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Summary. This note provides an elementary short proof of the Knaster-Kuratowski-Mazurkiewicz-Shapley (K-K-M-S) Theorem based on Brouwer's fixed point theorem. The usefulness of the K-K-M-S Theorem lies on the fact that it can be applied to prove directly Scarf's (1967) theorem, i.e., that any balanced game has a non-empty core. We also show that the K-K-M-S Theorem and the Gale-Nikaido-Debreu Theorem can be proved by the same arguments.

1 Introduction

In a seminal paper Scarf (1967) proved that any non-transferable utility (NTU) game whose characteristic function is balanced, has a non-empty core. His proof is based on an algorithm which approximates fixed points.

Shapley (1973) provided a generalization of the Sperner Lemma, which in turn was used to obtain a generalization of the classical Knaster-Kuratowski-Mazurkiewicz (K-K-M) Lemma. The latter result has been known as the K-K-M-S Theorem and it is useful because it allows to prove directly Scarf's Theorem, i.e., that any NTU-game whose characteristic function is balanced has a non-empty core. In other words, Shapley replaced the algorithm of Scarf by a generalized version of the Sperner Lemma.

Subsequently, Ichiishi (1981) gave an alternative short proof of the K-K-M-S Theorem which is based on the coincidence theorem of Fan (1969). It should be noted that the coincidence theorem was proved by means of an existence of maximal elements theorem which in turn was obtained by means of an infinite dimensional generalization of the classical K-K-M Theorem.

[1]We want to thank Roko Aliprantis for useful comments.

Ichiishi (1987) and Shapley-Vohra (1991) provided alternative proofs of the K-K-M-S Theorem based on the Kakutani fixed point theorem.[2]

The purpose of this note is to provide one more proof of the K-K-M-S Theorem which seems to us to be short and elementary. In particular, it is based on the Brouwer fixed point theorem for which by now we have completely elementary proofs (see for instance Kannai (1981)). Our proof has been inspired by reading Ichiishi (1981, 1987). In essence, we have replaced some of his arguments by a simple maximal element theorem which is a corollary of the Brouwer fixed point theorem.[3] Consequently, we do not need to appeal either to the powerful coincidence Theorem of Fan (1969) or to the generalization of the Brouwer fixed point theorem known as the Kakutani fixed point theorem (Kakutani (1941)) or to the generalization of the Sperner Lemma of Shapley (1973).

It should be pointed out that our mathematical tools used to prove the K-K-M-S theorem are the same as those needed to prove the Gale-Nikaido-Debreu theorem (see Debreu (1959)). Hence, the level of sophistication of these two basic theorems is the same.

2 Preliminaries

2.1 Notation

Let $N = \{1, 2, \dots, n\}$. Then Δ^N denotes the $(n-1)$-simplex $\Delta^N = \{x \in \mathbb{R}^n : x_i \geq 0, \text{ and } \sum_{i=1}^n x_i = 1\}$. For $S \subset N$ let $\Delta^S = \{x \in \Delta^N : \sum_{i \in S} x_i = 1\}$.

For any set X, let 2^X denote the set of all subsets of the set X. Denote by $|X|$ the cardinality of the set X, i.e., the number of elements in X, and by $\mathbb{R}^n_-$ the negative cone of $\mathbb{R}^n$.[4]

Let m^S be the center of the simplex Δ^S. Thus, $m^N = (1/n, \dots, 1/n)$ and $m^S = (m_1^S, \dots, m_N^S)$, where $m_i^S = 1/|S|$ if $i \in S$ and $m_i^S = 0$ if $i \notin S$. Let $e^S = (e_1^S, \dots e_n^S)$ where $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$, otherwise.

[2]A version of the Shapley and Vohra proof can also be found in Aliprantis, Brown and Burkinshaw (1989).

[3]Komiya (1993) has also used a similar argument to that of Ichiishi (1987) to provide a simple proof of the K-K-M-S Theorem. His proof is based on the Berge maximum theorem and the Kakutani fixed point theorem.

[4]That is, $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_i \leq 0 \text{ for } i = 1, \dots, n\}$.

Let $\mathcal{B}$ be a collection of subsets of N. Then $\mathcal{B}$ is *balanced* if and only if there exist weights λ_S, $S \in \mathcal{B}$ with $\lambda_S \geq 0$ and $\sum_{S \in \mathcal{B}} \lambda_S e^S = e^N$. It is easy to see that $\mathcal{B}$ is balanced if and only if $m^N \in \text{con}\{m^S: S \in \mathcal{B}\}$.[5]

2.2 Two Simple Facts

First, recall that a correspondence $\varphi: \Delta^N \to 2^{\mathbb{R}^n}$ has *open lower sections* if $\varphi^{-1}(y) = \{x \in \Delta^N: y \in \varphi(x)\}$ is open in Δ^N for all $y \in \mathbb{R}^n$.

Fact 1. Let Δ be a compact metric space (for example Δ^N). Assume that $\varphi: \Delta \to 2^{\mathbb{R}^n}$ is non empty, convex valued and has open lower sections. Then φ has a *continuous selection,* i.e., there exists a continuous function $f: \Delta \to \mathbb{R}^n$ with $f(x) \in \varphi(x)$ for all $x \in \Delta$.

Proof. Non-empty valuedness of φ implies that the set $\{\varphi^{-1}(y): y \in \Delta\}$ is an open cover of Δ. Since Δ is compact there exists a finite set of points $\{y_1, \ldots, y_m\}$ such that $\Delta \subset \bigcup_{i=1}^m \varphi^{-1}(y_i)$. For $i = 1, \ldots, m$ define $g_i: \Delta \to \mathbb{R}$ by $g_i(x) = \text{dist}(x, \Delta \setminus \varphi^{-1}(y_i))$. For each $x \in \Delta$ and for each $i = 1, \ldots, m$ let $\alpha_i(x) = g_i(x) / \left(\sum_{j=1}^m g_j(x)\right)$. Then $\alpha_i(x) = 0$ for $x \notin \varphi^{-1}(y_i)$, $0 \leq \alpha_i(x) \leq 1$, and $\sum_{i=1}^m \alpha_i(x) = 1$ for all $x \in \Delta$. Define $f: \Delta \to \mathbb{R}^n$ by $f(x) = \sum_{i=1}^m \alpha_i(x) y_i$. Then f is continuous and $f(x) \in \varphi(x)$ for $x \in \Delta$ because of convex valuedness of φ.

Fact 2. Let $\Delta \subset \mathbb{R}^n$ be compact, convex and non empty. Assume that $\varphi: \Delta \to 2^\Delta$ is convex valued and has open lower sections and that $x \notin \varphi(x)$ for all $x \in \Delta$. Then φ has a *maximal element,* i.e., there exists $x^* \in \Delta$ such that $\varphi(x^*) = \emptyset$.

Proof. Suppose otherwise, i.e., $\varphi(x) \neq \emptyset$ for all $x \in \Delta$. Then Fact 1 implies that there exists a continuous function $f: \Delta \to \Delta$ with $f(x) \in \varphi(x)$ for all $x \in \Delta$. By the Brouwer fixed point theorem there exists x^* with $x^* = f(x^*) \in \varphi(x^*)$, a contradiction to $x \notin \varphi(x)$ for all $x \in \Delta$.

Generalizations of Facts 1 and 2 can be found in Yannelis and Prabhakar (1983). However, in the present form both results are completely

[5]con denotes the convex hull.

elementary. It should be noted that Fact 2 is equivalent to the following finite dimensional version of Browder's fixed point theorem (see Yannelis and Prabhakar (1983)).

Fact 3. Let $\Delta \subset I\!R^n$ be compact, convex and non empty. Assume that $\varphi: \Delta \to 2^{\Delta}$ is convex, non-empty valued and has open lower sections. Then φ has a *fixed point,* that is there exists $x^* \in \Delta$ such that $x^* \in \varphi(x^*)$.

3 Statement and Proof of the K-K-M-S Theorem

Theorem (K-K-M-S). *Let $\{C_S: S \subset N\}$ be a family of closed subsets of Δ^N. Assume that $\Delta^T \subset \bigcup_{S \subset T} C_S$ for all $T \subset N$. Then there exists a balanced family $\mathcal{B}$ such that $\bigcap_{s \in \mathcal{B}} C_S \neq \emptyset$.*

Proof. For each $x \in \Delta^N$ let $I(x) = \{S \subset N: x \in C_S\}$, $F(x) = \text{con}\{m^S: S \in I(x)\}$ and note that $F(x)$ is u.s.c.[6] Suppose that the Theorem is false, i.e., $I(x)$ is not balanced and consequently $m^N \notin F(x)$ for all $x \in \Delta^N$. By the separating hyperplane theorem there exists $v \in I\!R^n$ such that $vF(x) > vm^N$. Define the set valued function $\varphi: \Delta^N \to 2^{I\!R^n}$ by $\varphi(x) = \{v: vy > vm^N$ for all $y \in F(x)\}$. Then φ is non-empty, convex valued and since F is u.s.c. it follows that φ has open lower sections. By Fact 1 there exists a continuous function $f: \Delta^N \to I\!R^n$ such that $f(x) \in \varphi(x)$ for all $x \in \Delta^N$, i.e.,

$$f(x)y > f(x)m^N, \text{ for all } x \in \Delta^N, \text{ for all } y \in F(x). \qquad (1)$$

Define $\psi: \Delta^N \to 2^{\Delta^N}$ by $\psi(x) = \{y: f(x)x > f(x)y\}$. It follows from the continuity of f that ψ has an open graph[7] (and hence open lower sections). Moreover, it is easy to see that ψ is convex valued and $x \notin \psi(x)$ for all $x \in \Delta^N$. Hence, by Fact 2, ψ has a maximal element, i.e., there exists $x^* \in \Delta^N$ such that $\psi(x^*) = \emptyset$ and hence

$$f(x^*)x^* \leq f(x^*)y, \text{ for all } y \in \Delta^N. \qquad (2)$$

[6]The set-valued function $F: \Delta^N \to 2^{I\!R^n}$ is said to be u.s.c. if for every open subset V of $I\!R^n$ the set $\{x \in \Delta^N: F(x) \subset V\}$ is open in Δ^N.

[7]I.e., the set $G_\psi = \{(x, y) \in \Delta^N \times \Delta^N: y \in \psi(x)\}$ is open in $\Delta^N \times \Delta^N$.

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$$f(x)y > f(x)m^N, \text{ for all } x \in \Delta^N, \text{ for all } y \in F(x). \tag{1}$$

Define $\psi: \Delta^N \to 2^{\Delta^N}$ by $\psi(x) = \{y: f(x)x > f(x)y\}$. It follows from the continuity of f that ψ has an open graph[7] (and hence open lower sections). Moreover, it is easy to see that ψ is convex valued and $x \notin \psi(x)$ for all $x \in \Delta^N$. Hence, by Fact 2, ψ has a maximal element, i.e., there exists $x^* \in \Delta^N$ such that $\psi(x^*) = \emptyset$ and hence

$$f(x^*)x^* \leq f(x^*)y, \text{ for all } y \in \Delta^N. \tag{2}$$

[6]The set-valued function $F: \Delta^N \to 2^{I\!R^n}$ is said to be u.s.c. if for every open subset V of $I\!R^n$ the set $\{x \in \Delta^N: F(x) \subset V\}$ is open in Δ^N.

[7]I.e., the set $G_\psi = \{(x, y) \in \Delta^N \times \Delta^N: y \in \psi(x)\}$ is open in $\Delta^N \times \Delta^N$.

Observe that there exists $T \subset N$ such that x^* is in the relative interior of Δ^T. Then

$$f(x^*)y = f(x^*)x^* \text{ for all } y \in \Delta^T. \tag{3}$$

By assumption[8] there exists $S \subset T$ such that $x^* \in C_S$ which implies that $S \in I(x^*)$ and therefore $m^S \in F(x^*)$. Combining (1) and (3) we get $f(x^*)x^* = f(x^*)m^S > f(x^*)m^N$, a contradiction to (2). This proves the Theorem.

4 Concluding Remarks

Remark 1. It should be pointed out that the mathematical tools used to prove the K-K-M-S theorem are the same as the ones needed to prove the Gale-Nikaido-Debreu (G-N-D) theorem (see Debreu (1959)). In that sense the level of sophistication required to prove the G-N-D theorem and the K-K-M-S theorem is the same. For the sake of completeness we reproduce the finite dimensional commodity space counterpart of the G-N-D theorem, given in Yannelis (1985, Theorem 3.1).

Let Δ^N now denote the (n-1)-price simplex in $\mathbb{R}^n$. An economy is described by an excess demand correspondence $\zeta: \Delta^N \to 2^{\mathbb{R}^n}$, which satisfies the *weak Walras law,* i.e., for every $p \in \Delta^N$ there exists $x \in \zeta(p)$ such that $px \leq 0$. The price vector $p \in \Delta^N$ is said to be a *free disposal equilibrium* if $\zeta(p) \cap \mathbb{R}^n_- \neq \emptyset$, (see Debreu (1959)).

Theorem (G-N-D). *Let $\zeta: \Delta^N \to 2^{\mathbb{R}^n}$ be an excess demand correspondence satisfying the following conditions:*
(i) ζ is u.s.c., convex, compact and non-empty valued.
(ii) For all $p \in \Delta^N$ there exists $x \in \zeta(p)$ such that $px \leq 0$.
Then there exists $p^* \in \Delta^N$ such that $\zeta(p^*) \cap \mathbb{R}^n_- \neq \emptyset$.

Proof. Suppose the Theorem is false, i.e., for all $p \in \Delta^N$, $\zeta(p) \cap \mathbb{R}^n_- = \emptyset$. Fix p in Δ^N. By the separating hyperplane theorem there exist $q \in \mathbb{R}^n$, $q \neq 0$, and $b \in \mathbb{R}$ such that

$$\sup_{y \in \mathbb{R}^n_-} qy < b < \inf_{x \in \zeta(p)} qx.$$

[8]This follows since $\Delta^T \subset \bigcup_{S \subset T} C_S$.

Notice that $b > 0$ and without loss of generality we can assume that $q \in \Delta^N$. Define $F: \Delta^N \to 2^{\Delta^N}$ by

$$F(p) = \left\{q \in \Delta^N : qx > 0 \text{ for all } x \in \zeta(p)\right\}.$$

Then F is convex, non-empty valued and it follows from the u.s.c. of ζ that F has open lower sections. By Fact 1 there exists a continuous function $f: \Delta^N \to \Delta^N$ such that $f(p) \in F(p)$ for all $p \in \Delta^N$. By the Brouwer fixed point theorem there exists $p^* \in \Delta^N$ such that $p^* = f(p^*) \in F(p^*)$, i.e., $p^*x > 0$ for all $x \in \zeta(p^*)$ a contradiction to condition (ii). This completes the proof of the theorem.

Remark 2. The assumption in the G-N-D theorem that the excess demand function ζ is u.s.c. can be weakened to *upper demicontinuity* (u.d.c), i.e., for every open half spaced V of $\mathbb{R}^n$ the set $\{p \in \Delta^N : \zeta(p) \subset V\}$ is open in Δ^N. The proof of the G-N-D theorem remains unchanged since it can be easily checked that the set-valued function $F: \Delta^N \to 2^{\Delta^N}$ defined by $F(p) = \{q \in \Delta^N : qx > 0 \text{ for all } x \in \zeta(p)\}$ has open lower sections. In that sense and in view of the weak version of the Walras law (condition (ii)) the G-N-D theorem above is slightly more general than the standard version found in the literature (e.g., Debreu (1959)).

Remark 3. Since any correspondence having open lower sections is also lower semicontinuous (l.s.c.), (see Yannelis and Prabhakar (1983, Proposition 4.1.)) one could appeal to the powerful Michael selection Theorem, (Michael (1953)) and complete the proof of Fact 1. However, we choose not to do so because constructing continuous selections from l.s.c. correspondences is more complicated than the simple standard argument used in the proof of Fact 1.

Remark 4. Border (1984) and Florenzano (1989) have proved core existence results for economies where agents' preferences need not be representable by utility functions. Their proofs are based on the Fan coincidence theorem and the Kakutani fixed point theorem, respectively. It would be of interest to know if the K-K-M-S theorem can be applied in a direct way to simplify their proofs.

References

C.D.Aliprantis, D.J. Bown and O. Burkinshaw (1989), "Existence and Optimality of Competitive Equilibria," Springer Verlag:

K. Border (1984), "A Core Existence Theorem for Games Without Ordered Preferences," *Econometrica* **52**, 1537–1542.

G. Debreu (1959), "Theory of Value," New York: Wiley.

K. Fan (1969), "Extensions of Two Fixed Point Theorems of F.E. Browder," *Math. Zeitschr.* **112**, 234–240.

M. Florenzano (1989), "On the Non-emptiness of the Core of a Coalitional Production Economy Without Ordered Preferences," *J. Math. Anal. Appl.* **141**, 484–490.

T. Ichiishi (1981), "On the Knaster-Kuratowski-Mazurkiewicz-Shapley Theorem," *J. Math. Anal. Appl.* **81**, 297–299.

T. Ichiishi (1987), "Weak Dominance of Cores," mimeo, Ohio State University.

Y. Kannai (1981), "An Elementary Proof of the No-retraction Theorem," *Amer. Math. Monthly* **88**, 262–268.

H. Komiya (1993), "A Simple Proof of the K-K-M-S Theorem," mimeo, University of California, Berkeley.

E. Michael (1956), "Continuous Selections I," *Ann. of Math.* **63**, 361–382.

H.E. Scarf (1967), "The Core of an n-person Game," *Econometrica* **35**, 50–69.

L. Shapley (1973), "On Balanced Games Without Side Payments," in *Mathematical Programming* T.C. Hu and S.M. Robinson (eds.), New York, Academic Press.

L. Shapley and R. Vohra (1991), "On Kakutani's Fixed Point Theorem, the K-K-M-S Theorem and the Core of a Balanced Game," *Econ. Theory* **1**, 108–116.

N.C. Yannelis (1985), "On a Market Equilibrium Theorem With an Infinite Number of Commodities," *J. Math. Anal. Appl.* **108**, 595–599.

N.C. Yannelis and N.D. Prabhakar (1983), "Existence of Maximal Elements and Equilibria in Linear Topological Spaces," *J. Math. Econ.* **12**, 233-245.