·43; $2x^4-5x^3+2=0$. Only necessary in (ii).
6. $-2{\cdot}73,\ 0{\cdot}73$; $x^2+2x-2=0$. **8.** $Rd=364600$.
9. $bx=0{\cdot}5918$. **11.** $x^2y=4{\cdot}00$.
12. $xy=4{\cdot}75x-1{\cdot}27y$. **13.** $xy=7{\cdot}88x-5{\cdot}23y$.
14. $xy=8{\cdot}39x+2{\cdot}60y$. **15.** $xy+6{\cdot}88x+24{\cdot}38y=986{\cdot}8$.
16. $F=\dfrac{18{\cdot}3}{d^2}+13{\cdot}4$. **17.** $KT=992T-5475$.
18. $x=4$; $y=8$. **19.** $x=4$; least perimeter is 16″.
20. $x=4$; $y=6$. **21.** 9. **22.** Radius $=6''$; Sum $=9''$.

Exercises. XV. Page 114.

1. 1·08, 1·55, 1·87. **3.** 1·43. **4.** 3·17.

5. 1·162. **6.** 1·466. **7.** $-0·851$.

8. $-0·67$, 1·42, 5·25. **9.** $-0·916$, 0·392, 1·858.

10. $-0·367$, 1·864. **11.** $-1·577$, 0·449.

12. (i) Neither max. nor min.
(ii) Min. $-0·385$ when $x=0·577$. Max. 0·385 when $x=-0·577$.
(iii) Neither max. nor min.
(iv) Max. 24·63 when $x=2·31$. Min. $-24·63$ when $x=-2·31$.
Central symmetry.

13. Raise or lower the x-axis:
(i) No turning values.
(ii) Min. $-20·3$ at $x=1·29$. Max. $-11·7$ at $x=-1·29$.
(iii) No turning values.

14. (i) Min. 0 at $x=0$. Max. 0·148 at $x=-0·667$.
(ii) Max. 0 at $x=0$. Min. $-0·148$ at $x=0·667$.
(iii) Min. 0 at $x=0$. Max. 0·148 at $x=0·667$.
(iv) Max. 0 at $x=0$. Min. $-4·63$ at $x=1·67$.

15. Max. 1·19 at $x=0·33$. Max. $1·19R^3$ at $x=0·33R$.

16. $x=0·33R$; max. vol. of cone$=1·24R^3$. **17.** 12.

19. Max. 3·85 at $x=1·42$. Min. $-3·85$ at $x=2·58$.

20. (i) Max. -4 at $x=-2$. Min. 4 at $x=2$.
(ii) No turning values.
(iii) Min. 7 at $x=2$. Max. -9 at $x=-2$.
(iv) No turning values.

21. (i) Min. 3 at $x=2$. (ii) Max. -3 at $x=-2$.
(iii) Min. 5 at $x=2$.

23. (i) Min. 0 at $x=0$.
(ii) Max. 0·25 at $x=0·71$.
Min. 0 at $x=0$.
Max. 0·25 at $x=-0·71$.
(iii) Min. -11 at $x=-1$.
Max. -10 at $x=0$.
Min. -11 at $x=1$.

24. $-0·96$, 1·38.

25. 7, 4·75, 3·31, 3·0301, $3+3h+h^2$; 3.

26. 1, 1·75, 2·71, 2·9701, $3-3h+h^2$; 3.

27. 19, 15·25, 12·61, 12·0601, $12+6h+h^2$; 12.

28. 15, 15·75, 15·99, 15·9999, $16-h^2$; 16.

29. -45, $-38·25$, $-33·21$, $-32·1201$, $-(32+12h+h^2)$; -32.

30. 15, 8·125, 4·641, 4·060401, $4+6h+4h^2+h^3$; 4.

31. $-\frac{1}{2}$, $-\frac{2}{3}$, $-\frac{10}{11}$, $-\frac{100}{101}$, $-\frac{1}{1+h}$; -1.

32. $-\frac{3}{4}$, $-\frac{10}{9}$, $-\frac{210}{121}$, $-\frac{20100}{10201}$, $-\frac{2+h}{(1+h)^2}$; -2.

33. $-\frac{1}{a(a+h)}$; $-\frac{1}{a^2}$.

Exercises. XVI. Page 120.

2. 3·94 **3.** 3·64. **4.** 0·057, 1·468.

5. (i) 3·80. (ii) 4·73. **8.** 2·87. **9.** 1·95, $-2·47$.

10. 15·98 at $x=0·434$. **11.** $-0·16$ at $x=0·37$.

12. 0·1065, 0·1130, 0·1175, 0·1190.

13. 9, 4·324, 2·59, 2·3. **14.** 90, 43·24, 25·9, 22.

Exercises. XVII. Page 125.

11. 1·8045. **12.** 2·79. **13.** 9·56 when $x=1·59$.

14. $pv^{1·065}=482·9$. **15.** $pv^{1·404}=501·4$.

16, 17, 18. In each case the value of n is approximately 0·5.

19. The simplest approximation is, $th^{1·5}=\text{constant}=1·97$, though some of the values do not satisfy it very well.

20. $pv=158$, roughly; more nearly $pv^{1·05}=171$.

21. $v=7·94h^{\frac{1}{2}}$. **22.** $V=2·26l^{\frac{1}{2}}$.

23. $T=8·1S^{0·6}$. **24.** $32y=x^3$. **25.** $y^3=32000x$.

Exercises. XVIII. Page 129.

3. 0·37 when $x=1$. **4.** Symmetry about the y-axis.

6. (i) 1·924, $-1·373$; (ii) 1·377, $-0·679$; (iii) 0·877, 4·814; (iv) 0·807.

8. $\frac{e-1}{e}Q$. T is the number of seconds after joining up before the charge reaches the fraction $\frac{e-1}{e}$ of its final value.

10. $v=14·5e^{-0·46t}$, *or* $vt^{\frac{9}{4}}=53$.

Exercises. XIX. Page 143.

1. (i), (ii), 180°; (iii), (iv), 120°; (v), (vi), 90°; (vii), (viii), 72°.

2. Move the origin (i) to $\left(\frac{A}{n}, 0\right)$, (ii) to $\left(-\frac{A}{n}, 0\right)$.

3. New x-unit is equal to (i) 2, (ii) 3, (iii) $\frac{1}{2}$, (iv) $\frac{1}{3}$, (v) n in old scale.

5. Max. 5·12 at $x=134°\ 47'$. Min. $-5·12$ at $x=314°\ 47'$.

6. Max. 111·8 at $x=116°\ 34'$. Min. $-111·8$ at $x=296°\ 34'$.

7. Max. 44·64 at $x=48°\ 37'$. Min. $-58·91$ at $x=259°\ 55'$.

8. Max. $55{\cdot}73$ at $x=16^\circ\ 4'$. Min. $-55{\cdot}73$ at $x=91^\circ\ 56'$.
Max. -16 at $x=144^\circ$. Min. $-55{\cdot}73$ at $x=196^\circ\ 4'$.
Max. $55{\cdot}73$ at $x=271^\circ\ 56'$. Min. 16 at $x=324^\circ$.

9. Max. $22{\cdot}56$ at $x=28^\circ\ 32'$. Min. $12{\cdot}67$ at $x=65^\circ\ 2'$.
Max. 15 at $x=90^\circ$. Min. $12{\cdot}67$ at $x=114^\circ\ 58'$.
Max. $22{\cdot}56$ at $x=151^\circ\ 28'$. Min. $-22{\cdot}56$ at $x=208^\circ\ 32'$.
Max. $-12{\cdot}67$ at $x=245^\circ\ 2'$. Min. -15 at $x=270^\circ$.
Max. $-12{\cdot}67$ at $x=294^\circ\ 58'$. Min. $-22{\cdot}56$ at $x=331^\circ\ 28'$.

10. Max. $1{\cdot}41$ at $x=25^\circ\ 45'$. Min. $-0{\cdot}08$ at $x=66^\circ\ 3'$.
Max. $1{\cdot}93$ at $x=111^\circ\ 12'$. Min. $-0{\cdot}64$ at $x=160^\circ\ 55'$.
Max. $0{\cdot}64$ at $x=199^\circ\ 5'$. Min. $-1{\cdot}93$ at $x=248^\circ\ 48'$.
Max. $0{\cdot}08$ at $x=293^\circ\ 57'$. Min. $-1{\cdot}41$ at $x=334^\circ\ 15'$.

11. Max. $13{\cdot}94$ at $x=60^\circ\ 38'$. Min. $3{\cdot}99$ at $x=95^\circ\ 11'$.
Max. $6{\cdot}87$ at $x=118^\circ\ 45'$. Min. $5{\cdot}63$ at $x=136^\circ\ 41'$.
Max. $9{\cdot}65$ at $x=162^\circ\ 28'$. Min. $-13{\cdot}94$ at $x=240^\circ\ 38'$.
Max. $-3{\cdot}99$ at $x=275^\circ\ 11'$. Min. $-6{\cdot}87$ at $x=298^\circ\ 45'$.
Max. $-5{\cdot}63$ at $x=316^\circ\ 41'$. Min. $-9{\cdot}65$ at $x=342^\circ\ 28'$.

24. $y=100\sin x+60\sin(3x-60^\circ)$. **25.** $y=50\sin x+25\sin(5x+230^\circ)$.

26. $y=100\left\{\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x\right\}$.

27. (i) $31^\circ\ 1'$, $65^\circ\ 21'$. (ii) $207^\circ\ 54'$, $299^\circ\ 55'$.

28. (i) $4{\cdot}493$, $7{\cdot}725$. (ii) $1{\cdot}166$, $4{\cdot}604$.

29. (i) $2{\cdot}279$, $-2{\cdot}279$. (ii) $0{\cdot}739$. **30.** $1{\cdot}895$.

31. $0{\cdot}0147$, $0{\cdot}0150$, $0{\cdot}0150$, $0{\cdot}0151$, $0{\cdot}0151$.

32. $0{\cdot}0118$, $0{\cdot}0122$, $0{\cdot}0122$, $0{\cdot}0123$, $0{\cdot}0123$.

33. $-0{\cdot}0094$, $-0{\cdot}0090$, $-0{\cdot}0089$, $-0{\cdot}0088$, $-0{\cdot}0087$.

34. $0{\cdot}0246$, $0{\cdot}0238$, $0{\cdot}0235$, $0{\cdot}0234$, $0{\cdot}0233$.

35. $0{\cdot}0147$, $0{\cdot}0164$, $0{\cdot}0169$, $0{\cdot}0172$, $0{\cdot}0174$.

Exercises. XX. Page 152.

2. (i) $\frac{3}{5}$; (ii) $\frac{\sqrt{41}}{5}$.

3. (i) Axes 6, 3, eccentricity $\frac{\sqrt{3}}{2}$, centre (3, 0);

(ii) Axes 6, 3, eccentricity $\frac{\sqrt{5}}{2}$, centre (3, 0).

4. (i) $\frac{(x-2)^2}{2^2}+\frac{y^2}{6^2}=1$, axes 12, 4, eccentricity $\frac{2\sqrt{2}}{3}$, centre (2, 0);

(ii) $\frac{(x+2)^2}{2^2}-\frac{y^2}{6^2}=1$, axes 4, 12, eccentricity $\sqrt{10}$, centre $(-2, 0)$.

5. (i) $\dfrac{\left(x-\dfrac{A}{B}\right)^2}{\dfrac{A^2}{B^2}}+\dfrac{y^2}{\dfrac{A^2}{B}}=1$; (ii) $\dfrac{\left(x+\dfrac{A}{B}\right)^2}{\dfrac{A^2}{B^2}}-\dfrac{y^2}{\dfrac{A^2}{B}}=1$.

7. (i) $x=3$, $x=-3$, $y=3\sqrt{2}$, $y=-3\sqrt{2}$.
(ii) $x=\frac{3}{2}$, $x=-\frac{3}{2}$, none parallel to the x-axis (a hyperbola).

9. (i) $(3, 2)$; $8x-3y=18$. (ii) $(5, 4)$; $4x+5y=40$.

10. $\left(\dfrac{35+m\sqrt{45m^2-20}}{5+m^2}, \dfrac{-7m+\sqrt{45m^2-20}}{5+m^2}\right)$,
$\left(\dfrac{35-m\sqrt{45m^2-20}}{5+m^2}, \dfrac{-7m-\sqrt{45m^2-20}}{5+m^2}\right)$, $m=\pm\frac{2}{3}$.

11. (i) $c=\pm\sqrt{(16m^2+9)}$; (ii) $c=\pm\sqrt{(16m^2-9)}$;
(iii) $c=\pm\sqrt{(a^2m^2+b^2)}$; (iv) $c=\pm\sqrt{(a^2m^2-b^2)}$.

12. (i) $c=-m^2$; (ii) $c=2+m-\dfrac{m^2}{4}$; (iii) $c=\dfrac{a}{m}$.

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