CHAMBERS'S
ELEMENTARY SCIENCE MANUALS

ELEMENTARY MECHANICS

INCLUDING

HYDROSTATICS AND PNEUMATICS

BY

OLIVER J. LODGE, D.Sc. LOND.

PROFESSOR OF EXPERIMENTAL PHYSICS IN UNIVERSITY COLLEGE,
LIVERPOOL

Revised Edition

W. & R. CHAMBERS
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GENERAL PLAN OF THE SERIES.

The subjects of these Manuals are for the most part, though not exclusively, the same as those of the Syllabus of the Science and Art Department, South Kensington, and the treatment will be found to meet the requirements for the Examinations held by that Department.

In their wider scope the Manuals are intended to serve two somewhat different purposes:

1. They are designed, in the first place, for SELF-INSTRUCTION, and will present, in a form suitable for private study, the main subjects entering into an enlightened education; so that young persons in earnest about self-culture may be able to master them for themselves.

2. The other purpose of the Manuals is to serve as TEXT-BOOKS IN SCHOOLS. The mode of treatment naturally adopted in what is to be studied without a teacher, so far from being a drawback in a school-manual, will, it is believed, be a positive advantage. The subject is made, as far as possible, to unfold itself gradually, as if the pupil were discovering the principles himself, the chief function of the book being, to bring the materials before him, and to guide him by the shortest road to the discovery. This is now acknowledged to be the only profitable method of acquiring knowledge, whether as regards self-instruction or learning at school.

PREFACE.

THE non-mathematical English works on Mechanics are mainly of two kinds—one the old-fashioned text-book, which every one now feels to be quite unsatisfactory; and the other, the comprehensive and powerful productions of Professors Thomson and Tait, Clerk Maxwell, and W. K. Clifford; which, though not confessedly mathematical, are yet far too difficult for ordinary beginners.

The present book aims at giving a clear knowledge of the principles of the subject, in as elementary and even popular a manner as is consistent with careful accuracy, and without assuming any mathematical knowledge beyond the most rudimentary algebra. At the same time it is hoped that students who use this manual will be able to master the elements of the science in such a way that they may rise from it to more advanced treatises, not only without having anything to unlearn, but with a very sound knowledge of principles. Copious illustrations and explanations have been all along inserted, and the general plan of the Series, of which this forms a volume, has been kept steadily in view.

The examples at the end of the chapters are typical ones, and are intended not only to be worked without looking at the answers, but also to be read almost as part of the book, because they frequently direct attention to important details. The solving of a few miscellaneous exercises such as those at the end of the book, is good practice, but it has not been thought well to fill up the book with a host of numerical questions which are often mere exercises in arithmetic; the time spent in solving such would often be more usefully employed in reading and thinking over fundamental principles.

The statements made in a book should be carefully criticised and not taken for granted—and all kinds of special cases should be thought of or tried, to see if an exception cannot be found. *It is by thinking one's-self on a subject that it becomes really known to one's-self; it will* *never be really known if we only try to understand and remember what the book says.* Any emendation or correction of statements in the following pages will be gratefully received.

The author has as a matter of course to acknowledge obligations to Thomson and Tait's *Natural Philosophy*, and to Professor Clerk Maxwell's little manual, *Matter and Motion.* To Deschanel (Part I.—obtainable separately) and Ganot (Books I.—IV.) such frequent reference has been made, that they need only be mentioned here in order to recommend real students to read one or other of them along with the present work, so as to fill up their knowledge in more detail. They may also be referred to Professor Garnett's *Elementary Dynamics* for a rather more mathematical treatment of certain subjects, and for numerous problems and exercises.

The author has to thank Dr Henrici for his kindness in revising the proofs of Chapters VII. and VIII., and for several valuable suggestions. His obligations to Professor Carey Foster are so great, that it is as impossible as it is unnecessary to express them. It is while he has been under Mr Foster's influence that he has learned everything of any accuracy that he knows on the subject, and more than half the book may be traced to his teaching, direct or indirect.

OLIVER J. LODGE.

SUGGESTIONS FOR READING.

Beginners are recommended to omit the following sections on a first reading: 15–17, 37–40, 50–52, 56, 57, 82, 83, 95, 104, 111, 112, 141–143, 147–150, 179; and then to return and read the omitted portions together, and finally to read the whole book through carefully without omitting anything. Students preparing only for London University matriculation, or for the elementary stage of the Science and Art Department, may pretty safely omit any of the above sections over which they experience much difficulty, until the examination is over.

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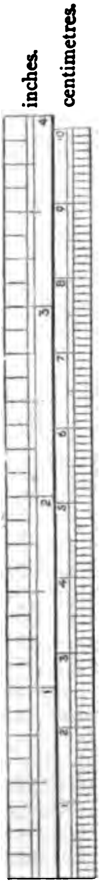


Comparison of Centimetres with English Inches.

METRIC SYSTEM OF UNITS.

The metric system of units, as now adopted throughout Europe, is as follows:

The *centimetre* is the unit of length.

A second is the unit of time.

The cubic centimetre is the unit of volume.

The mass of one cubic centimetre of distilled water at its temperature of maximum density is called a *gramme*, and is the unit of mass.

(For the derived units *dyne* and *erg*, see sects. 43 and 77.)

1 foot	=	30·4797 centimetres.
1 cubic inch	=	16·387 cubic centims.
1 pound	=	453·59 grammes.
1 gramme	=	15·43 grains.
The weight of a gramme	=	981 dynes.
1 centimetre	=	·3937 inch.
1 litre	=	1000 cubic centims.

A velocity of one mile per hour = 44·704 centimetres per second.

The weight of one grain = 63·57 dynes.

A pressure of one pound weight per square foot = 479 dynes per square centimetre.

An acceleration of 32·18 feet-per-second per second = 981 centimetres-per-second per second.

33 centimetres = 13 inches very nearly indeed.

For further details see Professor Everett's book, *Illustrations of the C. G. S. System of Units.*

ELEMENTARY MECHANICS.

INTRODUCTION.

ON FORCE.

1. MECHANICS is that branch of Natural Philosophy which treats of the different effects of force on matter.

Other branches of Natural Philosophy, summed up under the name *Physics*, treat of the different ways in which force may originate, and are concerned with different forms of *energy* (the force generator), just as *Chemistry* is concerned with different kinds of *matter;* Mechanics accepts both force and matter, and discusses only the effects of one on the other.

2. By the term *force* we are to understand muscular exertion, and whatever else is capable of producing the same effects.

Muscular action impeded gives us our primitive idea of force; our sense of muscular exertion itself is a primary one for which we have special nerves, and it is not resolved into anything simpler. When any inanimate agent produces an effect on bodies exactly similar to that which would be produced by muscular exertion on the part of an animal, it also is said to exert force. Thus, a steam-engine exerts force when propelling a carriage, or pumping water, or turning a mill; gunpowder exerts force on a cannon-ball during the time the ball is passing from the breech to the muzzle of the gun. But in order that an agent can exert force it must meet with some resistance; in other words, force is always the mutual action of *two* bodies against one another, and the amount of the force is precisely equal to the amount of resistance. Thus, a flying cannon-ball is not exerting force unless it meets with some resistance: if the air rub against it gently and resist its motion, it will exert a slight force against the air; but if it strike a target, it meets a very great resistance, and therefore exerts a very great force, possibly smashing the target. A running stream exerts very little force unless it meets with an obstacle; but if you resist its motion with your hand, it will press against your hand, or if you dip the vanes of a water-wheel in it, may force the wheel round.

It is meaningless, then, to speak of the force of a *freely* falling weight, though it is not at all meaningless to speak of the force exerted by a weight compelled to fall *gradually*, such as a clock-weight, or a bucket when being lowered down a well; because in any such case the falling is resisted, and the force exerted by the weight exactly equals the resistance offered, even if you apply so much resistance as to stop it altogether, or actually to pull it up again.

3. Force, then, is always a dual thing—an action taking place between *two* bodies, and the action (or push) of the one on the other is always precisely equal to the reaction of the other on the one. In other words, action and reaction are equal and opposite. If you want to tear a piece of paper, or break a string, or stretch a piece of elastic, or squeeze a bit of india-rubber, or crack a nut, it is no use pulling or pushing at one side only; you must apply the force to both ends or sides—that is, you must apply two forces, one opposite the other. This pair of forces which always go together it is convenient to have a name for, and it is called a *stress*. It may be either a *tension*, if the forces are acting away from each other, as in the first three of the above examples; or a *pressure*, if the forces are acting towards each other, as in the last two. (The effect which a stress produces in an ordinary solid before rupture is called a *strain*.)

It is often convenient to isolate one of the components of a stress between a pair of bodies, and consider only the force acting on one body; but we only do so by *attending* to this one and neglecting the other component, which always necessarily exists and is acting on the other body of the pair. Moreover, which of the forces we choose to call the direct action, and which the reaction, is merely a matter of convenience; but it will be obviously convenient to speak of that component which acts on the piece of matter we are dealing with as *the* force, or the *action* of the other piece of matter, while that component which affects this other piece of matter will of course be the *re*action of the first piece on it.

We have spoken of force as exerted *by matter*. Strictly speaking, this is hardly correct. Matter does not *of itself* exert force; it must be set in motion, or have some other form of *energy* conferred upon it, before it can exert force.*

Remembering this, however, we shall do no harm by habitually using the convenient phrase, 'the force exerted by such and such a body,' as indeed we have already frequently done. It follows, moreover, from what we have said, that a stone exerts precisely the same force on the earth as the earth exerts on the stone.

4. A book lying on a table is at rest. Why? Not because no force is acting on it, for the earth is pulling it; but because another and equal force is also acting on it in the opposite direction—namely, the resistance of the table. This is the condition of all bodies at rest near the surface of the earth; they have met with two or more forces which neutralise each other as far as *motion* is concerned, though they do not neutralise each other as regards *strain*.

Strictly speaking, *motion* appears to be the normal condition of matter at present; all known bodies are moving through space

* There is a universal, but probably only apparent, exception to this rule—namely, the force of gravitation—two pieces of matter (like the earth and a stone) do appear to pull towards each other of themselves. Now, the ultimate nature of gravitation is not at present known, and it *may* turn out to be a property really inherent in matter, and an exception to every known case; but it is more probable that it is not a pulling property inherent in matter at all, but a pushing property of some external energetic arrangement not at present understood, due probably to a strain in the medium in which all matter is immersed; so that a stone and the earth do not strictly draw each other together, but are pushed together by something else which extends from one to the other, and may be called the gravitation medium; its stress being called the gravitation stress, or crudely, the weight of a body.

with considerable speed, and no such thing as *absolute rest* is known. But it is convenient, in mechanics proper, to consider the earth as a body at rest; and to leave the study of the motion of it, and of the group of bodies to which it belongs, to Astronomy, which is really a branch of mechanics in a wide sense.

5. The effects of force on matter are: *

A. Change of motion, which is called *acceleration.*

B. Change of size or shape, which is called *strain* or deformation.

If only one force acts on a body, it must produce the effect A. If two or more forces act in different directions on different parts of a body, they must produce B, and they may produce A also.

6. The two kinds of effects, A and B, are distinct; and each would furnish a *measure* of force.

A force may be measured by the amount of motion it can produce in a given piece of matter in a given time; and this is the measure we shall mostly use.

Or a force may be measured by the amount of strain it can produce in a certain piece of matter: the amount it can bend a certain spring, for instance, as in a dynamometer; or the amount it can twist a certain wire, as in a torsion balance.

If we are not concerned with *measuring* forces absolutely, but simply wish to *compare* two forces, we may of course simply balance them one against the other, as is done in a balance or steelyard.

Of the two classes of effects, A and B, A is much the simpler, so we will proceed to consider it first. But before proceeding to our actual subject, the motive effect of force (called *Dynamics*, from δυναμις, force), it is convenient to study motion itself a little in the abstract, and without reference to either force or matter. (The subject of abstract motion is called *Kinematics*, from κινημα, motion.) We may

* Whatever other effects of force there may appear to be, are studied under *Physics*, and physicists are hoping to reduce all of them ultimately to the above two forms. Hence Physics is constantly tending to become more and more mechanical.

conceive a geometrical point or surface moving about in all sorts of ways without troubling ourselves with the cause of the motion, and the propositions which we so discover will be useful when we come to the motion of an actual piece of matter under the influence of a force.

CHAPTER I.

ON MOTION (*Kinematics*).

1. MOTION OF A POINT (TRANSLATION).

(*a*) *Rectilinear Motion.*

7. A body is said to *move* when it is in different positions at different times. This is to be regarded as the essential characteristic of motion—it involves a reference to both space and time. Geometry deals with space alone. Kinematics deals with both time and space. Now motion has two primary properties to be studied—Velocity and Direction. Let us take them in order.

When a body moves over equal spaces in equal times, its motion is said to be *uniform*, or its velocity is said to be *constant*.* For instance, the hand of a clock has such a motion.

When a body moves over *un*equal spaces in equal times, its velocity is said to be variable. As an example of *variable* velocity, we may take the case of a falling stone, which moves quicker and quicker as it descends; or of a stone thrown upward, which has a decreasing velocity till it reaches its highest point; or of the bob of a pendulum, which has a velocity alternately increasing and decreasing.

* It is probable that our idea of motion (that is, of free muscular action) precedes and suggests our idea of time; and that our notion of *equal intervals* of time depends on our recognition of *uniform motion*. Every measurer of time is simply a uniformly moving body. The most uniformly moving body we know is the earth, which rotates on its axis in a period of always the same duration; this period is taken as our fundamental unit of time, and the $\frac{1}{86400}$th part of it is called a *second*, of 'mean solar time,' and is used as the practical unit.

8. **Velocity** is defined as the rate of motion of a body. It is measured by the distance travelled over divided by the time taken in the journey.

Thus, if a train goes 80 miles in 4 hours, it is said to travel at the rate of 20 miles an hour; its velocity is $\frac{80}{4}$ in miles per hour, or $\frac{80 \times 1760 \times 3}{4 \times 60 \times 60} = \frac{88}{3}$ in feet per second; and we shall generally use feet per second. If a point move over s feet in t seconds, its velocity is $\frac{s}{t}$ feet per second, or

$$V = \frac{s}{t}.$$

Note that a velocity is *length per time*, and that it is not correct to speak (except as a well-understood ellipsis) of a velocity of so many *feet*. We speak of a *length* of so many feet—or a *time* of so many seconds—but a *velocity* of so many *feet per second.*

The *unit* of velocity is of course $\frac{\text{unit of length}}{\text{unit of time}}$; that is, $\frac{\text{one foot}}{\text{one second}}$ (read one foot per second). When we speak of a velocity 6 simply, we mean always 6 units of velocity as just defined.

9. The above measure of velocity as the ratio of s to t is independent of the size of s and t, so that it remains perfectly true when s and t are very small. Thus in the case of a body moving uniformly 6 feet every second, its velocity may be written either $\frac{6}{1}$ or $\frac{1}{\frac{1}{6}}$ or $\frac{\frac{1}{1000}}{\frac{1}{6000}}$; and any of these fractions represents its velocity *equally well so long as it be uniform.* But if the velocity were variable, the body might still go 6 feet in a second, so that its *average* velocity would still be 6; but its *actual velocity at each instant* might take all kinds of values, some greater and some less. Thus a train which had gone from London to York, 200 miles, in 5 hours, would have had an average speed of 40 miles an hour; but its actual speed would have varied greatly; sometimes

rising to 60 perhaps, sometimes falling to 0, as at a station. The whole distance travelled, divided by the whole time taken, will always give us the *average* velocity for that distance; and in the case of *uniform* motion, the average velocity coincides with the *actual* velocity at each instant. But to get information on the actual velocity, at any one place, of a thing whose speed varies continually, it is necessary to take a *small* distance at that place, and to divide that by the time taken to traverse it. Thus the speed of the train when passing any particular station might be stated pretty accurately by noting the time taken by the train in going 100 yards; because the speed for so short a distance would probably not vary greatly, and must therefore nearly coincide with the average speed. But to get rid of all *possibility* of variation, it is better to take a still smaller space, say a foot or an inch, and to divide this distance by the time taken to traverse it; and the smaller the distance the more necessarily accurate we are. Hence the actual velocity of any moving body at a given instant is the infinitely small distance then being described divided by the infinitely small time required for the purpose.

The facts are often expressed in a form which appears more simple, but which involves less important ideas—namely:

Uniform velocity is measured by the space described in unit time.

Variable velocity, by the space which *would* be described in a unit of time if at the given moment the velocity were to cease to vary.

So then, using little v to stand for actual velocity at any instant, $v = \frac{s}{t}$ is true when s and t are small; but, using big V to stand for average velocity throughout any time, $V = \frac{s}{t}$ is always true unconditionally.

10. **Acceleration.**—The rate of change of velocity is called *acceleration*. Velocity may change in magnitude and in direction. The rate of either change is called acceleration,

but the consideration of change in direction will be deferred till sect. 13.

Acceleration is measured by the velocity gained by the body in a certain time, divided by the time taken to gain it.

Thus, if a falling body acquire a velocity of 96 feet per second in three seconds, its average acceleration is said to be $\frac{96}{3}$ or 32.

Hence acceleration bears the same relation to velocity as velocity did to distance, and denoting it by a, we have

$$a = \frac{v}{t}$$

as the algebraic statement of the measure of acceleration; remembering that v stands for the *velocity gained by the body in the time t*, and need not stand for any velocity actually possessed by the body. Thus the above falling body, instead of simply falling from rest, might have been thrown down from a balloon with an initial velocity of 100 feet a second; but if at the end of three seconds, its velocity were 196, then its *gain* of velocity would be precisely the same as before, and its acceleration therefore still $\frac{96}{3}$ or 32. Hence, generally, v may be said to stand for the difference between the final and the initial velocities, which are conveniently denoted by v_1 and v_0 respectively, so that $v = v_1 - v_0$, and

$$a = \frac{v_1 - v_0}{t}.$$

11. Acceleration may be constant or variable; in other words, velocity may change at a uniform or a variable rate. When a body acquires equal increments of velocity in equal intervals of time, its acceleration is said to be constant: for instance, a falling stone has constant acceleration; its velocity uniformly increases. It gains in fact a velocity 32 every second of its motion. The study of motion under variable acceleration being rather complicated, we shall only consider constant acceleration in the present stage. Hence, in all that follows, the acceleration is supposed to be constant, unless it is otherwise stated.

Note that acceleration is *velocity per time*, and that it is absurd to speak of an acceleration of so many *feet*, or even of an acceleration of so many *feet per second*, for this last is a velocity. An acceleration is properly so many *feet-per-second per second;* and the unit of acceleration is

$$\frac{\text{unit of velocity}}{\text{unit of time}} \text{ which equals } \frac{\frac{\text{unit of length}}{\text{unit of time}}}{\text{unit of time}} \text{ or } \frac{\text{unit of length}}{(\text{unit of time})^2};$$

and an acceleration 32 stands for 32 units of acceleration as here defined.*

12. If then the velocity of a body *increases*, its acceleration is the gain of velocity in each second of time; but if its velocity *decreases*, then the acceleration is really a retardation, and it must be reckoned negative, but as numerically equal to the loss of velocity in each second. Thus, suppose that in 3 seconds the velocity of a body changes from 196 to 100, its acceleration is — 32. If the velocity of a body is constant, then of course the acceleration (or rate of change of velocity) is zero.

The use of the negative sign.—It is a well-known method to distinguish between opposite directions by opposite signs. Thus, if all distances measured to the right of any point be reckoned positive, any distance to the left will be negative, so that — 30 feet will mean 30 feet to the left.

It is usual to reckon distances *up* as positive, and hence distances *down* as negative. The same may be extended to velocities, and a velocity upward may be called a positive velocity, a velocity downward a negative one. Thus, the velocity of a falling stone is negative, and it is continually getting *numerically* greater (though algebraically less): so the acceleration produced by gravity ought properly to be called negative, because it is negative velocity which is added by it every second. In fact, an increasing negative quantity corresponds in algebra to a decreasing positive one, and *vice versâ*.

* The length of both these expressions, 'feet-per-second per second,' and 'units of acceleration,' often causes them to be abbreviated into some meaningless form, such as feet (one might almost as well say *quarts*), even in books intended for beginners. The use of some name for the unit of velocity, shorter than 'one foot per second,' would obviate this. Suppose, for instance, we agreed to call the unit of velocity a 'speed,' then an acceleration would be stated as so many *speeds per second*. If a name were occasionally wanted for the unit of acceleration, or one speed per second, it might perhaps be called a 'hurry.'

(*b*) *Curvilinear Motion of a Point.*

13. Besides change in the *magnitude* of velocity or rate of motion, there is another thing to be considered—namely, change in its *direction*. Hitherto we have only considered motion in a constant direction—that is, in a straight line; but when the direction of a point's motion is constantly changing, the path described is a curved line, or the motion is curvilinear. The rate of change of direction per unit length of a curve is called its *curvature;* and this again may be constant or variable. Most curves (the parabola, sect. 28, for instance) have variable curvature. A circle or helix has constant curvature. A straight line possesses zero curvature. The curvature of a circle is inversely proportional to its linear dimensions; because the angle which the direction of motion turns through in going once round any circle is 2π (see sect. 16, small type), and the curvature will be this angle divided by the distance travelled—that is, by the circumference, $2\pi r$; hence the curvature of a circle is numerically equal to the reciprocal of the radius, for

$$\frac{2\pi}{2\pi r} = \frac{1}{r} = \text{curvature of a circle.}$$

And the curvature at any point of any other curve is defined on the strength of this, as the reciprocal of the radius of that circle which coincides most closely with the curve at the point.

14. A point moving in a curve, besides any acceleration it may have *along* the curve increasing its velocity, possesses an acceleration *at right angles* to the curve, or normal to the direction of its motion; this acceleration being proportional to the curvature of the curve, and affecting only the direction and not the magnitude of the velocity. Its magnitude is the rate at which velocity normal to the curve is gained by the point. This normal acceleration is called centripetal acceleration, and is further discussed in sects. 54—57, where it will be found to be proportional to the square of the velocity of the point as well as to the curvature of the curve; to be equal, in fact, to $v^2 \times \frac{1}{r}$.

Although the point is always *gaining* velocity normal to the curve or along its radius at this rate, it does not follow that it ever *possesses* any such velocity. It is in fact impossible for a point to possess any velocity except that along the curve, or at right angles to the radius of curvature; for as fast as velocity *along* the radius is generated, so fast does the direction of the radius change; in the same sort of way that a promise for to-morrow need never be fulfilled, because 'to-morrow never comes.'

2. Motion of an Extended Body (Rotation).

15. A point can only move along, it cannot spin; or rather spinning makes no difference whatever to it or to its motion: but an extended body, whether it be a line, surface, or solid, may not only move bodily along or be translated; it may also turn round or rotate. The most general motion of an extended body is a combination of translation and rotation, but it is simpler to consider them separately. All that we have said about the motion of a point is equally true of the motion of an extended body so far as its *translation* is concerned; because, in simple translation, if we know the motion of any single point, we know that of the whole. Its rotation involves different ideas, which must now be considered briefly.

16. When a body rotates, every point of it describes a circle round some point or line which is the centre or axis of rotation.

The velocity of a point far from the axis is greater than that of a point nearer the axis, and in general every point has its own velocity, which is proportional to its distance from the axis, only points at the same distance having the same velocity; hence the 'velocity of a rotating body' is a meaningless expression. The number of times the body turns round in a second, however, is perfectly characteristic, and we must define some kind of *rotational* or *angular* velocity proportional to this.

To express the speed with which a body rotates, it is sufficient to give the velocity of any one point together with its distance from the axis; and the velocity of the particles at *unit* distance from the axis is that which is universally used, the velocity of these points being called the *angular velocity* of the rotating body. The velocity of every particle of the body is known in terms of this, for, being proportional to its distance from the axis, it is equal to this distance multiplied by the 'angular velocity;' or, denoting the velocity of a particle at unit distance from the axis of rotation, that is, the angular velocity, by the letter ω, as is customary, the velocity of any other particle at a distance r is

$$v = r\omega;$$

and this might be called the *moment* of the angular velocity with respect to the axis of rotation (see further on, sect. 37).

Angular velocity in rotations takes the place of ordinary velocity in translations. The name 'angular velocity' is given because it really represents the angle turned through per second by the whole body, as well as the distance travelled per second by particles at unit distance from the axis.

An example may render this more clear. Let us assume, what is well known, that the circumference of a circle is proportional to its diameter, the ratio between them being a constant number equal to about $3\frac{1}{7}$, and which it is usual to call π. The circle described by a particle at a distance r from the axis of a rotating body (say a nail on the circumference of a fly-wheel of r feet radius) is $2r$ feet in diameter, and hence $2\pi r$ feet in circumference. If the wheel turn round in T seconds, the velocity of the nail is $\frac{2\pi r}{T}$; hence the *angular* velocity of the wheel, or the velocity of any nail 1 foot from the axis, is $\frac{2\pi}{T}$, which is ω.

If, then, the angular velocity may be defined as the angle turned through per second, or, which is the same thing, the angle turned through in T seconds divided by the time, it must be because the angle corresponding to one complete rotation is equal to 2π, or about $6\frac{2}{7}$. And this is the case, though you may not see what it means till you come to read Trigonometry, as for most practical purposes the above angle is denoted by the number 360, and not by the number $6\frac{2}{7}$.

17. Of course angular velocity may be uniform or variable; and if the latter, its rate of change, or increase per second, is called the *angular acceleration* of the body. Denoting this by α, we have $\alpha = \frac{\omega}{t}$ (just as we had $a = \frac{v}{t}$ in sect. 10).

But $$\omega = \frac{v}{r}.$$

Hence $$\alpha = \frac{v}{rt} = \frac{\frac{v}{t}}{r} = \frac{a}{r};$$

that is, the angular acceleration of a body, or the acceleration of a particle at unit distance from the axis, is $\frac{1}{r}$th of that of a particle at a distance r. That is, Angular acceleration : acceleration : : angular velocity : velocity : : angle : distance : : 1 : r.

EXAMPLES IN RECTILINEAR MOTION.

(*I.*) *With Constant Velocity or Zero Acceleration.*

(1.) If a snail crawl at the rate of $\frac{1}{4}$ inch a second, how far will it go in an hour? *Ans.* 75 feet.

(2.) How long would a train take to go 100 yards at the rate of 20 miles an hour? *Ans.* $10 \cdot 2\dot{2}\dot{7}$ seconds.

(3.) With what velocity must I walk in order to go half a mile in five minutes? *Ans.* 8·8 feet per sec.

Hence, roughly, 4 miles an hour is 2 yards a second.

(*II.*) *With Constant Acceleration.*

(4.) A body starts from rest and acquires a velocity of 600 feet per second in half a minute; what is its acceleration? *Ans.* 20.

(5.) A body starts with a velocity 50, and in $6\frac{1}{2}$ seconds has acquired the velocity 102; what is its acceleration? *Ans.* 8.

(6.) A body moves with acceleration 32, starting at velocity 20; what is its velocity in 1, 2, 3, 6 seconds respectively? *Ans.* 52, 84, 116, 212.

(7.) A body starting with velocity 100 has only a velocity 52 in 4 seconds; what is its acceleration? *Ans.* — 12.

(8.) A body with acceleration — 32 starts with velocity 128; how soon is its velocity zero? and what is its velocity after 1, 3, 5, 7 seconds respectively?
Ans. 4 seconds; 96, 32, — 32, — 96 feet per second.

(9.) A body dropped from a stationary balloon falls with acceleration 32, and hits the ground with a velocity 512; how long was it in falling? *Ans.* 16 seconds.

All these are merely profit and loss questions. Velocity corresponds to capital, and acceleration to rate of gain. Thus Question 6 may be paraphrased thus: 'A man starts in business with £20, and gains £32 every year; how much has he got in 1, 2, 3, 6 years respectively?'

And No. 8 thus: 'A man starts with £128, and loses £32 annually; how soon will he have lost all? and what will he have in 1, 3, 5, 7 years?' Obviously he will have lost all in four years, and in seven years he will be £96 in debt.

A less simple kind of question is one that involves *distance*; for some examples, see end of next chapter.

EXAMPLES IN CURVILINEAR MOTION

(that is, motion with some acceleration perpendicular to the direction of motion).

(10.) What is the curvature of a circle 14⅔ yards in circumference?

It is numerically equal to the reciprocal of the radius in feet—that is, $\frac{2\pi}{44} = \frac{1}{7}$ nearly.

(11.) A point moves in the above circle with a constant velocity of 6 feet a second; what is its acceleration in magnitude and direction?

Its acceleration is always along that radius of the circle which passes through the moving point, and its magnitude is $\frac{36}{7}$.

(12.) A point moving in a circle 8 feet in diameter, has a velocity increasing by 18 every 3 seconds; what is the acceleration in magnitude and direction at different times?

There is a constant tangential acceleration equal to 6. The normal acceleration is zero at starting; at the end of the first second of motion it is $\frac{36}{4} = 9$; in two seconds it is 36; in three seconds 81, and in t seconds it is $\frac{1}{4}(6t)^2 = 9t^2$. The actual acceleration at any instant is the square root of the sum of the squares of the tangential and normal accelerations at that instant; hence its direction, which at first is tangential, gradually swings round, so that in a few seconds it nearly coincides with the radius.

This explains what happens when we whirl a stone at the end of a string: it is necessary to start it with some purely tangential acceleration, obtained either by the help of gravity, or by a tangential push or pull. When once started, however, the speed may be increased to any extent by simply pulling the string a little to one side of the centre of the circle of motion, so that the tension in the string has both a tangential and radial component; and since the faster the stone is going, the smaller need the former be in comparison with the latter, it follows, that at a high speed the hand remains very nearly steady in the centre of the circle; but it is really travelling round a small circle about a quadrant in advance of the stone—thus supplying the tangential force necessary to overcome the resistance of the air, even if the motion is not being accelerated.

Verify all this experimentally—whirling a weight in a horizontal circle on a flat table, in order to simplify matters by eliminating gravity.

CHAPTER II.

CONTINUATION OF THE SUBJECT OF RECTILINEAR MOTION.

DISCUSSION OF THE STATEMENTS MADE IN CHAPTER I.

18. We have now obtained two definite statements, each of the nature of a definition, namely:

$$\text{Average velocity} = \frac{\text{distance travelled}}{\text{time taken in the journey}} \quad \text{or } V = \frac{s}{t};$$

$$\text{and acceleration} = \frac{\text{velocity gained}}{\text{time taken in the acquisition}} \quad \text{or } a = \frac{v}{t}.$$

And we can proceed to reason on them, and trace their logical consequences, which will all be certainly true.

First, however, it may be well to explain what is meant by 'average' velocity. An *average*, or *mean*, of a set of numbers, is the number about which they all lie most symmetrically, and is found by adding them all together, and dividing the total by the number of them. Thus, to find the average of the five numbers, 12, 16, 25, 30, 32, add them all up, and you get 115; divide by the number of them, that is, by 5, and you obtain 23, which is the average or mean of the five numbers. But now if the numbers, instead of increasing irregularly, as in this set, had progressed by a common difference as in the following, 7, 11, 15, 19 (which increase regularly by 4), then, though we *may* use the same process, and find the mean to be 13, it is not necessary to do more than take the first and last of the set, and find the mean of *them*; that is, add them together, and divide by two, $\frac{19+7}{2} = 13$. This applies to all the cases of velocity we shall have to deal with. The velocity is to increase regularly (or the acceleration is to be constant), consequently the average velocity is obtained at once by halving the sum of the initial and final velocities, $V = \frac{v_1 + v_0}{2}$; using v_0 to stand for initial, and v_1 for final velocity (sect. 10).

Hence our first equation, which may be put into the form $s = Vt$, may be written more fully thus:

$$s = \frac{v_1 + v_0}{2} t.$$

Similarly the second equation may be written in the form $v = at$, or more fully,

$$v_1 = v_0 + at,$$

which signifies that the final velocity is equal to the initial velocity, plus the gain.

Of course v, or at, the gain of velocity, is equal to the difference between the final and initial velocities, $v = v_1 - v_0$; but in case the final velocity is less than the initial, the gain becomes a loss, or v is negative, and therefore also a is negative—that is, it is really a retardation, but it may still be called an acceleration, only a negative one.

19. Now let us study the two equations together, and see what we can get from them by any algebraical operation; remembering that algebra, like all other reasoning, never gives us anything really fresh; it only brings out explicitly what is already contained implicitly in the *physical* statements which we subject to reasoning. The physical statements must be the results of the observation of nature, which is the only way of arriving at fundamentally new truths. Mathematical reasoning will, however, serve to bring out and make manifest what was really involved in the statements themselves when put together, if only we had sufficient insight to perceive it.

Our two statements or equations, written out fully, are

$$s = \frac{v_1 + v_0}{2}t, \quad \text{and} \quad a = \frac{v_1 - v_0}{t}.$$

First multiply the two left-hand members together and double the product; then do the same with the two right-hand members, and write the two products equal to each other (as of course they must be), and you get the new equation,

$$2as = (v_1 + v_0)(v_1 - v_0) = v_1^2 - v_0^2.$$

This is a relation between a, s, and v, without explicit reference to t, and it will often be useful.

Now try again, and this time get a statement not involving v_1, which we can do by substituting the value of v_1 from the second equation, namely, $v_1 = v_0 + at$, in the first,

and we shall get $\qquad s = v_0t + \frac{1}{2}at^2.$

Similarly we can get a relation excluding v_0, and it is

$$s = v_1t - \tfrac{1}{2}at^2.$$

20. But before proceeding to study the two equations together, we might in this stage have first made a simplification. An obvious simplification would occur if the initial velocity were made zero ($v_0 = o$); in other words, if we agreed to consider only bodies starting from rest. In this case the gain of velocity v is equal to the final, v_1, and the average velocity V is equal to $\frac{1}{2}v_1$, which is now

the same as $\frac{1}{2}v$; and so the two fundamental equations reduce to

$$s = \frac{1}{2}vt, \quad \text{and} \quad a = \frac{v}{t};$$

and the three derived from them simplify in like manner. We thus obtain the following four equations between the distance travelled by a body *from rest*, the time taken in the journey, the acceleration, and the final velocity gained;

$$v = at.$$
$$s = \frac{1}{2}vt.$$
$$s = \frac{1}{2}at^2.$$
$$v^2 = 2as.$$

Of these, two are independent statements, and the other two are logical consequences of them. The first of the four reads thus: The velocity gained in t seconds equals t times the velocity gained in each second. The second one thus: The distance travelled over in t seconds equals t times the average distance travelled over in one second (for this last is the meaning of average velocity). Both these statements are perfectly obvious. The other two statements cannot be put in quite so obvious a form. Observe that there are only four quantities involved, s, v, a, t, and that one of them is absent from each of the four equations.

21. The meaning of the second derived equation in sect. 19 is now clear. The space described by a body with the constant velocity v_0 is v_0t, and by one with the uniform acceleration a is $\frac{1}{2}at^2$; so the whole space described by the body possessing the initial velocity v_0, and *also* subject to the acceleration a, is

$$s = v_0t + \frac{1}{2}at^2.*$$

This is really a case of the composition of motions in the same direction. See sects. 23 and 69.

22. The results expressed by these equations may be made to appeal to the eye more directly, and thus be rendered easier to grasp, if illustrated by their analogy with geometrical diagrams.

* If a and v_0 are of opposite sign, the subtraction is to be performed when the letters are arithmetically interpreted. The sign + means *algebraical* addition, which includes subtraction.

If a horizontal line be considered as representing by its length a definite lapse of time, say three seconds for every inch; and if a vertical line represent by its length a certain velocity (so many feet a second for every inch); then the product of velocity and time (that is, distance travelled) will be represented by the area of the rectangle contained by these two lines.

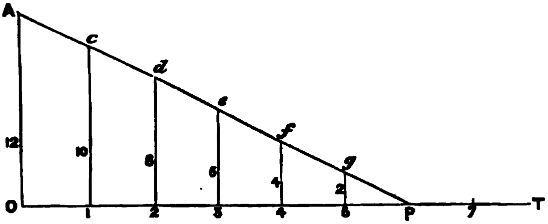


Fig. 1.

Thus in fig. 1, OT is the line of time, with the seconds marked off upon it. OA is a vertical line, and represents a velocity, say of 12 feet a second.

Let a body start with this velocity and lose 2 of it every second, then in one second its velocity will be represented by the line *c*1, in 2 seconds by the length of the line *d*2, and so on. Consequently, in 6 seconds the body will be at rest. The diagram thus represents, in a conventional and utterly non-pictorial fashion, a body starting with initial velocity 12, and going with a uniform negative acceleration — 2, till it stops. The average velocity would of course be 6, and would be represented by the length of the vertical line drawn in the middle of the time—namely, *e*3.

The distance traversed would be this average velocity multiplied by the time. That is, geometrically, *e*3 multiplied by OP, which is the area of the triangle OAP; for the area of a triangle is equal to the product of base and average height—in other words, to the product of half its height into its base.

Areas then in this figure represent distances. Or, more correctly, the ***number of units of area*** in one of these figures equals the ***number of linear units*** in the distance travelled.

The distance travelled in the first second is the area O1*c*A; in the second second, *c*12*d*; in the last second, *g*5P. The distance travelled in the three seconds between the first and fourth is represented by the area *c*14*f*, and so on—that is, each of these distances is equal to the number of units of area contained in the respective spaces.

The representation of a body starting from rest with a *positive* acceleration, is given in fig. 2.

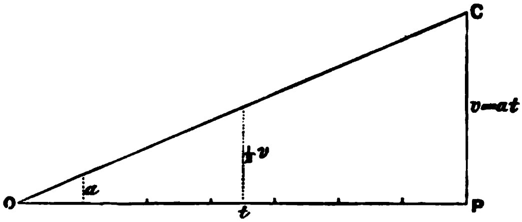


Fig. 2.

The line of time is divided to represent seven seconds. The velocity gained in one second is represented by the line marked a, which therefore represents the numerical value of the acceleration. The velocity gained in the whole time is marked v; it is obviously equal to $7a$ or at. The dotted line in the middle of the time is the average velocity, and it is evidently $\frac{1}{2}v$.

The area of the whole triangle represents the whole distance travelled, and it is half the height multiplied by the base, or $\frac{1}{2}v \cdot t$, or, what is the same thing, $\frac{1}{2}at \cdot t$, that is, $\frac{1}{2}at^2$.

The little left-hand triangle is numerically equal to $\frac{1}{2}a$ in area (its base being unity), and it is the distance travelled in the first second.

The velocity possessed by the body at any second or fraction of a second is found at once, simply by measuring the vertical height of the triangle at the place defined by the time. The whole problem is in fact geometrically represented.

If the body started with an initial velocity, and then went on with increasing velocity, its motion would be represented by fig. 3, which is supposed to represent what happens in three seconds. The initial velocity is marked v_0, and the final v_1; the latter being made up of two parts, the gain of velocity at, and the original velocity v_0. The rest is marked as before, and the whole area represents the whole distance, S, travelled in the three seconds.

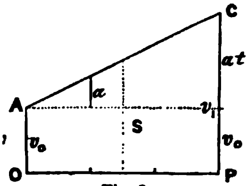


Fig. 3.

The dotted line in the middle is of height $v_0 + \frac{1}{2} at$, or, what is the same thing, $v_1 - \frac{1}{2} at$; and therefore it is $\frac{1}{2}(v_0 + v_1)$, or the average velocity. Either of these expressions for the average velocity multiplied by the time will of course give the distance travelled (cf. equations of sect. 19). If the initial velocity be negative, the line representing it must be drawn *down* from the line of time, instead of up.

These diagrams will be found exceedingly useful and conducive to clear ideas, as soon as a little practice has made you familiar with them. For some more illustrations of their use, see sect. 65, which can be read now.

Composition of Motions in General.

23. When a body has several motions given to it at the same time, its actual motion is a compromise between them, and the motions are said to be compounded, the actual path taken being called the resultant. Thus, suppose a fly to crawl along a tea-tray from A to B (fig. 4), while at the same time some one pushes the tray along a table a distance PQ; the fly will then have two motions, and its actual motion with reference to the table is the resultant of the two motions, AB and PQ. To find where

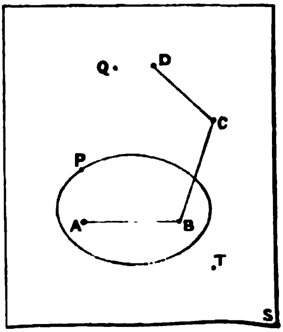


Fig 4.

the fly is at the end of the two motions, we must observe where the point B of the tray has gone to, for the fly has crawled to B; but B has been moved to a point C, such that BC = PQ. Hence the fly is at C, and its actual motion must have been along some path AC, not necessarily a straight line; and AC is therefore called the resultant of the two motions, AB and BC. If, besides these two, the table itself had been pushed in the direction ST, or what is the same thing, CD, then we should have had three motions to compound; and, as the fly would have got ultimately to D, AD would have been the resultant of the three motions. The *order* in which the steps are added evidently does not matter, for the same point D is arrived at by taking the table motion before that of the tray, as in 1, fig. 5; or the fly's

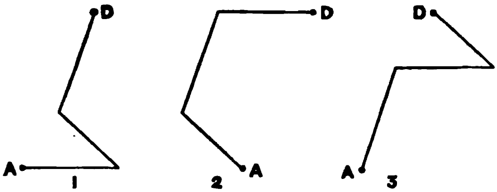


Fig. 5.

proper motion after both the others, as in 2; or the fly's motion between the other two; or in any of the six possible orders in which three motions can be compounded.

And so we readily see the rule for compounding any number of motions. Draw lines, or cut pieces of stick,† representing each motion in magnitude, direction, and sense,* and lay these lines or sticks in any order, with the end of one coinciding with the beginning of the next (the lines may be moved into any positions, provided each is kept parallel to itself); then some line joining the first

* That is, make some difference between the two ends of the line, indicating by an arrow-head or otherwise which *way* the motion takes place in the given direction.

† See footnote to page 96.

point of the first with the last point of the last, must be the resultant of the whole set of motions.

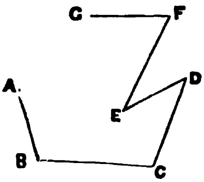


Fig. 6.

Thus, some line AG is the resultant of the six motions, AB, BC, CD, DE, EF, FG. This proposition is called the polygon of motions, because the resultant is represented by the line required to complete a polygon.

As a matter of fact, however, the sides of the polygon need not necessarily be straight lines. The end points of the line are the only essential matter when one is dealing with simple change of position without regard to time or speed.

24. The composition of *two* motions, AB, BC, into a third, AC, requires only a three-sided polygon, so it is often called the *triangle* of motions. Or if we choose to represent the two component motions, AB, BC, by lines, AB, AB′, drawn from the same point, we get the *parallelogram* of motions, which is merely a less simple, but sometimes convenient, way of regarding the triangle of motions. The resultant motion is the diagonal of a parallelogram whose two adjacent sides represent the component motions.

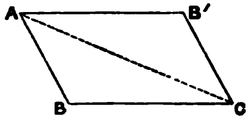


Fig. 7.

25. This law, by which two motions are compounded, is of very frequent occurrence in all parts of mechanics, and is referred to as *the parallelogram law.* It may be stated thus: If two causes act on a body at once, or if a body experience two simultaneous effects in different directions, then if these effects are represented in magnitude and direction by two adjacent sides of a parallelogram, the effect experienced by the body is called the *resultant* effect, and is represented, on the same scale, by the concurrent diagonal of the parallelogram—that which passes through the point of intersection of the two sides; or it is the same effect as would be produced by a *resultant cause* represented in magnitude and direction by the similarly situated

diagonal of a parallelogram whose two sides represent the component causes; provided always that the causes or the effects can be shewn to be of such a nature that this law is applicable to them.

Composition of Uniform Velocities.

26. So far we have only studied the composition of changes of position; now let us study the composition of velocities—first, when uniform. Remember that a velocity is the space described in one second.

Let a body start from O (fig. 8) with two velocities, one horizontal and of magnitude O*a*, the other vertical and of magnitude O*b*. Then O*a* and O*b* represent the distances travelled in one second in the respective directions, and consequently at the end of one second the body is at the point *c*, the opposite corner of a parallelogram with sides O*a* and O*b*; hence the body must really have travelled the distance O*c* in one second, therefore O*c* is its *resultant velocity* in magnitude and direction. In two seconds it will have travelled horizontally to *a′*, and vertically to *b′*, and therefore it will really have reached *c′*. And it is easy to see, by drawing or otherwise, that the straight line O*c′* passes through the point *c*, and that O*c′* equals twice O*c* (because O*a′* = twice O*a*, and O*b′* = twice O*b*); or the distance travelled in two seconds is twice the distance travelled in one; and so, generally, the resultant of two uniform velocities is another uniform velocity along the diagonal of the parallelogram whose adjacent sides represent the components.

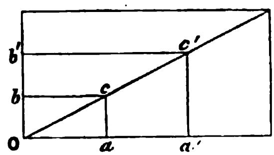


Fig. 8.

Hence the resultant of two velocities is obtained by precisely the same parallelogram law as the resultant of two simple motions. Similarly the 'polygon' law is applicable for compounding any number of velocities greater than two.

Composition of Uniform Accelerations.

27. Accelerations may evidently be compounded by the same law as velocities, because acceleration is the velocity gained per second. Thus let a body be subject to any two accelerations, say a horizontal one O*a*, and a vertical one O*b*; then O*a* and O*b* represent the velocities gained per second in these two directions respectively, and therefore the actual velocity gained in the second is O*c*; in other words, O*c*, the diagonal of the parallelogram, measures the resultant acceleration. Hence accelerations are compounded by the same law as velocities.

28. *Composition of a uniform Velocity with a Velocity uniformly accelerated in a constant direction.*—Let a body start from O with a uniform velocity u in some direction OV (fig. 9), and a uniform acceleration a in some other direction, such as OL vertically downwards; then in successive seconds the distances traversed in the first direction will be

$$u, \quad 2u, \quad 3u, \quad 4u, \quad \&c.;$$

so that, if this constant velocity u were the only one possessed by the body, the body would be at T after one second, at U after two, at V after three, and so on (fig. 9). But the uniform acceleration is acting at the same time, and causing the body to descend a height proportional to the square of the time ($\frac{1}{2}at^2$); hence in successive seconds the vertical distances traversed will be

$$\frac{a}{2}, \; 4\frac{a}{2}, \; 9\frac{a}{2}, \; 16\frac{a}{2}, \; \&c.,$$

bringing the body to the level N in one second, M in two, L in three, and so on, if it had acted alone.

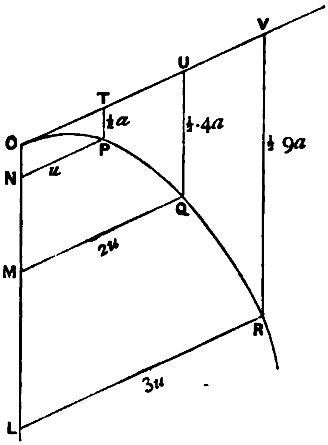


Fig. 9.

The *actual* position of the body, therefore, at the end of successive seconds will be found by completing the parallelograms,

OTPN,
OUQM,
OVRL,
&c.;

the result being that the body reaches the point P in one second, Q in two, R in three, and so on.

Now the simplest continuous curve which can be drawn through these points, OPQR, &c., is a parabola, and this is found to be the actual path of the body.

It will be shewn later (sect. 70) that this is the path of a projectile thrown *in vacuo* in the given direction with the velocity u, and subject to gravity. And it is well seen in the curve of a steady jet of water, for each drop of water takes this path. It is also illustrated by Morin's machine (sect. 71).

This example illustrates the fact, noted at the end of sect. 23, that the resultant motion (the diagonal of the parallelogram) need not be represented by a straight line. It will be straight if the two things compounded are of the same kind and both uniform; otherwise it will in general be curved.

EXAMPLES.

(Here acceleration is understood to be uniform unless otherwise stated.)

(1.) In all the questions involving uniform acceleration at the end of Chapter I., find the distance travelled by the different things in the times given.

Ans. In Question 4 the initial velocity is 0, the final is 600; ∴ the average is 300; ∴ in thirty seconds it will have gone 9000 feet. In Question 5 the average velocity is $\frac{50 + 102}{2} = 76$, so in $6\frac{1}{2}$ seconds it will go 494 feet. And so on.

Or we may do them by the formula $s = \frac{1}{2}at^2$.

(2.) A body starting from rest, and travelling 63 feet in a straight line, gains a velocity of 81 feet per second; what is its acceleration?

Ans. Since $v^2 = 2as$, $a = \frac{v^2}{2s} = 52\frac{1}{14}$ speeds per sec.

(3.) What is the acceleration of a body whose velocity changes from 7 to 21 while it travels 100 feet?

Ans. $a = \frac{v_1^2 - v_0^2}{2s} = 1{\cdot}96.$

N.B.—The arithmetic is often simplified by taking the difference of two squares in the form of the product of sum and difference.

(4.) Find the accelerations of the following bodies:

A,	whose velocity changes from	15 to 5	in going	50 ft.
B,	" "	15 to −5	"	50 ft.
C,	" "	−5 to 15	"	50 ft.
D,	" "	120 to 0	"	640 ft.

(Remember that the square of a negative number is positive.)

Ans. A and B, − 2; C, + 2; D, − 11·25.

The extreme distance from the starting-point attained by B is 56¼ feet, but 6¼ feet of this is retraced. It therefore takes longer in the journey than A did, but its acceleration happens to be the same.

Similarly with C, the first thing it does is to go 6¼ feet backwards and come to rest for an instant; then it retraces its path and goes 50 feet forwards, where the question leaves it; but it is still going on with a speed increasing by 2 in every second.

(5.) Find the time of the motion in all these cases, and draw a diagram for the several motions.

Begin by drawing the line of time; then draw verticals for the initial and final velocities, paying attention to sign, and join the extremities of these lines; then study every part of the diagram, and note its connection with the equations.

Ans. The diagram for C will look like fig. 18 upside down.

(6.) A train with the brakes on, moving with acceleration − 3, has a velocity 78 when passing a particular station; how much further will it go?

Ans. It will continue moving for 26 seconds, and therefore will go 1014 feet.

(7.) A point moves 16 feet in one second and 20 feet in the next; how long has it been moving with uniform acceleration since it started from rest, and what is the rate of the acceleration? Also, how far would it go in the next 12 seconds of its motion, and when will its velocity be 128?

Ans. The acceleration must be 4, because the average velocity during first second spoken of, and therefore the velocity in the middle of that second, is 16, while in the middle of the next second it is 20. The velocity at the beginning of the former of these seconds must therefore have been 14, and at the end of the latter second 22. To gain the velocity 14 with acceleration 4 required $3\frac{1}{2}$ seconds, which is therefore the time the point had been moving at the beginning of the first second spoken of. Starting with velocity 22, it would go 552 feet in the next 12 seconds; and its velocity would be 128 in 32 seconds from the original start, or $26\frac{1}{2}$ seconds from the time its velocity was 22.

Examples in the Composition of Motions.

(8.) A point has two motions, one east with a uniform velocity 30, the other north with a uniform velocity 40; what is its actual motion?

Ans. It moves with a uniform velocity 50 in a straight line nearly NE by N.

(9.) The banks of a river run north and south, and a boat is rowed at right angles to the stream half as fast again as the river flows: it reaches the opposite bank 2 miles below the starting-point; find the breadth of the river and the distance rowed.

Ans. 3 miles; $\sqrt{13}$ miles.

(10.) A point describes a circle with a constant velocity v, and at the same time the centre of the circle moves forward in a straight line with the same velocity. What is the motion of the point?

N.B.—This is the case of a nail on the circumference of a coach-wheel. The point describes a curve with cusps, called a cycloid; its velocity when at the top of the wheel is $2v$, and when on the ground is zero; its velocity at the extreme right and left points of the wheel is $v\sqrt{2}$; its velocity is v at two points whose distance from the ground is half the radius.

CHAPTER III.

ON QUANTITY OF MATTER AND QUANTITY OF MOTION.

A CHAPTER OF DEFINITIONS.

(I.) MOTION OF A PARTICLE, OR TRANSLATION.

(*Inertia and Momentum.*)

29. We have so far studied motion in the abstract, with reference to its direction and its rate, but without reference to the body moving, or to the amount of motion possessed by it. Let us now consider what is meant by this last phrase 'amount' or 'quantity of motion.'

First, it is plain that in any actual case of motion there must be some matter moving; and it will be sensible and consistent with the ordinary use of language to consider the quantity of motion in a body as proportional, first, to its speed, and, secondly, to its quantity of matter; and this is the scientific custom.

30. Now we understand what is meant by speed, but what do we mean by quantity of matter? First of all, of course, a large solid ball contains more matter than a small one of the same material; but quantity of matter does not depend on size alone, it depends also on the closeness or density of the substance. A small iron ball may contain more matter than a large cork one.

Now matter possesses a certain characteristic property called 'inertia,' or power of reacting against a force applied to change its state of motion. It is on account of this property that *force* is required to move matter or to check its motion—the passive resistance or reaction of the matter itself being called its *inertia-reaction* or *inertia*. Thus a railway truck has great inertia, because it is hard work to stop it or to set it going, quite independently of its weight or of any friction there may happen to be.

This 'inertia' or reluctance of matter to change its state, whether of rest or motion, was expressed by Newton in the following 'law' or axiomatic statement: '*Every body perseveres in its state of rest or of moving uniformly in a straight line, except in so far as it is made to change that state by external forces.*' This is often referred to as Newton's *First Law of Motion*, or as *the Law of Inertia.*

31. Since inertia then is a characteristic property of all matter, it will serve to *measure* the quantity of matter in any given mass, and it is always used for this purpose in Dynamics. Suppose you have a number of smooth cubes or blocks, each made of a different material but of the same size, resting on a perfectly smooth horizontal table, and you give them each a little push of exactly the same strength; the push will have the least effect on those which contain the greatest quantity of matter. Thus imagine four of the cubes to be of cork, wood, iron, and gold respectively, and that you give each a sudden knock. The cork block would be considerably affected, and would slide off the table; the block of wood would be affected next in extent; while the iron and gold blocks would perhaps hardly be stirred, but whatever movement there were would be greater in the iron than in the gold. We should hence conclude that the gold block contained most matter, the iron next, and the cork least. This is a perfectly direct and scientific method of comparing the masses of bodies, and more than *comparing*, for it is capable of affording a definite *measure* of the quantity of matter in a body. Thus either apply the same force for the same time to each body, and measure the velocity imparted (if the same velocity is imparted to a number of bodies by the same shock or impulse, they have all the same inertia, and therefore the same quantity of matter); or graduate the forces applied to the different bodies, so that each may move with the same acceleration, the forces required will measure the inertia of the several

bodies. The forces themselves must be measured by the strain method (see Introduction, sect. 6), as the other method would lead to reasoning in a circle.

Fig. 10 shews the experiment carried out as far as it is possible to carry it out without a perfectly smooth table. The blocks are mounted on rollers to diminish friction, and are attached each to a strong spring balance (like those used for weighing letters, and sometimes for weighing fish) which will yield in a very small but yet a measurable degree. These balances are then all tied to a rod, and are pulled quickly along, so that all the blocks have practically the same acceleration imparted to them. The springs indicate by their stretch the inertia-reaction of each body.

Fig. 10.

32. One often actually applies this method of comparing masses in common life. Suppose you see a cask lying on level ground, and wish to know whether it is full or empty; you give it a kick or a push with your foot, and if it yields and moves easily, you conclude that it contains very little matter—that is, that it is empty; whereas if it almost refuses to move, it must contain much matter; and if it contains *dense* matter, such as iron or lead, it will be harder to move than if it contained, say earthenware, and this again harder than if it were full of straw. Hence we find that the *quantity of matter* in a given body, as measured by its inertia, depends first on the *density* of its material; and secondly, on its size or *volume*. And we might define quantity of matter as the product of volume and density, giving this product the name of *mass*. The 'mass' of a body hereafter, then, shall stand for the quantity of matter in it, and shall equal its volume multiplied by its density. This last serves strictly for a definition of *density* rather than of mass, as thus:

$$\text{Density} = \frac{\text{quantity of matter in body}}{\text{volume of body}};$$

or more simply, ***density is the mass of unit volume.*** (The unit of volume is the cube of the unit of length—say a cubic foot.)

We see, then, that mass is measured, and must be held to be defined, by the property of inertness possessed by matter—that is, by its requiring force to move it if at rest, and to stop it if in motion. This idea of the muscular effort needed to set a body moving or to stop it, must be held to be the primitive idea of inertia. The greater the effort required to produce a given motion, the greater the inertia; and as every particle of matter possesses this property, the more particles there are the greater is the inertia, and inertia is the only *direct* measure of mass in mechanics.

To recapitulate, then, mass *means* quantity of matter, and *is measured by* inertia.

33. Just as the unit of length is an arbitrary distance (called a foot), so the unit of mass must be an arbitrary quantity of matter, and in this country the unit is called a *pound*. That is, the quantity of matter contained in a standard *pound avoirdupois* is taken as the unit of mass. Hence arises confusion. Because the pound happens to pull downwards with a certain force (*avoir*, in fact, *du pois*), people constantly think of this *pull*, or *force*, or *weight*, as the essential thing, whereas it is quite a secondary thing. When we speak of this force, we shall call it the pound *weight*, or the weight of a pound—it is not the pound itself (see sect. 60).

Suppose you wish to leave some flowers to be pressed all night in a book, and you put on the book for the purpose a few pound or other weights; what you are then concerned with is the *weight* of the pounds, or their pull downwards. But suppose you buy six pounds of sugar or of soap; what you are *then* concerned with is the quantity of matter or *mass* which you obtain, and the force with which the matter tends downward is a secondary, and sometimes a burdensome consideration. This confusion has arisen from the fact that the shopman measures the mass out to you, not by a direct method like that shewn in fig. 10, but by an indirect, though practically simpler method, founded on the attraction of gravitation, which Newton shewed was proportional to the mass of the attracting bodies within the limits of experimental error. The confusion is perpetuated by the absence of any word signifying the comparison of masses. 'Weighing' must mean the comparison of weights, but as there is no word 'massing,' one has to use weighing in the sense of comparing masses also. We must try, however, to avoid this confusion, even at the risk of a little pedantry, which may be necessary until we are quite clear on the subject. A pound, an ounce, a grain, a gramme, &c., then, represent quantities of matter, or masses, and *not* weights. The term a 'hundredweight' bears marks of the confusion on its surface, and is therefore better avoided for the present.

34. Now we have already seen (sect. 29) that it is reasonable to define *quantity of motion* as directly proportional to the quantity of matter (or mass) moving, and to its rate of motion (or velocity). Hence let us at once define quantity of motion as equal to the *product* of the mass in motion and its velocity. The name given to quantity of motion is *momentum;* so we have now the definition :

$$Momentum = mass \times velocity, \quad or\ \mu = mv,$$

where m stands for mass, and μ for momentum.

Momentum *means* quantity of motion, and *is measured by* the quantity of moving matter multiplied by its velocity.

35. The unit of mass being a pound, the unit of momentum must be that quantity of motion possessed by a pound of matter when moving with a velocity of one foot per second. The momentum of a $\frac{1}{4}$-lb. cricket-ball moving at the rate of 56 feet a second, is $\frac{1}{4} \times 56 = 14$—that is, fourteen units of momentum, as just defined.

The momentum of a 50-lb. cannon-ball moving with a velocity of 1612 feet per second, is 80,600.

That of a three-ton truck (the ton = 2240 lbs.), moving at the rate of 12 feet per second (roughly about eight miles an hour, see ex. 3, Chapter I.), would be 80,640, or nearly the same as that of the cannon-ball.

Now we shall find in the next chapter that a force is proportional to the quantity of motion it causes; hence we see that in some sense or other the same motive power was required to set the above cannon-ball going as was required to set the truck, for both possess the same quantity of motion. Yet the force exerted by the powder in the cannon was undoubtedly greater *while it lasted* than the force exerted by the horse or engine, or whatever started the truck; but then the former acted for the fraction of a second only, while the latter took perhaps a minute. What is called the *impulse* of the force was the same in the two cases. If you put an obstacle in the path of each body so as to stop both in the same time, they would each deal the same blow.

36. Before passing on to the action of force on matter, it will be well to explain that now we have come to deal with the motion of actual pieces of matter, we shall, if we wish to consider a piece so small that its parts may be neglected, use the term *particle* instead of 'point;' meaning by *particle* a point possessing inertia, or a *material point.* A 'particle' may have any finite *mass;* its *size*, indeed, is to be small (or at anyrate negligible), but its *density* may be anything—infinite if we like. A body whose parts are taken into account may still be called an 'extended body,' but if stress is wished to be laid on the fact that these parts are immovable relatively to each other, it will be called a *rigid body.* An extended body whose parts are capable of relative motion is called an elastic or a plastic body. (Chapter X.)

Also it will be well to point out that the parallelogram and polygon laws apply to the composition of momenta just as they do to the composition of velocities (sect. 26). For the momentum of a given mass is simply proportional to its velocity, and the resultant velocity of a particle when multiplied by its mass must be its resultant momentum.

(II.) SPINNING MOTION OF AN EXTENDED BODY, OR ROTATION.

(Moment of Inertia and Moment of Momentum.)

37. We have already partly seen (sect. 16) that when we come to consider the motion of a rotating body, the distance of each particle from the axis of rotation is always coming in as a factor, multiplying the term which previously had been sufficiently expressive. As this product so often occurs, it is convenient to have a name for it, and the name employed is *moment.* The moment of any physical quantity is the numerical measure of its importance. [This

must not be confounded with *momentum*, with which it has nothing to do.]

When any directed quantity is multiplied by a distance at right angles to itself, the product is called the *moment* of that quantity.

Now, in the case of a rotating body, distances measured from any point of it *to the axis of rotation* are necessarily at right angles to the motion of the body, for the same reason that the radius of a circle is at right angles to the tangent. So the actual velocity of a particle of a rotating body might be called the *moment* of the angular velocity, for it equals ωr, r being measured from the particle to the axis of rotation. The actual *acceleration*, again, is the *moment* of the angular acceleration—that is, it equals αr (see sect. 17).

It often happens that the distance from the axis comes in as a factor *twice*, so that we have a moment of a moment, which is called a *second moment*.

When any directed quantity is multiplied by the square of a distance at right angles to itself, the product is called the *second moment* of that quantity.

Thus for some purposes it is convenient to speak of the moment of the velocity of a particle of a rotating body—that is, vr; and this is the *second* moment of its *angular* velocity, being equal to ωr^2. The moment of *momentum* of such a particle is of course mvr, or as it may also be written $m\omega r^2$.

38. These terms being understood, we will proceed to consider how we must define the quantity of motion of a rotating body, or a system of circularly moving particles. Simple momentum, or product of velocity and quantity of matter, will not do, for the effect produced by a given shock depends not only upon this, but also upon how far distant from the axis the bulk of that matter is. For consider a fly-wheel; which you know is a large heavy wheel fixed to the crank-shaft of stationary engines and driven at a high speed, not for the purpose of communicating its motion to a lathe-band or anything, but simply for the

purpose of storing up a certain quantity of motion sufficient to carry the engine over its 'dead points,' and also over any accidental shocks or sudden impediments which the machinery may experience: it is made massive so as to have great inertia, it is also made to go fast so that it may possess great momentum; but besides this it is made large, and nearly all the mass is placed in its rim, so that the motion stored up in it may have a great *leverage*.

For just as the power or moment of a force depends not only on its magnitude but also on the place at which it is applied—not only on its strength but on its leverage—being equal to the product of the force into its distance from the fulcrum (for example, the longer a crowbar is, the more power it gives you; the more unequal the length of the arms of a steelyard, the bigger the weight which can be balanced by a little one; and so on, see sect. 137); so with the fly-wheel, the effect or power of its stored-up motion depends not only on the actual quantity of motion or momentum of the rim, but also on the distance this rim is from the axle —that is, on the radius of the wheel. It depends, in fact, on the *moment* of its momentum, μr.

39. Now if the wheel were a simple infinitely thin rim, the meaning of this would be simple enough; r would stand for the radius of the rim, and μ for the product of its mass and velocity mv (sect. 34); but any actual wheel must have a rim of some thickness, as well as some spokes and a nave, so the meaning of neither μ nor r is quite clear without further definition.

The moment of momentum of a rotating body is the sum of the moments of momenta of its several particles.

Let a wheel turn with the uniform angular velocity ω. A particle of mass m_1, at a distance r_1 from the axis, and moving with velocity $r_1\omega$ or v_1, has a momentum m_1v_1, and therefore a *moment* of momentum $m_1v_1r_1$, or what is the same thing, $m_1r_1^2\omega$. Similarly with a particle of mass m_2 at a distance r_2; and with one of mass m_3 at distance r_3, and so on; hence the *moment of momentum* of the whole wheel is the sum of these terms for all the particles in the body,

$$m_1r_1v_1 + m_2r_2v_2 + m_3r_3v_3 + \ldots\ldots\ldots$$

or as it is often written $\Sigma\,(mrv)$.

Since $v = r\omega$, and since ω is the same for every particle as for the whole body, we may write the above expression for the moment of momentum in this equivalent form,

$$\omega\,(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots) = \omega\Sigma(mr^2)\,;$$

or in words, the moment of momentum of the wheel is the angular velocity multiplied by the sum of the second moments of inertia of every particle in the wheel.

In this last form, $mr^2\omega$, the moment of momentum, is often called the *angular* momentum; because, instead of being simply the product of inertia and velocity (as momentum is), it is the product of a *moment* of inertia and *angular* velocity.

40. In the last paragraph we have the occurrence of the *second* moment of mass or inertia, mr^2, and indeed this occurs in Dynamics so much more frequently than the first moment (mr), that it is usually called *the* moment of inertia.

The moment of inertia of any rotating body about its axis of rotation is the sum of the second moments of the masses of all the particles in it about that axis; and we will denote it by M, so that $M = \Sigma\,(mr^2)$.

The angular momentum, or moment of momentum, of the above fly-wheel is thus simply $M\omega$.

CHAPTER IV.

ON FORCE AND MOTION (*Dynamics*).

It was stated in the Introduction that force produced two kinds of effects on matter—'acceleration' and 'strain.' In the present chapter we will consider only the first or motive effects of force—that is, the effects of force on rigid bodies or particles (see sect. 36); and first on particles moving in straight lines—

(I.) ON THE SPEED OF MOTION AS AFFECTED BY FORCE; OR, FORCE AND RECTILINEAR MOTION.

(*Dynamics of a Particle.*)

41. When a single force F is applied to a certain quantity of matter or mass, m, for a unit of time, a certain quantity of motion or momentum is generated in the mass. If the same force (for example, a piece of elastic stretched to the same extent as before) is applied to a greater quantity of matter for the same time, it will move with less velocity, but the *product* of the quantity of matter and the velocity—that is, the *quantity of motion* or the *momentum*—will be found to be the same; so the force may be measured by the momentum generated by it per second, since this is constant, and depends on nothing but the force. If the same force be applied for t seconds instead of one, t times the quantity of momentum will be generated; hence the *force*, or the momentum generated per second, is obtained by dividing the whole momentum generated, by the time taken to do it; or in symbols,

$$F = \frac{mv}{t};$$

and the *unit* force will be that which can generate unit momentum in unit time.

42. Force, then, by this definition comes to be *rate of*

change of momentum, just as acceleration was defined to be rate of change of velocity.

$$a = \frac{v}{t} \text{ (sect. 10).}$$

Hence force bears the same relation to acceleration as momentum does to velocity: each, in fact, equals the other multiplied by m, or

$$F = ma.$$

This last is a very convenient form of the definition, and may be expressed thus:

A force is numerically equal to the acceleration it can produce in unit mass; and in any case it is equal to the product of the mass acted on, and the acceleration produced in it; or concisely,

FORCE = MASS ACCELERATION.

This is, indeed, the fundamental relation of Dynamics, for it makes all that we have learned about motion in the abstract (Kinematics) available for dynamical problems—that is, for all problems involving force.

43. The unit of force may be expressed in these three different but equivalent ways:

The unit of force is that which causes unit acceleration in one pound of matter (unit mass);

Also, unit force is that which generates unit momentum in one second, as said above;

Also, it is that which, acting on unit mass for unit time, causes it to move with unit velocity. So, if the unit force act on a pound for a second, the pound at the end of that second will be moving at the rate of one foot per second.

It is often convenient to have a name for the unit of force as defined in any of these equivalent ways. The name *poundal* has been suggested in order to indicate a connection between the British unit of mass and the force unit (not by any means to signify that the force unit equals the *weight* of a pound: it is nearer the weight of half an ounce). A poundal is also called the *British* unit of force, to distinguish it from the unit founded on the metric system, which involves *grammes* and *centimetres* instead of pounds and feet. This metric unit of force is now very frequently called a *dyne*. It is, of course, that force which, acting on a gramme for a second, generates in it the velocity of one centimetre per second. It is a very small force indeed, only about the thousandth part of the weight of a gramme, which is itself only about 15 grains.

One poundal equals 13825·38 dynes.

44. The fundamental connection between force and acceleration, $F = ma$ (sect. 42), may be written, of course, in two other forms; and this one,

$$a = \frac{F}{m},$$

is an abbreviated statement of the fact that when a force F acts on a mass m, the acceleration produced in it is the ratio of the force to the mass.

Let us take an example to illustrate the application of this. Find the distance travelled in 8 seconds by a mass of 2 lbs. which starts from rest, and has a force of 6 poundals acting on it all the time.

The acceleration or velocity acquired per second is

$$a = \frac{F}{m} = \frac{6}{2} = 3.$$

The whole velocity acquired in the 8 seconds is therefore 24, and hence the average velocity is 12.

The distance travelled is the average velocity multiplied by the time, or 96 feet, which is the answer.

Or we might, without troubling about the velocity, have applied the formula $s = \frac{1}{2}at^2$ as soon as we knew the value of the acceleration $a = 3$, and of course we should have arrived at the same result. But all this latter part is simple Kinematics: the only dynamical part was the finding of the acceleration from the given force and mass;

$$a = \frac{F}{m}.$$

Whether the body is in motion or not when the force begins to act, matters nothing—the acceleration produced is precisely the same. Of course the distance travelled in a given time will be different, because of the initial velocity

(v_0t will have to be added to the $\frac{1}{2}at^2$); but all that was considered in Kinematics, Chapter II.

45. The following is Newton's statement of the above connection between force and motion:

'*Change of motion is proportional to impressed force, and takes place in the direction in which the force acts;*'

Or as it has been restated by Professor Clerk Maxwell, in equivalent modern language:

'THE CHANGE OF MOMENTUM OF A BODY IS NUMERICALLY EQUAL TO THE IMPULSE WHICH PRODUCES IT, AND IS IN THE SAME DIRECTION.'

By *impulse* is meant the *product* of the *force* acting, and the *time* it lasts; for it is on both these that the power of a force depends. Thus the blow of a hammer is a very great force while it lasts; but as it is only momentary, its *impulse* (or motive effect) may not be so great as a much smaller force applied continuously for some time* (cf. sect. 35). The motive effect or impulse is proportional both to the strength of the force, F, and to its duration, t; and hence it is defined as the product Ft. So the first portion of the above statement is, in symbols,

$$mv = \mathrm{F}t;$$

where v represents the velocity gained by the mass m owing to the action of the force F for a time t; it is, in fact, simply the fundamental relation of sect. 42 in another form.

46. The above statement is often called the second law of motion: it might with propriety be called *the* law of motion, or the law of force and motion. It is very general, and involves a great deal.

First, it shews that where there is no force there is no change of momentum—that is, that a body not acted upon by any external force, if in motion, will continue with that motion unaltered, and, if at rest, will remain at rest; a fact often stated separately as the law of inertia, or the first law of motion (sect. 30).

* This is best observed by first striking sharply, and then pushing steadily, a thing on wheels where the friction is small. The advantage of a blow is felt, not when you want to move a thing, but when you have a great force of friction to overcome, as in hammering a nail.

It further declares implicitly that if a force act on a body in motion, it produces just the same effect as if it had acted on the same body at rest—that is to say, the *state* of the body on which the force acts is immaterial, as nothing is said about it in the statement.

In some old-fashioned books this part of the law is set forth as the whole second law itself.

Moreover, it implies that if two or more forces act on a body, each produces its own change of motion in its own direction without regard to the others.

47. This last is an important aspect of the law, and tells us that the operation of compounding together a lot of forces is just the same as that of compounding together the motions which each force separately tends to produce in the same time.

Thus if AB represents the quantity of motion (that is, the momentum) which would be produced by one force by itself in a second, and BC the motion which would be produced by another force by itself; then AB and BC may also be taken to represent the two forces themselves. But we learn from Chapter II., and from sect. 36, that the resultant of the two motions AB and BC is the single motion AC, hence AC may be taken as representing the *resultant* force—that is, a force which, if acting by itself, would produce precisely the same effect as the other two forces acting together.

Hence all that we have said about the composition of motions applies equally well to the composition of forces. In other words, forces are compounded by the parallelogram and polygon laws just as motions are compounded (see Chapter VII.).

48. Moreover, we learn that in order to specify the translating power of a force, it is only necessary to specify the velocity it is able to produce in unit mass in a second, which is readily done by drawing a straight line anywhere of definite length in a definite direction. But we shall soon learn (sect. 51) that, as force has *rotating* as well as *translating* power, it is necessary, for the complete specification of a

force, to assign also its position or line of action; it is not necessary to assign it any definite place in that line.

Hence three things determine a force—Direction (with sign), Position, and Magnitude. As these things are possessed by an arrow-headed line of given length, such a line is often used to symbolise a force. This ↓, for instance, would be one force, and this → a force of the same magnitude as the first, but in a different direction; while this other one, equal and parallel to the first, ↓ would be equivalent to the first in translating power, for it has the same magnitude and direction, but different in rotating power, having a different position, that is, line of action. The only defect of this mode of representation is that it is a little too expressive—that is, it expresses a little more than is wanted. For → and →, though two distinct *lines*, represent the same *force* in every respect, having the same direction, magnitude, and line of action—the rotating and translating powers are the same (see end of sect. 53). For further development of this, see Chapter VIII.

49. There is one more thing about force which is very important, but in the present stage its full meaning can scarcely be appreciated, and that is the fact, mentioned in the Introduction, that force is always due to the *mutual* action of two bodies or systems of bodies; that *every* force, in fact, is one of a pair of equal opposite ones—one component, that is, of a *stress*—either like the stress exerted by a piece of stretched elastic, which *pulls* the two things to which it is attached with equal force in opposite directions, and which is called a *tension;* or like the stress of a pair of compressed railway buffers, or of a piece of squeezed india-rubber, which exerts an equal *push* each way, and is called a *pressure* (see sect. 3). Newton's law concerning this is what is called his *third law of motion:*

'*Reaction is always equal and opposite to action—that is to say, the actions of two bodies upon each other are always equal and in opposite directions.*'

This may be called the law of stress, and it has been shewn by Professor Tait to be susceptible of considerable development (see Thomson and Tait's *Natural Philosophy*, art. 269, and see also Chapter VI. of the present text-book). It is deducible from the first law of motion (see Maxwell, *Matter and Motion*, art. lviii.), for if the forces exerted by two parts of the same body on each other were not equal and opposite, they would not be in equilibrium; and consequently two parts of the same body might, by their mutual action, cause it to move with increasing velocity for ever, the possibility of which the first law denies.

We have already shewn (sect. 46) that the first law is a special case of the second, and now we have deduced the third from the first; hence all are really included in the second, which is therefore excessively important.

(II.) ON ANGULAR VELOCITY AS AFFECTED BY FORCE; OR, FORCE AND ROTATION.

(Dynamics of a Rigid Body.)

50. When force acts on an extended piece of matter, it produces in general both motion and strain (sect. 5). The latter we do not want to consider at present; so to exclude it, we suppose the body to be *rigid*—all its parts rigidly bound together and incapable of distortion or relative displacement. The effects of force on such a body are translation and rotation. If the former only, the body acts like a particle (sect. 36), as if all its mass were concentrated at a point (called its *centre of inertia*, or sometimes its centre of gravity), and the second law of motion as stated for particles applies to the rigid body; so that if R is the resultant of all the external forces acting on the body, and if m is its mass, the acceleration of its centre of inertia is $\frac{R}{m}$. When, however, rotation is allowed, the subject becomes much more complicated, especially if translation is possible as well. We can, however, consider rotation by itself, by supposing one line or point in the body to be fixed in position, so as to constitute an axis or centre of rotation.

51. All we can say about the subject here is, that in

estimating the rotating effect of a power, one must not only consider its impulse—that is, its magnitude multiplied by its duration (sect. 45)—but we must also consider its position; how far its line of action is from the fixed line or axis of rotation: the further it is off, the more effect it has; its *moment* (sect. 37) is greater.

Suppose a force acts on a body only capable of rotation, at a distance R from its fixed axis: the *moment of momentum*, or *angular momentum*, generated [$\Sigma(mvr)$, or $\Sigma(mr^2\omega)$, see sect. 39], equals the product of the moment of the force, FR, into the duration, t; in other words, it equals the *moment of the impulse* $Ft \cdot R$.

This is expressed by the following equation, where the *moment of inertia* of the body $\Sigma(mr^2)$ is denoted by M (see sect. 40):

$$M\omega = FRt,$$

or, *moment of momentum* = *moment of impulse*,

which is an extension of the simpler particle equation, (sect. 45), *momentum* equals *impulse*,

$$mv = Ft.$$

52. This equation may also be written (since $\omega = \alpha t$),

$$\text{angular acceleration} = \alpha = \frac{FR}{M} = \frac{\text{moment of force}}{\text{moment of inertia}},$$

which is evidently analogous to the simple, and, for particles, fundamental equation (sect. 44),

$$\text{acceleration} = a = \frac{F}{m} = \frac{\text{force}}{\text{inertia}},$$

and includes it as a special case.

For an application of this equation, see Chapter IX., sect. 142.

Read again, carefully, sections 37–40.

Moment of a Force.

53. The idea of the moment of a force is a very important one, and will occur again and again in statics (Chap. VIII.). It was from this particular case of it that the name *moment* arose, signifying that on which the power of a force in producing rotation depends. Thus, to close a door rotating

on its hinge, by a push, it is much more effectual to apply the push near the handle than near the hinge. In pulling at a lever, the further you are from the fulcrum the more power you have. Doubling the distance of the force from the fulcrum, is as good as doubling the force itself—doubling either, doubles the effect—doubling both, quadruples it: hence, distance and force enter equally into the effect—that is, the *moment* of the force is *proportional* to the product of force and distance FR, and may be defined as *equal* to it.

The distance called R here is always the *shortest* distance from the point or axis of rotation to the line of action of the force—that is, it is the length of the perpendicular drawn between these two lines, or let fall from the fixed point upon the line of action. Now, the area of a triangle is half the base multiplied by the perpendicular height; hence, if the force be taken as the base of a triangle, and the point of rotation as the vertex, the area of the triangle so formed will be half the moment of the force about that point. Or, in symbols: the moment of the force AB about the point O is AB×ON, where ON is the distance R, being the perpendicular let fall from O upon the line AB, produced if necessary (fig. 11); but AB×ON also equals twice the area of the triangle AOB; hence, twice the area of AOB represents geometrically the moment spoken of. The position of AB *in the line* is evidently of no consequence, as all triangles of equal heights and bases have the same area (cf. sect. 48).

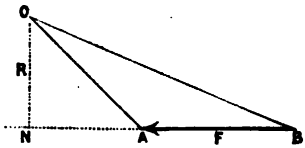


Fig. 11.

But to express a moment completely, we must also notice the *direction* of its rotative tendency. In the figure it happens to be like the hands of a watch, a direction it is convenient to call, with Professor Clifford, 'clockwise.' If AB were reversed, or if O had been on the other side of it, the direction of rotation would be also reversed, or 'counter-clockwise.' This last direction—namely, that opposite to the hands of a clock, it is customary to call positive—the clockwise rotation being therefore negative. So in the above figure the moment is equal to − 2.OAB.

Force Perpendicular to Path of Motion.

54. But there is another aspect of the subject. When a body (say a wheel) rotates round an axis, every point of it is describing a circle; and so, even when its motion is uniform, and not accelerated in the ordinary sense, still a force must act on each of its particles to compel them to move in the circle contrary to the first law of motion. This force is supplied by the strength of its material, and is often neglected; it is, however, very important. It may happen that the material of a wheel is not strong enough to exert the force required when the rotation is very rapid, and in that case the particles will cease to move in their circles, but will begin to move in straight lines: in other words, the wheel will fly to pieces. If a body revolve about a centre outside itself, this force must be supplied to it by a link or cord, or by some other constraining mechanism (a groove in the case of a solitaire marble running round its board).

In this, as in every case, the acceleration is proportional to the force, and a constant force produces a uniform acceleration (sect. 42); but the acceleration is here perpendicular to the direction of motion (see sect. 14). We will now proceed to investigate it further.

(III.) ON THE DIRECTION OF MOTION AS AFFECTED BY FORCE; OR, FORCE AND CURVILINEAR MOTION.

(*Dynamics of a particle continued.*) (*Centrifugal force.*)

55. The velocity of a particle of matter may be changed both in magnitude and direction by the action of force. Hitherto we have dealt only with change of magnitude; let us now proceed to change of direction; and consider a case where a force produces *only* curvature in the path of a particle without otherwise affecting the velocity.

Imagine a particle of matter moving round and round a circle with constant speed. Although there is no acceleration in the direction of its motion, yet nevertheless, a force must necessarily act continually in order that the *circular* motion may continue. The velocity is uniform indeed, but its direction is constantly changing. But, by the first law of motion, a particle of matter will move always in the same direction—that is, in a straight line—unless it is acted on by force: hence, force is necessary to change the direction.

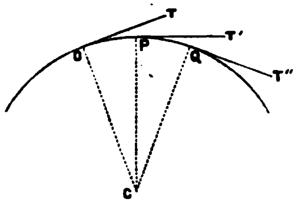


Fig. 12.

If the particle were at the point O (fig. 12), and the force were to cease to act, it would continue to move in the straight line OT, touching the circle at O. In order to go round the circle, it must then fall from this line normally, that is toward the centre of the circle; it thus arrives at the point P, and now it is going along PT′; but it falls a little towards the centre again and so reaches the point Q, and so on. A force then must constantly act drawing the particle towards the centre of the circle; and this force is called therefore the *centripètal* force. It is constant in magnitude, but continually changing in direction, being always at right angles to the direction of motion of the particle. And because it *is* at right angles to this direction, it can produce no acceleration in it. Whirl a stone round by a string: the tension in the string is this centripetal force, and you will find it greater as the stone is larger, and also as you whirl it quicker. The tension in the string, however, is really a stress (sects. 3 and 49), and has two aspects, one the *action* of the hand or central body on the revolving particle, which is the centripetal force proper; the other the *reaction* of the revolving particle on the central body, which is the force felt by the hand, and goes by the name of the *centrifŭgal* force. Of course the two are equal. The essential thing however is the stress, and which component we speak of matters little: but, as we are at present concerned more with the action on the particle than with the reaction on the centre, it will be convenient to attend more to the centripetal force than to the other.

Value of Centripetal Force and Centripetal Acceleration.

56. Now, to find the magnitude of this force, we must regard the motion of the particle as compounded of two—one a uniform velocity along the tangent to the circle; the other a uniform acceleration along the radius, produced by the uniform centripetal force F, according to the law

$$a = \frac{F}{m},$$

a being the centripetal acceleration.

We have then a case of composition of motions very like that discussed in sect. 28, where a uniform rectilinear motion was compounded with a uniform acceleration in a constant direction—that is, always parallel to itself; and where the path of the resultant motion was found to be a parabola. But, in the present case, we have to compound a uniform motion with a uniform acceleration at right angles to the path of motion at each instant, in fact along the radius of the circle, and by no means parallel to itself.

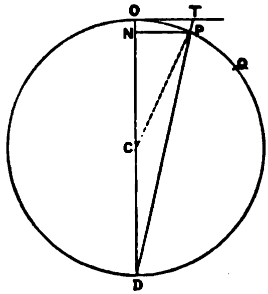


Fig. 13.

Drawing a figure similar in principle to that of sect. 28 (fig. 9, which see), let OP be the very minute portion of the circular path described in an infinitesimal portion of time t with the constant velocity v, so that

$$OP = vt;$$

and complete the figure as shewn in fig. 13, letting fall PN perpendicularly to the diameter of the circle OD.

Then OP is the diagonal of an infinitely small parallelogram* with sides OT and ON; wherefore the motion along

* The quadrilateral ONPT is not really a parallelogram, but it is more nearly one the smaller it is—that is, the nearer P is taken to O; and it is accurately one in the limit when it is infinitely small—that is, when P is the next consecutive point to O, which is supposed to be the case; for of course OPQ, &c., are really *consecutive* points of the circle, only they have to be spread out in the diagram. In the limit also OP and OT are equal, and hence OT is also equal to vt.

OP may be regarded as compounded of two motions—one with the constant velocity v along OT, which the particle would have if left to itself; the other, due to a constant pull of the centre C, and therefore uniformly accelerated, along ON, which is the distance travelled in that direction in the above small time t; wherefore

$$\text{ON} = \tfrac{1}{2}at^2.$$

It only remains to determine, from the geometry of the figure, the relation between ON and OP, in order to find the value of the centripetal acceleration for a point moving with given velocity in a circle of given size.

The angle OPD, being an angle in a semicircle, is a right angle (Euc. III. 31), and so is the angle at N; moreover the angle at O is common to the two triangles ONP and OPD; wherefore these triangles are similar (that is, one is like the other magnified), and their corresponding sides are therefore proportional; so

$$\text{ON} : \text{OP} : : \text{OP} : \text{OD};$$

or in symbols, if r is the radius, CP, of the circle,

$$\tfrac{1}{2}at^2 : vt : : vt : 2r;$$
$$\text{or, } a : v = v : r.$$

whence $v^2 = ra$, or v is a 'mean proportional' or 'geometric mean' between a and r.

The value of the centripetal acceleration is then

$$a = \frac{v^2}{r}; \text{ or (writing } v = \omega r) \ a = \omega^2 r = v\omega.$$

The centripetal *force* is of course simply m times this, m being the mass of the revolving particle of matter, or

$$\text{F} = \frac{mv^2}{r} = m\omega^2 r = mv\omega;$$

or it is proportional to the mass of the particle, the square of its velocity, and the curvature (see sect. 13) of its path: in other words, it is proportional not only to the momentum of the particle, but also to the rate at which its direction of motion revolves—that is, to its angular velocity.

Read again sect. 14 carefully, and also the examples on circular motion at the end of Chapter I., especially example 12.

57. As an example take a stone weighing 5 lbs., attach it to a string 3 feet long, and then whirl it round twice a second.

The length of one circumference being $2 \times \pi \times 3 = 6\pi$ feet, its velocity must be 12π feet per second; and the tension in the string, or the centripetal stress, must be

$$\frac{5 \times 144 \times \pi^2}{3} = \text{about } 2400 \text{ poundals}$$

(taking π^2 as equal to 10 instead of 9·98); equivalent to the weight of $\frac{2400}{32} = 75$ lbs.

This stress might easily be sufficient to break the string, and one would say then that the centrifugal force, exerted by the revolving mass on the string, broke it. This may be understood as an abbreviation for the following more expressive statement: The force required to continually deflect the mass from its natural rectilinear path, and cause it to move in the given circle at the given rate, is so great that the string was incompetent to exert it, but was torn asunder in the effort.

Take another example from astronomy, which, however, will be better appreciated after reading Chap. V. The moon revolves round the earth, in a path which is nearly a circle with the earth as centre, in a time of nearly 28 days. Hence it too is continually being deflected from its natural rectilinear path: the force which deflects it being its *weight*—that is, the earth's pull (or gravitative attraction). Call the mass of the moon m; then its weight must be mg' (see sect. 60), where g' is the intensity of terrestrial gravity at the distance of the moon.

The intensity of gravity at the moon's distance is much less than 32, its value near the surface of the earth, because it decreases in the same proportion as the square of the distance from the centre of the earth increases.

This force, mg', then, is the centripetal force which makes the moon describe its curved path, and hence it should equal

$$\frac{mv^2}{r}.$$

Now the radius (r) of the moon's orbit is about 240,000 miles, or about sixty times the earth's radius; and it goes once round in 2,360,000 seconds, or about 27 days 8 hours: hence its velocity (v) is

$$\frac{2\pi \times 240{,}000 \times 1760 \times 3}{2{,}360{,}000} = 3374 \text{ feet per second.}$$

So $\frac{v^2}{r}$, the centripetal acceleration, is $\frac{3374 \times 2\pi}{2{,}360{,}000} = \cdot 00898$.

This is the value of g', and the centripetal *force* is

$$\cdot 00898 m.$$

Now if this force be really due to gravity, and if gravity really diminishes with the square of the distance, then, the distance of the centre of the earth from the moon being sixty times as great as its distance from the surface of the earth (that is, the earth's radius), it would follow that g' at the distance of the moon should be the 3600th part of the value of g at the surface of the earth.

But the value of g is 32·2 (see next chapter), and the 3600th part of this is ·00894; so the weight of the moon should be $\cdot 00894m$; and this is as near $\cdot 00898m$ as our rough data can be expected to give it.

This is the sort of calculation which Newton went through when he proved that the force required to keep the moon in her orbit was just the same as would be exerted by the gravitative pull of the earth; supposing that the force which pulls down stones and apples extended so far, and decreased regularly all the way with the square of the distance from the centre of the earth; and hence concluded that this force *does* so extend, and is the actual force in operation.

EXAMPLES.

1. What is the acceleration when a force of 36 units acts on a mass 4; and how far will the mass move in 10 seconds? *Ans.* 9, and 450.
2. What is the least force necessary to cause 15 lbs. to move 30 feet from rest in 5 seconds? *Ans.* 36.

3. If a mass of 7 lbs. is acted on by two opposite forces of magnitudes 56 and 42 respectively, what is the acceleration; and what will be the momentum generated in 5 seconds? *Ans.* 2, and 70.
4. How long must a force of 8 units act on a mass 20 to change its velocity from 2 to 26 feet per second? *Ans.* 1 minute.
5. In what distance will a force of 2 poundals be able to stop a mass of 30 lbs., which at the time the force begins to act is moving 50 feet every second? *Ans.* 6250 yards.
6. Half a pound is whirled at the end of a string 18 inches long 3 times round per second: what is the tension in the string? *Ans.* $27\pi^2$ poundals.
7. If a string can stand a force of 1000 units without breaking, what is the greatest length of it which can be used to whirl a 5-lb. mass once round a second? *Ans.* (taking π^2 as equal to 10) 5 feet.
8. What is the smallest length of the same string which can be used to whirl a 5-lb. mass with a velocity of 10 feet a second? *Ans.* 6 inches.

CHAPTER V.

ON FORCE AND MOTION—*Continued.*

THE FORCE OF GRAVITATION.

58. Before proceeding farther, it will help our ideas to apply some of the general laws to a few special cases. The most universal force known is the force of gravitation, and it will be convenient to take illustrations from the action of this force; but we will, in the present stage, only consider it as a ***uniform*** action exerted by the earth, tending to pull every piece of matter down to the earth's surface with a force varying with the mass of the piece of matter, but with

nothing else. This is practically true in all common cases, for though the force really varies inversely with the square of the distance from the centre of the earth, yet the variation for ordinary heights is very small. For there is scarcely any difference in the distance of the centre of the earth from the sea-level and from the top of a mountain—one is say 4000, and the other perhaps 4001 miles.

59. This force is what is known as *weight;* it is measured like every other force by the acceleration it can produce in unit mass, or, in other words, by the momentum it can generate in a second. To measure the force, and see how it depends on the nature of the attracted body, we will first take the *same* mass of *different* bodies, and compare the accelerations which gravity is able to produce in them. Thus take a pound (see sects. 31 and 33) of lead, of iron, of stone, of wood, and of cork, and drop them all at the same instant from a high tower; then if every disturbing cause were absent—that is, if they were subject to no other force but that of gravitation—they would all be found to reach the ground at precisely the same instant, having all acquired the same velocities.

If, however, the experiment took place in air, they would be subject to disturbing causes, and nothing would be learned from it. The wood and cork would be retarded by the air more than the others, partly from the same cause as enables us to winnow chaff from grain, and partly for a reason which may be rendered more obvious by dropping the different things under water. The falling of the wood and cork would be then not only retarded but reversed into a rise. The air has a floating power, only it is less than that of water. The air must therefore be removed, and the bodies dropped *in vacuo*, an experiment often called the guinea and feather experiment, for a description of which you may refer to Ganot, Book II., chap. ii., page 51, or Deschanel, vol. i., page 49.

The above experiment, if carried out accurately, would prove that *the pull of gravity has nothing to do with the material or nature of the substances*, for all equal masses are equally accelerated whatever the material; and, since the masses *are* equal, this means that they are all pulled with equal force (sect. 42).

60. Next take unequal masses (either of the same material or not), say a swan shot and a cannon ball, and drop them from a height at the same instant. They will both reach the ground at the same time—that is, they each receive the same acceleration. This experiment was carried out by Galileo from the Tower of Pisa. It shews that *the earth's pull on a body is directly proportional to its mass.* For since force is equal to the product of mass and acceleration (sect. 42), and since the acceleration is found experimentally to be the same for all masses, it follows that the force is necessarily proportional to the mass.

If we denote by g the acceleration produced by gravity —that is, the velocity gained by a freely falling body in one second—the force pulling it down is g multiplied by its mass, and this force is termed its *weight;* so

$$W = mg;$$

or, the weight of a body is g times its mass. Hence g is often called the *intensity* of gravity.

This is simply a special case of the general relation

$$F = ma,$$

weight being a particular case of force, and g being a particular case of acceleration.

The weight of one pound is therefore g units of force (g poundals). (Read sect. 33 again.)

Falling Bodies.

61. To express all the laws of falling bodies, we have simply, first of all to find the value of the uniform acceleration g, and then to apply all the kinematics we know.

Thus a stone let drop is found to fall about 16 feet in one second (more accurately 16·09), so that 16 is its average velocity during that second; but the average velocity is half the final velocity; hence the velocity acquired in one second, or the acceleration, is 32 (more accurately 32·18), and this is the value of g. (Since 16·09 feet equal 490·5 centimetres nearly, the value of g is 981 in centimetres-per-second per second; and this therefore is the weight of a gramme in dynes.)

The velocity 32 *is gained in every second of the fall, so the velocity gained in t seconds is* $32t$ (*feet per second*).

The distance travelled, being proportional to the square of the time, is $16t^2$ *feet* ($h = \frac{1}{2}gt^2$).

The velocity gained while falling from a height of h feet from rest is (*by equation* $v^2 = 2gs$)

$$8\sqrt{h};$$

8 *being the square root of* 64 *or* $2g$.

The time taken to fall from rest at a height of h feet is

$$\tfrac{1}{4}\sqrt{h},$$

which follows at once from the equation $s = \frac{1}{2}gt^2$.

62. **Modes of diluting the Intensity of Gravity.**—The acceleration is equal to g for all bodies only on condition that they fall *freely*—that is, that the weight of each has only its own mass to move and nothing else; for then $a = \frac{F}{m}$, but as $F = mg$, $a = g$.

If, however, by any arrangement, we make a weight move another mass as well as its own, the acceleration must be less.

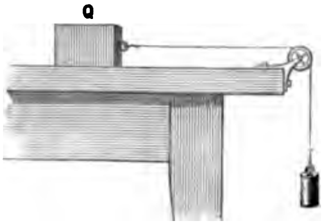


Fig. 14.

Thus suppose we tie a falling weight P (say 6 lbs.) to a mass Q of 18 lbs. resting on a *smooth* flat table, as in fig. 14; then the force causing the motion is the weight of the 6 lbs.—that is, $6g$—but the total mass moved is $18 + 6 = 24$ lbs.; hence the acceleration is

$$\frac{6g}{24} = \tfrac{1}{4}g = 8$$

feet-per second per second. Hence in the first second the combination would move 4 feet, and in t seconds $4t^2$ feet, while the velocity acquired in t seconds would be $8t$.

In a similar way, we can find the acceleration if two weights are connected by a string passed over a frictionless pulley, without inertia, as in fig. 15.

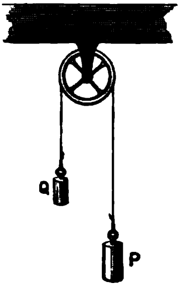


Fig. 15.

Let the masses Q and P be 7 and 9 lbs., their weights will be $7g$ and $9g$ units of force respectively, and the effective force will be the difference, that is $2g$. The mass moved is 16; hence the acceleration is

$$\frac{2g}{16} = \frac{1}{8}g = 4.$$

This arrangement of two unequal weights over a pulley is called 'Atwood's machine,' for determining the acceleration produced by gravity, and for experimenting on the laws of uniform acceleration.

The advantage gained by experimenting with it instead of with freely falling bodies is owing to the fact that the latter fall too quickly to be conveniently observed. Any acceleration whatever less than g can be obtained by the use of this simple machine. Gravity is as it were *diluted* (that is, its intensity is multiplied by a proper fraction), but the laws of falling remain the same.

63. **Mode of measuring the Intensity of Gravity.**—To use the machine for measuring g, we put on the string two nearly equal weights, masses P and Q; the effective force is then the difference of their weights $Pg - Qg$; the mass moved is $P + Q$; hence the acceleration is

$$\frac{Pg - Qg}{P + Q}, \text{ or } \frac{P - Q}{P + Q}g.$$

If this acceleration (call it a) is observed, g is easily calculated. To obtain a, you may observe the distance s fallen in t seconds, and then apply the formula $s = \frac{1}{2}at^2$.

For instance, let P be 21 oz. and Q be 23 oz., then the acceleration is

$$\frac{2}{44}g, \text{ or } \frac{g}{22}.$$

Let them move for six seconds, and observe that the heavier weight has fallen (and the lighter weight risen) a distance of 26 feet; then say $26 = \frac{1}{2}a \times 36$, so the acceleration is $\frac{13}{9}$; but it is also $\frac{g}{22}$, hence $g = 31\frac{7}{9}$.

An actual experiment with Atwood's machine would be hardly likely to give g so nearly correct as this. There are other methods of finding the value of g, which are much better practically, though not so theoretically simple. The most accurate method consists in observing the time of oscillation of a pendulum of measured length (see sect. 73).

64. It is easy to understand how experiments may be made with Atwood's machine on the laws of uniform acceleration. Thus, to take the case when the weights are 21 and 23, and the acceleration therefore $\frac{13}{9}$, we should find that the distances travelled in 1, 2, 3, 4, 5, 6 seconds respectively were

$$\frac{13}{18}, \quad \frac{4 \times 13}{18}, \quad \frac{9 \times 13}{18}, \quad \frac{16 \times 13}{18}, \quad \frac{25 \times 13}{18}, \quad \frac{36 \times 13}{18}$$

—that is, always half the acceleration multiplied by the square of the time ($\frac{1}{2}\,at^2$).

The distances travelled *during* each second would follow another law. They are easily obtained from the preceding numbers, for if we subtract the distance travelled in three seconds from the distance travelled in four, we should obtain the distance travelled *during the fourth second*, namely,

$$\frac{16 \times 13}{18} - \frac{9 \times 13}{18} = \frac{7 \times 13}{18};$$

and similarly, we get for the distance travelled in the first, second, third, fourth, and fifth seconds respectively,

$$\frac{13}{18}, \quad \frac{3 \times 13}{18}, \quad \frac{5 \times 13}{18}, \quad \frac{7 \times 13}{18}, \quad \frac{9 \times 13}{18};$$

a series ascending by the odd numbers; the distance travelled in the nth second being half the acceleration multiplied by the nth odd number $\left(\frac{1}{2}\,a(2n-1)\right)$.

65. All this may be readily remembered by observing its analogy with a simple geometrical diagram, as in sect. 22.

Draw any right-angled triangle, OPC (upside down does best for falling bodies); divide its base, OP, into any number of equal parts, and draw a vertical line at each division. You will thus cut up your triangle into trapeziums, of which the left-hand one degenerates into a triangle; and it is plain that, whatever be the area of this small triangle, the trapezium next to it has three times that area, the next five times, the next seven times, and so on, as may be seen from the four dotted lines drawn in fig. 16. Hence, if the first area represent the space travelled by a uniformly accelerated body in the first second, the second area will represent that described in the second second; and the sum of the two figures will be the space described in the two seconds together, and so on.

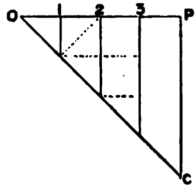


Fig. 16.

Moreover, the whole area of the triangle will represent the space travelled in the whole time of a number of seconds equal to the number of segments of the base. Thus, in the above figure, the whole area is the space described in four seconds.

The vertical height of the figure being nothing at its left-hand point, corresponds with the fact that the falling body starts from rest—that is, is *dropped*. But if the body is *thrown* either down or up with an initial velocity, this velocity must be represented by a line drawn at the left-hand point, either down or up, and the figure becomes as in fig. 17 or as in fig. 18, where OA represents a velocity downwards, and OA′ a velocity upwards.

In the first case the velocity continually increases, until in four seconds it becomes equal to PC. In the second case it at first decreases, becoming zero at the point E two seconds after starting, and then increases downwards until it becomes P′C′.

This second case exactly corresponds with what a ball thrown up in a vacuum against gravity does. In both cases the whole area of the figure represents the whole space travelled. In the second case we see that the area OA′E is the space or height the ball rose through, and EP′C′ the height it afterwards fell through. The ball was at its

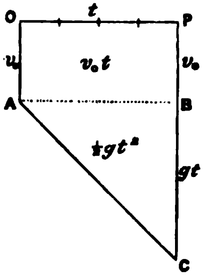


Fig. 17.

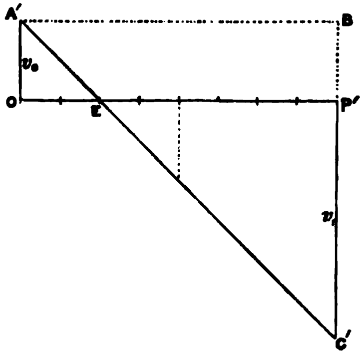


Fig. 18.

highest point two seconds after throwing up, having then no velocity.

In both cases the ball must have been thrown from the top of a tower or some other height, or it could not fall for so much as four seconds without striking the ground. The area OACP may represent the height of this tower, and OP the time taken to fall, in the first case—that is, when the ball was thrown downwards; but in the second case, when the ball was started upwards, the height of the tower is the difference of the areas EP′C′ and OA′E. OP′ is the whole time taken by the ball, first to rise a height above the tower equal to the area OA′E, and then to fall from this height to the ground. The lines AC and A′C′ are necessarily parallel, since the slope of each represents the rate of numerical gain of velocity, 32 feet-per-second per second. (It might, however, be anything less than this if Atwood's or some other 'diluting' machine were used.)

Supposing it is 32, and that OE is two seconds, and EP′ six; then, of course, the initial velocity OA′ must be 64, and P′C′ must be 192, feet per second. The area OA′E will be 64 units (its height being 64, and its base 2), and therefore the height the ball rises is 64 feet. The area EP′C′ is 576, and so the height of the tower is 512 feet. In the diagrams, the time represented by OP′ in the second diagram is greater than the time, OP, in the first, by twice OE; and the initial velocity OA is numerically equal to OA′; hence also the final velocity PC is equal to P′C′, and the area OACP represents 512 linear feet.

In each of these figures (neglecting dashes), OP is the line of time; OABP represents the space described due to the initial velocity; and ABC the space described due to gravity. Also BC represents the gain of velocity at; and PC the actual final velocity.

Refer to sect. 22 for some more statements concerning these diagrams, and practise drawing diagrams for all kinds of cases of rectilinear motion. Thus, draw diagrams for the motion of a railway train, which gets up speed, goes uniformly, slackens, stops, and goes on again, several times, and then comes back; for the motion of an india-rubber ball thrown down to the ground and then bouncing; for the motion of the bob of a very long pendulum; for the motion of a tilt hammer, &c.; and remember in drawing these diagrams that *time* never retrogrades, and hence that no part of a diagram can be vertically under or over another part, but the drawing must progress continually forwards.

66. To actually experiment on the velocity acquired by the falling weights in Atwood's machine, we must remember the definition of variable velocity at any instant (given in Chapter I., end of sect. 9), namely, the distance the body would go in the next second if at that instant the acceleration ceased. Now, the cause of the acceleration in this machine is the force $(P - Q)g$. If this force were suddenly removed—that is, if P and Q were suddenly made equal, there would be no farther acceleration, and the masses would continue to move uniformly forward with the velocities they had already acquired, until they were checked either by striking something or by friction.

This sudden removal of the inequality in the two weights is practically accomplished by making the extra weight by which P exceeds Q (2 ounces in the experiment of sect. 63), a loose metal bar too big to pass through a certain fixed ring placed in the path of P. When P passes through this ring the bar is removed; P and Q become equal, and move a distance in the next second which is numerically equal to the velocity they had acquired at the instant the bar was taken off. For a fuller description of Atwood's machine, and for many details of its actual construction, you may refer to Deschanel, vol. i., p. 57, or Ganot, sect. 69.

67. **Further illustration of the Fundamental Equation.**—This example of the two weights, one pulling up the other, illustrates the statement in sect. 50 that the second law of motion applies to other cases than those where the motion is perfectly free and unresisted; in fact, that it is quite general, if we always consider the force F as the *resultant* of all the forces acting on a body, and not simply that force which happens to be most obviously apparent to us.

Thus, go back to the mass of 18 lbs. resting on a table, and pulled along by a weight of 6 lbs. hanging over the edge by a string (fig. 14, sect. 62). The acceleration we saw ought to be 8, but suppose it was observed to be only 3, we should at once know that all the forces had not been taken into account. The table, perhaps, is rough, and retards the motion of the 18 lbs. with a force sufficient to reduce its acceleration to 3, and the force of friction may from these data be calculated.

So again when a 56-lb. bucket is dragged up a well with a force of 1920 units (the weight of 60 lbs.); if this were the only force acting, the acceleration of the bucket upward would be

$$\frac{1920}{56} = 34\tfrac{2}{7}$$

units—that is, it would gain this velocity per second; but if the experiment be tried, the velocity actually gained per second will be found to be nothing like so much as this—it will be only about $2\tfrac{2}{7}$ units. The reason obviously is that there is another force left out of account, opposing the pull of the rope, namely the pull of the earth, which is $56 \times 32 = 1792$ units; and the resultant force is the difference of these two, or 128. Hence the actual acceleration is

$$\frac{128}{56} \text{ or } 2\tfrac{2}{7}.$$

68. To take another very similar example, a cage of m lbs. is lowered by a rope down a coal-pit; what is the tension in the rope, at a time when the cage is gaining downward velocity at the rate of a feet a second every second? Well, the resultant force must equal the mass-acceleration, but this resultant force is the difference

between the weight of the cage, mg, and the pull of the rope, T, hence

$$mg - T = ma,$$

or

$$T = m(g - a),$$

which is the tension required.

If the tension in the rope were *equal* to mg, the weight of the cage, the cage would necessarily have a constant velocity; it might be moving either up or down, but there could be no acceleration (cf. sect. 122).

69. That aspect of the second law of motion which says that it makes no difference to the effect of a force on a body whether that body was in motion or not (sect. 46) is well illustrated by falling bodies (see sect. 21).

70. But the law is more strikingly illustrated when the direction of the initial velocity of a falling body (now called a *projectile*) is inclined at some angle to the force of gravity. The path of a projectile is shewn in fig. 9, sect. 28; the simplest case being where the initial velocity is at right angles to the force of gravity, or horizontal.

Thus a rifle bullet, starting with an initial horizontal velocity u, retains this velocity unaltered, if we neglect friction against the air, and therefore in t seconds it travels a horizontal distance ut; but its vertical velocity, which at first was zero, continually increases, and in t seconds is gt; the vertical space fallen through being $\frac{1}{2}gt^2$, or just the same as if the gravity had acted upon the body at rest. The whole circumstances of the motion of such a projectile have therefore been already worked out in sect. 28; which see, and read again.

If the rifle was fired horizontally from the top of a cliff of given height, say 144 feet, it is easy to find how far the bullet will go before striking level ground, its initial velocity being known. Let the initial horizontal velocity be 1200 feet per second. We must first find t, the time the bullet takes to fall from the top of the cliff to the ground, from the equation $144 = 16t^2$ (for it takes just the same time as if it had no horizontal velocity. Law II., section 46); this gives $t^2 = 9$, or $t = 3$. It goes therefore three seconds before striking the ground, so evidently the horizontal distance it travels is $3 \times 1200 = 3600$ feet.

And generally if h be the height of the cliff, and u the initial horizontal velocity of the bullet, its *range*, or horizontal distance, is $\frac{1}{4}u\sqrt{h}$.

71. The composition of these two motions, a uniform horizontal velocity with a uniform vertical acceleration, is well illustrated by Morin's machine, for a description of which see Deschanel, page 55, or Ganot, sect. 69 *a*.

It consists of a long drum or cylinder, capable of rotating by clock-work about a vertical axis. Down one side a weight can fall between guides, and can, by means of a pencil, mark a line on the drum as it falls. If the drum is stationary, the line drawn is of course straight and vertical; but if the drum rotates, it is spread out into a curve. This curve, when unwrapped from the drum, is precisely the same as that which is described by a projectile shot out horizontally *in vacuo* with a velocity equal to that imparted to the surface of the drum by its clock-work.

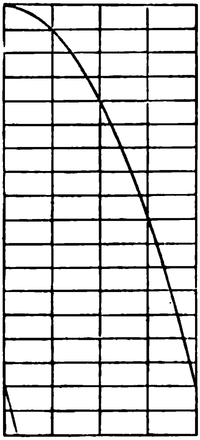


Fig. 19.

The drum is usually covered with paper, ruled into squares or oblongs, which can be detached and unrolled. The line traced on it may then present the appearance shewn in fig. 19. In successive seconds the horizontal distances are as 1, 2, 3, 4, 5,...... the vertical as 1, 4, 9, 16, 25, and so on. A curve with this property is called a parabola. It is the path of a projectile in a vacuum (compare sect. 28).

Curvilinear Motion and Rotation.

72. We have already illustrated, though in an excessively superficial manner, one case of the *curvilinear motion of a particle* (sect. 55) produced by the force of gravity, namely that of the moon, supposing it to be a particle and to move in a circle (see sect. 57, and read it again). The whole subject of the motion of the planets in their orbits comes properly in here, but of course we cannot attempt it at the present stage.

73. We might also discuss the subject of the *rotation of a rigid body* (sect. 50), as illustrated by gravity, by fixing a point of a rigid body, and then letting gravity act on it. We should thus get a very important set of physical laws known as those of the *pendulum*, but in this stage they are a little too elaborate for us to attempt to work out. Suffice it to say that a 'pendulum' is simply a rigid body, with either a point or a line in it fixed somehow relatively to the earth, and then the body displaced from its position of equilibrium, and left to swing under the action of gravity. The motion is periodic, and the rate of oscillation depends only on the length of the pendulum (whatever that may mean) and the intensity of gravity. The time of a complete swing to and fro is obtained by multiplying twice the ratio of the circumference of a circle to its diameter by the square root of the ratio of the length of the pendulum to the intensity of gravity—that is, in symbols,

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

Assuming this (which will be practically proved in sect. 139), one sees that, by measuring t and l, the value of g can be ascertained; and this is the most accurate means of determining g.

The practical use of a pendulum as a timekeeper depends on the time of an oscillation being almost invariable—that is, on its motion being *on the average* very uniform; and hence it is very largely used as a timekeeper, all the rest of the clock being, firstly, an apparatus to keep the pendulum going, notwithstanding friction, and, secondly, an apparatus to record (like a gas meter) how many times the pendulum has oscillated. For more about pendulums, see Chapter IX.; and also Deschanel, chapter viii.

EXAMPLES.

1. What is the weight of 20 lbs. at a place where a falling body travels 4 feet in the first second?

Ans. 160 poundals.

2. At what height above the earth's surface could such a place be found?

Ans. At a height equal to the earth's radius.

3. A curling weight is thrown on ice with a velocity 50; supposing the force of friction to be $\frac{1}{10}$th of the weight, how soon will it stop? *Ans.* In $15\frac{5}{8}$ secs.

4. In an Atwood's machine a 40-gramme weight on one side is drawn up by a 50-gramme weight on the other, 2·18 metres in two seconds; what is the value of g in centimetres-per-second per second? *Ans.* 981.

5. In the preceding question find the tension in the rope in grammes weight, and in dynes.

Ans. $44\frac{4}{9}$ grammes weight, or 43,600 dynes.

6. When a 3-lb. weight hanging over the edge of a smooth table drags a 45-lb. mass along it, find the acceleration and the tension in the string. *Ans.* 2, and 90.

7. Find also the acceleration if the coefficient of friction between the table and the weight is ·05. *Ans.* $\frac{1}{2}$.

8. A cannon ball is fired horizontally from a hill 900 feet high on the coast. Find the time which elapses before it strikes the sea, neglecting the resistance of the atmosphere. *Ans.* $7\frac{1}{2}$ seconds.

9. If the velocity of projection in the preceding question were 1320, find the horizontal distance travelled.

Ans. 1·875 miles.

10. A string, 2 feet long, able to sustain a weight of 104 lbs. without breaking, is attached to a stone weighing 4 lbs. and whirled in a vertical plane round a fixed centre 6 feet above the ground till it breaks. What happens to the stone?

Ans. The tension in the string being greatest at its lowest point (because of gravity), the string is most likely to break there. The centrifugal force being equivalent to the weight of 100 lbs., the velocity of the stone when the string breaks must be 40 feet per second. It will start forward horizontally with this velocity and describe a portion of a parabola—striking the ground after the lapse of half a second 20 feet away.

11. What is the value of g at a place where a simple

pendulum 2·4 inches long makes two complete oscillations a second?

12. A 'seconds pendulum' is one that makes ½ an oscillation per second. Find the length of the seconds pendulum in inches and in centimetres. *Ans.* $\frac{g}{\pi^2}$

13. An engine winds a three-ton cage up a coal-pit shaft at a uniform pace of 11 yards a second; what is the tension in the rope? *Ans.* The weight of 3 tons.

14. Instead of a uniform *velocity*, the above cage is wound up with a uniform *acceleration* 6; what is the tension in the rope? *Ans.* The weight of $3\frac{9}{16}$ tons.

CHAPTER VI.

WORK AND ENERGY.

74. The present chapter is to indicate a method of treating the effects of force on matter in a perfectly general manner; all consideration of how the force acts, or what it acts on, being regarded as accidental and of no consequence. Whether the body acted on is a particle, or a rigid solid, or an elastic solid, or a liquid, or a gas, matters nothing; and whether the effect produced is motion, or strain, or both, or neither, also matters nothing. It is to treat of the effects of force in general on any body whatever.

This part of the subject is sometimes called Energetics; it will be found to be a sort of combination of Kinematics and Dynamics, the ideas of both motion and force being necessarily involved.

75. Now, in order that an agent exerting a force may produce any effect on the body to which it is applied, it is necessary that the body shall yield somewhat—that is, that the point of application of the force shall move in the direction of the force; and whenever this happens—whenever the point of application of the force does move along its line of action—*some* effect is *necessarily* produced. Thus either the body is set rolling, or swinging, or moving in

some way, or its motion is checked, or it is squeezed into smaller compass, or bent out of shape, or it is lifted up against gravity, or it is merely shifted along against friction, or it is warmed or electrified ; no matter what the effect is, some effect is always produced, and the force, or more properly the agent exerting the force, is said to have *done work*. Moreover, a body upon which work has been done is found to have an increased power of doing work itself, that is, of producing physical changes in other bodies ; and it is therefore said to possess more *energy* than before. This increase of energy is indeed the most essential part of the effect produced in a body by an act of work.

76. *Energy* therefore is that part of the effect produced when work is done upon matter, which is not an accidental concomitant, but really owes its origin to the work, and could not, so far as we know, have been produced without it ; and which, moreover, confers upon the body possessing it an increased power of doing work.

77. The work done in any case is proportional both to the magnitude of the force and to the distance through which it moves. Unless the point of application moves, no work is done and no energy is produced, however great the force may be ; for instance, a pillar supporting a portico is doing no work, though it is manifestly exerting great force.

Work, then, is the act of producing an effect in bodies by means of a force moving through a distance in its own line of action, and it is measured by the product of the force into the distance,* or

$$W = Fs.$$

The work is reckoned positive, and is called simply 'work,' when the body acted on is moved in the same sense as the force ; if, however, by any means, it be caused to move in *opposition* to the force exerted by an agent, the work done by that agent must be reckoned negative—that is, work is done upon it.

Thus if a force of five units acts through a distance of six

* The *moment of a force* was also defined as a force multiplied by a distance, but by a distance measured *at right angles to the force*. It is therefore an entirely different thing from work ; it may be called 'imaginary' work.

feet in its own direction, it does thirty units of work; the unit of work being that which is done when unit force acts through unit distance. Hence, the British unit of work is that done by a poundal (sect. 43) acting through a foot, and may be called a *foot-poundal.*

If a *dyne* (sect. 43) be taken as the unit of force, then the unit of work is that done by a dyne acting through a centimetre, and is called in this country an *erg* (from ἔργον, work). There are 421,393·8 ergs in one foot-poundal.

78. The effects produced in material bodies when work is done upon them are various, and constitute the different *forms* of energy. The full discussion of the subject of energy belongs to the science of physics, so we can here only just roughly enumerate its principal forms.

(1.) Motion (whether translation or rotation). (2.) Strain (whether extension, compression, or distortion).

(3.) Vibration, including the particular kinds called Sound. (4.) Heat (sensible and latent).

(5.) Radiation (including the particular kinds which are able to affect the eye, and which are therefore called Light).

(6.) Electrification. (7.) Electricity in motion. (8.) Magnetisation.

(9.) Chemical separation. (10.) Gravitative separation.

To these we ought perhaps to add vital energy, only that it may be held to be included under head 9. It is quite possible that many of these may reduce to simpler forms; in fact all but Nos. 9 and 10 are already pretty well known to be special cases of Nos. 1 and 2 (cf. sect. 5).

It is usual to consider those forms of energy which are more directly connected with large and visible masses of matter as more particularly the province of mechanics; and we shall here discuss only these more mechanical forms of energy, Nos. 1, 2, and 10.

The essential nature of No. 10 is at present unknown (see Introduction, foot-note), but for most practical purposes it comes under the class indicated by No. 2.

79. Now the question arises—When work is done and energy produced, is it created out of nothing, or is it only

manufactured from previously existing materials? The latter is the truth, for it has been found, as the result of innumerable experiments on the subject of 'perpetual motion' and others, that it is as impossible to create *energy* as it is to create *matter*, and that whenever energy appears as the result of work, it is always at the expense of some other form of energy which was previously existing. This fact is popularly expressed by saying that 'perpetual motion is impossible'—a statement which requires interpretation, because if there is one thing more universal than another it is perpetual motion (see sect. 4). The statement, however, is understood to be an abbreviation for the following: It is impossible for us to construct any machine which shall move and do work (and therefore generate energy) of itself without consuming at least an equal quantity of pre-existing energy.

80. All this indeed, in a much more complete and accurate form—more complete, because it involves the *non-destruction* of energy as well as its non-creation—follows from Newton's third law of motion, sect. 49, provided we make the assumptions (justified by experiment as above), 1st, that just as something called energy is generated whenever *positive* work is done, so whenever *negative* work is done something so like the first as properly to be called energy too, is destroyed; and 2d, that quantity of energy is measured by the work (Fs) done or undone in producing or destroying it. For the third law tells us, that whenever force is exerted, and therefore (*a fortiori*) whenever work is done, the two things concerned—the body which acts, and the body which is acted upon or re-acts—exert equal and opposite forces; hence whatever quantity of work one body does, the other has done upon it; or *the positive and negative works are equal* (see sect. 77, small print).

The 'agent,' or body which does the positive work, *loses* a certain quantity of energy. The body which has the work done upon it *gains the same amount*. Hence, on the whole—that is, taking both bodies into account—no energy is lost, and, algebraically speaking, no work is done. The energy is merely *transferred*, and the act of transfer involves two equal opposite works.

The law that, on the whole, no energy is ever created or destroyed by any forces which we know of and have experimented upon, is called *the law of the 'Conservation of Energy.'*

81. Just in the same way then that a force is the partial aspect of a stress, so work is the partial aspect of a something which consists of action and re-action, in the sense of work and anti-work, but which neither has, nor as yet perhaps needs, any name; and whenever we speak of 'work done,' it will be by *attending* to the action of one body on another, and neglecting the reaction of that other on the one. To summarise then: Work creates energy, anti-work destroys it, so both together simply transfer it. If it were possible to have a force without its anti-force, it would also be possible to get work done without its anti-work, but as a fact of experience it is *not* possible.

82. The fact that work is done whenever energy is transferred, taken in connection with the experience that energy often manifests a tendency to transfer *itself* from one body to another, and thereby to do work, has caused energy to be defined as *the power of doing work.* Now certainly a body possessing energy thereby possesses the power of doing an equivalent amount of work, *provided* the energy is of such a sort that it can be transferred to some other body; and in this sense *energy and power of doing work* are equivalent, though it is truer to say that the possession of energy *confers upon a body* the power of doing work, than to say that energy *is* the power of doing work. It is quite possible, however, for a body to possess energy and yet have no power of doing work, for energy is not always *available.*

Thus, a stone lying on the ground possesses an amount of energy corresponding to its fall to the centre of the earth, but this energy confers on it no power of doing work, for it would be impossible to let it fall without first expending a great deal more energy in digging a hole.

Again, energy is indestructible, and a given quantity may be transferred from one body to another, from A to B, from B to C, from C to D, and so on and back again, each time conferring upon its possessor a power of doing work, which work is done at each transfer by the body losing it. Hence, if it were correct to speak of work as being done by the *energy*, instead of by the body possessing the energy, the working power of a given quantity of energy might be unlimited, and at anyrate would be wholly incommensurate with the quantity of energy. The power of doing work, in fact, does not depend on the absolute quantity of energy in a body or system, but on its variations.

83. There are, however, practical difficulties in effecting such a series of transfers of energy without loss of working power, for though the quantity is unalterable, yet the *quality* has a tendency to deteriorate.

These practical difficulties are very similar to those which you would experience if you attempted to transfer a given quantity of *water* down a series of vessels. For you might spill some, some would evaporate, some of the vessels might leak, and all would remain wet. The quantity of water would be unchanged—it would be all there—but some of it would be unavailable. It would be not lost—only useless. Just so with energy, whenever it is transferred from one body to another—that is, whenever work is done, some of it is pretty sure to pass into a less available and more useless form. Its *quantity* is not altered, but its *availability* is less.

This tendency of energy to become less available is called the law of the *Dissipation or Degradation of Energy*. It may be expressed thus: When energy is *transferred* from one body to another, it is also always *transformed* from one of its forms to another, and some portion of the new form is pretty sure to be lower in the scale of energy than the original form; because of friction, imperfect elasticity, and so on. It is, in fact, impossible by any known process to raise energy in the scale of availability *on the whole*. Any given quantity, indeed, may be raised, but some other greater quantity will in the operation be degraded. The average is usually lower, and cannot be higher.

The energy of the earth in its orbit is not available to us. The energy of a flying molecule is almost unavailable, because we have as yet no means of dealing with molecules singly; if we could see and handle them, their motion would be as high a form of energy to us as the motion of other visible masses. Hence the distinction between high and low forms of energy is a purely relative one.

Energy falls in availability usually by becoming molecular, that is, by being transferred from visible masses to their ultimate molecules. The transfer is effected by friction and viscosity.

84. Energy and work are not to be confounded together; and all such phrases as 'accumulated work,' 'conservation of work,' 'work consumed,' &c., should be eschewed. Energy is not work, but work can be got out of it if the proper condition be supplied. Energy might therefore be called *possible* work. For consider the two fundamental forms of energy:

(1) The free motion of masses of matter relatively to one another; and (2) The separation of masses of matter from one another against stress.

In the first case, the body possessing the energy is moving through a distance, but is not exerting any force. Supply a resistance, and work is immediately done. In the second case, the body possessing the energy is exerting force or pressure, but it is stationary. Allow it to move, and work is immediately done.

The two fundamental forms of energy, therefore, correspond to the two factors in the product called *work*, namely, F and *s*. The first form corresponds to *s*; there is motion through space, but no force. The second corresponds to F; there is force, but no motion.

The first is called *Kinetic Energy*, or the energy of motion; the second might be called *Dynamic Energy*, or the energy of force (properly stress); or it might be called *Static* Energy, to distinguish it from Kinetic. As a matter of fact, however, it is generally called *Potential Energy*, which is not a bad name so long as it is not misunderstood to mean *possible* energy—a phrase without sense. Neither is Kinetic ever to be called *Actual* Energy. All energy is *actual* and *real*—potential just as much as kinetic; and both represent possible *work*—that is, work that will become actual as soon as the other factor is supplied.

85. Whenever work is done, both factors must be present —that is, both kinetic and potential energy—and the energy is always passing from one of these forms into the other while the work is being done. For if the motion is *with* the force, the speed must increase, and if it is *against* the force, it must decrease; while in the first case the distance through which the force can act, or the *range* of the force, is decreasing, in the second increasing. The energy of a vibrating body is continually alternating from one form to the other.

Enough has now been said to shew that the energy method of treating forces and their effects is a very general one, and extends to the whole of Physics. But the branch of the subject concerning which we can here enter into any detail will be a very small one, and will only extend to giving some examples of the transformation of energy from form 1, that of motion, to some other form, especially that of gravitative separation, and back again.

Measure of Kinetic Energy.

86. First consider how to measure the energy of motion in the case of simple translation of a particle; remembering that its energy is defined as equal to the work done by the force which caused the motion.

Now when a force F is applied to a mass m, the acceleration is

$$a = \frac{F}{m} \quad \text{(Chapter IV.)},$$

and the velocity generated when a body moves a distance s, with the acceleration a, is given by

$$v^2 = 2as \text{ (Chapter II.), that is, } v^2 = 2\frac{F}{m}s;$$

an equation readily written in the form

$$Fs = \tfrac{1}{2}mv^2.$$

But Fs equals the work done by the force while it acts through the distance s; and as energy is measured by the work done in its production, it follows that the energy of a body of mass m moving with velocity v, is

$$\tfrac{1}{2} mv^2,$$

because v is the velocity generated in the body during the performance of the amount of work, Fs.

This expression, $\frac{1}{2}mv^2$, is a most important one, and its numerical value is called the *kinetic energy* of a particle due to its motion relatively to the body which is supposed to be at rest—usually, of course, the earth. It equals the number of units of work that have been done upon the body in setting it in motion, and also the amount of work which it must do in order to stop itself—that is, to transfer its energy to some other body, either to the earth or to anything else which happens to come in its way.

When one elastic ball impinges directly on an equal one at rest, the first stops dead, and the other receives the whole motion; the energy has been here *obviously* transferred. The transference takes place just as really, though not so obviously, in every case where a body comes to rest or starts moving.

The unit of kinetic energy is twice that possessed by unit mass moving with unit velocity; it is of course equivalent to the unit of work, and usually goes by the same name (sect. 77). For instance, the British unit of energy would be a foot-poundal, being the effect produced by the action of unit force through unit distance.

87. If a body, instead of being at rest when the force acted on it, had been moving with velocity v_0, it would have already possessed the energy $\frac{1}{2} mv_0^2$, and so the *gain* of kinetic energy, equivalent to the work done, would have been

$$Fs = \tfrac{1}{2} mv_1^2 - \tfrac{1}{2} mv_0^2;$$

where v_1 represents the final velocity possessed after the force has acted for a distance s. This immediately follows from the old equation $v_1^2 - v_0^2 = 2as$, if we write $\frac{F}{m}$ for a, and leave the term Fs on one side the equation alone.

ILLUSTRATIONS.

88. A truck of mass 2000 lbs. running along a level line at the rate of twenty feet a second, has an amount of energy equal to

$$\tfrac{1}{2} \times 2000 \times 20^2, \text{ or } 400{,}000 \text{ units.}$$

If it were required to stop it in a distance of 500 feet, we should have to apply a brake exerting 800 units of force; for the work done by the truck against this force in the given distance would be 800 × 500, or 400,000 units, which is precisely the energy of the truck required to be destroyed, or rather to be transferred to something else.

One can always find the force necessary in any such case therefore by dividing the work required by the distance given, for, of course, $F = \frac{Fs}{s}$.

Again, to propel a one-ounce rifle bullet ($\frac{1}{16}$th lb.) with a velocity of 1200 feet per second, will require work to be done upon it equal to the energy generated, namely,

$$\tfrac{1}{2} \times \tfrac{1}{16} \times (1200)^2, \text{ or } 45{,}000 \text{ units.}$$

(This energy, and a good deal more, was contained in the charge of powder in the form of chemical separation, No. 8 (sect. 78); a quantity is always wasted in the useless noise and flash attending the explosion, Nos. 3, 4, and 5.) This work must have been done by the powder while the bullet was travelling from the breech to the muzzle of the gun, a length of say four feet; hence the average force exerted by the powder must have been 45,000 divided by 4, or 11,250 units of force.

Suppose now in passing through the air it loses 400 of its velocity by friction, so that it reaches the target with the velocity of only 800 feet per second, then the energy of the blow will be

$$\tfrac{1}{2} \times \tfrac{1}{16} \times (800)^2, \text{ or } 20{,}000 \text{ units;}$$

while that which has been 'lost' by friction (that is, transferred, some to the air and some to the molecules of the bullet, but in any case debased into the form of heat) is

$$45{,}000 - 20{,}000, \text{ or } 25{,}000 \text{ units of energy;}$$

and this must be the number of units of work which have been done by the flying bullet against the resistance of the air. Hence if its *range*, or distance travelled, were 1500 feet, the average resistance exerted by the air must have been 25,000 divided by 1500, or $16\frac{2}{3}$ units of force.

Finally, let a target stop the bullet dead in the space of $\frac{1}{4}$ inch ($\frac{1}{48}$th of a foot), then, since the whole (negative) work it has to do is numerically equal to the energy of the blow—namely, 20,000 units—it follows that the average force of the blow on the target is 20,000 divided by $\frac{1}{48}$, that is, 960,000 units of force, a much greater force than even the powder exerted; and this is apparent in the results, for the bullet is flattened out by the target, while the force of the powder had but a slight effect upon its shape.

Very likely an iron target would not yield so much as $\frac{1}{4}$ inch; if it only yielded half as much, the force of the blow would be doubled. Whether the bullet bounces off or not, matters nothing; it must have been stopped before its motion can be reversed. The reverse motion would not alter the *force* required to stop the ball, but it would increase its *impulse* (sect. 45) by lengthening the time during which the force was exerted against the target. Thus, if the ball bounced off with its original speed, the time and therefore the impulse would be double what they would have been if it had stopped dead like dough.

89. Notice the distinction between the *energy* of a blow, the *impulse* of a blow, and the *force* of a blow.

The energy equals Fs, or $\frac{1}{2}mv^2$.

The impulse equals Ft, or mv.

The average force equals F, or $\frac{\frac{1}{2}mv^2}{s}$, or $\frac{mv}{t}$ (cf. sect. 42).

It will be a good exercise to find from this last equation, in all the above cases, the time taken to do the work—that is, to transfer the energy. For instance, find the time of flight of the bullet, and also how long it took to travel the length of the gun, and so on.

These two expressions for an average force $\frac{mv}{t}$ and $\frac{\frac{1}{2}mv^2}{s}$, are worth comparing. The first we know is expressed in words by saying that force is rate of change of momentum; *rate* here having a reference to time, and meaning the increase per second of time elapsed. Similarly the second may be expressed by saying that force is rate of change of energy, only *rate* here has a reference to distance, and means the increase per linear foot of distance travelled.

The whole subject of the rates of variation of things with respect to different variables, considered as a branch of pure mathematics, is called the differential calculus, a science the foundation of which was laid by Newton for the purpose of treating questions concerning velocity, acceleration, and the like.

Measure of Potential or Dynamic Energy.

90. Now let us consider how to measure potential energy, or the energy of stress, especially in the form of gravitative stress exerted between the earth and a raised weight. This is a very simple matter, for, suppose a stone is at a height of h feet, we have a constant force mg exerted on the body, and a distance h through which it can act, so the work it can do while the stone falls is simply mgh; and therefore mgh measures the energy due to the relative position of the earth and stone, and the numerical value of this expression is often called 'the potential energy of the raised weight.' It equals the number of units of work that have been done upon the weight in raising it, and also the amount of work it must do whenever it drops. The energy is often *called* that of the weight, but it really belongs to whatever agent is exerting the stress pressing the weight and earth together (see Introduction); and as the nature of this agent is unknown, it is better not to speak of the potential energy *of* anything.

91. The energy of a pound of matter one foot high is called a foot-pound, because it is the effect which has been produced by a force of one pound-weight acting through a foot; it equals thirty-two units of energy or foot-poundals, because the weight of a pound equals thirty-two units of force or poundals. The unit of work or energy about corresponds to the raising a half-ounce weight one foot high (cf. sect. 43), (half an ounce being the $\frac{1}{32}$d of a pound).

'Thirty-two' of course stands for the value of g, whatever it may happen to be: it is different in different latitudes, and not necessarily exactly thirty-two anywhere. In French measure the numerical value of g is 981 (sect. 61); so the energy of a gramme of matter, one centimetre high (called a gramme-centimetre), is 981 ergs, because the weight of a gramme is 981 dynes.

92. To keep a raised weight still, it must be supported,

and it will exert pressure on its support, because it is being pressed by something towards the earth. This something is not, however, yet doing work. Remove the support, and immediately the weight begins to move; hence, now work is done, and the potential energy of the agent which exerts the pressure is transformed gradually into kinetic energy, and transferred gradually to the moving mass—to the weight itself, if falling freely—to whatever strings and wheel-work it is connected with, if it is constrained to fall slowly like a clock weight. When half-way down, the energy is half kinetic and half potential; when $\frac{3}{4}$ down, it is $\frac{3}{4}$ kinetic and $\frac{1}{4}$ potential, and so on.

For the original energy was mgh; but when half-way down, the potential energy is $mg\,\frac{1}{2}\,h$, or only half what it was, so the kinetic must be equal to the other half. When $\frac{3}{4}$ down, the potential energy is only $mg\,\frac{1}{4}\,h$, and the remainder is kinetic.

When within an ace of the ground there is no potential energy, and therefore the body has kinetic $\frac{1}{2}\,mv^2$, equal to the original energy, mgh.

This equation,

$$\tfrac{1}{2}\,mv^2 = mgh,$$

gives us the velocity acquired by a body freely falling a height h, as $v = \sqrt{2gh}$; a fact we knew perfectly well before, only we formerly arrived at it in a different way (see sects. 20 and 61).

The instant the falling body touches the ground it compresses it, and so work is done again, though this time very rapidly; and the energy is again transformed, and transferred, some to the molecules of the earth and ball as heat, some to the air in the form we call sound; while the rest, after having existed for an instant as strain between the earth and ball, reappears as kinetic energy in the bouncing ball. No ball, however, is perfectly elastic, so after a few bounces it will come to rest, and will possess neither kinetic nor potential energy relatively to the earth (it will be a little hotter than it was—that is all). To raise it again, something else must do work upon it.

93. As another illustration, consider a body sliding down a rough inclined plane. Let a mass m slide from A to B

(fig. 20), a length l, against a force of friction f, the *vertical* descent being h. Then the work done against friction is fl; the work done upon the mass is $\frac{1}{2}mv^2$ if it reaches B with the velocity v; and all this work has been done by gravity. But the work done by gravity is the force mg multiplied by the distance moved through *in its own (vertical) direction*, namely, h; so we have the equation,

$$fl + \tfrac{1}{2}mv^2 = mgh,$$

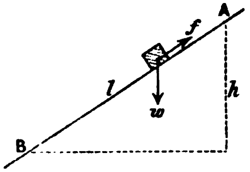


Fig. 20.

from which v can be readily found. The term fl represents the amount of energy which is transformed (degraded) into heat, or the 'mechanical equivalent' of the heat generated.

If $f = o$, that is, if the plane be smooth, the velocity v acquired in descending the vertical height h down the plane is the same as that found for a *freely* falling body in the last section, and it has no connection with the slope of the plane; shewing that the *path* of a falling body has no influence on the velocity acquired by it, provided everything be smooth.

94. The simplicity of *gravitation* examples is due to the fact that the force acting (the weight of the raised body) is constant and does not alter as the weight descends. But in every case, if s be the *range*—that is, the distance through which the force can act—and if F be the *average* value of this force, the potential energy is Fs.

Energy of Rotation.

95. So far we have only considered energy of motion in the form of translation, or the motion of a particle; but the energy of a rotating body can now be easily expressed since it is made up of particles, and the energy of the whole is the sum of their separate energies.*

Any particle of mass m, at a distance r from the axis of a

* Notice that the parallelogram law (sect. 25) does not apply to the composition of *energies*. Energy is not a directed quantity, and simple arithmetical addition applies to it.

body rotating with angular velocity ω, is revolving round and round a circle with velocity $v = r\omega$, and its energy is $\frac{1}{2}\,mv^2$, or, as it may be also written, $\frac{1}{2}\,mr^2\omega^2$. Now the energy of the whole body is the sum of the energies of all the particles in it; it is therefore

$$\Sigma(\tfrac{1}{2}\,mv^2) = \tfrac{1}{2}\,\Sigma(mr^2\omega^2) = \tfrac{1}{2}\,\omega^2\Sigma(mr^2)\,;$$

for, since the ω is constant, it may be taken outside the sign of summation; but $\Sigma(mr^2)$, the sum of the second moments of inertia of all the particles in the body, is the quantity we have called *the moment of inertia* of the rotating body (sect. 40), and denoted by M; hence the simplest expression for the energy of a rotating body, like a fly-wheel, is

$$\tfrac{1}{2}\,M\omega^2.$$

EXAMPLES.

1. A body slides down a rough plane, travelling 20 feet along the plane, but only descending 12 feet vertically. If the force of friction were equal to $\frac{1}{4}$th of the weight of the body, find the velocity gained during the descent. *Ans.* $16\sqrt{2}$ feet per second.
2. What is the work that must be done in order to propel a 3-lb. stone at the rate of 40 feet a second?
   Ans. 2400 foot-poundals, or about 75 foot-pounds.
3. A simple pendulum is pulled aside till its heavy bob is raised 3 inches and then let go: find its velocity when it passes its lowest point.
   Ans. 4 feet per second.
4. What initial velocity is necessary to make a rifle bullet strike a target placed 300 feet high vertically above the gun, with the velocity 600 feet per second, neglecting the resistance of the air?
   Ans. (by equating the initial and final energies) 615·8.
5. What would be the answer to the last question if the bullet weighed an ounce, and if the resistance of the air were taken to be equivalent to a drag of $11{\cdot}541\dot{6}$ poundals? *Ans.* 700 feet per second.
6. If a projectile were started in any direction with the velocity 80, and arrived at another point on the same

level with the velocity 30, after having travelled 150 feet, what must the average resistance of the air have been equal to?

Ans. $\frac{11}{80}$th of the weight of the projectile.

7. What force would be necessary in order to stop the projectile of Question No. 2 in the space of 6 feet, and how long would it take?

Ans. 400 poundals and ·3 second.

8. Find the mechanical equivalents of the heat generated by friction in the motions considered in Questions 1, 5, 6, and 7; assuming the mass to be 3 lbs. in each case.

CHAPTER VII.

COMPOSITION AND RESOLUTION OF FORCES.

(*Introduction to Statics.*)

96. Hitherto we have only considered the effect of a single force when it acts on a particle or rigid body, and we find that it may either pull the body along (translation), or turn it round (rotation), or do both at once. But in very few cases in practice do we have only one force acting in this way; often there are a great number of different forces, so that it becomes necessary to consider how the motive effect of a number of forces may be deduced. The simplest way is to reduce the forces in number.

When any number of forces act on a *particle* they may always be reduced to one—that is, they may be replaced by a single force which produces precisely the same effect as them all. This single force is called the *resultant;* and the operation of reducing the number of forces is called the *composition* of forces.

If a number of forces act on different points of a *rigid body*—that is, an assemblage of particles connected rigidly together—they cannot in general be reduced to one force, but they may always be reduced to two (sect. 110). We however will here only consider the cases where they may be reduced to one ; in other words, only the cases where the forces either all pass through the same point (that is, virtually act on a particle), or else all lie in the same plane (cf. sect. 126).

A picture hanging by a cord over a nail furnishes us with an example of a rigid body acted on by several forces, and the tension in the two parts of the cord is equivalent to the weight of the picture. A weight resting on a tripod-stand is another example, and the three stresses in the legs are equivalent to the one weight. Again, a table or chair is supported by as many forces as it has legs, unless some are too short. A teetotum is spun by forces, which may be reduced to two equal and parallel ones in opposite directions, and we have here a case of pure rotation without translation. A kite in the air is acted upon by the wind pressing it, by a tension in the string, and by the pull of gravity; and the kite moves about according to the direction of the resultant of all these forces.

97. Again. for some purposes, it is convenient to analyse or split up a single force acting on a body into two or three components, so as to study their effects separately. This operation is called the *resolution* of forces. Thus, suppose a weight resting on an inclined plane (see fig. 26, sect. 103), one may resolve its weight into two forces, one perpendicular to the plane, and therefore balanced by its resistance ; the other acting along the plane and producing motion, except in so far as it is balanced by friction. Again, in a windmill, it is convenient to resolve the wind's pressure on the sails into two components—one the effective one in the direction of motion ; the other a useless one in the direction in which, by the construction of the machine, no motion is allowed.* (This last component, therefore, only produces strain.)

Composition of Forces acting on a Particle.

98. The method of compounding forces into a resultant, or resolving them into components, is a very simple one, being the same as that by which motions were compounded and resolved.

* *N.B.*—A windmill always faces the wind.

This has been proved in sect. 47. In fact, it is an immediate consequence of the second law of motion, $F = ma$, that forces must be compounded like accelerations (see sect. 27); and therefore an elaborate proof of the parallelogram of forces founded upon axioms no more obvious than the thing to be proved, like Duchayla's, is unnecessary. The proof from the second law, however, does not establish the *position* of the resultant, but in the case of a particle this is obvious. The further proof necessary for an extended body is given in sect. 108.

The rule, then, is—Draw a set of lines one after the other, without taking the pen* off, parallel to, and in the same sense as the successive forces acting on the body, and proportional to them in magnitude; then the line required to complete the polygon, taken in the reverse sense (that is, drawn *from* the starting-point, not *to* it), will be the resultant in magnitude and direction. The forces may be taken in any order just as the motions might (sect. 23).

Since we are only dealing with a particle, this is the full and complete solution; for the resultant, of course, acts on the particle, and therefore its position is known; and the three things, magnitude, direction, and position, completely specify a force (see sect. 48).

The resultant of *two* forces is usually more conveniently expressed as the diagonal of the parallelogram whose sides represent the forces, than as equal to the third side of a triangle.

99. *Examples of the Composition of Two Forces.*—A particle of mass m is pulled along by two strings—one always pulling east, with a force P; the other always north, with a force Q. What is the acceleration and direction of motion?

Drawing the two forces P and Q (fig. 21), one finds the resultant R at once as equal to $\sqrt{(P^2 + Q^2)}$ by Euclid, I. 47; and since this is the resultant force, the acceleration is $\frac{R}{m}$ units along the diagonal of the parallelogram. If the two forces P and Q were equal, then R^2 would be simply $2P^2$: that is, $R = P\sqrt{2}$, a result worth remembering.

Suppose now that the two forces are at some acute angle,

* If the forces do not all lie in one plane, the polygon cannot be drawn on paper, but it may be constructed in wood.

say 60°, then to find R we may use Euclid, II. 12, which says that AD^2 exceeds $AB^2 + BD^2$ by twice the rectangle $AB \cdot BN$ (fig. 22).

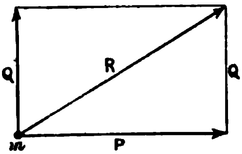


Fig. 21.

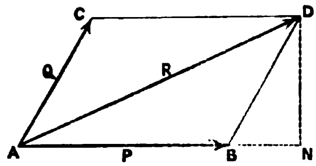


Fig. 22.

The angles CAB and DBN are always equal (I. 29), and if each equals 60°, BN is easily seen to be half BD, because the triangle BND is then half an equilateral triangle; so putting in this value for BN, and noting that BD = AC = Q, and therefore BN = ½ Q, we can write the general relation $AD^2 = AB^2 + BD^2 + 2AB \cdot BN$ in the form $R^2 = P^2 + Q^2 + PQ$ for the case when the angle between P and Q is 60°.

If the angle CAB between the forces had been an obtuse angle, such as 120°, we should have proceeded similarly, only using Euc. II. 13, and we should have arrived at $R^2 = P^2 + Q^2 - PQ$.

Similarly we might proceed for angles between P and Q of 45° or 135°, of 30° or 150°; but for angles in general, though the relation

$$AD^2 = AB^2 + BD^2 + 2AB \cdot BN,$$

will always apply—regard being paid to sign in the last term (see sect. 12)—yet it is not easy to determine the side BN in terms of the side BD (or Q)—the subject of the mensuration of triangles, or Trigonometry, not being supposed known at this stage.

Our resource is then to find the resultant by construction; and this indeed is often a very good way, even when one knows some trigonometry. You lay off on paper the two given forces to any scale, and inclined at the proper angle; then you complete the parallelogram, and measure the diagonal on the same scale—this gives you its magnitude;

and its direction referred to the given forces you get also from the figure.

100. Notice that in the parallelogram of forces, you really have two diagrams drawn together as one : a representation of the forces, and a geometrical construction ; but they should be understood to be essentially distinct. The proposition of the triangle of forces is really the geometrical part of the parallelogram by itself.

An example will render the meaning of this clearer. Let two forces, 6 and 8, act on a particle with an angle of 60° between them. Find resultant.

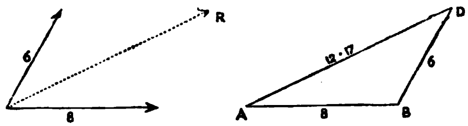


Fig. 23.

On the left of fig. 23 is a picture of the given forces. On the right is the geometrical figure—namely, a triangle in which AB represents the force 8, BD the force 6, and AD their resultant, in magnitude and direction. (AD equals 12·17 nearly, as may be found either by drawing and measuring, or by calculating it from sect. 99 as $\sqrt{(8^2 + 6^2 + 8 \times 6)}$. Its position is known, for of course it acts on the given particle ; so we return to the left-hand diagram, draw through the point of intersection of the two given forces a line equal and parallel to AD, and say this is the resultant. Obviously it is the diagonal of the parallelogram of forces—the triangle ABD is simply half the parallelogram ; compare fig. 22.

Observe that the geometrical construction is based upon only magnitude and direction : it does not give you position ; this must always be determined from the positions of the given forces in the force diagram. It is not usual to separate the two figures in simple cases, but as a matter of principle it is best always to keep them distinct.

101. The diagrams for one instance of the polygon of forces may be also given, just to make sure it is fully understood.

Forces in a plane, of magnitudes 4, 5, 3, 8, act on a particle, their directions making angles with each other of 75°, 45°, and 120° respectively. Find the resultant. In actual cases the angles between the forces are not specified numerically, but are indicated directly. In artificial questions, however, like the above, when the angles are specified in degrees, a protractor may be used to lay off the directions, though it is an objectionable instrument.

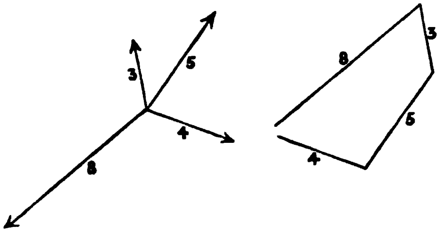


Fig. 24.*

It turns out to be ·504, so that the given forces are very nearly in equilibrium, or their resultant is very small.

Observe the reciprocity of these diagrams. In one the lines meet in a point, in the other they inclose an area.

Try drawing the sides of the polygon in some other order, and see that you always get the same result. Especially try the order 4, 3, 8, 5; for the polygon then happens to be a crossed one.

The bits of forces represented by the lines surrounding the inclosed space in a crossed polygon are in equilibrium (Chapter VIII.), and may be removed from the particle without disturbance.

In the above case the inclosed bit is an equilateral triangle, and the forces which may be removed are three threes—namely, three parts from force 8, three from 4, and all of 3; the forces left being 1, 5, 0, 5. Construct the polygon for this mutilated set, and see that you still get the same resultant. Notice the reason why the three removed forces were in equilibrium—namely, that they were equal and lay symmetrically, making angles of 120° with each other.

* In all these figures the lines are drawn *parallel* to the forces; this is the easiest, though not the essential plan. What *is* essential is, that the lines shall represent the directions of the forces in some understood manner. It is usually said that they will do either parallel or perpendicular; but they would do equally well if all were inclined at 45°, or at any other angle, to the forces which they respectively represent, provided this angle were the same for all. The interior angles of the polygon are supplementary to the angles between the corresponding forces.

You are strongly recommended at once to get out your instruments and a sheet of drawing-paper, and verify all this by careful drawing, as well as some of the examples at the end of this chapter. The instruments needed are a graduated scale of equal parts, a parallel ruler or T square, and a pair of compasses.

Resolution of Forces.

102. Every force may be split up into two definite components acting at given angles with it; but, if the angles are *not* given, a force may be resolved into two components in an infinite number of ways; in other words, the same line may be the diagonal of an infinite number of parallelograms (fig. 25). One chooses in each problem the particular pair of components which are most convenient, the most convenient being usually at right angles to each other. Often one is in the direction of possible motion, and the other perpendicular to it; again, in cases where gravity is concerned, one is often horizontal and the other vertical.

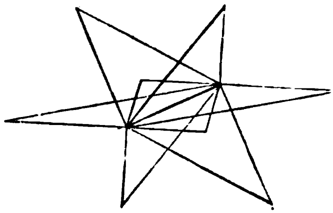


Fig. 25.

Verify, by drawing, the following: A force of 8 units is equivalent to two components of $\frac{8}{\sqrt{(2+\sqrt{3})}}$ each, acting one on each side the given force at angles of 15° with it; also to two of 8 each, if the angles be 60°; also to two of $\frac{8}{\sqrt{(2-\sqrt{3})}}$ each, if the angles be 75°; also to a component 4, acting at an angle of 60°, and another of $4\sqrt{3}$ at an angle of 30°; and so on.

$$(\sqrt{2} = 1{\cdot}4142\ldots\ldots;\ \sqrt{3} = 1{\cdot}732\ldots\ldots)$$

103. To illustrate the use of this, take a mass m, or say ¼ lb. for those who like numbers best, on a smooth inclined plane inclined to the horizon at an angle say of 30°. We

have, acting on the mass, the force w (= mg) due to gravity acting downwards, and the pressure of the plane, say R, acting normal to the plane. Now, resolve the downward force into two—one along the plane, that is, in the direction of motion, and call this component p; the other normal to the plane, as q (fig. 26).

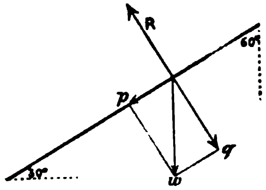


Fig. 26.

This you do by drawing a parallelogram with sides in these directions, and of such size that w (in the present example, $\frac{1}{4}g$ or 8) is the diagonal. The angle between p and w is 60°, that between w and q is 30°, and we have just found (nineteen lines above) that a force of eight units is equivalent to two forces, 4 and $4\sqrt{3}$, acting at 60° and 30° respectively; so then $p = 4$ and $q = 4\sqrt{3}$.

The motion is in the direction of p, the acceleration being $\frac{p}{m}$ or $\frac{4}{\frac{1}{4}} = 16$; whereas there is no motion in the direction of q, because it is balanced by R, hence $R = q = 4\sqrt{3}$.

104. **Friction.**—The reaction of surfaces will afford us other examples. When anything exerts pressure on a plane surface, the reaction of the surface is in general inclined in some direction or other to the surface—usually in that direction most likely to oppose relative motion; but it is convenient to resolve this reaction into two—one normal to the surface (normal merely means perpendicular), which is called the *normal pressure;* the other along the surface, which is called the *friction.* If either surface be perfectly *smooth*, this last component is absent, and all the reaction is normal. And even for rough surfaces this may be so too, as in the case of a ball resting on a level floor; but if any forces are tending to cause motion over rough surfaces, then there is some component along the surfaces, or friction, which always opposes the motion.

The force of friction is precisely equal and opposite to the forces tending to cause the motion, so long as the body does not move; but if the latter gradually increase, they will, at a certain instant, become too much for the friction, which reaches a maximum, and can increase no further; so then motion ensues, the effective or accelerative force being the motive force applied, minus the friction. For instance, in the above example of the inclined plane, suppose the force of friction to be called f, it would act up the plane in exact opposition to p. If the body were at rest, it would be so because $f = p$; if it were in motion, the acceleration would be $\frac{p-f}{m}$.

It is found experimentally that the maximum or critical value of f is proportional to the normal pressure R between the surfaces, the ratio between f and R depending on the nature of the surfaces in contact, and being called the *Coefficient of Friction.**

Say that this coefficient in the above case of the inclined plane is $\frac{1}{4}$, so that $f = \frac{1}{4}$ R, then the acceleration would be

$$\frac{4-\frac{1}{4}4\sqrt{3}}{\frac{1}{4}} = 16 - 2\sqrt{3}.$$

The actual pressure or reaction between the surfaces in contact, is of course the resultant of the two forces R and f, the normal pressure and the friction—that is, it is the square root of the sum of their squares; and the angle which its direction makes with the normal when the two surfaces are on the point of sliding over one another is called the *limiting angle of friction*, or (by a somewhat *lucus a non lucendo* name) the *angle of repose.*

105. It is often convenient to resolve motions and velocities. Thus a projectile shot up at any angle has a certain initial velocity imparted to it, which may be conveniently resolved into two—one a horizontal one unaffected by gravity, which therefore remains constant except for the resistance of the air; the other a vertical one, which is gradually diminished by gravity (at the rate of 32 units a second) until it is

* The coefficient of friction when the surfaces are in actual relative motion is usually less than when they are just going to move; but we cannot enter into details here, and, moreover, the value of the motion coefficient is not constant, but depends somewhat upon the speed.

converted into a negative, that is, a downward, velocity increasing at the same rate (32 'speeds' per second) till the body strikes the ground.

Again, take a north-east wind. This may be considered as made up of a north and an east wind, each $\frac{1}{\sqrt{2}}$th of the actual strength, and on any thin, flat, smooth surface facing the north only the northerly component can exert any pressure, the easterly component simply gliding over it.

Or suppose the surface faced N.N.W., and we wanted to find the pressure on it; the wind might be resolved into an N.N.W. component, $\frac{1}{2}\sqrt{2-\sqrt{2}}$ times its strength, and an E.N.E. one, $\frac{1}{2}\sqrt{2+\sqrt{2}}$ times its strength, and the surface would experience the pressure of the N.N.W. component only, the other being useless.

This is how one deals with kites and windmill- and boat-sails. They are all surfaces exposed in a skew fashion to the wind, so that the perpendicular pressure on the surface is a component only of the whole available force of the wind. The sails of a windmill are set so as to be inclined both to the direction of the wind and to the direction of possible motion; so also usually are the sails of a boat.

106. In the case of a kite the normal pressure of the wind is balanced by two other forces, the pull of gravity and the pull of the string, otherwise the kite would be blown about, scarcely experiencing any pressure at all. The sails of a windmill are not blown in the direction of the normal pressure on them, but in some other direction determined by the way they are set on the axle and on the sole direction in which this can turn, usually at right angles to the direction of the wind. So also with a boat; the reason why it is not blown in the direction of the normal pressure on its sails is that it is more easily moved through the water lengthways than breadthways because of its shape. Hence the normal pressure of the wind requires again resolving into two components, one along the direction of easy motion, the other at right angles to it. The first component is the active one in the case of both windmill and boat; the other component is entirely counteracted in the case of the wind-

mill, but in the case of the boat it does cause a slow broadside motion, which is called leeway.

Thus if BR (fig. 27) represents the plan of a boat, MS its sail, and W the relative direction and strength of the wind (represented also by the arrow i), P is the normal pressure and Q the useless

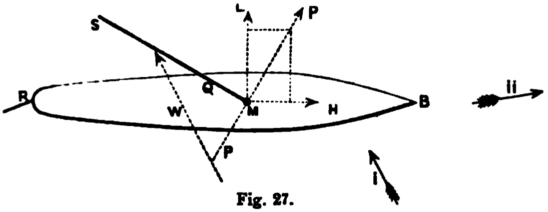


Fig. 27.

component or tail-wind. Producing P for convenience, and resolving it along and across the boat, H is the effective component producing headway, and L is the leeway component. The arrow ii shews the direction in which the boat would probably tend to sail.* The rudder R is represented as turned in the direction required to counteract the leeway and make it sail along the line RB produced.

The *rudder* also affords an illustration of the present subject. When turned, there is a normal pressure on its front surface due to its motion through the water, and this pressure is resolvable into two forces—one in a direction opposite to the boat's motion, which simply acts as a drag (hence in racing, the cockswain uses the rudder as little as possible), the other at right angles to the length of the boat, which pushes the stern round.

It is obvious that no force can directly exert pressure at right angles to itself, and yet it is easy for a ship to sail at right angles to the wind. The reason is, that the sails act as a mediary, being inclined to both wind and boat. The force directly urging the boat is a component of the pressure on the sails, this pressure again being due to a component of the wind's motion. Remember that the effective wind is compounded of the true wind and the speed of the ship. This explains why a ship can sail very close to the wind.

These examples will serve to illustrate the application of

* This assumes that the sail is set amidships. In practice there is always a preponderance of sail towards the stern, consequently an unsteered ship gets blown round, and 'sails up into the wind's eye.' The rudder would therefore more likely have to be turned the other way.

the principle; but other examples occur daily, and may be worked out in the same way as the preceding, drawing and measuring being often sufficient.

COMPOSITION OF FORCES ACTING ON A RIGID BODY.

107. For the case of a rigid body, in addition to the *magnitude* and *direction* of the resultant as determined by the polygon construction, sect. 98, it is necessary also to determine its *position*—that is, a point on its line of action. For observe that, though as regards translation a force in one place is as good as an equal parallel force in another, yet as regards rotating power its position is important. Thus, imagine a long trough of water lying on the ground with a string tied to it by which you wish to raise it. Any vertical force greater than the weight of the trough must needs raise it, wherever the string is tied; but if the string is tied anywhere except above one definite point, the trough will also turn round as it rises, and the contents will be upset.

Again, if you raise it by two parallel strings, one near each end, then when the pull of the two strings together is a little greater than the weight of the trough, it is raised; but if you want to raise it without rotation, the pull of each string must be carefully proportioned, so that the resultant of the two forces may pass through the point above spoken of, which is called the centre of gravity.

Again, in the case of a pivoted body, it is obvious that a force applied close to the pivot has much less effect than an equal one far off; and if applied *at* the pivot, it can have no motive effect whatever.

108. Now, the fundamental dynamical idea in rotation is the *moment of a force* (read sect. 53 again); and the following general statements are true, with their converses.

(1.) The moment of the resultant must equal the sum of the moments of the components about any point in every possible case, otherwise the resultant would not be truly the resultant, because unable to replace the components in rotating power.

That this condition is fulfilled by the diagonal of a parallelogram may be proved among other ways as follows:

To shew that the resultant given by the parallelogram of forces is equivalent to its components in rotating *as well as in translating power—that is, that its moment about any point in the plane is equal to the sum of the moments of the two components.* The moment of the force AB about a point O (fig. 28) is twice the area of the triangle OAB; the moment of AD is twice the triangle OAD, and that of AC is twice OAC: hence what we have to prove is the following equality between the areas, OAB + OAC = OAD; the point O being in the plane of the parallelogram.

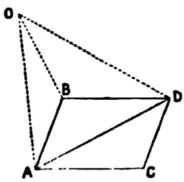


Fig. 28.

Now OAC = OBD + ADB, because the bases are equal, and the height of the single triangle is equal to the sum of the heights of the others (this is an easy extension of Euc. I. 38—analytically obvious, thus, $\frac{1}{2} b(h_1 + h_2) = \frac{1}{2} bh_1 + \frac{1}{2} bh_2$);

and by inspection, OAD = OAB + OBD + ADB;
therefore OAD = OAB + OAC;
which was to be proved.

(2.) The algebraic sum of the moments of any number of forces about a point on their resultant equals *o*; in other words, the sum of the positive moments equals the sum of the negative. (The moments of *two* forces about a point on their resultant are therefore numerically equal, but of opposite sign.)

For their resultant can have no rotating power about such a point, neither therefore can the components.

(3.) If the body on which the forces act has one point fixed, it will not be rotated by them, provided their resultant passes through the fixed point or pivot.

For instance, to keep the beam (fig. 31) steady, C is the point to fix. The pressure of the pivot or fulcrum is then equal and opposite to the resultant of all the forces.

Composition of Two Forces in General.

109. If the two forces are in one plane, the parallelogram is a complete solution, whether they act on a particle or a rigid body, for the forces must intersect somewhere, and the point of intersection fixes the position of the resultant.

Thus fig. 29 is the physical part of fig. 23 repeated for a rigid body, say a stone pulled by two strings. The geometrical part applies just as well as before. The direction of the resultant must pass through E, the point where the given forces produced backwards intersect, and it may be applied to the body at any point in a line EP parallel to AD (fig. 23).

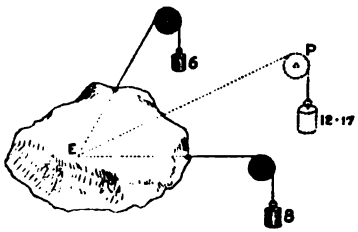


Fig. 29.

It may happen, however, that the point of intersection of the two forces is inconveniently distant, as off the paper for instance, or even at infinity when the forces are parallel. In such cases the general construction of sect. 112 is resorted to.

110. If the two forces are not in one plane they cannot intersect, and our construction for finding the resultant fails. The fact is *they have no resultant*, and cannot be farther reduced; they can only be put into the more convenient form of a force and a 'couple' (sect. 117) in a plane perpendicular to the force; so they tend to carry the body along and turn it round at the same time. This pair of forces is called a *wrench*, because it tends to twist the body about a certain screw; but the subject now becomes too complicated for us in this stage. This is what is meant in sect. 96 by the two forces to which any forces whatever acting on a rigid body can always be reduced, even when no more can be done. If, however, all the forces lie in one plane, no 'wrench' is possible, and they may then always be reduced to one simple resultant.

Composition of any number of Forces in a Plane.

111. The parallelogram construction may be applied several times in succession, reducing the number of forces by one each time. This is a complete but cumbrous solution.

The polygon construction is a solution as regards magnitude and direction, but requires supplementing in order to determine position. The supplementary construction employed is such an important one, that it seems well to introduce it here, although its full discussion would lead us beyond our present mark. It will be best understood by an example, and the case of only three forces will afford a sufficient illustration of the method. It depends on the fact that a single force may be resolved into a pair of components in an infinite variety of ways (fig. 25); so that, if the given forces are not convenient to find the resultant from, we can choose a more convenient pair out of the set which have the same resultant, and then draw the resultant of these. Expressed in another way, it may be said to depend on the fact that forces in equilibrium produce no disturbance, and hence may be introduced or removed at pleasure.

Construction for completely finding the Resultant of any number of Forces anywhere in a Plane.

(*Illustrated by the case of three forces.*)

112. Let P, Q, S be the forces. Draw the sides of the polygon ABCD parallel to, equal to, and in the same sense as the three forces; then the completion of the polygon, AD, is the resultant R in magnitude and direction. Where is it to be placed?

Choose *any* point O, join OB, and draw in the other diagram a line PQ parallel to it across the forces P and Q (the line is to be drawn across P and Q, because B is the meeting-point of the sides of the polygon which represent P and Q).

Then join OC, and draw a line QS parallel to it (C being the meeting-point of the sides representing Q and S).

Then join OD, and draw a line through S parallel to it, say SE; also join OA, and draw a line through P parallel to it.

The point E, where these last two lines intersect, shall be a point on the resultant, and its position is therefore determined. *Q.E.F.*

Proof (This may be omitted till after Chapter VIII. has been read).—A force represented by AB is the resultant of two forces represented by AO and OB, because it forms a triangle with them; but P is a force represented by AB, hence P is equivalent to two forces acting along the lines EP and PQ, equal to AO and OB respectively, and may be replaced by them. Let it be so

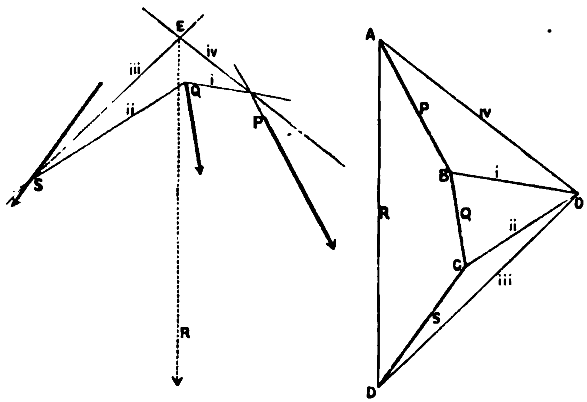


Fig. 30.
(Lines parallel to each other in the two diagrams are labelled similarly.)

replaced. Similarly the force S may be replaced by two forces acting along SQ and ES, equal to CO and OD respectively.

But Q is in equilibrium with two of these forces—namely, those along SQ and PQ, since their representative lines, BC, CO, OB form the sides of a triangle taken in order (sect. 124); hence this set of three forces may be removed; and there now remain, as the equivalents of the original forces, P, S, and Q, only forces along EP and ES, represented by AO and OD respectively. Hence these two forces have the same resultant as the three original forces had; but the resultant of these two forces passes through E, their point of intersection (sect. 109); therefore the resultant of the three original forces, P, Q, and S, passes through the point E.

Q.E.D.

It will be seen, therefore, that what we really do in the construction, is to compound with P, Q, and S, two sets of equilibrating forces—namely, two equal opposite forces in the line PQ of magnitudes OB and BO, and two in the line

QS of magnitudes OC and CO; and by their help to replace the given forces by two intersecting ones, ES and EP, the position of whose resultant is obvious.

This construction applies equally well to parallel forces, only then of course the polygon ABCD shuts up, the points B and C being on the straight line AD; but everything else remains without modification.

The use of the above construction may not be quite apparent perhaps, but it is put here as an indication of quite a large art—namely, *graphical statics*—which may well occupy the student's attention at a later stage.

Composition of Parallel Forces.

113. Parallel forces can only act on an extended body: forces which act on a particle of course cannot be parallel. The direction of the resultant of parallel forces is the same as the common direction of its components, while its magnitude is their algebraic sum—that is, their sum paying regard to sign—adding all that act in one direction, subtracting any that pull the other way. This is all that is required to be known for translation (sect. 107); but to discuss the *rotation* of a body under the influence of parallel forces, we must learn the *position* of the resultant, and this requires either a geometrical construction or an arithmetical calculation.

The general construction of sect. 112 applies to parallel just as well as to other forces, so we have only to give the method of calculating its position arithmetically.

114. The fact (No. 1, sect. 108) that the moment of the resultant equals the algebraic sum of the moments of all the components, though universally true, is most useful in its application to *parallel* forces, and it affords a ready method of finding the position of their resultant arithmetically.

Thus imagine a beam acted on by any parallel forces, say weights, 4, − 5, 6, − 2, &c., arranged anywhere on the beam (as shewn in fig. 31), at distances 4, 8, 16, 22 inches from some fixed point of reference O; then the resultant R is equal to

$$4 - 5 + 6 - 2 = 3,$$

and is at a distance x from O, such that

$$3x = 4 \times 4 - 5 \times 8 + 6 \times 16 - 2 \times 22 = 28;$$

wherefore $x = 9\frac{1}{3}$ inches. Mark off OC equal to this; then R acts at the point C, as shewn; and, to keep the bar in equilibrium,

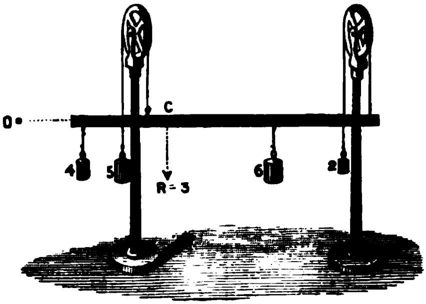


Fig. 31.

another pulley and string must be arranged to exert a force 3 upwards at this point.

And, generally, if the forces be w_1, w_2, w_3, ... at respective perpendicular distances x_1, x_2, x_3, ... from any point O, then the distance of the resultant from the same point is

$$x = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots}{w_1 + w_2 + w_3 + \ldots\ldots}.$$

This is a constantly occurring form of fraction, and is a more general sort of average. If $w_1 = w_2 = w_3 =$ &c., then it would be the ordinary expression for finding the average of the distances x_1, x_2, x_3, &c.—that is, it would give the average distance of all the weights from O, for it would add all the distances together and divide by the number of them (see sect. 18).

Composition of Two Parallel Forces.

115. When we have only two forces to deal with, the general statements and constructions are of course equally applicable, but they may be put into a more simple form.

The following two simple constructions may be given for finding the position of the resultant geometrically.

First Construction (fig. 32).—Take a point M half-way between the forces P and Q, and draw two lines through it; one parallel to the forces, the other not, but cutting them in A and B respectively. Lay off from M two lengths in the former of these lines, in the same sense as the respective forces, MC equal to P, and MD equal to Q, and join AC and BD; the resultant shall pass through E, the intersection of AC and BD.

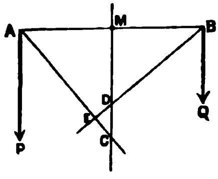


Fig. 32.

Proof.—We may suppose that we have here compounded P with a force AM, and Q with an equal, opposite, and therefore equilibrating, force BM; and AC, BD, are the diagonals of parallelograms, and have the same resultant as P and Q have.

Second Construction (fig. 33).—Anywhere on the line of P take a length equal to Q, and on the line of Q a length equal to P. The lines joining the extremities of these two lengths will intersect in a point on the resultant.

If the forces have the same sense, they are to be joined crosswise, and E is the point.

If they have contrary sense, they are to be joined without crossing, and F is the point.

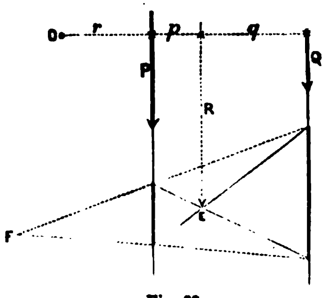


Fig. 33.

This is the best construction.

Proof.—Observe that the two triangles side by side with common vertex E are similar (their bases being parallel), hence their heights and bases are proportional. But their bases are Q and P; so, calling their heights p and q,

$$\frac{p}{q} = \frac{Q}{P},$$

or $$Pp = Qq.$$

This fact proves the proposition; for, by (2) sect. 108, the moments Pp and Qq of the two forces about their resultant must be equal and opposite; but they are evidently *opposite*, from the figure, and this equation states their *equality*, about the point E. Wherefore the resultant does pass through E if the forces have the same sense; and similarly it may be shewn to pass through F if they are contrary.

116. The following propositions concerning parallel forces are now at once seen to be true, being little more than repetitions in a compact form of what has gone before.

(1.) The distances between each force and the resultant are inversely as the forces—that is, $p : q = Q : P$. This is the formula to use when you want to find the resultant arithmetically.

(2.) If two parallel forces have the same sense, their resultant is equal to their sum, and lies between them, nearer the bigger one. If, however, they are of contrary sense, their resultant equals their difference, and lies outside them on the side of the bigger one, agreeing with the bigger one in direction.

(3.) If two forces are equal, the resultant must be equidistant from both.

If they are of contrary sense, this means that the resultant is at infinity; but its magnitude is zero, being equal to the difference of the components.

117. Hence, two equal contrary parallel forces have a resultant zero at infinity; or, as it is sometimes expressed, they have no resultant at all. (In any of the constructions the lines whose intersection gives the position of the resultant will for this case be found to be parallel.) Such a pair of forces cannot be further simplified, hence they are taken together and called a *couple*. The moment of the couple about any point will be easily seen to be independent of the position of that point, and to equal either force

multiplied by the perpendicular distance between the two forces, this distance being called the *arm* of the couple.

A couple is not properly to be regarded as *two* forces, but as a particular case of *one*—namely, an infinitely small force at an infinitely great distance. It obviously possesses only *rotating* power. (Read again sect. 96.)

The Composition of Parallel Forces as Illustrated by Gravity.

(Centre of Gravity.)

118. The force of gravity illustrates the subject of parallel forces very well. A rigid body is made up of particles, every one of which is pulled towards the centre of the earth with a force proportional to its mass, and equal to its mass multiplied by g (sect. 60). Now, since the centre of the earth is such a long way off, these converging forces are for bodies of ordinary size practically parallel. Hence the whole pull of gravity on a table or a book is really the resultant of an infinite number of parallel forces—the attractions on the several particles.

To find the *magnitude* of this resultant, you hang up the body on a spring balance—in ordinary language, you *weigh* it.

To find its *position*, the easiest way is to hang up the body by a bit of string; the line of the resultant is then a continuation of the string, since it must pass through the point of suspension. Or you may balance the body on your finger; the line of the resultant is always the vertical through the point of support whenever the body is in equilibrium.

Its *direction* is a fixed one—namely, always pointing to the centre of the earth, no matter how you turn the body.

Now when a body exposed to the action of a number of parallel forces is turned about, there is one point in the body through which their resultant always passes in every position of the body—and this point is called the *centre of the parallel forces;* or, if the parallel forces are due to gravity, it is called the *centre of gravity.*

Determination of the Centre of Gravity by Experiment.

119. If this point be directly supported, the body is in equilibrium in *every* position necessarily; and conversely, if a body is in such equilibrium, it must be because its centre of gravity is directly supported. A coach wheel, for instance, should be pivoted at this point. Hence this gives one way of finding it. Another way of experimentally determining its position is to find out the line of the resultant in some two positions of the body by hanging it up twice in different ways (see fig. 34); then the centre of gravity must be the

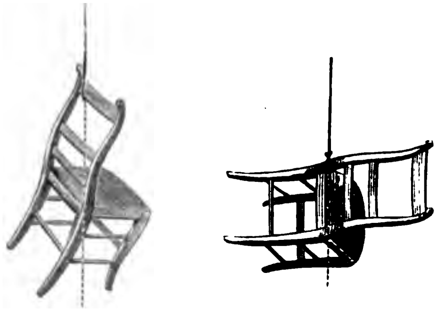


Fig. 34.

point common to the two lines—that is, it must be where they cross. However the body be hung up by a single point, the centre of gravity will always, when at rest, be vertically under or over the point of suspension; that is, the line of the resultant will always pass through the fixed point.

The whole weight of a body, then, may be considered to act at its centre of gravity; in other words, it is as if the whole mass of the body were concentrated at this point.

Determination of the Centre of Gravity by Calculation.

120. The centre of gravity is always the most symmetrical point in a body. In a sphere it is the centre; so it is also in a cube or an ellipsoid, and in a square or circular plate.

In a parallelogram or a parallelopiped (that is, solid parallelogram), it is the intersection of diagonals. In a rod of uniform thickness and material, it is the middle, and so on.

But it is easy to calculate its position in less uniform cases by any process which will determine the position of the resultant of a number of parallel forces, for it is simply the point through which the resultant *always* passes.

Thus let this rod with middle point M (fig. 35), be of weight two pounds, and let a ring A, weighing half a pound, be placed on it four inches to the left of M, and another ring B, weighing three pounds, six inches to the right; then evidently the centre of gravity of the whole must lie somewhere in the length of the rod. To find whereabouts, we need only calculate the position of the resultant of the three weights (the two rings and the rod itself) in any position except the vertical one, say when horizontal. The magnitude of the resultant is plainly $5\frac{1}{2}$. Take moments about any point, say about A; let the resultant act at some unknown point C, such that that AC $= x$. Then we have $5\frac{1}{2}x = 3 \times 10 + 2 \times 4 + \frac{1}{2} \times 0 = 38$; wherefore $x = \frac{76}{11} = 6\frac{10}{11}$; or the point C is $2\frac{10}{11}$ inches to the right of M, and it is the centre of gravity.

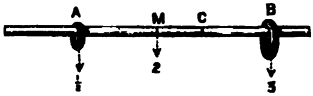


Fig. 35.

Try now taking moments about M, also about B, also about O (anywhere), and see that you always get the same result (when interpreted properly), remembering to allow for negative moments.

It often happens, as in this example, that the line of the resultant in one position of the body (in this case when the rod is vertical) is perfectly obvious.

The arithmetical determination of the position of the centre of gravity of a body therefore, depends on precisely the same principle as the experimental method, and consists simply in finding the line of the resultant in any two positions of the body, and noting their point of intersection. It therefore scarcely needs further exposition; but it is probably necessary to shew how this same principle is applicable to cases rather less obvious.

For instance, to find the centre of gravity of a body made of two parts, each part having a known centre of gravity; say two flat oblong plates, of known weights, w_1 and w_2, joined end to end. G_1 and G_2 being the centre of each separately, their weights may be considered as acting here, and so the resultant passes through a point G, which divides the line G_1G_2 in the ratio $w_2 : w_1$ (sect. 116)—that is, so that $G_1G : GG_2 :: w_2 : w_1$; hence G is the centre of gravity of this combination.

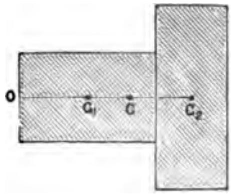


Fig. 36.

Or we might take moments about any point O, and say

$$w_1 \cdot OG_1 + w_2 \cdot OG_2 = (w_1 + w_2)\ OG,$$

whence the distance OG, and therefore the position of G, is determined.

The same method applies if a bit is taken away instead of added on. Suppose, for instance, a square plate with a round hole in it anywhere (fig. 37). The operation of finding the centre of gravity in such a case may be regarded as the same as that of finding the position of the resultant of two contrary forces—the weight W of the whole square acting downwards at G_1 and the weight w of the missing bit acting *upwards* at G_2. The centre of gravity G must evidently be somewhere in the line G_1G_2; so, taking moments about any point O in this line, the equation $W \cdot OG_1 - wOG_2 = (W - w)\ OG$ determines its position.

Or again, the centre of gravity of a trapezium (that is, a quadrilateral with two parallel sides), which may be regarded as a triangle with the top missing, can be found in precisely the same way. The last equation applies as it stands, in fact, provided we know the positions of G_1 and G_2, the centres of gravity of the whole and of the missing triangles (see fig. 38).

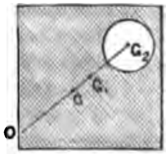


Fig. 37.

The centre of gravity of a triangular plate is in the line joining a vertex to the bisection of the opposite base (because this line bisects every line in the triangle parallel to the base). Three such lines can be drawn, because there are three vertices. Therefore these three lines, joining each vertex to the middle point of the opposite side, meet in a point, and that point is the centre of gravity. It is easily seen to divide each line in the ratio 1 : 2—that is, it is one-third of the way up from the base to the vertex.

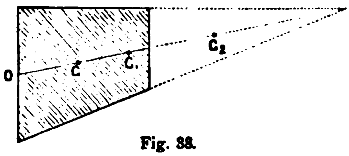


Fig. 88.

EXAMPLES.

1. Find the resultant of two equal forces each equal to 10 units for each of the following cases—namely, when the angle between them is 120°, 90°, 60°, 45°, 30°, respectively.

   Ans. 10, $10\sqrt{2}$ or 14·142, $10\sqrt{3}$ or 17·32, $10\sqrt{(2+\sqrt{2})}$, $10\sqrt{(2+\sqrt{3})}$.

2. Resolve the force 12 into two forces, making angles of 45° with the given force on either side of it.

   Ans. Both equal $6\sqrt{2}$.

3. Resolve a force 20 into two parallel forces, one of them 3 times as far from the given force as the other.

   Ans. 5 and 15, or 30 and — 10.

4. A weightless curtain rod has 4 equal rings on it, so that the 2 end rings are 5 feet apart, and the 2 middle rings are 1 foot apart, one of the end rings being 18 inches from the nearest middle one. Find the centre of gravity. *Ans.* 3 inches from the middle.

5. Where would the centre of gravity in the last question be, if the rod itself were 5 feet long, and weighed twice as much as a ring?

   Ans. 2 inches from the middle.

6. A uniform circular disk has a circular hole punched out of it, extending from the circumference half way to

the centre. Find the centre of gravity of the remainder.

Ans. The diameter of the hole being $\frac{1}{4}$, its area is $\frac{1}{16}$ of that of the whole disk; so the centre of gravity is $\frac{1}{60}$th of the radius of the disk away from its centre, on the side opposite the hole.

CHAPTER VIII.

ON EQUILIBRIUM (*Statics*).

121. Before leaving the subject of motion as affected by force, there is one important part to be considered—namely, the conditions under which forces may act on a body without affecting its motion in any way whatever. One force cannot satisfy these conditions, but a combination of any number of forces greater than one may; and it is interesting, and for many practical purposes important, to be able to specify these conditions, and to decide in any given case whether they are satisfied or not. This part of the subject is called 'Statics,' and it is a branch of the more general science of Dynamics. Its treatment will depend upon the ideas illustrated at length in the last chapter, which may be regarded as an introduction to Statics; indeed, they are usually considered as a part of it, and often are made to follow, or are mixed up with, the subject of the present chapter.

122. When all the forces applied to any mass of matter are so balanced that they produce no acceleration in it of any kind, the forces are (or the body is) said to be *in equilibrium*, and the conditions which they then necessarily satisfy are called the *conditions of equilibrium*.

Observe that equilibrium does not mean *rest* or zero *velocity*, it simply means zero *acceleration*—that is, constant velocity. There is no occasion for the velocity to be nothing; all that is meant is that it keeps the same value, whatever that may happen to be. Thus in the case of a bucket lowered down a well, suppose that it is descending with a constant velocity of 20 feet a second; then, its acceleration being *o*, the resultant force acting on it (being equal to mass-acceleration, sect. 42) must also be *o*. Now the actual forces acting on it are the pull of the earth downwards, and the pull of the rope upwards; and the resultant of these two being zero, it follows that they are equal. Whether the bucket is descending or ascending or standing still, matters nothing, the tension in the rope is always equal to the weight of the bucket so long as its velocity is not *changing*. The conditions of equilibrium are therefore the conditions under which acceleration is impossible; or, as it is often correctly expressed, they are the conditions under which *rest* is *possible*.

123. This being clear, we will proceed to state the conditions of equilibrium for any number of forces, and first of all

The Conditions of Equilibrium for Two Forces.

The conditions which two forces have to satisfy in order to balance each other and have no effect on the motion of the body to which they are applied, are very simple and obvious—namely: (1) The forces must both lie in the same straight line; (2) They must act in opposite directions; and (3) They must be equal.

This is all usually expressed by saying simply that the two forces must be *equal and opposite*, the acting in the same straight line being understood. The phrase *non-concurrent* has been used to express parallel opposition not in the same line, but we have preferred the word *contrary*.*

If any number of forces are in equilibrium, the resultant of any number of them must be equal and opposite to the resultant of all the rest. For obviously all the rest are

* In many text-books parallel forces with the same sense are called *concurrent*, and with opposite sense *non-concurrent*; but Johnson gives as the meaning of the word *concurrent*, 'meeting in a point,' and this appears to be its correct meaning, and it is used in this sense in the 1878 London University Calendar, page 90. Two trains are concurrent, in this sense, not when they run alongside one another, but when they run *into* one another. Words are wanting, therefore, to distinguish agreement from opposition of sense. Consensient or prosensient, and contrasensient are too long; but the abbreviations *pro* and *con* might be used.

equivalent to their resultant, and that resultant is balanced by a force equal and opposite to it.

124. Let us see how this gives us the equilibrium conditions for *three* forces, for instance. Any one force must be equal and opposite to the resultant of the other two. Now any two of them, as A and B, in order to have a resultant, must lie in one plane, in other words, must meet in a point, and through this point their resultant must pass, being the diagonal of the parallelogram of forces; the third force, C, in order to maintain equilibrium, must, by the above statement, be a prolongation of this diagonal, and hence it too passes through the same point as the other two, and is in the same plane—namely, the plane of the parallelogram; it must also be equal to the diagonal in magnitude; in other words, it must be equal to the third side of a triangle, two of whose sides represent the other forces, such as OAR (fig. 39). Its magnitude, direction, and position are thus completely determined.

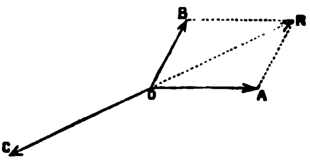


Fig. 39.

Let us restate these:

The Conditions of Equilibrium for Three Forces.

(1.) The three forces must all be in the same plane.

(2.) Their lines of action must all pass through the same point.

(3.) It must be possible to draw a triangle with sides parallel (or perpendicular, see foot-note, sect. 101) to the forces, and proportional to them in magnitude. The sides of the triangle must all be drawn in the same *sense* as the forces (thus in the figure, OA, AR, RO are the senses), and it must be possible to draw the triangle without taking the pen off. This is usually expressed by saying that the three forces must be representable by the sides of a triangle *taken in order*. The last two conditions together really include the first.

Any number of forces greater than three need neither meet in a point, nor lie in the same plane, in order to be in equilibrium.

Conditions of Equilibrium of a Particle.

125. Any number of forces acting on a particle will evidently be in equilibrium if they are representable by the sides of a closed polygon (plane or otherwise) drawn parallel to the respective forces and taken in order.

This is the same as saying that the forces must have no resultant; for the line required to complete the polygon represents the resultant (sect. 98), but no line is required to complete a *closed* polygon, hence there is no resultant.

The converse is also true—namely, that if forces acting on a particle are in equilibrium, they must be representable by the sides of a polygon taken in order. This proposition obviously includes the triangle of forces, for a triangle is only a three-sided polygon.

Conditions of Equilibrium of a Rigid Body.

126. If the condition just stated for a particle is satisfied by the forces acting on a rigid body, they can produce no translation, only rotation; hence a rigid body will evidently be in equilibrium if the above condition for a particle be satisfied, and also if the directions of all the forces pass through a single point; for a set of forces which intersect in one point cannot possibly rotate anything. But this last condition, though *sufficient*, is not *necessary*—that is, the converse is not true: if the forces acting on a rigid body are in equilibrium, they must indeed be representable by the sides of some closed polygon (plane or otherwise), but they need not meet in a point. The more general condition for no rotation is that the moments of all the forces about every possible point or axis of rotation must add up to zero.

If this and the particle condition are satisfied, equilibrium is complete; and conversely, wherever there is equilibrium, these must be satisfied. So these are the necessary and sufficient conditions, though not in a very simple form to apply practically. It will be sufficient for us, however, to consider at length, and put into a more practical form, only the case where all the forces act *in one plane;* and we will proceed to this from a fresh point of view. (Compare this limitation with that introduced in Chapter VII., sect. 96.)

General Conditions of Equilibrium of a Rigid Body acted on by Forces in a Plane.

127. The motions possible to a rigid body are translation or rotation or both, hence the conditions for equilibrium really involve the conditions for no translation and for no rotation (strictly speaking, for no rectilinear and for no angular *acceleration;* but the words translation and rotation are used instead of these more accurate terms for shortness; and the error is not great, for the conditions of equilibrium render entire rest *possible*, though they do not in any way enforce it).

Now, having assumed that the body can only move in a plane (say a vertical plane), and that the forces only act in this plane, it is obvious that all translations must be up or down, or right or left, or else a motion compounded of the two, which may be analysed into up or down and right or left components. Hence, in order that there need be no translation at all, the forces must have no resultant either up or down or right or left: this being a practically convenient form of saying that they have no resultant at all at a finite distance. Still, however, they might spin the body (sect. 117); hence, in addition to the above, the condition is necessary that the sum of their moments about every point in the plane must vanish; and then the forces will be unable to cause any motion at all.

So *the general conditions of equilibrium for a body moving in a plane* are:

(1.) That the sum of the components of all the forces in any two directions at right angles to each other shall vanish.

(2.) That the sum of the moments of all the forces about any one* point in the plane shall vanish.

(1) is the condition for no translation (properly speaking, for no rectilinear acceleration).

(2) is the condition for no rotation (properly speaking, for no angular acceleration).

If (1) is satisfied without (2), there is rotation, but no translation.

If (2) is satisfied without (1), there is translation, but no rotation.

If neither is satisfied, there must be both translation and rotation.

If both are satisfied, there must be complete equilibrium.

The converse of each of these statements is also true.

In case the body on which the forces act has one point fixed so as to be incapable of translation, the necessary and sufficient condition for equilibrium is simply that the resultant of all the forces must pass through the fixed point or pivot (see sect. 108, statement 3).

ILLUSTRATIONS.

128. Consider a ladder standing on rough ground, and resting against a perfectly smooth wall. What forces are acting upon it? There is the weight of the ladder W acting downwards at its centre of gravity; there is the pressure of the ground R acting in some unknown upward direction (fig. 40), and the pressure of the wall P acting normal to the wall (sect. 104) or horizontally; and that is all. But the ladder is in equilibrium, hence these three forces must pass through a point (sect. 124).

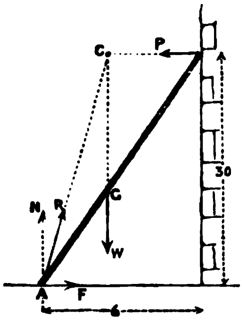


Fig. 40.

Now W and P, whose direc-

* *One* point is sufficient because the moment of a couple about every point is the same (sect. 117); hence, if it is zero about any one point, it is zero altogether.

tions are known, intersect when produced in the point C; hence R also passes through the point C (fig. 41).

This determines its direction.

Moreover, when three forces are in equilibrium, they must be proportional to the sides of any triangle which are drawn respectively parallel to the forces.

Such a triangle is ABC (fig. 41); CB is parallel to W, and represents it; BA is parallel to P, and represents it; and AC is parallel to R, and represents it. If, then, the position of the ladder were given us, and also its weight, we should simply have to draw the above diagram, and measure the sides of the triangle ABC, in order to determine the pressures P and R in terms of W; the direction of R being also given by measuring either the angle BAC or BCA.

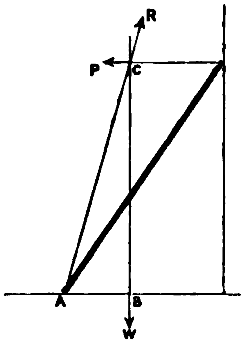


Fig. 41.

This would be solving the problem by construction.

129. But suppose we wished to do it by calculation, applying the general conditions of sect. 127: we should first consider the inclined force R resolved into two, a normal pressure N, and a friction F (the friction being always in such direction as best hinders slipping, sect. 104), and then say that, since there is equilibrium as regards translation, there can be first no up or down resultant, or N and W must be equal and opposite; and then that there can be no horizontal components, or F and P must be equal and opposite.

But to determine either F or P, in terms of W, we must make use of the second condition—the condition for no rotation—namely, that the forces can have no rotating power, or resultant moment, about any point. Take it numerically:

Let us suppose that we are told the weight of the ladder is 60 lbs., and that its centre of gravity is $\frac{1}{3}$ of its length up, that the foot of the ladder stands six feet from the wall, and the top of the ladder thirty feet from the ground; then, as the condition for no translation, we have already found

$$N = W = 60g,$$

and

$$F = P.$$

But we don't know either F or P yet; we must find them by taking moments about some point—any point we like, for we know that since there is no rotation the sum of the moments about every point must add up to zero.

Suppose we take moments about the point A, then neither N nor F has any moment; so the moment of P, $P \times 30$, must be equal and opposite to the moment of W, $W \times \frac{1}{3} \times 6$, or $2W$;

hence

$$15P = W = 60g,$$

or

$$P = 4g, \qquad \text{the weight of 4 lbs.}$$

And we already know that F and P are equal; so then N, F, and P are all known, and now too we know R, because $R^2 = N^2 + F^2$—that is, $R = 60 \cdot 13g$, or the weight of 60·13 lbs.

See if this agrees with a determination by measurement, and then repeat the whole process with the wall rough instead of the ground, and then with both wall and ground rough.

130. Next consider a weightless rod resting against a smooth wall over a smooth rail, and with a weight stuck somewhere on it, as shewn in fig. 42. (The *end* only of the rail supporting the rod is shewn as a small circle.) To determine where the weight must be for equilibrium. The forces acting are: the weight, W; the normal pressure of the wall, P; and the normal pressure of the rail, R.

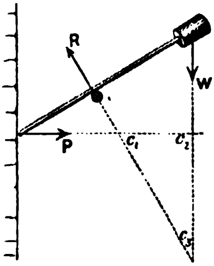


Fig. 42.

Now, here again are three forces, so to be in equilibrium they ought to intersect in a point; but in fig. 42 they *do not* intersect in a point, produce them as much as you like; their direction incloses a triangle $c_1 c_2 c_3$

instead. Hence there is no equilibrium,* and W must be shifted until the three points c_1, c_2, c_3, coincide in one point c.

To find where it ought to be shifted to, draw a fresh figure, and from C, the intersection of P and R, draw a vertical; this will cut the rod at the point where the weight ought to be for equilibrium (fig. 43).

To measure the relation between the magnitudes of the forces when the weight is in this place, we can produce R till it cuts the wall in B; then the triangle ABC has its sides parallel to the three forces,

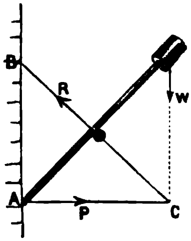


Fig. 43.

BA to W,
AC to P,
CB to R;

hence the length of these sides will give the forces, if one of them, say W, is known.

It is easy to see that $R^2 = P^2 + W^2$.

In the former figure the rod would be slipping up the wall and falling over the peg; this is because the line of W falls to the right of the point C, when P and R intersect. If the line of W fell to the left of this point, the rod would slip down the wall, and drop between it and the peg. There is just one position where it does not slip either way.

131. Now consider a body on an inclined plane held still by some force P acting in any given direction. There are three forces, P, R, and W, in equilibrium (R being the normal pressure of the plane), hence P must be in the plane of the other two. To find its magnitude: take off a length AB to represent the weight of the body, and from B draw a line parallel to P, till it cuts R produced in the

* Here is an instance of forces represented in magnitude and direction by the sides of a triangle (c_1 c_2 c_3), and yet not in equilibrium, the reason being that they act on a rigid body instead of a particle, and so their positions are not necessarily right. Such forces, however, can only produce rotation (sect. 126), and hence can be reduced to a 'couple' (wherefore the resultant of R and P is a force equal and contrary to W), the moment of the couple being twice the area of the triangle c_1 c_2 c_3. The moment of the couple in such cases is *always* twice the area of the polygon, which represents the forces not only in magnitude and direction but also in *lines of action*.

point C. Then we have a triangle ABC (fig. 44); BC represents P, and AC represents R; and it is easy to measure these lengths on the same scale as AB was drawn.

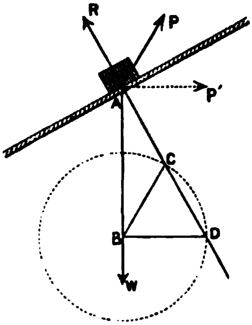


Fig. 44.

There are in general two positions in which the same force P can hold up the body. For draw a circle with centre B and radius BC, it will cut R in two points, C and D; hence the same force P would be just as effective if it acted in a direction shewn dotted as P′ parallel to BD; but the pressure on the plane would then be greater than AC, namely, AD.

There is one case when only one direction will do, and that is when the radius of the circle is so small that it only just *touches* R. This radius then represents the minimum force possible, and shews that it must act perpendicularly to R and therefore parallel to the plane, and must have the same relation to the weight that the height of the plane has to its length. If the plane be rough, the friction is such a force.

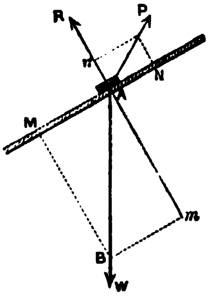


Fig. 45.

If the force P is smaller than this—that is, so small that the line representing it is unable to reach across from B to the line R, then there cannot be equilibrium; and even if P is greater than this, but does not act in the best direction, there need not be equilibrium, and the body will slide down, as in fig. 45: the accelerative force being the component of W along the plane, namely AM, minus the component of P along the plane, namely AN. The pressure on the plane is the component of W at right angles to the plane, minus the component of P at right angles to the plane; that is, A*m*—A*n*.

132. We have only here considered the slipping of the body; but if it were a ball it would roll, and if it were a block it might topple over, before it began to slide. Let us just see how soon a rectangular block on a rough inclined plane will topple over.

Stability of Equilibrium.

We know that the resultant of all the forces which gravity exerts on the particles of the body passes through the centre of gravity—that is, the body acts statically as if its weight were all concentrated at the centre of gravity. Hence if this point be supported, the whole body is supported. The line of W is the vertical through G; and if this line falls inside the base* it cannot topple over; it can only slide down. To upset the body, it must be tilted through the angle AGO (fig. 46); and if it be momentarily tilted through less than this, it will return to its old position. The equilibrium is therefore said to be *stable;* and the angle AGO is a measure of the 'stability:' the larger this angle, the more stable is the body. If the vertical through G fell outside the point O, there could be no equilibrium at all, but the body would topple over; and this applies universally.

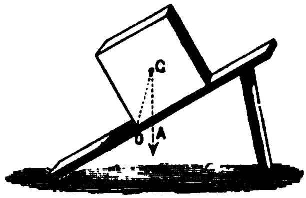


Fig. 46.

A waggon going along with one wheel in the gutter does not upset so long as the vertical through its centre of gravity falls inside the wheel-base; but the act of going over a stone may tilt it sufficiently to make this line pass beyond the base, and then it upsets.

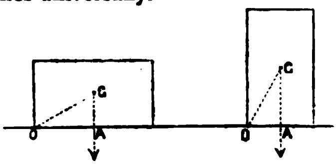


Fig. 47.

* By 'the base' must be understood the area inclosed by a string stretched round that part of the body which touches the plane: consider, for example, the case of a retort stand with a forked foot.

The two bodies in fig. 47 resting on a flat plane are both evidently in stable equilibrium, but the stability of the first is much greater than that of the second; and this for two reasons, firstly, because its base is wider, secondly, because its centre of gravity is lower.*

The centre of gravity of an omnibus full outside, but with no inside passengers, must be very high up; and a moderate shock might be sufficient to destroy its stability and upset it.

133. A body in equilibrium with infinitely small stability, is said to possess *unstable* equilibrium; the least shock must upset it.

Thus, if you narrow the above block till its base is nothing, there remains only a plane or line standing on its edge, and though, when vertical, the centre of gravity of this does not fall without its base, and therefore it is in equilibrium, yet the slightest breath will upset it.

A pyramid or cone standing on its base has very stable equilibrium; but on its vertex, very unstable.

It is quite possible for a body to possess an equilibrium which is neither stable nor unstable—that is, the body, when disturbed, neither topples over nor returns to its original position. All that is necessary is that the vertical through G shall *always* pass through the point of support, as in the case of a sphere on a flat table; or that the centre of gravity itself shall be supported, as in a fly-wheel. The body will then remain steady, however you place it, and its equilibrium is called *neutral*.

A cone or cylinder lying on its side has neutral equilibrium.

134. In the case of a body balanced on a point, if the point is *above* the centre of gravity, the equilibrium is perfectly stable; if *at* the centre of gravity, it is neutral; and if *below*, it is unstable.

EXAMPLES.—The nearer the centre of gravity of the beam of a

* The most useful measures of stability are 1st, *the moment of stability*; namely, the moment of the couple required to upset the body, or the weight of the body multiplied by the distance OA; and 2d, the *dynamic stability*, namely, the work that is required to upset it, or the weight of the body multiplied by the difference of the distances AG and OG (fig. 47).

balance is to the point of support, the more sensitive is the balance; but it is necessary to have the centre of gravity slightly lower than the point of support, or the equilibrium would not be stable.

A compass needle is always made with a little central cap, into which the point supporting the needle passes from below, so as to be above the centre of gravity of the needle. See fig. 48.


Fig. 48.

Again, it is easy to balance a curved beam on a knife edge, while a straight one will not remain steady for more than a few seconds, unless loaded. Compare the diagrams in fig. 49. The weights in the third must be attached to the beam by rigid wires, not by strings.


Stable. Unstable. Stable.
Fig. 49.

135. In the case of a body with a spherical base standing on a level plane, its centre of gravity cannot help being above the point of contact with the plane, and yet the equilibrium may be stable or neutral; as, for instance, in a sphere the equilibrium is neutral, and in a hemisphere it is stable; or again, it may be unstable, as in an egg balanced on one end.

The centre of the sphere, of which the base forms a part, is in these cases to be regarded as the real point of support, and then the former rules apply. Thus, if G be the centre of gravity of the combination shewn in fig. 50, and if C be the centre of the sphere of which the base forms a part, the whole will oscillate in stable equilibrium.

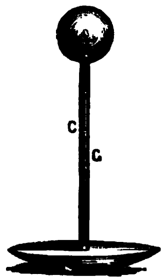


Fig. 50.

When a body rolls along any surface, its centre of gravity in general describes a curve with crests and hollows; every hollow corresponds to a position of stable equilibrium (the centre of gravity is then in one of its lowest positions); every crest corresponds to a position of unstable equilibrium, and a measure of the *instability* is the curvature (see sect. 13) of the path of the centre of gravity. For instance, in the case of a body balanced on a point, the higher the centre of gravity above the point the less curved will be its path, and the less unstable will be the equilibrium: for example, it is easy to balance a stick loaded at one end on one's finger if the load be at the top of the stick, but if the stick be inverted it is not easy.

EXAMPLES.

1. When a weight is supported on an inclined plane by a force acting along the plane, shew that the ratio of the force to the weight is the same as the ratio of the height of the plane to its length.
2. And shew that the ratio of the supporting force to the normal pressure on the plane is the same as the ratio of the height of the plane to its base.
3. Hence shew that if a body is supported on a plane only by friction, it will begin to slide down when the ratio of the height of the plane to its base is equal to the coefficient of friction (see sects. 131 and 104).
4. A picture-frame weighing 10 lbs. is hung by a cord passing over a nail, the two parts of the cord making an angle of 120° with each other. Find the tension in the cord. *Ans.* 10 lbs. weight.
5. If the two parts of the cord included an angle of 90°, what would then be the tension? *Ans.* $5\sqrt{2}$ lbs. weight.
6. If a rod rests inside a smooth spherical shell, its centre of gravity must be vertically under the centre of the sphere. Hence, if the rod be uniform, it can only lie horizontally, unless it is equal in length to the diameter of the sphere.
7. It is wished to upset a tall column by means of a rope of given length, pulled by men on the ground; at what height above the base of the column will it be best to attach the rope?

   Ans. At a height $\frac{1}{\sqrt{2}}$th of the length of the rope, because then the perpendicular distance of the rope from the base of the column will be greatest, and therefore the moment of any stress in the rope about it will be a maximum.

8. A uniform pendulum-rod is pulled aside from the vertical by a horizontal force equal to ½ its weight applied at its lower end; at what angle will it be in equilibrium? *Ans.* 45°.

CHAPTER IX.

ON MACHINES AND OTHER CONTRIVANCES ILLUSTRATING FOREGOING PRINCIPLES.

136. A machine is an instrument for transferring energy in such a manner that certain useful or desirable work is done. The agent which loses the energy used often to be called the 'power,' the body which receives it being called the resistance or the 'weight.' The machine is simply a mediary by which the energy is indirectly transferred from one body to the other.

The quantity of energy gained by the one body is equal to that lost by the other; in other words, no change in quantity of energy is ever effected by any machine.

Numerous attempts have been made to construct a machine able to effect this: such attempts are called the search after perpetual motion, and always result in failure (cf. sect. 79). All that one can do by means of any machine is to vary the ratio of the two factors, F and s, occurring in the product *work*, the product itself remaining unalterable. But just as the number 12 may be split up into various pairs of factors, 12 and 1, 6 and 2, 3 and 4, or more generally $x\sqrt{12}$ and $\frac{\sqrt{12}}{x}$, where x may be any number, whole or fractional; so the factors of the constant product work may be varied at will: and this is the use of a machine. Given a force, and a distance through which it can act, a machine can always be devised to overcome any other force whatever through some definite distance, such that the product of the first force and distance is equal to the product of the second force and distance. The greater the force required to be overcome, the smaller the distance is through which it can be overcome by a given force. This is often expressed by saying that what is gained by any machine in power is lost in time (or in distance). Or again, by saying that the 'mechanical advantage' of a machine—the ratio of the resistance overcome to the least force required—is equal also to the ratio of the distance travelled by the 'power' to the distance travelled by the 'weight.'

This condition may also be expressed by saying that if any system in equilibrium under the action of any number of forces receive a slight displacement, then the total work done by the whole of the forces, or the total loss of potential energy, is zero. In other words, the sum of the products of all the forces into the respective distances they have simultaneously moved, or, what is the same thing, into the respective velocities of their points of application measured along their lines of action, is zero. This is frequently a useful mode of finding the condition of equilibrium of a system, and shall be applied in a very elementary form in the present chapter.

It is often referred to as the principle of *virtual work* or *virtual velocities*; the meaning of the word 'virtual' being merely that the displacement or shift supposed to take place is an imaginary one, and need not really occur.

Simple Machines.

137. A pulley is a simple machine by which a weight may apparently be supported by means of a force only half as great as itself; the obvious reason being that the other half of the force necessary to support the weight is supplied by the hook fixed in the ceiling, to which one end of the cord is attached (fig. 51). If the force P exceeds in the slightest degree half the weight, it must raise it; but only half as fast as itself descends. To raise it at the same rate would require *both* parts of the loop of cord in which W is slung to be lifted. If only one end is lifted, the wheel or pulley rotates, and W only rises at half the rate.

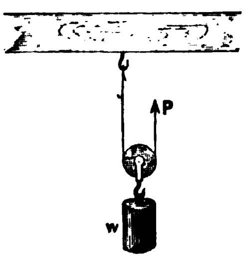


Fig. 51.

The *mechanical advantage* of a simple pulley is thus 2.

An inclined plane is another simple machine on which a weight may be apparently supported by a force less than its own weight; the reason being that the rest of the necessary force is supplied in a stationary manner by the pressure of the plane. If the sustaining force or 'power,' P, is applied as shewn in fig. 52, it is evident that a descent of P through a vertical height l, equal to the whole length of the plane, would pull W all the way up the plane indeed, but would only *raise* it a vertical height h; hence the mechanical advantage of this machine is $\frac{l}{h}$; and if P exceeds $W\frac{h}{l}$ in the slightest degree, it must raise the weight; provided, of course, that there is no friction.

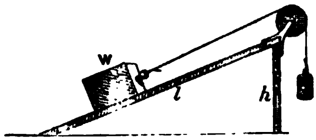


Fig. 52.

A lever, a wheel and axle, and a capstan are simple machines in which a weight applied at a great distance from an axis of rotation may apparently support a greater weight nearer the axis; the reason being that the rest, or the whole, of the sustaining force is supplied by the support of the axis, or the *fulcrum*.

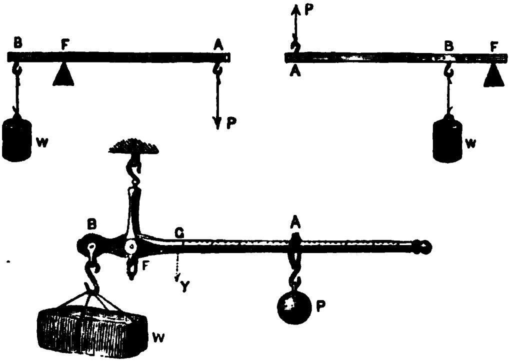


Fig. 53.

Thus, in two diagrams of fig. 53, P and W are both really supported by the fulcrum F; the pressure on it being always W + P, if the plus be understood algebraically. All that P does is to balance the *rotation* tendency of W; and for this purpose its moment, P × AF, must equal the moment of W, W × BF. Hence the mechanical advantage of a lever, the ratio of W to P, is always $\frac{AF}{BF}$, or the $\frac{\text{'power's arm'}}{\text{'weight's arm'}}$.

In the case of the steelyard (fig. 53), the weight of the 'yard,' Y, acting at its centre of gravity, helps the power, so that W . BF = P . AF + Y . GF.

A lever cannot, however, be used to raise weights *far;* but an easy modification, securing continuous action, is to make the fulcrum F into a pivot, and to apply P and W at the circumference of circles or wheels, with common centre F. Thus we get the wheel and axle, or capstan (fig. 54), of which the mechanical

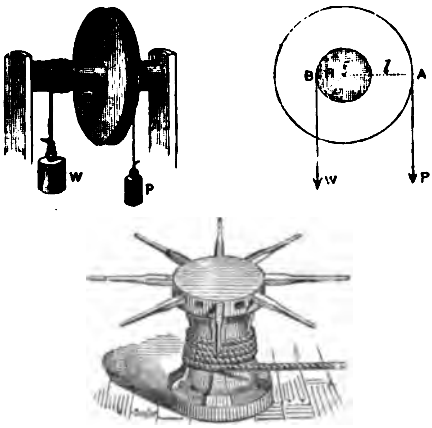


Fig. 54.

advantage is, as before, the ratio of the distance of P from the pivot to the distance of W from the pivot—that is, the radius of the wheel divided by the radius of the axle.

Combinations of Simple Machines.

138. Any of these machines may be combined together, so that the resistance of one machine constitutes the 'power' of the next, and the mechanical advantage of the combination will be the product of their separate mechanical advantages.

Thus three pulleys are shewn combined in fig. 55, and the mechanical advantage of the combination is 2 × 2 × 2, or 8, if the pulleys are weightless. If W is raised one foot, P must rise eight feet. The whole pull on W is here the pull of the beam above plus P; hence the pull on the beam is W − P. The arrangement may evidently be turned upside down, so that the beam becomes the weight, and the weight the beam (fig. 55). In this

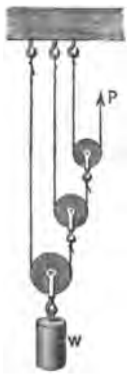


Fig. 55.

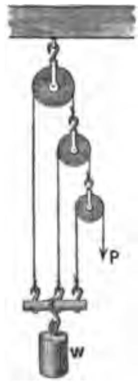


Fig. 56.

case the weight supported is less by P than it was in the former case. Fig. 55 is often referred to as the first system of pulleys, and fig. 56 as the third.

If the weight of the pulleys is not small enough to be neglected, call them w_1 w_2 &c., and consider fig. 55. The lowest pulley is attached to the weight, and rises at the same rate as it does; the second pulley rises at twice, and the third at four times this speed. Now, if any weight w be raised a height h, the work done is wh: so if W is raised one foot,

$$W + w_1 + 2w_2 + 4w_3$$

represents the whole work done by P, in moving through a distance of 8 feet, that is, by the expenditure of 8 P units of potential energy; hence, in general when there is equilibrium, the mechanical advantage W : P must be determined from the equation,

$$2^n P = W + w_1 + 2w_2 + 4w_3 + \dots + 2^{n-1} w_n,$$

if there are n pulleys. An equation which expresses the fact that the algebraical total of the work done is nothing.

The only system of pulleys frequently employed for hoisting is what is called the second system, where there are two blocks of pulleys, one attached to the weight, and the other to the beam; and where the same rope passes round all (fig. 57). The mechanical advantage in this case is simply equal to the number of strings supporting the weight: which in the figure happens to be four.

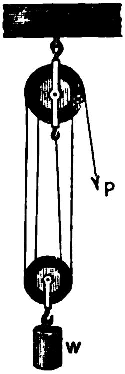


Fig. 57.

139. A combination of levers is sometimes used, but more often for the purpose of magnifying small motions than for exerting great force; that is, for increasing the factor s in the product work at the expense of the factor F. In fig. 58 the motion of the screw is magnified, the pointer describing a considerable arc for one turn of the screw. A lever and an inclined plane may be combined together into a screw-press, the inclined plane being coiled up into a spiral or screw-thread (fig. 59). For every complete revolution of the lever, the weight is raised a distance equal to that between the spires of the screw-threads; hence the mechanical advantage of such a press is

Fig. 58.

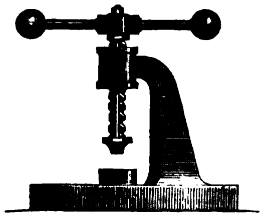


Fig. 59.

the circumference of the circle traversed by the force applied at right angles to the lever, divided by the distance between successive spires of the screw. Wheels and axles are usually combined by means of cogs, as is well seen in the wheel-work of a clock.

A pulley is often used in conjunction with a capstan, the rope passing round a pulley attached to the weight, and the mechanical advantage of the capstan is thereby doubled. Moreover, the free end of the rope, instead of being rigidly fixed, may be coiled round another smaller axle with the same centre F, so that its tension shall help the force P (fig. 60). By this means the mechanical advantage can be increased to any desired extent, for the weight is now wound up only because the cord wraps itself on to one, the larger, axle faster than it unwraps itself from the other smaller axle; and the two axles may be as nearly the same size as one pleases. The mechanical advantage is the radius of the wheel (or the length of P's arm) divided by the difference of the radii of the two axles, the whole being multiplied by two because of the pulley.

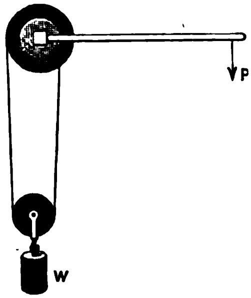


Fig. 60.—Chinese Capstan.

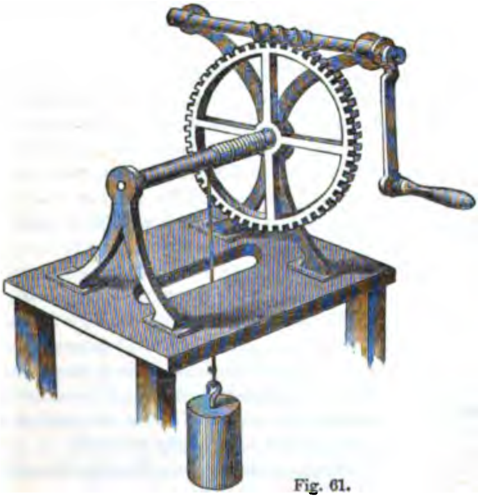


Fig. 61.

A wheel and axle may be combined with a screw, as shewn in the contrivance of fig. 61. When the handle is turned, the screw-thread on its axle sends the cog-wheel forward one tooth for every revolution. Such a screw, which itself does not advance in a nut, but which merely rotates in ordinary bearings, is called an 'endless' screw. If l is the length of the handle arm, n the number of teeth in the wheel, and r the radius of the axle on which the rope winds itself, the mechanical advantage of the whole machine is $\frac{2\pi nl}{2\pi r}$ or $\frac{nl}{r}$.

140. To drive a machine an agent must expend energy upon it, and its rate of expenditure of energy is called its 'power.'* But when the agent is inanimate (like running water or compressed steam), its utilised power is often spoken of as the power of the machine driven by it. The *power* of a machine, then, means its *rate of doing work;* in other words, it equals the work done in any short time divided by that time—so many foot-poundals per second. A machine is said to have one 'horse-power' when it can do 17,600 units of work every second; which is equivalent to raising 33,000 lbs. of matter one foot high against gravity every minute.

Pendulums.

141. *Conical Pendulum and Governor Balls.*—Let AB (fig. 62) be a vertical axis of rotation, and P a massive ball at the end of an arm AP, capable of rotation about this vertical axis and pivoted at A; then it is well known that AP will fly out from the vertical more and more as it revolves faster and faster. Let it be revolving with a constant angular velocity ω, and let it perform every revolution in T seconds, so that $2\pi = \omega T$.

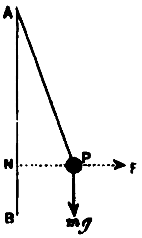


Fig. 62.

The centripetal force which must be acting on P in the direction PN to keep it moving in the circle (sect. 56) is $m\omega^2 r$, where r is the radius PN of the circle in which P moves; and if the rotation were to cease, this is the force which must be applied in the opposite direction PF, in order to keep the ball in its position without letting it fall back to the axis AB.

Hence in the diagram (fig. 62), we may regard P as stationary and in equilibrium under the action of three forces—the force $F = m\omega^2 r$, its weight $W = mg$, and the tension in its supporting arm. The

* Or sometimes its *activity*. The word 'power' is frequently used to express the maximum activity of which an engine is capable. Its use to denote a force applied to a lever, is simple misuse.

triangle APN has its sides parallel to these forces, and hence represents them; so, calling the vertical distance AN, h, we have

$$m\omega^2 r : mg :: r : h;$$

or

$$h = \frac{g}{\omega^2};$$

that is, the vertical distance of the governor ball below the pivot A, is inversely proportional to the square of the angular velocity of rotation.

The time of one revolution is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$; and such an arrangement is sometimes used as a measurer of time, when it is called a 'conical pendulum,' because the arm AP traces out a cone.

If the radius of the circle in which P moves is very small, the height h is practically equal to the length of the pendulum, AP, which we will call l. Moreover, if you try swinging a weight at the end of a string, you will find that the time of a complete *small* motion is the same whether the pendulum simply oscillates in a nearly straight line or whether it revolves in a horizontal circle; that is, the time of an oscillation (to and fro) of a simple pendulum equals the time of rotation of a conical one, provided the motion of both is small; and each period is very approximately

$$2\pi\sqrt{\frac{l}{g}}.$$

By a *simple* pendulum is meant one about whose length there can be no ambiguity. It is a heavy *particle*, swinging at the end of a perfectly light cord attached to a fixed point (cf. sect. 73).

142. *Compound Pendulum.*—The time of oscillation of a compound pendulum, that is of a rigid body of any size fixed at one point O, and swinging slightly under gravity, may now be calculated.

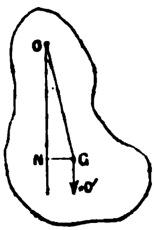


Fig. 68.

Let G be the centre of gravity of the mass, and call the distance OG, a; the small angle of displacement from the vertical NOG, call θ; and the distance NG, call x; the latter is practically equal to $a\theta$, the arc of a circle with centre O.

Then, if m be the mass of the whole body, the force restoring the body to its position of equilibrium is mg acting at G, so that its moment about O is

mgx; and the angular acceleration produced by this is (see sect. 52)

$$\alpha = \frac{mgx}{M} = \frac{mga\theta}{M},$$

where M is the moment of inertia of the body about the point O. For the particular case of a simple pendulum when the whole mass is concentrated into a particle at G, and when $a = l$ and $M = ml^2$, this equation becomes

$$\alpha = \frac{mgl\theta}{ml^2} = \frac{g\theta}{l}.$$

Now we can choose a simple pendulum of such length that its angular acceleration at every instant, and therefore its whole motion, is the same as for the compound pendulum. Let L be the length of such an equivalent simple pendulum, then the equation

$$\frac{mga\theta}{M} = \frac{g\theta}{L}$$

is satisfied; and the length of the equivalent simple pendulum (sometimes called the 'length' of the compound pendulum itself, see sect. 73) is

$$L = \frac{M}{ma}.$$

But the time of a small oscillation of this simple pendulum is

$$2\pi\sqrt{\frac{L}{g}};$$

therefore the time of a small oscillation of the compound pendulum is

$$T = 2\pi\sqrt{\frac{M}{mga}},$$

where M stands for its moment of inertia Σmr^2 about the centre of suspension O, and a is the distance between this point and its centre of gravity.

The above equation $M = maL$ gives a simple means of experimentally determining the moment of inertia of any body about any point. Hang it up by this point and measure a, the distance from it to the centre of gravity; then set it swinging slightly, and observe the length of a simple pendulum which keeps time with it: multiply the product of these two lengths by the mass of the body (in lbs. or grammes), and you have its moment of inertia under those circumstances.

A point O′ in the body at a distance L from the centre of suspension O, is called the *centre of oscillation*, because the body oscillates as if all its mass were concentrated there. It may be easily shewn that the body will swing in just the same period if suspended at this point as if it were suspended at O. For it can be shewn, if M be its moment of inertia about a point O at a distance a from G, and M′ its moment of inertia about a point O′ at a distance a', such that $a + a' = L$, that then $M : a = M' : a'$; or the length L is the same for both points.

The centre of oscillation O′ is also sometimes called the *centre of percussion*, because this is the place where the body strikes things best without any jar on its support. A cricket bat drives the ball best if it strikes it at this point, and it does not then jar the hand.

143. *Ballistic Pendulum.*—A heavy block of wood hung up as a pendulum by two strings so that it can swing without any rotation, is sometimes used to measure the impulse (mv) of a blow, such as that of a rifle bullet fired into the wood. The block will be displaced and will rise a vertical height, h, which must be observed (either directly or by calculation from the angle of swing); and, if the mass of the block be m', the velocity v' imparted to it is measured as $\sqrt{(2gh)}$. The velocity v with which the rifle bullet struck the wood can then be found if its mass m is known, from the equations,

$$mv = (m + m')v'$$

and

$$v' = \sqrt{(2gh)}.$$

EXAMPLES.

1. Apply the principle of 'virtual velocities,' to determine the condition of equilibrium of a body resting on a rough inclined plane.

   The principle is that, if the body receive a slight displacement, the total work done must be zero. The limiting condition required is given in Ex. 3, Chap. VIII.

2. Shew that a body on a plane tilted to the 'angle of repose' (see sect. 104) is on the point of sliding.
3. If a hundredweight be hung on to the hook W in fig. 55,

what force P is required to support it, the pulleys being weightless?

Ans. 448 poundals, or a weight of 14 lbs.

4. If each pulley weighed 4 lbs., what force would be necessary? *Ans.* 17½ lbs. weight.

5. In fig. 56 shew that if the pulleys are weightless the mechanical advantage is 7; but that if they each weigh $\frac{1}{14}$th as much as the weight, then the mechanical advantage is 8¼.

6. If in fig. 57 W = 20 lbs. and P = 6, find the velocity of W when it has risen one foot, neglecting friction.

Ans. 1·486 nearly.

7. Find also the accelerations of W and of P, and the time required for P to descend 16 feet.

Ans. Acceleration of $W = \frac{1}{29}g$; acceleration of $P = \frac{4}{29}g$; time = 7¼ seconds.

8. If a weight be attached to a string 4 feet long, and is then caused to describe a horizontal circle, so that the string is inclined at 60° to the vertical, find its angular velocity, its actual velocity, and the time of one revolution.

Ans. $\omega = 4$; $v = 8\sqrt{3}$; $t = \frac{1}{2}\pi$ seconds.

9. A chair weighing 20 lbs. is hung by a point 2½ feet from its centre of gravity, and is found to oscillate in precisely the same way as a simple pendulum 3 feet long. Find the moment of inertia of the chair about the point of suspension. *Ans.* 150.

10. Find the time the chair would take to complete a small oscillation. *Ans.* $\frac{\pi}{4}\sqrt{6}$ seconds.

11. A one-ounce rifle bullet is fired into a suspended block of wood weighing 30 lbs.; if the blow causes the wood to rise a vertical height of 1⅓ inches without any rotation, find the velocity of the bullet just before it struck the wood. *Ans.* 1280.

12. Find the correct position of the weight W in fig. 56, so that the rod on which it hangs may be horizontal. (The figure is not quite correct.)

CHAPTER X.

ON PROPERTIES AND STATES OF MATTER.

(*Rudiments of Elasticity, and Introduction to Fluid Mechanics.*)

144. The particular kind of effect which a given force will produce in a given piece of matter when it does work on it, depends not on the nature of the *force*, for forces can only differ in amount and not in kind, but on the nature of the *matter*. Matter exists in various states, and has very different properties in each state; and though the principal effects of work, or forms of energy, may be summed up, as stated in the introduction (sect. 5), under the heads Motion and Strain, yet the kind of motion and the kind of strain produced in different sorts of matter may be very different; and we must now proceed to consider briefly some of the peculiar properties possessed by matter in its different states; inertia and apparently gravitative attraction being properties common to all.

145. Hitherto we have only considered matter in a rigid form insusceptible of strain, but it is time now to say what little can be said in so elementary a book on the production of strains in non-rigid matter by the action of forces.

Strain means either change of size or change of shape.

Change-of-size strain is called *Compression* or *Dilatation*, and the active resistance of matter to it is called *Elasticity of Volume*, or *Incompressibility*.

Change-of-shape strain is called *Distortion*, and the active resistance to it is called *Elasticity of Figure*, or *Rigidity*.

The adjective 'rigid' is applied to all bodies which strongly resist *any* kind of strain; but the term 'rigidity' is used to denote the measure of the resistance to change of *shape*, while the term 'incompressibility' represents the measure of the resistance to change of *size*.

146. Bodies with high rigidity are called *Solids*. The incompressibility of solids is generally still greater than their rigidity. Cork, however, is an exception to this rule. By the term 'rigid body' in previous chapters, we have always meant a *perfectly* rigid solid. Such a solid it would be impossible to strain by any finite forces; all its particles would maintain their relative positions unchanged, unless the body were *broken*—for this would be possible; perfectly *rigid* does not mean perfectly *strong*.

Such a solid does not exist, though it is approximated to by rocks and metals. All actual solids are capable of being strained—that is, they all yield somewhat to the action of external forces applied to them; and they are divided into two extreme classes, according to the *way* in which they yield.

They may yield *actively;* the stress exerted by their particles in opposition to the distorting force continuing constant, no matter how long that force is applied, and restoring the body to its old shape the instant the distorting force is removed, without the least *permanent* strain or *set;* in which case they are called *perfectly elastic*. Glass and steel are practically so.

Or they may yield *passively;* passing into any shape without exerting any *continuous* stress in opposition to the distorting forces, and therefore not recovering their form at all when these forces are removed. In this case they are called *perfectly plastic* or *inelastic;* putty, wet clay, and dough are practically so.

Most solids (strictly speaking, *all* existing ones) lie between these two extremes; they have a certain amount of elasticity combined with a certain amount of plasticity, partly yielding permanently and partly springing back; as you see at once if you bend iron, wood, paper, &c.

147. A great number of things are elastic when the distorting forces are small, but experience a 'set' when

they are too great. These are said to be elastic between certain limits, called the *limits of elasticity*. If strained above those limits, they are more or less plastic, and if still more strained, they are torn asunder or broken. The greatest longitudinal stress (sect. 151) which a material can bear is called its *tenacity*.

148. When a solid is strained, both its elasticity of volume and its elasticity of figure are called out, for both size and shape usually change. For instance, if you stretch a piece of india-rubber, it alters greatly in shape, but it also expands a little. The strains practically produced in solids may be conveniently considered under the heads of—(1) *longitudinal elongation* or compression; and (2) *shear*.

The first is produced when a rod is either stretched or squeezed lengthways by a simple stress, and the elasticity involved is called *longitudinal rigidity*, or sometimes *Young's modulus of elasticity*.

Shear is produced by couples, as when you twist a rod or cut anything with a pair of scissors. It involves the sliding over one another of parallel planes in the body—thus a book is sheared when its top cover is either pressed sideways or turned round, while its lower cover is held still. The sliding of the parallel planes (or leaves of the book) is then well seen, especially if you use a thick book like a London Directory. There is in a pure shear no change of size, only of shape. The elasticity involved in a shear is called *torsional rigidity*, or simply *rigidity*.

When a beam is bent, say by a weight resting on its middle, its lower or convex surface is elongated, and its upper concave surface is compressed, hence longitudinal rigidity only is called out; unless indeed its horizontal planes slide over one another to some extent, in which case simple rigidity will also be brought into play. If you bend a book, you will see that the leaves slide.

149. All resistances to strain are included under the general name Elasticity (the term *elastic* having a slightly different meaning from *elasticity*, just as *rigid* has from *rigidity*).

A body which exerts a great stress when subject to a given strain, is said to have a high elasticity, but if a small stress, a low elasticity; in fact, elasticity is defined as the ratio of the stress called out to the strain which calls it out; or shortly,

$$\text{elasticity} = \frac{\text{stress}}{\text{strain}};$$

or what is the same thing, *elasticity equals the stress called out by unit strain.* 'Stress' is here short for *pressure or tension per unit area.*

150. The *kind* of elasticity depends on the nature of the strain; if it is simple dilatation or compression, the ratio of the stress to the strain is *elasticity of volume;* if it is an elongation or contraction, the ratio is *Young's modulus;* if it is a twist or shear, the ratio is *simple rigidity;* and the most general kind of strain that can possibly be given to a body can be compounded of these three elements, or can be resolved into them.

Moreover, a shear may be analysed into two longitudinal strains, a stretch and a squeeze, at right angles to one another; similarly a shearing stress may be resolved into a pressure and an equal tension perpendicular to it.

151. *Strain* is always measured as a ratio; the ratio of a change to an original. The first sort of strain, simple change of size, is best illustrated by gases. See Chapter XIII., Part ii. This strain is measured as the ratio $\frac{\text{change of volume}}{\text{original volume}}$. The second kind, or longitudinal strain, is measured by the ratio of the change of length of a rod to the original length.

Stress is measured by the pressure or tension *per unit area*—for instance, the force applied to either end of a rod divided by the area of the cross section of the rod.

152. Notice that elasticity is measured not by the ratio of *distorting force* to strain, but by the ratio of *internal stress* to strain; for a body may be quite inelastic, and yet require a considerable force to distort it. You would find it hard work to flatten out or to punch a hole through a mass of wet clay for instance; but no active internal stress would be exerted capable of restoring the body to its old shape when

the distorting force is removed—the resistance would have been produced by friction between the different parts which have slid over one another, and friction, we know, is a passive force which can destroy motion, but not generate it.

Such bodies as these then, though *plastic*, are *viscous*—that is, there is friction between their particles, so that energy is converted into heat when they are distorted. Elastic bodies also may be viscous—that is, there may be some friction between their particles whenever *shear* or sliding of parts occurs. Even steel is very slightly viscous, and when bent becomes infinitesimally warmer, otherwise a tuning-fork *in vacuo* could go on vibrating for ever.

153. Matter, however, is known to exist in a perfectly plastic state, which is not viscous at all, but *limpid;* and in this state it is termed *fluid.* A *perfect fluid* is a body with zero rigidity and zero viscosity—in other words, it has infinite plasticity and infinite limpidity. No force whatever is required to alter its shape, but it takes the shape of whatever vessel contains it. Many *actual* fluids come very near to this, but they all have more or less trace of viscosity. Ether has a little less than water, while oil has more, treacle has more than oil, Canada balsam still more, and pitch or sealing-wax a great deal—so much that it is practically a solid except for very long-continued forces. The only elasticity possessed by fluids is *elasticity of volume;* in other words, no stress is called out in them by any strain except simple expansion or contraction.

154. All fluids are perfectly elastic as regards *volume*—that is, they all regain their size perfectly when the compressing stress is removed. Nevertheless the values of their elasticities vary very much, for some are nearly incompressible, while others are readily compressed; and they are divided into two great groups on this ground.

The group of fluids which have a very high volume-elasticity, or are nearly incompressible, are termed *liquids*—type, water. A *perfect liquid* might be defined as an *incompressible perfect fluid.*

The other or compressible group have an elasticity not depending on themselves at all, but simply on the pressure to which at the time they are subject—the elasticity being equal to the pressure; these are termed *gases*—type, air. From this it follows that the volume occupied by a gas also depends, not upon itself, but upon the pressure to which it is subject. Gases in fact take not only the *shape*, as all fluids do, but also the *size* of their containing vessel, no matter how large this may be.

We may sum up shortly thus:

Solids have both size and shape.
Liquids have size, but not shape.
Gases have neither size nor shape.

Matter exists in all kinds of states, some approximating closely to one of these three types, others lying between them and passing almost insensibly from one type to another.

155. The only forms of matter which can be treated in a simple manner, besides perfectly rigid and perfectly elastic solids, are *perfect liquids* and *perfect gases;* and also ordinary liquids and gases when at *rest*. It remains now to see what special mechanics is necessary for matter in these two fluid states.

The special mechanics for liquids is called *Hydrodynamics;* the branch of it treating of liquids at rest being *Hydrostatics*.

The branch of hydrodynamics relating to fluids in motion, or *Hydrokinetics*, is not an easy subject, and has not as yet made much progress.

The special mechanics for gases is called *Pneumatics*, or sometimes Aërodynamics.

156. The essential difference between the mechanics of solids and the mechanics of fluids is based upon the different ways in which they transmit pressure. Thus, take a rigid stick standing on the ground, and press downwards upon the upper end of it; the pressure is transmitted unchanged to the other end, which therefore presses the ground with an equal force; but not the slightest pressure is exerted sideways, say against a tube surrounding and

fitting the stick. But place some liquid in a closed tube, and press one end of the liquid with a piston; then, though the pressure is still transmitted to the other end, it is also transmitted sideways to every part of the tube just as much; and, moreover, the pressure on the closed end of the tube is not now necessarily equal to the pressure of the piston, unless the area of the closed end equals the area of the piston; if the area is greater, the pressure is greater, and if less, less. Every portion of the surface of the tube which exposes to the liquid a surface equal to the area of the piston, experiences a pressure equal to that exerted by the piston; a fact which is briefly expressed thus:

Fluids transmit pressure equally in all directions.

This is entirely because of their plasticity, or the perfect mobility of their particles. The structure of a liquid might be imitated roughly by a number of exceedingly small well-oiled shot. A bag full of such shot, if compressed in any way, would experience the pressure in every part of it.

CHAPTER XI.

ON THE PRESSURE OF GRAVITATING LIQUIDS AT REST.

(*Hydrostatics.*)

157. We conceive a perfect liquid as an incompressible fluid, that is, a body all whose particles are capable of free motion among themselves without the slightest friction, whose shape therefore is wholly indefinite, but whose volume it is impossible to change. Water is an imperfect liquid, partly because it is slightly compressible, but principally because it is slightly viscous—that is, because its particles experience, when they slide over one another, a certain amount of resistance analogous to friction, called viscosity.

Hence it is that a basin full of water which has been stirred round and round and left to itself, will after a time come to rest. The energy of motion will be wasted by 'friction' against the wet sides of the vessel—that is, it will be expended in warming the water. But because the friction is very small, a particle of water can travel against it a long way before its energy is expended—that is, before the work done, Fs, is equal to the energy to be got rid of, $\frac{1}{2}mv^2$.

158. The friction due to viscosity differs from ordinary friction in that it depends very greatly on the speed of the relative motions; it seems, in fact, to be about proportional to the square of the velocity, and as the velocity vanishes, so does the viscosity-friction. The properties of water, or any other actual liquid *in motion*, are therefore very different from those of the ideal perfect liquid; but when water is *at rest*, there is no friction among its particles, their reactions are all normal, and its behaviour is then identical with that of the perfect liquid. Hence it is that the mechanics of liquids *at rest* (even such liquids as treacle) is so simple; the simple laws of the perfect liquid are applicable to them, for their viscosity may be neglected.

Pressure of Fluids in General at Rest.

159. The general law of pressure common to all fluids, and following at once from the mobility of their particles, is that they act like perfectly smooth bodies (cf. sect. 104); or,

The pressure of a fluid at rest is always perpendicular to every surface on which it acts.

For if the reaction of the surface had any component *along* it, it would be able to move the fluid, which would therefore be *not* at rest.

A second general law may also be stated thus: If a pressure is applied to any area of the surface of a fluid in a full closed chamber, that same pressure is transmitted to every portion of the walls of the chamber of equal area (sect. 156).

Thus imagine a closed cistern quite full of water, with tubes or cylinders let into the sides anywhere, and plungers or pistons, A, B, C, D, fitting these tubes quite freely, but yet water-tight (fig. 64); and let A have an area of 1 square inch; B, 2 square inches; C, 3; and D, 4 square inches. Now push A in with a force say of 20 poundals; every square inch of the interior surface of the cistern will experience this pressure, and therefore B will experience a force of 40, C of 60, and D of 80 poundals. Of the three larger pistons, let D be the only one free to move, and let a constant external pressure of 80 poundals be applied to it; then if A is pressed in with a force the least exceeding 20, D will move out and overcome the force 80. But it would only move ¼th as fast as A. This is evident; for suppose A were pushed in 1 foot, it would throw 12 cubic inches of water into the cistern, and therefore into the cylinder of the other movable piston, D; but as this cylinder is 4 square inches in area, the 12 cubic inches of water would only cause D to move out 3 inches, the quarter of

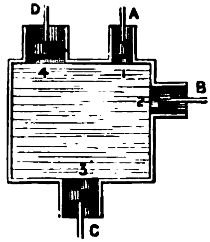


Fig. 64.

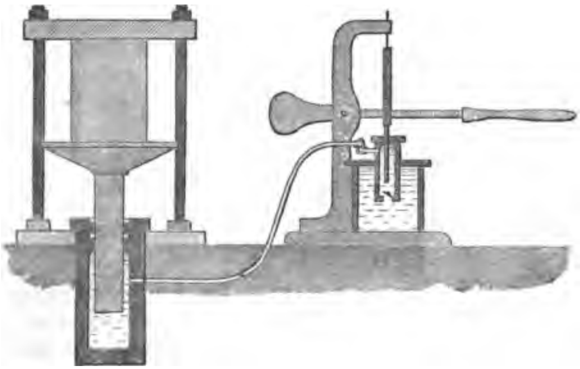


Fig. 65.—Hydraulic Press.

a foot. In other words, the work (Fs) done by the piston A, 20×12, is equal to the work done upon the piston D, 80×3.

So that we have here simply a machine subject to the universal law of machines, that 'what is gained in power is lost in speed,' or more accurately, that there is no gain of energy in a *hydraulic* machine any more than in any other.

The machine just described, put into a working form, is known as the hydraulic or Bramah press (fig. 65). It consists fundamentally of two cylinders of different sizes, with pistons or plungers fitting them, and a pipe connecting them. Water fills both cylinders, and the mechanical advantage of the machine is the ratio of the areas of the two pistons $A : a$, so that a 50-lb. pressure on the small piston balances $50 \frac{A}{a}$ lbs. on the large one. The liquid acts only as an incompressible plastic medium for transmitting pressure. For a fuller account of the machine, see Ganot, sect. 99, or Deschanel, page 221.

160. So far we have supposed the pressure to be produced only by pistons which endeavour to compress the liquid, but it is important to consider also pressures due to the *weight* of the liquid. Every particle of a liquid is attracted to the centre of the earth, and will tend to get there by percolation unless prevented by being inclosed in some vessel with impervious sides; in other words, water must be kept in non-porous vessels. The vessel, however, need not have a lid, for a *liquid* occupies an unchangeable volume, and therefore may have its upper surface free; it keeps at the bottom of the vessel as the nearest accessible position to the centre of the earth. But it will press on the bottom and sides of the vessel with a certain force which will always be normal to those surfaces, and whose magnitude we have now to consider.

Pressure of Liquids due to their Weight.

The first simple law is that the upper or free surface of a liquid at rest is horizontal; that is, is normal to the vertical force of gravity on each particle. Such a surface is said to be *level*, and it is practically flat or plane, because the forces on the several particles are practically parallel.

Inasmuch, however, as these forces are not really parallel, but intersect at the centre of the earth, the level surface of a liquid at rest is not really plane, but is curved round the centre of the earth; in other words, it forms part of a sphere with the radius of the earth as its radius. The curvature is too small to be appreciable in a bucketful of water, but it is apparent enough in the ocean.

Another law, that the pressure of a liquid varies directly with the depth, is what we must now establish.

161. Consider a cylindrical bucket with a flat bottom, filled with water; the base of the vessel has to support the whole of the water, as if it were a rigid mass slipped into the bucket with its sides well oiled. For although certainly the sides are pressed, and therefore exert reactionary pressure on the water, yet they, being upright, press it horizontally only, and so can have nothing to do with sustaining its weight. The pressures of the sides simply maintain the shape of the water in opposition to the force of gravity, which tends to flatten it out.

The pressure on one side is equal and opposite to the pressure on the other, and therefore there is equilibrium, unless part of one side be removed by boring a hole through it. In that case the water will flow out, and the uncompensated pressure on the side opposite the hole will force the vessel bodily along in a direction opposed to the stream of water. This is the principle of Barker's mill, turbines, Catherine wheels, rockets, &c. See Deschanel, page 92; or Ganot, sect. 193.

In an upright cylindrical vessel, then—that is, any vessel with vertical sides—the pressure on the base is equal to the whole weight of water contained in the vessel. But the cubic contents of a cylinder are obtained by multiplying its height by the area of its base always, whether that base be round or square, or any other shape; and the weight of water a vessel can contain is, of course, its contents in cubic feet multiplied by the weight of each cubic foot. Hence, the pressure on the base of an upright-sided vessel, A square feet in area, filled to a height of h feet with a liquid of which a cubic foot weighs s lbs., is in lbs. weight, $P = sAh$.

Thus, suppose an oblong-based plane-sided cylinder (also called a *prism*) with base 10 inches by 5 inches, and height 15 inches; the contents would be $10 \times 5 \times 15 = 750$ cubic inches, and the pressure on its base when full of water would be the weight of 750 cubic inches of water; which happens to be about $27\frac{1}{8}$ lbs. weight.

162. If we are speaking about water, this *s* is often written *w*, meaning the weight of a cubic unit of water, just as it might be written *m* if we were speaking of mercury. Whether *w* stand for the weight of a cubic inch, a cubic foot, or a cubic centimetre, is wholly immaterial, being only a matter of custom or convenience; only we must keep to one unit all through. Hence, we use the word, a *cubic unit*, as expressing the cube of whatever arbitrary length happens to be taken as the unit of length in other parts of the book, or question, or problem under consideration.

A cubic foot is found to contain 62·33 lbs. avoirdupois of water, which is not far off 1000 ounces.

A cubic inch contains the $\frac{1}{1728}$th part of this—namely, $252\frac{1}{2}$ grains.

A cubic centimetre contains one *gramme* of water; and this is one reason why the French system of weights founded on the gramme makes calculations simpler: the unit of mass, or unit quantity of matter, is defined as that of unit volume of water.

Mercury is 13·6 times as heavy as water. Hence 1 cubic inch contains about $\frac{13600}{1728}$ ounces of mercury; and a cubic centimetre 13·6 grammes.

163. Suppose now that, instead of a cylindrical vessel, we consider a conical one, set up like a tumbler, with the wider end uppermost: then the pressure on the sides, being still perpendicular to them, is no longer horizontal, but has more or less of a vertical component as well as a horizontal one; hence, we can no longer say that the pressure on the base is the whole weight of water in the vessel, for the sides may and do support some.

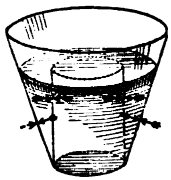


Fig. 66.

How much the sides support, and how much the base, may be readily seen by imagining an infinitely thin circular drum of the same diameter as the base of the vessel to be let into the water, as shewn by the dotted lines (fig. 66). Or you may suppose a thin circular drum of the liquid to freeze or become rigid, as indicated by the dotted lines.

The pressure across the walls of this imaginary drum is horizontal; and inside the drum we have what is equivalent to a rigid cylinder, with well-oiled sides, resting on and entirely supported by the base (just as we had in the cylindrical vessel); while outside we have a ring-shaped mass of water which is not supported by the base at all, and therefore must be supported by the sides. It is, in fact, supported by the vertical component of the pressure of the sides, and therefore it has nothing to do with the pressure on the base, which is wAh as before.

So also, if we turned the conical vessel the other way up, with the wide end as base; the pressure on the base would then be greater than the whole weight of water in the vessel, because of the vertical component of the pressure of the sides, which now acts downwards. And, as the pressure of the water on the sides would, if the sides were removed, be able to sustain the ring-shaped mass of water completing a drum set up on the base, it follows that the whole pressure on the base is still the weight of a volume of liquid filling a cylinder whose base is the actual base, and whose height is the height to which the vessel is filled; or again, wAh as before.

Notice particularly that none of this reasoning is impaired or affected if the sides of the vessel, instead of being plane, are curved or zigzag, or indeed any shape whatever, as in figs. 67 and 68. The pressure on the base is always simply sAh, or the weight of a cylinder of the given liquid with the

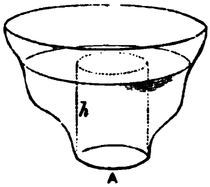


Fig. 67.

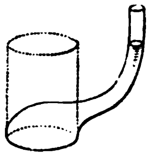


Fig. 68.

given base as base, and the given height as height; for the base supports this cylinder, the sides support the rest.

164. The vessel shewn in fig. 68 is supposed to be flexible like an india-rubber tube, and its base can be turned into different positions as in fig. 69; but, since liquids transmit pressure equally in all directions, the pressure on

it will not vary except in so far as the bending of the tube alters the height of the liquid in it. The only difficulty is the knowing what point to measure the depth to. The pressure on the lower part of the base is greater than that on the upper portion; but since the pressure is simply proportional to the depth, the average or mean pressure will be simply the pressure at the average or mean depth (compare average velocity in sect. 22)—that is, the pressure at the middle point of the base. Hence, the pressure on any surface of area A, immersed under a liquid to the *mean depth h*, is always sAh.

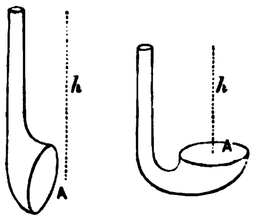


Fig. 69.

The surface plainly need not form the base of a vessel, but may be immersed anyhow.

Thus, let a rectangular plate 5 inches long by 4 inches broad be immersed slantingly under water, so that its upper edge is 8, and its lower edge 10 inches below the surface. Then evidently its mean depth, or depth of its middle point, is 9 inches; and the pressure on its surface, being equal to wAh, is $w \times 5 \times 4 \times 9 = 180w = 180 \times \frac{1000}{1728}$ ounces weight.

If the liquid had been mercury, this pressure would have been 13·6 times as great.

To find the mean depth of a bent or curved plate of irregular shape requires calculation, and the calculation required is just the same as that which would be used to find the centre of gravity of the plate (indeed, the centre of gravity is the most middle point in a body); hence the *mean depth* of a surface is often spoken of as *the depth of its centre of gravity.*

So we get the perfectly general result for liquids subject only to gravity:

The pressure on any surface whatever, due to the weight of a liquid under which it is immersed, is its area, multiplied by the vertical depth of its centre of gravity below the free surface of the liquid, multiplied by the weight of a cubic unit of the liquid;

or in symbols, $P = sAh$.

There is nothing more to explain. This simple formula contains it all.

165. Since the pressure of a liquid does not depend upon the quantity of the liquid, but only upon its depth, we may make a small quantity of liquid exert any pressure we please by putting it in a long narrow vertical tube, and giving it a large area to press upon.

This is the principle of the 'Hydrostatic Bellows;' which consists of a pair of circular boards joined water-tight by corrugated leather like ordinary blow-bellows, with a long tube opening into the cavity between the boards, which rises a good height, and finishes off with a funnel. See fig. 70. A man may stand on the upper board of the bellows, and raise his own weight slowly by simply pouring water down the tube.

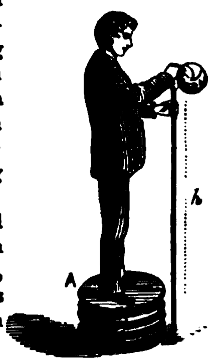


Fig. 70.

For if A be the area of the upper board of the bellows, and h its vertical depth below the surface of the water in the tube, all that is necessary to balance the man is that wAh shall be equal to or greater than his weight, say 200 lbs. or 3200 ounces.

Suppose A is a square foot, then to find the necessary height h to which the tube must be filled, we have $1000 \times 1 \times h = 3200$; or $h = 3{\cdot}2$ feet, a very moderate height indeed.

The man is, in fact, equal to a cylinder of water standing on A as base, and of height 3·2 feet; for this quantity of water would be balanced by the column of water in the tube (see sect. 167 and fig. 71), and the board and man take its place. The man rises so soon as this imaginary cylinder of water is equal to himself in weight; and it will be equal to him in weight just about the same time as it is equal to him in bulk, for a man is just about able to float in water (see Chapter XII.).

Hence the average cross section of a man is equal to the area of the board of a hydrostatic bellows, on which he would just be supported by a column of liquid equal to himself in height; for instance, if his height were 6 feet, and his weight 15 stone (210 lbs.), his average cross section would be ·56 square foot, or 80·64 square inches, because 1000 × ·56 × 6 = 210 × 16.

166. The total pressure on a surface under a liquid depends partly on itself—namely, on its area; but the pressure *per square inch* of surface depends not at all on itself, but on external conditions—namely, how deep it is immersed, and what it is immersed in: hence it is convenient to distinguish these, and to call the pressure per unit of surface the *intensity* of the pressure, and to denote it by p, so that $p = \frac{P}{A}$; or of course, $p = sh$.

One often speaks simply of 'the pressure of a liquid' at such and such a depth, without specifying the surface on which the pressure is exerted; for instance, the pressure of the ocean at a depth of one hundred fathoms, and so on. In such cases the *intensity* of pressure is always meant, or the pressure which would be experienced by a surface of unit area if placed at that depth—that is, simply sh.

The pressure of an incompressible fluid (or liquid) therefore varies directly with the depth (for s is constant); being nothing at the surface, and increasing uniformly as you descend.

167. When any number of communicating vessels are filled with the same liquid, the level of the liquid in all is the same. See fig. 71.

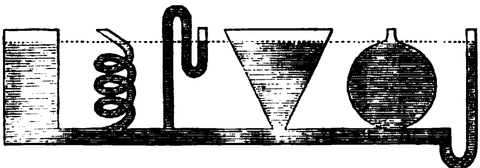


Fig. 71.

For the intensity of the pressure at any point due to every column of liquid must be the same, or there could not be equilibrium; and this pressure is proportional to the depth.

Further, when communicating vessels contain different liquids which do not mix, the heights of the columns of liquid are inversely as their specific weights.

For take any two, one full of mercury say, the other of water. Call the area of the surface of contact of the two liquids (fig. 72) A, and let the vertical height of the surface of the water B above A be called h, while the vertical height of C, the surface of the mercury above A, is called h'; then the pressure on each side of the area A must be the same, as soon as there is equilibrium and the columns have ceased to oscillate; but the pressure on its upper side is wAh, and on its lower, mAh', hence $wh = mh'$, or $h : h' :: m : w$.

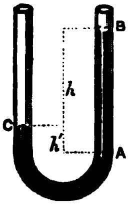


Fig. 72.

Centre of Pressure.

168. The whole pressure on a surface under a liquid may be considered as composed of a number of parallel forces —the pressures on each individual small area of the surface —and all these parallel forces will have a resultant equal to their sum, passing through a certain point of the surface which is called the *centre* of the parallel forces (cf. sect. 118), or the 'centre of pressure.'

EXAMPLES.

1. The small plunger or pump-piston of a Bramah press is half an inch, and the large one is 8 inches, in diameter; the pump is worked by a handle 5 feet long, the fulcrum being one inch from the point of attachment of the plunger; what is the greatest weight that a man of 15 stone can lift by this machine if he sits on the end of the handle?

   Ans. The mechanical advantage of the lever is 60, and of the press itself 256; hence the total mechanical advantage is 15,360, and the greatest weight the man can raise is 1440 tons.

2. Find the pressure on the bottom and sides of a cubical vessel 10 centimetres in the side full of mercury.

   Ans. 13,600 grammes weight on the bottom, and 6800 on each side.

3. Find the pressure on one side of the above cubical vessel if half full of water and half full of mercury.

Ans. 2075 grammes weight.

4. What is the pressure of water at a depth of 1020 feet?

Ans. 30 'atmospheres' or about 440 lbs. weight per square inch.

5. A sphere one metre in radius is just immersed under water; what is the pressure on its whole surface?

Ans. 12,566·4 kilogrammes weight.

CHAPTER XII.

FLOATING BODIES (*Hydrostatics continued*).

169. We shall now proceed to consider what happens when a solid is wholly or partially immersed in a liquid. Most of what we shall state will be true of fluids in general, but receives its most obvious illustration in the case of liquids.

When you dip your hand in the water, you displace some of the water; in other words, a portion of space below the surface which was formerly occupied by water is now occupied by your hand. The volume or bulk of the water displaced is, of course, equal to the volume or bulk of your hand.

All solids, then, when immersed either wholly or partially in a liquid, displace a volume of that liquid equal to the bulk of that part of them which is immersed. This is perfectly obvious.

170. Further, when your hand is immersed you can feel, if you attend, a certain pressure urging it up out of the water. This upward pressure is more apparent if you immerse your whole body; indeed the upward pressure is then so great as nearly to counteract the weight of your body altogether, consequently, in a bath you weigh apparently next to nothing.

This upward pressure is what we must now discuss.

Take an ordinary chemical test-tube of very thin glass, and plunge it in water with the closed end downward. You will feel a very distinct upward pressure, and the tube will be forced up if you let go. Keep, however, the tube immersed, and slowly fill it with water. You will find that it is forced up gradually less and less, until, when the level of the liquid inside and out is the same, the tube will weigh almost exactly the same as it did before it was immersed at all. The displaced water has been restored.

If you perform this experiment accurately with a balance, you will find the tube does not *quite* recover its original weight, even when the level is the same inside and out. This is evidently because some little water is still displaced by the walls of the tube, which, however, are very thin, and in what follows will be assumed to be infinitely thin.

Now imagine the glass tube annihilated; the water it contained will remain occupying the place the tube had occupied, and experiencing the same pressures as the tube did; because the same quantity of water is displaced as before, only now not by the glass tube but by the liquid water which had been poured into it. Obviously, however, this water will be in equilibrium, as all water in water at rest is; hence the two forces under whose influence it is—namely, its weight downwards, and the pressure of the surrounding water upward—are equal and opposite. But the pressure upward is the same as that the tube experienced before its annihilation; therefore the pressure on the tube was equal to the weight of its own volume of water—that is, the weight of the water it displaced—and acted in the same straight line, namely, through the centre of gravity of the water displaced.

This result is perfectly general, and is known as the principle of Archimedes.

When any solid is immersed either wholly or partially in a fluid, it is pressed up with a force equal to the weight of the fluid displaced; and this force may be considered to act at the centre of gravity of the fluid displaced.

The fluid displaced is equal in volume to the solid, hence the upward force is the weight of an equal bulk of the fluid.

To shew this by means of our symbols, consider a special case, say a cubical block of stone, a inches in the side, immersed in water, so that its upper surface is at a depth h below the surface of the water, and therefore, of course, its lower surface at a depth $h + a$ (fig. 73). The area of any of its faces is a^2. The pressure on its upper face (sect. 162) is wa^2h, on its under face is $wa^2(h + a)$, and on each of its sides $wa^2(h + \frac{1}{2}a)$. The pressures on its four sides are horizontal, and are in equilibrium among themselves two and two. The pressures on its upper and lower faces are opposite but are not equal, and therefore are not in equilibrium: their resultant is

Fig. 73.

$$wa^2(h + a) - wa^2h = wa^3 \text{ units of force}$$

acting upwards. But a^3 is the volume of the block, and wa^3 is the weight of this volume of water—that is, the water displaced by the block; so then the resultant of all the pressures on its entire surface is a single force upwards equal to the weight of the water displaced.

If we did not care about simplicity, the same might be shewn by the symbols for a solid of any irregular shape whatsoever, and a most important mathematical theorem it would be. You may make its acquaintance hereafter in a more general form under the name of 'Green's theorem.'

171. But now we know that if the cube in fig. 73 were really a block of stone it would not stay where it is; it would sink. This is because it is only pressed *up* by the weight of an equal bulk of *water*, whereas it is pulled *down* by the weight of its own bulk of *stone*—which is greater. The resultant force pulling it down, or its apparent weight under water, is

$$sa^3 - wa^3, \text{ or more generally } (s - w)v;$$

if s stand for the specific weight of stone, and v for the volume of the block, whatever shape it may happen to be. It still weighs downward, therefore, but it has lost weight equal to the weight of its own volume of water. If, on the other hand, it were a block of wood, it would be pulled down only by the weight of the wood, whose specific weight d is less than that of water; consequently it is forced upwards with a resultant force

$$wa^3 - da^3, \text{ or } (w - d)v.$$

And so generally, an immersed body is always urged up or down with a force proportional to the difference of the specific gravities of itself and the liquid in which it is immersed—up, if its specific gravity be the less; down, if it be the greater. Only when the specific gravities of the solid and liquid are equal, does the solid remain floating wholly immersed in any position—that is, in neutral equilibrium.

172. When a light body rises in a liquid, the resultant force urging it up is constant so long as it is wholly immersed; but it decreases as soon as some of the body begins to emerge, and it vanishes as soon as the weight of the water displaced equals the weight of the body. Hence, a body whose specific gravity is less than that of a liquid can float in that liquid, and does float in stable equilibrium when it has displaced a quantity of liquid equal to itself in weight.

A piece of floating wood, for instance, whose whole bulk is 9 cubic inches, and which is $\frac{2}{3}$ds as heavy as water, must float with 6 cubic inches immersed; for 6 cubic inches of water will be as heavy as 9 cubic inches of wood. And so generally,

$$\frac{\text{immersed volume of a floating body}}{\text{whole volume}} = \frac{\text{weight of unit vol. of solid}}{\text{weight of unit vol. of liquid}}$$

$=$ relative specific gravity of solid.

Since ice, for instance, has a specific gravity of $\frac{8}{9}$, that is, since 9 cubic feet of ice weigh the same as 8 cubic feet of water, it follows that an iceberg must have $\frac{8}{9}$ths of its whole bulk immersed; hence, the visible berg is only $\frac{1}{9}$th of the whole mass, there being eight times as much underneath the water. So also a floating cork whose specific gravity is $\frac{1}{4}$ has $\frac{3}{4}$ of its volume projecting above the water.

Determination of Specific Gravities.

173. The foregoing principles are all remarkably well illustrated by their practical application to the determination of specific gravity.

First, let us define what we mean by *specific gravity*. Refer to sect. 32, and you will find *density* defined as the mass of unit volume, or the mass of any volume divided by that volume,

$$\rho = \frac{m}{v};$$

similarly we might define specific gravity as the *weight* of unit volume, or the weight of any volume divided by that volume,

$$s = \frac{w}{v};$$

which would make specific gravity be to density as weight is to mass; or, as weight is g times mass (sect. 60), the specific gravity of a substance would be g times its density.

This, however, is not the definition of the term specific gravity as ordinarily used; it is the definition of what is called *absolute* specific gravity, which for distinction has been here called 'specific weight,' whereas the ordinary or *relative* specific gravity is the weight of any volume of a substance compared with the weight of an equal volume of some standard substance. The relative specific gravity of mercury with reference to water, for instance, is 13·6; of wood is, say ·6, and so on.

When one speaks of *the* relative specific gravity of any body, without stating the standard substance to which reference is made, it is understood that that standard substance is water; and so we may define *the* relative specific gravity, or *the* specific gravity of a substance, as the weight of any volume of it divided by the weight of the same volume of water.

Its relative *density* is precisely the same thing, both being simple numbers of equal value, but one having a direct reference to weight, the other to mass.

We have, in the preceding chapter, used s as standing for the absolute specific gravity or 'specific weight' of substances in general, m for that of mercury, and w for that of water; so the relative specific gravities of the three things are, of course, $\frac{s}{w}$, $\frac{m}{w}$, and $\frac{w}{w}$; the relative specific gravity of water itself being of course unity.

French measure.—In the metric system of weights and measures, the absolute and relative specific gravity of a thing are represented by the same number, because the unit volume of water is defined to be the unit of mass (cf. sect. 162). The absolute specific gravity of water, or the weight of 1 cubic centimetre, is 1 gramme; and if a thing is three times as heavy as water, a cubic centimetre of it weighs 3 grammes, and so on.

To compare the Specific Gravities of two Liquids.

174. 1*st Method.*—If they do not mix, place them one in each of the two legs of a U tube, and measure the heights of their respective columns (sect. 167); then $\frac{s_1}{s_2} = \frac{h_2}{h_1}$.

This method is not often used, as it is not very convenient, but it has been employed to compare with great accuracy the relative densities of mercury at the boiling and freezing points of water. (Dulong and Petit's method for absolute coefficient of expansion of mercury; see Ganot, art. 273, or Deschanel, Part II., page 287.)

2d Method.—Weigh a bottle full of the first liquid, and then the same bottle full of the second; deduct from each the weight of the bottle, and you will have the weight of the same volume of the two liquids to compare. In symbols, if b is the weight of the empty bottle,

$$\frac{s_1}{s_2} = \frac{w_1 - b}{w_2 - b}.$$

This method is often used. Flasks ('specific-gravity bottles') are made for the purpose (fig. 74). They are very light, and are arranged so that they can be accurately filled always to the same extent. For this purpose their neck has a constriction with a ring drawn round it with a diamond, and they are always filled up to this ring. This is done by filling them at first too full and then extracting the surplus with a scrap of blotting-paper or a capillary tube. The stopper is then inserted to prevent evaporation, and the whole is weighed in a delicate balance. The weight of the empty bottle, b, must have been previously ascertained.

Fig. 74.

3d Method.—Take any solid heavier than both liquids and insoluble in either of them, weigh it first in air (or vacuum), then immerse it wholly in one of the liquids (hanging it from the pan of the balance by a fine wire or hair), and weigh it in that. It now weighs less by the weight of the liquid it displaces—note this loss of weight. Now weigh it in the other liquid, and note its loss of weight in that. The same volume of each liquid has been displaced, and the first loss of weight was the weight of this volume of the first liquid; the second loss, the weight of the same volume of the second liquid; so the specific gravity of the first liquid referred to the second, is the ratio of the first loss to the second loss. Or in symbols, if w is the weight of the solid in air, and w_1 and w_2 its weight in the two liquids respectively,

$$\frac{s_1}{s_2} = \frac{w - w_1}{w - w_2}.$$

This method has been used by Matthiessen to determine the coefficient of expansion of water. (See Balfour Stewart's *Heat*, page 51.)

The operation of weighing a solid under a liquid is conducted by an ordinary balance with one of its pans replaced by a much shorter one with a hook under it, to which the solid can be hung by a fine platinum wire (fig. 75). When so arranged, it is often called a hydrostatic balance.

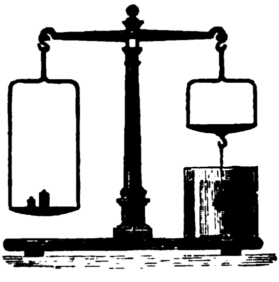


Fig. 75.—Hydrostatic Balance.

4th Method.—Take an insoluble and non-porous solid lighter than all the liquids you have to compare, and float it in each of them; ascertaining in each case the volume of it immersed. The weight of this volume of the liquid must in each case be equal to the weight of the solid, which is constant; so we obtain a set of different volumes all of the same weight. Call these volumes v_1 v_2 v_3 &c., and let w be the weight of the solid; then

$$\text{since} \quad w = v_1 s_1 = v_2 s_2 = v_3 s_3 = \&c.;$$

the ratios of the specific gravities to one another are inversely as the immersed volumes. Instruments for carrying this out are made of glass or metal, and sold under the name of *hydrometers* (see sect. 177).

5th Method.—Take a solid lighter than all the liquids, and float it in each, loading it so as to immerse the same volume in all; that is, always make it sink to a fixed mark. The weight of this volume of the liquid is the weight of the solid plus the load, so the specific gravities of the liquids are as the numbers representing this total weight in the different cases.

An instrument for carrying this out is called Fahrenheit's hydrometer, but it is seldom now used.

Another method is given in Ganot, art. 121.

To determine the absolute Specific Weight of a Liquid.

175. *1st Method.*—Weigh a known volume of the liquid in a gauged specific-gravity bottle (fig. 74), and divide the weight by the volume.

2d Method.—Weigh a solid of known volume before and after immersion in the liquid, say a sphere of measured diameter. Its loss of weight will be the weight of its own volume of the liquid, so the weight of unit volume of the liquid is $\dfrac{\text{loss of weight of solid}}{\text{volume of solid}}$.

To determine the absolute Specific Weight of a Solid.

Weigh a known volume of it, say a sphere or a cube or something easily gauged, and divide the weight by the volume.

To compare the Specific Gravities of a Solid and a Liquid.

176. *1st Method.*—If the solid be heavier than the liquid. Weigh it in air and in the liquid, and divide the weight in air by the loss of weight in the liquid; the quotient is the relative specific gravity of the solid referred to the liquid; $s = \frac{w}{w - w'}$.

2d Method.—Applicable only if the solid be lighter than the liquid. Float it in the liquid, and take the ratio of the volume immersed to the whole volume (sect. 172).

If the solid is a cylinder floating upright, volumes are proportional to lengths; and the specific gravity is then $\frac{\text{length immersed}}{\text{whole length}}$.

3d Method.—If the solid be lighter than the liquid. Weigh it first in air; then immerse it in the liquid by attaching a heavy body to it to sink it, and weigh the two together. Also weigh the sinker by itself in air and in the liquid. The loss of weight of the two together gives the weight of liquid displaced by both; the loss of weight of the sinker alone gives the weight of liquid it displaces; therefore the difference of the two losses gives the weight of the liquid displaced by the body itself—that is, the weight of an equal volume of the liquid. So the relative specific gravity of the solid is its weight in air divided by the difference of the two losses.

A liquid must always be chosen in which the solid is not soluble. Thus, for a piece of rock-salt, one must not use water, but either some such liquid as turpentine or benzol, or a saturated solution of salt: and the specific gravity of the salt referred to this liquid must be multiplied by its specific gravity to give the specific gravity of the solid with reference to water.

Another, though essentially similar method, is given under Nicholson's hydrometer, sect. 177, which see.

4th Method.—Useful when the solid is in the form of a powder. The difficulty with a powder is that it is impossible to gauge the volume of the solid particles directly, and also difficult to suspend the powder in water so as to determine its loss of weight. A specific-gravity bottle with a wider neck than that shewn in fig. 74 is used. Ascertain the weight of the bottle when empty, and also the weight of water it will contain when full up to the mark. Put a known weight of the powder into the bottle, and fill up with water; the powder displaces some water, so it will not now hold so much as before the powder was in; but the weight of the whole, minus the weight of the powder and bottle, gives the weight of the water now in. The difference between this weight and the weight of water the empty bottle originally contained, gives the weight of water displaced by the solid powder; so the specific gravity of the solid is

$$\frac{\text{weight of powder}}{\begin{array}{c}\text{weight of water required to}\\ \text{fill empty bottle}\end{array} - \begin{array}{c}\text{weight of water required to fill up}\\ \text{bottle after the powder is in.}\end{array}}$$

If the powder be soluble in water, of course some other liquid must be used: the result can be multiplied by the specific gravity of this liquid, if the specific gravity of the powder referred to water be required.

Hydrometers.

177. A hydrometer is a light body loaded so as to float in stable equilibrium at the surface of a liquid, and of a shape which renders it easy to observe accurately how much of its volume is immersed; and its use is to compare the specific gravities of liquids, or of solids and liquids. See methods 4 and 5, sect. 174, and methods 2 and 3, sect. 176. They are of two classes.

1st, Hydrometers of variable immersion or common hydrometers (Twaddell's, Beaumé's, Sykes', &c.).

2d, Hydrometers of constant immersion (Nicholson's and Fahrenheit's).

1st Class.—Common hydrometers are glass cylinders or 'stems,' loaded and arranged so as to float upright. This is done by making them terminate below in a couple of bulbs, one full of air, the other full of mercury or shot (fig. 76). They must be of such weight as to float in a liquid with part of the cylindrical stem projecting; hence they are usually sold in sets, say a set of three, one for heavy liquids, one for medium, and one for light. The heavier the liquid the more of the stem projects, but in a light liquid they sink pretty deep—always sinking until they have displaced their own weight of the liquid. A thin stem makes the instrument sensitive, a wide stem diminishes its sensitiveness, but increases its range.

Fig. 76.

The specific gravity of the liquid is (see sect. 172),

$$\frac{\text{the whole volume of the instrument}}{\text{the volume immersed}} \times \frac{\text{the weight of the instrument}}{\text{the weight of an equal volume of water}}$$

(the last fraction being the average specific gravity of the instrument); that is, the specific gravity of the liquid varies inversely with the volume immersed. The stem, however, is graduated so that the specific gravity is read off directly from the numbers on it.

2d Class.—Hydrometers of constant immersion will serve not only to compare the specific gravity of liquids, but also to determine the specific gravity of any solid, whether heavier or lighter than water, and this is their principal use; they will, moreover, make a very good substitute for a common balance. They consist, like the others, of a floating cylinder, which, however, is usually made very thin, and instead of being graduated, has one fixed mark on it, to which the cylinder is always sunk. The appendages to the cylinder are, a tray, A; a large light bulb, B; and a heavy bulb, or tray and cage, C. Fahrenheit's has only a shotted bulb below, and is made of glass. Nicholson's is made of metal, so that it cannot be used in corrosive liquids. It is, in fact, only used floating in water to determine the specific gravity of solids: it is the one which has the tray and cage C, and is shewn in fig. 77.

Fig. 77.
Nicholson's Hydrometer.

To sink it down to the fixed mark *m* on the fine cylindrical stem, some extra weights must be put on the tray A; let 20 grammes be the weight required.

To use it as a common balance, you place on the upper tray the body you wish to weigh, and then add weights, say $6\frac{1}{2}$ grammes, till it has sunk to *m*; one then knows that the body weighs $20 - 6\frac{1}{2} = 13\frac{1}{2}$ grammes.

To use it as a hydrostatic balance, you place the body in the lower tray; and now it requires say 3 more grammes to sink it to the mark, shewing that the solid has lost 3 grammes of weight by being immersed in water, hence this is the weight of the water it displaces; and its specific gravity is $\frac{13\frac{1}{2}}{3} = 4\frac{1}{2}$ (cf. Method 3, sect. 176).

(Its weight when under water is of course $10\frac{1}{2}$ grammes.)

Suppose the solid had been lighter than water, and that when it was in the upper tray 12 grammes had been required to sink the instrument, whereas, when placed in the lower tray (where of course it would tend to float upward, and have to be confined by the cage), 30 grammes were required; then the loss of weight in water would be 18 grammes, and as its weight was 8, its specific gravity would be $\frac{4}{9}$.

(Its weight when under water is − 10 grammes—that is, 10 grammes upwards.)

Equilibrium of Floating Bodies as regards Rotation.

178. We have now learned that a body necessarily floats in a liquid whenever it displaces its own weight of that liquid—that is, that under these circumstances the two contrary forces, its own weight and the resultant of all the fluid pressures on its surface, are equal, and are hence in equilibrium as far as *translation* is concerned. But in order that there may be also equilibrium as regards *rotation*, these two equal contrary forces must act along the same straight line; in other words, since the weight of the body passes through its centre of gravity, the resultant of the fluid pressures must also pass through this point; or, again in other words, the centre of pressure (sect. 168) of the immersed surface must lie vertically under the centre of gravity of the body.

When this condition is satisfied there is complete equilibrium; but there remains the question whether this equilibrium is stable or not.

It is manifestly stable if the point of application of the upward force is above the point of application of the downward one.

Now, just as the downward force, the weight of the solid, may be considered as acting at its centre of gravity, so the upward force, the weight of the liquid displaced, may be considered as acting at *its* centre of gravity; and this point, the centre of gravity of the liquid displaced, is the real *centre of buoyancy or flotation*; the term 'centre of pressure' being commonly applied only to simple surfaces which displace no water. The centre of pressure is always a point on the *surface*—namely, that point where the line of resultant pressure meets the surface. This line of resultant pressure, which is vertical, and which always passes through both the centre of pressure and the centre of buoyancy, may be called the *line of buoyancy*.

If, then, the centre of gravity of the water displaced be above the centre of gravity of the solid, the equilibrium is certainly stable.

This, however, cannot be the case with *homogeneous* solids; it can only be satisfied by loading the floating body. And it is satisfied in all the above hydrometers; their centre of gravity being down near the shotted bulb, while the centre of gravity of the water displaced is up near the centre of the air-bulb; consequently their equilibrium is very stable.

But unless the floating body is totally immersed, it is quite possible to get stable equilibrium without satisfying the above condition; in other words, this condition is *sufficient*, but not *necessary*, for bodies floating at the surface of a liquid.

For instance, in a canoe, the joint centre of gravity of canoe and occupant is much higher than that of the water displaced by it; and so it is in ships and boats generally, though ballast is used to keep the centre of gravity of a vessel as low as possible.

The higher the centre of gravity of a vessel is, the less is its stability; and by making it high enough, the equilibrium is sure to become unstable, so that the least disturbance will cause the body to rotate or turn over into some more stable position.

You will find an example of unstable equilibrium if you try to float an empty bottle or a common pencil upright. A penholder, however, or a bottle half full, will float upright one way, because loaded.

A long cylinder like a pencil or wine-cork floats in stable equilibrium on its side; but a short cylinder like a flat plate or a collar-box will float with its length vertical. A sphere rests in neutral equilibrium in any position; and so does a totally immersed homogeneous body of any shape whatever.

179. To investigate fully the conditions of stability or instability of equilibrium, it is no use taking the body just in its position of equilibrium with the two equal forces acting along the same vertical line; but one must imagine the body tilted a little, so that the equal forces act along different though parallel lines—that is, form a couple—and observe whether the effect of this couple is such as to restore the body to its original position, or whether it tends to increase the displacement more and more. In the former

case the equilibrium in the original position was stable; in the latter, it was unstable.

Let 1 (fig. 78) be a hemisphere floating in water in equilibrium, and therefore with the two centres of gravity, G of the body, and C of the displaced water, in the same vertical line—the line of buoyancy or resultant pressure. And let 2 be the same body disturbed from equilibrium into a new position, and therefore with a new centre of buoyancy, C_1. We have then the downward force w acting at G, and the upward force equal to w acting at C_1; the two constituting a couple of moment $w \times ab$, whose tendency is to restore the body into its original position; which was therefore one of *stable* equilibrium. In 3, this fig. 2 is repeated, but the old centre of buoyancy, C, of fig. 1 is indicated in the body as well as the new one, C_1; and the old line of buoyancy, CG, is produced till it cuts the new one through C_1 in the point M; which, in the case supposed, happens to be the centre of the sphere.

This point M is called the *metacentre*.

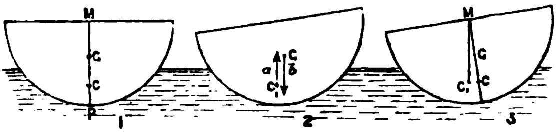


Fig. 78.

The *metacentre* is defined as the intersection of the old line of buoyancy, drawn in the body when in equilibrium, with the new line of buoyancy when the body is slightly disturbed from its position of equilibrium; and the rule for stability is:

If the metacentre M is above the centre of gravity G, the equilibrium is stable.

If it is below G, the equilibrium is unstable.

If M coincides with G, it is neutral.

And the height of M above G measures the *stability*.

All this will be seen at once if one just considers the couple as in fig. 2 above. For consider the upward force acting through the point M on the line GC fixed to the body (fig. 3); if M is above G, the upthrust will tend to restore the body and to bring GC upright again, the moment of the couple being proportional to the length MG; whereas, if M is below G, it tends to topple the body over more, and to turn the line GC more and more from the vertical.

The position of M depends on that of the new centre of buoyancy, and this depends on the shape of the floating body about the water-line. The shape of a ship or boat is devised so as to make the metacentre as high as possible, see fig. 79. Outriggers would raise it still more.

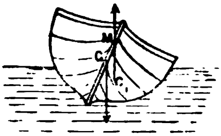


Fig. 79.

Strictly speaking, the disturbance from equilibrium ought to be infinitely small in order to give the correct position of M, and the correct measure of the stability MG. If the disturbance be great, the metacentre will in general be in a different position. If a ship lurches too much, the metacentre comes down very low, and may even pass below G; in which case, unless all the men can rush to one side, so as to alter the position of G, or unless an opportune wave comes to right the vessel, it must heel over, like the *Captain*.

Thus, in a floating body in equilibrium, there are four points vertically over one another (see fig. 78, No. 1):

M, the metacentre;
G, the centre of gravity of the floating body;
C, the centre of gravity of the fluid displaced; and
P, the centre of pressure of the immersed surface.

Of these P is always the lowest; and M is always above C (hence if C happens to be above G, much more is M); and the stability or instability of the equilibrium depends on whether M is above or below G.

As a matter of fact, a ship, like many floating bodies, has two metacentres; one, the one ordinarily spoken of as *the* metacentre, concerned in rolling; the other, very high up and of no practical account, concerned in pitching. It would be next to impossible to upset a ship by tilting it at the bows. In the *circular* Russian ironclads the two coincide. In an ordinary wine-cork floating on its side, one metacentre, the rolling one, coincides with the centre of gravity of the cork; the other, the pitching one, is a good height up.

In bodies of irregular shape the two lines of buoyancy, CG and the vertical through C_1, need not intersect at all, for they may lie in different planes: such bodies have no metacentre at all.

The whole subject of the metacentre, however, is not one that can be treated in an elementary book like the present; and it will be sufficient to have indicated the sort of ideas connected with the stability of equilibrium of floating bodies.

EXAMPLES.

1. Find the force with which a sphere one metre in radius is urged upward, if it is totally immersed in water (cf. Ex. 5, Chap. XI.). *Ans.* 4188·8 kilogrammes weight.

   N.B.—Observe that the *depth* to which it is immersed is now immaterial.

2. Find the apparent weight of a decimetre cube of stone in water, if its specific gravity is 2·5.

Ans. 1500 grammes.

3. How much of this block of stone would project above the surface of mercury in which it was floating?

Ans. 816 cubic centimetres.

4. A solid which weighs 35 grammes in air, weighs when immersed in water only 5 grammes, while in another liquid it weighs 14 grammes; find the specific gravity of this liquid. *Ans.* ·7.

5. The stem of a common hydrometer is graduated into 100 equal parts. The bulb and immersed portions, when it is sunk to the division 0, are equal to 3 times the stem in bulk. If it sinks to 20 in water, what will be the specific gravity of liquids in which it sinks to 80 and to 0 respectively?

Ans. ·8421 and 1·0̇6̇.

6. How deep would the hydrometer of the last question sink in a liquid of specific gravity ·8?

Ans. To the division 100.

7. If a floating body projects $\frac{1}{5}$th of its bulk above water, what will be the specific gravity of a liquid from which $\frac{1}{3}$d of its bulk projects? *Ans.* 1·2.

8. If a centimetre cube of metal weighs 8·5 grammes under water, what is its true weight? *Ans.* 9·5 grammes.

9. A Nicholson hydrometer which will sink to the fixed mark if 20 grammes be placed on the upper tray, requires 5 grammes more if the weights are placed on the lower tray beneath the surface of the water instead of on the upper one. What is the specific gravity of the metal of which the weights are made? *Ans.* 5.

10. A body A weighing 3 grammes is attached to another body B weighing 6 grammes, and the whole immersed under water, when they are found to weigh 2 grammes. The body B under water alone weighs 4 grammes, so what is the specific gravity of A and of B?

Ans. Of B the specific gravity is 3, of A it is ·6.

11. A specific-gravity bottle, when empty, weighs 15 grammes; when full of mercury, it weighs 151 grammes; and when full of another liquid, it weighs 33 grammes; what is the specific gravity of this liquid? *Ans.* 1·8.

12. The above bottle, when 8 grammes of a certain sand have been introduced, and the rest filled up with water, weighs altogether 30·5 grammes; what is the specific gravity of the sand? *Ans.* 3·2.

CHAPTER XIII.

ON THE PRESSURE OF THE ATMOSPHERE, AND ON THE PROPERTIES OF GASES.

(*Pneumatics.*)

180. Most of what we have said in the last two chapters about liquids is equally true of all fluids. Gases have the same mobility of particles, and therefore transmit pressure equally in all directions. Gases are subject to gravity, and therefore press upon all surfaces in them with a pressure depending on their depth and density; and they exert a sustaining force on bulky bodies equal to the weight of the gas displaced by them, thus causing them to lose weight, and if very light to float upwards. Hence, the only part of the two preceding chapters which does not apply to gases is that which relates directly or indirectly to the free surface of a liquid—a free surface being precisely the thing which a perfect gas never has. It is infinitely expansible.

This and all other peculiarities of gases as distinguished from liquids are due to the fact that their elasticity of volume is not constant or dependent on the gas itself, but is simply equal to the pressure to which at the time the gas happens to be subject; but all the special properties of gases, *qua* gases, we will reserve for consideration in sect. 189 *et seq.;* at present we will only deal with those properties which they possess in common with all fluids.

PART I.—THE PRESSURE OF THE ATMOSPHERE.

181. Now we live immersed in an ocean of air of unknown and indefinite depth, and hence we and all terrestrial surfaces experience its weight just as if it were an ocean of liquid; and many phenomena of common life depend upon this pressure. Its intensity may be expressed in pounds-weight per square inch, or grammes-weight per square centimetre, or units of force per unit area; it is not quite constant at any one place, varying with many apparently accidental and local circumstances, but its average value is 1033 grammes weight (or 981 times this number of dynes), per square centimetre, or 14·6 lbs. weight per square inch, or roughly, a ton weight per square foot.

Hence, a man's body experiences a total pressure of about 18 tons weight, for we found his average cross section (sect. 165) to be 80 square inches, which is that of a rectangle 8" × 10", whose periphery is 3 feet; so, if the man be 6 feet high, his surface, without allowing much for irregularities, is 18 square feet.

The pressure is exerted with perfect uniformity on all sides, and not only on the outside but on the inside too, so that it is not felt. The only way to make it appreciated is to destroy its uniformity by partial removal. If the pressure be removed from one side of any surface, then the other side experiences the whole uncompensated pressure of 14½ lbs. per square inch. If the air be withdrawn from any closed vessel, the outside experiences a crushing pressure, and if not very strong it will collapse.

Again, if the air be removed from a vessel whose mouth is beneath the surface of a liquid, that liquid is forced up into the vessel by the atmospheric pressure on the rest of the surface, the weight of the air sustaining the weight of the liquid, and completely filling it if the vessel is not too high. The product sh which expresses the intensity of pressure of the liquid (sect. 166) at the mouth of the vessel, must therefore be about 1033 grammes weight per square centimetre, if the liquid is supported by the average pressure of the air. Now, if the liquid be water, s equals 1 gramme, consequently h cannot be much greater than 1033 centimetres (or about 34 feet); if the vessel were taller than this, it would not be full. Of mercury, which is 13·6 times as heavy as water, the atmosphere can only support a column 76 centimetres (about 30 inches) high. ($34 \times 12 = 30 \times 13{\cdot}6$.)

Modes of Removing the Air from Vessels.

182. One way of exhausting a vessel is to drive out the air by steam, and then condense the steam.

Experiment 1.—Boil water in an air-tight tin canister and cork it up: remove the lamp and pour cold water over it: the uncompensated pressure outside will crush it.

Experiment 2.—Take a long tube closed at the top and bent as shewn in fig. 80; fill it with steam, and dip its open end under mercury. As the steam condenses, the mercury is forced up to a height of nearly 30 inches, and the tube may then be removed from the basin of mercury and carried about. The weight of liquid in one limb of the tube is balanced by the weight of the atmosphere in the other, which may be supposed to be extended to the top of the atmosphere (compare fig. 72, Chap. XI.).

Fig. 80.

A still simpler way of removing air from a tube is to fill it with a liquid. This is the way in which Torricelli originally performed the experiment and measured the pressure of the atmosphere. He filled a long tube with mercury without air-bubbles, and then inverted it with its mouth under mercury in a basin (fig. 81). On removing his finger, he saw the mercury descend till its surface was 29 or 30 inches above that of the liquid in the basin, and there come to rest after a few oscillations.

Above the mercury was a nearly perfect vacuum, now called a Torricellian vacuum. If any gas or vapour be introduced into this, it will depress the column more or less against the force of the atmosphere. For instance, the water vapour left in the cold tube after the experiment of fig. 80 will depress the column half an inch or so.

Pumps.—Another mode of removing air or any fluid from a vessel is by means of an arrangement of valves which only open and permit egress one way, combined with some method of squeezing the fluid so as to make it move in one direction or other. Such a combination is called a *pump*, and three kinds are shewn in fig. 82. The valves in each are self-closing flaps (shewn open in the figures for clearness), which will open upwards by pressure from beneath, but which only close more tightly if any pressure be exerted on them from above. (Such valves exist in the veins, and cause whatever flow there is to take place in one direction.) The compressing apparatus to cause motion in the fluid is in (1) an elastic bag to be alternately squeezed and relaxed by the hand—such an apparatus is the lung of an animal; in (2) and (3) it is a piston fitting a cylinder which is to be pushed to and fro, or up and down; the peculiarity in (3) being that one valve is in the piston itself. All three arrangements evidently tend to transfer any fluid they may contain from A to B, producing an exhaustion in any vessel screwed on to the end A, and a condensation in any vessel screwed on to B.

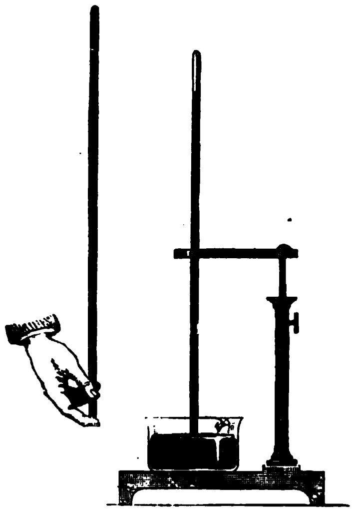
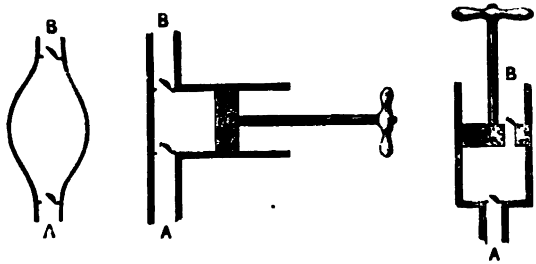


Fig. 81.—Torricelli's Experiment.

No. 1 is a pump used in surgery for producing injections or for delivering a strong jet of liquid. The heart of an animal acts on the same principle; so does a pair of blow-bellows imperfectly, for though it has only one valve, the narrowness of the jet acts partially as a second one.

No. 2 is a mere modification of No. 1, and is used in garden and fire engines. Both these are called *force-pumps*.



No. 1. No. 2. No. 3.

Fig. 82.—Pumps.

No. 3 is called a lift-pump, because it gets the fluid above the piston and then lifts it up when the piston is raised. It used also to be called a 'suction' pump.

Modes of Measuring the Atmospheric Pressure.

183. *Barometers.*—The pressure may be measured and its variations indicated by exhausting a strong metal box with a thin and flexible (corrugated) top, supported by a spring against the weight of the atmosphere, as shewn in fig. 83. If the pressure increases, the spring is compressed a little more; if the pressure decreases, the spring recovers itself a little; and so the box lid indicates variations of pressure by moving in or out, and its motions may be magnified by a rack and pinion and long index as shewn. Such an instrument is called a 'barometer' (weight-measurer), and being made without mercury this form of it is called 'aneroid.'

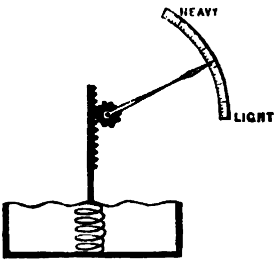


Fig. 83.

Mercury Barometers.—The mercury column (fig. 81) is a convenient measure of the pressure of the air, and is the original form of barometer. If the pressure increases, the column is forced higher up; if it decreases, the column descends.

It is found to oscillate on different days between 31 and 28 inches, being usually high when the weather is fine, and low when the air contains much moisture (aqueous vapour being lighter than air): moreover, since any sudden local rarefaction of the air which lets the column down may also enable air from surrounding localities to flow rapidly in, a barometer often falls before a gale.

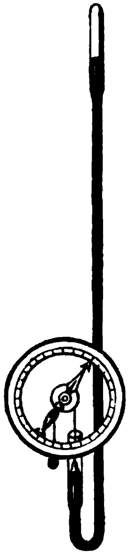


Fig. 84.
Weather-glass.

These facts cause a barometer to be used as a weather-glass; and a convenient form is that of fig. 80, arranged as in fig. 84, where the motion of the mercury in the short open tube is used as the indicator instead of that in the long tube, and its motion is magnified by a float counterpoised over a pulley with an index; or else by a rack and pinion as in fig. 83. The advantage of this form is that the friction prevents very prompt motion, so that the accumulated changes of the last hour or two are indicated by the needle whenever you go and tap the instrument. As an accurate measurer of pressure, however, it is worthless.

The cistern form (fig. 81) is always used for accuracy, and some arrangement is added by which the level of the mercury in the cistern can either be kept constant or can be read off; for, of course, when the mercury falls in the tube it rises in the cistern, and it is the *difference* of levels which really measures the pressure. If a barometer be carried up a mountain, the mercury column must descend, because some of the column of air which formerly balanced it is left below. By this decrease of atmospheric pressure, the height of the mountain may be calculated. For more about barometers, see Deschanel, chap. xvii., or Ganot, sects. 146–160.

184. *Manometers.*—Columns of liquid may be used to measure pressures other than those of the atmosphere—such pressure gauges are called manometers. Fig. 85 shews a gauge for measuring the pressure of the steam in a boiler over and above that of the atmosphere by the height of a column of mercury; and the pressure may be stated as equal to so many inches or centimetres of mercury, or if very large, it may be stated as so many 'atmospheres'—every 30 inches of mercury being called one atmosphere.* Metal manometers are, however, always preferred in practice. See Deschanel, fig. 128.

Fig. 85.

By 'a pressure of 76 centimetres' on any area, then, is meant the pressure which would be produced by a column of mercury 76 centimetres high with that area as base. The intensity of pressure in grammes per square centimetre of a column of *water*, is equal to its vertical height in centimetres (because 1 cubic centimetre of water weighs 1 gramme); or in absolute measure (dynes) its pressure is 981 times its height. The pressure of any other liquid of specific gravity s is s times as great; so '76 centimetres of mercury' means a pressure of $76 \times 13{\cdot}6 \times 981$ dynes per square centimetre.*

The height of the column of course means the *vertical* height (cf. fig. 71); hence if a manometer or barometer tube be inclined, the mercury will flow further up the tube, but so that the vertical height of its surface is the same as before (fig. 86).

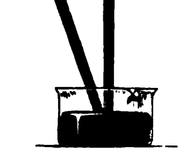


Fig. 86.

* In the metric system of measures, a million dynes (or a megadyne) per square centimetre is conveniently called an 'atmosphere.' It is very nearly equal to 75 centimetres of mercury. Regnault employed 76 centimetres as his standard pressure.

Modes of Raising Water.

185. The most obvious mode of raising water is to get something underneath it, and lift it up. This is the old method of a bucket and windlass. Since, however, the atmosphere can support a column of water about 34 feet high, it may be used to force water up from wells not much more than 30 feet deep. For this purpose, a tube is let down into a well, and then exhausted of air, either by filling it with steam and condensing it (which is nearly the oldest form of steam-engine or steam-pump, and was set up by Captain Savery at the water-works, York Buildings, Charing Cross, and used from 1698 to 1706), or by screwing the end A of one of the pumps of fig. 82 on to the tube, and working the pump. Fig. 87 shews pump No. 3 so applied, and is a common house-pump. First the air, and then the water, is transferred from A to B, and the water finds egress at the spout.

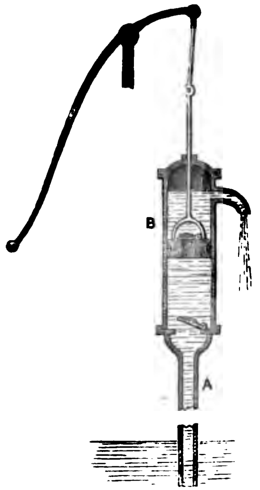


Fig. 87.—House-pump.

It is often required to raise water from mines several hundred yards deep. Atmospheric pressure is of course quite incompetent to effect this: the only plan is to get something under the water and lift it. Pump No. 3 is still used, only it is arranged at the bottom of the mine, within 20 or 30 feet of the water, and its spout is transferred higher up, so that it delivers the water at the top of the shaft. Water may be thus raised any height whatever. Such pumps are called lift-pumps, and are usually worked by engines at the top of the shaft; long rods connecting the piston of the pump with the beam of the engine.

A house-pump can also be used to lift water up to a cistern on the top of the house. The piston-rod of such a lift-pump works through a water-tight stuffing-box, as in fig. 87, but the spout has a tap by which it can be closed when desired; and a pipe leads from the upper portion, B, of the pump-barrel to the cistern.

Force-pumps Nos. 1 and 2 (fig. 82) are not used to raise water from any depth, but to deliver a strong jet; and fig. 88 shews the arrangement in a garden-engine. The stream of water is rendered continuous instead of intermittent, either by an elastic bag, or by an air-chamber. C is the air-chamber which contains air compressed by the over-supply of water, so that, if the pump stops working, the jet continues for a few seconds, only gradually diminishing in strength as the compressed air expands.

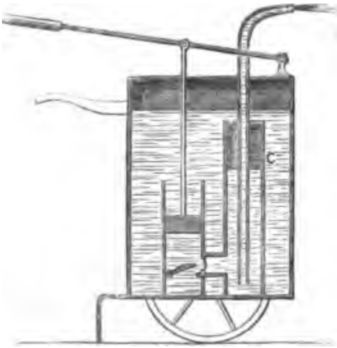


Fig. 88.—Garden Force-pump.

Fig. 65 shews a force-pump applied in the hydraulic press, with plungers instead of pistons. Plungers are indeed generally used in force-pumps; they act precisely like pistons of equal area, the only difference is that they fit the stuffing-box instead of the cylinder.

186. In any kind of lift-pump, the piston has, during its up-stroke, virtually to support a column of water reaching from the surface of the water in the well to the highest surface of water in the pipe. Calling this height h, and the area of the piston A, the pressure on it is wAh. To work the pump, a force somewhat greater than this must therefore be applied to the piston. In force-pumps, the pressure during the up-stroke corresponds to a column of water from piston to well; and during the down-stroke to

a column from the piston to the highest point reached by the water, whether it be a free jet or confined in a tube (neglecting the friction of the moving water in all cases).

Mode of Lowering Water.

187. The force of gravity renders the lowering of water a very easy matter. If we have a liquid in a vessel, and wish to transfer it to another at a lower level, all that is needed is two holes in the vessel—one to let the liquid out, which must be below the surface, and the other to let the air in, which is best above the surface of the liquid; if it is beneath the surface, it may act, but it will do so irregularly, letting the air in by bubbles. One hole half beneath and half above the surface will act as two holes, and this is the way one empties a jug or bottle, rotating it till its one hole occupies this position. If the hole be large, it will act as two even if wholly beneath the surface, but the flow will be very irregular. The beer in a cask with the tap open, but without a vent-hole, is kept in by the atmospheric pressure, unless it is fermenting and forcing itself out by means of its own gas, or unless you blow up the tap. A pipette (fig. 89) is a vessel with two holes, and the flow of liquid from it can be stopped by closing either of them with the finger.

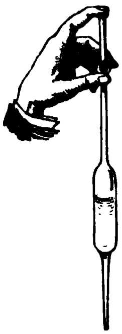


Fig. 89.—Pipette.

Siphon.—In an open glass vessel, however, it is not convenient to bore a hole through the glass beneath the surface of the liquid, neither is it always convenient to rotate the vessel till part of its mouth is below the surface. In such cases the necessary second hole may be introduced beneath the surface as one end A of a bent tube, whose other end, B, is at a lower level—say is immersed in another vessel at a lower level (fig. 90). If this tube be once exhausted of air, either by sucking liquid into it with the mouth, or by filling it at a tap before inverting it, the atmospheric pressure will afterwards keep it full of water; and the column of water in one leg, being longer than that in the other, will overbalance it, and a steady flow from A to B will be kept up till either the water sinks below the opening A, or till the level in both vessels is the same. Such a tube is called a 'siphon.' Its shape is wholly immaterial, provided that no part of it is at a height above the surface in either vessel greater than the column of liquid which the atmosphere can support, otherwise the action will cease. So also it would cease if it were put under the receiver of an air-pump and the air exhausted.*

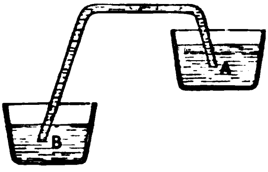


Fig. 90.—Siphon.

While the air was being exhausted, the flow would go on *with undiminished speed* until the air pressure became too weak to sustain the longer of the two columns; the liquid would soon then snap at the highest point, and the longer column would fall till it was the same length as the other. As the air pressure still further diminished, the two columns would slowly sink, like barometers, until, when there was no pressure left, the level of the liquid inside and outside the tube would be the same. On readmitting the air, the action would commence again, unless either end A or B was not fully submerged.

Fig. 91.

The shape of the siphon tube being immaterial, it might pass straight through the wall of the vessel from A to B (fig. 91), and such a pipe would empty the vessel to just the same extent, and at the same rate, as the tube of fig. 90; only it does not obviate the necessity of a hole through the side of the vessel as the tube bent over the

* It is probable that, in a perfect vacuum, a siphon of moderate height would work perfectly well, because the cohesion of water free from air is pretty strong, and might maintain the continuity of the column of liquid in spite of gravity. Under these circumstances, the cause of the flow would be exactly like that of a chain over a pulley with one end longer than the other; and the analogy will be complete if the chain be supposed to uncoil itself from a table, and to coil itself up on the floor.

edge does; neither, of course, would it cease to act in a vacuum.

Floating of Bodies in Air.

188. All things which displace any air (that is, which have any bulk) are pressed or buoyed up with a force equal to the weight of the air whose place they occupy (sect. 171), and so everything weighs less in air than it would in a vacuum. The true weight of a thing is its weight *in vacuo*, and this equals its apparent weight plus the weight of an equal bulk of air. The bulkier a thing is, the more does its apparent weight differ from its true; and if a very light body be also very large it may have no apparent weight at all, but may float about in equilibrium, or even be forced upwards, like a balloon.

What is called a pound of cork is therefore really more than a true pound, for it has been weighed against metal weights which are not so bulky as itself and displace much less air. A little balance is sometimes made to hold a ball of cork and another of lead of the same apparent weight, so that they equilibrate each other in air; but if the buoyant power of the air be withdrawn by putting the whole under an air-pump, the cork will descend, shewing that it is really the heavier of the two.

A thin copper or glass sphere with a tap may be used to measure this buoyant power. When the tap is open, very little air is displaced by the sphere; if you weigh it then, you get its true weight very nearly. But exhaust it and shut the tap. It now displaces a quantity of air, and accordingly is buoyed upwards, and will be found to be apparently lighter than before. The difference between its true and apparent weights gives the weight of an equal volume of air.

In this way 1 cubic centimetre of ordinary air, when the barometer stands 76 centimetres high and the thermometer stands at zero centigrade, is found to weigh ·001293 gramme. (This number ·001293 is therefore the sp. gr. of air referred to water.) Or 11·2 litres weigh about 14·4 grammes. Or 1 cubic inch weighs ·31 grain; or a cubic foot weighs about an ounce and a quarter.

A sphere of brass a yard in diameter displaces, if exhausted, rather more than half a cubic yard of air, say 14 cubic feet, which weighs 17½ ounces about. If then its own weight were only a pound or so it would ascend slowly like a balloon. But if so light as this, its walls could not be strong or regular enough to resist the pressure, and it would collapse. Such balloons are therefore impracticable. To sustain very thin walls against the air pressure, it is necessary to fill the balloon with some gas; and hydrogen, being the lightest gas known, is always used. Hydrogen enough to fill the above sphere would weigh only 1¼ ounce, so it would not add very greatly to the weight, and its presence enables the walls to be of thin oil-silk instead of metal. The first balloons were filled with hot air, which occupies more room and therefore displaces more than its own weight of cold air (see Deschanel, chap. xxi., or Ganot, art. 169).

PART II.—ON PROPERTIES PECULIAR TO GASES.

189. A perfect fluid whose elasticity of volume (see Chap. X., sects. 145, 149) is equal to the pressure upon it, provided the temperature is constant, is called a *perfect gas*. We have now to consider what properties a gas possesses in consequence of this peculiarity.

First of all, gases must be very compressible: any additional pressure produces a corresponding change of volume. The increase of pressure (sect. 166) is the stress; the ratio of the change of volume to the original volume is the strain (sect. 151). Let the original pressure be P, and the new pressure P′; then the stress is $P'-P$. Let the original volume of the gas be V, the new volume V′, then the strain is $\frac{V-V'}{V}$. Its elasticity, when in the compressed state, by definition (sect. 149), is $\frac{P'-P}{\frac{V-V'}{V}}$; and for gases this is now stated to equal the pressure on it when in that state, namely, P′.*

* If the strain takes place very suddenly, the elasticity is greater than P, being 1·4 times P. This is because the temperature does *not* then remain constant—heat is generated by the compression which has not time to escape. We will suppose, however, that all our compressions and expansions take place slowly enough to allow the temperature of the gas to remain without change.

$$\text{Hence} \quad \frac{P' - P}{V - V'} = \frac{P'}{V}, \quad \text{or} \quad PV = P'V',$$

$$\text{or } P : P' :: V' : V ;$$

or, in words, the volume of a given quantity of a perfect gas varies inversely with the pressure, other things being equal. If the pressure be doubled, the volume is halved; if the pressure be halved, the volume is doubled. This is called Boyle's law, and may be verified by the bent tube of fig. 92. Its short leg is closed, its long leg open. Mercury poured down the long leg confines some air in the short one and compresses it, the whole pressure on the air in the tube being that of the atmosphere plus that of the column of mercury in the tube.

Fig. 92. Boyle's Tube.

If the mercury stands 30 inches higher in the long leg than in the short, the original volume of the air will be found to be halved: for the original pressure it sustained was one atmosphere, and now it is two. Another 30 inches of mercury will make it shrink into one-third its original bulk, and so on. Under ordinary atmospheric pressure, 14·4 grammes of air occupy 11·2 litres (see sect. 188); but under a pressure of two atmospheres they shrink to 5·6 litres.

The shortest statement of Boyle's law is that, *ceteris paribus*,

$$PV = \text{constant};$$

but remember that *cetera* must be *paria;* the temperature must not change, ***neither must the quantity*** (that is, mass) ***of gas.***

One gramme of hydrogen under a pressure of 76 centimetres of mercury, and at 0° centigrade, occupies 11·2 litres, or 11,200 cubic centimetres. Hence the value of the above constant PV for 1 gramme of hydrogen is in absolute measure (centimetre, gramme, seconds), (see sects. 181 and 184):

$$76 \times 13{\cdot}6 \times 981 \times 11{,}200.$$

Call this K. It is the same constant for 16 grammes of oxygen, 14 of nitrogen, 22 of carbonic anhydride, and so on. For 5 grammes of hydrogen or 80 grammes oxygen, the constant is 5K; it is, in fact, proportional to the mass of a gas, but varies for different gases with their molecular weights. A better statement of Boyle's law is that the ratio of pressure to *density*, $\frac{P}{\rho}$, is constant; for this is independent of everything but the nature of the gas and the temperature. If the pressure of any gas be stated in gravitation units, say grammes weight per square centimetre, this constant is called *the height of the homogeneous atmosphere* of that gas (see example 8).

190. The density of a gas, therefore (the mass of unit volume, see sect. 32), is directly proportional to the pressure. One consequence of this is, that as one ascends in the atmosphere, the pressure does not decrease uniformly as in the case of a liquid, but it decreases at first at a more rapid rate, and afterwards more slowly. At a height of only three miles, for instance, the intensity of pressure is half what it is at the sea-level. For the pressure decreases not only by reason of the elevation, but also by reason of the diminution of density accompanying the decrease of pressure. Both causes combine, and the pressure diminishes upwards in what is called *geometrical* instead of in *arithmetical* progression.

191. But just as no actual liquids were perfect, so no actual gas is a perfect gas. They all deviate slightly from Boyle's law; they are probably not infinitely expansible, and certainly not infinitely compressible, for many of them, if squeezed very much, condense into liquids; and as they approach their condensing point, they deviate from Boyle's law a good deal, becoming more and more compressible. Air and hydrogen are the most perfect gases, but in 1878 these too were liquefied by M. Pictet of Geneva, and by M. Cailletet. Still they are, at ordinary pressures and temperatures, a very long way off their condensing point, and they obey Boyle's law with considerable accuracy. They cannot, indeed, be condensed by any amount of simple squeezing; they have to be cooled enormously as well.

Air-pumps.

192. Air-pumps differ in no respect from other pumps except in details of arrangement. Their peculiarity is that the vessels they are used to exhaust or to fill contain always the same volume of fluid ; its density and pressure, however, are diminished or increased to any extent.

Pump No. 3 (fig. 82) is generally used for exhaustion, but pump No. 2 can also be used, and it will at the same time produce condensation in any vessel screwed on to its end B. It is then called a condensing syringe. If it obtains its air from the atmosphere, the same mass of air will be injected at every stroke, and consequently the pressure in a vessel screwed on to B will increase by a fixed amount at each stroke, that is, it will increase in arithmetical progression.

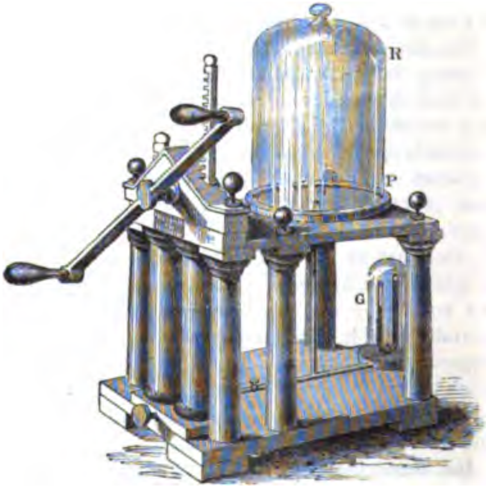


Fig. 93.—Air-pump.

Fig. 93 shews a double-barrelled air-pump with two of the No. 3 pumps arranged to exhaust a glass vessel known (for no very apparent reason) as the 'receiver.' At every stroke the air in the receiver expands to fill both receiver and pump-barrel, and the portion filling the latter is at the reverse stroke expelled into the atmosphere.

Call the volume of the receiver V, and that of a pump-barrel v; the same *volume* of air, v, is extracted at every stroke, but not the same mass, because its density keeps on diminishing. If the pressure of the air in the receiver to start with, is P_0, and after the first stroke, P_1; the product of pressure and volume being constant, we have

$$P_0 V = P_1 (V + v).$$

During the second stroke the volume V again expands to fill the volume $V + v$, without the quantity of the air changing; so, if P_2 is the pressure after the second stroke,

$$P_1 V = P_2 (V + v).$$

Similarly P_3, the pressure after the third stroke, is given by

$$P_2 V = P_3 (V + v);$$

and so on.

The pressure after three strokes may therefore be written

$$P_3 = \left(\frac{V}{V+v}\right) P_2 = \left(\frac{V}{V+v}\right)^2 P_1 = \left(\frac{V}{V+v}\right)^3 P_0;$$

similarly, the pressure after n strokes is

$$P_n = \left(\frac{V}{V+v}\right)^n P_0.$$

The pressures $P_0\ P_1\ P_2\ P_3 \ldots\ldots P_n$ decrease, therefore, in a geometrical progression with the common ratio $\frac{V}{V+v}$

Hence perfect exhaustion (or pressure equal zero) cannot be obtained, even with a perfect pump, without an infinite number of strokes.

193. To indicate the degree of exhaustion, a mercury gauge is commonly used, which may be simply a long tube reaching from the receiver into a cistern of mercury, like a barometer; or it may be of the form shewn separately in

fig. 94, and attached to the pump at G in fig. 93. The closed limb of the U tube is completely full of mercury, and remains so till the air pressure in the little bell jar which is exhausted with the receiver gets unable to support it; it then gradually descends as the exhaustion proceeds, and the pressure of the residual air in the receiver is measured by the difference of level between the mercury in the two limbs.

Fig. 94.

194. *Compressed-air Manometers.* — The diminution of the volume of a gas under pressure will measure that pressure in a more compact way than the mercury gauges of sect. 184 (compare the length of the two branches of the tube, fig. 92), and a manometer on this principle is shewn in fig. 95. Faraday used to measure high pressures in his glass vessels by inserting little conical glass tubes, with one end sealed, containing air and a globule of mercury (fig. 96). As the pressure of the gas in which they were, increased, the globule moved up

Fig. 95.

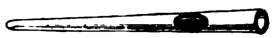


Fig. 96.

and compressed the air in the tube more and more; and the diminution of volume measured the increase of pressure, $P' = P\frac{V}{V'}$.

Sir William Thomson has applied the same principle to ocean sounding, for, since every 34 feet of water adds another atmosphere to the pressure, if the pressure of the water be known, its depth can be calculated. A tube closed at one end is lowered into the sea, like a diving-bell mouth downwards; and a registering arrangement records how far the water has entered the tube, and therefore how far the air in it has been compressed.

EXAMPLES.

1. What is the height of the mercury barometer when the intensity of the atmospheric pressure is a megadyne per square centimetre? (A million dynes is called a megadyne.) *Ans.* 75 centimetres.
2. If a mercury barometer falls one inch, what will be the fall of a water barometer? *Ans.* 13·6 inches.
3. Shew that the oscillation of the column in a 'siphon' barometer, with its long and short limbs of equal cross section, is only half that of the column of a cistern barometer with an infinitely large cistern.
4. Shew that the motion of the top of the mercury in a barometer may be doubled by inclining the upper part of the tube at an angle of 30° to the horizon.
5. What is the total pressure inside a steam boiler when the mercury gauge (fig. 85) stands at 150 centimetres and the barometer at 75?

   Ans. 3 megadynes per square centimetre.
6. A barometer in a diving-bell indicates a pressure of 45 inches of mercury, the height of the barometer at the surface of the earth being 30 inches. What is the depth of the diving-bell? *Ans.* 17 feet.
7. The piston of a lift-pump is 7 inches in diameter, and the depth of the water in the mine below the spout where the water is discharged is 533 yards. Find the least force which can raise the piston.

   Ans. About 12 tons weight.
8. Find the height of the homogeneous atmosphere at zero centigrade. (This means the height an atmosphere must have, if it were made of incompressible fluid, of the same density as the real atmosphere at any point, and if it exerted the same pressure

as the real atmosphere does at that point. See sect. 189.)

Ans. 76 × 13·6 ÷ ·001293 centimetres, about 8000 metres (or roughly about 5 miles).

N.B.—Notice that this does not vary with the barometric height.

9. If a rectangular mass of cork, dimensions 10 × 8 × 5 centimetres, is counterpoised in air by 80 grammes of platinum, find the mass of the cork (neglecting the floating power of the air on the platinum).

Ans. 80·517 grammes.

10. A mass of wood (sp. gr. ·6) is counterpoised by 105 correct grammes of iron (sp. gr. 7·5); find the mass of the wood (or its true weight *in vacuo*).

Ans. The volume of the iron is 14 c. c., so its apparent weight is 105 − (14 × ·001293); and this is equal to the apparent weight of the wood, which is $x - (\frac{5}{3}x \times \cdot 001293)$, where x is the number of grammes of the wood; hence $x = 105{\cdot}294$.

11. A piece of metal weighs 2·4 grammes in mercury and 9 grammes in water; what would be its weight *in vacuo?* *Ans.* 9·523 grammes.

12. A siphon barometer which has a little air in its 'vacuum' only indicates a pressure of 72 centimetres; and on pouring more mercury into the open limb until the vacuum is diminished to half its former bulk, the difference of levels becomes 70 centimetres; what is the true height of a proper barometer?

Ans. 74 centimetres.

13. The cylinder of an air-pump barrel has a capacity $\frac{1}{10}$th of that of the receiver it is used to exhaust. Find the pressure in the receiver after 1 and after 2 strokes of the pump, if the original pressure was 77 centimetres.

Ans. After one stroke 70 centimetres; after two, $63{\cdot}\dot{6}\dot{3}$; after three strokes $57{\cdot}\dot{9}42\dot{0}$, and so on; each time dividing by 11 and multiplying by 10.

14. If a quantity of air is squeezed up in a closed tube into $\frac{1}{100}$th of its original volume which it occupied when the barometer was at 30 inches, what pressure does it indicate; and at what depth under water would this pressure be experienced?

Ans. ·652 ton weight per square inch. At a depth of 3366 feet.

15. A diving-bell, 6 feet high and weighing half-a-ton, weighs apparently $\frac{1}{3}$d of a ton at a depth of 86 feet under water. What would be its apparent weight when just immersed?

Ans. (Neglecting the thickness of its wall), 0.

[*For Miscellaneous Exercises, see pages* 198 *to* 202.]

MISCELLANEOUS EXERCISES.

SET I.

1. A bullet is fired vertically upwards with a velocity of 1600 ft. per second; find how high it will rise, and how soon it will hit the ground.

2. If you throw 2 balls up with a velocity of 192 feet per second, one 2 seconds after the other, when and where will they meet?

3. The intensity of gravity on Jupiter is 2·6 times as much as on the earth. How long would a body take to fall on Jupiter from a height of 167 feet?

4. A man in a stationary balloon throws a ball up with a velocity of 96; where is the ball in 10 seconds, and what velocity has it?

5. A man on a cliff 300 feet high throws a stone down to the ground in 3 seconds; with what velocity did he throw it?

6. A balloon is going up at the rate of 80 feet per second, and when at a height of 1000 feet a halfpenny is dropped over its edge; what does the halfpenny do? When and with what velocity does it hit the ground?

7. A cannon is fired horizontally at a height of 10 feet above a lake; how soon does the ball hit the water?

8. From a cliff 400 feet high one stone is dropped from the top, and at the same moment one thrown up from the bottom with a velocity sufficient to carry it to the top of the cliff; when and where do they meet?

9. A weight of 6 lbs. is attached to one end of a string and 10 lbs. to the other, and the string is hung over a freely movable pulley; find the tension in the string; and how long 6 feet of the string take to pass over the pulley.

10. A knife is dropped from the middle of the ceiling of a railway-carriage going 50 miles an hour; how does it fall?

11. A train is going 50 miles an hour; a man throws a ball up vertically at 32 feet a second. What becomes of it, and how long will it take before it comes back to his hand?

12. An iron cage descends a mine. The tension in the rope equals the weight of 200 lbs.; when at rest it was 225 lbs. Find the time of descending 100 feet from rest.

13. Find the tension on a rope which draws a carriage of 8 tons weight up a smooth incline of 1 in 5, and causes an increase of velocity of 3 feet per second. If on the same incline the rope breaks when the carriage has a velocity of 48·3, how far will it continue to move up the incline?

SET II.

1. Six forces act on a point making angles of 60° with each other. Their magnitudes are 4, 6, 5, 1, 10, 7. Find, by drawing, the magnitude and direction of the resultant.

2. Three forces, 10, 10, 36, act on a point at angles of 120°; find resultant.

3. A hundredweight is hung to 2 hooks in the ceiling by 2 cords, one 3 times as long as the other; find by construction the tension in each.

4. A weight of 42 lbs. is balanced at a height of 6 feet above the ground on 2 inclined rods meeting in a point under the weight. One rod supports 36 lbs., the other, 20 lbs.; find the length of each rod by a construction.

5. Two boys sitting at the ends of a plank 11 feet long see-saw over a log; the log sustains 2 cwt., and one boy is 4 feet off it. What are the boys' weights?

6. Forces of 5, -3, 4, -2, 6, are arranged along a rod at equal distances (2 inches); find resultant.

7. A uniform rod weighing 4 lbs. has 12 lbs. at one end and 18 at the other. The centre of gravity of the whole is 9 inches from the middle; what is the length of the rod?

8. Two men carry a block of iron weighing 176 lbs. suspended from a pole 14 feet long; each man is 1 foot 6 inches from his end of the pole. Where must the block hang in order that one man may bear $\frac{3}{5}$ of the weight borne by the other?

SET III.

1. State the characteristic difference between solids and fluids in relation to the transmission of pressure, and explain clearly what is meant by the equal transmission of pressure by fluids in all directions.

2. Shew what the pressure exerted by a liquid on any part of the surface of the containing vessel depends upon, and explain how to calculate the amount of this pressure when the necessary data are given.

3. Describe and explain an experiment proving that the pressure on the base of a vessel may be greater or less than the weight of liquid in the vessel.

4. Prove that the resultant pressure on an immersed body acts vertically upwards, and is equal to the weight of a quantity of fluid equal in bulk to the body.

5. Describe experiments proving that the air has weight, and shew how the weight of a given volume of air can be approximately ascertained.

6. Explain the construction and action of the barometer, and shew how to ascertain the pressure per unit of surface exerted by the air.

7. In a barometer which contains a little air in the space above the mercury, this space amounts to 20 c. c. when the mercury in the tube is 70 centimetres above the mercury in the cistern; on lowering the tube, so that the mercury in the tube is only 67 centimetres above that outside, the space above the mercury measures 12·5 c. c. Find true barometric height.

SET IV.

1. Define the terms 'force,' 'work,' 'power,' 'energy,' 'momentum.'

2. What experiments or observations can you adduce to prove that the weight of a body is proportioned to its inertia?

3. A light frictionless pulley with a string over it has 17 ounces hanging on one end of the string and 15 on the other. Calculate the tension in the string, and the acceleration of either mass.

4. Discuss the direct impact of two small spheres on one another in the light of Newton's law of motion, shewing what happens to their separate momenta and energies—(*a*) When inelastic; (*b*) When elastic.

5. Calculate the position of the centre of gravity of a light square frame, six inches in the side, with weights at its four corners, proportioned to 5, 2, 7, 4.

6. Explain the common air-pump, and show how to calculate the pressure of the residual air after a specified number of strokes.

7. A pressure is often specified as equal to 70 centimetres of mercury. Express this in absolute measure, for example, in dynes, per square centimetre.

8. How can the specific gravity of sand be practically determined? Illustrate by an example.

9. A ladder, weighing half a hundredweight and 30 feet long, rests against a smooth wall, with its foot 15 feet from the bottom of the wall. Find the pressure on the wall and ground, taking the centre of gravity of the ladder as one-third of its length up.

10. A stone is thrown up with a velocity of 192 feet a second. Find how high it ascends, and how long it takes before returning to the hand. Find also its position three seconds after throwing.

11. A body, moving with uniform acceleration, describes 180 feet in the fifth second of its motion. Find its acceleration and the distance travelled in the five seconds.

12. A tricycle, weighing 80 lbs., moving along a level road at the rate of 12 miles an hour, is stopped by the friction in the space of 60 yards. What must the average resistance to its motion have been?

13. Of three blocks of wood one is pivoted on a point, another rests on an inclined plane, and the third floats in water. Discuss the conditions necessary for equilibrium and for stable equilibrium in each.

14. How would you determine the coefficient of friction between two given flat surfaces?

SET V.

1. Define Acceleration, Inertia, Force, and Work, shewing how each is measured, and giving the most important standards or units of each in present use. What is meant by the statement that $g = 981$?

2. It is found experimentally that *in vacuo* all bodies fall through a given distance in the same time: what consequences can be deduced from this fact? If the distance were doubled, how much more time would the fall require?

3. A falling body is observed to describe 100 feet in the last second of its motion; find how far it must have fallen, and also the time taken. Consider $g = 32$, and neglect the resistance of the air.

4. What is meant by *centripetal acceleration*? Find the force necessary to cause a planet of mass m to revolve in a circle of radius r in a time T.

5. How have the masses (*a*) of the Earth and (*b*) of the Sun been ascertained?

6. Define moment of inertia. Find the time a solid cylinder will take to roll down an inclined plane 20 feet long, inclined at 30° to the horizon; the moment of inertia of the cylinder being $\frac{1}{2} mr^2$.

7. Explain clearly what is meant by the centre of oscillation of a swinging rigid body; and determine its position in the case of a uniform rod swinging about one end, its moment of inertia being $\frac{1}{3} mr^2$.
How has the intensity of gravity been accurately measured?

8. Three weightless rods are jointed together, the two free ends are pivoted to firm supports, and the middle rod is loaded at any point; sketch the position of equilibrium which the system will take up, and show how to determine by construction the stress in each of the unloaded rods.

9. A weight rests on an inclined plane of given roughness; find by construction the least force which will suffice to pull the weight up the plane, showing the angle at which it must act. Also find how much the plane must be tilted in order that the weight may slide down.

10. A rectangular block weighing 20 lbs. with a square base 8 inches in the side, is set up on a level table; and it is found that a force of 5 lbs. weight, if applied below a certain point, is just able to make it slide, while if it is applied above that point the block topples over. Find the position of this critical point, and also the coefficient of friction between the block and the table.

11. Define density and specific gravity.
A piece of iron weighing 84 grammes is put into a beaker, which is then filled with water up to a certain mark above the level of the iron, and the whole is found to weigh 128 grammes. The iron is then turned out, and water poured in till the beaker is again full up to the same mark; it now weighs 56 grammes. Find the specific gravity of the piece of iron, and its weight when under water.

12. Describe an accurate form of barometer; and give some of the contrivances which have been employed to make barometers more sensitive to slight changes of pressure.

13. A balance is arranged under water, and a mass of iron-ore in one of its pans is counterpoised by 3 kilogrammes of lead in the other. What is the mass of the iron-ore, its specific gravity being 7, while that of lead is 11?

14. A ladder standing on rough horizontal ground rests against a rough vertical wall. Find its position when just not slipping down.

15. A solid regular triangular pyramid is drawn along a rough table by a horizontal force. What is the greatest height at which the force can be applied without upsetting the pyramid?

16. A weight is swung round and round in a vertical circle by a rope of given length; determine the conditions that the rope may keep tense.

SET VI.

1. Describe fully a method of comparing the specific gravities, (*a*) of a solid and liquid, (*b*) of two liquids.

2. Explain clearly why a diving-bell which is not supplied with additional air appears to get heavier as it descends in water; and shew how its depth might be ascertained, either by reading a barometer inside the bell, or by noticing the height to which the sea-water had risen into its interior.

3. A gun is fired horizontally, at a height of 169 feet above a lake, with an initial velocity of 1000 feet a second. Find how soon, and how far away, the ball will first strike the lake, neglecting the resistance of the air and taking the acceleration produced by gravity as 32 feet-per-second per second.

4. What is the principal of *virtual work* (or *virtual velocities*)? Illustrate it by applying it to find the mechanical advantage of any system of pulleys when the weights of the pulleys are not neglected.

5. A mass of 3 lbs., hanging vertically, drags a mass of 17 lbs. along a perfectly smooth level table by means of a string over the edge. Find the acceleration, and the distance travelled in five seconds.

6. A projectile is shot up *in vacuo*,

(*a*) with a given initial velocity,
(*b*) in a given direction.

Find in the first case the necessary direction, and in the second case the necessary velocity, that the projectile may hit an object at a given horizontal distance from the gun, and at a given elevation. Show specially in each case what are the conditions that the body may be beyond the range.

7. Find the tension in a flexible rope which is passed round a single movable pulley supporting 20 lbs., while to the free end of the rope 12 lbs. is hung; and find the acceleration upwards of the 20-lb. weight (neglecting the mass of the pulley and of the rope).

Show that this may be done either by direct application of Newton's Second Law, or by a work-and-energy method.

8. State Newton's Third Law. If two spheres of given masses and coefficients of restitution impinge directly on each other with known velocities, show how to find the velocities after the impact.

Consider specially the case when the masses are equal.

9. A ball let fall on to a stone slab from a height of 16 feet, bounces the first time to a height of 9 feet. What is the coefficient of restitution, neglecting the resistance of the air? and how high will the ball bounce next time?

Find also the total distance it will travel before coming to rest.

10. Given the moment of inertia of a body about an axis through its centre of gravity, determine it about any other parallel axis.

11. A uniform rod of given length is swung as a pendulum about a given point in it. Find the length of the equivalent simple pendulum, and find for what point of suspension the time of swing will be a minimum. [The moment of inertia of the rod about its centre is $\frac{1}{12} ml^2$.]

12. A conical pendulum or governor ball (considered as a particle) is spinning round a vertical axis 20 times a second. Find its distance in inches or centimetres below a horizontal plane through its hinge.

13. How has the value of *g* been determined accurately? Explain its variation with latitude.

14. Define the unit of force. How has the force of attraction between two pound masses one foot apart been determined?

Shew how a knowledge of this would enable us to express the mass of the earth in terms of its bulk, and also would tell us the mass of the sun in terms of its distance.

15. What is the law of acceleration to which a body is subject if it is dropped into a deep hole in the earth? How long would it take to reach the centre, if the density of the earth is 5·7. Shew that this time is independent of the size of the earth.

16. A body slides down a plane inclined at a given angle to the horizon; determine its acceleration, and the time taken to slide down, supposing the coefficient of friction constant.

Determine also the least force necessary to support the body, and the direction in which it must act.

17. A uniform narrow beam is pivoted at one end at a given height above a pond, and the other end rests in the water. Determine its position of equilibrium, the specific gravity of the wood being known.

Discuss its change of position if the level of the pond is gradually rising towards, and ultimately above, the pivot.

SPECIMENS OF EXAMINATION PAPERS.

South Kensington Examination, 1875.

First Stage. *Examiner*—Rev. J. F. Twisden, M.A.

The value attached to each question is shewn in brackets after the question. But a full and correct answer to an easy question will in all cases secure a larger number of marks than an incomplete or inexact answer to a more difficult one. *Three hours are allowed for this paper.*

1. Three equally heavy points are placed at the angular points of a triangle; shew how to find the position of their centre of gravity. (8.)

2. A cube is placed on a horizontal plane; in what positions is it in stable, and in what positions in unstable equilibrium? (8.)

3. If forces of 5, 7, and 10 units act on a point, shew by a diagram how they must be adjusted so as to be in equilibrium.

State the principle in statics called the triangle of forces. (10.)

4. The length, base, and height of an inclined plane are 13 feet, 12 feet, and 5 feet respectively; if a body weighing 100 lbs. is placed upon it, what force acting along the plane will support it—the plane being smooth—and what will be the pressure on the plane? (12.)

5. What is in general the relation between two forces, P and Q, which are in equilibrium on a straight lever?

What is meant by the moment of a force with reference to a point? (10.)

6. Find the relation between the power and the weight in a single movable pulley, the parts of the cord being parallel. Shew by a diagram how the pulley is supported, and find what force is exerted on the beam or fixed point. (8.)

7. What is meant by a unit of work, and by a horse-power? How many foot-pounds of work are required to raise 30,000 lbs. of water from a depth of a furlong; and how many horse-powers to do it in five minutes? (8.)

8. What is meant by the inertia of matter?

State the first law of motion; and give illustrations of it. (8.)

9. A body moving from rest under the action of a constant force acquires in each second an additional velocity of 12 feet per second; find (1) the distance it passes over in the first five seconds of its motion; (2) the velocity it has after passing over 96 feet from its starting-point. (12.)

10. It is found that a body (considered as a point) has its velocity increased by 7 feet a second in any second of its motion; it is known that the body weighs 23 lbs.; what is the magnitude of the force producing this acceleration?

How many pounds of matter would this force support against gravity in a place where $g = 32·2$? (10.)

11. State the rule which enables us to determine the amount of the pressure exerted by a fluid against a plane area.

A reservoir has one of its walls vertical: a circle a yard in radius is described on that wall; when the water just covers the circle, what is the amount of the pressure exerted by the water on the portion of the wall within the circle? (A cubic foot of water may be taken to weigh 1000 ounces.) (10.)

12. Mention a way in which you could determine the specific gravity of a fluid by the balance.

A glass ball weighs 3000 grains, and has a specific gravity 3; what will be its apparent weight when immersed in a liquid whose specific gravity is 0·92? (10.)

13. Describe briefly the action of the common force-pump. If the plunger has a cross section of 8 square inches, and works 50 feet below the cistern, what pressure is required to force it down? (10.)

14. Describe briefly the mercurial barometer in its simplest form.

When a barometer is taken up a lofty hill, why should it fall?

What would it do if taken down a deep mine, and why? (8.)

Second Stage, or Advanced Examination.

INSTRUCTIONS.

Read the General Instructions at the head of the Elementary paper.

You are not permitted to *attempt* more than *ten* questions. You may select these from any part of the paper.

21. State and prove the rule by which, when the position of the centre of gravity of a plane area is known, the volume can be found of the solid formed by the revolution of the area round an axis in its plane (Guldinus' rule).

Apply the rule to find the volume of a right cone on a circular base. (20.)

22. When any number of parallel forces act in one plane on a rigid body, investigate the equations which serve to determine the magnitude of their resultant, the line along which it acts, and its direction along the line.

How can you ascertain from these equations when the forces reduce to a couple, and when they are in equilibrium? (15.)

23. AB is a lever without weight acted on at A and B by two equal forces, P and Q, whose directions contain an angle of 60°; P acts at right angles to AB. Find where the fulcrum must be situated that P and Q may be in equilibrium, and the pressure they exert on the fulcrum. (15.)

24. What is the angle of friction or limiting angle of resistance?

When a body urged against a rough fixed plane by certain forces is at rest, to what extent is the direction of the reaction of the plane against it known? (15.)

25. If a machine is in a state of uniform motion, what relation must exist between the 'power' and the 'weight'—passive resistances being put out of the question?

What is the meaning of the statement, that in a machine what is gained in power is lost in velocity? Illustrate the statement briefly by reference to any simple machine such as a wheel and axle. (20.)

26. A fly-wheel weighing 7 tons turns on a horizontal axle 1 foot in diameter; if the coefficient of friction between the axle and its bearing is 0·075, what number of foot-pounds of work must be done against friction while the wheel makes 10 turns? (15.)

27. State and illustrate briefly the relation which holds good between the mutual actions of two bodies on each other. (Newton's third law of motion.) (15.)

28. In uniformly accelerated motion shew that $s = \frac{1}{2}ft^2$. A body moving from rest under the action of a constant force describes 50 feet in the first 5 seconds of its motion; what distance does it describe in the first 7 seconds of its motion, and with what velocity is it moving at the end of the 7 seconds? (20.)

South Kensington Examination, 1876.

First Stage. *Examiner*—Rev. J. F. Twisden, M.A.

1. What is meant by the density of a body? How are the densities of bodies compared? If 5 cubic inches of mercury weigh 2·45 lbs., and 2 cubic inches of cast-iron weigh 0·52 lbs.; what ratio does the density of mercury bear to that of cast-iron? (8.)

2. Define the centre of gravity of a body. State how to find the centre of gravity of two points of known masses. A weight of 4 oz. is placed at each of three of the corners of a square; where must a fourth weight of 8 oz. be placed that the centre of gravity of the four weights may be at the intersection of the diameters of the square? (12.)

3. A cube is placed on a plane with an edge parallel to a horizontal line round which the plane is capable of turning. At what inclination of the plane will the cube topple over? (The cube is supposed to be so rough as not to slide.) (8.)

4. A, B, C is a thread suspended from the end A; a weight of 5 oz. is fastened to the other end C; and a weight of 10 oz. to an intermediate point B. What are the magnitudes of the tensions to which the parts AB and BC of the thread are exposed, and of the reaction of the supporting point at A? (10.)

5. Two parallel forces of 3 and 4 units act on a body in opposite directions; specify the force required to balance them, and shew by a diagram how the three forces act. (8.)

6. What is meant by the resolution of a force into rectangular components? A force of 18 units acts along a line making an angle of 30° with a given line; find, by construction or otherwise, its components along and at right angles to that line. (8.)

7. Describe briefly the wheel and axle, and find the relation which holds good in it between the power and the weight.

If a power of 15 lbs. will support a weight of 50 lbs. on this machine, what ratio must the radius of the wheel bear to the radius of the axle? Would your answer be affected by any circumstances of which you have taken no account? (10.)

8. A weight of 500 lbs., by falling through 36 ft., lifts, by means of a machine, a weight of 60 lbs. to a height of 200 ft. How many units of work have been expended on friction, and what proportion does the expenditure bear to the whole amount of work done? (10.)

9. It is found that a body moving in a straight line has its velocity increased by equal amounts in equal times; what inference can be drawn as to the force or forces acting on the body during its motion?

A body moves from rest and has its velocity uniformly accelerated; if it describes 30 ft. in the first second and a half of its motion, what distance does it describe in the next second? (12.)

10. When a body moves in a circle, what is meant by its centrifugal force? On what does it exert its centrifugal force?

A body weighing 4 lbs. tied to one end of a string is whirled round 50 times a minute, in a circle 3 ft. in radius; what is the amount of the centrifugal force of the body? (12.)

11. At a depth of one mile below the surface of the sea, what is the pressure of the water in pounds per square inch? (A cubic foot of sea water weighs 1025 oz.) (8.)

12. State the relation between the pressure and volume of a gas at a given temperature, called Boyle's or Mariotte's law. Is this law exactly or only approximately true? Give briefly a reason for your answer. (10.)

13. Describe the mercurial gauge by which the degree of exhaustion in the receiver of an air-pump is measured. (8.)

UNIVERSITY COLLEGE, LONDON.—*June* 1879.

Junior Class.

1. Distinguish between the *mass* of a body and its *weight*, and give examples of phenomena which depend upon one or the other respectively.

2. A stone dropped from the ceiling of an hotel lift, which is ascending with a constant *velocity* of 2 feet a second, takes $\frac{3}{4}$ second to strike the floor of the lift. How long would it have taken to drop if the lift had been descending with a uniform *acceleration* of 1 foot-per-second per second?

3. When is a force said to do *work*? and how is work measured? Write down an equation expressing the relation between the work done upon a body or system of bodies, and the general results (such as change of velocity, &c.) produced, and apply it to determine the motion in the case of a single fixed and single movable pulley when the power is 12 lb., and the weight (including the movable pulley) 29 lb., the inertia of the pulleys being neglected.

4. What is meant by the *moment* of a force about a point? Prove that, if the forces acting on a body are in equilibrium, the sum of their moments about any point is equal to nothing.

5. State the relation between weight, volume, and specific gravity. Describe and explain in detail a method of finding experimentally the weight of a cubic inch of water.

6. Describe 'Nicholson's Hydrometer,' and explain the method of determining the weight or specific gravity of a solid body by its means.

7. Describe the construction and explain the action of the common air-pump. Shew what essential limit there is to the degree of exhaustion producible by it, and also what further limitations arise in practice from the construction of the instrument.

8. Explain how to determine the absolute intensity of atmospheric pressure at any time and place, and point out any corrections or practical precautions required for accuracy.

CAMBRIDGE, ST JOHN'S COLLEGE.—*April* 1881.

HYDROSTATICS AND DYNAMICS—First Year.

1. Calculate, in pounds, the amount of pressure exerted by water on a circular area 3′ in diameter, its centre being at a depth of 20 feet below the highest point of the liquid.

2. A triangular area with its vertex in the surface and base horizontal is pressed by water: divide it into two portions by a line parallel to the base, on which the pressures are equal.

3. Divide it into n portions by horizontal lines, on which the pressures are equal.

4. State and prove the general case of the principle of Archimedes.
Describe any method of raising a sunken ship.

5. A heavy heterogeneous body rests, wholly immersed in water, on two smooth vertical props; calculate the pressure on each prop.

6. Prove, from the laws of Boyle and Dalton, the fundamental equation, $\frac{pv}{T}$ = Const., for a given mass of gas.

7. Assuming Earth's radius = 4000 miles, Moon's distance = 240,000 miles, what is the acceleration value of the Earth's attraction at the position of the Moon?

8. If a particle falls from a point A to another B, prove that the change in the square of its velocity is the same, whatever be the (smooth) curve or surface on which the motion takes place.

9. A spring balance in a balloon marks o when the balloon is at rest; what will be its indication when the balloon is rising with constant velocity? Shew how to find the acceleration of the balloon's velocity when the balance marks n.

10. Prove graphically the equations of uniformly accelerated motion.

UNIVERSITY OF LONDON MATRICULATION EXAMINATION.
June 1883.

1. Define Acceleration, Force, Energy. A train which is uniformly accelerated starts from rest, and at the end of 3 seconds has a velocity with which it would travel through 1 mile in the next 5 minutes; find the acceleration.

2. Describe Attwood's Machine.

Two scale-pans, each weighing 2 oz., are suspended by a weightless string over a smooth pulley. A mass of 10 oz. is placed in one and 4 oz. in the other. Find the tension of the string, and the pressure on each scale-pan.

3. Distinguish between *mass* and *weight.*

A certain force acting on a mass of 10 lb. for 5 seconds, produces in it a velocity of 100 feet per second. Compare the force with the weight of 1 lb., and find the acceleration it would produce if it acted on a ton.

4. The horizontal and vertical components of a certain force are equal to the weights of 5 and 12 lbs. respectively; what is the magnitude of the force?

Supposing this force to act for 10 seconds on a mass of 8 lbs., which is also exposed to the action of gravity and is initially at rest, what velocity will be communicated to the mass, the vertical component of the force acting upwards?

5. The arms of a bent lever are at right angles to one another, and their lengths are in the ratio of 5 to 1. The longer arm is inclined 45° to the horizon, and carries at its extremity a weight of 10 lbs. The end of the shorter arm presses against a smooth horizontal plane. Draw a figure shewing the forces in action, and find the pressure between the shorter arm and the plane.

6. What is the centre of gravity of a body?

A uniform plate of metal 10 inches square has a hole 3 inches square cut out of it, the centre of the hole being $2\frac{1}{2}$ inches distant from the centre of the plate. Find the position of the centre of gravity of the plate.

7. A body is the equilibrium under the action of three forces whose directions are not parallel. State fully the conditions which must be fulfilled.

A heavy uniform ladder rests with its upper end pressing against a smooth vertical wall: shew by a figure how to determine the direction of the resultant force acting upon the foot of the ladder.

8. What is meant by the Specific Gravity of a substance?

A body floats with one tenth of its volume above the surface of pure water. What fraction of its volume would project above the surface if it were floating in liquid of specific gravity 1·25?

9. Explain the principle of action of the common pump. How may it be converted into a lift pump? A lift pump is employed to raise water through a vertical height of 200 feet. If the area of the piston be 100 square inches, and a cubic foot of water contain $62\frac{1}{2}$ lbs., what force (in addition to its own weight) will be required to lift the piston?

10. Describe the common Air-pump.

In the process of exhausting a certain receiver, after 10 strokes of the pump the mercury in the gauge stands at 20 inches, the barometer standing at 30 inches. At what height will the mercury in the gauge stand after 20 more strokes?

11. A beaker of water with a wooden sphere floating on it is placed under the receiver of an air-pump: explain how the sphere will be affected on exhausting the air from the receiver.

If the density of water be 800 times that of air at ordinary pressure, state precisely what will happen supposing the sphere originally to have been immersed to the depth of its centre.

University of London Matriculation Examination.
June 1884.

1. Define velocity, acceleration, force, and momentum. What is meant by uniformly accelerated motion?

A heavy body slides down a smooth plane inclined 30° to the horizon; through how many feet will it fall in the fourth second of its motion (taking 'g' to be 32 foot-second units)?

2. Two heavy bodies are connected by a flexible string which passes over a fixed pulley. Show how to find the acceleration with which the heavier body will descend.

If the masses of the two bodies are respectively 17 oz. and 15 oz., find the tension of the string.

3. What is meant by the *mechanical advantage* of a system of pulleys or other machine?

Find the relation between the power (P) and the weight (W) in a system of five movable pulleys, in which each pulley hangs by a separate string, and the weight of each pulley is equal to P.

4. What is meant by the centre of gravity of a body? Shew how to find the centre of gravity of a system of heavy particles lying in one plane.

Weights of 1 lb., 2 lbs., 3 lbs., 4 lbs. are placed at the angular points A, B, C, D respectively of a square, A B C D; find the distance of the centre of gravity of the system from the centre of the square.

5. Distinguish between *mass* and *weight*.

A cannon ball whose mass is 60 lbs. falls through a vertical height of 400 feet; what is its energy? With what velocity must such a cannon ball be projected from a cannon to have initially an equal energy?

6. Distinguish between a solid and a fluid. What is the special characteristic of a *perfect* fluid?

A cube of one foot edge is suspended in water with its upper face horizontal, and at a depth of $2\frac{1}{2}$ feet below the surface; find the pressure on each face of the cube, assuming that the mass of a cubic foot of water is 1,000 oz.

7. Define specific gravity, and explain the principle of the common hydrometer. The stem of a common hydrometer is cylindrical, and the highest graduation corresponds to a specific gravity of 1·000, while the lowest corresponds to 1·200. What specific gravity corresponds to the point which is exactly midway between these two divisions?

8. What conditions limit the height to which water can be raised by means of a common pump? How must the pump be modified when it is required to raise water to a height greater than the limit thus assigned? Explain the contrivance in the lift pump for enabling the piston rod to work water-tight through the cylinder cover.

9. How would you determine the specific gravity of a gold medal by means of a hydrostatic balance furnished with brass weights?

Explain how each weighing and the final result will be affected by the presence of the air, if no correction is made for the air displaced.

10. Describe some experimental method by which the weight of a cubic foot of air may be determined. How does the density of the air vary with the pressure to which it is exposed, and how may this relation be experimentally determined?

University of London Matriculation Examination.

1. Explain a convenient method of representing geometrically the velocity of a body moving according to a known law, and the distance passed over by it.

Employ the method to find the distance traversed in ten minutes by a train which has a velocity of 20 miles an hour, and which has its speed diminished at a uniform rate of 5 miles an hour.

2. Define force, and explain how it is measured.

What units of force are employed in this country for scientific purposes?

How would you determine the number of British absolute units of force in the weight of a pound?

3. A particle is acted upon by any number of forces in one plane. How would you express the conditions necessary for its equilibrium?

If the system of forces be reduced to three, what simple geometrical relation must exist between their magnitude and their directions?

4. A particle, whose mass is M pounds, moves from rest under the action of a force of P units, which is constant in magnitude and direction. How far will the particle move in n seconds, and what space will it describe in the n seconds?

If the force be the weight of the body, and the particle traversed 176·99 feet during the sixth second of its motion, find the value of 'g.'

5. A body is capable of turning freely about its centre of gravity, which is fixed. If the body is displaced in any manner, subject to this condition, and be then left to itself, how will it behave?

A uniform rod, AB, is 4 feet long, and weighs 3 lbs. One pound is then attached to the end A, 2 lbs. at a point distant 1 foot from A, 3 lbs. at 2 feet from A, 4 lbs. at 3 feet from A, and 5 lbs. at end B. Find the distance from A of the centre of gravity of the system.

6. A quantity of heavy liquid is at rest under the action of gravity, its surface extending over a considerable area. State what you know respecting

(1) The form of the free surface;

(2) The relation between the pressures in different directions at any point;

(3) The magnitude of the pressure at different depths below the surface.

How might the value of 'g' in different latitudes be determined by means of a delicate pressure gauge immersed at a constant depth, say 20 feet, beneath the surface of still water?

7. A cubic foot of water weighs 1000 ounces. A cylindrical test tube is held in a vertical position, and immersed mouth downwards in water. When the middle of the tube is at a depth of 32·75 feet, it is found that the water has risen half-way up the tube. Find the atmospheric pressure in pounds weight per square inch.

8. Define Specific Gravity.

Suppose that a cubic foot of air weighs 1·2 oz., and a cubic foot of water 1000 oz. A balloon so thin that the volume of its substance may be neglected, contains 1·5 cubic feet of coal-gas, and the envelope, together with the car and appendages, weighs 1 oz. The balloon just floats in the middle of the room, without ascending or descending; find the specific gravity of coal-gas, (1) compared with air, (2) compared with water.

9. Explain the principles on which the use of Bramah's press depends; and show how to find the relation between the power and the pressure, when the areas of the pistons are given.

If a pressure of 1 ton is produced by a power of 5 pounds, and the diameters of the pistons are in the ratio of 8 to 1, find the ratio of the lengths of the arms of the lever employed to work the piston.

THE END.

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