# ELEMENTARY LESSONS IN ALGEBRA

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# ELEMENTARY LESSONS

IN

# ALGEBRA

*A TEXT-BOOK FOR GRAMMAR SCHOOLS*

BY

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AND

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# PREFACE.

It is believed that the time has come for the introduction of the study of Algebra into Grammar Schools. The elementary processes of Algebra follow so closely the fundamental processes of Arithmetic, that the teaching of both in the grammar grades is at once rational and feasible. New text-books, especially adapted to this purpose, are, therefore, necessitated. Such books should consist of series of elementary lessons, inculcating a thorough knowledge of algebraic processes and giving facility in the use of algebraic symbols.

*Elementary Lessons in Algebra* aims to fulfil such a demand. The Introductory Lessons set simply, but accurately, before the learner, the combinations of literal quantities into sums, differences, products, and quotients, with little reference to arithmetical processes, and without associating number values to the letters — often a source of confusion to the beginner, who, for the first time, meets with the use of letters as symbols. The fundamental processes are placed after Simple Equations; and Equations containing Two Unknown Quantities involving Elimination, follow Multiplication. This sequence is believed to be more logical than placing the fundamental processes after Equations of Two Unknown Quantities, Elimination being in that case a merely mechanical process to the pupil. The treatment of Factoring is particularly full, as its importance demands.

The development throughout is inductive, and is believed to be simple and logical.

# CONTENTS.

# ELEMENTARY LESSONS IN ALGEBRA.

## INTRODUCTORY LESSONS.

### LESSON I.

**1.** What is the sum of $a$ and $a$?

The sum of $a$ and $a$ equals $2a$; read *two a*. 2 is called the **Coefficient** of $a$.

NOTE. When a letter is written without a coefficient, as in $a + a$, the coefficient is understood to be 1. Thus, $a + a = 2a$ is equivalent to $1a + 1a = 2a$.

**2.** What is the sum of $a + a + a$?

The sum equals $3a$. What is the coefficient of $a$? $b + b + b = ?$ $2a + 3a = ?$ $a + 4a = ?$ $2b + b = ?$

**3.** $5a+3a=?$ $3b+5b=?$ $8c+10c=?$ $10y+8y=?$ $4x + 3x = ?$ $6a + 3a + 4a = ?$ $9a + 2a + 5a = ?$

**4.** $8b+4b+6b= ?$ $12x+3x+5x= ?$ $4c+6c+5c=?$ $8y + 4y + 9y + 3y = ?$ $4x + 6x + 5x + 10x = ?$

**5.** What is the value of $2a - a$?

SOLUTION. $2a - a = a.$

What is the value of $3b - b$?

SOLUTION. $3b - b = 2b.$

$3a - a = ?$ $4a - 2a = ?$ $5a - 3a = ?$ $6b - 2b = ?$

**6.** $5a - 4a = ?$ Name the coefficients in this problem. $10c - 8c = ?$ $12x - 7x = ?$ $14y - 3y = ?$ $11y - y = ?$

**7.** $2a + 3a - a = ?$

SOLUTION. $2a + 3a = 5a.$ $5a - a = 4a.$

$6x - 3x + 2x = ?$ $5y + 4y - 3y = ?$ $8x - 4x + 5x = ?$

**8.** $10a - 3a + 4a = ?$ $4a - a + 5a = ?$ $9a - 4a + 2a = ?$ $9a + 5a - 3a = ?$ $60a - 56a + 22a = ?$ $35c + c - 29c = ?$

**9.** $12a - 3a - 5a = ?$

SOLUTION. $12a - 3a = 9a.$ $9a - 5a = 4a.$

$8a - 2a - a = ?$ $10a - 6a - a = ?$ $7a - 4a - 3a = ?$

**10.** $16x - 5x - x = ?$ $14x - 7x - 2x = ?$ $18y - 3y - 9y = ?$ $17b - 10b - 5b = ?$ $9c - 2c + 4c = ?$ $21c - 14c - 5c = ?$

**11.** What is the sum of $a$ and $b$?

SOLUTION. The sum of two numbers is expressed by a third number; as $5 + 3 = 8$, $4 + 5 = 9$. But the sum of different letters cannot be expressed by a third letter; hence the sum of $a$ and $b$ is written $a + b$ because there is no simpler form.

**12.** What is the difference of $a$ and $b$?

SOLUTION. The difference of two letters cannot be expressed by a third letter. The difference of $a$ and $b$ is, therefore, expressed as $a - b$ when $a$ is the greater, or as $b - a$ when $b$ is the greater, because there is no simpler form.

**13.** Write the sums of the following: $a$ and $c$; $x$ and $y$; $5a$ and $3b$; $12y$, $3x$, and $2a$; $5a$, $3b$, and $7c$.

**14.** Write the difference of $x$ and $y$ when $x$ is the greater; when $y$ is the greater. Of $5a$ and $4b$ when $5a$ is the greater; when $4b$ is the greater.

15. $5a + 4b + 2b = ?$ *Ans.* The sum is $5a + 6b$.
$3x + 2y - x = ?$ *Ans.* The value is $2x + 2y$.
$5x + 2x - y = ?$ $3a + 3b + 2c = ?$ $10y + 2y + 4x - 3x = ?$

16. $12x - 3x + 7a = ?$ $15y + 3y + 9y - 2y = ?$
$10x - 2c + 4a = ?$ $9a + 10c - 4a - 3c = ?$
$18a + 21c - 4a = ?$ $6y - 2y + y = ?$

17. $19x - 13y - c + 20y = ?$ $15a + 13a - 4a + 2c = ?$
$7x + 4y - 3x = ?$ $21a + 7b - 3b + 8a = ?$

18. $4c + 3a - 9b - 2x = ?$ $16y - 12y - 3y - x = ?$
$18b - 14b + 6b = ?$ $4x + 2y + 5y - 3y = ?$

## Lesson II.

1. What is the sum of 2 and $x$? What is their product? The product is $2x$. What is 2?

2. What is the product of 3 and $x$? 4 and $b$? 10 and $y$?

3. $2 \times 2a = ?$ $3 \times 2b = ?$ $2x \times 5 = ?$ $3 \times 2 \times a = ?$

4. $6y \times 2 \times 4 = ?$ $2 \times 2y \times 3 = ?$ $2c \times 7 \times 2 = ?$

5. Write the sum of each set of factors in Nos. 3 and 4.

6. What is the product of $a$ and $b$?

Solution. $a \times b = ab$, read *a-b*.

Note. In algebra the product of unlike letters is denoted by writing them in succession. The order in which the letters stand does not alter the value of the product; *i.e.*, $ab = ba$; $abc = bac = acb = bca = cab = cba$.

7. Write the product, sum, and difference of $x$ and $y$; $b$ and $c$; $x$ and $3y$; $3b$ and 3.

**8.** $2a \times 4b = ?$ $6c \times x = ?$ $7y \times 3a = ?$ $4c \times 4y = ?$ Name the coefficients of the letters in these examples.

**9.** What is the product of $a$, $b$, and $c$?

SOLUTION. $a \times b \times c = abc$, read *a-b-c*.

**10.** $c \times x \times y = ?$ $a \times c \times x = ?$ $b \times x \times y = ?$ $2x \times y \times a = ?$

**11.** Write the sum of each set of factors in No. 10.

**12.** $2a \times c \times x = ?$ $2b \times 2c \times 2y = ?$ $4b \times 5x \times 2y = ?$

**13.** $a + a = ?$ $a \times a = ?$

SOLUTION. $a \times a = (aa) = a^2$, read *a square*.

NOTE. When a letter is used two or more times as a factor, the product is expressed by writing a small figure above and to the right of the letter to show the number of times the letter is used as a factor of the product. This figure is called an **Exponent**. When a letter is written without an exponent, the exponent is understood to be 1; thus, $a \times a = a^1 \times a^1 = a^2$.

When the exponent is 2, the product is called a **Square.**
When the exponent is 3, the product is called a **Cube.**

**14.** Write the sum and product of $b$ and $b$; $2y$ and $y$; $2x$ and $2x$; $3c$ and $2c$; $5a$ and $4a$; $7x$ and $7x$; $10b$ and $3b$.

**15.** $a \times a \times a = ?$ $a + a + a = ?$ $x \times x \times x = ?$
$2x \times x = ?$ $6c \times 3c \times 2c = ?$ $4b \times 5b \times b = ?$

**16.** Write the product of $2x$, $2x$, and $x$; $2a$ and $2a$; $4c$, $3c$, and $2c$. Write their sums.

**17.** Express the products and sums of the following:

$a$, $b$, $b$, and $a$. $b$, $c$, $c$, and $c$.
$x$, $y$, $x$, and 2. 4, $x$, $x$, $y$, and 2.

## LESSON III.

1. What are the factors of 4? Of 9? Of 27?

2. What are the factors of $2a$? *Ans.* 2 and $a$.

3. What are the factors of $3c$? Of $5x$? Of $7y$? Of $13b$?

4. Separate the following products into their prime factors: $4a$; $6b$; $10x$; $16a$; $12y$; $28c$; $27y$; $14a$.

5. What are the factors of $ab$? Of $ay$? Of $2cx$? Of $3by$?

6. Factor $4abc$; $6bxy$; $27cxy$; $14acx$; $16bcy$.

7. Factor $a^2$.

SOLUTION. $a^2 = aa = a \times a$.

8. Factor $a^2b$; $b^3$; $4c^2$; $16c^2x$; $5x^2y$; $27b^3$.

9. Divide 24 by 4.

SOLUTION. $24 \div 4 = \frac{24}{4} = 6$.

10. Divide $4a$ by 2.

SOLUTION. $4a \div 2 = \dfrac{4a}{2} = \dfrac{2 \times 2a}{2} = 2a$.

NOTE. The process of division is indicated in the form of a fraction, the dividend being the numerator, and the divisor the denominator. The process of dividing thus becomes equivalent to the cancellation of the equal factors in the numerator and denominator.

11. Indicate in fractional form the division of 27 by 3. Of $x$ by 2. Of $2a$ by $a$. Of 4 by $2c$. Of $2ac$ by $ay$.

12. $\dfrac{6b}{2} = ?$ $\dfrac{10x}{5} = ?$ $\dfrac{12y}{y} = ?$ $\dfrac{27cx}{9} = ?$ $\dfrac{bcx}{b} = ?$

13. $\frac{6b}{2b} = ?$

SOLUTION. $\frac{6b}{2b} = \frac{3 \times 2b}{2b} = 3.$

14. $\frac{14b}{7b} = ?$ $\frac{16ax}{4a} = ?$ $\frac{24cy}{xy} = ?$ $\frac{64bcx}{16bx} = ?$

15. $\frac{b^2}{b} = ?$ $\frac{b^3}{b} = ?$

SOLUTION. $\frac{b^2}{b} = \frac{bb}{b} = b.$ $\frac{b^3}{b} = \frac{bbb}{b} = b^2.$

16. $\frac{c^2}{c} = ?$ $\frac{4x^2}{2x} = ?$ $\frac{9y^2}{3y} = ?$ $\frac{27b^3}{3b} = ?$ $\frac{14a^2c}{7a} = ?$

17. $\frac{acx}{c^2} = ?$

SOLUTION. $\frac{acx}{c^2} = \frac{acx}{cc} = \frac{ax}{c}.$

18. $\frac{7ab}{21b} = ?$ $\frac{12a^3}{24a^2} = ?$ $\frac{16c^2x}{4cx} = ?$ $\frac{27b^2cy}{9bc^2y} = ?$

## LESSON IV.

1. A boy spends $b$ cents for a top and $x$ cents for string. How much money does he spend?

2. In a schoolroom there are $4c$ girls and $2c$ boys. How many pupils are there? How many more girls than boys? Compare the number of girls and boys.

3. Mr. A. had $4x$ acres and bought 20 acres more. How many acres has he now?

4. From a farm of $c$ acres, 50 acres were sold. How many acres were left unsold?

5. A yard of silk costs $x$ dollars. What is the cost of 3 yards? Of 6 yards? Of half a yard? Of $b$ yards?

6. From a roll of carpet containing $25a$ yards, $10a$ yards were sold. How many yards were left?

7. What will be the total cost of 4 yards of calico at $a$ cents a yard, and of 5 yards of gingham at $b$ cents a yard?

8. How much more expensive is silk, at $3y$ dollars a yard, than woolen goods, at $2x$ dollars a yard?

9. Land bought for $60 an acre was sold for $c$ dollars an acre. How much was gained on the acre?

10. I sold 16 cattle at a gain of $y$ dollars per head. How much money did I gain?

11. How many square feet are there in a room $b$ feet long and $c$ feet wide?

12. A square field is $b$ rods long. What is its area? What would be the cost of fencing the field at $c$ dollars a rod?

13. How many square feet of plaster are there in the walls and ceiling of a room $a$ feet long, $b$ feet wide, and $c$ feet high?

14. How many pencils at 5 cents each can be bought for $x$ cents?

15. What is the price per acre if $x$ acres of land cost $1000?

16. What is the interest of $b$ dollars for one year at 6%?

17. I bought a horse for $c$ dollars and sold him at a gain of 10%. What was the selling price?

## SIMPLE EQUATIONS.

### Lesson V.

The expressions $7+12=19$, $3a+2a=5a$, and $3a^2=12$ are called **Equations.** An equation is made up of two parts or values joined by the **Sign of Equality** $(=)$.

All written at the left of the sign of equality is known as the **First Member** of the equation; all written at the right, as the **Second Member** of the equation. In the first of the three equations given above, $7+12$ is the first member; 19 is the second member.

1. What is the first member of the second equation? Of the third?

2. Write several equations having $6a$ for the second member.

3. Supply the second member in the following:

$4a \times 5a = \quad . \qquad \frac{16cx}{4c} = \quad . \qquad \frac{20a^2}{5a} = \quad . \qquad 5a-3a= \quad .$

Note. In the equation $6a+4b=12$, the first member contains two quantities to be added. In the equation $7a-4a=3a$, the first member has two quantities, the one to be subtracted from the other. These quantities are called **Terms.** Thus, in the equation $7a+4a-6a=5a$, the first member has three terms, *i.e.*, $7a$ and $4a$ to be added, $6a$ to be subtracted. The second member has but a single term.

4. Name the terms in each of the following equations:

$4a+4b-3c=15. \qquad 6ax-\frac{b}{2}=10. \qquad 5b^2+5=30.$

Which terms have coefficients? Which exponents?

In algebra, *any letter* used may represent *any number value.* The pupil must remember, however, that no letter has a fixed value; thus $a$ in one problem may stand for the number 2, but in the next problem may represent 4 or $\frac{1}{2}$, etc.

5. When $a$ equals 2, what is the number value of $a+a$?

SOLUTION. $a + a = 2a =$ *two* 2's $= 4$.

6. Let $a = 3$. $a + 4 = ?$ $2a + 3a = ?$ $4a - 2a = ?$

7. Let $b = 4$. $4b + 3b - 2b = ?$ $6b - 6 + b = ?$ $14 - 2b = ?$ $24 - 4b = ?$

8. Let $c = 6$. $c \times 4 = ?$ $5 \times 2c = ?$ $c^2 = ?$ $4c^2 - 6c = ?$ $36 - c^2 + 2c - 8 = ?$

9. Let $x = 10$. $\frac{x}{2} = ?$ $\frac{5x}{25} = ?$ $\frac{50}{x} = ?$ $\frac{2x^2}{2x} = ?$ $\frac{5x^2}{5x} + \frac{3x}{5} = ?$

In the following examples, let $a = 2$, $b = 1$, $c = 4$, $x = 3$, $y = 5$, and find the number value:

10. $a + b = ?$ $c - b = ?$ $x + a - c = ?$ $y + 2a - 2x = ?$ $2c - x + y = ?$

11. $bc = ?$ $5 \times 3ax = ?$ $2a \times x = ?$ $axy - 5x = ?$ $2acy - cx - 20 = ?$

12. $c^2 - a^2 = ?$ $\frac{3a}{x} = ?$ $\frac{4c}{8} = ?$ $\frac{2abx}{c} = ?$ $\frac{5c}{y} = ?$

$x^2 - b^2 = ?$ $\frac{4y}{c} = ?$ $\frac{bx}{1} = ?$ $\frac{abcy}{10} = ?$ $\frac{7y}{b} = ?$

13. $\frac{2bc}{a} - x = ?$ $\frac{2xy}{c} + ax = ?$ $\frac{c^2}{a} - \frac{ay}{2} = ?$ $c^2 - b^2 + \frac{2x}{3} = ?$

## Lesson VI.

1. Stewart has twice as many books as Charles. They have together 21 books. How many has each?

Solution. Since a letter has no fixed numerical value, but may stand for any number, we may represent the number of books Charles has by $x$. Then since Stewart has twice as many as Charles, his number will be represented by $2x$.

Since they both have 21 books, we may say:

$$2x + x = 21.$$

From this we find, $3x = 21$;

$x = 7$, Charles's number;

$2x = 14$, Stewart's number.

Note. It is important to notice that when a value is given to a letter, this value does not change throughout the problem.

2. The sum of two numbers is 40. The greater is three times the less. Find the numbers.

3. The sum of two numbers is 75. The greater is four times the less. Find the numbers.

4. A man paid $30 for a hat and coat. The coat cost five times as much as the hat. Find the cost of each.

5. Frank and Mary picked 36 quarts of cherries. Frank picked twice as many as Mary. How many quarts did each pick?

6. A and B together have a capital of $10,000. A's share is four times B's. What is the share of each?

7. In a flock of 441 sheep there are 20 white sheep to every black one. Find the number of each.

8. The greater of two numbers is seven times the less. Their sum is 256. Find the numbers.

9. A certain number added to twice itself equals 510. What is the number?

10. Mr. B. is three times as old as his son John. The sum of their ages is 60 years. Find the age of each.

## Lesson VII.

1. Frank, Fred, and Harry together caught 36 fish. Fred caught twice as many as Harry. Frank caught three times as many as Fred. How many did each catch?

Solution. Let $x =$ the number Harry caught;
then $2x =$ the number Fred caught;
and $6x =$ the number Frank caught.

$$x + 2x + 6x = 36;$$
$$9x = 36.$$
$x = 4$, the number Harry caught;
$2x = 8$, the number Fred caught;
$6x = 24$, the number Frank caught.

2. A farmer owns three farms. The second is worth twice as much as the first, the third is worth four times as much as the second. The three together are worth \$7700. What is the value of each?

3. Three boys together caught 24 fish. John caught twice as many fish as Harry. Frank caught as many as both John and Harry caught. Find the number caught by each.

4. Divide 400 into three parts, such that the second is four times the first, and the third five times the second.

5. Divide 60 into three parts, such that the second is five times the first, and the third equal to the sum of the other two.

6. An estate of $1000 was divided among two boys and one girl. Each boy received the same amount; the girl received three times as much as either of the boys. What was the share of each?

7. A man doubled his capital every 5 years for 15 years. He then had $25,600. What was his capital at first?

8. A merchant bought 600 bushels of grain. There were twice as many bushels of wheat as of rye, and the number of bushels of oats equaled the sum of the number of bushels of rye and of wheat. Find the number of bushels of each.

9. A merchant earned three times as much on dry goods as on shoes, and twice as much on shoes as on notions. His entire profits were $486. Find his profit on each.

## Lesson VIII.

1. A's age is to B's as 2 is to 3. The sum of their ages is 50 years. Find the age of each.

Suggestion. Let $2x =$ A's age;
then $3x =$ B's age.

2. Divide 75 into two parts, which shall be to each other as 4 is to 1.

3. A and B are in partnership with a capital of $8500. A's capital is to B's as 9 is to 8. Find the share of each.

4. A and B together have 500 sheep. A's number is $\frac{2}{3}$ of B's. How many has each?

Suggestion. Let $2x =$ A's number;
then $3x =$ B's number.

5. Separate 615 into three parts, which shall be to each other as 3, 5, and 7.

6. The sum of three numbers is 527. The first is to the second as 2 is to 3, the third is four times the second. Find the value of each.

7. In a library of 540 volumes, one half of the volumes are works of fiction. The remainder are books of history or of science. The proportion of the books of history to the books of science is as 4 is to 5. Find the number of books of each kind.

8. John is $\frac{5}{8}$ as old as Mary, and Henry is twice as old as Mary. The sum of their ages is 28 years. How old is each?

9. Divide 72 into three parts, such that the first is to the second as 4 is to 5, and the third equals the sum of the other two parts.

10. Three books together cost \$4.80. The first cost $\frac{4}{9}$ as much as the second; the third, $\frac{1}{3}$ as much as the second. Find the cost of each.

11. Three men received \$20, each getting a share as follows: B received $\frac{3}{5}$ as much as A, and C $\frac{2}{3}$ as much as B. How much did each receive?

12. A, B, and C have \$95. $\frac{6}{5}$ of A's money equals B's. $\frac{4}{3}$ of B's money equals C's. How much money has each?

## LESSON IX.

1. The sum of two numbers is 20. The larger number is 8 more than twice the smaller. Find the numbers.

SOLUTION. Let $x =$ the smaller number;
then $2x + 8 =$ the larger number.

$$x + 2x + 8 = 20.$$
$$3x + 8 = 20.$$

Observe that in order to find the value of $x$, the first member of the equation must consist of the term containing $x$ only, and the second member must contain *all* the numbers, and nothing but numbers.

$$3x + 8 = 20.$$

Subtract 8 from each member.

$$3x + 8 - 8 = 20 - 8;$$
$$3x = 12;$$
$$x = 4, \text{ the smaller number;}$$
$$2x + 8 = 16, \text{ the larger number.}$$

NOTE. Since the two members of an equation are equal, we may *add to* or *subtract from* each member any number, and the members will still be equal.

2. A horse and carriage are worth \$480. The horse is worth \$150 more than the carriage. What is the value of each?

3. The sum of three numbers is 92. The first is to the second as 2 is to 5. Three times the second is 4 less than the third. Find the numbers.

4. The yield of an orchard was 70 bushels of fruit. Three times the number of bushels of apples is 6 more than the number of bushels of pears. Find the number of bushels of each.

SOLUTION. Let $x =$ number of bushels of apples;
then $3x - 6 =$ number of bushels of pears.

$$x + 3x - 6 = 70;$$
$$4x - 6 = 70.$$

Add 6 to each member of the above equation. (See Note above.)

$$4x - 6 + 6 = 70 + 6.$$
$$4x = 76.$$
$$x = 19.$$

5. A horse, a cow, and a sheep together cost $106. The cow cost sixteen times as much as the sheep, and the horse cost $40 more than the cow. What was the cost of each?

6. The sum of two numbers is 73. Their difference is 13. Find the numbers.

7. Two men received $25 for 10 days' work. The first received 50 cents per day more than the second. Find the daily wages of each.

## REDUCTION OF SIMPLE EQUATIONS.

### LESSON X.

1. Find the value of $x$ in the following equation:

$$7x + 5 = 9 - 4x. \qquad (1)$$

SOLUTION. Remember that in order to do this, all the terms containing $x$, the unknown quantity, must be in the first member, and the other terms must be in the second member.

Subtract 5 from each member of the equation.

$$7x + 5 - 5 = 9 - 4x - 5. \qquad (2)$$

$$7x = 9 - 4x - 5. \qquad (3)$$

Add $4x$ to each member of equation (3).

$$7x + 4x = 9 - 4x - 5 + 4x. \qquad (4)$$

$$7x + 4x = 9 - 5. \qquad (5)$$

$$11x = 4.$$

$$x = \tfrac{4}{11}.$$

Compare equation (5) with equation (1).

$$7x + 4x = 9 - 5. \qquad (5)$$

$$7x + 5 = 9 - 4x. \qquad (1)$$

The term 5 in equation (1) appears as $-5$ in equation (5). The term $-4x$ in equation (1) appears as $+4x$ in equation (5).

**Note.** A term may, therefore, be transferred from one member of an equation to the other, if at the same time the sign of the term is changed.

This operation is called **Transposition.**

2. $8x-8=7x+20$. Find the value of $x$.

Solution. Transposing the 8 to the second member and the $7x$ to the first member, we have

$$8x-7x=20+8.$$

$$x=28.$$

Find the value of $x$ in the following equations:

3. $7x+5=26$.
4. $13x-4=35$.
5. $2x-2=6+x$.
6. $3x-4=x+6$.
7. $3x-5=x+13$.
8. $4+8x=3x+39$.
9. $5+6x=21-2x$.
10. $7-3x=42-10x$.
11. $8x-25=14-5x$.
12. $3x-3+10=14+2x$.

In the following, unite terms as far as possible before transposing, and find the value of $x$:

13. $3x-6=x+14-4$.
14. $4x+13+38=10x-3x$.
15. $7+x-9=3x+10-8x$.
16. $5x+6+3x=12-4x+18$.
17. $5x-3+4x-2=6x+16$.

## Lesson XI.

1. $\frac{4x}{3} = x + 5.$ Find the value of $x$.

Suggestion. In solving this equation, the first step is to change the fraction to an integer.

Since the members of an equation are equal, if both members are multiplied by the same number, we shall still have an equation.

Multiply both members by 3.

$$\frac{3 \times 4x}{3} = 3x + 15. \qquad 4x = 3x + 15.$$

$$x = ?$$

2. $\frac{4x}{5} + 7 = \frac{3x}{10} + 49.$ Find the value of $x$.

Suggestion. To change these fractions to integers, multiply both members by 10. The equation may then be reduced to

$$8x + 70 = 3x + 490.$$

$$x = ?$$

3. $7x + \frac{3x}{5} = \frac{x}{4} + 9.$ Find the value of $x$.

Suggestion. Multiply both members by 20.

Note. From these three problems, observe that the fractions of an equation are changed to integers by multiplying both members by the *Least Common Multiple* of the denominators of the fractions. This operation is called **Clearing the Equation of Fractions.**

Find the value of $x$ in the following equations:

4. $\frac{x}{4} + \frac{2x}{3} = 11.$

5. $\frac{5x}{6} + \frac{x}{4} = 52.$

6. $\frac{x}{3}-\frac{x}{4}=2.$

7. $\frac{x}{2}+\frac{x}{3}=5.$

8. $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10.$

9. $\frac{2x}{3}+\frac{x}{4}-\frac{4x}{5}=1.$

NOTE. From Ex. 9, observe that the value of $x$ may be fractional.

10. $\frac{x}{4}+\frac{x}{8}-\frac{x}{6}=\frac{5}{12}.$

11. $\frac{2x}{2}-4=\frac{x}{3}+6-8.$

SUGGESTION. In Ex. 11, transpose $-4$, and unite the terms before clearing of fractions.

12. $\frac{x}{2}-2=5-\frac{x}{5}.$

13. $x+\frac{x}{2}=18-\frac{3x}{4}.$

14. $\frac{x}{2}-14=\frac{x}{4}-\frac{x}{3}.$

15. $\frac{4x}{3}-\frac{3x}{4}=\frac{3}{5}.$

16. $\frac{3x}{4}-\frac{7}{10}=\frac{2x}{5}.$

17. $\frac{x}{6}+\frac{x}{5}-\frac{3x}{4}=5-\frac{3x}{10}-4.$

SUGGESTION. The equation in Ex. 17 reduces to the form
$$-5x=60. \qquad (1)$$
Transpose both terms,
$$-60=5x; \qquad (2)$$
from which we obtain $x=-12.$

NOTE. Observe that the value of $x$ may be a number preceded by a minus sign.

In Ex. 17, the same result is obtained, by reducing equation (1) to the form $5x=-60$,

whence $x=-12.$

NOTE. From the above, observe that the *signs* of all the terms of an equation may be *changed*, and the members still remain equal. This operation is of frequent service.

18. $\frac{2x}{5}-2-\frac{6x}{3}=\frac{4}{5}-3x.$

20. $\frac{2x}{3}+\frac{1}{6}-7=\frac{x}{2}.$

19. $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}.$

21. $\frac{x}{3}+\frac{5x}{9}+15=\frac{3x}{6}+18\frac{1}{2}.$

## LESSON XII.

Find the value of $x$ in the following equations:

1. $\frac{2x}{1}-1=\frac{4}{3}+\frac{x}{12}.$

4. $6x-23=\frac{3x}{2}+7+2x.$

2. $x+\frac{3}{5}=\frac{2x}{8}+\frac{x}{2}.$

5. $\frac{13}{3}+3x-\frac{x}{2}=\frac{3x}{4}-\frac{5x}{6}-6.$

3. $\frac{5x}{8}+\frac{1}{4}=\frac{11}{6}+\frac{7x}{12}.$

6. $\frac{x+2}{3}-\frac{1}{6}=\frac{x}{4}+2.$

SUGGESTION In multiplying a quantity made up of two or more terms, we must multiply each term. Thus, multiplying the fraction $\frac{x+2}{3}$ in Ex. 6 by 12, we have

$$\frac{12x+24}{3}, \text{ or } 4x+8.$$

7. $\frac{5x+1}{3}+x=\frac{6x}{5}+\frac{4}{3}.$

11. $\frac{x-2}{2}=\frac{x-1}{7}+\frac{x+2}{5}.$

8. $\frac{x}{4}+\frac{x}{10}+\frac{x}{8}=19.$

12. $\frac{7x+2}{10}-12=\frac{3x+3}{5}-\frac{x}{2}.$

9. $2+\frac{4x-5}{3}=\frac{x+3}{2}+\frac{3x}{5}.$

13. $\frac{3x-4}{2}=\frac{6x-5}{8}+\frac{3x-1}{16}.$

10. $\frac{x-12}{x}+5=\frac{3}{4}.$

14. $\frac{2x-3}{x}-\frac{1}{4}=\frac{4x+6}{3x}.$

15. $x - 7 + \frac{x-7}{2} + \frac{3x-21}{4} = 1\frac{3}{4}$.

16. $\frac{3}{2} - \frac{5x}{3} = \frac{5-3x}{4} + \frac{3-5x}{3}$.

17. $\frac{x+3}{2} + \frac{x+4}{3} + \frac{x+5}{4} = 16$.

## Lesson XIII.

1. A man bought a horse and a cow for $150. Twice the cost of the horse is three times the cost of the cow. Find the cost of each.

Suggestion. Let $x =$ the cost of the horse;

then $\frac{2x}{3} =$ the cost of the cow.

2. The sum of two numbers is 50, and $\frac{1}{3}$ of the greater equals $\frac{1}{2}$ of the less. Find the numbers.

Suggestion. Let $x =$ the greater;

then $50 - x =$ the less.

3. If $\frac{2}{3}$ of A's age is added to $\frac{5}{4}$ of his age, the result will equal 69. What is his age?

4. Mr. Clark is 32 years older than his son John, and $\frac{4}{5}$ of Mr. Clark's age added to $\frac{1}{4}$ of John's age equals 34 years. Find the age of each.

5. Frank is three times as old as Harry. Ten years hence, Frank will be twice as old as Harry. Find the age of each.

Suggestion. Let $x =$ Harry's age;

then $x + 10 =$ his age 10 years hence.

6. Stephen is $\frac{5}{4}$ as old as Lucy. Ten years ago he was $\frac{3}{2}$ as old as Lucy. Find the age of each.

7. One number is three times another, and $\frac{2}{5}$ of the greater and 5 is 14 more than $\frac{3}{4}$ of the less. Find the numbers.

8. The sum of $\frac{1}{4}$, $\frac{2}{3}$, and $\frac{1}{9}$ of a number is 4 more than $\frac{11}{12}$ of the number. Find the number.

9. The sum of two numbers is 42. The quotient arising from dividing the greater by the less is 5. Find the numbers.

Suggestion. Let $x =$ the less number.

10. The difference between two numbers is 24. The greater divided by the less equals 5. Find the numbers.

11. Four times the number of hours until noon equals $\frac{4}{7}$ of the number of hours since midnight. What time is it?

Suggestion. Let $x =$ number of hours until noon;

then $12 - x =$ number of hours since midnight.

## Lesson XIV.

1. A number is represented by three digits. The digit in hundreds' place is twice that in tens' place, and that in tens' place is three times that in units' place. The sum of the digits is 10. Find the number.

2. A number is represented by two digits. The digit in units' place is three times the digit in tens' place. The entire number and 6 equals five times the digit in units' place. Find the number.

Suggestion. Let $x =$ the digit in tens' place;

then $3x =$ the digit in units' place,

and $10x + 3x =$ the entire number.

3. In a number of two digits, the digit in tens' place is to the digit in units' place as 3 is to 2. The sum of the digits multiplied by 6 equals 6 less than the entire number. Find the number.

4. In the number representing the soldiers of a certain regiment, the number of hundreds is three times the number of units, and the number of units is one half the number of tens. Add 27 to the entire number, divide the sum by the sum of the digits, and the quotient equals 55. How many soldiers in the regiment?

5. A can do a piece of work in 3 days. B can do the same work in 5 days. How long will it take them to do it working together?

SUGGESTION. Let $x =$ the number of days it will take them when working together;

then $\frac{1}{x} =$ the part of the work both will do in one day.

A can do $\frac{1}{3}$ of the work in one day.

B can do $\frac{1}{5}$ of the work in one day.

Therefore $\frac{1}{3}+\frac{1}{5}=\frac{1}{x}$.

6. If A can do a piece of work in 6 days, and B can do it in 10 days, in how many days can both do it?

7. A, B, and C can do a piece of work in 5, 6, and 4 days respectively. Find the time they will need when working together.

8. B and C together do a piece of work in 7 days. B can do the same work in 10 days. How long will it take C to do the work alone?

9. The number of hours since noon divided by the number of hours until midnight equals 8. What time is it?

# ADDITION.

## Lesson XV.

**Before beginning Addition a rapid review of Lesson I. is suggested.**

1. $5x - 3x + 2x = ?$

Solution. $5x - 3x = 2x$; $2x + 2x = 4x$.

Or, $5x + 2x = 7x$; $7x - 3x = 4x$.

Observe that the result is the same in whatever order the quantities are added.

2. $5x - 3x + 5x - 2x = ?$

Solution. Of the four terms in the first member of this equation two are to be added, two to be subtracted. Performing the addition, we have

$$5x + 5x = 10x.$$

From this we may subtract first $3x$ and then $2x$, or we may combine $3x$ and $2x$, making $5x$, and subtract this from $10x$. Thus,

$$10x - 3x = 7x;\ 7x - 2x = 5x.$$

Or, $$10x - 5x = 5x.$$

Note. Quantities whose signs indicate addition are called **Positive Quantities**, and quantities whose signs indicate subtraction are called **Negative Quantities**.

3.
$$\begin{array}{r} 7x \\ -4x \\ 6x \\ 4x \\ \underline{-3x} \\ 10x \end{array}$$

Solution. The positive quantities combined equal $17x$; the negative quantities combined equal $7x$.

Subtracting the negative quantities from the positive, we obtain $10x$.

This result is called the *Algebraic Sum*.

**Note.** Observe that finding the sum in algebra includes both the process of addition and of subtraction as designated in arithmetic. Thus the **Algebraic Sum** is the result of addition and subtraction, according to the sign, of the coefficients of like literal quantities.

Find the sums of the following columns:

| 4. | 5. | 6. | 7. |
|---|---|---|---|
| $8x$ | $12y$ | $17ac$ | $21ab^2$ |
| $4x$ | $-7y$ | $-3ac$ | $14ab^2$ |
| $-6x$ | $4y$ | $7ac$ | $-12ab^2$ |
| $2x$ | $-6y$ | $-14ac$ | $-7ab^2$ |

| 8. | 9. | 10. |
|---|---|---|
| $16xy^3$ | $22a^2b^2c$ | $14c^2x$ |
| $-4xy^3$ | $-11a^2b^2c$ | $15c^2x$ |
| $15xy^3$ | $14a^2b^2c$ | $-9c^2x$ |
| $-14xy^3$ | $-9a^2b^2c$ | $-25c^2x$ |

| 11. | 12. | 13. |
|---|---|---|
| $3a^2b^2$ | $4axy$ | $9xyz$ |
| $17a^2b^2$ | $-5axy$ | $-5xyz$ |
| $-5a^2b^2$ | $3axy$ | $4xyz$ |
| $-4a^2b^2$ | $-4axy$ | $8xyz$ |
| $-6a^2b^2$ | $8axy$ | $-10xyz$ |
| $-2a^2b^2$ | $-6axy$ | $-6xyz$ |

## Lesson XVI.

1. $10x - 5x + 2x - 12x = ?$

Solution. The positive quantities combined equal $12x$; the negative quantities combined equal $17x$. Since negative quantities are quantities whose signs indicate subtraction, we have $17x$ to subtract from $12x$. From $12x$, $12x$ only can be taken, leaving $5x$ to be subtracted. Since there remains nothing from which to take the $5x$, we write $5x$ as a negative quantity, thus: $-5x$.

**Note.** Observe that an *algebraic sum* may be a *negative quantity.*

Find the algebraic sum of the following columns:

| 2. | 3. | 4. | 5. | 6. |
|---|---|---|---|---|
| $11x$ | $17y$ | $19ac$ | $17b^2x$ | $26ab^2c$ |
| $-12x$ | $4y$ | $-4ac$ | $-15b^2x$ | $-13ab^2c$ |
| $-5x$ | $-20y$ | $5ac$ | $3b^2x$ | $22ab^2c$ |
| $4x$ | $-5y$ | $-12ac$ | $-6b^2x$ | $-20ab^2c$ |

**Note.** Examine the terms in any one column given above, and observe that every term has the same number of literal factors. Thus each term in column 6 has the literal factors $a$, $b$, $b$, $c$. Such terms are called **Similar Terms.** Similar terms may have different signs and coefficients.

In the addition of algebraic quantities of more than one term, arrange the similar terms in columns.

7. Find the algebraic sum of:

$$4a+b,\ 4b-2a,\ 3a-6b,\ -5a-3b,\ a+b.$$

$$\begin{array}{r} 4a+\ b \\ -2a+4b \\ 3a-6b \\ -5a-3b \\ a+\ b \\ \hline a+2b \end{array}$$

Solution. In arranging the similar terms in columns, each term must be given its proper sign. When a term has no sign before it, it is understood to be a positive quantity.

The algebraic sum of the first column is $a$; the algebraic sum of the second column is $2b$. Since these are quantities to be added, we connect them by the sign $+$.

Find the algebraic sums of the following:

8. $5a-2b,\ 3a+4b,\ 3b-4a,\ 3b-2a.$

9. $5b-2x+c,\ 4x-2b+2c,\ 3b+2x-c.$

10. $2a+3b+6y,\ 2b-2a+y,\ a-4b-6y,\ a+b+y.$

11. $2a^2+4b-2c,\ 3c-2b,\ 3c+b-a^2,\ a^2+c.$

## Lesson XVII.

1. What is the algebraic sum of:

$7a - 5b + 6c,\ 6b - 7c - 5a,\ 3a - 4b - c,\ a - b + c?$

$$\begin{array}{r} 7a - 5b + 6c \\ -5a + 6b - 7c \\ 3a - 4b - \phantom{7}c \\ a - \phantom{4}b + \phantom{7}c \\ \hline 6a - 4b - \phantom{7}c \end{array}$$

Solution. The algebraic sums of the second and third columns are found to be negative quantities and must be connected by the sign $-$.

Find the sums of the following:

2. $8c - 7b + 6a,\ 4b - 7a,\ 7b - 10a + 4c,\ 3a - 6c.$
3. $7a - 2b,\ a^2 - 3b,\ 5a + 3b,\ 4a^2 + 5a,\ 6b - 3a.$

Suggestion. Observe that in arranging the similar terms in columns, the terms containing $a^2$ are not similar to those containing $a$. For the term $a^2$ contains the factor $a$ twice, and $a$ contains this factor but once. Similar terms must have *like literal quantities* with *like exponents*.

Find the sums of the following:

4. $7b^2 - 2a + b,\ 6b + 4a - 3b^2,\ 7a - 10b,\ b - 3b^2 - 3a.$
5. $6c^2 - 4a + 3c,\ 4a - 3c - 3c^2,\ a + c + c^2,\ c^2 - a - c.$
6. $\frac{1}{4}x + \frac{2}{3}y + \frac{1}{2}c,\ \frac{1}{3}y + c + \frac{3}{4}x,\ \frac{1}{2}c + x + y.$
7. $4ac + 2b^2 - 3x + 4b^2 - 6ac + 4x + 3ac - 8b^2 = ?$

Suggestion. Combining the similar terms, we obtain $ac - 2b^2 + x$. This result is called the *simplest form* of the above algebraic expression.

Express the following in their simplest forms:

8. $6x - 3y + 4a - 5x + x - 10a - 4y + 2a + 6y.$
9. $7a^2c - 8cx + 4y^2 - 3a^2c + 4y^2 + 10cx - 8y^2 + a^2c.$
10. $24c - \frac{1}{4}ax + \frac{3}{4}by + \frac{1}{4}ax + \frac{1}{4}by - 20c + 5ax + by.$

## SUBTRACTION.

### LESSON XVIII.

1. $9ax-7ax=?$ $19b^2+4b^2-3c=?$ $15c-4y-8c=?$

Name the differences at sight:

| 2. | 3. | 4. | 5. |
|---|---|---|---|
| $9c^2$ | $18x^3$ | $28bcy$ | $21c^2xy^2$ |
| $4c^2$ | $6x^3$ | $14bcy$ | $7c^2xy^2$ |

NOTE. Observe that the sum of the **subtrahend** and the **difference**, or **remainder**, must equal the **minuend.**

6. From $16c+21x+7y$ subtract $7x+3y+8c$.

SOLUTION.

$$\begin{array}{r} 16c+21x+7y \\ 8c+\ \ 7x+3y \\ \hline 8c+14x+4y \end{array}$$

NOTE. As in addition, first arrange the similar terms in column form. Connect the differences of the several columns by the proper signs.

7. From $15b^2+7x+14a$ subtract $3x+7a+10b^2$.

8. From $17bc^2+5xy+7a$ subtract $10bc^2+a+xy$.

9. From $12bc^2+7b^2c+3xy$ subtract $6b^2c+3xy+12bc^2$.

10. From $8ac+7b$ subtract $4ac+9b$.

$$\begin{array}{r} 8ac+7b \\ 4ac+9b \\ \hline 4ac-2b \end{array}$$

SOLUTION. In column 2 we have $9b$ to be subtracted from $7b$. But $7b$ only can be taken from $7b$, leaving $2b$ to be subtracted. Therefore we connect this difference to that of column 1 by the minus sign. Observe that the algebraic sum of $9b$ and $-2b$ equals $7b$.

11. From $9ac+7x+6y$ subtract $6ac+8x+3y$.

12. From $5bc+18y+a^2$ subtract $3a^2+20y+bc$.

13. From $10c^2 + 7b + 5x$ subtract $8b + 5c^2 + 6x$.

14. From $6x^2 + 2y$ subtract $4x^2 + 2b$.

$$\begin{array}{l} 6x^2 + 2y \\ \underline{4x^2 \qquad + 2b} \\ 2x^2 + 2y - 2b \end{array}$$

SOLUTION. As there is nothing to subtract from $2y$, it is written as a part of the remainder. Since there is no similar term from which to subtract $2b$, it is written as a *negative* quantity in the remainder.

15. From $6x^2y + 7b$ subtract $3x^2y + 3a$.

16. From $7x^2 + 6c$ subtract $5x^2 + 8c + 2b$.

17. From $16x + 5a^2y + 7c$ subtract $4x + a + 5a^2y$.

## LESSON XIX.

The algebraic sum of $8a$ and $-3a$ equals $5a$. Therefore the difference, when $-3a$ is subtracted from $5a$, must equal $8a$. For the sum of the subtrahend, $-3a$, and the difference, $8a$, equals $5a$, the minuend.

The algebraic sum of $-8a$ and $3a$ equals $-5a$. Therefore the difference, when $3a$ is subtracted from $-5a$, must equal $-8a$.

The algebraic sum of $-8a$ and $-3a$ equals $-11a$. Therefore the difference, when $-3a$ is subtracted from $-11a$, must equal $-8a$.

These subtractions are shown as follows:

| | | | |
|---|---|---|---|
| Minuend, | $5a$ | $-5a$ | $-11a$ |
| Subtrahend, | $-3a$ | $3a$ | $-3a$ |
| Difference, | $8a$ | $-8a$ | $-8a$ |

Observe that in each problem the algebraic sum of the subtrahend and difference equals the minuend.

NOTE. In the foregoing problems, the same result would be obtained *by changing* the *signs* of the *terms* in the *subtrahend* and *taking the algebraic sum.*

1. From $8ab + 7c + 3x$ subtract $5ab - 3c + 7x$.

| | | |
|---|---|---|
| SOLUTION. | Minuend, | $8ab + 7c + 3x$ |
| | Subtrahend, | $\pm 5ab \mp 3c \pm 7x$ |
| | Difference, | $3ab + 10c - 4x$ |

Changing the signs of the terms as indicated, we have to find in the first column the algebraic sum of $8ab$ and $-5ab$, which equals $3ab$; in the second column we have to find the algebraic sum of $7c$ and $3c$, which equals $10c$; in the third column we have to find the algebraic sum of $3x$ and $-7x$, which equals $-4x$.

| | | |
|---|---|---|
| PROOF. | Subtrahend, | $5ab - 3c + 7x$ |
| | Difference, | $3ab + 10c - 4x$ |
| | Minuend, | $8ab + 7c + 3x$ |

NOTE. Observe that the algebraic sum of the subtrahend with signs unchanged, and the difference found above, equals the minuend.

2. From $6x^2 + 2c + 9a$ subtract $3x^2 - 3c + 7a$.

3. From $9b + 8x + 4y$ subtract $7b - 12x - 3y$.

4. From $19c + 9y + 29a$ subtract $-a - c - y$.

5. From $3bc + 4y + ax$ subtract $2y - bc + ax$.

6. From $14c + 7a + 21xy$ subtract $6a - 7c - 7xy$.

7. From $5bx$ subtract $2bx + 6c + y$.

8. From $ax^2 + 3ay - 4b$ subtract $2ay - 4ax^2 + 5b$.

9. From $xy - 2ab + y^2$ subtract $3xy + 5y^2$.

10. From $3ax + 2by - 7ab$ subtract $2ax - 4by + x$.

11. From $2a^2y + 3xy - 20$ subtract $15 + 2xy$.

12. From $30y - 16b + 4a$ subtract $20y - 3b$.

13. From $3x^2 - 2x - 2$ subtract $-2x^2 - 2 - 2x$.

## Lesson XX.

1. From $7c - 4x - 6y$ subtract $5c + 2x + 3y$.

**Solution.**

| | |
|---|---|
| Minuend, | $7c - 4x - 6y$ |
| Subtrahend, | $\pm 5c \pm 2x \pm 3y$ |
| Difference, | $2c - 6x - 9y$ |

**Proof.**

| | |
|---|---|
| Subtrahend, | $5c + 2x + 3y$ |
| Difference, | $2c - 6x - 9y$ |
| Minuend, | $7c - 4x - 6y$ |

2. From $4a - 2bc + x$ subtract $a + bc - 2x$.

3. From $14b - 10x + 3y$ subtract $5x - 4b - 4y$.

4. From $6c^2 + 7c - 15y$ subtract $3c^2 - 3c + 7y$.

5. From $17bc^2 - ax$ subtract $c + 2ax - 3bc^2$.

6. From $15c - 12b - 32a$ subtract $5c - 18b - 8a$.

**Solution.**

| | |
|---|---|
| Minuend, | $15c - 12b - 32a$ |
| Subtrahend, | $\pm 5c \mp 18b \mp 8a$ |
| Difference, | $10c + 6b - 24a$ |

**Proof.**

| | |
|---|---|
| Subtrahend, | $5c - 18b - 8a$ |
| Difference, | $10c + 6b - 24a$ |
| Minuend, | $15c - 12b - 32a$ |

7. From $10x^2 - 7x - 4b$ subtract $2x^2 - 3b - 8x$.

8. From $22a + 10by - 3c$ subtract $2a - 2by - 2c$.

9. From $7y - 24a - a^2$ subtract $4y - 12a - 8a^2$.

10. From $\frac{1}{2}a - \frac{3}{4}c - \frac{2}{3}y$ subtract $\frac{1}{4}a - \frac{7}{4}c - y$.

11. Find the sum of $3b+2c$, $a+7c-6b$, $4b-8c+4a$, and $-4a$.

12. From the above sum subtract $b+c-a$.

13. From the sum of $14a-2b^2$, $3c+9b^2$, $4c-7a$, and $a+b^2-c$, subtract the sum of $4a+2c$, $b^2-2a-c$, and $3a-6c+5b^2$.

14. Find the simplest form of:

$$8x+3a-7y+3x-4a+2y-6x-x+3a+8y.$$

15. From the simplest form of the above expression, subtract the sum of:

$$x-4a+7y,\ a-3y,\ 4x+a,\ 3a-4x-y.$$

## Lesson XXI.

*Problems in Addition and Subtraction.*

1. From the sum of $2a^2-4b+c$, $3c+6b-a^2$, and $3a^2-b+c$, subtract the sum of $3b-4a^2$, $c+2a^2+b$, $b-c$, and $3a^2$.

2. Simplify: $4x+2a-3x+8b-3a-a+4x-10b+2a+x$.

3. From $5a-3b-2x+b+5x-4a+2a+b$ subtract $2b-4a+3x+2a-5b-3x+5x+a$.

Suggestion. Simplify the expressions before subtracting.

4. Find the sum of $4x-2by$, $3c+4by-3x$, $3by-6c+x$, $c-x$.

5. From $\frac{1}{2}by+\frac{1}{4}c-\frac{1}{3}x$ subtract $\frac{1}{4}by-\frac{3}{4}c-\frac{4}{3}x$.

6. From $\frac{1}{3}c-\frac{2}{3}b+\frac{1}{4}y$ subtract $\frac{5}{4}y+\frac{1}{3}b-\frac{2}{3}c$.

7. From $a^2 + b^2$ subtract $b^2 - a^2$.

8. From $4ab$ subtract $2ab - 2x + c$.

9. From $8cx$ subtract $3a - 4b + 2cx$.

10. From $a + 2b$ subtract $2a - b - x$.

11. Simplify $\frac{1}{4}x - \frac{1}{3}b - \frac{2}{3}y + b - 2y + 1\frac{3}{4}x + \frac{1}{3}b + 2\frac{1}{3}y$.

12. From the sum of $5ac + 7by$, $4x - 3ac - by$, $by - 5x + ac$, and $2x$, subtract the sum of $ac - by + x$, $5by - 3ac$, $4x - 2ac + by$, and $x + ac$.

13. $5x - 2x + \frac{1}{4}x + \frac{3}{2}x - 6x + 3x - \frac{3}{4}x + 2x = 3$. Find the value of $x$.

14. $2y + 1\frac{1}{4}y - \frac{3}{4}y + 7y - \frac{1}{2}y + 6y - \frac{2}{3}y + 6y + \frac{5}{3}y = 23$. Find the value of $y$.

## MULTIPLICATION.

### Lesson XXII.

1. What is the product of $3a$ and $4b$? $4c$ and 2? $5x$ and $y$? $4a$ and $2x$? 5, $2b$, and $3y$? $2a$, $3b$, and $2c$? $3a$ and $2a$? $4c$ and $8c$?

2. What is the product of $5a^2bc$ and $2ab^2c^2$?

Solution. The factor $a$ occurs twice in the **multiplicand** and once in the **multiplier**; the factors $b$ and $c$ occur once each in the multiplicand and twice each in the multiplier. Therefore the product of the literal factors equals $aaabbbccc$ or $a^3b^3c^3$.

| | |
|---|---|
| Multiplicand, | $5a^2bc$ |
| Multiplier, | $2ab^2c^2$ |
| Product, | $10a^3b^3c^3$ |

NOTE. Observe that the product of any like literal factors is found by taking the *sum* of the exponents of the factors in the *multiplicand* and *multiplier* for the exponent of the literal quantity in the *product*.

Find the products of the following:

3. $\begin{array}{l} 8axy \\ \underline{3ax^2y} \end{array}$ 4. $\begin{array}{l} 15b^2cy \\ \underline{3b^2c^2y^2} \end{array}$ 5. $\begin{array}{l} 7a^2cb \\ \underline{4a^2b^2c^2} \end{array}$ 6. $\begin{array}{l} 4c^3xy^4 \\ \underline{2x^4y} \end{array}$

7. What is the product of $4ac + 3x$ and $4cx$?

SOLUTION: Multiplicand, $4ac + 3x$
Multiplier, $4cx$
Product, $16ac^2x + 12cx^2$

NOTE. In algebra, the term to the left, or first term, of the multiplicand is first multiplied.

Find the products of the following:

8. $\begin{array}{l} 5a^2b + 2c \\ \underline{2bc} \end{array}$ 9. $\begin{array}{l} 10cy + 4bx \\ \underline{5bc} \end{array}$ 10. $\begin{array}{l} 7c^2 + 6ax \\ \underline{3ac} \end{array}$

11. $\begin{array}{l} 6bc + 2xy + 3a \\ \underline{4ab} \end{array}$ 13. $\begin{array}{l} 5a^2c + 3bx + 2y \\ \underline{3b^2xy} \end{array}$

12. $\begin{array}{l} 12a^2b + 6c + 5ax \\ \underline{5a^2c} \end{array}$ 14. $\begin{array}{l} 10abc + 4x + 5axy \\ \underline{4a^2bx^2} \end{array}$

## LESSON XXIII.

1. What is the product of $-5a^2b$ and $3a$?

SOLUTION. Since $-5a^2b$ indicates that $5a^2b$ is to be subtracted, multiplying $-5a^2b$ is equivalent to subtracting $5a^2b$ $3a$ times, or to subtracting the product of $5a^2b$ and $3a$ once. Therefore we write, $-15a^3b$.

2. What is the product of $5a^2b$ and $-3a$?

Solution. Since $-3a$ is a negative quantity, multiplying $5a^2b$ by $-3a$ is equivalent to subtracting $5a^2b$ $3a$ times, or to subtracting the product of $5a^2b$ and $3a$ once. Therefore we write, $-15a^2b$.

3. What is the product of $-5a^2b$ and $-3a$?

Solution. Multiplying $-5a^2b$ by $-3a$ is equivalent to subtracting $-5a^2b$ $3a$ times. But since in subtraction we change the signs of the subtrahend, subtracting $-5a^2b$ $3a$ times becomes equivalent to adding $5a^2b$ $3a$ times, or to adding the product of $5a^2b$ and $3a$ once. Therefore we write, $15a^3b$.

The four operations in multiplication are shown thus:

| | | | |
|---|---|---|---|
| $5a^2b$ | $-5a^2b$ | $5a^2b$ | $-5a^2b$ |
| $3a$ | $3a$ | $-3a$ | $-3a$ |
| $15a^3b$ | $-15a^3b$ | $-15a^3b$ | $15a^3b$ |

Note. Observe that when the *signs* in the multiplicand and multiplier are *alike*, the *product* is a *positive quantity;* when the *signs* in the multiplicand and multiplier are *unlike*, the *product* is a *negative quantity*

4. What is the product of $2x - y - b$ and $3x$?

Solution.

$$\begin{array}{l} 2x - y - b \\ 3x \\ \hline 6x^2 - 3xy - 3bx. \end{array}$$

Find the products of:

5. $$\begin{array}{l} 7ab - 8c + y \\ 3acy \\ \hline \end{array}$$

6. $$\begin{array}{l} 4c - 5x + 2y^2 \\ 4cy \\ \hline \end{array}$$

7. $3\,cx + 4\,by + a^2$
   $-5\,bx$

8. $4\,bx + 5\,c^2y + 10\,a^3$
   $-6\,c$

9. $5\,c^2 - 4\,bx + 5\,y$
   $-9\,by$

10. $7\,a^2y - 8\,b^2x - 4\,c^2$
    $-6\,b$

11. $4\,b^2cx - 4\,x - 8\,y^2$
    $-2\,cx$

12. $6\,b - 15\,x^2 + 4\,c$
    $10\,a$

## Lesson XXIV.

When an algebraic expression, consisting of two or more terms, is used as a whole, *i.e.* as a factor or as a divisor, etc., it is inclosed in a parenthesis, thus $(2\,a + 3\,b)$.

As two literal quantities are written in succession to indicate their product, so two parentheses are written in succession to indicate the product of the quantities inclosed in the parentheses. Thus $(2\,a + 3\,b)(a + 2\,b)$ indicates that the sum of $2\,a$ and $3\,b$ is to be multiplied by the sum of $a$ and $2\,b$.

Performing the multiplication indicated is called **Expanding the Expression.**

1. Expand $(2\,a + 3\,b)(a + 2\,b)$.

| | | |
|---|---|---|
| Solution. | Multiplicand, | $2\,a + 3\,b$ |
| | Multiplier, | $a + 2\,b$ |
| | 1st Partial Product, | $2\,a^2 + 3\,ab$ |
| | 2d Partial Product, | $4\,ab + 6\,b^2$ |
| | Product, | $2\,a^2 + 7\,ab + 6\,b^2$ |

Beginning at the left, the *first partial product* is obtained by multiplying the multiplicand by $a$, the first term of the multiplier; the *second partial product* is obtained by multiplying the multiplicand by $2\,b$, the second term of the multiplier.

Arranging the similar terms of the partial products in column form, and taking the algebraic sum, we obtain the final **Product.**

Expand the following:

2. $(a+b)(a+b)$.
3. $(3x+2y)(x+2y)$.
4. $(4c+2x)(2c+2x)$.
5. $(3y+2a)(y+3a)$.
6. $(2b+3x)(3b+2x)$.
7. $(2a+x)(2a+x)$.
8. Expand $(2a+3b)(2a-3b)$.

SOLUTION.

$$\begin{array}{lll} 2a & +3b & \\ 2a & -3b & \\ \hline 4a^2 & +6ab & \\ & -6ab & -9b^2 \\ \hline 4a^2 & & -9b^2 \end{array}$$

Expand:

9. $(x+y)(x-y)$.
10. $(2a+3x)(2a-3x)$.
11. $(2c+y)(2c-y)$.
12. $(3x+2y)(3x-2y)$.
13. $(x+2)(2x-3)$.
14. $(3a-x)(2a+x)$.
15. $(x+2a)(3x-4a)$.

## LESSON XXV.

1. Expand $(a^2+ab+b^2)(a-b)$.

SOLUTION.

$$\begin{array}{llll} a^2 & +ab & +b^2 & \\ a & -b & & \\ \hline a^3 & +a^2b & +ab^2 & \\ & -a^2b & -ab^2 & -b^3 \\ \hline a^3 & & & -b^3 \end{array}$$

Expand the following:

2. $(x^2 + xy + y^2)(x - y)$.

3. $(4c^2 + 2cy + y^2)(2c - y)$.

4. $(25a^2 + 10ay + 4y^2)(5a - 2y)$.

5. $(9b^2 + 6b + 4)(3b - 2)$.

6. Expand $(a^2 - ab + b^2)(a + b)$.

SOLUTION.

$$\begin{array}{l} a^2 - ab + b^2 \\ a + b \\ \hline a^3 - a^2b + ab^2 \\ \qquad a^2b - ab^2 + b^3 \\ \hline a^3 \qquad\qquad\quad + b^3 \end{array}$$

Expand the following:

7. $(x^2 - xy + y^2)(x + y)$.

8. $(4c^2 - 2cy + y^2)(2c + y)$.

9. $(9y^2 - 6ay + 4a^2)(3y + 2a)$

10. $(25b^2 - 10b + 4)(5b + 2)$.

There are certain products which occur so frequently as to deserve special attention.

Expand: 11. $(a + b)(a + b)$

12. $(a - b)(a - b)$.

13. $(a + b)(a - b)$.

EXPLANATION. Since in Ex. 11 the sum of $a$ and $b$ is to be used twice as a factor, we may indicate the product as follows: $(a + b)^2$. This is read, *the square of the sum of a and b*. For the same reason, Ex. 12 is written $(a - b)^2$, and read, *the square of the difference of a and b*. Ex. 13 is read, *the product of the sum and difference of a and b*.

Let the pupil verify the following results:

(11) $(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab.$

(12) $(a-b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab.$

(13) $(a+b)(a-b) = a^2 - b^2.$

From these equations, we reach the following conclusions, called **Theorems.**

THEOREM 1. The square of the sum of two quantities, $(a+b)^2$, equals the sum of their squares, $(a^2+b^2)$, and twice their product, $(+2ab)$.

THEOREM 2. The square of the difference of two quantities, $(a-b)^2$, equals the sum of their squares, $(a^2+b^2)$, less twice their product, $(-2ab)$.

THEOREM 3. The product of the sum and difference of two quantities, $(a+b)(a-b)$, equals the difference of their squares, $(a^2-b^2)$.

In accordance with the above theorems, expand:

14. $(a+c)^2.$
15. $(c-a)^2.$
16. $(4a+x)^2.$
17. $(x-4a)^2.$
18. $(2b+c)(2b-c).$
19. $(5x-2c)(5x+2c).$
20. $(5a+5b)^2.$
21. $(4a-5x)(4a+5x).$
22. $(3y+9a)^2.$
23. $(6x+7b)(6x-7b).$
24. $(\frac{1}{2}x+\frac{1}{3}y)^2.$
25. $(\frac{1}{2}a+\frac{1}{3}x)(\frac{1}{2}a-\frac{1}{3}x).$

## LESSON XXVI.

Expand:

1. $(a^2+ab+b^2)(a-b).$
2. $(a^2-ab+b^2)(a+b).$

Note carefully the following facts:

In Ex. 1, the first factor consists of the sum of the squares of $a$ and $b$, plus the product of $a$ and $b$; the second factor is the difference of $a$ and $b$.

In Ex. 2, the first factor consists of the sum of the squares of $a$ and $b$, less the product of $a$ and $b$; the second factor is the sum of $a$ and $b$.

Verify the following results:

$$(1)\quad (a^2 + ab + b^2)(a - b) = a^3 - b^3.$$

$$(2)\quad (a^2 - ab + b^2)(a + b) = a^3 + b^3.$$

From the above facts and equations we derive the following theorems:

THEOREM 4. The sum of the squares and product of two quantities, multiplied by the difference of the quantities, equals the difference of their cubes.

THEOREM 5. The sum of the squares of two quantities less the product of the quantities, multiplied by the sum of the quantities, equals the sum of their cubes.

Expand:

3. $(x^2 + xy + y^2)(x - y)$.
4. $(x^2 - xy + y^2)(x + y)$.
5. $(4a^2 + 6ab + 9b^2)(2a - 3b)$.
6. $(4x - 3y)^2$.
7. $(3a + 5c)(3a - 5c)$.
8. $(4a^2 + 6y)^2$.

NOTE. When a single term is written directly before a parenthesis, it indicates that all the terms within the parenthesis are to be multiplied by the term without. Thus $3a(2b - 3c)$ equals $6ab - 9ac$.

Expand the following:

9. $2a(3b + 3c + a)$.
10. $(a + 10bx)^2$.
11. $(16y^2 - 8by + 4b^2)(4y + 2b)$.
12. $(25a^2x^2 + 10ax + 4)(5ax - 2)$.

## DIVISION.

### Lesson XXVII.

Before beginning this lesson a brief review of Lesson III. is suggested.

Reduce the following to the simplest form:

1. $\frac{4a}{2}$. 2. $\frac{4a}{a}$. 3. $\frac{4a}{2a}$. 4. $\frac{10abc}{5ac}$.

5. $\frac{25a^2b}{5ab}$. 6. $\frac{33cx^2}{11c^2x}$. 7. $\frac{14x^2y}{28xy^2}$.

8. What is the value of $27a^3b^4x^6 + 18a^6b^3x^3$ divided by $9a^2b^2x^3$?

Solution. The exponent of any literal quantity in a product equals the sum of the exponents of that quantity in the factors. (See Lesson XXII.)

Therefore the exponent of any literal quantity in a quotient equals the *difference* between the exponents of that quantity in the dividend and divisor.

$$\frac{27a^3b^4x^6 + 18a^6b^3x^3}{9a^2b^2x^3} = 3ab^2x^3 + 2a^4b.$$

Since in multiplication the first term (that to the left) of the multiplicand is first multiplied, in division the term to the left of the dividend is first divided.

Find the quotients of the following:

9. $\frac{32x^3 + 16x^2y + 64x^4y^2}{16x^2}$.

10. $\frac{6a^2b + 12ab^2}{3ab}$.

11. $\frac{14c^2x^2 + 21cx + 49c^2x^2y}{7cx}$.

12. $\frac{14b^2cx^3 + 28bcx^2 + 42b^3c^2x^4}{7bcx^2}$.

Ans

13. What is the quotient of $-10a^2b^3c^4$ divided by $-2abc$, or of $10a^2b^3c^4$ divided by $2abc$?

SOLUTION. Since the dividend and divisor have like signs, the quotient must be positive, $5ab^2c^3$.

14. What is the quotient of $-10a^2b^3c^4$ divided by $2abc$?

SOLUTION. Since a product is a negative quantity only when the factors have unlike signs (see Lesson XXIII.), and since the divisor is a positive quantity, the quotient must be a negative quantity, $-5ab^2c^3$.

15. What is the quotient of $10a^2b^3c^4$ divided by $-2abc$? Since the signs of the factors must be alike to produce a positive quantity as a product, the quotient, as well as the divisor, must be negative, $-5ab^2c^3$.

The operations in division are shown below:

(13) $\dfrac{-10a^2b^3c^4}{-2abc} = 5ab^2c^3$, or $\dfrac{10a^2b^3c^4}{2abc} = 5ab^2c^3$.

(14) $\dfrac{-10a^2b^3c^4}{2abc} = -5ab^2c^4$. (15) $\dfrac{10a^2b^3c^4}{-2abc} = -5ab^2c^3$.

NOTE. Observe that when the signs of the dividend and divisor are alike, the quotient is a *positive quantity;* when the signs of the dividend and divisor are unlike, the quotient is a *negative quantity.*

16. $\dfrac{15a^2x^3 - 6a^3b^2x}{3ax} = ?$

17. $\dfrac{6c^2y^3 - 12c^3y^2 - 8cy}{2cy} = ?$

18. $\dfrac{-4a^2x^2 + 2a^3x^3 - 6a^4x^4}{-2ax} = ?$

19. $\dfrac{14c^2x^4 + 49cx^2y + 21x^4}{-7x} = ?$

20. $\dfrac{27c^2x^3 - 9c^2x^2y^4 + 15c^4x^4y^2}{3c^2x^2} = ?$

21. $\dfrac{-56a^4 + 14cx + 21by}{-7} = ?$

## Lesson XXVIII.

1. Multiplicand, $2a + 3b$
   Multiplier, $a + b$

   1st Partial Product, $2a^2 + 3ab$
   2d Partial Product, $2ab + 3b^2$

   Product, $2a^2 + 5ab + 3b^2$.

| | | | |
|---|---|---|---|
| Dividend, | $2a^2 + 5ab + 3b$ | $2a + 3b$ | Divisor. |
| 1st Partial Product, | $2a^2 + 3ab$ | $a + b$ | Quotient. |
| Remainder, | $2ab + 3b^2$ | | |
| 2d Partial Product, | $2ab + 3b^2$ | | |

Observe the following facts:

(1) The first term of a product is obtained by multiplying the first term of the multiplicand by the first term of the multiplier.

Therefore, dividing the first term of the dividend by the first term of the divisor, we obtain the first term of the quotient. Multiplying the divisor by the first term of the quotient, we obtain the first partial product.

(2) The first term of the second partial product is obtained by multiplying the first term of the multiplicand by the second term of the multiplier.

Therefore, dividing the first term of the remainder by the first term of the divisor, we obtain the second term of the quotient. Multiplying the divisor by this, we obtain the second partial product.

Note. From these observations we reach the following conclusions:

(1) Each term of the quotient is found by dividing the first term of the dividend, or remainder, by the first term of the divisor.

(2) All the terms of the divisor are multiplied by each term of the quotient, when found, for a partial product.

Find the quotients of the following:

2. $x^2 + 10x + 24 \div x + 4$. 3. $x^2 - 11x + 30 \div x - 5$.

Suggestion. The pupil must use great care in subtracting the partial product.

4. $y^2 - y - 42 \div y + 6$.
6. $a^2 + 2ab + b^2 \div a + b$.
5. $c^4 + 3c^2 - 10 \div c^2 - 2$.
7. $c^2 - 2cy + y^2 \div c - y$.
8. $4a^2 + 8ab + 4b^2 \div 2a + 2b$.
9. $9c^2 - 24cy + 16y^2 \div 3c - 4y$.
10. $9a^2 - 18ay + 9y^2 \div 3a - 3y$.
11. $25b^4 + 60b^2x^2 + 36x^4 \div 5b^2 + 6x^2$.
12. $y^2 - 12y + 35 \div y - 5$.

## Lesson XXIX.

1. Divide $a^3 + b^3$ by $a + b$.

Solution.

$$
\begin{array}{lll|l}
a^3 & & + b^3 & a + b \\
\underline{a^3} & \underline{\pm a^2b} & & \overline{a^2 - ab + b^2} \\
 & -a^2b & + b^3 & \\
 & \underline{-a^2b} & \underline{\mp ab^2} & \\
 & & ab^2 + b^3 & \\
 & & \underline{ab^2 + b^3} &
\end{array}
$$

Suggestion. Observe that when there is no term in the dividend or remainder, similar to one in the partial product, the term in the partial product is brought down with its sign changed.

2. Divide $a^3 - b^3$ by $a - b$.

SOLUTION.

$$\begin{array}{l|l} a^3 \qquad\qquad - b^3 & a - b \\ a^3 \mp a^2b & a^2 + ab + b^2 \\ \hline \quad a^2b - b^3 & \\ \quad a^2b \qquad \mp ab^2 & \\ \hline \qquad ab^2 - b^3 & \\ \qquad ab^2 - b^3 & \\ \hline \end{array}$$

Divide:

3. $b^3 - y^3$ by $b - y$.

4. $9x^2 - 16y^2$ by $3x + 4y$.

5. $27c^3 + 8y^3$ by $3c + 2y$.

6. $25a^4 - 4c^4$ by $5a^2 - 2c^2$.

7. $c^6 + x^6$ by $c^2 + x^2$.

8. $4c^2 - 4c - 24$ by $2c + 4$.

9. $4y^4 - 4c^2y^2 - 3c^4$ by $2y^2 + c^2$.

10. $3a^2 - 6ax + 3x^2$ by $a - x$.

*Problems in Multiplication and Division.*

11. Expand $(a + 2)(a - 2)$.

12. $(b + y)^2 = ?$

13. $\dfrac{2a^2 - 8a + 8}{a - 2} = ?$

14. Expand $(a + 4)(a + 5)$.

15. $(2 - x)^2 = ?$

16. $(a + 7)(a - 6) = ?$

17. $\dfrac{x^2 - x - 20}{x + 4} = ?$

18. Expand $(x^2 + 4)(x^2 - 4)$.

19. $\dfrac{27 + y^3}{9 - 3y + y^2} = ?$

20. $\dfrac{16 - 8b + b^2}{4 - b} = ?$

21. Expand $(5 + 4a)^2$.

22. $(\frac{1}{2}a + \frac{1}{3}b)(\frac{1}{2}a - \frac{1}{3}b)$.

23. Expand $(49 - 7a + a^2)(7 + a)$.

24. Expand $(a^2 + b^2)(a + b)(a - b)$.

25. $\dfrac{2ax^2 - 2ay^2 + 4bx^2 - 4by^2}{2a + 4b} = ?$

# EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

## Lesson XXX.

1. $\left.\begin{cases} x + y = 12. \\ x - y = \ \ 2. \end{cases}\right\}$ Find the values of $x$ and $y$.

Solution. From the equation $x + y = 12$, it is evident that we cannot find the values of either $x$ or $y$, for there are a great many pairs of numbers whose sum is 12.

Also, in the equation $x - y = 2$, the values of $x$ and $y$ are indefinite, since there are a great many pairs of numbers whose difference is 2.

But if we know that $x + y = 12$, and that in the same problem $x - y = 2$, we may find the values of $x$ and $y$ as follows:

$$x + y = 12 \quad (1)$$
$$x - y = \ \ 2 \quad (2)$$
$$2x \quad = 14 \quad (3)$$

Add equations (1) and (2) member to member. Since we are adding equals to equals, the result will be a new equation, (3).

$$2x = 14; \quad (3)$$
$$x = \ \ 7.$$

Substitute for $x$ in equation (1) its number value.

$$7 + y = 12 \quad (4)$$
$$y = \ \ 5$$

Note. The process above shown, by which two equations containing two unknown quantities are combined to produce a third containing one unknown quantity, is called **Elimination**.

It will be observed, that in order to find the values of two unknown quantities, there must be two *independent equations.*

**Independent Equations** are such as cannot be reduced to the same form.

Thus, $2x + 3y + 2 = 7$ (1)
and $4x + 6y = 10$ (2)

are not independent equations; for, subtracting 2 from each member of (1), and dividing each member of (2) by 2, both equations reduce to the form:—

$$2x + 3y = 5.$$

But $2x + 3y = 5$ (1)
and $2x - 3y = 3$ (2)

are independent equations, for in no way can they be reduced to the same form.

2. Find the values of $x$ and $y$ in the following equations:

$$2x + 3y = 13. \quad (1)$$
$$4x + y = 11. \quad (2)$$

SOLUTION. Multiply (1) by 2, and subtract (2) from the result.

$$4x + 6y = 26 \quad (3)$$
$$4x + y = 11 \quad (2)$$
$$5y = 15 \quad (4)$$
$$y = 3.$$

Substitute for $y$ in equation (2) its number value.

$$4x + 3 = 11. \quad (5)$$
$$4x = 8. \quad (6)$$
$$x = 2.$$

3. Find the values of $x$ and $y$ in the following equations:

$$2x + 10y = 50. \quad (1)$$
$$3x + 7y = 43. \quad (2)$$

Solution. Multiply (1) by 3, and (2) by 2, and subtract (4) from (3).

$$6x + 30y = 150 \qquad (3)$$
$$6x + 14y = 86 \qquad (4)$$
$$16y = 64 \qquad (5)$$
$$y = 4.$$

Find the value of $x$ as in Problem 2.

Note. From these problems we learn the following **Method of Elimination**:

Multiply or divide the terms of each equation by such numbers as will make the coefficients of one of the unknown quantities equal in the two equations. Then add the terms of the equations, if the quantities made equal have unlike signs; or subtract the terms of one equation from those of the other, if they have like signs.

Eliminate the letter whose coefficients may most easily be made equal.

Find the values of $x$ and $y$ in the following equations:

4. $\begin{cases} 4x + 5y = 61. \\ 5x + 6y = 75. \end{cases}$

5. $\begin{cases} x + 2y = 7. \\ x + y = 5. \end{cases}$

6. $\begin{cases} 2x + 6y = 10. \\ 3x + 2y = 8. \end{cases}$

7. $\begin{cases} 4x + 3y = 7. \\ 2x - 3y = -1. \end{cases}$

8. $\begin{cases} 4x - 5y = 3. \\ 3x + 5y = 11. \end{cases}$

9. $\begin{cases} 8x + 3y = 22. \\ 4x + 5y = 18. \end{cases}$

10. $\begin{cases} 3x + 4y = 25. \\ 4x + 3y = 24. \end{cases}$

11. $\begin{cases} 10x - 2y = 2. \\ 3x + 4y = 42. \end{cases}$

12. $\begin{cases} 3x - y = 3. \\ 2x + y = 17. \end{cases}$

13. $\begin{cases} 5x + y = 17. \\ 5x - y = 3. \end{cases}$

## Lesson XXXI.

Find the values of $x$ and $y$ in the following equations:

1. $\begin{cases} 6x - y = 27. \\ 9x - 5y = 30. \end{cases}$

2. $\begin{cases} 10x + 2y = 36. \\ 3x + 4y = 38. \end{cases}$

3. $\begin{cases} 4x + 7y = 47. \\ 7x - 2y = 11. \end{cases}$

4. $\begin{cases} x - y = 4. \\ 5x - 11y = -10. \end{cases}$

5. $\begin{cases} 5x + 5y = 55. \\ 6x - 3y = 3. \end{cases}$

6. $\begin{cases} 3x - 7y = 0. \\ 4x + 2y = 34. \end{cases}$

Transpose before eliminating, and find the values of $x$ and $y$:

7. $\begin{cases} 8x = 3y + 33. \\ 5x = 20 + 2y. \end{cases}$

8. $\begin{cases} 10x = 37 - 3y. \\ 5x = 14 - y. \end{cases}$

9. $\begin{cases} 3x + 5y = 38. \\ x = 14 - 2y. \end{cases}$

10. $\begin{cases} 4x - 6y = 14 - y. \\ 3x + 3y = 24. \end{cases}$

11. $\begin{cases} 10 + 2x = 17y - 14. \\ 5x + 3y = 31. \end{cases}$

12. $\begin{cases} 9x + 20 = 63 - 5y. \\ 4y = x + 18. \end{cases}$

13. $\begin{cases} 2y + 8 = 3x - 23. \\ 4x = 12 + 9y + 23. \end{cases}$

14. $\begin{cases} 3x = 5y. \\ 4x = 7y - 3. \end{cases}$

Suggestion. Transposing $5y$ in the first equation of Ex. 14, we have $3x - 5y = 0$.

In the following, first clear both equations of fractions:

15. $\begin{cases} \dfrac{2x + 8}{y} = 3. \\ \dfrac{3x}{2} = y + 1\frac{1}{2}. \end{cases}$

16. $\begin{cases} \dfrac{3x}{5y} = 2\frac{2}{5}. \\ \dfrac{x + 1}{9} = \dfrac{2y}{3} - \dfrac{1}{3}. \end{cases}$

17. $\begin{cases} \dfrac{5y + 3}{2} = 4x - \dfrac{1}{2}. \\ 4y - 9 = \dfrac{5x}{3} + 2. \end{cases}$

18. $\begin{cases} \dfrac{2y}{x} + \dfrac{4}{7} = 2. \\[2ex] \dfrac{2y + 2x}{3} = x + 1. \end{cases}$

19. $\begin{cases} 5x - \dfrac{5y}{2} = 50. \\[2ex] \dfrac{x}{5} + \dfrac{6y}{2} = 33. \end{cases}$

20. $\begin{cases} \dfrac{4x}{8} + \dfrac{1}{6} = 5x - 4\frac{1}{3}. \\[2ex] \dfrac{y}{2} - x = \dfrac{2x + 4}{2}. \end{cases}$

## Lesson XXXII.

Find the values of $x$ and $y$:

1. $\begin{cases} \dfrac{1}{x} - \dfrac{2}{y} = 0. \\[2ex] \dfrac{x + y}{5} = \dfrac{2x}{3} - \dfrac{2}{15}. \end{cases}$

2. $\begin{cases} \frac{2}{3}x + \frac{3}{5}y = 17. \\ \frac{3}{4}x + \frac{2}{3}y = 19. \end{cases}$

3. $\begin{cases} 5x - 4.9y = 1. \\ 3x - 3.9y = 1. \end{cases}$

4. $\begin{cases} \dfrac{x + y}{5} + \dfrac{y - x}{2} = 3. \\[2ex] \dfrac{x - y}{2} + \dfrac{x + y}{10} = 0. \end{cases}$

5. $\begin{cases} \dfrac{5x + 7}{2} + \dfrac{1}{3} = \dfrac{y}{6} + 10\frac{2}{3}. \\[2ex] 3x - 9y = -27. \end{cases}$

6. $\begin{cases} \dfrac{x + 2y}{11} = x - 2y. \\[2ex] \dfrac{6x + 2y}{9} + y = x + 2\frac{1}{3}. \end{cases}$

$\begin{cases} \dfrac{x+2y}{5} + \dfrac{3x-2y}{6} = \dfrac{36}{5}. \\[2ex] \dfrac{2x+4y}{16} + \dfrac{5x-14y}{8} = 3. \end{cases}$

8. $\begin{cases} \dfrac{3x-2y}{2} + \dfrac{y}{5} = \dfrac{2y-16}{3}. \\[2ex] \dfrac{3x}{12} - \dfrac{4y}{10} = \dfrac{5x}{20} - \dfrac{x}{2}. \end{cases}$

9. $\begin{cases} \dfrac{7x+5y}{6}+\dfrac{3x-5y}{2}=\dfrac{2x+y}{3}. \\ \dfrac{6x+5}{3}=\dfrac{7y+4}{5}+\dfrac{22}{15}. \end{cases}$

10. $\begin{cases} \dfrac{x}{5}-\dfrac{y}{3}+4=\dfrac{x+y}{6}+\dfrac{x-3y}{2}. \\ \dfrac{x+2y+4}{8}=\dfrac{4y-x+9}{4}+1\frac{5}{8}. \end{cases}$

## Lesson XXXIII.

We have heretofore solved problems by using one unknown quantity only. But many problems are much more easily solved by means of two unknown quantities.

1. A grocer sold to one customer 5 lb. of sugar and 3 lb. of coffee for \$1.20, and to another, at the same rates, 2 lb. of coffee and 6 lb. of sugar for \$.96. Find the price of 1 lb. of each.

Suggestion.

Let $x =$ the cost of 1 lb. of sugar,
and $y =$ the cost of 1 lb. of coffee;
then $5x =$ the cost of sugar bought by first customer,
$3y =$ the cost of coffee bought by first customer,
$6x =$ the cost of sugar bought by second customer,
and $2y =$ the cost of coffee bought by second customer.

$$\therefore \begin{aligned} 5x+3y &= \$1.20, \\ 6x+2y &= \ \ .96, \end{aligned}$$

whence the values of $x$ and $y$ may be found as in the preceding lessons.

2. Find two numbers, of which the sum divided by the less equals $1\frac{1}{2}$, and the difference divided by the less equals 2.

3. A dealer sold to one person 5 horses and 6 cows for \$192. At the same prices, he sold 4 horses and 3 cows for \$132. Find the price he charged for each horse and cow.

4. Find two numbers, such that $\frac{1}{2}$ of the less plus $\frac{1}{3}$ of greater equals 22, while $\frac{2}{3}$ of the less plus $\frac{4}{5}$ of the greater equals 40.

5. Find two numbers, such that twice the greater plus 4 equals three times the less, and 4 times the less equals three times the greater.

6. Find two numbers, such that their sum divided by the less equals $2\frac{1}{2}$, and their difference divided by 5 equals 1.

7. The less of two numbers divided by the greater gives a quotient of $\frac{2}{7}$. The greater divided by the less gives a quotient of 3 plus a remainder of 3. Find the numbers.

SUGGESTION. Let $x =$ the less number,

and $y =$ the greater number;

then $$\frac{x}{y} = \frac{2}{7},$$

and since in division we indicate a remainder by writing it over the divisor,

$$\frac{y}{x} = 3 + \frac{3}{x}.$$

8. The greater of two numbers and 2, divided by 7, equals the greater number divided by 6; and the larger number divided by the smaller equals 1 plus a remainder of 3. Find the numbers.

9. A grocer mixed two grades of tea, taking 9 lb. of the poorer quality and 3 lb. of the better, and this mixture was worth \$1 per pound. Again he mixed 4 lb. of the better with 4 lb. of the poorer, and this mixture was worth \$1.20 per pound. Find the cost of 1 lb. of each grade of tea.

SUGGESTION. Let $x$ = the cost of 1 lb. of better grade,
and $y$ = the cost of 1 lb. of poorer grade;

then $$\frac{9y+3x}{12} = \$1,$$

and $$\frac{4x+4y}{8} = \$1.20.$$

10. Two grades of sugar are mixed. A mixture of 12 lb. of the better with 4 lb. of the poorer is worth \$.05½ per lb. A mixture of 3 lb. of the poorer with 15 lb. of the better is worth \$.05⅝ per lb. Find the price of 1 lb. of each.

11. In a mixture of two grades of coffee worth \$.36 per pound, 30 lb. of the better are mixed with 20 lb. of the poorer. In a mixture of the same grades worth \$.31 per pound, 18 lb. of the poorer are taken to 2 lb. of the better Find the price per pound of each grade of coffee.

## LESSON XXXIV.

1. In a mixture of 40 lb. of sugar worth \$.07 per lb., two grades of sugar were used worth \$.08 and \$.04 respectively. Find the number of pounds of each used.

SUGGESTION.
Let $x$ = number of pounds of better grade,
and $y$ = number of pounds of poorer grade;

then $$\frac{8x+4y}{40} = 7.$$

2. A barrel of sugar, made up of two grades of sugar worth \$.04½ and \$.07 per lb. respectively, and weighing 250 lb., is worth \$12.50. How many pounds of each grade were used?

SUGGESTION. Since the price per pound of each grade of sugar is given in cents, in making the equations from this problem \$12.50 must be taken as 1250 cents.

3. A sack of coffee weighs 300 lb. and is worth $ 75. It is made up of coffees worth $ .24 and $ .30 per pound respectively. How many pounds of each were used?

4. A junk dealer bought 540 lb. of scrap iron, paying $ .04 per pound for part of it, and $ .01¾ per pound for the remainder. He paid for the whole $10.80. Find the number of pounds of each sort that he bought.

5. A peddler sold a crate of 25 qt. of blackberries and a crate of 20 qt. of strawberries, receiving for the two $ 3.50. The amount he received for the blackberries was to the amount he received for the strawberries as 3 is to 4. What was the price of 1 quart of each?

6. A cistern is supplied by two pipes. The first will fill it in 3 hours; the second will fill it in 4 hours. How long will it require to fill the cistern, if both pipes are open?

Suggestion. Solve with one unknown quantity. $\frac{1}{3}$ of the cistern is filled by the first pipe in 1 hour. $\frac{1}{4}$ of the cistern is filled by the second pipe in 1 hour.

Let $x =$ the number of hours needed for the two pipes to fill the cistern;

then $\frac{1}{x} =$ the part they will fill in 1 hour.

$$\frac{1}{3} + \frac{1}{4} = \frac{1}{x}.$$

7. A cistern is supplied with two pipes. The first will fill it in 5 hours; the second will empty it in 6 hours. How long will it take to fill the cistern if both pipes are left open?

8. One pipe will fill a cistern in 7 hours; a second will fill it in 8 hours. The first is kept open for 3 hours. The

second is then opened. How long will it take the two to fill the remainder of the cistern?

SUGGESTION. First find how long it will take both pipes to fill the whole cistern.

9. A tank can be filled by one pipe in 10 hours, and emptied by a second pipe in 8 hours. The first is left open 5 hours. The second is then opened. How long will it take to empty the cistern?

10. If a pipe which can fill a tank in 10 hours, and another which can empty it in 12 hours, are both left open for $3\frac{1}{2}$ hours, what will be the state of the water in the tank?

## LESSON XXXV.

1. Two numbers are to each other in the ratio of 2 to 7, and 6 times the less plus 8 is to the greater as 2 is to 1. Find the numbers.

SUGGESTION. Let $x =$ the less number,
and $y =$ the greater number.

$$x : y :: 2 : 7,$$

whence $$7x = 2y;$$

$$6x + 8 : y :: 2 : 1,$$

whence $$6x + 8 = 2y.$$

2. Two numbers are to each other as 3 to 5, and 4 times the less minus 6 is to 3 times the greater as 2 is to 3. Find the numbers.

3. The sum of two numbers is to their difference as 5 is to 1. The less number plus 2 divided by 4 equals the greater number plus 1 divided by 5. Find the numbers.

4. The sum of two numbers is to the greater as 5 is to 3. Their difference is to the less number plus 2 as 2 is to 5. Find the numbers.

5. Four times the greater of two numbers minus 5 times the less equals 2, and the quotient obtained by dividing twice the less minus 3 by the less equals the quotient given by dividing twice the sum of the two numbers by 10 times the less number. Find the numbers.

6. In a certain election, 375 persons voted for two candidates. The one chosen received a majority of 91. How many voted for each?

7. A gentleman bought a gold watch, a silver watch, and a gold chain. The chain cost $25, the chain and the gold watch are together worth $3\frac{1}{2}$ times as much as the silver watch alone. The chain and the silver watch together are worth $15 more than half the value of the gold watch alone. What is the value of each watch?

8. A and B together buy a house for $1200. A says to B, "Lend me $\frac{2}{3}$ of your money, and I can buy the house alone." B replies, "Lend me $\frac{3}{4}$ of your money, and I can buy it alone." How much money has each?

9. Seven years ago the age of A was three times that of B, and 7 years hence A's age will be twice that of B. What is the present age of each?

10. A number consisting of two digits is equal to 5 times the sum of the digits. If 9 is added to the number, the order of the digits will be reversed. Find the number.

11. The sum of two numbers is to their difference as 11 to 1. The greater number minus 10 divided by 2 equals the less number plus 5 divided by 3. Find the numbers.

12. Five years ago, a father was 5 times as old as his son. Eight years hence, their ages will be as 12 to 5. Find their ages.

## LESSON XXXVI.

1. A certain number consists of tens and units. If we divide the number by 2, we have 6 more than twice the sum of the digits. If we add 9 to the number, we reverse the order of the digits. Find the number.

2. A number is made up of tens and units. If we multiply the number by 2, we have 6 more than twelve times the sum of the digits. If we subtract 18 from the number, we reverse the order of the digits. Find the number.

3. A boy spent $.84 for lemons and oranges, giving $.03 each for lemons and $.05 each for oranges. He sold $\frac{1}{2}$ of the lemons and $\frac{1}{4}$ of the oranges for $.40 and cleared $.08 on what he sold. How many of each did he buy?

4. The same boy bought peaches at the rate of 4 for $.05, and apples at the rate of 8 for $.05, paying $.35 for them all. He sold, at the same rate that he bought, $\frac{1}{3}$ of his peaches and $\frac{1}{4}$ of his apples for $.10. How many of each did he buy?

5. A man laid out a rectangular lawn. If he adds 10 yd. to the shorter side, the lawn will be square. If he adds 10 yd. to the longer side, the length of the lawn will be twice the width. What is the area of the lawn?

6. A plot of ground is rectangular. If the longer side is shortened $\frac{1}{10}$, the plot will be square. The sum of the sides minus 16 rd. equals 15 times the difference of the sides. Find the area of the plot.

7. The altitude of a triangle is to its base as 5 is to 4. The difference between the altitude and the base plus 4 is to their sum as 1 is to 3. Find the area of the triangle.

8. The length of a rectangle is to its breadth as 3 is to 2. Twice the length plus 6 ft. is $3\frac{1}{2}$ times the breadth. Find the area of the rectangle.

9. Two men are 34 miles apart and walking toward each other. If A walks for 4 hours, and B for 5 hours, they will meet. Or if A walks for 6 hours, and B walks for 3 hours, they will still lack one mile of meeting. How fast does each man walk?

10. Two men are 40 miles apart and walking toward each other. If A walks for 6 hours, and B for 8 hours, they will lack 4 miles of meeting. But if A walks for 10 hours, and B for 8 hours, they will have passed each other and again be 4 miles apart. How fast does each man walk?

## LESSON XXXVII.

1. A sum of money is made up of pounds and shillings. The number of pounds equals $\frac{3}{4}$ the number of shillings, and the entire sum in shillings divided by 16 equals 12 shillings. What is the sum of money?

SUGGESTION. Let $x =$ number of pounds,

$y =$ number of shillings;

$20x + y =$ the entire sum in shillings, since there are 20*s.* in a pound.

2. A sum is made up of shillings and pence. Three times the number of pence equals 2 more than the number of shillings. Add 12 to the entire sum in pence, divide the result by 10, and the quotient is 1 less than the sum of the numbers representing the shillings and pence. What is the amount of money?

3. A sum is made up of pounds, shillings, and pence. Three times the number of pounds equals the number of shillings. The number of shillings is to the number of pence as 6 is to 5. The entire sum minus 7 divided by the number of pence equals 110. What is the sum of money?

**SUGGESTION. Solve by one unknown quantity.**

4. If we add 8 to the numerator of a certain fraction, its value becomes $\frac{4}{3}$. If we add 3 to the denominator of the same fraction, its value becomes $\frac{1}{3}$. What is the fraction?

SUGGESTION. Let $\frac{x}{y}$ = the fraction.

5. If we add 7 to the numerator of a fraction, its value becomes 2. If we multiply the denominator by 3, and at the same time add 2 to the numerator, the value of the fraction becomes $\frac{1}{3}$. Find the fraction.

6. Two numbers are to each other as 2 to 5. Four times the less minus 2 is to the greater as 3 to 2. Find the numbers.

7. Two numbers are to each other as 4 to 5, and $\frac{1}{3}$ of the less plus 5 is to twice the greater minus 3 as 1 to 3. Find the numbers.

8. If to the numerator of a certain fraction 9 is added, and the denominator is multiplied by 3, its value becomes $\frac{2}{3}$. If we multiply the numerator by 5, and subtract 3 from the product, the value of the fraction becomes 4. What is the fraction?

9. A owes \$ 60, B owes \$ 75; neither can pay his debt. B says to A, "Lend me $\frac{7}{10}$ of your money, and I can pay my debt." A replies, "Lend me $\frac{1}{4}$ of your money, and I can pay my account." How much money has each?

10. Find two numbers in the ratio of 5 to 7, to which two other numbers in the ratio of 3 to 5 being added, the sums will be to each other as 9 to 13, and the difference of the sums be 16.

SUGGESTION. Let $5x + 7x$ = the first pair of numbers;

then $3y + 5y$ = the second pair of numbers.

# FACTORING.

## Lesson XXXVIII.

Compare the numerical and algebraic quantities in (*a*) with those in (*b*).

(*a*) 5, 7, $x$, $a$, 11, $a+b$, $a+b+c$.

(*b*) 4, 6, $x^2$, $a^3$, 9, $a+ab$, $x^2+xy+xy^2$.

Observe that no single quantity in (*a*) is exactly divisible by any other quantity than itself and 1. Such quantities are called **Prime Quantities.**

Each quantity in (*b*) is exactly divisible by some other quantity than itself and 1. Such quantities are called **Composite Quantities.**

A **Prime Factor** is one which is a prime quantity.

1. What are the prime factors of $12a^2b^2$?

Solution. Note that there are several sets of factors for this product, such as 12 and $a^2b^2$, 6 and $2a^2b^2$, $4a^2$ and $3b^2$, etc., but that these factors are not prime quantities.

The prime factors are 2, 2, 3, $a$, $a$, $b$, $b$.

Note. The prime factors of a quantity multiplied together must produce the quantity.

An algebraic quantity consisting of one term is called a **Monomial.** Thus, $a^2$, $ax$, $-b^3$, $6ab$, $-12a^2x^2y^2$ are monomials.

An algebraic expression of more than one term is called a **Polynomial.** Thus, $a+b$, $7ax+6b$, $a^2-2ax+x^2$, $a^2+b^2+c^2+2ac+2ab+2bc$, are polynomials.

A polynomial consisting of two terms is called a **Binomial.** Thus, $a+b$, $7ab^2+14a^2b$, are binomials.

A polynomial consisting of three terms is called a **Trinomial.** Thus, $a^2-2ab+b^2$ and $7ax+6ay+3az$ are trinomials.

2. What are the prime factors of: $12a^4b$; $6a^2x^2y^2$; $-9ab$; $-4x^3y^2$; $c^2x^2y^2$?

3. What are the prime factors of $ab + b^2$?

SOLUTION. The prime factors are $b$ and $a + b$. The product may be indicated thus: $b(a + b)$.

4. What are the prime factors of $6a + 3b$?

SOLUTION. The prime factors are 3 and $2a + b$. The product may be indicated thus: $3(2a + b)$.

5. Separate $4a^2 + 2ab + 6ac$ into its prime factors, one of which shall be a monomial.

SOLUTION. $4a^2 + 2ab + 6ac = 2 \times a(2a + b + 3c)$.

NOTE. Observe that if a monomial is a factor of a polynomial, it must occur as a factor in each term of the polynomial. Thus, in Example 5, 2 and $a$ are each of them factors of $4a^2$, $2ab$, and $6ac$, the three terms of the given polynomial, and hence are factors of the polynomial $4a^2 + 2ab + 6ac$.

Separate the following into their prime factors, one of which shall be a monomial:

6. $2a + 2ax$.

7. $axy + aby$.

8. $2x^2y + 3xy^2$.

9. $6ab^2 + 12a^2bc$.

10. $25a^4 - 40a^3b + 20a^2b^2$.

11. $5a^2x + 10abx + 5b^2x$.

12. $48a^2b^2 - 8abcy + 16aby$.

13. $2ab^2c + 5ab^3 + 3ab^2c^2$.

14. $3cx + 9cxy - 2cx^2$.

15. $4bc^4x^3 - 5c^3x^4 + 2c^2x^3y$.

16. $2abx^2 + 4abx + 2ab$.

17. $3ab^2x^3 - 6ab^2x^2 - 12a^2b^3x$.

18. $2x^3y + 6x^2y^2 + 6xy^3 + 2y^4$.

19. $5ax^3 - 15ax^2y + 15axy^2 - 5ax$.

**20.** What are the prime factors of $a(b+x)+b(b+x)$?

Solution. Consider this expression as a binomial in each term of which $(b+x)$ is a factor.

The factors are then seen to be $(a+b)$ and $(b+x)$, and the product is indicated thus: $(a+b)(b+x)$.

**21.** Separate $ac(x+y)+b(x+y)$ into its prime factors.

**22.** Separate $ab(x+y)+ac(x+y)$ into its prime factors.

**23.** Separate $2\,ab(c+x)+2\,ab(x+y)$ into its prime factors.

**24.** Separate $ax+ay+bx+by$ into its prime factors.

Suggestion. This expression may be written:

$$a(x+y)+b(x+y).$$

**25.** Separate $cx-ca+bx-ba$ into its prime factors.

**26.** Separate $3\,ax+3\,by+a^2x+aby$ into its prime factors.

**27.** Separate $6\,a^2b+6\,ab^2+a+b$ into its prime factors.

## Lesson XXXIX.

The product obtained by using a quantity two or more times as a factor is called a **Power** of that quantity, and the quantity so used is called the **Root** of the power.

Thus $(2\,ab)^2$ indicates that $2\,ab$ is to be used twice as a factor, and the product $4\,a^2b^2$ is the **Square** or **Second Power** of $2\,ab$, while $2\,ab$ is the **Square Root** of $4\,a^2b^2$.

In the same way, $(2\,ab)^3$ indicates that $2\,ab$ is to be used three times as a factor, and the product $8\,a^3b^3$ is the **Cube** of $2\,ab$, while $2\,ab$ is the **Cube Root** of $8\,a^3b^3$.

Thus the expression $x^2-y^2$ indicates the difference of two squares, the roots of which are $x$ and $y$.

The expression $x^3-y^3$ indicates the difference of two cubes, the roots of which are $x$ and $y$.

1. $x^2-y^2=(x+y)(x-y)$. (See Lesson XXV., Theorem 3.)

The converse of the theorem referred to gives the following:

A binomial which is the difference of two squares may be separated into two binomial factors. One of these factors is the *sum* of the *roots* of the *squares*; the other is the *difference* of the *same roots.*

Factor:

2. $a^2-b^2$.
3. $a^2b^2-c^2$.
4. $4a^2-1$.
5. $9b^4-4$.
6. $16-9y^2$.
7. $1-49x^2y^4$.
8. $25-a^2x^2$.
9. $4x^2-81$.
10. $1-x^2$.
11. $1-x^4$.

SUGGESTION. This quantity is first separated into the factors $(1+x^2)$ and $(1-x^2)$; then $1-x^2$ is factored as above.

12. $x^4-y^4$.
13. $25x^2-a^4b^2$.
14. $9a^2b-9b^3$.

SUGGESTION. In factoring Example 14, all monomial factors should first be removed.

15. Factor $a^2-(c+y)^2$.

SOLUTION. The factors are $a+(c+y)$ and $a-(c+y)$. To remove the parentheses, the operation indicated by the signs preceding them must be performed.

$$a+(c+y)=a+c+y.$$

Since in subtraction the signs of the subtrahend are changed,

$$a-(c+y)=a-c-y.$$

NOTE. When a parenthesis is preceded by the sign plus, $(+)$, the parenthesis may be removed without affecting the signs of the terms included.

When a parenthesis is preceded by the sign minus, (−), the parenthesis may be removed if, at the same time, all the signs of the included terms are changed. If the first term of the parenthesis has no sign, it is understood to be a positive quantity.

Therefore, it follows that: (1) Any number of terms may be inclosed in a parenthesis preceded by the sign plus, (+), without changing the signs of the terms so inclosed. (2) Any number of terms may be inclosed in a parenthesis preceded by the sign minus, (−), if, at the same time, the signs of all the terms so inclosed are changed. Thus,

$$a + b - c + d = a + (b - c + d).$$

$$a - b - c + d = a - (b + c - d).$$

Factor:

16. $a^4 - (x + y)^2$.
17. $a^2 - (a + b)^2$.
18. $(a + b)^2 - (a - b)^2$.
19. $4a^2 - (a + b)^2$.
20. $(x + a)^2 - 4x^2$.

## Lesson XL.

Factor the following, first removing all monomial factors:

1. $2x^2 - 8y^2$.
2. $ab^2 - ay^2$.
3. $4ac^2x^2 - 16ay^2$.
4. $5x^4 - 20y^4$.
5. $5x^4 - 5$.
6. $20a^5 - 5a$.
7. $9a^2x^3 - 25b^2x$.
8. $45a^3x^5 - 20ax$.
9. $a^2(b + x) - b^2(b + x)$.
10. $a^2b^2(x + y) - 4(x + y)$.
11. $4(x + y) - a^2(x + y)$.
12. $9c^2(a + b) - (a + b)$.
13. $x(a^2 - b^2) - y(a^2 - b^2)$.

14. $a^2x^3(a^2b^2 - x^2y^2) - a^2x^3(a^2b^2 - x^2y^2)$.

15. $4a^3 - a(x + y)^2$.

16. $a^2 - a^4 - b^2 + a^2b^2$.

SUGGESTION. Inclose the four terms in two parentheses, remembering the law governing the signs of the terms inclosed:

$$(a^2 - a^4) - (b^2 - a^2b^2).$$

Remove the binomial factor from each parenthesis:

$$a^2(1 - a^2) - b^2(1 - a^2).$$

Factor into two binomials:

$$(a^2 - b^2)(1 - a^2) = (a + b)(a - b)(1 + a)(1 - a).$$

Factor:

17. $a^2b + a^2 - b^3 - b^2$.

18. $4a - 4b - ax^2 + bx^2$.

19. $(a + b)^4 - (c + x)^4$.

20. $a^2 + b^2 - a^2x - b^2x$.

21. $a^2 - b^2 - a^2x^2 + b^2x^2$.

22. $3a^3 + 3ab - 3a^3x^2 - 3ab^2x^2$.

23. $5cy^2 + 5b^2y - 5c^2x^2y^2 - 5b^2x^2y$.

24. $a^8 - b^8$.

25. $10x^8 - 10$.

26. $8x - 8y - 2a^2x + 2a^2y$.

27. $8 - 32y^4$.

28. $16y^3 - y(a + b)^2$.

29. $1 - b^6$.

30. $a(x^2 - y^2) - b(x^2 - y^2)$.

31. $b(1 - x^4) - y(a^2 - 1)$.

## Lesson XLI.

(a) $x^2 + 2xy + y^2 = (x + y)(x + y)$.

(b) $x^2 - 2xy + y^2 = (x - y)(x - y)$.

(See Lesson XXV., Theorems 1 and 2.)

The converse of the theorems referred to gives the following:

A trinomial composed of two terms that are *squares* and *positive*, and a third term which is *twice the product of the roots of the squares*, has *two equal binomial factors.*

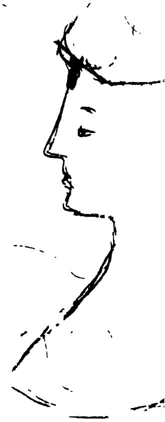
Note. To obtain one of these equal binomial factors, extract the roots of the squares and connect them by the sign of the other term.

Factor:

1. $c^2 - 4ac + 4a^2$.
2. $a^2 + 2ab + b^2$.
3. $b^2 - 2bc + c^2$.
4. $a^2b^2 + 2abc + c^2$.
5. $9a^2 + 12ab + 4b^2$.
6. $1 + 2a^2 + a^4$.
7. $4a^4 - 4a^2b^2 + b^4$.
8. $x^2 + 2x + 1$.
9. $4x^2 + 4x + 1$.
10. $4x^4 + 12x + 9$.
11. $16a^2 + 40ab + 25b^2$.
12. $7a^2 + 14ab + 7b^2$.
13. $5a^3 + 10a^2b + 5ab^2$.
14. $7a^5 - 14a^3 + 7a$.
15. $49 - 14x^3 + x^6$.
16. $16 + 16a + 4a^2$.
17. $32y^2 - 16y + 2$.
18. $27a^5 - 18a^3 + 3a$.

19. Factor $a^2 - x^2 + 2xy - y^2$.

Solution. $a^2 - x^2 + 2xy - y^2 = a^2 - (x^2 - 2xy + y^2)$;
$a^2 - (x^2 - 2xy + y^2) = a^2 - (x - y)^2$.



Factor:

20. $a^2 + 2ab + b^2 - c^2$.

21. $9a^2 - a^2b^2 - 6abc - 9c^2$.

22. $(a+b)^2 + 2(a+b) + 1$.

23. $ax^2 - 4a^3 + 4a^2b - ab^2$.

24. $a^2 + 2a(x+y) + (x+y)^2$.

25. $a^2(a^2-1) - (a^2-1)$.

## Lesson XLII.

1. Expand $(x+2)(x-3)$.

Solution.

$$\begin{array}{l} x + 2 \\ \underline{x - 3} \\ x^2 + 2x \\ \underline{\quad\ \ -3x - 6} \\ x^2 + (2-3)x - 6 = x^2 - x - 6. \end{array}$$

Observe that the first term of the trinomial product is the square of the term common to the two binomial factors.

The last term of the trinomial is the product of the unlike terms of the two factors.

The second term of the trinomial consists of the common term of the binomial factors, multiplied by the algebraic sum of the unlike terms of those factors.

2. Expand $(x+5)(x+7)$.

Solution. $x^2 =$ the square of the common term.

$35 =$ the product of the unlike terms.

$12 \times x =$ the common term multiplied by the algebraic sum of the unlike terms.

$$(x+5)(x+7) = x^2 + 12x + 35.$$

Expand $(x+3)(x-1)$.

Expand $(x-2)(x+3)$.

Expand $(c-4)(c+5)$.

Expand $(y+4)(y-6)$.

Expand $(x+a)(x+b)$.

SOLUTION. $(x+a)(x+b) = x^2 + ax + bx + ab$

$$= x^2 + (a+b)x + ab.$$

Note that the second term consists of the common term $x$, multiplied by $(a+b)$, the algebraic sum of the unlike terms.

8. Expand $(x+c)(x+y)$.

9. Expand $(x+a)(x-c)$.

10. What are the factors of $x^2 + 7x - 8$?

SOLUTION. From the principles given above, the following method of factoring trinomials of the above form is derived:

A trinomial of the form $x^2 + 7x - 8$ may be resolved into two binomial factors having a common term.

For the common term take the root of the square, $(x)$.

For the unlike terms take two quantities whose product equals the third term of the trinomial and whose algebraic sum multiplied by the common term equals the second term of the trinomial.

The product of 8 and $-1$ equals $-8$.

The algebraic sum of 8 and $-1$ equals 7, and this multiplied by the common term $x$ equals $7x$.

$$x^2 + 7x - 8 = (x+8)(x-1).$$

Factor the following:

11. $a^2 + 3a + 2$.

12. $a^2 + 2a - 8$.

13. $a^2 + 5a + 6$.

14. $x^2 - 5x + 6$.

15. $a^2+7a+6$.

16. $a^2-a-6$.

17. $a^2+a-6$.

18. $x^2+8x+15$.

19. $a^2-7a+6$.

20. $x^2+5x+4$.

21. $x^2+6x+8$.

22. $a^2-7a+12$.

23. $y^2-13y+40$.

24. $x^2+2x-15$.

25. $a^2+a-12$.

26. $b^2-b-12$.

27. $c^2+13c+12$.

28. $x^2-13x+12$.

29. $a^2+7a+12$.

30. $y^2+4y-12$.

31. $b^2-4b-12$.

32. $y^2-8y+12$.

## Lesson XLIII.

Factor the following:

1. $2a^2+12x-32$.

2. $3x^2+18x+24$.

3. $a^3-9a^2+14a$.

4. $5a^2x^3-20a^2x^2-25a^2x$.

5. $a^2b-2ab-4b$.

6. $x^2+5ax+6a^2$.

Suggestion. $6a^2$ may be resolved into the factors $3a$ and $2a$, whose algebraic sum, $5a$, together with the root of the square, $x$, makes up the second term.

7. $x^2+4ax+3a^2$.

8. $x^2-2ax-15a^2$.

9. $y^2-4ay-21y^2$.

10. $x^2-7ax+12a^2$.

11. $2a^2b-4ab^2-8$.

12. $a^2-ac-6c^2$.

13. $a^2+3abc+2b^2c^2$.

14. $4x^2+8x+3$.

Solution. The root of the square, $4x^2$, is $2x$, and this enters as a factor into the second term, $8x$. Dividing $8x$ by $2x$, the quotient 4 is obtained, and this is the algebraic sum of 3 and 1, the factors of the last term of the trinomial.

$$4x^2+8x+3=(2x+3)(2x+1).$$

15. $4x^2 + 14x + 12$.

16. $9x^2 + 9x + 2$.

17. $4x^2 - 8ax + 3a^2$.

18. $9y^2 + 6yb + b^2$.

19. $36x^2 + 24bx - 5b^2$.

20. $4a^2x^2 - 4ax - 15$.

21. $x^2 + ax - bx - ab$.

SUGGESTION. $x^2 + ax - bx - ab = x^2 + (a - b)x - ab$.

22. $x^2 + 2ax + 3bx + 6ab$.

23. $x^2 + a^2x + ax + a^3$.

24. $x^4 + 2ax^2 - 3bx^2 - 6ab$.

25. $a^2x^2 + 2abx + acx + 2bc$.

## LESSON XLIV.

(This lesson may be omitted at the discretion of the teacher.)

1. Expand $(2a + 3b)(3a + 5b)$.

SOLUTION.

$$\begin{array}{r} 2a + 3b \\ 3a + 5b \\ \hline 6a^2 + 9ab \phantom{+ 15b^2} \\ 10ab + 15b^2 \\ \hline 6a^2 + 19ab + 15b^2 \end{array}$$

Observe that: (1) The first term of the trinomial product is the product of the first terms of the binomial factors.

(2) The last term of the trinomial product is the product of the last terms of the binomial factors.

(3) The second term of the trinomial product is the algebraic sum of two products, the first of which is obtained by multiplying the second term of one binomial by the first term of the other; the second of which is obtained by multiplying the first term of one binomial by the second term of the other: this process is known as cross-multiplication.

Apply these observations in expanding the following:

**2.** $(5x+3y)(3x+5y)$. **4.** $(3x+4y)(2x-y)$.

**3.** $(2a-3b)(3a+4b)$. **5.** $(7b-8c)(3b-4c)$.

**6.** Separate $2x^2+5x+2$ into two binomial factors.

SOLUTION. By reversing the principles involved in the above expansions, the following method is obtained for factoring any trinomial, which is the product of two binomial factors:

The first terms of the binomial factors will be factors of the first term of the trinomial product.

These are (1) 2 and $x^2$.

(2) $2x$ and $x$.

(3) $2x^2$ and 1.

The second terms of the binomial factors will be factors of the last term of the trinomial product.

These are 2 and 1.

These factors must be combined into two binomials, such that, by cross-multiplication, two products are obtained whose algebraic sum equals the second term of the trinomial.

The sets of factors in (1) and (3) contain $x^2$, and this does not occur as a factor in the second term of the trinomial. Therefore $2x$ and $x$ are the first terms of the binomial factors.

Trying the two possible methods of cross-multiplication, the factors are found to be

$$(2x+1) \text{ and } (x+2).$$

**7.** Factor $2x^2+11x+12$.

SUGGESTION. The possible factors of the first term are

$$2x \text{ and } x.$$

The possible pairs of factors of the last term are:

(1) 12 and 1.

(2) 6 and 2.

(3) 4 and 3.

Each pair of these latter factors must be tried with the factors of the first term, until products are found whose algebraic sum equals $11x$.

Factor the following:

8. $6x^2 + 7x + 2$.
9. $6x^2 + 11x + 3$.
10. $2x^2 - 7x + 6$.
11. $2x^2 + x - 6$.
12. $2x^2 - x - 15$.
13. $12x^2 + 11x - 15$.
14. $6a^2 + a - 15$.
15. $15a^2 + 20a - 35$.
16. $2a^2 + ab - b^2$.

## Lesson XLV.

(a) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(b) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(See Lesson XXVI., Theorems 4 and 5.)

The pupil should memorize formulas (a) and (b).

From the theorems above referred to, it follows that a binomial consisting of the sum or difference of any two cubes may be factored as follows:

Let it be required to factor $8a^3 + b^3$.

The cube roots of the two terms are $2a$ and $b$. Substitute in formula (a) $2a$ for $x$, and $b$ for $y$.

$$8a^3 + b^3 = (2a + b)(4a^2 - 2ab + b^2).$$

Let it be required to factor $8a^3 - 27b^3$.

The cube roots of the terms given are $2a$ and $3b$. Substitute these roots in formula ($b$) for $x$ and $y$.

Then $8a^3 - 27b^3 = (2a - 3b)(4a^2 + 6ab + 9b^2)$.

Factor the following:

1. $a^3 + b^3$.
2. $c^3 + x^3$.
3. $a^3 - b^3$.
4. $a^3 - y^3$.
5. $a^3 - 1$.
6. $1 + x^3$.
7. $8 + y^3$.
8. $27 + x^3$.
9. $8a^3 + 8y^3$.
10. $a^3b^3 + c^3$.
11. $4a^3 - 4y^3$.
12. $27x^3 - 1$.
13. $1 - a^3$.
14. $27 + x^3y^3$.
15. $24x^3 - 3y^3$.
16. $16a^3b^6 + 2b^3y^3$.
17. $54xy^3 - 16xz^3$.
18. $x^2yz^3 + 27x^5y$.

19. Factor $a^6 - 1$.

SUGGESTION. Note that $a^6$ is the square of $a^3$. This binomial, therefore, is the difference of two squares, and may be separated into the factors $(a^3 + 1)$ and $(a^3 - 1)$. See Lesson XXXIX.

Each of these factors may be again factored as in ($a$) and ($b$).

20. Factor $x^6 + 1$.

SUGGESTION. $x^6$ is the cube of $x^2$.

Factor the following:

21. $x^6 + 8$.
22. $x^4 - xy^3$.
23. $a^3b^3 - c^3x^3$.
24. $(a+b)^3 - c^3$.
25. $x^4 - x(a+c)^3$.
26. $(a+b)^3 + (c+x)^3$.
27. $(x+y)^3 - (a+b)^3$
28. $x^{12} - y^{12}$.
29. $27 - (a+b)^3$.
30. $1 - 64y^3$.
31. $3 - 24(x+1)^3$.
32. $a^4 - a(8 + b^3)$.
33. $54 - 2(a^3 - 8)$.
34. $3b^3 - 3(x+y)^3$.
35. $2a^3 + 2(a^6 - b^6)$.

## Lesson XLVI.

Factor the following:

1. $a^3 - a.$
2. $5ax^2 + 10axy + 5ay^2.$
3. $3ax^2 - 18ax + 27a.$
4. $36a - 24ax + 4ax^2.$
5. $3a^3b - 3ab^3.$
6. $3x^2 + 9x - 12.$
7. $5x^3 + 10x^2 - 175x.$
8. $ac + ad - bd - bc.$
9. $ab + 2b$[illegible] $+ 2ax + b$[illegible]
10. $x^3 - xy^2.$
11. $5x^2 - 7x + 2.$
12. $x^3 - x^2 - 2x.$
13. $x^4 - 10x^2 + 9.$
14. $x^4 - 17x^2 + 16.$
15. $x^4 - xy^3.$
16. $a^8 - b^4.$
17. $xy^4 + x^4y.$
18. $2a^3b^2c - 8a.$
19. $a^4 - 4b^4.$
20. $a^4 - 2a^2b^2 + b^4.$
21. $a^2 + b^2 - c^2 + 2ab.$
22. $a^2 - (b - c)^2.$
23. $(x + 7)^2 - (x + 2)^2.$
24. $x^2y - y^9.$

## Lesson XLVII.

Factor the following:

1. $x^5y - xy^5.$
2. $2a^4 - 2ab^3.$
3. $3ax^5 - 6ax^3y^2 + 3axy^4.$
4. $4y^7 + 8y^4 + 4y.$
5. $x^3y - 9x^2y + 20xy.$
6. $a^3 - a(b + c)^3.$
7. $a^5 + a^2(b + c)^3.$
8. $x^2 - y^2 - x + y.$
9. $4a^3b + 4a^2b - 168ab.$
10. $b^2 - 12b - 45.$
11. $x^4 + x^2 + 1.$
12. $c^4 - 8c^2 + 16.$
13. $4a^6 - 4b^6.$
14. $a^2 - a - 90.$
15. $a^2 + b^2 - c^2 - 2ab.$
16. $(x + y)^2 - x - y.$
17. $(a + b)^2x^2 - c^2x^2.$
18. $a^3y^3 + 1.$
19. $4a^3c^3 - 32b^3x^3.$
20. $1 - (a + b)^3.$
21. $x^4 - (y + 2)^4.$
22. $x^3 - (y + 2)^3.$
23. $ab + ac + 2bx + 2cx.$
24. $a^2b^2 - a^2c^2 - b^2x^2 + c^2x^2.$

## HIGHEST COMMON DIVISOR.

### Lesson XLVIII.

(a) $a^2b,\ bcx,\ abc.$

(b) $ax^2y,\ bxy^2,\ 25\,y.$

(c) $a^2x + a^2y,\ x^2 - y^2,\ x^2 + 2\,xy + y^2.$

What factor is common to the three quantities in (*a*)? In (*b*)? In (*c*)?

Note. A factor which is found in two or more quantities is called a **Common Factor**, or **Common Divisor**, of those quantities.

The product of all the prime factors common to two or more quantities is called their **Highest Common Divisor**, abbreviated **H. C. D.**

1. Find the H. C. D. of $6\,a^2b$ and $9\,ab^2$.

Solution. The common prime factors are 3, $a$, and $b$.

$3 \times a \times b = 3\,ab$, the Highest Common Divisor.

2. Find the H. C. D. of $10\,a^3bc$ and $18\,ab^2c$.

3. Find the H. C. D. of $55\,a^2c^2xy$ and $77\,acx^3y^3$.

4. Find the H. C. D. of $16\,a^2b^3$ and $24\,ab^2c$.

Suggestion. Note that the common prime factors are *three* factors 2, *one* factor $a$, and *two* factors $b$.

$2 \times 2 \times 2 \times a \times b \times b = 8\,ab^2$, the Highest Common Divisor.

Find the H. C. D. of the following:

5. $4\,a^2x^2,\ 10\,ax^3.$

6. $a^3b^2x^5,\ 24\,a^5b^5x^2.$

7. $4\,a^3x^3,\ 6\,a^3x^2,\ 10\,a^4x.$

8. $8\,ax^2y^4,\ 12\,x^5y^6,\ 24\,a^3x^3.$

9. $3\,ab,\ 5\,a^2b,\ 15\,ab^3.$

10. $21\,a^3x,\ 49\,ax^3y,\ 9\,xy^3.$

11. $8x^2y,\ 12xy^2,\ 24x^2y^2$.

12. $15a^2b^2c^2,\ 25a^2b^2,\ 30a^2c^2$.

13. $a^3+1,\ a^2-1$.

14. $a^2+2ab+b^2,\ ab+b^2$.

15. $20axy^2,\ 30ax^2y,\ 4a^2xy$.

16. $20a^2b^3,\ 25a^3b^2c,\ 35a^4b^3c^2,\ 15a^2b^2$.

17. $a^2+ab,\ 3a(a+b)$.

18. $(a+b)^2,\ (a+b)^3,\ (a+b)^4$.

19. $3(x+y)^3,\ 6(x+y)^4,\ 9(x+y)^5$.

20. $3(x+y)^2,\ 6(x^2+2xy+y^2)^2,\ 6x^2-6y^2$.

21. $3x^2-3b^2,\ 3x^2+6bx+3b^2$.

22. $ax^3-a^4,\ x^2-a^2$.

23. $x^2-2x+1,\ x^2-5x-6$.

24. $x^2-4x+4,\ x^2+8x+15$.

## Lesson XLIX.

Find the H. C. D. of:

1. $2(x^2-3abx-10a^2b^2),\ 4x^2+8abx$.
2. $x^2+ax+a^2,\ 3x^3-3a^3$.
3. $x^2+9ax+14a^2,\ x^2-4a^2$.
4. $4x^3(a+x)^2,\ 10(a^2x-x^3)^2$.
5. $x^2(a^2-x^2)^2,\ a^2x+ax^2$.
6. $(a^2b-ab^2)^2,\ ab(a^2-b^2)^2$.
7. $6(x^2-1),\ 8(x^2-3x+2)$.
8. $(x^2+x)^2,\ x^3(x^2-x-2)$.
9. $4(x^3+a^3),\ 6(x^2-2ax-3a^2)$.
10. $a^3(x^2+12x+11),\ a^2x^2-11a^2x-12a^2$.

11. $x^4-y^4$, $x^3-xy^2-x^2y+y^3$.

12. $3x^3-3$, $3x^2+12x-15$.

13. $x^6-y^6$, $x^2-xy+y^2$.

14. $x^3+1$, $x^2-4x-5$.

15. $9(a^2x^2-4)$, $12(a^2x^2+4ax+4)$.

16. $a^3-b^3$, $a^3+a^2b+ab^2$.

17. $a^4-ab^3$, $4a^3+4a^2b+4ab^2$.

18. $3a^2+6ab+3b^2$, $a^3b+b^4$.

19. $x^2+(a+b)x+ab$, $x^2+(a+c)x+ac$.

20. $a^2-ax-ab+bx$, $3a^3-3x^3$.

21. $a(a+b)^2$, $a^2(a+b)^3$, $a^3(a+b)$.

22. $2ax(x^2+y^2)$, $4ax^2(x^2+y^2)^2$, $6a^2x^2(x^2+y^2)^3$.

23. $4(a-b)^2$, $6(a-b)^3$, $8(a-b)^4$.

24. $xy(a-b)^3$, $x^2y(a-b)^4$, $a^2y^2-aby^2+b^2y^2$.

25. $x^2-4x$, $x^2-8x+16$, $x^2-2x-8$.

26. $x^2+x-6$, $x^2+7x+12$, $x^2-2x-15$.

27. $x^3+27$, $x^2+5x+6$, $x^2+6x+9$.

28. $x^2+ax+bx+ab$, $x^2+ax+cx+ac$.

29. $x^2-(a+b)^2$, $a^2-(x+b)^2$, $b^2-(x+a)^2$.

## LEAST COMMON MULTIPLE.

### Lesson L.

The quantity $a^2+ax$ may be exactly divided by $a$. Therefore $a^2+ax$ is called a **Multiple** of $a$. $a^2$, $a^3b$, $a^3-ab^2$ are also *Multiples* of $a$.

The quantity $a^2+ax$ is also exactly divisible by $a+x$ and 1. It is, therefore, called a **Common Multiple** of $a$, $a+x$, and 1.

Find all the quantities of which $a^2b^2$ is a common multiple. Of which $4ab$ is a common multiple. Of which $x^2 - y^2$ is a common multiple. Of which $3ax^2 - 3a^3$ is a common multiple.

NOTE. Observe that any quantity is a multiple of all its factors and of its factors only. Therefore, a common multiple of two or more quantities must contain all the factors of each of these quantities.

Find common multiples for:

**1.** $4,\ 6,\ a^2,\ b^2,\ a.$

**2.** $6a^2,\ a,\ x+y,\ (x+y)^2.$

**3.** $9,\ a^2,\ 3,\ b^3,\ a^3.$

**4.** $x+a,\ x^2-a^2,\ 2.$

NOTE. Observe that for each of these groups of quantities there are a great many products which are common multiples of the quantities. Of these multiples the least is called the **Lowest Common Multiple**, abbreviated **L. C. M.** This must contain *all the prime factors* of its divisors *and no other factors.*

To find the L. C. M. of any set of quantities, separate each quantity into its prime factors, and form a product which shall contain all the prime factors of each of the quantities and no other factors.

**5.** Find the L. C. M. of $a^2,\ 3b,\ 4c^2$.

SOLUTION. The prime factors of these three quantities are

$$a,\ a,\ 3,\ b,\ 2,\ 2,\ c,\ c.$$

Their product equals $12a^2bc^2$, the L. C. M. of the quantities.

**6.** Find the L. C. M. of $x^3,\ ay,\ 10x - 10y$.

**7.** Find the L. C. M. of $abc,\ x+y,\ 4$.

8. Find the L. C. M. of $a^2$, $abc$, $3ac^2$.

Solution. The L. C. M. of these three quantities is $3a^2bc^2$, for this contains the factors $a$, $a$ of the first quantity, the factors $a$, $b$, $c$ of the second quantity, and the factors 3, $a$, $c$, $c$ of the third quantity; and it contains no other factors.

Find the L. C. M. of the following:

9. $6a^2b$, $9ab^2$.

10. $10x^3y$, $18ax^3y$.

11. $3ab$, $5a^2b$, $15ab^3$.

12. $10x^2y$, $15xy^2$, $20x^2y^2$.

13. $12a^2b^2$, $18ab^2c$, $24a^2c^3$.

14. $24ax^3$, $32bxy$, $48cy^2$

15. $22x^2y^3$, $33x^2a^2$, $44y^2a^2$.

16. $24ab^2x^2$, $12a^2x^3c$, $8b^3c^3$.

17. $9$, $a^3y^2$, $12a^2xy$, $18a^4x^3$.

18. $2a$, $(a+x)$, $(a+x)^2$.

19. $3c$, $c+cx$, $(1+x)^2$.

20. $2(x-a)$, $3cx^2-3a^2c$.

21. $2(x+a)$, $8(x-a)^2$, $4(x^2-a)^2$.

22. $6(y-3a)$, $3(y+3a)$, $9(y^3-9a^3)$.

23. $(a+b)^3$, $a^3-b^3$, $a^2-b^2$.

24. $(x+y)^2$, $x^2-y^2$, $x^2+y$.

25. $x^3-y^3$, $(x-y)^3$, $(x+y)^3$.

## Lesson LI.

Find the L. C. M. of the following:

1. $a(b^2-c^2)$, $x(b+c)$, $y(b-c)$.

2. $x^2+ax+a^2$, $3(x^3-a^3)$, $x-a$.

3. $21(x^4-a^4)$, $14(x^2+a^2)$, $18x^2-18a^2$.

4. $(a+b)^2$, $(a+b)^3$, $(a+b)^4$.

5. $25(x+y)^2,\ 50(x+y)^3,\ 10(x+y)^5.$

6. $a^3(x-y),\ a^2(x-y)^2,\ a^4(x-y)^3.$

7. $12a^2(x+c)^4,\ 8ab(x+c)^3,\ 4x^2+8cx+4c^2.$

8. $a^3(x-y),\ a^2(x-y)^2,\ a^4(x-y)^3.$

9. $(a+b)^2,\ 3(a^2-b^2).$

10. $x-y,\ x^2-y^2,\ (x+y)^2.$

11. $x^2+2xy+y^2,\ x^2-2xy+y^2.$

12. $(x-y)^2,\ x-y,\ x^2-y^2.$

13. $x+3,\ 2(x+5),\ x^2+8x+15.$

14. $(x+a)(x+b),\ (x+a)(x+c),\ (x+b)(x+c).$

15. $x^3+y^3,\ x^3-y^3,\ 2x^2-2xy+2y^2.$

16. $x^2+5x+6,\ x^2-2x-8,\ x^2-x-12.$

17. $x^2+3x-4,\ x^2-6x+5,\ x^2-x-20.$

18. $x^2+8x+15,\ x^2+7x+12,\ x^2+9x+20.$

19. $x^2+2x-15,\ x^2-7x+12,\ x^2+x-20.$

20. $x^2+x-20,\ x^2-5x-4,\ x^2+4x-5.$

21. $a^2-(b+c)^2,\ b^2-(a+c)^2,\ c^2-(a+b)^2.$

22. $2x^2+11x+15,\ 2x^2+x-10,\ x^2+x-6.$

23. $6x^2+13x+6,\ 6x^2-5x-6,\ 4x^2-9.$

24. $ax^2-ay^2,\ x^4-xy^3,\ x^3y+y^4.$

25. $x^2+y^2,\ x^4-x^2y^2-2y^4,\ x^4-y^4.$

26. $x^3+y^3,\ x^6-x^3y^3+y^6,\ x^2-y^2.$

27. $a^2-b^2,\ 3(a-b)^3,\ a^3-b^3.$

28. $6(a+b)^2,\ 3(a-b)^2,\ a^4-b^4.$

## FRACTIONS.

### Lesson LII.

In Lesson III., it was shown that one method of indicating the operation of division is by writing the divisor beneath the dividend with a line between them.

Thus $a \div b = \frac{a}{b}$. This last expression is called a **Fraction**.

The fraction, $\frac{a}{b}$, indicates the quotient obtained by dividing the quantity $a$ by the quantity $b$, and is the simplest form in which the quotient can be expressed.

$$\frac{6a}{2a} = \frac{24a}{8a} = \frac{3}{1} = 3.$$

Note. Multiplying a dividend and its divisor by the *same* quantity does not alter the value of the quotient. Thus, the divisor, $2a$, is $\frac{1}{3}$ of the dividend, $6a$, and four times $2a$ is $\frac{1}{3}$ of four times $6a$.

$$\frac{6a}{2a} = \frac{3a}{a} = \frac{3}{1} = 3.$$

Dividing a dividend and its divisor by the same quantity does not alter the value of the quotient. Thus, the divisor, $2a$, is $\frac{1}{3}$ of the dividend, $6a$, and one half of $2a$ is $\frac{1}{3}$ of one half of $6a$. Hence, the numerator and denominator of any fraction may be multiplied by the same quantity, or divided by the same quantity, without altering the value of the fraction.

Thus, $$\frac{6a}{2a} = \frac{24a}{8a} = \frac{3a}{a} = \frac{3}{1} = 3.$$

1. Reduce $\frac{ax + ay}{ab}$ to its simplest form.

SOLUTION. Find a factor common to the numerator and denominator of the above fraction, and divide both terms of the fraction by the common factor.

$$\frac{ax + ay}{ab} = \frac{x + y}{b}.$$

Since the terms of the fraction $\frac{x+y}{b}$ have no common factor, the fraction is said to be in its *simplest form.*

2. Reduce $\frac{x+y}{x^2 - y^2}$ to its simplest form.

SOLUTION. A fraction is in its simplest form when its terms have no common factors. Since the H. C. D. of two quantities contains all their common factors, a fraction may be reduced to its simplest form by dividing both its terms by their H. C. D. Divide both terms of the above fraction by their H. C. D., $x + y$.

$$\frac{x + y}{x^2 - y^2} = \frac{1}{x - y}.$$

Reduce the following fractions to their simplest form:

3. $\frac{4a^3x^2}{6a^4}$.

4. $\frac{6a^2x^2}{8ax^3}$.

5. $\frac{9x^4y^3c^4}{12x^3y^4c^5}$.

6. $\frac{12x^2y^2b^4}{8x^2b^2}$.

7. $\frac{2(a+b)}{3a(a+b)}$.

8. $\frac{(a+b)^3}{2a(a+b)^2}$.

9. $\frac{x^2-1}{2(x+1)}$.

10. $\frac{x^3-y^3}{2a(x-y)}$.

11. $\frac{5(x^3+y^3)}{25a(x+y)}$.

12. $\frac{a^4-b^4}{5a(a-b)}$.

13. $\frac{4a^5-4ab^4}{6a^2c+6b^2c}$.

14. $\frac{a^3-x^3}{(a-x)^2}$.

15. $\frac{a^2+2a-3}{a^2+6a-7}$.

16. $\frac{2a^2-2}{a^2-2a+1}$.

17. $\frac{x^2-a^2}{5(x^3-a^3)}$.

18. $\dfrac{3a(x^2+2ax+a^2)}{4c(x^2-a^2)}$.

19. $\dfrac{4(a+x)(a^2+x^2)}{8a^4b-8bx^4}$.

20. $\dfrac{2a^2-6a-8}{2a^2-8a-10}$.

21. $\dfrac{4a^2-4b^2}{a^2-2ab+b^2}$.

22. $\dfrac{y^3-b^2y}{y^2+2by+b^2}$.

23. $\dfrac{a^2-(b+c)^2}{a^2+ab+ac}$.

24. $\dfrac{(1-x^2)^2}{(1+x)^3}$.

25. $\dfrac{2x^3+2y^3}{4x^2-4xy+4y^2}$.

26. $\dfrac{16a^4+4b^3}{8(a+b)^2}$.

27. $\dfrac{(a+1)(a^2-a+1)}{a^6-1}$.

## Lesson LIII.

1\. Reduce $\dfrac{x^2-y^2}{x+y}$ to its simplest form.

Suggestion. In this fraction the denominator is a factor of the numerator; hence both terms are exactly divisible by the denominator, and the quotient is an integer.

Note. Observe that a fraction may be reduced to an integer when the denominator is a factor of the numerator.

Reduce the following fractions to integers:

2. $\dfrac{2(a^2-b^2)}{a-b}$.

3. $\dfrac{a^6-b^6}{a^2+b^2}$.

4. $\dfrac{x^3+y^3}{x^2-xy+y^2}$.

5. $\dfrac{3(a^2+2ax+x^2)}{3a+3x}$.

6. $\dfrac{8(a^3-b^3)}{4a^2+4ab+4b^2}$.

7. $\dfrac{x^2-5x+4}{x-4}$.

8. $\dfrac{x^2+12x+35}{x+7}$.

9. Reduce $2a + b$ to a fraction having the denominator $ac$.

SOLUTION. $2a + b = \frac{ac(2a + b)}{ac}$.

In division, the product of the quotient and divisor must equal the dividend. Consider $2a + b$, the given integer, as the quotient of the required fraction. Then the product of $2a + b$, the quotient, and $ac$, the divisor or denominator, must be the dividend or numerator. This product is indicated above, thus: $ac(2a + b)$. Hence, an integer may be reduced to a fraction having a given denominator by taking for the numerator of the required fraction *the product of the integer and the required denominator.*

10. Reduce $a^2x - by$ to a fraction with the denominator $ax$.

11. Reduce $x^2 - cx$ to a fraction with the denominator $5c$.

12. Reduce $a + b$ to a fraction with the denominator $a - b$.

13. Reduce $x - y$ to a fraction with the denominator $x + y$.

14. Reduce $\frac{a^2 + c}{a}$ to its simplest form.

$$\frac{a^2 + c}{a} = a + \frac{c}{a}.$$

SUGGESTION. Dividing both terms of the fraction by its denominator, the quotient is the integer $a$, and a remainder $c$, the division of which is indicated in the form of a fraction.

NOTE. An expression consisting of an integer and a fraction is called a **Mixed Quantity.**

15. Reduce $\frac{bx + cy}{b}$ to a mixed quantity.

16. Reduce $\frac{a^2b^2 + abc + c^2}{ab}$ to a mixed quantity.

17. Reduce $\frac{a^2+ab-c}{a}$ to a mixed quantity.

SUGGESTION. In this fraction the remainder, $-c$, is a negative quantity, and, therefore, the fraction, $\frac{c}{a}$, must be subtracted from the integer, $a+b$.

$$\frac{a^2+ab-c}{a}=a+b-\frac{c}{a}.$$

Reduce to mixed quantities:

18. $\frac{2by+3ax+c}{2b}$.    19. $\frac{bx-3a}{b}$.    20. $\frac{(a-b)^2}{a}$.

21. Reduce $a+\frac{a}{x}$ to a fraction.

$$a+\frac{a}{x}=\frac{ax+a}{x}.$$

SOLUTION. Since any integer may be made the numerator of a fraction having a given denominator by multiplying it by the given denominator, the integer, $a$, is multiplied by the given denominator, $x$, and the product added to the numerator of the fraction $\frac{a}{x}$.

Reduce the following mixed quantities to fractions:

22. $ab+\frac{c}{a}$.    23. $9c^2+\frac{2ab}{y}$.    24. $c-\frac{c}{y}$.

SUGGESTION. In Example 24, since the fraction is a negative quantity, the numerator, $c$, must be subtracted from the product of $c$ and $y$. Therefore, $c-\frac{c}{y}=\frac{cy-c}{y}$.

25. $a+\frac{x+a}{x}$.    27. $a-x+\frac{x^2}{a+x}$.

26. $a+x+\frac{a^2}{a+x}$.    28. $x-a+\frac{x^2+a^2}{x+a}$.

## LESSON LIV.

1. Reduce $\frac{2a-b}{c}$ to an equivalent fraction with the denominator $ac^2$.

SOLUTION. Observe that the denominator, $c$, of the given fraction must be multiplied by $ac$ to equal $ac^2$, the required denominator. Since the value of the fraction is to remain unchanged, the numerator, $2a-b$, must also be multiplied by $ac$. Thus,

$$\frac{2a-b}{c}=\frac{(2a-b)\times ac}{c\times ac}=\frac{ac(2a-b)}{ac^2}.$$

NOTE. To reduce a fraction to an equivalent fraction with a given denominator, multiply both terms of the fraction by such a quantity as will make the denominator of the fraction equal the required denominator.

2. Reduce $\frac{x-2}{x+3}$ to an equivalent fraction with the denominator $x^2-9$.

SUGGESTION. The factors of $x^2-9$ are $(x+3)$ and $(x-3)$. Therefore both terms of the fraction must be multiplied by $x-3$.

NOTE. To find the quantity by which to multiply both terms of the given fraction, divide the required denominator by the denominator of the fraction.

3. Reduce $\frac{x+y}{a+b}$ to an equivalent fraction with the denominator $4a^2+4ab$.

4. Reduce $\frac{5ax}{3cy}$ to an equivalent fraction with the denominator $9cy^2$.

5. Reduce $\frac{8(a-b)}{a+b}$ to an equivalent fraction with the denominator $a^2 - b^2$.

6. Reduce $\frac{5a+b}{2}$, $\frac{4x+b}{4a}$, $\frac{7y+3c}{6ab}$ to equivalent fractions having the common denominator $12a^2b$.

7. Reduce $\frac{1}{ab}$, $\frac{1}{bc}$, $\frac{1}{ac}$ to equivalent fractions having the common denominator $a^2b^2c$.

8. Reduce $\frac{1}{x+y}$, $\frac{a}{x-y}$, $\frac{b}{2(x^2-y^2)}$ to equivalent fractions having the common denominator $4(x^2-y^2)$.

NOTE. Fractions having a common denominator are called **Similar Fractions.**

9. Reduce to similar fractions: $\frac{ax}{by}$, $\frac{bx}{ay}$, $\frac{cx}{ab}$.

NOTE. Any common multiple of the three denominators may be taken for a common denominator, but the simplest result is obtained by taking the L. C. M. of the three denominators for a common denominator. This denominator is called the **Least Common Denominator,** abbreviated **L. C. D.**

Reduce to similar fractions having the L. C. D:

10. $\frac{3a}{bc}$, $\frac{4c}{ab}$, $\frac{9b}{2ac}$.

11. $\frac{3ac}{2x^2y}$, $\frac{2bc}{3ax^2}$, $\frac{a}{x^2y^2}$.

12. $\frac{a}{a+b}$, $\frac{b}{a-b}$, $\frac{c}{a^2-b^2}$.

13. $\frac{a+x}{a-x}$, $\frac{a-x}{a+x}$, $\frac{a^2+x^2}{a^2-x^2}$.

14. $\frac{a}{a+x}$, $\frac{b}{a^3+b^3}$, $\frac{c}{a^2-ax+x^2}$.

**15.** $\dfrac{3}{x^2+5x+6}$, $\dfrac{2a}{x^2-3x-10}$. **16.** $\dfrac{3a}{a^2-ab}$, $a+\dfrac{b^2}{a+b}$.

SUGGESTION. In Example 16, reduce the first fraction to its simplest form and reduce the mixed number to a fraction.

**17.** $\dfrac{ab}{x^2-3x+2}$, $\dfrac{bc}{x^2+4x-12}$.

**18.** $\dfrac{x}{4x^2-1}$, $\dfrac{2x-1}{2x+1}$, $\dfrac{2x+1}{2x-1}$.

**19.** $a+b+\dfrac{a^2}{b}$, $\dfrac{b}{ab^2-b^3}$.

**20.** $\dfrac{3}{1-2x}$, $\dfrac{7}{1+2x}$, $\dfrac{4a}{1-4x^2}$.

**21.** $\dfrac{1}{a+x}$, $\dfrac{2a}{a-x}$, $\dfrac{3b}{(a+x)^2}$, $\dfrac{4c}{(a-x)^2}$.

## ADDITION AND SUBTRACTION OF FRACTIONS.

### LESSON LV.

**1.** Find the sum of $\frac{2}{8}$ and $\frac{3}{8}$.

SOLUTION. $\frac{2}{8}$ may be considered 2 times $\frac{1}{8}$. $\frac{3}{8}$ may be considered 3 times $\frac{1}{8}$. Then $2\times\frac{1}{8}+3\times\frac{1}{8}=(2+3)\times\frac{1}{8}=\frac{5}{8}$.

**2.** Find the sum of $\dfrac{2}{x}$ and $\dfrac{3}{x}$.

SOLUTION. $\dfrac{2}{x}=2\times\dfrac{1}{x}$. $\dfrac{3}{x}=3\times\dfrac{1}{x}$.

$$2\times\frac{1}{x}+3\times\frac{1}{x}=(2+3)\times\frac{1}{x}=\frac{5}{x}.$$

3. Find the sum of $\frac{a}{x}$ and $\frac{b}{x}$.

SOLUTION. $\frac{a}{x}=a\times\frac{1}{x}.\quad \frac{b}{x}=b\times\frac{1}{x}.$

$$a\times\frac{1}{x}+b\times\frac{1}{x}=(a+b)\times\frac{1}{x}=\frac{a+b}{x}.$$

4. Find the difference of $\frac{a}{x}$ and $\frac{b}{x}$.

SOLUTION. $\frac{a}{x}=a\times\frac{1}{x}.\quad \frac{b}{x}=b\times\frac{1}{x}.$

$$\frac{a}{x}-\frac{b}{x}=(a-b)\times\frac{1}{x}=\frac{a-b}{x}.$$

NOTE. The *sum* of similar fractions equals *the algebraic sum of their numerators divided by the common denominator.*

The *difference* of similar fractions equals the *difference of their numerators divided by the common denominator.*

Let the pupil verify the above statements from the operations in the foregoing examples.

Find the sums of the following fractions:

5. $\frac{ab}{xy}$, $\frac{ac}{xy}$, and $\frac{2ab}{xy}$.

6. $\frac{2a}{x+y}$, $\frac{3c}{x+y}$, and $\frac{x}{x+y}$.

7. $\frac{a^2}{ac+bc}$, $\frac{ab}{ac+bc}$, $\frac{ab}{ac+bc}$, and $\frac{b^2}{ac+bc}$.

SUGGESTION. The sum should always be reduced to its simplest form.

8. Find the sum of $\frac{ab}{c}$, $\frac{ac}{b}$, and $\frac{bc}{a}$.

SUGGESTION. Only similar fractions can be added.

9. Add $\frac{1}{1+a}$, $\frac{1}{1-a}$, and $\frac{2}{1-a^2}$.

10. Add $\frac{a}{x^2+y^2}$, $\frac{a}{x^2-y^2}$, and $\frac{a}{x^4-y^4}$.

11. From $\frac{5\,ac}{2\,xy}$ subtract $\frac{2\,ab}{3\,xy}$.

12. From $\frac{2\,ab}{4\,y}$ subtract $\frac{3\,ab}{4\,y^2}$.

13. From $\frac{x+y}{x-y}$ subtract $\frac{x^2+y^2}{x^2-y^2}$.

14. From $\frac{x^2+ax+a^2}{x^2+ax+a^2}$ subtract $\frac{x^3+a^3}{x^3-a^3}$.

NOTE. In place of the *parenthesis* a straight line called a **Vinculum** ($\overline{\phantom{xx}}$) is often used. Thus $(a+b)$ is equivalent to $\overline{a+b}$, and the product, $(a+b)(a-b)$, may be written $\overline{a+b}\;\overline{a-b}$.

The horizontal line in a fraction *acts as a vinculum for both numerator and denominator.*

15. Find the value of $\frac{3\,a+b}{xy}-\frac{a+2}{xy}$.

SOLUTION. The sign before the fraction $\frac{a+2}{xy}$ indicates that this fraction is to be subtracted. Since the horizontal line acts as a vinculum, or parenthesis, for the numerator, and since the fraction is preceded by the minus sign, the signs of the terms in the numerator must be changed.

$$\frac{3\,a+b}{xy}-\frac{a+2}{xy}=\frac{3\,a+b-a-2}{xy}=\frac{2\,a+b-2}{xy}.$$

NOTE. From the above it follows that, in combining similar fractions, when a fraction is preceded by the *minus sign*, the signs of all the terms in the numerator of the fraction must be changed.

Find the value of:

16. $\frac{1}{1+a}+\frac{1}{1-a}$.

17. $\frac{a}{a+x}-\frac{a}{a-x}$.

18. $\frac{x}{a^2-b^2}+\frac{b}{a+b}$.

19. $\frac{1}{ab}+\frac{1}{bc}-\frac{1}{bc}$.

20. $\frac{a}{a+b}+\frac{b}{a-b}+\frac{a^2+b^2}{a^2-b^2}$.

21. $\frac{1}{x-1}-\frac{x}{x+1}+\frac{y}{x^2-y^2}$.

## LESSON LVI.

Find the value of:

1. $\frac{1}{x-1}-\frac{2}{x+1}$.

2. $\left(2a-2x+\frac{a-x}{a}\right)-\left(2a-4x+\frac{x-a}{x}\right)$.

3. $\frac{3x}{4}-\frac{y}{3}+\frac{7x}{3}+\frac{4y}{5}$.

4. $\frac{a}{a-x}+\frac{a}{a+x}-\frac{a^2}{a^2-x^2}$.

5. $\frac{x}{x-3}-\frac{x+3}{x}$.

6. $\frac{2(a^2+b^2)}{a^2-b^2}-\frac{a-b}{a+b}$.

7. $\frac{1+a^2}{1-a^2}-\frac{1-a^2}{1+a^2}$.

8. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

9. $\frac{4a^2}{1-a^4}+\frac{1-a^2}{1+a^2}$.

10. $\dfrac{a}{b}+\dfrac{a-3b}{cx}+\dfrac{a^2-b^2-ab}{bcx}$.

11. $\dfrac{b+c}{bc}+\dfrac{c-a}{ac}-\dfrac{a-b}{ab}$.

12. $\dfrac{a-x}{a}-\dfrac{a^2-x^2}{ax}$.

13. $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}$.

14. $\left(a+\dfrac{a}{x}\right)+\left(2a-\dfrac{b}{x}\right)-\left(3a+\dfrac{c}{x}\right)$.

15. $\left(a+b-\dfrac{b}{ax}\right)-\left(2a-3b+\dfrac{a}{bx}\right)$.

16. $\dfrac{1}{x+y}+\dfrac{xy}{x^3+y^3}+\dfrac{x-y}{x^2-xy+y^2}$.

17. $\dfrac{x}{x+y}-\dfrac{xy}{(x+y)^2}-\dfrac{x^3}{(x+y)^3}$.

18. $\dfrac{2}{x}-\dfrac{3}{2x-1}-\dfrac{2x-3}{4x^2-1}$.

19. $\dfrac{2}{x^2-7x+12}-\dfrac{4}{x^2-6x+8}$.

20. $\dfrac{1}{a+x}+\dfrac{1}{a-x}-\dfrac{x}{(a+x)^2}+\dfrac{x}{(a-x)^2}$.

21. $\dfrac{3}{a(x-y)}+\dfrac{2}{b(x+y)}+\dfrac{x}{c(x^2-y^2)}+\dfrac{y}{d(x^4-y^4)}$.

22. $\dfrac{ab+1}{a^2b^2-1}+\dfrac{a^2b^2-1}{a^4b^4-1}+\dfrac{a^4b^4+1}{a^8b^8-1}+\dfrac{x}{y}$.

23. $\dfrac{3ax(c+1)^2}{c^2-1}-\dfrac{x^2+y^2}{c^2+9c+8}+\dfrac{a-b}{c^2-7c-8}$.

## MULTIPLICATION AND DIVISION OF FRACTIONS.

### Lesson LVII.

Multiply the dividend in $\frac{abx}{bx} = a$ by $b$; then

$$\frac{ab^2x}{bx} = ab.$$

Observe that the quotient has also been multiplied by $b$.

Divide the divisor in $\frac{abx}{bx} = a$ by $b$; then

$$\frac{abx}{x} = ab.$$

Observe that the quotient has again been multiplied by $b$.

Note. Since a fraction is an indicated division, and, in its simplest form, is also the quotient of the division, it follows from the above that a fraction is multiplied by an integer *when its numerator is multiplied by the integer or its denominator is divided by the integer.*

Find the value of:

1. $\frac{xy}{a} \times x.$
2. $\frac{xy}{a^2} \times a.$
3. $\frac{4x^2}{3y} \times 3.$
4. $\frac{ax}{by} \times y.$
5. $\frac{a+c}{x^2-y^2} \times x+y.$
6. $\frac{7y^3}{3(a+b)^2} \times 3a+3b.$

Divide the dividend in $\frac{ab^2x}{ax} = b^2$ by $b$; then

$$\frac{abx}{ax} = b.$$

Observe that the quotient is also divided by $b$.

Multiply the divisor in $\frac{ab^2x}{ax} = b^2$ by $b$; then

$$\frac{ab^2x}{abx} = b.$$

Observe that the quotient has again been divided by $b$.

NOTE. A fraction is divided by an integer *when its numerator is divided by the integer, or its denominator is multiplied by the integer.*

Find the value of:

7. $\frac{ab}{c} \div a.$

8. $\frac{abc}{x} \div x.$

9. $\frac{ac^2}{x^2} \div ac.$

10. $\frac{x^2 - y^2}{x^2} \div x - y.$

11. Find the value of $\frac{x + y}{2(a^2 - b^2)} \times 3(a + b).$

SOLUTION. Multiply the numerator of the fraction by the integer.

Then $\frac{3(x + y)(a + b)}{2(a^2 - b^2)} = \frac{3(x + y)}{2(a - b)}$ when reduced to its simplest form.

This result is more easily obtained by removing the common factors from the denominator of the fraction and from the integer before multiplying.

Thus $$\frac{x + y}{2\underset{a-b}{\cancel{(a^2 - b^2)}}} \times 3\cancel{(a + b)} = \frac{3(x + y)}{2(a - b)}.$$

This operation is called **Cancellation**, and should be used when possible.

12. Find the value of $\dfrac{2(x+a)}{4(c-y)} \times c^2 - y^2$.

SOLUTION. $\dfrac{\overset{}{\cancel{2}}(x+a)}{\underset{2}{\cancel{4}}\cancel{(c-y)}} \times \overset{c+y}{\cancel{c^2-y^2}} = \dfrac{(x+a)(c+y)}{2}$.

13. Find the value of $\dfrac{ab}{c} \times \dfrac{cx}{a+b} \times \dfrac{a^2-b^2}{4ab}$.

SOLUTION. $\dfrac{\cancel{ab}}{\cancel{c}} \times \dfrac{\overset{x}{\cancel{cx}}}{\cancel{a+b}} \times \dfrac{\overset{a-b}{\cancel{a^2-b^2}}}{\underset{4}{\cancel{4ab}}} = \dfrac{x(a-b)}{4}$.

Find the value of:

14. $\dfrac{a+b}{4x-2y} \div 2x+y$.

15. $\dfrac{x^2-4y^2}{a^2-b^2} \div 3x-6y$.

16. $\dfrac{(x-y)^2}{4(a-x)} \div x^2-y^2$.

17. $\dfrac{a-c}{x^4-y^4} \div x^2-y^2$.

18. $\dfrac{x^2+2xc+c^2}{4a(x+c)} \div x+c$.

19. $\dfrac{a^2-b^2}{a+b} \div a-b$.

20. $\dfrac{x^2-5x+6}{2(x-3)} \div x-2$.

21. $\dfrac{x^2+ax+a^2}{x-a} \times (x-a)^2$.

22. $\dfrac{5ab^2}{a^2+2ab+b^2} \times 5(a+b)$.

23. $\dfrac{2x^2y}{x^2-2x-6} \times x^2+2x-24$.

24. $\dfrac{x-y}{a+b} \times x(a^2-b^2)$.

25. $\dfrac{x^2y-a^2y}{1-b} \div axy+a^2y$

Pauline Lauper



## LESSON LVIII.

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

SOLUTION. Multiply $\frac{a}{b}$ by $c$. $\frac{a}{b} \times c = \frac{ac}{b}$.

This product is $d$ times too large, since it was required to multiply only by $c \div d$. Therefore, the fraction $\frac{ac}{b}$ must be divided by $d$.

$$\frac{ac}{b} \div d = \frac{ac}{bd}.$$

Therefore, $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

NOTE. Observe that the product of the numerators of the factors is the numerator of the product, and that the product of the denominators of the factors is the denominator of the product.

2. Multiply $\frac{a}{c}$ by $\frac{b}{y}$.

SOLUTION. $$\frac{a}{c} \times \frac{b}{y} = \frac{a \times b}{c \times y} = \frac{ab}{cy}.$$

Find the value of:

3. $\frac{ab}{cd} \times \frac{ad}{bc}$.

4. $\frac{x^2y}{ab} \times \frac{3bc}{ax}$.

5. $\frac{x^2}{y^2} \times \frac{a}{x} \times \frac{b}{x}$.

6. $\frac{c^2}{xy} \times \frac{ab}{y} \times \frac{xy^2}{ac}$.

7. $\frac{a+b}{a-b} \times \frac{x+y}{2a^2-2b^2} \times \frac{a-b}{2x}$.

8. $\frac{2a^2-4ab+2b^2}{x-2} \times \frac{x^2-x-2}{4x+4}$.

9. Divide $\frac{ac}{bd}$ by $\frac{c}{d}$.

In Ex. 1, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, of which $\frac{a}{b}$ and $\frac{c}{d}$ are the factors.

Therefore, $$\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}. \quad (1)$$

Observe that the numerator of the quotient equals the numerator of the dividend divided by the numerator of the divisor, and that the denominator of the quotient equals the denominator of the dividend divided by the denominator of the divisor.

Also $$\frac{ac}{bd} \div \frac{a}{b} = \frac{c}{d}. \quad (2)$$

But $$\frac{ac}{bd} \times \frac{b}{a} = \frac{c}{d},$$

and $$\frac{ac}{bd} \times \frac{d}{c} = \frac{a}{b}.$$

NOTE. From the above it may be seen that dividing by a fraction is equivalent to *multiplying by the fraction inverted.*

Find the value of:

10. $\frac{x^2 - y^2}{x^2a^2} \div \frac{x + y}{a}$.

11. $\frac{2\,ax}{a^2 - b^2} \div \frac{2\,a}{a + b}$.

12. $\frac{7\,axy}{3\,x^2a^2} \div \frac{7\,x}{3\,x^2a}$.

13. $\frac{25\,axy}{27\,bc} \div \frac{5\,xy}{9\,bc}$.

14. $\frac{5\,x^2}{7} \div \frac{x}{3}$.

15. $\frac{x - 1}{6} \div \frac{3\,x}{3}$.

16. $\frac{x}{x + 1} \div \frac{x}{2}$.

17. $\frac{(a - b)^2}{a + b} \times \frac{b}{x(a - b)}$.

Find the value of:

18. $\frac{2a}{x} \times \frac{3b}{c} \times \frac{4ac}{d}$.

19. $\frac{a+x}{20} \times \frac{4a}{3(a+x)}$.

20. $\frac{4ax}{y} \times \frac{3xy}{2a} \times \frac{2}{x}$.

21. $\left(b + \frac{bx}{a}\right)\frac{a}{x}$.

22. $\frac{x^2 - b^2}{bc} \times \frac{x^2 + b^2}{b + c}$.

23. $\frac{x^2 - 2xy + y^2}{ab} \div \frac{x - y}{bc}$.

24. $\frac{x^4 - b^4}{x^2 - 2bx + b^2} \div \frac{x^2 + bx}{x - b}$.

25. $\left(1 + \frac{1}{a}\right) \div \left(1 - \frac{1}{a^2}\right)$.

SUGGESTION. In Example 21, reduce the mixed quantity to a fraction before multiplying.

26. $\left(\frac{1}{1+x} + \frac{x}{1-x}\right) \div \frac{(1+x^2)^2}{(1-x^2)^2}$.

SUGGESTION. In Example 26, perform the addition before dividing.

27. $\frac{(a-x)^3}{2a} \times \frac{3ab}{a-x} \times \frac{2c}{(a-x)^2}$.

28. $\frac{x^2}{y^2} \times \frac{a+b}{x^2 - y^2} \div \frac{4(a+b)}{x^2 + 2xy + y^2}$.

SUGGESTION. Invert the last fraction in Example 28, and take the continued product.

## LESSON LIX.

### *Problems in Fractions.*

1. Reduce $\frac{1 - x^2 - 2}{1 + x}$ to a mixed quantity.

2. Reduce $\frac{x^2 - xy + y^2}{x + y}$ to a mixed quantity.

3. Reduce $\frac{x^3+y^3}{x+y}$ to an integer.

4. Reduce $2a - y + \frac{(x-y)^2}{y}$ to a fraction.

5. Reduce $2 - \frac{(x-y)^2}{x^2+y^2}$ to a fraction.

Reduce to the simplest form:

6. $\frac{a^2+2ab+b^2-c^2}{2a+2b+2c}$.

7. $\frac{x^2-y^2}{x^2-ax-xy+ay}$.

8. $\frac{x^3-y^3}{x^2-xy+xy^2-y^3}$.

9. $\frac{(x+y)^2-(x-y)^2}{8xy}$.

10. $\frac{2x^2-10x-28}{10x-35}$.

Find the value of the following:

11. $\frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{a^2+b^4}$.

12. $\left(\frac{a}{x}+\frac{b}{y}\right)\left(\frac{x}{b}+\frac{y}{a}\right)$.

13. $\frac{x^2+y^2}{x^2-y^2}+\frac{2x^2y}{x^3+y^3}+\frac{2xy^3}{y^3-x^3}$.

14. $\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)^2\left(\frac{x}{x^2-1}\right)$.

15. $\frac{3}{1-2a}-\frac{7}{1+2a}-\frac{4-20a}{4a^2-1}$.

16. $\frac{a^2+b^2}{ab}-\frac{a^2}{ab+b^2}-\frac{b^2}{a^2+ab}$.

17. $\frac{(x+a)(x-a)}{x^3+a^3} \div \frac{2a^2-2ax+2x^2}{(x+a)^2}$.

18. $\frac{3ax}{2by} \times \frac{(a+x)(a-x)}{c^2-x^2} \times \frac{bc+bx}{a^2+ax} \times \frac{c-x}{a-x}$.

19. $\left(\frac{x+a}{x-a}-\frac{x-a}{x+a}-\frac{4a^2}{x^2-a^2}\right)\left(\frac{x+a}{2a}\right)$.

20. $\left(\frac{1}{(a-b)^2}-\frac{1}{c^2}\right)\left(\frac{a^2-b^2}{a-b-c}\right)$.

# INVOLUTION.

## Lesson LX.

Before beginning this lesson, let the pupil re-read carefully the beginning of Lesson XXXIX.

1. $x \times x = ?$ $2a \times 2a = ?$ $9c^2 \times 9c^2 = ?$
   $3a^2b \times 3a^2b = ?$ $xy \times xy = ?$ $7c^2y^2 \times 7c^2y^2 = ?$
   $(a+b)(a+b) = ?$ $(a-b)(a-b) = ?$ $(x+2)(x+2) = ?$

What name is given to the products of the above pairs of factors? How many equal factors are there in each product?

2. $a \times a \times a = ?$ $2a^2 \times 2a^2 \times 2a^2 = ?$ $b^3 \times b^3 \times b^3 = ?$

What name is given to the products of the above sets of factors? How many equal factors are there in each product?

Note. (For the definition of *Power* and *Root*, see Lesson XXXIX.) A power takes its name from the exponent, which shows how many times the given quantity is to be used as a factor. Thus, $(b^2c^2y^4)^5$ is read: *The fifth power of* $b^2c^2y^4$. The second power and third power of a quantity are more commonly called the *Square* and *Cube* of the quantity.

3. How many times is a quantity used as a factor in producing the fourth power? The sixth power?

4. What is the cube of $a^2y^2$? The fourth power of $xy$? The sixth power of $abc$?

5. Find the square of $2a^2b^3$.

Solution. $(2a^2b^3)^2 = 2a^2b^3 \times 2a^2b^3$
$= 4a^{2+2}b^{3+3} = 4a^{2\times 2}b^{3\times 2} = 4a^4b^6.$

6. Find the cube of $2a^2b^3$.

SOLUTION. $(2a^2b^3)^3 = 2a^2b^3 \times 2a^2b^3 \times 2a^2b^3$
$= 8a^{2+2+2}b^{3+3+3}$
$= 8a^{2\times 3}b^{3\times 3} = 8a^6b^9$.

NOTE. From the above operations, it is evident that the exponents of the several literal quantities in a power are obtained by multiplying the exponents of the corresponding literal quantities in the root by the exponent of the power.

Find the values of the following:

7. $(2xy)^3$.
8. $(3c^2y^2)^2$.
9. $(a^2c^2x^4)^4$.
10. $(b^3c^2x)^5$.
11. $(3a^2b^4)^3$.
12. $(2a^2b^3x^4)^4$.
13. $(3xy^2a^3)^5$.
14. $(2abc)^6$.

15. Verify the following results by multiplication:

$(2ab)^2 = 4a^2b^2$. $(-2ab)^2 = 4a^2b^2$.

$(2ab)^3 = 8a^3b^3$. $(-2ab)^3 = -8a^3b^3$.

$(2ab)^4 = 16a^4b^4$. $(-2ab)^4 = 16a^4b^4$.

$(2ab)^5 = 32a^5b^5$. $(-2ab)^5 = -32a^5b^5$.

From the above series of products it is evident that,

(1) *All powers of a positive quantity are positive.*
(2) *All even powers of a negative quantity are positive.*
(3) *All odd powers of a negative quantity are negative.*

Find the value of the following:

16. $(3bc^2x^3)^2$.
17. $(-5c^2x^3)^2$.
18. $(2b^2x^3)^3$.
19. $(3b^4c^2y^3)^7$.
20. $(-3x^2y^3z^4)^8$.
21. $(-ax^2c)^2$.
22. $(-ax^2c^3)^3$.
23. $(-3ab^3c^2)^5$.
24. $(x^2y^4z^6)^4$.
25. $(-3ab^3x^6)^5$.
26. $(-3ab^2)^2$.
27. $(-4a^2cx^3)^3$.
28. $(2b^2cx)^7$.
29. $(-4a^2b^6y^7)^5$.
30. $(7c^4d^3x^7)^6$.

## Lesson LXI.

1. Indicate the square of the sum of $a$ and $b$. Find its value.

2. Indicate the square of the difference of $a$ and $b$. Find its value.

How many terms are there in each of the above powers?

Of what does the square of the sum of two quantities consist?

Of what does the square of the difference of two quantities consist? (See Lesson XXV., Theorems 1 and 2.)

3. What is the square of the sum of $2a$ and $3b$?

Solution. When the literal quantities have coefficients, it is convenient to represent the terms by $x$ and $y$ and restore the proper values in the square as follows:

Let $x = 2a$,

and $y = 3b$.

Then $(2a + 3b)^2 = (x + y)^2 = x^2 + 2xy + y^2$.

$$x^2 = 4a^2.$$

$$2xy = 2 \times 6ab = 12ab.$$

$$y^2 = 9b^2.$$

$$(2a + 3b)^2 = 4a^2 + 12ab + 9b^2.$$

Find the squares of the following:

4. $x + 1$.
5. $2 - x$.
6. $4 - c^2$.
7. $2x + 2y$.
8. $x - 10$.
9. $a^2 + 2$.
10. $3b - 2y$.
11. $x^3 - 2$.
12. $2 - x^2$.
13. $3c - 4x$.
14. $x^2 + y^2$.
15. $7y - 4c$.
16. $6 - 2b^3$.
17. $2x - 3y^2$.
18. $3x^2 + 3$.

**19.** What is the square of $\frac{2a^2b}{3xy^2}$?

$$\left(\frac{2a^2b}{3xy^2}\right)^2 = \frac{2a^2b}{3xy^2} \times \frac{2a^2b}{3xy^2} = \frac{4a^4b^2}{9x^2y^4}.$$

NOTE. In finding the square of a fraction the square of both terms of the fraction must be found.

Find the squares of the following:

**20.** $\frac{2xy}{5a^2b}$. **22.** $\frac{2}{x^2y^3}$. **24.** $\frac{1+y}{1+x}$.

**21.** $\frac{3a^2by^2}{2cxy}$. **23.** $\frac{x+a}{x-a}$. **25.** $\frac{a+2}{x-1}$.

**26.** What is the square of $a+b+c$?

SOLUTION. Let $x = a + b$;

then $(a+b+c)^2 = (x+c)^2 = x^2 + 2xc + c^2$.

$$x^2 = a^2 + 2ab + b^2.$$

$$2xc = 2 \times c(a+b) = 2ac + 2bc.$$

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2.$$

Find the squares of the following:

**27.** $a+y+1$. **29.** $c+a-1$. **31.** $2b-2c+2$.

**28.** $1-a+b$. **30.** $2a+2b+c$. **32.** $1+2b+c$.

**33.** $\frac{1-c}{1+x+a}$. **36.** $\frac{a+y+b}{2-y}$.

**34.** $\frac{x^2-1}{x+y}$. **37.** $\frac{a^2-ab+b^2}{a+b}$.

**35.** $\frac{x-a+1}{5-6y}$. **38.** $\frac{1+2a}{2a+1+y}$.

# SQUARING OF NUMBERS.

## Lesson LXII.

Any number above ten is a sum of tens and units. Thus, 27 is the sum of 2 tens (20) and 7 units; 123 is the sum of 12 tens (120) and 3 units.

Let $t$ represent the tens of any number, and $u$ the units of the number.

Then $t+u$ equals the number, and $(t+u)^2$ equals the square of the number.

$$(t+u)^2 = t^2 + 2tu + u^2.$$

Substitute for $t$ and $u$ their number values.

$$27^2 = (20)^2 + 2(20 \times 7) + 7^2 = 400 + 280 + 49 = 729.$$

$$123^2 = (120)^2 + 2(120 \times 3) + 3^2 = 14{,}400 + 720 + 9 = 15{,}129.$$

Find the squares of the following numbers by the above formula:

| 1. | 2. | 3. | 4. | 5. | 6. |
|---|---|---|---|---|---|
| 25 | 28 | 14 | 18 | 63 | 113 |
| 35 | 32 | 17 | 31 | 72 | 111 |
| 45 | 44 | 56 | 42 | 48 | 119 |
| 55 | 47 | 81 | 59 | 110 | 105 |
| 65 | 49 | 67 | 71 | 112 | 109 |
| 75 | 34 | 98 | 77 | 126 | 129 |

The formula

$$(x+y)(x-y) = x^2 - y^2$$

may be used to advantage in the squaring of numbers as follows:

Transpose $y^2$.

$$x^2 = (x + y)(x - y) + y^2.$$

7. Find the square of 34.

SOLUTION. Let $x = 34,$

and $y = 4;$

then $x^2 = (x + y)(x - y) + y^2.$

$$34^2 = (34 + 4)(34 - 4) + 4^2 = 38 \times 30 + 16 = 1156.$$

8. Find the square of 47.

SOLUTION. Let $x = 47,$

and $y = 3;$

then $47^2 = (47 + 3)(47 - 3) + 3^2 = 50 \times 44 + 9 = 2209.$

NOTE. By the above formula, numbers less than 100 can be readily squared by letting $x$ equal the number to be squared, and $y$ equal the difference between this number and the nearest multiple of 10.

Find the squares of the following:

| 9. | 10. | 11. | 12. | 13. | 14. |
|---|---|---|---|---|---|
| 22 | 23 | 31 | 71 | 54 | 14 |
| 18 | 27 | 42 | 63 | 91 | 25 |
| 16 | 29 | 53 | 65 | 45 | 98 |
| 97 | 56 | 73 | 25 | 83 | 36 |
| 84 | 72 | 69 | 51 | 49 | 61 |

## EVOLUTION — SQUARE ROOT.

### Lesson LXIII.

1. Separate $4a^2b^4$ into two equal factors.

Solution. The factors are $2ab^2$ and $2ab^2$, or $-2ab^2$ and $-2ab^2$. These quantities may be abbreviated by writing them thus:

$$\pm 2ab^2 \text{ and } \pm 2ab^2.$$

$\pm 2ab^2$ is called the **Square Root** of $4a^2b^4$.

Note. Whenever a quantity can be separated into two equal factors, the quantity is a **Perfect Square,** and one of the equal factors is the *Square Root* of the given quantity. The sign $\sqrt{\quad}$, called the radical sign, indicates that a root of the quantity written under the sign is to be taken. Thus, $\sqrt{4a^2b^4}$ indicates that the square root of $4a^2b^4$ is to be found. $\sqrt[3]{8a^3}$ indicates that the cube root of $8a^3$ is to be found. The number written in the angle of the radical sign indicates the root that is to be taken.

Find the value of the following:

| | | |
|---|---|---|
| 2. $\sqrt{9}$. | 4. $\sqrt{a^4b^2}$. | 6. $\sqrt{a^2+2ab+b^2}$. |
| 3. $\sqrt{25}$. | 5. $\sqrt{x^2y^2}$. | 7. $\sqrt{a^2-2ab+b^2}$. |

8. What is the square root of $4a^2+8ab+4b^2$?

Solution. When the square root is not evident on inspection, separate the quantity into its prime factors; then combine the factors into two equal groups. The product of one of these groups is the root sought.

Thus, $4a^2 + 8ab + 4b^2 = 2 \times 2 \times (a+b) \times (a+b)$. Combine into groups $2(a+b)$ and $2(a+b)$. The square root is $2(a+b)$.

9. $\sqrt{9a^2x^2 - 18a^2xy + 9a^2y^2}$. 10. $\sqrt{16b^2 + 16by + 4y^2}$.

11. Extract the square root of $a^2 + 2ab + b^2$.

When the quantity of which the square root is to be found is such that the root cannot be seen on inspection, or found readily by separating it into factors, proceed as follows:

| SOLUTION. | | $a^2 + 2ab + b^2$ | $a + b$ |
|---|---|---|---|
| | | $a^2$ | |
| Trial Divisor, | $2a$ | $2ab + b^2$ | |
| Complete Divisor, | $2a + b$ | $2ab + b^2$ | |

The first term of the quantity is $a^2$. Therefore its square root, $a$, is the first term of the root sought. Subtracting $a^2$ from the quantity, the remainder is $2ab + b^2$.

The first term of the remainder, $2ab$, divided by $2a$, twice the first term of the root, equals $b$, the second term of the root. $2a$ is called the *trial divisor*. By adding $b$, the second term of the root, the *complete divisor* is obtained, and this multiplied by $b$ equals the remainder, $2ab + b^2$.

This method may be extended to roots, which contain more than two terms, in the following manner:

12. Extract the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

| SOLUTION. | | $a^2+2ab+b^2+2ac+2bc+c^2$ | $a+b+c$ |
|---|---|---|---|
| | | $a^2$ | |
| Trial Divisor, | $2a$ | $2ab+b^2$ | |
| Complete Divisor, | $2a+b$ | $2ab+b^2$ | |
| Trial Divisor, | $2a+2b$ | $2ac+2bc+c^2$ | |
| Complete Divisor, | $2a+2b+c$ | $2ac+2bc+c^2$ | |

The first two terms of the root are found as in the previous problem. These two terms are then taken *as one term*, and the *succeeding term of the root* is found *in the same manner as was the second term.*

From these problems, the following method of extracting the square root of a polynomial is derived:

NOTE. (1) Extract the square root of the first term of the polynomial for the first term of the root.

(2) Subtract the square of this term from the polynomial.

(3) Double the term of the root, and by this product (the trial divisor) divide the second term of the polynomial. Annex the quotient to the root and to the trial divisor.

(4) Multiply the complete divisor by the term of the root last found, and subtract the product from the polynomial.

(5) If there is a remainder, regard the root already found as a single term, double it for a trial divisor, and proceed as before.

Extract the square root of each of the following:

**13.** $4a^2 + 20a + 4$.

**14.** $36x^4 + 48x^2a^2 + 16a^4$.

**15.** $9 - 30b^2 + 25b^4$.

**16.** $c^2 + cx + \frac{1}{4}x^2$.

**17.** $9y^2 + 6y + 1$.

**18.** $x^2 + 2xy + y^2 - 2xc - 2yc + c^2$.

**19.** $4a^4 - 12a^3 + 13a^2 - 6a + 1$.

**20.** $b^6 - 4b^5 + 10b^4 - 12b^3 + 9b^2$.

**21.** $49c^4 - 28c^3 - 17c^2 + 6c + \frac{9}{4}$.

**22.** $16c^2 + 16cy + 4y^2 + 24ca + 12ay + 9a^2$.

**23.** $4a^2 - 12ab + 9b^2 - 8a - 12b + 4$.

**24.** $x^6 - 24x^3 + 144$.

**25.** $\dfrac{4x^4 - 12x^2 + 9}{x^2 + 4ax + 4a^2}$.

## SQUARE ROOT OF NUMBERS.

### Lesson LXIV.

$1^2 = 1.$ $10^2 = 100.$ $100^2 = 10,000.$

Observe that the square of any number between 1 and 10 must be a number *of one or two places of figures.*

The square of any number between 10 and 100 must be a number *of three or four places of figures.*

To find how many places of figures there are in the root of any number, count the number of places in the number beginning at units, and allow two places of figures in the number to each figure in the root.

Thus, in the number 729 there are three places of figures; therefore there are two figures in the root. The square root of 1225 has also two figures. The square root of 11,025 has three places of figures.

It is usual to divide a number, of which the square root is to be found, into periods of two figures each, by placing a dot over each alternate figure beginning with tens. The number of dots equals the number of figures in the root, or the number of figures in the root less one.

1. Find the square root of 2809.

Solution. The root will consist of two places of figures, and hence is the sum of a number of tens and a number of units.

Let $t =$ the tens of the root,

and $u =$ the units of the root;

then $t + u =$ the root of 2809,

and $(t + u)^2 = t^2 + 2\,tu + u^2 = 2809.$

Since $t^2$ is the square of a number of tens, it must be wholly within the period to the left, or second period.

$$\begin{array}{lrl} & & t + u \\ & \dot{2}8\dot{0}9 & \underline{|50 + 3} = 53 \\ & t^2 = 25 & \\ \text{Trial Divisor,} & 2t = 100 & |309 = 2tu + u^2. \\ \text{Complete Divisor,} & 2t + u = 103 & |\underline{309} = (2t + u)u. \end{array}$$

The greatest square in the period to the left is 25. This, therefore, is $t^2$, and its root 5 is $t$ and the first figure of the root sought. Subtracting $t^2$ or 25 from the left period, and bringing down the next period, leaves a remainder, 309, which equals $2tu + u^2$.

Since $2tu$ is a product of tens and units, it cannot enter as a factor into units' place.

Therefore we may find $u$ by dividing the remainder, 309, by $t$ or 5. $u$ is then added to $2t$, making $2t+u$, the complete divisor, and also added to the root, giving the entire root sought, $t + u$. Multiplying the complete divisor by $u$, the product is $2tu + u^2$, or 309.

2. Find the square root of 105,625.

$$\begin{array}{lrl} \text{SOLUTION.} & & \dot{1}0\dot{5}6\dot{2}5 \ \underline{|\ 325} \\ & & 9 \\ \text{Trial Divisor,} & 60 & |\ 156 \\ \text{Complete Divisor,} & 62 & |\ \underline{124} \\ \text{Final Divisor,} & 640 & |\ \ 3225 \\ \text{Complete Divisor,} & 645 & |\ \ \underline{3225} \end{array}$$

Proceed, as in Example 1, to find the first two figures of the root. Multiplying the first complete divisor, 62, by 2, and subtracting the product from the first remainder, 156, leaves a second remainder, 3225. Regard the root so far found as tens, double it for a trial divisor and proceed as before.

*The pupil must remember that in doubling the root already found, he is using tens, and not units.* Thus, in the above problem, the first trial divisor $2t$ equals 60, not 6; and the second trial divisor equals 640, not 64.

The following is the method of extracting the square root of numbers:

NOTE. (1) Divide the given number into periods of two figures each, beginning at units.

(2) Extract the square root of the highest perfect square in the period to the left. This root will be the first figure in the root sought.

(3) Subtract the highest perfect square from the period to the left, and to the remainder annex the next period; regard this as a new dividend.

(4) Double the root found, regarding it as tens, for a trial divisor. Divide the last remainder by the trial divisor, and the quotient of this division is the second figure of the root sought. Add this figure to the trial divisor for a complete divisor.

(5) Multiply the complete divisor by the figure of the root last found, and subtract the product from the dividend.

(6) To the remainder annex the next period, double the root already found, regarded as tens, for a trial divisor, and proceed as before.

Find the square roots of the following:

3. 1225.

4. 2025.

5. 2209.

6. 5786.

7. 3969.

8. 70,756.

9. 13,225.

10. 272,474.

11. 55,225.

12. 48,841.

13. $\frac{625}{1225}$.

14. $\frac{225}{1024}$.

15. $\frac{169}{196}$.

$.1^2 = .01.$ $\qquad .01^2 = .0001.$ $\qquad .001^2 = .000001.$

$.9^2 = .81.$ $\qquad .09^2 = .0081.$ $\qquad .009^2 = .000081.$

From the above, it is evident that any decimal contains twice as many places as its square root. Hence, if the number whose root is to be extracted is a decimal, or contains a decimal, point off periods of two places each, beginning at the decimal point, completing the period to the right with a cipher, if necessary, thus:

$$.0\dot{6}2\dot{5}.$$

$$.7\dot{2}2\dot{5}1\dot{0}.$$

Extract the square root of:

**16.** .16. **20.** .0256. **24.** 144.0196.

**17.** .225. **21.** 25.0441. **25.** 10000.01.

**18.** 6.25. **22.** 7.29. **26.** .763876.

**19.** 12.25. **23.** 1.96. **27.** 9.011025.

**28.** Find the square root of 2 to three decimal places.

```
       2.       |1.414+
       1
    20 |1.00
    24 |  96
      280 |400
      281 |281
     2820 |11900
     2824 |11296
              604
```

**29.** Find the square root of 5 to three decimal places.

**30.** Find the square root of .9 to four decimal places.

## MISCELLANEOUS PROBLEMS.

### Lesson LXV.

1. Simplify $3ab + 5c - 7ab + 8c + 8ab - 14c - 2ab + c$.

2. Simplify $5x^2y - 3y^2 + 4x^2y - 8x^2y + 5y^2 + 2y^2$.

3. $x - 4x + 2\frac{2}{3}x + \frac{1}{6}x - 1\frac{5}{6}x + 3x + \frac{4}{3}x + 2\frac{2}{3}x = ?$

4. $x - 5\frac{2}{5}x + \frac{3}{5}x + 1\frac{4}{5}x + 4x - 2x + \frac{6}{5}x - 1\frac{1}{5}x = ?$

5. Find the sum of: $3b + 4x - y^2$, $5b + 7x + 3y^2$, $b - 6x + 4y^2$, $-4b + 9x - 8y^2$.

6. Find the sum of: $a^3 - 3a^2b + 3ab^2 - b^3$, $3a^3 + 7a^2b - 5ab^2 - 4b^3$, $a^3 - ab^2 + 5b^3$.

7. Find the sum of: $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$, $5x^3y - 12x^2y^2 + 12xy^3 - 5y^4$, $6xy^3 - 6y^4$, $8x^2y^2 - 12xy^3 + 8y^4$, $y^4$.

8. Find the sum of: $\frac{3}{4}ax^2 + \frac{2}{3}a^2 + 2x^3y + b^3$, $3ax^2 + \frac{1}{4}x^3y + 3a^2 - 2b^3$, $2ax^2 + 3x^3y - a^2 - \frac{2}{3}b^3$, $\frac{1}{16}ax^2 + \frac{1}{8}x^3y + 3a^2 - \frac{2}{5}b^3$.

9. From $cx - 14aby + 7a^2b^2$ subtract $9x - 14aby + 15b^2$.

10. From $8x^3y^2 + 16xy^3 + 10xy$ subtract $14x^3y^2 - 8xy^3 - 4xy$.

11. From $15x^3 + 10y^3 + 8c^3 - b^3$ subtract $5y^3 + 4c^3 + 6b^3$.

12. From $3x^2y^4 - 5x$ subtract $2x^2y^4 - 2xy + 4x - 5b + 6$.

13. Simplify $3x + 2y - (2x - 2y)$.

14. Simplify $7a + 4b - (a + 2b + 4c)$.

15. Simplify $4x - 3ay^2 - (x + y^2 - 4ay^2)$.

16. Simplify $4a^2 + 3b^2 - (-5a^2 - 2b^2 - 3c^3)$.

17. Simplify $4ax^2 - 4ay^2 - (-4ax^2 - 5ay^2)$.

18. Simplify $(x + y) + (x - y) - (2x - 2y)$.

19. Simplify $(x+y-a-b)-(y-a-b+x)+(a-b)$.

20. Simplify $(3a-4b)+(a-3b)-(4a-7b-10)$.

21. Simplify $(3b^2-2b^2x-7)-(7+3b^2-4b^2c+2)-3$.

22. Simplify $(x^2+2ax+a^2)-(x^2-2ax+a^2)-(-4ax)$.

23. Simplify $4x^3-2x^2+x+1-(-7-x-x^2+3x^3)-(-4x^2+2x+x^3+8)$.

24. Simplify $\left(7a+\frac{2a}{b}\right)-\left(3a-\frac{a-3c}{b}\right)$.

## Lesson LXVI.

1. Expand $(4x+5y)(2a+3x)$.

2. Expand $(3a^2+2b^2)(2a^2+3b^2)$.

3. Expand $(a^2+ab+b^2)(a+b)$.

4. Expand $(x^2+2xy+y^2)(x+y)$.

5. Expand $(a^2+2ab+b^2)(a^2-2ab+b^2)$.

6. Expand $(2x^2-3xy+y^2)(x^2-5xy)$.

7. Expand $(2a^3+2a^2x+2ax^2+2x^3)(3a-3x)$.

8. Expand $(x^2+xy+y^2)(x^2-xy+y^2)$.

9. Expand $(x-4)(x-5)(x+4)(x+5)$.

10. Expand $(a+c)(a-c)(a+b)(a-b)$.

11. Expand $(x^2+x+1)(x^2+x+1)(x-1)(x-1)$.

12. Multiply $\frac{a-b}{b}$ by $\frac{bc}{a^2-b^2}$.

13. Divide $8a^4-8b^4$ by $2a^2-2b^2$.

14. Divide $ax+bx-ay-by$ by $a+b$.

Yes you can, tell them to take



15. Divide $a^3 + b^3 + 5ab^2 + 5a^2b$ by $a^2 + 4ab + b^2$.

SUGGESTION. In dividing, the terms of the dividend and the divisor should be arranged according to the powers of some common letter. The dividend in the above problem should be arranged as follows:

$$a^3 + 5a^2 + 5ab^2 + b^3.$$

16. Divide $x^3 - y^3 + 3xy^2 - 3x^2y$ by $x - y$.

17. Divide $4x^4 - 64$ by $2x - 4$.

18. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.

19. Divide $4a^6 - 25a^2x^4 - 4x^6 + 20ax^5$ by $2a^3 - 5ax^2 + 2x^3$.

20. Divide $a^6 - 2a^3b^3 + b^6$ by $(a^2 + ab + b^2)^2$.

21. Divide $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $(x + b)(x + c)$.

22. Divide $(a - 2)(a - 1)a(a + 1)(a + 2)$ by $a^2 - a - 2$.

23. Multiply $\frac{x}{a + x}$, $\frac{a^2 - x^2}{x^2}$, and $\frac{a}{a - x}$ together.

24. Multiply $c + \frac{cx}{c - x}$ by $\frac{c^2 - x^2}{x + 1}$.

25. Divide $\frac{x - y}{x^2 - xy + y^2}$ by $x + y$.

26. Divide $\frac{a}{a^2 - 1}$ by $\frac{a + 1}{a - 1}$.

27. Expand $\left(\frac{1 + a}{1 - a}\right)^2$.

28. Divide $\frac{x^3 + a^3}{3}$ by $\frac{x^2 - xa + a^2}{6}$.

29. Divide $\frac{(a + b)^3}{7ab}$ by $\frac{a^2 + 2ab + b^2}{14b}$.

## Lesson LXVII.

1. Reduce $\dfrac{5a^3 - 10a^2x + 5ax^2}{8a^3 - 8a^2x}$ to its simplest form.

2. Reduce $\dfrac{a^2 - (b+c)^2}{a^2 + ab + ac}$ to its simplest form.

3. Reduce $\dfrac{(x^3 - y^3)(x^3 + y^3)}{(x^2 - y^2)^2}$ to its simplest form.

4. Reduce $a^2 + ab + b^2 + \dfrac{c^3 - a^3}{a - b}$ to a fraction.

5. Reduce $\dfrac{a^2 - 3b^2}{a + b}$ to a mixed quantity.

6. Simplify $\dfrac{3}{x - 2a} - \dfrac{7}{x + 2a} - \dfrac{4 - 20a}{x^2 - 4a^2}$.

7. Simplify $\dfrac{xy}{a - x} + \dfrac{bx^2}{a + x} + \dfrac{x^3}{a + x}$.

8. Simplify $\left(2a - 3x + \dfrac{a - x}{a}\right) - \left(a - 5x + \dfrac{x - a}{x}\right)$.

9. Simplify $\dfrac{3a - 4b}{2} + \dfrac{15a - 4x}{12} - \dfrac{2a - b - x}{3}$.

Find the value of:

10. $\dfrac{(x - y)^2}{x + y} \times \dfrac{y}{a(x - y)}$.

11. $\left(\dfrac{c}{x} + \dfrac{b}{y}\right)\left(\dfrac{x}{b} + \dfrac{y}{c}\right)$.

12. $\dfrac{2a^2}{x^3 + a^3} \div \dfrac{a}{x + a}$.

13. $\left(\frac{a+b}{3}-\frac{b}{2}\right)\div\left(\frac{a}{c}-\frac{b}{2c}\right).$

14. $\left(a+\frac{a}{b}\right)\left(a-\frac{a}{b}\right)^2\div\left(a-\frac{a}{b}\right).$

15. $\left(\frac{x^2+y^2}{2x^2}-\frac{2y^2}{x^2+y^2}\right)\div\left(\frac{x^2+y^2}{2y^2}-\frac{2x^2}{x^2+y^2}\right).$

## Lesson LXVIII.

Find the value of $x$ in the following equations:

1. $\frac{2x-6}{4}+\frac{x}{3}=20-\frac{3x+57}{6}.$

2. $\frac{x}{12}-\frac{24-3x}{24}-\frac{10+2x}{8}+\frac{11}{4}=0.$

3. $\frac{x}{7}-\frac{8x}{9}-\frac{x}{5}+\frac{3}{5}=-12\frac{29}{45}.$

4. $\frac{11x+4x+2}{11}-\frac{3x-5}{13}=x+1.$

5. $2x-\frac{8x-4}{10}=\frac{3x-1}{2}.$

6. $\frac{4x}{5-x}-\frac{20}{x}+\frac{12x}{3x}=\frac{15}{x}.$

7. $\frac{4x+13}{15}-\frac{9x+13}{5x-25}+9=\frac{2x+14}{5}.$

8. $\frac{3}{x+1}+\frac{x+1}{x-1}=\frac{x^2}{x^2-1}.$

9. $9+10x-6x\left(\frac{1}{x}-\frac{1}{3}\right)=27.$

10. $\frac{2x+12}{2}-\frac{3(x+6)}{4}-\frac{1}{3}(x+6)=-6.$

11. $\left(\frac{2x+5}{2}\right)\left(\frac{2x-3}{2}\right)=(x+5)(x-3)-\frac{3}{4}.$

12. $\frac{x}{2}-\frac{5}{6}-\left(\frac{x}{3}-\frac{5}{12}\right)=\frac{1}{4}-\frac{x}{4}.$

13. $\frac{x}{6}+\frac{x+1}{3}=\frac{1}{3}\left(\frac{2}{5}-\frac{x}{3}\right).$

14. $\frac{2x+5}{6}+\frac{x+6}{5}=\frac{3x}{6}+\frac{121}{30}.$

15. $\frac{x+1}{x-1}=\frac{x-1}{x+1}+\frac{3}{x-1}.$

## LESSON LXIX.

Find the values of $x$ and $y$ in the following equations:

1. $\begin{cases}\frac{y}{4}-\frac{x}{7}+5=6.\\ \frac{2y}{10}+4=\frac{x}{14}+6.\end{cases}$

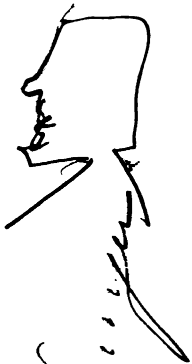
2. $\begin{cases}\frac{2x-3y}{5}=x-2\frac{2}{5}.\\ x-\frac{y+1}{2}=0.\end{cases}$

3. $\begin{cases}\frac{3x}{2}-36=\frac{3y}{4}-30.\\ \frac{2(x+y)}{5}+\frac{2x}{3}=16-\frac{2x-x}{2}-11.\end{cases}$

4. $\begin{cases}\frac{x-y}{2}+\frac{2x+2y}{5}=4\frac{2}{5}.\\ \frac{1}{2}(x-y)+4\frac{1}{4}y=8\frac{1}{4}.\end{cases}$

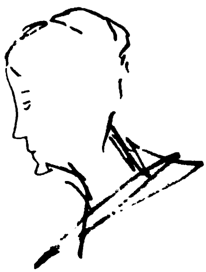
5. $\begin{cases}\frac{5(3y-x)}{6}=25-\frac{10x-5y}{4}.\\ 6x-y+\frac{8-2x}{4}=43\frac{1}{2}.\end{cases}$

6. $\begin{cases} 4x - 4 + 7 = \dfrac{3(y-5)}{4} + 27. \\ \frac{1}{2}x - \frac{1}{8}y + \dfrac{y-4}{3} = \dfrac{5}{3}. \end{cases}$

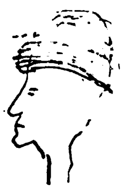


7. $\begin{cases} \dfrac{x}{4} + 8 = \dfrac{y}{2} - 17. \\ \dfrac{x+y}{5} + \dfrac{y}{3} = \dfrac{2x-y}{4} + 41. \end{cases}$

8. $\begin{cases} \dfrac{2x}{7} + \dfrac{21x-y}{7} = 2y + 8. \\ \dfrac{y+3}{5} + \dfrac{y-x}{6} = 2x - 12. \end{cases}$



9. $\begin{cases} \dfrac{2x-4}{5} + \dfrac{30-3x}{6} = \dfrac{y-10}{4}. \\ \dfrac{2y+24}{3} = \dfrac{4x+y+34}{4}. \end{cases}$



10. $\begin{cases} \dfrac{4}{7}\left(\dfrac{2x}{3} - \dfrac{5y}{12}\right) - \dfrac{2}{23}\left(\dfrac{3x}{2} - \dfrac{y}{3}\right) = \dfrac{1}{3}. \\ 1 - \dfrac{2y}{x+y} = \dfrac{1}{5}. \end{cases}$

11. $\begin{cases} \dfrac{x}{2} - \dfrac{2y+12}{16} = \dfrac{x+y}{8} - 1. \\ 4x - 2y = 4. \end{cases}$

12. $\begin{cases} x - y = 3y - \dfrac{2x}{5} + \dfrac{8}{5}. \\ \dfrac{3x}{4} + 3y = 2y + \dfrac{x}{2} + 2. \end{cases}$

## Lesson LXX.

1. The tail of a fish weighs 9 pounds. His head weighs as much as his tail and half his body, and his body weighs as much as his head and tail together. What is the weight of the fish?

2. A third part of a barrel of molasses leaked away. Twenty-one gallons were then drawn, leaving the barrel half full. How many gallons did the barrel hold?

3. A workman received $2 per day and his board for each day that he worked. When he was idle, he paid $1 per day for board. At the end of 48 days he received $42. How many days did he work, and how many days was he idle?

4. One half of a gentleman's estate was left to his widow, one sixth to each of his two daughters, and one twelfth to a servant. The remainder, which was $500, was left to the poor. What was the value of the estate?

5. Mr. A.'s estate was divided as follows: his widow received one third of the estate and $1000; his son received one half as much as the widow; his daughter received twice as much as the son. What was the value of the estate, and what was the share of each?

6. Find three consecutive numbers whose sum is 84.

7. Find five consecutive numbers whose sum is 125.

8. Find five consecutive numbers, such that the least divided by the greatest equals $\frac{2}{3}$.

9. A sum of money is divided between two persons, A and B, so that as often as A receives $9 B receives $4. A receives $20 more than B. What are their respective shares of the money?

## Lesson LXXI.

1. What number is that whose third part less 10 exceeds its sixth part plus 15 by 5?

2. Find two numbers in the ratio of 2 to 3, to each of which, if 4 be added, the sums will be to each other as 5 to 7.

3. A gentleman divided \$2 among 12 children, giving to certain children \$.18 each, and to the others \$.14 each. To how many children did he give \$.18? To how many, \$.14?

4. A grocer has two kinds of tea, one worth \$.72 per pound, the other \$.40 per pound. How many pounds of each must be taken, to form a chest of 48 lb., which shall be worth \$.60 per pound?

5. What number is that to which if 1, 4, and 10 are severally added, the first sum will be to the second as the second is to the third?

6. What number is that to which if 2, 5, and 9 are severally added, the first sum will be to the second as the second is to the third?

7. Find a number, such that one third thereof added to one fourth of the same will equal one sixth of it increased by 30.

8. Two persons, A and B, invested equal sums of money in trade. During the year, A gained \$504, and B lost \$348, when A's money is twice B's. How much money did each invest?

9. A train of cars, moving at the rate of 20 miles per hour, had been gone three hours when a second train followed at the rate of 25 miles per hour. In what time did the second train overtake the first?

## LESSON LXXII.

1. Multiply a certain number by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, and the result is the original number. What is the number?

2. A pile stands one third in the ground, one half in the water, and 3 feet above water. How long is the pile?

3. If 3 is added to the numerator of a certain fraction, and at the same time, 3 is subtracted from the denominator, the value of the fraction is 2. Also, twice the numerator plus 6 divided by three times the denominator minus 4 equals 1. Find the fraction.

4. If 2 is subtracted from the numerator of a certain fraction, and at the same time 3 is added to the denominator, the value of the fraction equals $\frac{1}{10}$. And fourteen times the numerator equals six times the denominator. Find the fraction.

5. The sum of two numbers is 12, and the difference of their squares divided by the sum of the numbers is 2. Find the numbers.

6. The weights of two loaded wagons were in the ratio of 4 to 5. Parts of their loads, in the ratio of 6 to 7, being taken out, the weights of the loaded wagons were then in the ratio of 2 to 3, and the sum of their weights was 10 tons. Find their weights at first.

7. A mast of a ship consists of two parts. One sixth of the lower part added to one half of the upper equals 35 feet. Three times the lower part diminished by six times the upper equals 30 feet. Find the height of the mast.

8. There are two numbers whose difference is 5. The quotient of the greater divided by the less is also 5. Find the numbers.

## LESSON LXXIII.

1. Seven years ago A was seven times as old as B, and three years hence A will be three times as old as B. Find their ages.

2. Find two numbers, such that if the first be divided by 6 and the second by 5, the sum of the quotients will be 52; and if the first be divided by 8 and the second by 12, the sum of the quotients will be 31.

3. A gentleman has two horses and one carriage. The first horse is worth $ 120. If the first horse is harnessed to the carriage, together they will be worth twice as much as the second horse. But if the second horse is harnessed to the carriage, the horse and carriage will be worth three times the value of the first horse. Find the value of the second horse and of the carriage.

4. A gentleman employed 5 men and 4 boys to work for one day, and paid them all together $ 10.50. The next day he hired, at the same wages, 8 men and 6 boys, and paid them all together $ 16.50. What were the wages of each man and boy?

5. A number is expressed by two digits. If to the sum of the digits 7 is added, the result will be three times the left-hand digit. If 18 is subtracted from the number, the digits are inverted. Find the number.

6. A and B together had a capital of $ 9800. A invested one sixth of his capital, and B one fifth of his, whereupon each had the same sum left. How much had each before the investment?

7. A farmer bought 100 acres of land for $ 2450. For part of it he paid $ 20 per acre, and for the rest he paid $ 30 per acre. How many acres were there in each part?

Pauline Sawyer



8. A grain dealer sold 47 bushels of corn and 18 bushels of wheat to one man for $45.26, and 3 bushels of corn and 63 bushels of wheat to another at the same price for $64.74. Find the cost of each per bushel.

9. The hour and minute hands of a clock are together at noon. At what time will they be together again?

## Lesson LXXIV.

1. The square of the sum of two numbers equals 144, and the square of their difference equals 4. Find the numbers.

2. The sum of the squares of two numbers less 4 equals twice their product. Twice the greater plus 2 divided by three times the less minus 3 equals 2. Find the numbers.

3. The sum of the cubes of two numbers divided by the sum of their squares less their product equals 14. One half of the first number plus 1 equals one half of three times the second. Find the numbers.

4. The product of the sum and difference of two numbers is 75, and the sum of their squares equals 125. Find the numbers.

5. A party who had hired a coach found that if there had been three more persons they would each have had to pay $1 less than they did pay, and if there had been two less, they would each have had to pay $1 more. How many persons were there, and how much did each pay?

6. A and C can do a piece of work in 6 days. B and C can do it in 8 days. If A can do $\frac{3}{2}$ as much as B, in what time can they all do it, working together?

7. A man has a rectangular lot, such that twice the length increased by 6 yards equals four times the width diminished by 4 yards, and the distance around it is 50 yards. Find the area of the lot.

8. A fishing rod consists of two parts. The length of the upper part is $\frac{3}{2}$ of the lower part, and four times the upper part plus five times the lower part equals 8 more than four times the length of the rod. How long is the rod?

9. What number is that which, being divided by 9, gives the same quotient as 16 divided by the number?

10. Divide the number 73 into two such parts, that the difference between the greater and 77 equals three times the difference between the less and 40.

aimez-vous la
nouvelle fille.
Non, pas du tout.
l'aimez-vous.

très drôle.
Je pense qu'elle
a une très bonne
figure. Elle a très
jolie n'est pas très
beaucoup

aimez-vous algebra

Je la detest

Moi aussi

Pourquoi n'avez-vous pas dit que vous avez fait tous vos examples

Parceque elles sont toutes mal.

Je parle française très bien aussi pas

Oui, je avez très

[illegible]

~~Jean~~

J'ai vu Monsieur Meaux ce matin.

[illegible]

[illegible]

Ah, il ne me regarde pas. Peut-être malheureuse, très triste

R. H. S. A.

d ½ cent
4 farth = 1 penny = 2 cents

2

s. 12 pence = 1 shilling = 24 cents

5.00

£ 20 shillings = 1 pound = 4.84