SCOTT'S

BRIDGE'S

ALGEBRA.

FIRST BOOKS IN SCIENCE

SCOTT'S FIRST BOOKS IN SCIENCE.

AN

ELEMENTARY TREATISE

ON

ALGEBRA;

BY THE LATE

REV. B. BRIDGE, B.D., F.R.S.,

PROFESSOR OF MATHEMATICS

AND NATURAL PHILOSOPHY IN THE EAST INDIA COLLEGE, HERTFORD.

A NEW EDITION, IMPROVED AND SIMPLIFIED,

BY THOMAS ATKINSON, M.A.

MATHEMATICAL MASTER IN THE ROYAL GRAMMAR SCHOOL, GUILDFORD,

AND LATE SCHOLAR OF CORP. CH. COLL., CAMBRIDGE.

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ADVERTISEMENT.

The excellence of 'Bridge's Algebra,' as an elementary treatise, has long been well known and extensively recognised. In the Preface to the Second Edition the author expressly states, that "great pains were taken to give to it all the perspicuity and simplicity which the subject would admit of, and to present it in a form likely to engage the attention of young persons just entering on their mathematical studies." The design, which he thus proposed to himself, was accomplished with singular felicity,—for not one of the many publications on Algebra, which have during a period of forty years issued from the press, with the professed object of producing a more simple and appropriate introduction to the study of the science, has evinced such merits as justly entitle it to be placed in comparison with the performance of Mr. Bridge. These publications have accordingly failed to secure for themselves the same measure of public approbation.

This *small* compendium embraces all which is comprised in the former *large and expensive* editions, that is either practically useful or theoretically valuable. By introducing Equations and Problems at the earliest stage possible, a novel and instructive feature—which the editor is persuaded cannot fail to excite the curiosity and stimulate the ardour of the young algebraist, so as to induce him to pursue his studies with more than usual alacrity, intelligence, and success—has been given to the work. A great variety of new, easy, and interesting problems, which are not contained in former editions, have thus been interspersed through the several chapters. Besides these additions many alterations have been made, either for the sake of uniformity of arrangement, or of rendering the subject still more easy and accessible to youthful minds. Mr. Bridge, nearly forty years ago, "was not without a hope that his '*Elementary Treatise on Algebra*,' would find its way into our *Public Schools;* where, it was very well known, this branch of education was [then] but little attended to:" and it is now confidently hoped, that this *new edition* will (in consequence of its cheap and improved form) find its way into *many schools*, where this science is not *yet* sufficiently attended to, and thus be the means of rendering this instructive study a subject of general education.

T. A.

Guildford, December, 1847.

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BRIDGE'S ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. ALGEBRA is a general method of computation, in which number and quantity and their several relations are expressed by means of written signs or symbols. The symbols used to denote numbers or quantities are the letters of the alphabet.

2. *Known* or *determined* quantities are generally represented by the *first* letters of the alphabet; as, a, b, c, &c.

3. *Unknown* or *undetermined* quantities are usually expressed by the *last* letters of the alphabet; as x, y, z, &c.

4. The *multiples* of quantities, that is, the *number of times* quantities are to be taken, as *twice* a, *three times* b, *five times* ax, are expressed by placing numbers before them, as $2a$, $3b$, $5ax$. The numbers, 2, 3, 5, are called the *coefficients* of the quantities a, b, ax, respectively. When there is no *coefficient* set before a quantity, 1 is always understood: thus a is the same as $1a$.

5. The symbol $=$ (read *is equal to*) placed between two quantities means that the quantities are equal to each other. Thus, 12 pence $=$ 1 shilling; 3 added to $5 = 8$; $2a$ added to $4a = 6a$. This symbol is called the *sign of equality*.

6. The sign $+$ (read *plus*) signifies that the quantities before which it is placed are *to be added*. Thus $3 + 2$ is the same thing as 5; and $a + b + x$, means the sum of a, b, and x, whatever be the values of a, b, and x.

---

1. Define Algebra. What symbols are used to denote *numbers* or quantities?—2. What letters of the alphabet are employed to represent *known* or *determined* quantities?—3. How are *unknown* or *undetermined* quantities represented?—4. What are coefficients? When is the coefficient omitted?—5. What is the sign of equality?—6. What is the use of the sign $+$?

7. The sign $-$ (read *minus*) signifies that the quantity to which it is prefixed is *to be subtracted*. Thus $3-2$ is the same thing as 1; $a-b$ means the *difference* of a and b, or b taken from a; and $a+b-x$, signifies that x is to be subtracted from the sum of a and b.

8. Quantities which have the sign $+$ prefixed to them are called *positive*, and those which have the sign $-$ set before them are termed *negative* quantities. When there is no sign before a quantity $+$ is understood; thus a, stands for $+a$.

9. The symbol $\times$ (read *into*) is the *sign of multiplication*, and signifies that the quantities between which it is placed are to be multiplied together. Thus, 6×2 means that 6 is to be multiplied by 2; and $a\times b\times c$, signifies that a, b, c, are to be multiplied together. In the place of this symbol a *dot* or *full-point* is often used. Thus, $a.b.c$, means the same as $a\times b\times c$. The product of quantities represented by letters is usually expressed by placing the letters *in close contact*, one after another, according to the position in which they stand in the alphabet. Thus, the product of a into b is denoted by ab; of a, b, and x, by abx, and of $3a$, x, and y, by $3axy$.

10. In algebraical computations the word *therefore* often occurs. To express this word the symbol $\therefore$ is generally made use of. Thus the sentence "therefore $a+b$ is equal to $c+d$," is expressed by "$\therefore a+b=c+d$."

EXAMPLES.

Ex. 1. In the algebraical expression, $a+b-c$, let $a=9$, $b=7$, $c=3$; then

$$\begin{aligned} a+b-c &= 9+7-3 \\ &= 16-3 \\ &= 13 \end{aligned}$$

Ex. 2. In the expression $ax+ay-xy$, let $a=5$, $x=2$, $y=7$; then, to find its value, we have

$$\begin{aligned} ax+ay-xy &= 5\times 2+5\times 7-2\times 7 \\ &= 10+35-14 \\ &= 45-14 \\ &= 31 \end{aligned}$$

---

7. How is the symbol $-$ read?—8. What is meant by *positive* and what by *negative* quantities?—9. Write down the *sign of multiplication?* Is any other *mark* used to denote multiplication? When is no symbol used?—10. What symbol is used to denote the word *therefore?*

Ex. 3. If $a=5$, $b=4$, $c=3$, $d=2$, $x=1$, $y=0$, find the numerical values of the following expressions:

(1.) $a+b+c+x$	Ans. 13.
(2.) $a-b+c-x+y$	... 3.
(3.) $ab+3ac-bc+4cx-xy$	... 65.
(4.) $abc-abd+bcd-acx$	... 29.
(5.) $3abc+4acx-8bdx+axy$	... 176.

11. The symbol $\div$ (read *divided by*) is the *sign of division*, and signifies that the former of two quantities between which it is placed, is to be divided by the latter. Thus, $8 \div 2$ is equivalent to 4. But this division is more simply expressed by making the former quantity the numerator, and the latter the denominator of a fraction; thus $\frac{a}{b}$ means *a divided by b*, and is usually, for the sake of brevity, read *a by b*.

EXAMPLES.

Ex. 1. If $a=2$, $b=3$; then, we find the values of

(1.) $\frac{3a}{5b}=\frac{3\times 2}{5\times 3}=\frac{6}{15}=\frac{2}{5}$

(2.) $\frac{2a+b}{8a-3b}=\frac{2\times 2+3}{8\times 2-3\times 3}=\frac{4+3}{16-9}=\frac{7}{7}=1$

Ex. 2. If $a=3$, $b=2$, $c=1$, find the numerical values of

(1.) $\frac{3a+c}{4b+a}$	Ans. $\frac{10}{11}$
(2.) $\frac{a+2b-c}{3a+b-5c}$	... 1
(3.) $\frac{ab+ac-bc}{2ab-2ac+bc}$	... $\frac{7}{8}$

12. When a quantity is multiplied into itself any number of times, the product is called *a power* of the quantity.

---

11. By what symbol is division denoted? What is its name? Is division ever expressed in any other manner?—12. What is meant by the *power* of a quantity?

13. *Powers* are usually denoted by placing above the quantity to the right a *small figure*, which indicates how often the quantity is multiplied into itself. Thus

a ... ... the *first* power of a is denoted by a (a^1)
$a \times a$ ... ... the 2[d] power or *square* of a ... a^2
$a \times a \times a$ ... ... the 3[d] power or *cube* of a ... a^3
$a \times a \times a \times a$ ... the 4th power of a ... a^4

The small figures ^{2}, ^{3}, ^{4}, &c., set over a, are respectively called the *index* or *exponent* of the corresponding power of a.

14. The *roots* of quantities are the quantities from which the powers are by successive multiplication produced. Thus, the root of the square number 16 is 4, because $4 \times 4 = 16$, and the root of the cube number 27 is 3, since $3 \times 3 \times 3 = 27$.

15. To express the *roots* of quantities the symbol $\sqrt{}$, (a corruption of *r*, the first letter in the word *radix*) with the proper index, is employed. Thus,

$\sqrt[2]{a}$ or $\sqrt{a}$, expresses the *square* root of a.
$\sqrt[3]{a}$ ... ... " the *cube* root of a.
$\sqrt[4]{a}$ ... ... " the *fourth* root of a.
&c. &c.

EXAMPLES.

Ex. 1. If $a = 3$, $b = 2$; then $a^2 = 3 \times 3 = 9$, $a^3 = 3 \times 3 \times 3 = 27$, $b^4 = 2 \times 2 \times 2 \times 2 = 16$.

Ex. 2. If $a = 64$; then $\sqrt{a} = \sqrt{64} = 8$, $\sqrt[3]{a} = \sqrt[3]{64} = \sqrt[3]{4 \times 4 \times 4} = 4$, $\sqrt[6]{a} = \sqrt[6]{64} = 2$.

Ex. 3. In the expression $\dfrac{ax^2 + b^2}{bx - a^2 - c}$, let $a = 3$, $b = 5$, $c = 2$, $x = 6$. What is the numerical value?

Here $ax^2 + b^2 = 3 \times 6 \times 6 + 5 \times 5 = 108 + 25 = 133$, and $bx - a^2 - c = 5 \times 6 - 3 \times 3 - 2 = 30 - 9 - 2 = 19$

$$\therefore \frac{ax^2 + b^2}{bx - a^2 - c} = \frac{133}{19} = 7.$$

---

13. How are *powers* denoted?—14. What are the *roots* of quantities?—15. By what symbol are the *roots* of quantities expressed? What is the origin of this symbol?

Ex. 4. If $a=1$, $b=3$, $c=5$, $d=0$, find the values of

(1.) a^2+2b-c. Ans. 2.

(2.) $a^2+3b^2-c^2$ ... 3.

(3.) $a^2+2b^2+3c^2+4d^2$ ... 94.

(4.) $3a^2b-2b^2c+4c^2-4a^2d$ ... 19.

(5.) a^3+b^3 ... 28.

(6.) $\frac{a^3}{3}+\frac{b^3}{3}+\frac{c^3}{3}$ ... 51.

Ex. 5. Let $a=64$, $b=81$, $c=1$: find the values of

(1.) $\sqrt{a}+\sqrt{b}$ Ans. 17.

(2.) $\sqrt{a}+\sqrt{b}+\sqrt{c}$ ... 18

(3.) $\sqrt{abc}$ ... 72.

16. When several quantities are to be taken as *one quantity* they are enclosed in *brackets*, as ( ), { }, [ ]. Thus, $(a+b-c).(d-e)$ signifies that the quantity represented by $a+b-c$, is to be multiplied by that represented by $d-e$; if then $a=3$, $b=2$, $c=1$, $d=5$, $e=2$, $a+b-c=4$, $d-e=3$, and $\therefore (a+b-c).(d-e)=4\times 3=12$.

Great care must be taken in observing how brackets are employed, and what effects arise from the use of them. Thus $(a+b).(c+d)$, $(a+b)\,c+d$, $a+bc+d$, are three very different expressions; for if $a=3$, $b=2$, $c=3$, $d=5$,

(1.) $(a+b).(c+d)=(3+2).(3+5)=5\times 8=40.$

(2.) $(a+b)\,c+d=(3+2)\,3+5=5\times 3+5=20.$

(3.) $a+bc+d=3+2\times 3+5=3+6+5=14.$

17. Instead of brackets, a line called a *vinculum* is sometimes used, and is drawn over quantities, which are taken collectively. Thus, $a-\overline{b-c}$ is the same as $a-(b-c)$.

The line which separates the numerator and denominator of a fraction may be regarded as a sort of vinculum, cor-

---

16. When are *brackets* employed?—17. What is a *vinculum?* May the line which separates the numerator and denominator of a fraction be regarded as a vinculum?

responding, in fact, in *Division* to the bracket in *Multiplication*. Thus, $\frac{a+b-c}{5}$ implies that the whole quantity $a+b-c$ is to be divided by 5.

18. *Like* quantities are such as consist of the *same letter*, or the *same combination of letters*; thus, $5a$, and $7a$; $4ab$ and $9ab$; $2bx^2$ and $6bx^2$, &c. are called *like* quantities; and *unlike* quantities are such as consist of *different letters*, or of *different combinations of letters*; thus, $4a$, $3b$, $7ax$, $5bx^2$, &c., are *unlike* quantities.

19. Algebraic quantities have also different denominations according to the number of terms (connected by the sign $+$ or $-$) of which they consist: thus,

a, $2b$, $3ax$, &c., quantities consisting of one term, are called *simple* quantities.

$a+x$, a quantity consisting of two terms, is called a *binomial*.

$bx+y-z$, a quantity consisting of three terms, is called a *trinomial*.

CHAPTER II.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION, OF ALGEBRAIC QUANTITIES.

ADDITION.

20. ADDITION consists in collecting quantities that are *like* into one sum, and connecting by means of their proper signs those that are *unlike*. From the division of algebraic quantities into *positive* and *negative*, *like* and *unlike*, there arise three cases of Addition.

CASE I.

To add like quantities with like signs.

21. In this case, the rule is "To add the coefficients of the several quantities together, and to the result annex the

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18. What are *like* and what are *unlike* quantities?—19. What is a *simple* quantity? What is a *binomial* and what a *trinomial*?—20. In what does addition of algebra consist? Into how many cases is it divided?

common sign, and the common letter or letters;" for it is evident from the common principles of Arithmetic, if $+2a$, $+3a$, and $+5a$, be added together, their sum must be $+10a$; and if $-3b^2$, $-4b^2$, and $-8b^2$, be added together, their sum must be $-15b^2$.

Ex. 1.

$$\begin{array}{r} 2x+3a-4b \\ 3x+2a-5b \\ 4x+8a-7b \\ 9x+4a-6b \\ 5x+7a-9b \\ \hline 23x+24a-31b \\ \hline\hline \end{array}$$

Ex. 2.

$$\begin{array}{r} 7x^2+3xy-5bc \\ 9x^2+2xy-7bc \\ 11x^2+5xy-4bc \\ x^2+4xy-bc \\ x^2+9xy-2bc \\ \hline 29x^2+23xy-19bc \\ \hline\hline \end{array}$$

Ex. 3.

$$\begin{array}{r} 4a^3-3a^2+1 \\ 2a^3-a^2+17 \\ 5a^3-2a^2+4 \\ 3a^3-7a^2+3 \\ a^3-a^2+10 \\ \hline 15a^3-14a^2+35 \\ \hline\hline \end{array}$$

Ex. 4.

$$\begin{array}{r} 3x^3+4x^2-x \\ 2x^3+x^2-3x \\ 7x^3+2x^2-2x \\ 4x^3+x^2-x \\ \hline\hline \end{array}$$

Ex. 5.

$$\begin{array}{r} 7a^3-3a^2b+2ab^2-3b^3 \\ 4a^3-a^2b+ab^2-b^3 \\ a^3-2a^2b+3ab^2-5b^3 \\ 5a^3-3a^2b+4ab^2-2b^3 \\ \hline\hline \end{array}$$

Ex. 6.

$$\begin{array}{r} 2x^2y-3x+2 \\ 4x^2y-2x+1 \\ 3x^2y-5x+10 \\ x^2y-x+15 \\ \hline\hline \end{array}$$

In these Examples it may be observed that some of the quantities have *no coefficient*. In this case, *unity* or 1 *is always understood*. Thus, in adding up the *first* column of Ex. 2. we say, $1+1+11+9+7=29$; in the *third*, $2+1+4+7+5=19$; and so of the rest.

Case II.

To add like quantities with unlike signs.

22. Since the compound quantity $a+b-c+d-e$ &c. is positive or negative, according as the sum of the positive terms is greater or less than the sum of the negative ones, the aggregate or sum of the quantities $2a-4a+7a-3a$ will be $+2a$, and of the quantities $7b^2-5b^2+2b^2-8b^2$ will be $-4b^2$; for in the former case

---

22. State the rule in the 1st case.

the excess of the sum of the positive terms above the negative ones is $2a$, and in the latter $4b^2$. Hence this general rule for the addition of like quantities with unlike signs, "Collect the coefficients of the *positive* terms into one sum, and also of the *negative;* subtract the *lesser* of these sums from the *greater;* to this *difference*, annex the sign of the *greater* together with the common letter or letters, and the result will be the sum required."

If the aggregate of the positive terms be *equal* to that of the negative ones, then this *difference* is equal to 0; and consequently the sum of the quantities will be equal to 0, as in the *second* column of Ex. 2. following.

Ex. 1.	Ex. 2.	Ex. 3.
$4x^2-3x+4$	$-7ab+3bc-xy$	$-5x^3+13x^2$
$-2x^2+x-5$	$-ab+2bc+4xy$	$-2x^3-4x^2$
$3x^2-5x+1$	$3ab-bc+2xy$	$7x^3+x^2$
$7x^2+2x-4$	$-2ab+4bc-3xy$	$9x^3-14x^2$
$-x^2-4x+13$	$5ab-8bc+xy$	$-13x^3-2x^2$
$11x^2-9x+9$	$-2ab \quad * \quad +3xy$	$-4x^3-6x^2$

Ex. 4.	Ex. 5.	Ex. 6.
$4x^3-2x+3y$	$5a^3-2ab+b^2$	$4x^2y^2+2xy-3$
$-x^3+4x-y$	$-a^3+ab-2b^2$	$-x^2y^2-xy-1$
$7x^3-x+9y$	$4a^3-3ab+b^2$	$3x^2y^2+4xy-5$
$9x^3+21x-2y$	$2a^3+4ab-4b^2$	$-9x^2y^2-2xy+9$

Case III.

23. There now only remains the case where *unlike* quantities are to be added together, which must be done by collecting them together into one line, and annexing their proper signs; thus, the sum of $3x, -2a, +5b, -4y$, is $3x-2a+5b-4y$; except when *like* and *unlike* quantities

23. State the rule in the 2d and 3d cases.

are mixed together, as in the following examples, where the expressions may be simplified, by collecting together such quantities as will coalesce into one sum.

Ex. 1.

$$\begin{array}{l} 3ab + x - y \\ 4c - 2y + x \\ 5ab - 3c + d \\ 4y + x^2 - 2y \\ \hline 8ab + 2x - y + c + d + x^2 \\ \hline\hline \end{array}$$

Collecting together *like* quantities, and beginning with $3ab$, we have $3ab + 5ab = 8ab$; $+x + x = +2x$; $-y - 2y + 4y - 2y = -y$; $4c - 3c = +c$; besides which there are the two quantities $+d$ and $+x^2$, which do not coalesce with any of the others; the sum required therefore is $8ab + 2x - y + c + d + x^2$.

Ex. 2.

$$\begin{array}{l} 4x^2 - 2xy + 1 - 3y + 4x^3 \\ 4y + 3x^3 - y^2 + xy - x^2 \\ 5x^3 - 2x + y - 15 + y^2 \\ \hline 3x^2 - xy - 14 + 2y + 12x^3 - 2x \\ \hline\hline \end{array}$$

Here $4x^2 - x^2 = 3x^2$
$-2xy + xy = -xy$
$+1 - 15 = -14$
$-3y + 4y + y = +2y$
$+4x^3 + 3x^3 + 5x^3 = +12x^3$
$-y^2 + y^2 = 0$
$-2x = -2x$.

SIMPLE EQUATIONS.

24. When two algebraic quantities are connected together by the sign of equality (=), the expression is called *an equation*. Equations, in their application to the solution of problems, consist of quantities, some of which are *known* and others *unknown*. Thus $2x + 3 = x + 7$ is an equation in which x is an *unknown* quantity, and its value is such a number as will make $2x + 3$ and $x + 7$ equal to each other. The number which here *satisfies* the equation, or is the value of x, is manifestly 4, since $2 \times 4 + 3 = 11$, and $4 + 7 = 11$. The value of the unknown quantity, which has in this example been found by inspection, is usually obtained by a direct calculation, which is called *the solution* of the equation.

24. What is an equation? What is meant by *the solution* of an equation?

25. In effecting the solution the several steps of the process must be conducted by means of the following axioms, and in strict accordance with them :—

(1.) Things which are equal to the same thing are equal to one another.

(2.) If equals be added to the same or to equals, the sums will be equal.

(3.) If equals be subtracted from the same or from equals, the remainders will be equal.

(4.) If equals be multiplied by the same or by equals, the products will be equal.

(5.) If equals be divided by the same or by equals, the quotients will be equal.

(6.) If equals be raised to the same power, the powers will be equal.

(7.) If the same roots of equals be extracted, the roots will be equal.

These axioms, exclusive of the first, may be *generalized*, and all included in *one very important principle*, which should in every investigation in which equations are concerned be carefully borne in mind; viz. *that whatever is done to one side of an equation the same thing must be done to the other side, in order to keep up the equality.*

26. If an equation contain no *power* of the unknown quantities higher than the *first*, or those quantities in their simplest form, it is called a *Simple Equation.*

ON THE SOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

27. The rules which are absolutely necessary for the solution of simple equations, containing only one unknown quantity, may be reduced to four, each of which will in its proper place be formally enunciated and exemplified.

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25. What are the axioms employed in the solution of equations, and state the general principle which is based upon them?—26. What is a simple equation?

Rule I.

"If the unknown quantity has a coefficient, then its value may be found by dividing each side of the equation by that coefficient;" and the foundation of the Rule is, that "if equals be divided by the same, the quotients arising will be equal."

Ex. 1. Let $2x=14$; then *dividing* both sides of the equation by 2, we have $\frac{2x}{2}=\frac{14}{2}$; but $\frac{2x}{2}=x$, and $\frac{14}{2}=7$, $\therefore x=7$.

Ex. 2. Let $ax=b+c$; then $\frac{ax}{a}=\frac{b+c}{a}$; but $\frac{ax}{a}=x$; $\therefore x=\frac{b+c}{a}$.

Ex. 3. Let $x+2x+4x+6x=52$

Collecting the terms, $13x=52$

Dividing both sides of the equation by 13,

$$x=4.$$

Ex. 4. Let $6x-4x+3x-x=36$

The terms being added as in Case 2d of Addition,

$$4x=36$$

Dividing each side of the equation by 4,

$$x=9.$$

Ex. 5.	$10x=150$	Ans.	$x=15$
Ex. 6.	$3x+4x+7x=84$	...	$x=6$
Ex. 7.	$8x-5x+4x-2x=25$	...	$x=5$
Ex. 8.	$12x-3x-4x-x=24$	...	$x=6$

28. Arithmetical questions may with great ease be exhibited under the form of an equation, and it will be seen by the subjoined examples in what relation arithmetical and algebraic operations stand to each other: as for instance,

If 30*s.* be given for 5lbs. *of tea*, what is the price of 1lb.?

(1.) The *price of* 1lb. is that which is to be found.

(2.) Then it is clear that *the price of* 1 lb $\times$ 5 must give the price of 5 lbs.

(3.) But the cost of 5 lbs. by the question $= 30s$.

(4.) Therefore, the price of 1 lb. in shillings $\times 5 = 30s$.

(5.) And therefore by dividing by 5, we obtain the price of 1 lb. $= 6s$.

The several steps of this solution expressed algebraically would take the following more compendious form:

(1.) Let $x =$ the price of 1 lb. in shillings.

(2.) Then $5x =$ the price of 1 lb. in shillings $\times$ 5.

(3.) But the cost of 5 lbs. is by the question $= 30s$.

(4.) $\therefore 5x = 30s$.

(5.) and $\therefore x = 6s$. which is the price of 1 lb., as was required.

It will be seen by steps (2) and (3) of this example, that there are two distinct expressions for the same thing, and that in step (4) these expressions are made equal to each other. In framing equations from problems, this will in all cases take place. As a second example let this problem be taken:—

A house and an orchard are let for £28. a year, but the rent of the house is 6 times that of the orchard. Find the rent of each.

The rent of the house is equal to that of 6 orchards; we may therefore change the house into 6 orchards, and we shall have

Rent of the orchard + 6 times *rent of the orchard* = £28.
Taking the sum of the rents of the orchard, we get

7 times *the rent of the orchard* = £28,

and the 7th part of each side of the equation being taken,

The rent of the orchard = £4;

and $\therefore$ *the rent of the house* = 6 times the rent of the orchard
= 6 times £4
= £24.

Now to give to these operations an algebraical shape,

Let $x =$ *the rent of the orchard* in £

then $6x =$ ..... *house* .....

But by the condition of the question,

Rent of the orchard + 6 times *rent of the orchard* = £28.

$$\therefore x + 6x = £28.$$

$$\text{or,} \quad 7x = £28.$$

and, dividing each side of the equation by 7,

x = £4, the rent of the orchard,

and $\therefore 6x = 6 \times £4 = £24$, the rent of the house.

Again, suppose the following arithmetical question was proposed for solution; viz. "To divide the number 35 into two such parts, that one part shall exceed the other part by 9." A person unacquainted with algebra might with no great difficulty solve this question in the following manner:—

(1.) It appears, in the first place, that there must be a *greater* and a *less* part.

(2.) The greater part must exceed the less by 9.

(3.) But it is evident that the greater and less parts added together must be equal to the whole number 35.

(4.) If then we substitute for the greater part its *equivalent*, viz. "*the less part increased by 9*," it follows, that the less part increased by 9, with the *addition* of the said less part is equal to 35.

(5.) Or, in other words, that *twice* the less part with the addition of 9, is equal to 35.

(6.) Therefore, *twice the less part* must be equal to 35, *with* 9 *subtracted from it.*

(7.) Hence, twice the less part is equal to 26.

(8.) From which we conclude, that the *less part* is equal to 26 *divided by* 2; i.e. to 13.

(9.) And consequently, as the *greater* part exceeds the *less* by 9, it must be equal to 22.

But by adopting the method of algebraic *notation, the different steps of this solution may be much more briefly expressed as follows:*

(1.) Let the *less* part ..... $= x$.

(2.) Then the *greater* part ..... $= x + 9$.

(3.) But greater part + less part..... $= 35$.

(4.) $\therefore\ x+9+x$ ..... $=35.$

(5.) or $2x+9$..... $=35.$

(6.) $\therefore 2x$..... $=35-9.$

(7.) or $2x$..... $=26.$

(8.) $\therefore x$ (*less* part) ..... $=\frac{26}{2}=13.$

(9.) and $x+9$ (*greater* part) ..... $=13+9=22.$

29. Having thus explained the manner in which the several steps in the solution of an arithmetical question may be expressed in the language of Algebra, we now proceed to its exemplification.

PROBLEMS.

Prob. 1. A dessert basket contains 30 apples and pears, but 4 times as many pears as apples. How many are there of each sort?

Let $x=$ the number of apples;
then, as there are 4 times as many pears as apples,
$4x=$ the number of pears.
But by the question the apples and pears together $=30$,
$\therefore x+4x=30.$
Adding the terms containing x,
$5x=30.$
Dividing each side of this equation by 5,
$x=6$, the number of apples,
$\therefore$ the number of pears $=4x=4\times 6=24.$

Prob. 2. In a mixture of 16lbs. of black and green tea, there was 3 times as much black as green. Find the quantity of each sort.

Let $x=$ the number of lbs. of green tea,
then $3x=$ .....black.
But the black tea + the green tea $=16$lbs.
$\therefore x+3x=16$lbs.
Collecting the terms which contain x,
$4x=16$lbs.
Dividing each side of this equation by 4,
$x=4$lbs. of green tea,
$\therefore$ the black tea $=3x=3\times 4=12$lbs.

PROB. 3. An equal mixture of black tea at 5 shillings a lb. and of green at 7 shillings a lb. costs 4 guineas. How many lbs. were there of each sort?

Let $x =$ the number of lbs. of each sort;
then $5x =$ the cost of the black in shillings,
and $7x =$ ..... green .....

But cost of black + the cost of green $= 4$ guineas $= 84s$.

$$\therefore 5x + 7x = 84$$
$$12x = 84$$
$$\therefore x = 7 \text{ lbs. of each sort.}$$

PROB. 4. The area of the rectangular floor of a school-room is 180 square yards, and its breadth 9 yards. What is its length?

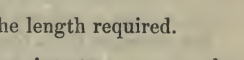
Let $x =$ the length in yards; then since the area is the length multiplied by the breadth, we have

$$x \times 9 = \text{the area of the floor,}$$
$$\therefore 9x = 180$$

and $\therefore$ $x = 20$ yards, the length required.

PROB. 5. Divide a rod 15 feet long into two parts, so that the one part may be 4 times the length of the other.

Let $x =$ the less part,
then $4x =$ the greater part



Now these two parts make together 15 ft.

$$\therefore x + 4x = 15 \text{ ft.}$$
$$5x = 15 \text{ ft.}$$
$$\therefore x = 3 \text{ ft. the less part,}$$

and the greater part $= 4$ times 3 ft. $= 12$ ft.

PROB. 6. A horse and a saddle were bought for £40., but the horse cost 9 times as much as the saddle. What was the price of each? Ans. £36. and £4.

PROB. 7. Divide two dozen marbles between Richard and Andrew, so that Richard may have three times as many as Andrew. Ans. Richard's share $= 18$
... Andrew's ... $= 6$.

PROB 8. A boy being asked how many marbles he had, said, If I had twice as many more, I should have 36. How many had he? Ans. 12.

PROB. 9. A bookseller sold 10 books at a certain price,

and afterwards 15 more at the same rate, and at the latter time received 35*s.* more than at the former: what was the price per book?

Let $x =$ the price of a book in shillings;
then $10x =$ the price of the 1st lot in shillings,
and $15x =$ .....2d .....

Now, if the price of the 1st lot be taken from that of the 2d, there remains a difference in price of 35*s.*

$$\therefore\ 15x - 10x = 35s.$$

Subtracting the $10x$ from the $15x$, we have

$$5x = 35s.$$

$$\therefore\ x = 7s.\ \text{the price of a book.}$$

Prob. 10. Divide £300 amongst A, B, and C, so that A may receive twice as much as B, and C as much as A and B together.

Let $x =$ B's share in £
then $2x =$ A's share ...
and $x + 2x$ or $3x =$ C's share in £.

But amongst them they receive £300;

$$\therefore\ x + 2x + 3x = £300,$$
$$6x = £300;$$
$$\therefore\ x = £50\ \text{B's share};$$
$$\therefore\ \text{A's share} = £100,\ \text{and C's share} = £150.$$

Prob. 11. If to nine times a certain number, three times the number be added, and four times the same number be taken away, there will then be obtained the number 48. What is the number?

Let $x =$ the number;
then $9x =$ nine times the number,
$3x =$ three times the number,
and $4x =$ four times the number;

$$\therefore\ 9x + 3x - 4x = 48,$$
$$8x = 48;$$
$$\therefore\ x = 6,\ \text{the number required.}$$

Prob. 12. The sum of £100 is to be divided among 2 men, 3 women, and 4 boys, so that each man shall have twice as much as each woman, and each woman three times as much as each boy. Find the share of each.

Let each boy's share $= x$;
then each woman's share $= 3x$,
and each man's share $= 2$ times $3x = 6x$;

Hence we have,

the share of the 4 boys $= 4x$,

the share of the 3 women $= 3$ times $3x = 9x$,

and the share of the 2 men $= 2$ times $6x = 12x$:

But the sum of all these shares is to amount to £100;

$$\therefore 4x + 9x + 12x = £100,$$

$$25x = £100;$$

$$\therefore x = £4;$$

$\therefore$ each boy's share $=$ £4; each woman's $= 3$ times £4 $=$ £12; and each man's $= 6$ times £4 $=$ £24.

PROB. 13. A gentleman meeting 4 poor persons gave five shillings amongst them; to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

Ans. 6*d.*, 12*d.*, 18*d.*, 24*d.*, respectively.

PROB. 14. Divide a line 12 ft. long into three parts, such that the middle one shall be double the least, and the greatest triple the least part. Ans. 2, 4, 6.

PROB. 15. Divide 40 into three such parts, that the first shall be 5 times the second, and the third equal to the difference between the first and second. Ans. 20, 4, 16.

PROB. 16. A grocer mixed three kinds of tea, Bohea at 3*s.* per lb., Twankay at 5s., and Souchong at 7*s.* per lb. The mixture contains the same quantity of each, and cost £6. How many lbs. are there of each kind? Ans. 8 lbs.

PROB. 17. A bill of £700 was paid in sovereigns, half-sovereigns, and crowns, and an equal number of each was used. Find the number. Ans. 400.

PROB. 18. Two travellers set out at the same time from Guildford and London, a distance of 27 miles apart; the one walks 4 miles an hour, and the other 5 miles. In how many hours will they meet? Ans. 3 hours.

PROB. 19. A person bought a horse, chaise and harness, for £120; the price of the horse was twice the price of the harness, and the price of the chaise twice the price of both horse and harness; what was the price of each?

Answer $\left\{ \begin{array}{lcl} \text{Price of harness} & = & £13 \,.\, 6 \,.\, 8 \\ \text{..... horse} & = & 26 \,.\, 13 \,.\, 4 \\ \text{..... chaise} & = & 80 \,.\, 0 \,.\, 0 \end{array} \right.$

SUBTRACTION.

30. SUBTRACTION is the finding the difference between two algebraic quantities, and the connecting them by proper signs, so as to form one expression: thus, if it were required to subtract $5-2$ (i.e. 3) from 9, it is evident that the remainder would be *greater* by 2 than if 5 only were subtracted. For the same reason, if $b-c$ were subtracted from a, the remainder would be greater by c, than if b only were subtracted. Now, if b is subtracted from a, the remainder is $a-b$; and consequently, if $b-c$ be subtracted from a, the remainder will be $a-b+c$. Hence this general Rule for the subtraction of algebraic quantities, "Change the signs of the quantities *to be subtracted*, and then place them one after another, as in Addition."

Ex. 1. From $5a+3x-2b$ take $2c-4y$. The quantity to be subtracted *with its signs changed*, is $-2c+4y$; therefore the remainder is $5a+3x-2b-2c+4y$.

Ex. 2. From $7x^2-2x+5$ take $3x^2+5x-1$;

The remainder is $7x^2-2x+5 \quad -3x^2-5x+1$;

or $7x^2-3x^2-2x-5x+5+1=4x^2-7x+6$.

But when *like* quantities are to be subtracted from each other, as in Ex. 2, the better way is to set one row under the other, and apply the following Rule; "*Conceive* the signs of the *quantities to be subtracted* to be *changed*, and then proceed as in Addition."

Ex. 3.	Ex. 4.	Ex. 5.
From $7x^2-2x+5$	$12a^2-3a+b-1$	$5y^2-4y+3a$
Subtract $3x^2+5x-1$	$6a^2+a-2b+3$	$6y^2-4y-a$
Remainder $4x^2-7x+6$	$6a^2-4a+3b-4$	$-y^2 \quad * \quad +4a$

Ex. 6.	Ex. 7.	Ex. 8.
From $7xy+2x-3y$	$14x+y-z-5$	$13x^3-2x^2+7$
Subtract $2xy-x+y$	$x+y+z-11$	$-x^3+x^2-6$

30. What is *subtraction?* State the rule for the subtraction of algebraic quantities, and explain the principle on which it rests.

ON THE SOLUTION OF SIMPLE EQUATIONS, CONTAINING ONLY ONE UNKNOWN QUANTITY.

Rule II.

31. "Any quantity may be transferred from one side of the equation to the other, by changing its sign;" and it is founded upon the axiom, that "if equals be *added* to or *subtracted* from equals, the *sums* or *remainders* will be equal."

Ex. 1. Let $x+8=15$; *subtract* 8 from each side of the equation, and it becomes $x+8-8=15-8$; but $8-8=0$, $\therefore x=15-8=7$.

Ex. 2. Let $x-7=20$; *add* 7 to each side of the equation, then $x-7+7=20+7$; but $-7+7=0$; $\therefore x=20+7=27$.

Ex. 3. Let $3x-5=2x+9$; *add* 5 to each side of the equation, and it becomes $3x-5+5=2x+9+5$, or $3x=2x+9+5$. *Subtract* $2x$ from each side of this latter equation, then $3x-2x=2x-2x+9+5$; but $2x-2x=0$, $\therefore 3x-2x=9+5$. Now $3x-2x=x$, and $9+5=14$; hence $x=14$.

On reviewing the steps of these examples, it appears

(1.) That $x+8=15$ is identical with..... $x=15-8$.

(2.) ..... $x-7=20$ ..... with..... $x=20+7$.

(3.) .....$3x-5=2x+9$ ..... with $3x-2x=9+5$.

Or, that "the equality of the quantities on each side of the equation, is not affected by removing a quantity from one side of the equation to the other and *changing its sign*."

From this rule also it appears, if the same quantity with the same sign be found on *both* sides of an equation, it may be left out of the equation; thus, if $x+a=c+a$, then $x=c+a-a$; but $a-a=0$, $\therefore x=c$.

It further appears, that the signs of all the terms of an equation may be changed from $+$ to $-$, or from $-$ to $+$, without altering the value of the unknown quantity. For let $x-b=c-a$; then, by the Rule, $x=c-a+b$; change

the signs of *all* the terms, then $b-x=a-c$, in which case $b-a+c=x$, or $x=c-a+b$, as before.

Ex. 4.	$2x+3=x+17$	Ans.	$x=14$
Ex. 5.	$5x-4=4x+25$	...	$x=29$
Ex. 6.	$7x-9=6x-3$	...	$x=6$
Ex. 7.	$4x+2a=3x+9b$	...	$x=9b-2a$
Ex. 8.	$15x+4=34$	...	$x=2$
Ex. 9.	$8x+7=6x+27$	...	$x=10$
Ex. 10.	$9x-3=4x+22$	...	$x=5$
Ex. 11.	$17x-4x+9=3x+39$	...	$x=3$
Ex. 12.	$ax-c=b+2c$	...	$x=\dfrac{b+3c}{a}$
Ex. 13.	$5x-(4x-6)=12$		

The sign $-$ before a bracket being the sign of the whole quantity enclosed, indicates that the quantity is *to be subtracted;* and therefore, according to the Rule, when the brackets are removed the sign of each term must be changed. Thus, the signs of $4x$ and of 6 are respectively $+$ and $-$, but when the brackets are removed they must be changed to $-$ and $+$ respectively. The equation then becomes

$$5x-4x+6=12.$$

By transposition, $5x-4x=12-6$;

$$\therefore\ x=6.$$

Ex. 14. $6x-(8+x)=4x-(x-10)$

By removing the brackets, and changing the signs of the terms which they enclose, the equation becomes

$$6x-8-x=4x-x+10.$$

Transposing, $6x-x-4x+x=10+8$;

$$\therefore\ 2x=18.$$

Dividing both sides of the equation by 2,

$$x=9.$$

Ex. 15.	$4x-(3x+4)=8$	Ans.	$x=12$
Ex. 16.	$8x-(6x-8)=9-(3-x)$	...	$x=-2$
Ex. 17.	$4x-(3x-6)-(4x-12)=12-(5x-10)$	Ans.	$x=2$
Ex. 18.	$5x-(3+3x)=8-(-x-1)$	...	$x=12$

PROBLEMS.

PROB. 1. There are two numbers whose difference is 15 and their sum 59. What are the numbers?

As their *difference* is 15, it is evident that the greater number must exceed the less by 15.

Let, therefore $x =$ the less number;
then will $x + 15 =$ the greater:
But their sum $= 59$;
$\therefore\ x + x + 15 = 59,$
or $2x + 15 = 59.$
And, transposing 15, $2x = 59 - 15,$
or $2x = 44;$
$\therefore\ x = 22$ the less number,
and $x + 15 = 22 + 15 = 37$ the greater.

PROB. 2. I gave to Richard and James 27 marbles, but to Richard 5 more than to James. How many did I give to each?

Let $x =$ the number I gave to James;
then $x + 5 =$ ..... Richard:
But together they receive 27;
$\therefore\ x + x + 5 = 27,$
or $2x + 5 = 27.$
Transposing, $2x = 27 - 5,$
or $2x = 22;$
$\therefore\ x = 11$, the No. James received,
and $x + 5 = 16$ ..... Richard received.

PROB. 3. Four times a number is equal to double the number increased by 12. What is the number?

Let $x =$ the number;
then $4x = 4$ times the number,
$2x =$ double the number,
and $2x + 12 =$ double the number increased by 12.
Therefore, by the equality stated in the question,
$4x = 2x + 12.$
By transposition, $4x - 2x = 12$
$2x = 12;$
$\therefore\ x = 6.$

PROB. 4. At an election 420 persons voted, and the suc-

cessful candidate had a majority of 46. How many voted for each candidate? Ans. 187 and 233.

PROB. 5. One of two rods is 8 feet longer than the other, but the longer rod is three times the length of the shorter. What are their lengths? Ans. 4 ft. and 12 ft.

PROB. 6. Five times a number diminished by 16, is equal to three times the number. What is the number? Ans. 8.

PROB. 7. A horse, a cow, and a sheep, were bought for £24; the cow cost £4 more than the sheep, and the horse £10 more than the cow. What was the price of the sheep?

Let $x =$ the price of the sheep in £;

then $x + 4 =$ ..... cow .....

and $x + 4 + 10 =$ ..... horse .....

But these three prices taken together amount to £24;

$$\therefore x + (x + 4) + (x + 4 + 10) = 24.$$

Adding together like terms,

$$3x + 18 = 24.$$

By transposition, $3x = 24 - 18$,

$$3x = 6;$$

$\therefore x =$ £2, the price of the sheep.

PROB. 8. A draper has three pieces of cloth, which together measured 159 yards; the second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What is the length of each piece?

Ans. 35, 50, and 74 yds.

PROB. 9. Divide £36 among three persons, A, B, and C, in such a manner that B shall have £4 more than A, and C £7 more than B. Ans. £7, £11, and £18.

PROB. 10. A gentleman buys 4 horses; for the second he gives £12 more than for the first; for the third £5 more than for the second; and for the fourth £2 more than for the third. The sum paid for all the horses was £240. Find the price of each. Ans. £48, £60, £65, and £67.

PROB. 11. What number is that whose double is as much above 21 as it is itself less than 21?

Let $x =$ the number;

then $2x =$ double the number,

$2x - 21 =$ what double the number is above 21,

and $21 - x =$ what the number is less than 21:

But by the question these two values are equal to each other; $\therefore 2x - 21 = 21 - x$.

By transposition, $2x + x = 21 + 21$,

$$3x = 42;$$
$$\therefore x = 14.$$

The answer may easily be proved to be correct, for $2x - 21 = 28 - 21 = 7$, and $21 - x = 21 - 14 = 7$; that is, twice 14 is as much above 21, as 21 is above 14, namely 7.

PROB. 12. In dividing a lot of oranges among a certain number of boys I found that by giving 4 to each boy I had 6 to spare, and by giving 3 to each boy I had 12 remaining. How many boys were there?

Let $x =$ the number of boys;

then, if I gave to each boy 4 oranges, I should give away 4 times x oranges;

$\therefore 4x =$ number of oranges distributed at first:

But the total number of oranges is 6 more than this number;

$\therefore$ *Total number* of oranges $= 4x + 6$:

Again, if each boy received 3 oranges, there were 12 oranges left;

$\therefore$ *Total number* of oranges $= 3x + 12$.

These two values for *the number of oranges* expressed in terms of x must necessarily be equal; Axiom (1.)

$$\therefore 4x + 6 = 3x + 12.$$

By transposition, $4x - 3x = 12 - 6$;

$\therefore x = 6$, the number of boys.

PROB. 13. An express set out to travel 240 miles in 4 days, but in consequence of the badness of the roads, he found that he must go 5 miles the second day, 9 the third, and 14 the fourth day less than the first. How many miles must he travel each day?

Let $x =$ the number of miles on the 1st day;

then $x - 5 =$ ..... 2d ...

$x - 9 =$ ..... 3d ...

and $x - 14 =$ ..... 4th ...

Now the number of miles which he travels in 4 days $= 240$;

$$\therefore x + x - 5 + x - 9 + x - 14 = 240.$$

Collecting the terms, $4x - 28 = 240$.

By transposition, $4x = 240 + 28$,
$4x = 268$;
$\therefore x = 67$, the number of miles he goes on 1st day,
$x - 5 = 62$ ..... 2d ...
$x - 9 = 58$ ..... 3d ...
and $x - 14 = 53$ ..... 4th ...

Prob. 14. It is required to divide the number 99 into five such parts that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

Ans. The parts are 17, 14, 27, 8, 33.

Prob. 15. Two merchants entered into a speculation, by which A gained £54 more than B. The whole gain was £49 less than three times the gain of B. What were the gains? Ans. A's gain = £157; B's = £103.

Prob. 16. In dividing a lot of apples among a certain number of boys I found that by giving 6 to each I should have too few by 8, but by giving 4 to each boy I should have 12 remaining. How many boys were there?

Ans. 10.

MULTIPLICATION.

32. Multiplication is the finding the product of two or more algebraic quantities; and in performing the process, the four following rules must be observed.

(1.) When quantities having *like* signs are multiplied together, the sign of the *product* will be +; and if their signs are *unlike*, the sign of the product will be —.*

* This rule for the multiplication of the Signs may be thus explained:—

I. If $+a$ is to be multiplied by $+b$, it means, that $+a$ is to be *added* to itself as often as there are units in b, and consequently the product will be $+ab$.

II. If $-a$ is to be multiplied by $+b$, it means, that $-a$ is to be *added* to itself as often as there are units in b, and therefore the product is $-ab$.

---

32. What is multiplication, and what are the Rules to be observed in multiplication?

(2.) The coefficients of the *factors* must be multiplied together, to form the coefficient of the *product*.

(3.) The letters of which they are composed must be set down, one after another, *according to their order in the Alphabet*.

(4.) If the *same* letter is found in both factors, the indices of it must be *added* together, to form the index of it in the *product*.

Thus, $+a$ multiplied by $+b$ is equal to $+ab$, and $-a$ multiplied by $-b$ is also equal to $+ab$; $+3x \times -5y = -15xy$; $-3ab \times +4cd = -12abcd$; $-4a^2b^2 \times -3abd^2 = +12a^3b^3d^2$; &c. &c.

From the division of algebraic quantities into *simple* and *compound*, there arise three cases of Multiplication. In performing the operation, the Rule is, "To multiply *first* the signs, *then* the coefficients, and *afterwards* the letters."

Case I.

33. When *both* factors are *simple* quantities; for which the Rule has been already given.

III. If $+a$ is to be multiplied by $-b$, it means, that $+a$ is to be *subtracted* as often as there are units in b, and consequently the product is $-ab$.

IV. If $-a$ is to be multiplied by $-b$, it means, that $-a$ is to be *subtracted* as often as there are units in b; and, since to *subtract a negative quantity* is the same as *to add a positive one*, the product will be $+ab$.

Or, these four Rules might be all comprehended in *one*; thus,

To multiply $a-b$ by $c-d$, is to add $a-b$ to itself as often as there are units in $c-d$; now this is done by *adding* it c *times*, and *subtracting* it d *times*;

But $a-b$, *added* c *times* . . . $= ac - bc$,
and $a-b$, *subtracted* d *times* $= -ad + bd$,
$\therefore \overline{a-b} \times \overline{c-d}$ . . . . . $= ac - bc - ad + bd$.

i.e. $+a \times +c = +ac$
$-b \times +c = -bc$
$+a \times -d = -ad$
$-b \times -d = +bd$.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$4ab$	$2axy$	$-3abc$	$-5a^2bc$
$3a$	$-3y$	$5a^2b$	$-2b^2x^2$
$12a^2b$	$-6axy^2$	$-15a^3b^2c$	$+10a^2b^3cx^2$

Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
$4abc$	$9x^2y^2$	$-4cdx$	$-7ax^2y$
$3ac$	$-2y$	$2c$	$-2ac^2x$

Case II.

34. When one factor is *compound* and the other *simple*, "Then *each term* of the compound factor must be multiplied "by the simple factor as in the last Case, and the result will be the product required."

Ex. 1.

Multiply $3ab-2ac+d$
by $4a$
Product $12a^2b-8a^2c+4ad$

Ex. 2.

$3x^3 \quad -2x^2+4$
$-14ax$
$-42ax^4+28ax^3-56ax$

Ex. 3.

Multiply $7x^2 \quad -2x \quad +4a$
by $-3a$
Product $-21ax^2+6ax-12a^2$

Ex. 4.

$12a^3-2a^2+4a-1$
$3x$

Ex. 5.

Multiply $9a^2x+3a-x+1$
by $-x^2$
Product

Ex. 6.

$4x^2y+3x-2y$
$-3xy$

Case III.

35. When *both* factors are *compound* quantities, each term of the multiplicand must be multiplied by each term of the

multiplier; and then placing *like quantities under each other*, the sum of all the terms will be the product required.

Ex. 1.

$$\begin{array}{rl}
\text{Multiply } & a + b \\
\text{by } & a + b \\
\hline
\text{1st, by } a \ldots & a^2 + ab \\
\text{2d, by } b \ldots & \phantom{a^2+{}} ab + b^2 \\
\hline
\text{Product} & a^2 + 2ab + b^2 \\
\hline\hline
\end{array}$$

Ex. 2.

$$\begin{array}{l}
a + b \\
a - b \\
\hline
a^2 + ab \\
 - ab - b^2 \\
\hline
a^2 \quad * \quad - b^2 \\
\hline\hline
\end{array}$$

Ex. 3.

$$\begin{array}{l}
a^2 + ab + b^2 \\
a - b \\
\hline
a^3 + a^2b + ab^2 \\
 - a^2b - ab^2 - b^3 \\
\hline
a^3 \quad * \quad\quad * \quad - b^3 \\
\hline\hline
\end{array}$$

Ex. 4.

$$\begin{array}{l}
3x^2 + 2x \\
4x + 7 \\
\hline
12x^2 + 8x^2 \\
 + 21x^2 + 14x \\
\hline
12x^3 + 29x^2 + 14x \\
\hline\hline
\end{array}$$

Ex. 5.

$$\begin{array}{l}
3x^2 - 2x + 5 \\
6x - 7 \\
\hline
18x^3 - 12x^2 + 30x \\
 - 21x^2 + 14x - 35 \\
\hline
18x^3 - 33x^2 + 44x - 35 \\
\hline\hline
\end{array}$$

Ex. 6.

$$\begin{array}{l}
14ac - 3ab + 2 \\
ac - ab + 1 \\
\hline
14a^2c^2 - 3a^2bc + 2ac \\
 - 14a^2bc \qquad + 3a^2b^2 - 2ab \\
 + 14ac \qquad - 3ab + 2 \\
\hline
14a^2c^2 - 17a^2bc + 16ac + 3a^2b^2 - 5ab + 2 \\
\hline\hline
\end{array}$$

Ex. 7.

$$\begin{array}{l}
x^2 - \frac{1}{2}x + \frac{2}{3} \\
\frac{1}{3}x + 2 \\
\hline
\frac{1}{3}x^3 - \frac{1}{6}x^2 + \frac{2}{9}x \\
\phantom{\frac{1}{3}x^3} + 2x^2 - x + \frac{4}{3} \\
\hline
\frac{1}{3}x^3 + \frac{11}{6}x^2 - \frac{7}{9}x + \frac{4}{3} \\
\hline\hline
\end{array}$$

Ex. 8. Multiply $a^3 + 3a^2b + 3ab^2 + b^3$ by ... $a + b$.
Ans. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Ex. 9. Multiply $4x^2y+3xy-1$ ..... by $2x^2-x$.
Ans. $8x^4y+2x^3y-2x^2-3x^2y+x$.

Ex. 10. ..... x^3-x^2+x-5 ..... by $2x^2+x+1$.
Ans. $2x^5-x^4+2x^3-10x^2-4x-5$.

Ex. 11. ..... $3a^2+2ab-b^2$ ..... by $3a^2-2ab+b^2$.
Ans. $9a^4-4a^2b^2+4ab^3-b^4$.

Ex. 12. ..... $x^3+x^2y+xy^2+y^3$ ... by $x-y$.
Ans. x^4-y^4.

Ex. 13. ..... $x^2-\frac{3}{4}x+1$ ..... by $x^2-\frac{1}{2}x$.
Ans. $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$.

ON THE SOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

Ex. 1. $3x+4(x+2)=36$.

The term $4(x+2)$ means, that $x+2$ is to be multiplied by 4, and the product by Case 2[d] is $4x+8$;

$$\therefore 3x+4x+8=36.$$

Adding together the terms containing x, and transposing 8,

$$7x=36-8,$$
$$7x=28;$$
$$\therefore x=4.$$

Ex. 2. $8(x+5)+4(x+1)=80$.

Performing the multiplication,

$$8x+40+4x+4=80.$$

Collecting the terms, $12x+44=80$.

Transposing,

$$12x=80-44,$$
$$12x=36;$$
$$\therefore x=3.$$

Ex. 3. $6(x+3)+4x=58$. Ans. $x=4$.

Ex. 4. $30(x-3)+6=6(x+2)$. ... $x=4$.

Ex. 5. $5(x+4)-3(x-5)=49$. ... $x=7$.

Ex. 6. $4(3+2x)-2(6-2x)=60$. ... $x=5$.

Ex. 7. $3(x-2)+4=4(3-x)$. ... $x=2$.

Ex. 8. $6(4-x)-4(6-2x)-12=0$. ... $x=6$.

PROBLEMS.

PROB. 1. What two numbers are those whose difference is 9, and if 3 times the greater be added to 5 times the less, the sum shall be 35?

Let $x =$ the less number;
then $x + 9 =$ the greater.
And 3 *times* the greater $= 3(x + 9) = 3x + 27$,
5 *times* the less $= 5x$.
But by the problem, 3 times the greater + 5 times the less $= 35$;

$$\therefore 3x + 27 + 5x = 35,$$
$$8x + 27 = 35.$$

Transposing, $8x = 35 - 27 = 8$;
$\therefore x = 1$, the less number,
and $x + 9 = 10$, the greater.

PROB. 2. A courier travels 7 miles an hour, and had been dispatched 5 hours, when a second is sent to overtake him, and in order to do this, he is obliged to travel 12 miles an hour. In how many hours does he overtake him?

Let $x =$ the number of hours the 2d travels;
then $x + 5 =$ ..... 1st .....
$\therefore 12x =$ the number of miles the 2d .....
and $7(x + 5) =$ ..... 1st .....
But by the supposition the couriers both travel the same number of miles;

$$\therefore 12x = 7(x + 5),$$
$$12x = 7x + 35.$$

Transposing, $12x - 7x = 35$,
$5x = 35$,
$x = 7$, the number of hours the second courier is in overtaking the first.

PROB. 3. In a railway train 15 passengers paid £3 12*s.*; the fare of the first class being 6*s.*, and that of the second 4*s.* How many passengers were there of each class?

Let $x =$ the number of passengers of the 1st class,
then $15 - x =$ ..... 2d .....
$\therefore 6x =$ sum in shillings paid by 1st class passengers,
and $4(15 - x) =$ ..... 2d .....

But these two sums amount to £3 12*s*., or to 72*s*.

$$\therefore\ 6x + 4\,(15 - x) = 72,$$
$$6x + 60 - 4x = 72.$$

By transposition, $6x - 4x = 72 - 60,$

$$2x = 12;$$
$$\therefore\ x = 6 \text{ No. of } 1^{st} \text{ class passengers;}$$
$$\therefore \text{ the number of } 2^{d} \text{ class passengers} = 15 - x = 9.$$

Prob. 4. What number is that to which if 6 be added twice the sum will be 24? Ans. 6.

Prob. 5. What two numbers are those whose difference is 6, and if 12 be added to 4 times their sum, the whole will be 60? Ans. 3 and 9.

Prob. 6. Tea at 6*s*. per lb. is mixed with tea at 4*s* per lb., and 16 lbs. of the mixture is sold for £3 18*s*. How many lbs. were there of each sort? Ans. 7 lbs. and 9 lbs.

Prob. 7. The speed of a railway train is 24 miles an hour, and 3 hours after its departure an express train is started to run 32 miles an hour. In how many hours does the express overtake the train first started? Ans. 9 hours.

Prob. 8. A mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants taken together measured 142 yards. What was the length of each piece?

Let $x =$ the length of each piece in yards;

$\therefore\ x - 19 =$ the length of each of the 3 remnants,

and $x - 17 =$ the length of the other remnant;

then $3\,(x - 19) + x - 17 = 142,$

or $3x - 57 + x - 17 = 142,$

$$4x - 74 = 142.$$

Transposing, $4x = 142 + 74,$

$$4x = 216;$$
$$\therefore\ x = 54.$$

Prob. 9. Divide the number 68 into two such parts, that the difference between the greater and 84 may equal 3 times the difference between the less and 40.

Let $x =$ the less part,

then $68 - x =$ the greater;

$\therefore 84 - (68 - x) =$ difference between 84 and the greater, and $3 \,.\, (40 - x) = 3$ times the difference between the less and 40.

But by the question the differences are equal to each other;

$$\therefore 84 - (68 - x) = 3 \,.\, (40 - x),$$
$$\text{or } 84 - 68 + x = 120 - 3x.$$

By transposition, $x + 3x = 120 + 68 - 84,$

$$4x = 104;$$
$$\therefore x = 26, \text{ the less part};$$

and $\therefore$ the greater $= 42$.

PROB. 10. A man at a party at cards betted three shillings to two upon every deal. After twenty deals he won five shillings. How many deals did he win?

Let $x =$ the number of deals he won;

$\therefore 20 - x =$ the number he lost;

$\therefore 2x =$ the money won;

and $3 \,.\, (20 - x) =$ the money lost.

But the difference between the money won and the money lost was $5s$.

$$\therefore 2x - 3 \,.\, (20 - x) = 5,$$
$$2x - 60 + 3x = 5,$$
$$5x - 60 = 5,$$
$$5x = 65;$$
$$\therefore x = 13.$$

PROB. 11. A and B being at play cut packs of cards so as to take off more than they left. Now it happened that A cut off twice as many as B left, and B cut off seven times as many as A left. How were the cards cut by each?

Suppose A cut off $2x$ cards;

then $52 - 2x =$ the number he left,

and $x =$ the number B left;

$\therefore 52 - x =$ the number he cut off.

But the number B cut off was equal to 7 times the number A left;

$$\therefore 52 - x = 7 \,.\, (52 - 2x)$$
$$52 - x = 364 - 14x.$$

Transposing, $14x - x = 364 - 52,$

$$13x = 312;$$
$$\therefore x = 24;$$

$\therefore$ A cut off 48, and B cut off 28 cards.

PROB. 12. Some persons agreed to give sixpence each to a waterman for carrying them from London to Greenwich; but with this condition, that for every other person taken in by the way, threepence should be abated in their joint fare. Now the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but fivepence each. How many persons were there at first?

Let $4x$ represent the number of passengers at first; then 3 more than a fourth part of this number $= x + 3$, and they paid $3(x + 3)$ pence.

$\therefore$ the original passengers paid $6 \times 4x - 3(x + 3)$ pence.

But the original passengers paid $5 \times 4x$ pence;

$\therefore$ by equalizing these two values, we get

$$6 \times 4x - 3(x + 3) = 5 \times 4x,$$
$$24x - 3x - 9 = 20x.$$

Transposing, $24x - 3x - 20x = 9$;

$$\therefore x = 9;$$

and $\therefore$ the No. of passengers were $= 4 \times 9 = 36$.

PROB. 13. There are two numbers whose difference is 14, and if 9 times the less be subtracted from 6 times the greater, the remainder will be 33. What are the numbers?

Ans. 17 and 31.

PROB. 14. Two persons, A and B, lay out equal sums of money in trade; A *gains* £120, and B *loses* £80; and now A's money is *treble* of B's. What sum had each at first?

Ans. £180.

PROB. 15. A rectangle is 8 feet long, and if it were two feet broader, its area would be 48 feet. Find its breadth.

Ans. 4 feet.

PROB. 16. William has 4 times as many marbles as Thomas, but, if 12 be given to each, William will then have only twice as many as Thomas. How many has each?

Ans. 24 and 6.

PROB. 17. Two rectangular slates are each 8 inches wide, but the length of one is 4 inches greater than that of the other. Find their lengths, the longer slate being twice the area of the other.

Let $x =$ the length in inches of the less;

then $x + 4 =$ ..... greater.

Now the area of a rectangle is its length multiplied by its breadth;

$\therefore 8x$ and $8(x+4)$ are the areas of the slates.

But the larger slate is twice the area of the less;

$$\therefore 8x \times 2 = 8(x+4),$$
$$16x = 8x + 32;$$
$$\therefore 8x = 32;$$
$$\therefore x = 4, \text{ the length of the less slate,}$$
$$\text{and } x + 4 = 8, \text{ ..... greater slate.}$$

PROB. 18. Two rectangular boards are equal in area; the breadth of the one is 18 inches, and that of the other 16 inches, and the difference of their lengths 4 inches. Find the length of each and the common area.

Ans. 32, 36, and 576.

PROB. 19. A straight lever (without weight) supports in equilibrium on a fulcrum 24 lbs. at the end of the shorter arm, and 8 lbs. at the end of the longer, but the length of the longer arm is 6 inches more than that of the shorter. Find the lengths of the arms.

Let x = length in inches of the shorter arm;

then $x + 6 =$ ..... longer ...

Now the lever will be in equilibrium, when the weight at one end multiplied by the length of the corresponding arm is equal to the weight at the other end, multiplied by its corresponding arm;

$$\therefore 24x = 8(x+6),$$
$$24x = 8x + 48,$$
$$16x = 48;$$
$$\therefore x = 3 \text{ inches, the length of the shorter arm;}$$
$$\text{and } x + 6 = 9 \text{ ..... longer ...}$$

PROB. 20. A weight of 6 lbs. balances a weight of 24 lbs. on a lever (supposed to be without weight), whose length is 20 inches; if 3 lbs. be added to each weight, what addition must be made to each arm of the lever, so that the fulcrum may preserve its original position, and equilibrium still be retained?

This problem resolves itself into two other problems:—

(1.) To find the lengths of the arms in the original position:

Let x = the length in inches of the shorter arm;

then $20 - x =$ ..... longer ...

Now, in order that there may be equilibrium, $24x$ and $6(20-x)$ must be equal to each other;

$$\therefore 24x = 120 - 6x,$$
$$30x = 120;$$
$$\therefore x = 4, \text{ the length of the shorter arm;}$$
$$\text{and } 20 - x = 16, \text{ ..... longer ...}$$

(2.) To find the *addition* to be made to each arm, so that there may again be equilibrium on the fulcrum in its original position, after 3 lbs. have been added to each weight:

Let x = number of inches to be added to each arm; then the lengths of the arms become $4 + x$, and $16 + x$ inches respectively: and the weights at the arms have been respectively increased to 27 lbs. and 9 lbs.

But by the principle of the equilibrium of the lever, $27(4+x)$ and $9(16+x)$ must be equal to each other;

$$\therefore 27(4+x) = 9(16+x).$$

Divide each side of the equation by 9, and

$$3(4+x) = 16 + x,$$
$$12 + 3x = 16 + x,$$
$$3x - x = 16 - 12$$
$$2x = 4;$$
$$\therefore x = 2.$$

Prob. 21. The conditions being the same as in the last problem, how many inches must be added to the shorter arm in order that the lever may in its original position retain its equilibrium? Ans. $1\frac{1}{3}$ inch.

Prob. 22. A garrison of 1000 men was victualled for 30 days; after 10 days it was reinforced, and then the provisions were exhausted in 5 days; find the number of men in the reinforcement. Ans. 3000.

Prob. 23. Two triangles have each a base of 20 feet, but the altitude of one of them is 6 feet less than that of the other, and the area of the greater triangle is twice that of the less. Find their altitudes. Ans. 6 and 12.

N.B. The area of a triangle $= \frac{1}{2}$ base $\times$ altitude.

Prob. 24. A and B began to play with equal sums; A won 12*s.*; then 6 times A's money was equal to 9 times B's. What had each at first? Ans. £3.

PROB. 25. A company settling their reckoning at a tavern, pay 4 shillings each, but observe that if there had been 5 more they would only have paid 3 shillings each? How many were there. Ans. 15.

PROB. 26. Two persons, A and B, at the same time set out from two towns 40 miles apart, and meet each other in 5 hours, but B walks 2 miles an hour more than A. How many miles does A walk in an hour? Ans. 3 miles.

DIVISION.

36. The *Division* of algebraic quantities is the finding their quotient, and in performing the operation the same circumstances are to be taken into consideration as in their *multiplication*, and consequently the four following Rules must be observed.

(1.) That if the signs of the dividend and divisor be *like*, then the sign of the quotient will be $+$; if *unlike*, then the sign of the quotient will be $-$.*

(2.) That the coefficient of the *dividend* is to be divided by the coefficient of the *divisor*, to obtain the coefficient of the *quotient*.

(3.) That all the letters *common* to both the dividend and the divisor must be *rejected* in the quotient.

(4.) That if the same letter be found in both the dividend and divisor with *different* indices, then the index of

* The Rule for the *signs* follows immediately from that in Multiplication; thus,

$$\left.\begin{array}{l} \text{If } +a\times+b=+ab, \text{ then } \dfrac{+ab}{+a}=+b, \text{ and } \dfrac{+ab}{+b}=+a \\ +a\times-b=-ab,\ldots\ \dfrac{-ab}{+ab}=-b, \text{ and } \dfrac{-ab}{-b}=+a \\ -a\times-b=+ab,\ldots\ \dfrac{+ab}{-a}=-b, \text{ and } \dfrac{+ab}{-b}=-a \end{array}\right\} \begin{array}{l}\text{i.e. } \textit{like} \text{ signs} \\ \text{produce } +, \\ \text{and } \textit{unlike} \\ \text{signs } -.\end{array}$$

36. What is meant by the division of algebraic quantities? State the Rules in division.

that letter in the divisor must be *subtracted* from its index in the dividend, to obtain its index in the quotient. Thus,

(1.) $+abc$ divided by $+ac$..... or $\frac{+abc}{+ac} = +b.$

(2.) $+6abc$ ..... $-2a$..... or $\frac{6abc}{-2a} = -3bc.$

(3.) $-10xyz$ ..... $+5y$..... or $\frac{-10xyz}{+5y} = -2xz.$

(4.) $-20a^2x^2y^3$ ..... $-4axy$ ...or $\frac{-20a^2x^2y^3}{-4axy} = +5axy^2.$

Of *Division*, also, there are three Cases; the same as in *Multiplication*.

Case I.

37. When dividend and divisor are both *simple* terms.

Ex. 1.

Divide $18ax^2$ by $3ax$.

$$\frac{18ax^2}{3ax} = 6x.$$

Ex. 2.

Divide $15a^2b^2$ by $-5a$.

$$\frac{+15a^2b^2}{-5a} = -3ab^2.$$

Ex. 3.

Divide $-28x^2y^3$ by $-4xy$.

$$\frac{-28x^2y^3}{-4xy} = +7xy^2.$$

Ex. 4.

Divide $25\,a^3c^2$ by $-5a^2c$.

$$\frac{+25a^3c^2}{-5a^2c} =$$

Ex. 5.

Divide $-14a^3b^2c$ by $7ac$.

$$\frac{-14a^3b^2c}{+7ac} =$$

Ex. 6.

Divide $-20x^2y^2z^3$ by $-4yz$.

$$\frac{-20x^2y^2z^3}{-4yz.} =$$

Case II.

38. When the dividend is a *compound* quantity, and the divisor a *simple* one; then each term of the dividend must be divided separately, and the resulting quantities will be the quotient required.

---

38. State the rule for Case 2d.

Ex. 1.

Divide $42a^2+3ab+12a^2$ by $3a$.

$$\frac{42a^2+3ab+12a^2}{3a}=14a+b+4a.$$

Ex. 2.

Divide $90a^2x^3-18ax^2+4a^2x-2ax$ by $2ax$.

$$\frac{90a^2x^3-18ax^2+4a^2x-2ax}{2ax}=45ax^2-9x+2a-1.$$

Ex. 3.

Divide $4x^3-2x^2+2x$ by $2x$

$$\frac{4x^3-2x^2+2x}{2x}=$$

Ex. 4.

Divide $-24a^2x^2y-3axy+6x^2y^2$ by $-3xy$.

$$\frac{-24a^2x^2y-3axy+6x^2y^2}{-3xy}=$$

Ex. 5.

Divide $14ab^3+7a^2b^2-21a^2b^3+35a^3b$ by $7ab$.

$$\frac{14ab^3+7a^2b^2-21a^2b^3+35a^3b}{7ab}$$

Case III.

39. When dividend and divisor are *both compound* quantities. In this case, the Rule is, "to arrange both dividend and divisor according to the powers of the same letter, beginning with the *highest;* then find how often the first term of the divisor is contained in the first term of the dividend, and place the result in the quotient; multiply each term of the divisor by this quantity, and place the product under the corresponding (i. e. *like*) terms in the dividend, and then subtract it from them; to the remainder bring down as many terms of the dividend as will make its number of

39. When dividend and divisor are both *compound* quantities, what is the rule?

terms equal to that of the divisor; and then proceed as before, till all the terms of the dividend are brought down, as in common arithmetic."

Ex. 1.

Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

$$
\begin{array}{r|l}
a - b & a^3 - 3a^2b + 3ab^2 - b^3 \;\big(a^2 - 2ab + b^2 \\
 & \underline{a^3 - a^2b} \\
 & * \; - 2a^2b + 3ab^2 \\
 & \quad \underline{- 2a^2b + 2ab^2} \\
 & \qquad * \qquad\quad ab^2 - b^3 \\
 & \qquad\qquad\quad \underline{ab^2 - b^3} \\
 & \qquad\qquad\qquad * \quad\; *
\end{array}
$$

In this Example, the dividend is arranged according to the powers of a, the first term of the divisor. Having done this, we proceed by the following steps:—

(1.) a is contained in a^3, a^2 times; put this in the quotient.

(2.) Multiply $a-b$ by a^2, and it gives a^3-a^2b.

(3.) Subtract a^3-a^2b from a^3-3a^2b, and the remainder is $-2a^2b$.

(4.) Bring down the next term $+3ab^2$.

(5.) a is contained in $-2a^2b$, $-2ab$ times; put this in the quotient.

(6.) *Multiply* and *subtract* as before, and the remainder is ab^2.

(7.) Bring down the last term $-b^3$.

(8.) a is contained in ab^2, $+b^2$ times; put this in the quotient.

(9.) *Multiply* and *subtract* as before, and nothing remains; the *quotient* therefore is $a^2-2ab+b^2$.

Ex. 2.

$$
\begin{array}{l}
a^2+2ax+x^2\,\big)\,a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\,\big(\,a^3+3a^2x+3ax^2+x^3 \\
\quad a^5+2a^4x+\ a^3x^2 \\
\hline
\quad * \quad 3a^4x+9a^3x^2+10a^2x^3 \\
\quad\quad 3a^4x+6a^3x^2+\ 3a^2x^3 \\
\hline
\quad\quad * \quad 3a^3x^2+\ 7a^2x^3+5ax^4 \\
\quad\quad\quad 3a^3x^2+\ 6x^2x^3+3ax^4 \\
\hline
\quad\quad\quad * \quad a^2x^3+2ax^4+x^5 \\
\quad\quad\quad\quad a^2x^3+2ax^4+x^5 \\
\hline
\quad\quad\quad\quad * \quad * \quad * \\
\hline\hline
\end{array}
$$

Ex. 3.

$$
\begin{array}{l}
4x^2-7x\,\big)\,12x^5-13x^4-34x^3+39x^2\,\big(\,3x^3+2x^2-5x+\dfrac{4x^2}{4x^2-7x}\ ^{(a)} \\
\quad 12x^5-21x^4 \\
\hline
\quad\quad +\ 8x^4-34x^3 \\
\quad\quad +\ 8x^4-14x^3 \\
\hline
\quad\quad\quad * \ -20x^3+39x^2 \\
\quad\quad\quad\quad -20x^3+35x^2 \\
\hline
\quad\quad\quad\quad * \ +\ 4x^2 \\
\hline\hline
\end{array}
$$

Ex. 4.

$$
\begin{array}{l}
3x-6\,\big)\,6x^4-96\,\big(\,2x^3+4x^2+8x+16 \\
\quad 6x^4-12x^3 \\
\hline
\quad * \ +12x^3-96 \\
\quad\quad +12x^3-24x^2 \\
\hline
\quad\quad * \ +24x^2-96 \\
\quad\quad\quad +24x^2-48x \\
\hline
\quad\quad\quad * \ +48x-96 \\
\quad\quad\quad\quad +48x-96 \\
\hline
\quad\quad\quad\quad * \quad * \\
\hline\hline
\end{array}
$$

[a] When there is a *remainder*, it must be made the *numerator* of a Fraction whose denominator is the *divisor;* this Fraction must then be placed in the *quotient* (with its proper sign), the same as in common arithmetic.

Ex. 5.

$$
\begin{array}{l}
x^2+x-1\,\big)\,x^6-x^4+x^3-x^2+2x-1\,\big(\,x^4-x^3+x^2-x+1 \\
\qquad\quad x^6+x^5-x^4 \\
\qquad\quad \overline{\ *-x^5+x^3-x^2} \\
\qquad\qquad\quad -x^5-x^4+x^3 \\
\qquad\qquad\quad \overline{\ *\ +x^4-x^2+2x} \\
\qquad\qquad\qquad\quad +x^4+x^3-\ x^2 \\
\qquad\qquad\qquad\quad \overline{\ *\ -x^3+2x-1} \\
\qquad\qquad\qquad\qquad -x^3-\ x^2+x \\
\qquad\qquad\qquad\qquad \overline{\ *\ +\ x^2+x-1} \\
\qquad\qquad\qquad\qquad\quad +\ x^2+x-1 \\
\qquad\qquad\qquad\qquad\quad \overline{\ *\quad *\quad *} \\
\qquad\qquad\qquad\qquad\quad \overline{\overline{\qquad\qquad\qquad}}
\end{array}
$$

Ex. 6.

$$
\begin{array}{l}
x^2-\frac{1}{2}x\,\big)\,x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x\,\big(\,x^2-\frac{3}{4}x+1 \\
\qquad\quad x^4-\frac{1}{2}x^3 \\
\qquad\quad \overline{\ *-\frac{3}{4}x^3+\frac{11}{8}x^2} \\
\qquad\qquad -\frac{3}{4}x^3+\frac{3}{8}x^2 \\
\qquad\qquad \overline{\ *\ +\ x^2-\frac{1}{2}x} \\
\qquad\qquad\qquad +\ x^2-\frac{1}{2}x \\
\qquad\qquad\qquad \overline{\ *\quad *} \\
\qquad\qquad\qquad \overline{\overline{\qquad\qquad}}
\end{array}
$$

Ex. 7. Divide $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ by $a+b$.

Ans. $a^3+3a^2b+3ab^2+b^3$.

Ex. 8. ..... $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^3-3a^2x+3ax^2-x^3$.

Ans. $a^2-2ax+x^2$.

Ex. 9. ..... $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$

Ans. $5x^3+4x^2+3x+2$.

Ex. 10..... $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$ by $a+2x$.

Ans. $a^3+6a^2x+12ax^2+8x^3$.

Ex. 11. Divide a^5-x^5 by $a-x$.

Ans. $a^4+a^3x+a^2x^2+ax^3+x^4$.

Ex. 12. ..... $6x^4+9x^2-20x$ by $3x^2-3x$.

Ans. $2x^2+2x+5-\dfrac{5x}{3x^2-3x}$.

Ex. 13. ..... $9x^6-46x^5+95x^2+150x$ by x^2-4x-5.

Ans. $9x^4-10x^3+5x^2-30x$.

Ex. 14. ..... $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

Ans. $\frac{3}{4}x^3-\frac{1}{2}x^2+1$.

PROBLEMS PRODUCING SIMPLE EQUATIONS, CONTAINING ONLY ONE UNKNOWN QUANTITY.

PROB. 1. A fish was caught the tail of which weighed 9 lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh?

Let $2x =$ weight of the body in lbs.;

$\therefore\ 9+x =$ weight of tail $+\ \frac{1}{2}$ body $=$ weight of head.

But the body weighs as much as the head and tail;

$$\therefore\ 2x=(9+x)+9,$$
$$2x=x+18;$$
$$\therefore\ x=18,$$

and $\therefore\ 2x=36$, the weight of body in lbs.,

$9+x=27$, the weight of head in lbs.

and the weight of fish $=36+27+9=72$ lbs.

PROB. 2. A servant agreed to serve for £8 a year and a livery, but left his service at the end of 7 months, and received only £2 13*s.* 4*d.* and his livery; what was its value?

Let $12x =$ the value of the livery in *d.*

But £8 $=1920d.$, and £2 13*s.* 4*d.* $=640d.$;

then, the wages for 12 months $=12x+1920$;

$$\therefore \text{ the wages for 1 month} = \frac{12x+1920}{12} = x+160.$$

and $\therefore$ the wages for 7 months $=(x+160)\,7$.

But the wages actually received for 7 months $= 12x + 640$;

$$\therefore 12x + 640 = 7x + 1120;$$
$$\therefore 5x = 480,$$
$$x = 96;$$

and $\therefore 12x = 1152d. = £4\ 16s.$, value of livery.

From the solutions of the two preceding problems, it will be seen, that by assuming *for the unknown quantity, x with a proper coefficient*, an equation *free* from fractions will be obtained. It is frequently not only convenient to make such an assumption, but a more elegant solution is generally thereby obtained. The coefficient of x must be a *multiple* of the denominator of all the fractions involved in the problem.

Prob. 3. A cistern is filled in 20 minutes by 3 pipes, one of which conveys 10 gallons more, and another 5 gallons less, than the third *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?
Ans 22, 7, and 12 gallons, respectively.

Prob. 4. A and B have the same income: A lays by $\frac{1}{5}$ of his; but B spending £60 a year more than A, at the end of 4 years finds himself £160 in debt. What did each annually receive? Ans £100.

Prob. 5. A met two beggars, B and C, and having a certain sum in his pocket, gave B $\frac{1}{6}$ of it, and C $\frac{5}{5}$ of the remainder: A now had 20*d.* left; what had he at first?
Ans. 5*s.*

Prob. 6. A person has two horses, [and a saddle worth £60]; if the saddle be put on the first horse, his value will become double that of the second; but if it be put on the second, his value will become triple that of the first. What is the value of each horse? Ans. £36 and £48.

Prob. 7. A gamester at one sitting lost $\frac{1}{3}$ of his money, and then won 18*s.*; at a second he lost $\frac{1}{3}$ of the remainder, and then won 3*s.*, after which he had 3 guineas left. How much money had he at first?

Let $15x =$ the number of shillings he had at first; having lost $\frac{1}{3}$ of his money, he had $\frac{4}{3}$ of it, or $12x$ remaining; he then won 18*s.*, and therefore had $12x + 18$ in

hand; losing $\frac{1}{3}$ of this, he had $\frac{2}{3}$ of it, or $8x+12$ left; he then won 3*s*., and so had $(8x+12)+3$ shillings, which was to be equal to 3 guineas, or 63*s*.

$$\therefore (8x+12)+3=63,$$
$$8x+12=60;$$
$$\therefore 8x=48,$$
$$x=6;$$
$$\text{Hence } 15x=90s.=£4\ 10s.$$

PROB. 8. A person at play lost a third of his money, and then won 4*s*.; he again lost a fourth of his money, and then won 13*s*.; lastly, he lost an eighth of what he then had, and found he had 28*s*. left. What had he at first? Ans. £1 12*s*.

CHAPTER III.

ON ALGEBRAIC FRACTIONS.

40. THE Rules for the management of Algebraic Fractions are the same as those in common arithmetic. The principles, on which the rules in both sciences are established, are the following:—

(1.) If the numerator of a fraction be multiplied, or the denominator divided by any quantity, the fraction is rendered so many times greater in value.

(2.) If the numerator of a fraction be divided, or the denominator multiplied by any quantity, the fraction is rendered so many times less in value.

(3.) If both the numerator and denominator of a fraction be multiplied or divided by any quantity, the fraction remains unaltered in value.

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40. Are the same rules employed in the management of algebraic fractions as in those of common arithmetic? Enumerate the principles which are the foundation of the rules in both sciences.

ON THE REDUCTION OF FRACTIONS.

41. *To reduce a mixed Quantity to an improper Fraction.*

RULE. "Multiply the *integral* part by the denominator of the fraction, and to the *product* annex the numerator with its proper sign; under this *sum* place the former denominator, and the result is the improper fraction required."

Ex. 1.

Reduce $3a + \frac{2x}{5a^2}$ to an improper fraction.

The integral part $\times$ the *denominator* of the fraction $+$ the *numerator* $= 3a \times 5a^2 + 2x = 15a^3 + 2x$;

Hence, $\frac{15a^3 + 2x}{5a^2}$ is the fraction required.

Ex. 2.

Reduce $5x - \frac{4x}{6a^2}$ to an improper fraction.

Here $5x \times 6a^2 = 30a^2x$; to this add the numerator with its proper sign, viz. $-4x$; then $\frac{30a^2x - 4x}{6a^2}$ is the fraction required.

Ex. 3.

Reduce $5x - \frac{2x - 3}{7}$ to an improper fraction.

Here $5x \times 7 = 35x$. In adding the numerator $2x - 3$ *with its proper sign*, it is to be recollected, that the sign $-$ affixed to the fraction $\frac{2x - 3}{7}$ means that the *whole* of that fraction is to be *subtracted*, and consequently that the signs of each term of the numerator must be *changed* when it is combined with $35x$; hence the improper fraction required is $\frac{35x - 2x + 3}{7} = \frac{33x + 3}{7}$.

---

41. How is a mixed quantity reduced to an improper fraction?

Ex. 4. Reduce $4ab + \frac{2c}{3a}$ to an improper fraction.

Ans. $\frac{12a^2b + 2c}{3a}$.

Ex. 5. ..... $3b^2 - \frac{4a}{5x}$ to an improper fraction.

Ans. $\frac{15b^2x - 4a}{5x}$.

Ex. 6. ..... $a - x + \frac{a^2 - ax}{x}$ to an improper fraction.

Ans. $\frac{a^2 - x^2}{x}$.

Ex. 7. ..... $3x^2 - \frac{4x - 9}{10}$ to an improper fraction.

Ans. $\frac{30x^2 - 4x + 9}{10}$.

42. *To reduce an improper Fraction to a mixed Quantity.*

RULE. "Observe which terms of the *numerator* are divisible by the *denominator* without a remainder, the quotient will give the *integral* part; to this annex (with their proper signs) the remaining terms of the numerator with the denominator under them, and the result will be the mixed quantity required."

Ex. 1.

Reduce $\frac{a^2 + ab + b^2}{a}$ to a mixed quantity.

Here $\frac{a^2 + ab}{a} = a + b$ is the *integral* part,

and $\frac{b^2}{a}$ is the *fractional* part;

$\therefore a + b + \frac{b^2}{a}$ is the mixed quantity required.

---

42. What is the rule for reducing an improper fraction to a mixed quantity?

Ex. 2.

Reduce $\frac{15a^2+2x-3c}{5a}$ to a mixed quantity.

Here $\frac{15a^2}{5a}=3a$ is the *integral* part,

and $\frac{2x-3c}{5a}$ is the *fractional* part;

$\therefore 3a+\frac{2x-3c}{5a}$ is the mixed quantity required.

Ex. 3. Reduce $\frac{4x^2-5a}{2x}$ to a mixed quantity.

Ans. $2x-\frac{5a}{2x}$.

Ex. 4. ..... $\frac{12a^2+4a-3c}{4a}$ to a mixed quantity.

Ans. $3a+1-\frac{3c}{4a}$.

Ex. 5. ..... $\frac{10x^2y+3x^3-2b^2}{x^2}$ to a mixed quantity.

Ans. $10y+3x-\frac{2b^2}{x^2}$.

43. *To reduce Fractions to a common Denominator.*

Rule. "Multiply each numerator into every denominator *but its own* for the new numerators, and *all the denominators together* for the common denominator."

Ex. 1.

Reduce $\frac{2x}{3}$, $\frac{5x}{b}$, and $\frac{4a}{5}$, to a common denominator.

$$\left.\begin{array}{l} 2x\times b\times 5=10bx \\ 5x\times 3\times 5=75x \\ 4a\times 3\times b=12ab \end{array}\right\}\text{ new numerators;}$$

$$3\times b\times 5=15b \text{ common denominator;}$$

Hence the fractions required are $\frac{10bx}{15b}$, $\frac{75x}{15b}$, $\frac{12ab}{15b}$.

43. How are fractions reduced to a common denominator?

Ex. 2.

Reduce $\frac{2x+1}{5}$, and $\frac{3x}{4}$, to a common denominator.

$$\text{Here } \left.\begin{array}{l}(2x+1)\times 4 = 8x+4\\ 3x\times 5 = 15x\end{array}\right\}\text{ new numerators; } \left\{\begin{array}{l}\text{Hence the fractions required are}\\ \frac{8x+4}{20}, \text{ and } \frac{15x}{20}.\end{array}\right.$$

$$5\times 4 = 20 \text{ common denominator;}$$

Ex. 3.

Reduce $\frac{5x}{a+x}$, $\frac{a-x}{3}$, and $\frac{1}{2x}$, to a common denominator.

$$\text{Here } \left.\begin{array}{l}5x\times\ 3\times 2x = 30x^2\\ (a-x)\ \ (a+x)\times 2x = 2a^2x-2x^3\\ 1\ \ \times(a+x)\times 3\ = 3a\ \ +3x\end{array}\right\}\ \therefore \text{ the new fractions are } \left\{\begin{array}{l}\frac{30x^2}{6ax+6x^2},\\ \frac{2a^2x-2x^3}{6ax+6x^2}, \text{ and } \frac{3a+3x}{6ax+6x^2}.\end{array}\right.$$

$$(a+x)\ \times 3\ \times 2x\ \ = 6ax+6x^2$$

Ex. 4. Reduce $\frac{3x}{5}$, $\frac{4bx}{3a}$, and $\frac{5x^2}{a}$, to a common denominator.

Ans. $\frac{9a^2x}{15a^2}$, $\frac{20abx}{15a^2}$, and $\frac{75ax^2}{15a^2}$.

Ex. 5. Reduce $\frac{2x+3}{x}$, and $\frac{5x+1}{3}$, to a common denominator.

Ans. $\frac{6x+9}{3x}$, and $\frac{5x^2+x}{3x}$.

Ex. 6.

Reduce $\frac{4x^2+2x}{5}$, $\frac{3x^2}{4a}$, and $\frac{2x}{3b}$, to a common denominator.

Ans. $\frac{48abx^2+24abx}{60ab}$, $\frac{45bx^2}{60ab}$, and $\frac{40ax}{60ab}$.

Ex. 7.

Reduce $\frac{7x^2-1}{2x}$, and $\frac{4x^2-x+2}{2a^2}$, to a common denominator.

Ans. $\frac{14a^2x^2-2a^2}{4a^2x}$, and $\frac{8x^3-2x^2+4x}{4a^2x}$.

44. *To reduce a Fraction to its lowest terms.*

RULE. "Observe what quantity will divide all the terms both of the numerator and denominator *without a remainder;* Divide them by this quantity, and the fraction is reduced to its lowest terms."

Ex. 1.

Reduce $\frac{14x^3+7ax+21x^2}{35x^2}$ to its lowest terms.

The coefficient of every term of the numerator and denominator of this fraction is divisible by 7, and the letter x also enters into every term; therefore $7x$ will divide both numerator and denominator without a remainder.

Now $\frac{14x^3+7ax+21x^2}{7x}=2x^2+a+3x$,

and $\frac{35x^2}{7x}=5x$;

Hence, the fraction in its lowest terms is $\frac{2x^2+a+3x}{5x}$.

Ex. 2.

Reduce $\frac{20abc-5a^2+10ac}{5a^2c}$ to its lowest term.

Here the quantity which divides both numerator and denominator without a remainder is $5a$; the fraction therefore in its lowest terms is $\frac{4bc-a+2c}{ac}$.

Ex. 3.

Reduce $\frac{a-b}{a^2-b^2}$ to its lowest terms.

Here $a-b$ will divide both numerator and denominator, for by Ex. 2. Case III. page 27. $a^2-b^2=(a+b)\ (a-b)$; hence $\frac{1}{a+b}$ is the fraction in its lowest terms.

Ex. 4. Reduce $\frac{10x^3}{15x^2}$ to it lowest terms.

Ans. $\frac{2x}{3}$.

---

44. Show how fractions are reduced to their lowest terms.

Ex. 5. Reduce $\frac{3abx^2}{6ax}$ to its lowest terms.

Ans. $\frac{bx}{2}$.

Ex. 6. ..... $\frac{14x^2y^2-21x^3y^2}{7x^3y}$ to its lowest terms.

Ans. $\frac{2y-3xy}{x}$.

Ex. 7. ..... $\frac{51x^3-17x^2+34x}{17x^5}$ to its lowest terms.

Ans. $\frac{3x^2-x+2}{x^4}$.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION, OF FRACTIONS.

45. *To add Fractions together.*

RULE. "Reduce the fractions to a common denominator, and then add their numerators together; bring the resulting fraction to its lowest terms, and it will be the sum required."

Ex. 1.

Add $\frac{3x}{5}$, $\frac{2x}{7}$, and $\frac{x}{3}$, together.

$$\left.\begin{array}{r} 3x\times7\times3=63x \\ 2x\times5\times3=30x \\ x\times5\times7=35x \\ \hline 5\times7\times3=105 \end{array}\right\} \therefore \frac{63x+30x+35x}{105}=\frac{128x}{105}$$ is the fraction required.

Ex. 2. Add $\frac{a}{b}$, $\frac{2a}{3b}$, and $\frac{5b}{4a}$, together.

$$\left.\begin{array}{r} a\times3b\times4a=12a^2b \\ 2a\times b\times4a=\ 8a^2b \\ 5b\times b\times3b=15b^3 \\ \hline b\times3b\times4a=12ab^2 \end{array}\right\} \therefore \frac{12a^2b+8a^2b+15b^3}{12ab^2}=\frac{20a^2b+15b^3}{12ab^2}$$

$$=\text{(dividing by } b) \ \frac{20a^2+15b^2}{12ab}$$ is the sum required.

---

45. State the rule for adding fractions.

Ex. 3. Add $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$, and $\frac{4x}{7}$, together.

$$\left.\begin{array}{r}(2x+3)\times 2x\times 7=28x^2+42x\\ (3x-1)\times 5\ \times 7=105x-35\\ 4x\times 5\times 2x=40x^2\\ \hline 5\times 2x\times 7=70x\end{array}\right\}\therefore \frac{28x^2+42x+105x-35+40x^2}{70x}=\frac{68x^2+147x-35}{70x}$$ is the sum required.

Ex. 4. Add $\frac{3x}{7}$, $\frac{5x}{9}$, and $\frac{4x}{11}$, together.

Ans. $\frac{934x}{693}$.

Ex. 5. ..... $\frac{3a^2}{2b}$, $\frac{2a}{5}$, and $\frac{3b}{7a}$, together.

Ans. $\frac{105a^3+28a^2b+30b^2}{70ab}$.

Ex. 6. ..... $\frac{2x+1}{3}$, $\frac{4x+2}{5}$, and $\frac{x}{7}$, together.

Ans. $\frac{169x+77}{105}$.

Ex. 7. ..... $\frac{5a^2+b}{3b}$, and $\frac{4a^2+2b}{5b}$, together.

Ans. $\frac{37a^2+11b}{15b}$.

Ex. 8. ..... $\frac{2x-5}{3}$, and $\frac{x-1}{2x}$, together.

Ans. $\frac{4x^2-7x-3}{6x}$.

Ex. 9. ..... $\frac{x}{x-3}$, and $\frac{x}{x+3}$, together.

Ans. $\frac{2x^2}{x^2-9}$.

Ex. 10. ..... $\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}$, together.

Ans. $\frac{2a^2+2b^2}{a^2-b^2}$.

46. *To Subtract Fractional Quantities.*

RULE. "Reduce the fractions to a common denominator; and then subtract the numerators from each other, and under the difference write the common denominator."

Ex. 1.

Subtract $\frac{3x}{5}$ from $\frac{14x}{15}$.

$$\left.\begin{array}{r} 3x\times 15=45x \\ 14x\times\ \ 5=70x \\ \hline 5\ \ \times 15=75 \end{array}\right\} \therefore \frac{70x-45x}{75}=\frac{25x}{75}=\frac{x}{3} \text{ is the difference required.}$$

Ex. 2.

Subtract $\frac{2x+1}{3}$ from $\frac{5x+2}{7}$.

$$\left.\begin{array}{r} (2x+1)\times 7=14x+7 \\ (5x+2)\times 3=15x+6 \\ \hline 3\times 7=21 \end{array}\right\} \therefore \frac{15x+6-14x-7}{21}=\frac{x-1}{21} \text{ is the fraction required.}$$

Ex. 3.

From $\frac{10x-9}{8}$ subtract $\frac{3x-5}{7}$.

$$\left.\begin{array}{r} (10x-9)\times 7=70x-63 \\ (\ 3x-5)\times 8=24x-40 \\ \hline 8\ \times 7=56 \end{array}\right\} \therefore \frac{70x-63-24x+40}{56}=\frac{46x-23}{56} \text{ is the fraction required.}$$

Ex. 4.

From $\frac{a+b}{a-b}$ subtract $\frac{a-b}{a+b}$

$$\left.\begin{array}{r} (a+b)\ (a+b)=a^2+2ab+b^2 \\ (a-b)\ (a-b)=a^2-2ab+b^2 \\ \hline (a-b)\ (a+b)=a^2-b^2 \end{array}\right\} \therefore \frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2} \text{ is the fraction required.}$$

Ex. 5. Subtract $\frac{4x}{5}$ from $\frac{9x}{10}$ ..... Ans. $\frac{x}{10}$.

---

46. Give the rule for subtracting fractions.

Ex. 6. Subtract $\frac{5x+1}{7}$ from $\frac{21x+3}{4}$. Ans. $\frac{127x+17}{28}$.

Ex. 7. ..... $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$. Ans. $\frac{4x^2-11x-5}{5x+5}$.

Ex. 8. ..... $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$. Ans. $\frac{4x^2+3}{3x}$.

Ex. 9. ..... $\frac{1}{a+b}$ from $\frac{1}{a-b}$. Ans. $\frac{2b}{a^2-b^2}$.

Ex. 10. ..... $\frac{3x-7}{8}$ from $\frac{4x}{7}$. Ans. $\frac{11x+49}{56}$.

47. *To Multiply Fractional Quantities.*

RULE. "Multiply their numerators together for a new numerator, and their denominators together for a new denominator, and reduce the resulting fraction to its lowest terms."

Ex. 1.

Multiply $\frac{2x}{7}$ by $\frac{4x}{9}$

$$\left.\begin{array}{l} 2x\times 4x=8x^2 \\ 7\ \times 9\ =63 \end{array}\right\} \therefore \text{ the fraction required is } \frac{8x^2}{63}.$$

Ex. 2.

Multiply $\frac{4x+1}{3}$ by $\frac{6x}{7}$.

$$\left.\begin{array}{l} \text{Here} \\ (4x+1)\times 6x=24x^2+6x \\ \text{and} \\ 3\times 7\ =21 \end{array}\right\} \begin{array}{l} \therefore \frac{24x^2+6x}{21} = \text{(dividing the nu-} \\ \text{merator and denominator by 3)} \\ \frac{8x^2+2x}{7} \text{ is the fraction required.} \end{array}$$

---

47. State the rule for the multiplication of fractions.

Ex. 3.

Multiply $\frac{a^2-b^2}{5b}$ by $\frac{3a^2}{a+b}$.

By Ex. 2. Case III. page 27, $(a^2-b^2)\times 3a^2=(a+b)(a-b)\times 3a^2$; hence the product is $\frac{3a^2\times(a+b)(a-b)}{5b\times(a+b)}=$ (dividing the numerator and denominator by $a+b$) $\frac{3a^2\times(a-b)}{5b}=\frac{3a^3-3a^2b}{5b}$.

Ex. 4.

Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$.

Here $(3x^2-5x)\times 7a=21ax^2-35ax$ and $(2x^3-3x)\times 14=28x^3-42x$ $\therefore \frac{21ax^2-35ax}{28x^3-42x}=$ (dividing the numerator and denominator by $7x$) $\frac{3ax-5a}{4x^2-6}$ is the fraction required.

Ex. 5. Multiply $\frac{2x}{x-1}$ by $\frac{3x}{7}$. Ans. $\frac{6x^2}{7x-7}$.

Ex. 6. ..... $\frac{3x^2-x}{5}$ by $\frac{10}{2x^2-4x}$ Ans. $\frac{3x-1}{x-2}$.

Ex. 7. ..... $\frac{2a}{a-b}$ by $\frac{a^2-b^2}{8}$. Ans. $\frac{a^2+ab}{4}$.

Ex. 8. ..... $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$. Ans. $\frac{9x}{2}$.

48. *On the Division of Fractions.*

RULE. "*Invert* the divisor, and proceed as in Multiplication."

---

48. Enunciate the rule for division of fractions.

Ex. 1.

Divide $\frac{14x^2}{9}$ by $\frac{2x}{3}$.

Invert the divisor, and it becomes $\frac{3}{2x}$; hence $\frac{14x^2}{9}\times\frac{3}{2x}$ $=\frac{42x^2}{18x}=\frac{7x}{3}$ (dividing the numerator and denominator by $6x$) is the fraction required.

Ex. 2.

Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$.

$$\frac{14x-3}{5}\times\frac{25}{10x-4}=\frac{(14x-3)\times 5}{10x-4}=\frac{70x-15}{10x-4}.$$

Ex. 3.

Divide $\frac{5a^2-5b^2}{2a}$ by $\frac{4a+4b}{6b}$.

$$\left.\begin{array}{l}\frac{5a^2-5b^2}{2a}=\frac{5\times(a+b)\,(a-b)}{2a}\\ \frac{4a+4b}{6b}=\frac{4\times(a+b)}{6b};\end{array}\right\}\begin{array}{l}\therefore\frac{5\times(a+b)\,(a-b)}{2a}\times\frac{6b}{4\times(a+b)}\\ =\frac{30b\times(a-b)}{8a}=\frac{15ab-15b^2}{4a}\text{ is}\\ \text{the fraction required.}\end{array}$$

Ex. 4. Divide $\frac{4x}{7}$ by $\frac{9x}{5}$. Ans. $\frac{20}{63}$.

Ex. 5. ..... $\frac{4x+2}{3}$ by $\frac{2x+1}{5x}$. Ans. $\frac{10x}{3}$.

Ex. 6. ..... $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$ Ans. $\frac{4x-12}{5}$.

Ex. 7. ..... $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. Ans. $\frac{9x-3}{x}$.

ON THE SOLUTION OF SIMPLE EQUATIONS, CONTAINING ONLY ONE UNKNOWN QUANTITY.

RULE III.

49. An equation may be cleared of fractions by multiplying each side of the equation by the denominators of the fractions in succession.

Or, an equation may be cleared of fractions by multiplying each side of the equation by the *least common multiple* of the denominators of the fractions.

This Rule is derived from the axiom (4), that, if equal quantities be multiplied by the same quantity (or by equal quantities), the products arising will be equal.

Ex. 1. Let $\frac{x}{3}=6$.

Multiply each side of the equation by 3; then (since, the multiplication of the fraction $\frac{x}{3}$ by 3 just takes away the denominator and leaves x for the product) we have

$$x=6\times 3=18.$$

Ex. 2. Let $\frac{x}{2}+\frac{x}{5}=7$.

Multiply each side of the equation by 2, and we have

$$x+\frac{2x}{5}=14.$$

Again, *multiply* each side of this equation by 5, and it becomes

$$5x+2x=70,$$
$$7x=70;$$
$$\therefore\ x=10.$$

Ex. 3. Let $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$.

Multiply each side by 2, then $x+\frac{2x}{3}=26-\frac{2x}{4}$.

..... 3, and $3x+2x=78-\frac{6x}{4}$.

..... 4, ... $12x+8x=312-6x$.

By transposition, $12x+8x+6x=312,$

$$26x=312;$$

$$\therefore \quad x=12.$$

This example might have been solved more simply, by multiplying each side of the equation by the *least common multiple* of the numbers 2, 3, 4, which is 12.

Thus, $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}.$

Multiply each side by 12, $\frac{12x}{2}+\frac{12x}{3}=156-\frac{12x}{4},$

or, $6x+4x=156-3x.$

By transposition, $6x+4x+3x=156,$

$$13x=156;$$

$$\therefore x=12.$$

Ex. 4. Let $\frac{2x}{3}+\frac{x}{4}=22.$ Ans. $x=24.$

Ex. 5. ... $\frac{7x}{4}-\frac{5x}{6}=\frac{55}{6}.$ ... $x=10.$

Ex. 6. ... $\frac{x}{2}+\frac{x}{3}=31-\frac{x}{5}.$ ... $x=30.$

Ex. 7. ... $\frac{2x}{5}-\frac{x}{6}+\frac{x}{2}=44.$ ... $x=60.$

50. In the application of the Rules to the solution of simple equations in general containing only one unknown quantity, it will be proper to observe the following method.

(1.) To clear the equation of fractions by Rule III.

(2.) To collect the *unknown* quantities on one side of the equation, and the *known* on the other, by Rule II.

(3.) To find the value of the unknown quantity by dividing each side of the equation by its coefficient, as in Rule I.

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50. Enumerate the three steps by which a simple equation containing only one unknown quantity may be solved.

Ex. 1.

Find the value of x in the equation $\frac{3x}{7}+1=\frac{x}{5}+\frac{13}{5}$.

Multiply by 7, then $3x+7=\frac{7x}{5}+\frac{91}{5}$.

..... by 5, ... $15x+35=7x+91$.

Collect the unknown quantities on *one* side, and the known on the other; $\left.\right\}$ $15x-7x=91-35$,

or $8x=56$.

Divide by the coefficient of x, $x=\frac{56}{8}=7$.

Ex. 2.

Find the value of x in the equation $\frac{x+3}{5}-1=2-\frac{x}{7}$.

Multiply by 5, then $x+3-5=10-\frac{5x}{7}$;

..... by 7 ... $7x+21-35=70-5x$.

Collect the *unknown* quantities on *one* side, and the known on the *other;* $\left.\right\}$ $7x+5x=70-21+35$,

or $12x=84$;

$\therefore x=\frac{84}{12}=7$.

Ex. 3.

Find the value of x in the equation

$$4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24.$$

Multiply by the *least common multiple* (10), $\left.\right\}$ $40x-5x+5=10x+4x-4+240$.

By transposition, $40x-5x-10x-4x=240-4-5$.

or $40x-19x=231$,

i.e. $21x=231$;

$\therefore x=\frac{231}{21}=11$.

As the *first* step in this Example involves the case "where the sign — stands before a fraction," when the unmerator

of that fraction is brought down into the same line with $40x$, the signs of both its terms must be *changed*, for the reasons assigned in Ex. 3, page 44; and we therefore make it $-5x+5$, and not $5x-5$.

Ex. 4.

Find the value of x in the equation $2x-\frac{x}{2}+1=5x-2$.

Multiply by 2, then $4x-x+2=10x-4$.

By transposition, $4+2=10x-4x+x$,

or $6=7x$;

$\therefore \frac{6}{7}=x$.

or $x=\frac{6}{7}$.

Ex. 5.

What is the value of x in the equation $3ax+2bx=3c+a$?

Here $3ax+2bx=(3a+2b)\times x$;

$\therefore (3a+2b)\times x=3c+a$.

Divide each side of the equation by $3a+2b$, which is the coefficient of x; then $x=\frac{3c+a}{3a+2b}$.

Ex. 6.

Find the value of x in the equation $3bx+a=2ax+4c$.

Bring the *unknown* quantities to *one* side of the equation, and the *known* to the *other*; then,

$3bx-2ax=4c-a$;

but $3bx-2ax=(3b-2a)\times x$;

$\therefore (3b-2a)x=4c-a$.

Divide by $3b-2a$, and $x=\frac{4c-a}{3b-2a}$.

Ex. 7.

Find the value of x in the equation $bx+x=2x+3a$.

Transpose $2x$, then $bx+x-2x=3a$,

or $bx-x=3a$;

but $bx-x=(b-1)x$;

$\therefore (b-1)x=3a$,

and $x=\frac{3a}{b-1}$.

Ex. 8. $x+\frac{x}{2}+\frac{x}{3}=11$ Ans. $x=6$.

Ex. 9. $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}=\frac{x}{2}+17$ ... $x=60$.

Ex. 10. $4x-20=\frac{3x}{7}+\frac{110}{7}$ ... $x=10$.

Ex. 11. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$ ... $x=\frac{6}{7}$.

Ex. 12. $3x+\frac{1}{9}=\frac{x+3}{3}$ ... $x=\frac{1}{3}$.

Ex. 13. $\frac{3x}{7}-5=29-2x$ ... $x=14$.

Ex. 14. $6x-\frac{3x}{4}-9=5x$ ... $x=36$.

Ex. 15. $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$ ... $x=12$.

Ex. 16. $\frac{x-2}{2}+\frac{x}{3}=20-\frac{x-6}{2}$ ... $x=18$.

Ex. 17. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$ ... $x=8$.

Ex. 18. $2ax+b=3cx+4a$ ... $x=\frac{4a-b}{2a-3c}$.

Ex. 19.

$$3x-4-\frac{4}{5}\cdot\frac{7x-9}{3}=\frac{4}{5}\left(6+\frac{x-1}{3}\right);\text{ find } x.$$

Mult. by 15, $45x-60-28x+36=72+4x-4$.

$$45x-28x-4x=72-4+60-36,$$
$$13x=92;$$
$$\therefore x=7\tfrac{1}{13}.$$

Ex. 20.

$$\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}; \text{ find } x.$$

Mult. by 18, $8x+6+\frac{126x-522}{5x-12}=8x+19.$

$$\frac{126x-522}{5x-12}=13.$$

Mult. by $5x-12$, $126x-522=65x-156,$

$$126x-65x=522-156,$$
$$61x=366;$$
$$\therefore x=6.$$

Ex. 21.

Given $\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)}$. to find x.

Mult. by $7(x-1)$, $7-\frac{14(x-1)}{x+7}=1.$

$$6=\frac{14(x-1)}{x+7}.$$

Divide by 2, $3=\frac{7(x-1)}{x+7}.$

$$3x+21=7x-7,$$
$$7x-3x=21+7,$$
$$4x=28;$$
$$\therefore x=7.$$

Ex. 22.

Let $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$; find x.

Mult. by 28, $16x+10+\frac{196x-84}{6x+2}=16x+15+9.$

$$\frac{196x-84}{6x+2}=14.$$
$$196x-84=84x+28,$$
$$112x=112;$$
$$\therefore x=1.$$

Ex. 23. $\frac{3x-3}{4}-\frac{3x-4}{3}=5\frac{1}{3}-\frac{27+4x}{9}$. Ans. 9.

Ex. 24. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$. ... 8.

Ex. 25. $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$. ... 4.

Ex. 26. $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$. ... 4.

Ex. 27. $4(5x-3)-64(3-x)-3(12x-4)=96$. ... 6.

Ex. 28. $10(x+\frac{1}{2})-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23$. ... 2.

Ex. 29. $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$. ... 3.

PROBLEMS.

PROB. 1. What number is that to which 10 being added, $\frac{3}{5}$ths of the sum shall be 66?

Let $x=$ the number required;

then $x+10=$ the number, with 10 added to it.

Now $\frac{3}{5}$ths of $(x+10)=\frac{3}{5}(x+10)=\frac{3(x+10)}{5}=\frac{3x+30}{5}$.

But, by the question, $\frac{3}{5}$ths of $(x+10)=66$;

Hence, $\frac{3x+30}{5}=66$.

Multiply by 5, then $3x+30=330$;

$\therefore\ 3x=330-30=300$; or $x=\frac{300}{3}=100$.

PROB. 2. What number is that which being multiplied by 6, the product increased by 18, and that sum divided by 9, the quotient shall be 20?

Let $x=$ the number required;

then $6x=$ the number multiplied by 6;

$6x+18=$ the product increased by 18,

and $\frac{6x+18}{9}=$ that sum divided by 9.

Hence, by the question, $\frac{6x+18}{9}=20.$

Multiply by 9, then $6x+18=180,$

or $6x=180-18=162$; or $x=\frac{162}{6}=27.$

PROB. 3. A post is $\frac{1}{5}$th in the earth, $\frac{3}{7}$ths in water, and 13 feet out of the water. What is the length of the post?

Let $x=$ length of the post in ft.;

then $\frac{x}{5}=$ the part of it in the earth,

$\frac{3x}{7}=$ the part of it in the water,

$13=$ the part of it out of the water.

But part in earth + part in water + part out of water = whole post;

$$\therefore \left(\frac{x}{5}\right)+\left(\frac{3x}{7}\right)+13=x.$$

Multiply by 5, then $x+\frac{15x}{7}+65=5x$;

..... by 7, ... $7x+15x+455=35x,$

or $455=35x-7x-15x=13x.$

Hence $x=\frac{455}{13}=35$ length of post in ft.

PROB. 4. After paying away $\frac{1}{4}$th and $\frac{1}{7}$th of my money, I had £85 left in my purse. What money had I at first?

Let $x=$ money in purse at first;

then $\frac{x}{4}+\frac{x}{7}=$ money paid away.

But money at first − money paid away = money remaining.

Hence $x-\left(\frac{x}{4}+\frac{x}{7}\right)=85,$

i. e. $x-\frac{x}{4}-\frac{x}{7}=85.$

Multiply by 4, then $4x-x-\frac{4x}{7}=340$;

..... by 7, ... $28x-7x-4x=2380.$

$\therefore 17x=2380;$

or $x=\frac{2380}{17}=£140.$

PROB. 5. What number is that to which if I add 20, and from $\frac{2}{3}$ds of this sum I subtract 12, the remainder shall be 10? Ans. 13.

PROB. 6. What number is that, of which if I add $\frac{1}{3}$d, $\frac{1}{4}$th, and $\frac{2}{7}$ths together, the sum shall be 73? Ans. 84.

PROB. 7. What number is that whose $\frac{1}{3}$d part exceeds its $\frac{1}{5}$th by 72? Ans. 540.

PROB. 8. There are two numbers whose sum is 37, and if 3 times the lesser be subtracted from 4 times the greater, and this difference divided by 6, the quotient will be 6. What are the numbers? Ans. 21 and 16.

PROB. 9. There are two numbers whose sum is 49; and if $\frac{1}{7}$th of the lesser be subtracted from $\frac{1}{5}$th of the greater, the remainder will be 5. What are the numbers? Ans. 35 and 14.

PROB. 10. To divide the number 72 into three parts, so that $\frac{1}{2}$ the *first* part shall be equal to the *second*, and $\frac{3}{5}$ths of the *second* part equal to the third. Ans. 40, 20, and 12.

PROB. 11. A person after spending $\frac{1}{5}$th of his income *plus* £10, had then remaining $\frac{1}{2}$ of it *plus* £35. Required his income. Ans. £150.

PROB. 12. A gamester at *one sitting* lost $\frac{1}{5}$th of his money, and then won 10 shillings; at a *second* he lost $\frac{1}{3}$d of the remainder, and then won 3 shillings; after which he had 3 guineas left. What money had he at first? Ans. £5.

PROB. 13. Divide the number 90 into four such parts, that the first *increased* by 2, the second *diminished* by 2, the third *multiplied* by 2, and the fourth *divided* by 2, may all be equal to the same quantity. Ans. 18, 22, 10, 40.

PROB. 14. A merchant has two kinds of tea, one worth 9*s.* 6*d.* per ℔., the other 13*s.* 6*d.* How many ℔s. of each must he take to form a chest of 104 ℔s. which shall be worth £56? Ans. 33 at 13*s.* 6*d.*
71 at 9*s.* 6*d.*

PROB. 15. Three persons, A, B, and C, can separately

reap a field of corn in 4, 8, and 12 days respectively. In how many days can they conjointly reap the field?

Let x = No. of days required by them to reap the field; then if 1 represent the work, or the reaping of the field,

$\frac{1}{4}$ = the part reaped by A in 1 day,

$\frac{1}{8}$ = ... B .....

$\frac{1}{12}$ = ... C .....

$\therefore \frac{1}{4}+\frac{1}{8}+\frac{1}{12}=$ ..... all three .....

But the part reaped by all three in 1 day multiplied by the number of days they took to reap the field, is equal to the whole work, or 1;

$$\therefore \left(\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{12}\right)x=1:$$

Clearing of fractions by multiplying by 24,

$$(6+3+2)\,x=24,$$
$$11x=24;$$
$$\therefore x=2\tfrac{2}{11} \text{ days.}$$

Prob. 16. A man and his wife usually drank a cask of beer in 10 days, but when the man was absent it lasted the wife 30 days; how long would the man alone take to drink it? Ans. 15 days.

Prob. 17. A cistern has 3 pipes, two of which will fill it in 3 and 4 hours respectively, and the third will empty it in 6 hours; in what time will the cistern be full, if they be all set a-running at once? Ans. 2h. 24m.

Prob. 18. A person bought oranges at 20*d.* per dozen; if if he had bought 6 more for the same money, they would have cost 4*d.* a dozen less. How many did he buy?

Let x = the number of oranges;

then $x+6=$ ..... at 4*d.* less per dozen.

Price of each orange in 1st case $=\frac{20}{12}=\frac{5}{3}d.$

and ..... 2d ... $=\frac{16}{12}=\frac{4}{3}d.$

$\therefore$ *the cost* of the oranges $=\dfrac{5x}{3}$.

But we have also

the cost of the oranges $=\frac{4}{3}(x+6)$.

Two independent values have therefore been obtained for

the cost of the oranges; these values must necessarily be equal to each other;

$$\therefore \frac{5x}{3} = \tfrac{4}{3}(x+6).$$

Multiplying each side of the equation by 3,

$$5x = 4(x+6),$$
$$5x = 4x + 24;$$
$$\therefore x = 24, \text{ the No. of oranges.}$$

PROB. 19. A market-woman bought a certain number of apples at two a penny, and as many at three a penny, and sold them at the rate of five for two-pence; after which she found that instead of making her money again as she expected, she lost fourpence by the whole business. How much money had she laid out? Ans. 8s. 4d.

PROB. 20. A person rows from Cambridge to Ely, a distance of 20 miles, and back again, in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 with it. Find the time of his going and returning.

Let $3x =$ No. of miles rowed per hour with the stream;
$\therefore 2x =$ ..... against .....

Now the distance divided by the rate per hour gives the time;

$\therefore \frac{20}{3x} =$ the No. of hours in going down the river,

and $\frac{20}{2x} =$ ..... coming up .....

But the whole time in going and returning is 10 hours;

$$\therefore \frac{20}{3x} + \frac{20}{2x} = 10.$$

Dividing by 10, $\frac{2}{3x} + \frac{1}{x} = 1.$

Multiplying each term of the equation by $3x$,

$$2 + 3 = 3x;$$
$$\therefore x = \frac{5}{3} = 1\tfrac{2}{3}.$$

and $\therefore 3x = 5$, miles per hour down,

$\therefore$ the time in going down the river $= \frac{20}{5} = 4$ hours, and consequently the time of returning $= 10 - 4 = 6$ hours.

PROB. 21. A lady bought a hive of bees, and found that the price came to 2*s*. more than $\frac{3}{4}$ths and $\frac{1}{5}$th of the price. Find the price. Ans. £2.

PROB. 22. A hare, 50 leaps before a greyhound, takes 4 leaps for the greyhound's 3; but two of the greyhound's leaps are equal to three of the hare's. How many leaps will the greyhound take to catch the hare?

Let x be the No. of leaps taken by the greyhound; then $\frac{4x}{3}$ will be the corresponding number taken by the hare.

Let 1 represent the space covered by the hare in 1 leap; then $\frac{3}{2}$ ..... greyhound ...

$\therefore \frac{4x}{3} \times 1$ or $\frac{4x}{3}$ will be the whole space passed over by the hare before she is taken; and $x \times \frac{3}{2}$ or $\frac{3x}{2}$ will be the space passed over in the corresponding time by the greyhound. Now, by the problem, the difference between the spaces respectively passed over by the greyhound and hare is 50×1, or 50 leaps;

$$\therefore \frac{3x}{2} - \frac{4x}{3} = 50,$$

$$9x - 8x = 300;$$

$$\therefore x = 300 \text{ leaps.}$$

ON THE SOLUTION OF SIMPLE EQUATIONS, CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

51. For the solution of equations containing two or more unknown quantities, as many *independent* equations are required as there are *unknown* quantities. The two equations necessary for the solution of the case, when two unknown quantities are concerned, may be expressed in the following manner:

$$ax + by = c$$
$$a'x + b'y = c'.$$

Where a, b, c, a', b', c', represent *known* quantities, and

x, y, the *unknown* quantities whose values are to be found in terms of these known quantities.

There are three different methods by which the value of one of the unknown quantities may be determined.

FIRST METHOD.

Find the value of one of the unknown quantities in terms of the other, and the known quantities by the rules already given. Find the value of the same unknown quantity from the second equation.

Put these two values equal to each other; and we shall then have a simple equation, containing only one unknown quantity, which may be solved as before.

Ex. 1. Given $\left.\begin{array}{ll} x+y=8 & \ldots\ldots\ (1) \\ x-y=4 & \ldots\ldots\ (2) \end{array}\right\}$ to find x and y.

From (1) $y=8-x$ ..... (α)

..... (2) $y=x-4$

Putting these two values equal to each other, we get

$$x-4=8-x,$$
$$2x=12;$$
$$\therefore x=6.$$

By (α) $y=8-x=8-6=2.$

Ex. 2. Let $x+4y=16$ ..... (1)

$4x+y=34$ ..... (2)

From equation (1), we have $x=16-4y$.

..... (2) ..... $x=\dfrac{34-y}{4}$,

Hence by the rule, $\dfrac{34-y}{4}=16-4y$,

$$34-y=64-16y,$$
$$15y=30;$$
$$\therefore y=2.$$

It has already been shewn that $x=16-4y=$ (since $y=2$; and $\therefore 4y=8$) $16-8=8$.

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51. For the solution of equations containing two or more unknown quantities, how many independent equations are necessary? State the *first method* of solution.

Ex. 3. $\left.\begin{array}{l}5x+3y=38\\4x-\ \ y=10\end{array}\right\}$ ..... Ans. $\left\{\begin{array}{l}x=4\\y=6.\end{array}\right.$

Ex. 4. $\left.\begin{array}{l}2x-3y=-1\\3x-2y=6\end{array}\right\}$ .... $\left\{\begin{array}{l}x=4\\y=3.\end{array}\right.$

SECOND METHOD.

From either of the equations find the value of one of the unknown quantities in terms of the other and the known quantities, and for the same unknown quantity substitute this value in the other equation, and there will arise an equation which contains only one unknown quantity. This equation can be solved by the rules already laid down.

Ex. 1. $y-x=2$ ..... (1)

$x+y=8$ ..... (2)

From (1) $y=2+x.$ (α)

This value of y being substituted in (2), gives

$$x+2+x=8,$$
$$2x=6;$$
$$\therefore x=3.$$

And by (α) $y=2+x=2+3=5.$

Ex. 2. $\frac{x+2}{3}+8y=31$ (1)

$\frac{y+5}{4}+10x=192$ (2)

Clearing equation (1) of fractions,

$x+2+24y=93$, or $x+24y=91$ (α).

Clearing equation (2) of fractions,

$y+5+40x=768$, or $y+40x=763$ (β).

From (α) $x=91-24y.$

Substitute this value of x, according to the rule in equation (β); and

$$y+40\,(91-24y)=763,$$
$$\text{or, } y+3640-960y=763;$$
$$\therefore 959y=3640-763=2877,$$
$$\text{and } y=3.$$

By referring to equation (α) we have $x=91-24y$ (since $y=3$; and $\therefore 24y=72$), $91-72=9.$

Enunciate the *second method* of solution.

Ex. 3. $\left.\begin{array}{l}4x+3y=31\\3x+2y=22\end{array}\right\}$ .....Ans. $\left\{\begin{array}{l}x=4\\y=5.\end{array}\right.$

THIRD METHOD.

Multiply the first equation by the coefficient of x in the second equation, and then multiply the second equation by the coefficient of x in the first equation; subtract the *second* of these resulting equations from the *first*, and there will arise an equation which contains only y and known quantities, from which the value of y can be determined.

It must be observed however, that if the terms, which in the *resulting* equations are the same, have *unlike* signs, the resulting equations must be added, instead of being subtracted, in order that x may be *eliminated* (i. e. expelled from the equations).

Ex. 1. Given $5x+4y=55$ ..... (1)
$3x+2y=31$ ..... (2)

To find the values of x and y

Mult. (1) by 3, then $15x+12y=165$
..... (2) by 5, ... $15x+10y=155$

$\therefore$ by subtraction, we have $2y=10$
$\therefore y=5.$

Now from equation (1) we have

$$x=\frac{55-4y}{5}$$
$$=\frac{55-20}{5}$$
$$=\frac{35}{5}$$
$$=7.$$

Ex. 2. Let the proposed equations be
$ax+by=c$ ..... (1)
$a'x+b'y=c'$ ..... (2)

Mult. (1) by a', and $aa'x+a'by=a'c$
..... (2) by a, ... $aa'x+ab'y=ac'$;

How are equations solved by the *third method?*

$\therefore$ by subtraction, $(a'b-ab')y=a'c-ac'$

$$\therefore\ y=\frac{a'c-ac'}{a'b-ab'}.$$

Mult. (1) by b', and $ab'x+bb'y=b'c$

..... (2) by b, ... $a'bx+bb'y=bc'$

By subtraction, $(ab'-a'b)x=b'c-bc'$;

$$\therefore\ x=\frac{b'c-bc'}{ab'-a'b}.$$

Ex. 3. Let $3x+4y=29$ (1)

$17x-3y=36$ (2)

Mult. (1) by 3, then $9x+12y=87$

..... (2) by 4, ... $68x-12y=144$.

The signs of $12y$ in the two equations are *unlike*; $\therefore$ to *eliminate* y from them, the two last equations must be added together; and then

$$77x=231;$$
$$\therefore\ x=3.$$

From (1) we have $4y=29-3x$,

$=29-9$, (since $x=3$; and $\therefore 3x=9$)

$=20$;

$\therefore\ y=5$.

Ex. 4. Let $\left.\begin{array}{r}6x+3y=33\\13x-4y=19\end{array}\right\}$ ..... Ans. $\left\{\begin{array}{l}x=3\\y=5.\end{array}\right.$

Ex. 5. $\left.\begin{array}{r}4x+3y=31\\3x+2y=22\end{array}\right\}$ .... $\left\{\begin{array}{l}x=4\\y=5.\end{array}\right.$

Ex. 6. $\left.\begin{array}{r}3x+2y=40\\2x+3y=35\end{array}\right\}$ .... $\left\{\begin{array}{l}x=10\\y=5.\end{array}\right.$

Ex. 7. $\left.\begin{array}{r}5x-4y=19\\4x+2y=36\end{array}\right\}$ .... $\left\{\begin{array}{l}x=7\\y=4.\end{array}\right.$

Ex. 8. $\left.\begin{array}{r}3x+7y=79\\2y-\frac{1}{2}x=\ \ 9\end{array}\right\}$ .... $\left\{\begin{array}{l}x=10\\y=7.\end{array}\right.$

Ex. 9. $\left.\begin{array}{r}\frac{x+y}{3}+1=\ 6\\ \\ \frac{x-y}{7}+3=\ 4\end{array}\right\}$ .... $\left\{\begin{array}{l}x=11\\ \\y=4.\end{array}\right.$

Ex. 10. $\left.\begin{array}{l}\dfrac{x+y}{3}-2y=2\\[2ex] \dfrac{2x-4y}{5}+y=\dfrac{23}{5}\end{array}\right\}$ ..... Ans. $\left\{\begin{array}{l}x=11\\ y=1.\end{array}\right.$

Ex. 11. $\left.\begin{array}{l}\dfrac{2x-3}{2}+y=7\\[2ex] 5x-13y=\dfrac{67}{2}\end{array}\right\}$ .... $\left\{\begin{array}{l}x=8\\ y=\frac{1}{2}.\end{array}\right.$

Ex. 12. $\left.\begin{array}{l}\dfrac{3x-7y}{3}=\dfrac{2x+y+1}{5}\\[2ex] 8-\dfrac{x-y}{5}=6\end{array}\right\}$ .... $\left\{\begin{array}{l}x=13\\ y=3.\end{array}\right.$

52. When *three* unknown quantities are concerned, the most general form under which equations of this kind can be expressed, is

$$ax+by+cz=d \quad (1)$$
$$a'x+b'y+c'z=d' \quad (2)$$
$$a''x+b''y+c''z=d'' \quad (3),$$

and the solution of these equations may be conducted as in the following example:

Ex. 1. Let $\left.\begin{array}{ll}2x+3y+4z=29 & (1)\\ 3x+2y+5z=32 & (2)\\ 4x+3y+2z=25 & (3)\end{array}\right\}$ to find the values of x, y, z.

I. Multiply (1) by 3, then $6x+9y+12z=87$ (4)
..... (2) by 2, ... $6x+4y+10z=64$ (5).

Subtract (5) from (4) ... $5y+2z=23$ (α).

Multiply (2) by 4, then $12x+8y+20z=128$
..... (3) by 3, ... $12x+9y+6z=75$

Subtract ..... $-y+14z=53$ (β).

II. Hence the given equations are reduced to,

$$5y+2z=23 \quad (\alpha)$$
$$-y+14z=53 \quad (\beta).$$

Again ..... $5y+2z=23$

Mult. (β) by 5, then $-5y+70z=265$

By *addition* .....$72z=288$, or $z=\frac{288}{72}=4$

From equation (β)..... $y=14z-53=56-53=3.$

III. From equation (1) ... $x=\frac{29-3y-4z}{2}=\frac{29-25}{2}=2.$

Ex. 2. $\left.\begin{array}{l} x+y+z=90 \\ 2x+40=3y+20 \\ 2x+40=4z+10 \end{array}\right\}$ ..... Ans. $\left\{\begin{array}{l} x=35 \\ y=30 \\ z=25. \end{array}\right.$

Ex. 3. $\left.\begin{array}{l} x+y+z=53 \\ x+2y+3z=105 \\ x+3y+4z=134 \end{array}\right\}$ .... $\left\{\begin{array}{l} x=24 \\ y=6 \\ z=23. \end{array}\right.$

PROBLEMS.

Prob. 1. There are two numbers, such, that 3 times the greater added to $\frac{1}{3}$rd the less is equal to 36; and if twice the greater be subtracted from 6 times the less, and the remainder divided by 8, the quotient will be 4. What are the numbers?

Let $x=$ the *greater* number,
$y=$ the *less* number;

$$\left.\begin{array}{l} \text{Then } 3x+\frac{y}{3}=36 \\ \frac{6y-2x}{8}=4 \end{array}\right\} \text{ or, } \begin{array}{l} 9x+y=108 \\ 6y-2x=32; \end{array}$$

$$\text{Or, } \begin{array}{ll} y+9x=108 & (1) \\ 6y-2x=32 & (2) \end{array}$$

Mult. equation (1) by 6, then $6y+54x=648$

Subtract ..... (2) ... $6y-2x=32;$

then $56x=616;$

$\therefore x=\frac{616}{56}=11.$

From equation (1) $y=108-9x=108-99=9.$

Prob. 2. There is a certain fraction, such, that if I add 3 to the numerator, its value will be $\frac{1}{3}$rd; and if I subtract

one from the denominator, its value will be $\frac{1}{3}^{\text{th}}$. What is the fraction?

$$\left.\begin{array}{r}\text{Let } x = \text{its numerator,}\\ y = \text{ denominator,}\end{array}\right\}\text{ then the fraction is } \frac{x}{y}.$$

$$\left.\begin{array}{l}\textit{Add } 3 \text{ to the numerator, then } \dfrac{x+3}{y}=\dfrac{1}{3}\\ \textit{Subtract} \text{ one from denom}^{\text{r}}\text{., and } \dfrac{x}{y-1}=\dfrac{1}{5}\end{array}\right\}\text{or, } \begin{array}{l}3x+9=y\\ 5x=y-1.\end{array}$$

$$\text{By } \textit{transposition, } \begin{array}{ll} y-3x=9 & (1)\\ y-5x=1 & (2).\end{array}$$

Subtract equation (2) from (1), and we have

$$2x=8;$$

$$\therefore x=\frac{8}{2}=4, \text{ the numerator.}$$

From equation (1) $y=9+3x=9+12=21$, the denominator.

Hence the fraction required is $\frac{4}{21}$.

PROB. 3. A and B have certain sums of money; says A to B, Give me £15 of your money, and I shall have 5 times as much as you will have left; says B to A, Give me £5 of your money, and I shall have exactly as much as you will have left. What sum of money had each?

$$\left.\begin{array}{l}\text{Let } x = \text{A's money}\\ y = \text{B's .....}\end{array}\right\}\text{ then } x+15=\left\{\begin{array}{l}\text{what A would have after}\\ \quad\text{receiving £15 from B.}\end{array}\right.$$

$$y-15=\text{what B would have left.}$$

$$\text{Again, } y+5=\left\{\begin{array}{l}\text{what B would have after}\\ \quad\text{receiving £5 from A.}\end{array}\right.$$

$$x-5=\text{what A would have left.}$$

$$\text{Hence, by the question, } \left.\begin{array}{l}x+15=5\times(y-15)=5y-75,\\ \text{and } y+5=x-5.\end{array}\right\}.$$

$$\text{By transposition, } \left.\begin{array}{ll}5y-x=90 & (1),\\ \text{and } y-x=-10 & (2).\end{array}\right\}$$

Set down equation (1) $5y-x=90$.

Multiply eq^{n}. (2) by 5, $5y-5x=-50$.

Subtract (2) from (1) $4x=140$;

$$\therefore x = \frac{140}{4} = 35, \text{ A's money.}$$

From equation (1) $5y = 90 + x = 90 + 35 = 125$;

$$\therefore y = \frac{125}{5} = 25, \text{ B's money.}$$

PROB. 4. What two numbers are those, to one-third the sum of which if I add 13, the result shall be 17; and if from half their difference I subtract *one*, the remainder shall be two? Ans. 9, and 3.

PROB. 5. There is a certain fraction, such, that if I add *one* to its numerator, it becomes $\frac{1}{2}$; if 3 be added to the denominator, it becomes $\frac{1}{3}$. What is the fraction?

Ans. $\frac{5}{12}$.

PROB. 6. A person was desirous of relieving a certain number of beggars by giving them 2*s.* 6*d.* each, but found that he had not money enough in his pocket by 3 shillings; he then gave them 2 shillings each, and had four shillings to spare. What money had he in his pocket; and how many beggars did he relieve?

Let x = money in his pocket (in *shillings*);
y = number of beggars.

Then $2\frac{1}{2} \times y$, or $\frac{5y}{2} = \left\{\begin{array}{l}\text{No. of } \textit{shills.} \text{ which would have} \\ \quad\text{been given at } 2s.\ 6d. \text{ each.}\end{array}\right.$

and $2 \times y$, or $2y =$ ..... at 2*s.* each.

Hence, by the question, $\frac{5y}{2} = x + 3$ (1)

and $2y = x - 4$ (2).

Sub^{t}. (2) from (1), then $\frac{y}{2} = 7$, or $y = 14$, the No. of beggars.

From eq^{n}. (2), $x = 2y + 4 = 28 + 4 = 32$ shillings in his pocket.

PROB. 7. A person has two horses, (and a saddle worth £10); if the saddle be put on the *first* horse, his value becomes *double* that of the *second;* but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by £13. What is the value of each horse?

Ans. 56 and 33.

PROB. 8. There is a certain number, consisting of two digits. The *sum* of those digits is 5; and if 9 be added to

the number itself, the digits will be inverted. What is the number?

Here it may be observed that every number consisting of two digits is equal to 10 times the left-hand digit plus the right-hand digit: thus, $34 = 10 \times 3 + 4$.

Let $x =$ *left-hand* digit.
$y =$ *right-hand* digit.
Then $10x + y =$ the number itself,
and $10y + x =$ the number with digits *inverted*.

Hence, by the question, $x + y = 5$ (1),
and $10x + y + 9 = 10y + x$, or $9x - 9y = -9$, or $x - y = -1$ (2).

Subtract (2) from (1), then $2y = 6$, and $y = 3$,
$x = 5 - y = 5 - 3 = 2$;
$\therefore$ the number is $(10x + y) = 23$.

Add 9 to this number, and it becomes 32, which is the number with the *digits inverted*.

Prob. 9. There are two numbers, such, that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the less is 13; and if $\frac{1}{2}$ the less be taken from $\frac{1}{3}$ the greater, the remainder is nothing. What are the numbers? Ans. 18 and 12.

Prob. 10. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be *inverted*. What is the number?
Ans. 53.

Prob. 11. A merchant has two kinds of tea, one worth 9*s.* 6*d.* per lb., the other 13*s.* 6*d.* How many pounds of each must he take to form a chest of 104 lbs. which shall be worth £56? Ans. 33 at 13*s.* 6*d.*
71 at 9*s.* 6*d.*

Prob. 12. A vessel containing 120 gallons is filled in 10 minutes by two spouts running *successively*; the one runs 14 gallons in a minute, the other 9 gallons in a minute. For what time has *each* spout run?
Ans. 14 gallon spout runs 6 minutes.
9 gallon spout ... 4 minutes.

Prob. 13. To find three numbers, such, that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the *second* with $\frac{1}{5}$th the difference of the *third* and *first* shall be 70; and $\frac{1}{2}$ the sum of the three numbers shall be 95.
Ans. 50, 65, 75.

CHAPTER IV.

ON INVOLUTION AND EVOLUTION.

ON THE INVOLUTION OF NUMBERS AND SIMPLE ALGEBRAIC QUANTITIES.

53. *Involution*, or "the raising of a quantity to a given power," is performed by the continued multiplication of that quantity into itself till the number of factors amounts to the number of units in the index of that given power. Thus, the *square* of $a=a\times a=a^2$; the *cube* of $b=b\times b\times b=b^3$; the *fourth power* of $2=2\times 2\times 2\times 2=16$; the *fifth power* of $3=3\times 3\times 3\times 3\times 3=243$; &c. &c.

54. The operation is performed in the same manner for simple algebraic quantities, except that in this case it must be observed, that the powers of *negative* quantities are alternately $+$ and $-$; the *even* powers being positive, and the *odd* powers *negative*. Thus the *square* of $+2a$ is $+2a\times +2a$ or $+4a^2$; the square of $-2a$ is $-2a\times -2a$ or $+4a^2$; but the *cube* of $-2a=-2a\times -2a\times -2a=+4a^2\times -2a=-8a^3$.

The several powers of $\frac{a}{b}$ are,

$$Squ.=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2},$$

$$Cube=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3},$$

$$\text{4th power}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^4}{b^4},$$

&c.=&c.

And the several powers of $-\frac{b}{2c}$,

$$Squ.=-\frac{b}{2c}\times-\frac{b}{2c}=+\frac{b^2}{4c^2},$$

$$Cube=-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}=-\frac{b^3}{8c^3},$$

$$\text{4th power}=-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}=+\frac{b^4}{16c^4}$$

&c. = &c.

ON THE INVOLUTION OF COMPOUND ALGEBRAIC QUANTITIES.

55. The powers of compound algebraic quantities are

---

53. What is involution? How is it performed?—54. In what manner is involution performed for simple algebraic quantities?—55. How are the powers of compound quantities raised?

raised by the mere application of the Rule for Compound Multiplication (Art. 34). Thus,

Ex. 1. What is the square of $a+2b$?

$$\begin{array}{l} a+2b \\ a+2b \\ \hline a^2+2ab \\ \quad +2ab+4b^2 \\ \hline \end{array}$$

Square $= a^2+4ab+4b^2$

Ex. 2. What is the cube of a^2-x?

$$\begin{array}{l} a^2-x \\ a^2-x \\ \hline a^4-a^2x \\ \quad -a^2x+x^2 \\ \hline \end{array}$$

Square $= a^4-2a^2x+x^2$

$$\begin{array}{l} a^2-x \\ \hline a^6-2a^4x+a^2x^2 \\ \quad -a^4x+2a^2x^2-x^3 \\ \hline \end{array}$$

Cube $= a^6-3a^4x+3a^2x^2-x^3$

Ex. 3.

What is the 5th power of $a+b$?

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 = \text{Square} \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad +a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3 = \text{Cube} \\ a+b \\ \hline a^4+3a^3b+3a^2b^2+ab^3 \\ \quad +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline a^4+4a^3b+6a^2b^2+4ab^3+b^4 = 4^{\text{th}}\ \text{Power} \\ a+b \\ \hline a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\ \quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \hline a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 = 5^{\text{th}}\ \text{Power.} \end{array}$$

Ex. 4. The 4th *power* of $a+3b$ is $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$.

Ex. 5. The *square* of $3x^2+2x+5$ is $9x^4+12x^3+34x^2+20x+25$.

Ex. 6. The *cube* of $3x-5$ is $27x^3-135x^2+225x-125$.

Ex. 7. The *cube* of x^2-2x+1 is $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

Ex. 8. The *square* of $a+b+c$ is $a^2+2ab+b^2+2ac+2bc+c^2$.

ON THE EVOLUTION OF ALGEBRAIC QUANTITIES.

56. *Evolution*, "or the rule for extracting the root of any quantity," is just the reverse of *Involution;* and to perform the operation, we must inquire what quantity multiplied into itself, till the number of factors amount to the number of units in the index of the given root, will generate the quantity whose root is to be extracted. Thus,

(1.) $49=7\times7$; $\therefore$ the *sq. root* of 49 (or by Def^n 15, $\sqrt{49}$)$=7$.

(2.) $-b^3=-b\times-b\times-b$; $\therefore$ *cube root* of $-b^3$ ($\sqrt[3]{-b^3}$)$=-b$

(3.) $\frac{16a^4}{81b^4}=\frac{2a}{3b}\times\frac{2a}{3b}\times\frac{2a}{3b}\times\frac{2a}{3b}$; $\therefore$ $\sqrt[4]{\frac{16a^4}{81b^4}}=\frac{2a}{3b}$,

(4.) $32=2\times2\times2\times2\times2$; $\therefore \sqrt[5]{32}=2$

(5) $a^6=a^2\times a^2\times a^2$; $\therefore \sqrt[3]{a^6}=a^2$.

Hence it may be inferred, that any *root of a simple quantity can be extracted, by dividing its index, if possible, by the index of the root.*

57. If the quantity under the radical sign does not admit of resolution into the number of factors indicated by that sign, or, in other words, if it be not a *complete power*, then its exact root cannot be extracted, and the quantity itself, with the radical sign annexed, is called a *Surd.* Thus $\sqrt{37}$, $\sqrt[3]{a^2}$, $\sqrt[4]{b^3}$, $\sqrt[5]{47}$, &c. &c. are Surd quantities.

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56. What is *Evolution?* How is it performed?—57. What is a *Surd* quantity?

58. In the involution of *negative* quantities, it was observed that the *even* powers were all $+$, and the odd powers $-$; there is consequently no quantity which, multiplied into itself in such manner that the number of factors shall be *even*, can generate a negative quantity. Hence quantities of the form $\sqrt{-a^2}$, $\sqrt[4]{-10}$, $\sqrt[6]{-a^3}$, $\sqrt{-5}$, $\sqrt[4]{-a^4}$, &c. &c. have no real root, and are therefore called *impossible*.

59. In extracting the roots of *compound* quantities, we must observe in what manner the terms of the *root* may be derived from those of the power. For instance (by Art. 55, Ex. 3), the square of $a+b$ is $a^2+2ab+b^2$, where the terms are arranged according to the powers of a. On comparing $a+b$ with $a^2+2ab+b^2$, we observe that the first term of the power (a^2) is the square of the first term of the root (a). Put a therefore for the first term of the root, square it, and subtract that square from the first term of the power. Bring down the other two terms $2ab+b^2$, and *double* the first term of the root; set down $2a$, and having divided the first term of the remainder ($2ab$) by it, it gives b, the other term of the root; and since $2ab+b^2=(2a+b)b$, if to $2a$ the term b is added, and this sum multiplied by b, the result is $2ab+b^2$; which being subtracted from the two terms brought down, nothing remains.

$$
\begin{array}{l|l}
a^2+2ab+b^2 & a+b \\
\cline{2-2}
a^2 & \\
\cline{1-1}
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\cline{2-2}
 & * \quad * \\
\hline\hline
\end{array}
$$

60. Again, the square of $a+b+c$ (Art. 55. Ex. 8.) is $a^2+2ab+b^2+2ac+2bc+c^2$; in this case the root may be derived from the power, by continuing the process in the last Article. Thus, having found the two first terms ($a+b$) of the root as before, we bring down the remaining three terms $2ac+2bc$

$$
\begin{array}{l|l}
a^2+2ab+b^2+2ac+2bc+c^2 & a+b+c \\
\cline{2-2}
a^2 & \\
\cline{1-1}
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\cline{2-2}
2a+2b+c & 2ac+2bc+c^2 \\
 & 2ac+2bc-c^2 \\
\cline{2-2}
 & * \quad * \quad * \\
\hline\hline
\end{array}
$$

58. Explain the nature of an *impossible* quantity.—59. How are the roots of *compound* quantities extracted?

$+c^2$ of the power, and dividing $2ac$ by $2a$, it gives c, the third term of the root. Next, let the last term (b) of the preceding divisor be doubled, and add c to the divisor thus increased, and it becomes $2a+2b+c$; multiply this new divisor by c, and it gives $2ac+2bc+c^2$, which being subtracted from the three terms last brought down, leaves no remainder. In this manner the following Examples are solved.

Ex. 1.

$$4x^4+6x^3+\frac{89}{4}x^2+15x+25\Big(2x^2+\frac{3}{2}x+5.$$

$$\begin{array}{l}
\underline{4x^4} \\
4x^2+\frac{3}{2}x\Big)6x^3+\frac{89}{4}x^2 \\
\qquad\qquad \underline{6x^3+\frac{9}{4}x^2} \\
4x^2+3x+5\ \Big)\ \ 20x^2+15x+25 \\
\qquad\qquad\qquad\ \ \underline{20x^2+15x+25} \\
\qquad\qquad\qquad\qquad\ \ * \quad * \quad *
\end{array}$$

Ex. 2.

$$x^6+4x^5+2x^4+9x^2-4x+4\big(x^3+2x^2-x+2$$

$$\begin{array}{l}
\underline{x^6} \\
2x^3+2x^2)4x^5+2x^4 \\
\qquad\quad \underline{4x^5+4x^4} \\
2x^3+4x^2-x)-2x^4+9x^2-4x \\
\qquad\qquad\quad \underline{-2x^4-4x^3+\ x^2} \\
2x^3+4x^2-2x+2)\ +4x^3+8x^2-4x+4 \\
\qquad\qquad\qquad\quad \underline{+4x^3+8x^2-4x+4} \\
\qquad\qquad\qquad\qquad\ \ * \quad * \quad * \quad *
\end{array}$$

Ex. 3. The square root of $4x^2+4xy+y^2$ is $2x+y$.

Ex. 4. ..... $25a^2+30ab+9b^2$ is $5a+3b$.

Ex. 5. Find the square root of $9x^4+12x^3+22x^2+12x+9$.
Ans. $3x^2+2x+3$.

Ex. 6. Extract the square root of $4x^4-16x^3+24x^2-16x+4$.

Ans. $2x^2-4x+2$.

Ex. 7. Find the square root of $36x^4-36x^3+17x^2-4x+\frac{4}{9}$.

Ans. $6x^2-3x+\frac{2}{3}$.

Ex. 8. Extract the square root of $x^4+8x^2+24+\frac{32}{x^2}+\frac{16}{x^4}$.

Ans. $x^2+4+\frac{4}{x^2}$.

ON THE INVESTIGATION OF THE RULE FOR THE EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

Before we proceed to the investigation of this Rule, it will be necessary to explain the nature of the common arithmetical notation.

61. It is very well known that the value of the figures in the common arithmetical scale increases in a tenfold proportion from the right to the left; a number, therefore, may be expressed by the addition of the *units*, *tens*, *hundreds*, &c. of which it consists. Thus the number 4371 may be expressed in the following manner, viz. $4000+300+70+1$, or by $4\times1000+3\times100+7\times10+1$; hence, if the digits* of a number be represented by a, b, c, d, e, &c. beginning from the left hand, then,

A No. of 2 figures may be expressed by $10a+b$.
..... 3 figures ..... by $100a+10b+c$.
..... 4 figures ..... by $1000a+100b+10c+d$.
&c. &c. &c.

62. Let a number of three figures (viz. $100a+10b+c$) be

* By the *digits* of a number are meant the figures which compose it, considered independently of the value which they possess in the arithmetical scale. Thus the digits of the number 537 are simply the numbers 5, 3, and 7; whereas the 5, considered with respect to its place in the numeration scale, means 500, and the 3 means 30.

61. Explain the common arithmetical scale of notation. What is a digit?—62. Show the relation between the algebraical and numerical method of extracting the square root, and that they are identical.

squared, and its root extracted according to the Rule in Art. 60, and the operation will stand thus;

$$
\begin{array}{l}
\text{I.}\ 10000a^2+2000ab+100b^2+200ac+20bc+c^2(100a+10b+c \\
\quad 10000a^2 \\
\hline
200a+10b)\ 2000ab+100b^2 \\
\qquad\qquad 2000ab+100b^2 \\
\hline
\qquad\qquad 200a+20b+c)\ 200ac+20bc+c^2 \\
\qquad\qquad\qquad\qquad\qquad\quad 200ac+20bc+c^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad * \quad * \quad * \\
\hline\hline
\end{array}
$$

II. Let $\left.\begin{array}{l} a=2 \\ b=3 \\ c=1 \end{array}\right\}$ and the operation is transformed into the following one;

$$
\begin{array}{l}
40000+12000+900+400+60+1\big(200+30+1 \\
40000 \\
\hline
400+30\big)\ 12000+900+400 \\
\qquad\qquad 12000+900 \\
\hline
\qquad\qquad 400+60+1\big)\ 400+60+1 \\
\qquad\qquad\qquad\qquad\qquad\quad 400+60+1 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad * \quad * \quad * \\
\hline\hline
\end{array}
$$

III. But it is evident that this operation would not be affected by collecting the several numbers which stand in the same line into one sum, and leaving out the ciphers which are to be *subtracted* in the several parts of the operation. Let this be done; and let two figures be brought down at a time, after the square of the first figure in the root has been subtracted; then the operation may be exhibited in the manner annexed; from which it appears that the square root of 53,361 is 231.

$$
\begin{array}{l}
\dot{5}3\dot{3}6\dot{1}\ \big(231 \\
\ \ 4 \\
\hline
43\,|\,133 \\
\qquad\ |\,129 \\
\hline
461\,|\,461 \\
\qquad\ |\,461 \\
\hline
\qquad\ \ *
\end{array}
$$

63. To explain the division of the given number into *periods* consisting of two figures each, by placing a dot over every second figure beginning with the units (as exhibited in the foregoing operation), it must be observed, that, since the square root of 100 is 10; of 10,000 is 100; of 1,000,000 is 1000, &c. &c.; it follows that the square root of a number *less than* 100 must consist of *one* figure; of a number *between* 100 *and* 10,000, of *two* figures; of a number between 10,000 and 1,000,000, of *three* figures, &c. &c.; and consequently the number of these dots will show the number of figures contained in the square root of the given number. Thus in the case of 53361 the square root is a number consisting of *three* figures.

Ex. 1. Find the square root of 105,625. Ans. 325.

Ex. 2. Find the square root of 173,056. Ans. 416.

Ex. 3. Find the square root of 5,934,096. Ans. 2436.

CHAPTER V.

ON QUADRATIC EQUATIONS.

64. Quadratic Equations are divided into *pure* and *adfected*. *Pure* quadratic equations are those which contain only the *square* of the unknown quantity, such as $x^2=36$; $x^2+5=54$; $ax^2-b=c$; &c. *Adfected* quadratic equations are those which involve both the *square* and *simple power* of the unknown quantity, such as $x^2+4x=45$; $3x^2-2x=21$; $ax^2+2bx=c+d$; &c. &c.

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63. Explain the principle of the rule and the object of pointing off in extracting the square root of numbers.—64. How are *quadratic equations* divided? What is an *adfected* quadratic equation?

ON THE SOLUTION OF PURE QUADRATIC EQUATIONS.

65. The Rule for the solution of pure quadratic equations is this, "Transpose the terms of the equation in such a manner, that those which contain x^2 may be on one side of the equation, and the *known quantities* on the other; divide (if necessary) by the coefficient of x^2; then extract the square root of each side of the equation, and it will give the values of x."

Ex. 1.

Let $x^2+5=54$.

By transposition, $x^2=54-5=49$.

Extract the square root of both sides of the equation, then $x=\pm\sqrt{49}=\pm 7$.

Ex. 2.

Let $3x^2-4=71$.

By transposition, $3x^2=71+4=75$.

Divide by 3, $x^2=\frac{75}{3}=25$.

Extract the square root, $x=\pm\sqrt{25}=\pm 5$.

Ex. 3.

Let $ax^2-b=c$;

then $ax^2=c+b$,

and $x^2=\frac{c+b}{a}$

$$\therefore x=\pm\sqrt{\frac{c+b}{a}}.$$

Ex. 4.	$5x^2-1$	$=244$	Ans.	$x=\pm 7$.
Ex. 5.	$9x^2+9$	$=3x^2+63$	...	$x=\pm 3$.
Ex. 6.	$\frac{4x^2+5}{9}$	$=45$	...	$x=\pm 10$.
Ex. 7.	bx^2+c+3	$=2bx^2+1$	...	$x=\pm\sqrt{\frac{c+2}{b}}$.

65. State the rule for solving pure quadratic equations.

ON THE SOLUTION OF ADFECTED QUADRATIC EQUATIONS.

66. The most general form under which an adfected quadratic equation can be exhibited is $ax^2+bx=c$; where a, b, c may be any quantities whatever, *positive* or *negative*, *integral* or *fractional*. Divide each side of this equation by a, then $x^2+\frac{b}{a}x=\frac{c}{a}$. Let $\frac{b}{a}=p$, $\frac{c}{a}=q$; then this equation is reduced to the form $x^2+px=q$, where p and q may be any quantities whatever, positive or negative, integral or fractional.

67. From the twofold form under which adfected quadratic equations may be expressed, there arise two Rules for their solution.

Rule I.

Let $x^2 \pm px=q$.

Add $\frac{p^2}{4}$ to each side of the equation, then $\left\{ x^2 \pm px+\frac{p^2}{4}=\frac{p^2}{4}+q=\frac{p^2+4q}{4}. \right.$

Extract the square root of each side of the equation, then $\left\{ \begin{array}{l} x \pm \frac{p}{2}=\frac{\pm\sqrt{p^2+4q}}{2}* \\ \text{and } x=\frac{\pm\sqrt{p^2+4q} \mp p}{2}. \end{array} \right.$

Hence it appears, that "if to each side of the equation there be added the *square of half the coefficient of x*, there will arise, on the left-hand side of the equation, a quantity which is *a complete square*; and by extracting the square root of each side of the resulting equation, we obtain a *simple* equation, from which the value of x may be determined."

* Since the square of $+a$ is $+a^2$, and of $-a$ is *also* $+a^2$, the *square root* of $+a^2$ may be either $+a$ or $-a$; hence the square root of p^2+4q may be expressed by $\pm\sqrt{p^2+4q}$.

---

66. What is the most general form of a quadratic equation? Can it be reduced to another form?—67. Enunciate the 1st Rule.

68. From the form in which the value of x is exhibited in each of these Rules, it is evident that it will have *two* values; one corresponding to the sign $+$, and the other to the sign $-$, of the radical quantity.

Ex. 1.

Let $x^2+8x=65$.

Add the square of 4 (*i.e.* 16) to each side of the equation, then . . . $x^2+8x+16=65+16=81$.

Extract the square root of each side of the equation, then

$$x+4=\pm\sqrt{81}=\pm 9,$$
$$\text{and } x=9-4=5;$$
$$\text{or } x=-9-4=-13.$$

Ex. 2.

Let $x^2-4x=45$.

Add the square of 2 (*i.e.* 4), then $\left.\right\}$ $x^2-4x+4=45+4=49$.

Extract the square root, and $x-2=\pm\sqrt{49}=\pm 7$,

$$\text{and } x=7+2=9;$$
$$\text{or, } x=2-7=-5.$$

Ex. 3.	$x^2+12x=108$	...	Ans.	$x=6$ or -18.
Ex. 4.	$x^2-14x=51$	...	...	$x=17$ or -3.
Ex. 5.	$x^2-8x=20$	...	...	$x=10$ or -2.

Ex. 6. $x^2-5x=6$.

In this example the *coefficient* of x is 5, an odd number. Its half is $\frac{5}{2}$; and $\therefore$ adding to each side of the equation $\left(\frac{5}{2}\right)^2$ or $\frac{25}{4}$, we get

$$x^2-5x+\left(\frac{5}{2}\right)^2=6+\frac{25}{4}=\frac{24+25}{4}=\frac{49}{4}.$$

Extracting the square root, $x-\frac{5}{2}=\pm\frac{7}{2}$;

$$\therefore x=\frac{5}{2}\pm\frac{7}{2}=6, \text{ or } -1.$$

Ex. 7. $x^2-x=6$.

Here the coefficient of x is 1; adding therefore $(\frac{1}{2})^2$ or $\frac{1}{4}$ to both sides, we get

$$x^2-x+(\tfrac{1}{2})^2=6+\tfrac{1}{4}=\frac{25}{4}.$$

Extracting the square root, $x-\frac{1}{2}=\pm\frac{5}{2}$;

$$\therefore x=\tfrac{1}{2}\pm\frac{5}{2}=3 \text{ or } -2.$$

Ex. 8. $x^2+7x=78$. ..... Ans. $x=6$ or -13.

Ex. 9. $x^2+3x=28$. .... $x=4$ or -7.

Ex. 10. $x^2-3x=40$. .... $x=8$ or -5.

Ex. 11. $x^2+x=30$. .... $x=5$ or -6.

Ex. 12. Let $7x^2-20x=32$; find x.

Dividing by 7, $x^2-\frac{20}{7}x=\frac{32}{7}$.

Compl^{s}. the sq. $\Big\}$ $x^2-\frac{20}{7}x+\left(\frac{10}{7}\right)^2=\frac{32}{7}+\frac{100}{49}=\frac{224}{49}+\frac{100}{49}=\frac{324}{49}$.

$$\text{Hence, } x-\frac{10}{7}=\pm\sqrt{\frac{324}{49}}=\pm\frac{18}{7};$$

$$\text{and } x=\frac{10}{7}\pm\frac{18}{7}=4 \text{ or } -1\tfrac{1}{7}.$$

Ex. 13. $5x^2+4x=273$.

Dividing by 5, $x^2+\frac{4}{5}x=\frac{273}{5}$.

To each side add $\Big\}$ $\left(\frac{2}{5}\right)^2$ or $\frac{4}{25}$ and $x^2+\frac{4}{5}x+\frac{4}{25}=\frac{273}{5}+\frac{4}{25}=\frac{1369}{25}$.

Extracting the square root, $x+\frac{2}{5}=\pm\frac{37}{5}$.

$$\therefore x=\pm\frac{37}{5}-\frac{2}{5}=7, \text{ or } -7\tfrac{4}{5}.$$

Ex. 14. $3x^2+2x=161$ ..... Ans. $x=7$ or $-7\frac{2}{3}$.

Ex. 15. $2x^2-5x=117$ ..... Ans. $x=9$ or $-\frac{13}{2}$.

Ex. 16. $3x^2-2x=280$ .... $x=10$ or $-9\frac{1}{3}$.

Ex. 17. $4x^2-7x=492$ .... $x=12$ or $-10\frac{1}{4}$.

A quadratic equation seldom appears in a form so simple as those of the preceding examples; it is therefore generally found necessary to employ in the solution of a quadratic the following reductions.

(1.) Clear the equation of fractions.

(2.) Transpose the terms involving x^2 and x to the left hand, and the numbers to the right-hand side of the equation.

(3.) Divide all the terms of the equation by the *coefficient* of x^2.

(4.) Complete the square.

(5.) Extract the square root of both sides, and there will arise a simple equation, from which the value of x may be found.

Ex. 1. $$\frac{4x^2}{3}-11=\frac{1}{3}x.$$

Multiply by 3, and $4x^2-33=x$.

By transposition, $4x^2-x=33$.

Divide by 4, and $x^2-\frac{1}{4}x=\frac{33}{4}$.

Complete the square, $\left\}\ x^2-\frac{1}{4}x+\frac{1}{64}=\frac{33}{4}+\frac{1}{64}=\frac{528}{64}+\frac{1}{64}=\frac{529}{64}\right.$.

Extracting the sq. root, $x-\frac{1}{8}=\pm\frac{23}{8}$.

$$\therefore x=\frac{1}{8}\pm\frac{23}{8}=3 \text{ or } -2\tfrac{3}{4}.$$

Ex. 2. $$\frac{9}{x+1}+\frac{4}{x}=5.$$

$$9+\frac{4x+4}{x}=5x+5.$$

$$9x+4x+4=5x^2+5x,$$
$$5x^2-8x=4,$$
$$x^2-\frac{8}{5}x=\frac{4}{5},$$
$$x^2-\frac{8}{5}x+\frac{16}{25}=\frac{4}{5}+\frac{16}{25}=\frac{36}{25};$$
$$\therefore x-\frac{4}{5}=\pm\frac{6}{5},$$
$$\text{and } x=\frac{4}{5}\pm\frac{6}{5}=2 \text{ or } -\frac{2}{5}.$$

Ex. 3. $\frac{x^2}{6}-1=x+11.$ ..... Ans. $x=12$ or -6.

Ex. 4. $\frac{2x}{3}+\frac{1}{x}=\frac{7}{3}.$ .... $x=3$ or $\frac{1}{2}$.

Ex. 5. $\frac{x^2}{3}-\frac{x}{2}=9.$ .... $x=6$ or $-\frac{9}{2}$.

Ex. 6. $\frac{6}{x+1}+\frac{2}{x}=3.$ .... $x=2$ or $-\frac{1}{3}$.

Ex. 7. $x^2-34=\frac{1}{3}x.$ .... $x=6$ or $-5\frac{2}{3}$.

Ex. 8. $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}.$ .... $x=25$ or 1.

Ex. 9. $x+\frac{24}{x-1}=3x-4.$ .... $x=5$ or -2.

Ex. 10. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}.$ .... $x=2$ or -3.

Ex. 11. $\frac{3x}{x+2}-\frac{x-1}{6}=x-9.$ ... $x=10$ or $-\frac{11}{7}$.

Ex. 12. Given $x^2+8x=-31$; find x.

$$x^2+8x+16=16-31=-15,$$
$$x+4=\pm\sqrt{-15};$$
$$\therefore x=-4+\sqrt{-15}, \text{ \& } x=-4-\sqrt{-15},$$

both of which are *impossible* or *imaginary* values of x.

Ex. 13. $x^2-2x=-2.$ ..... Ans. $x=1\pm\sqrt{-1}.$

Ex. 14. $x^2-16x=-15.$ .... $x=15$ or $1.$

Ex. 15. Let $13x^2+2x=60.$

Divide by 13, $x^2+\frac{2x}{13}=\frac{60}{13}.$

$$\left.\begin{array}{r}\text{Add the}\\ \text{square of } \frac{1}{13}\end{array}\right\} x^2+\frac{2x}{13}+\frac{1}{169}=\frac{60}{13}+\frac{1}{169}=\frac{780}{169}+\frac{1}{169}=\frac{781}{169}.$$

$$\left.\begin{array}{r}\text{Extract the}\\ \text{square root}\end{array}\right\} x+\frac{1}{13}=\pm\frac{\sqrt{781}}{13}=\pm\frac{27.94}{13}$$

$$\therefore x=\frac{\pm 27.94-1}{13}=\frac{26.94}{13}=2.07 \text{ or } -2.226.$$

Ex. 16. $x^2-6x+19=13.$ Ans. $x=4.732$ or $1.268.$

Ex. 17. $5x^2+4x=25.$ ... $x=1.871.$

Any equation, in which the unknown quantity is found only in two terms, with the index of the higher power double that of the lower, may be solved as a quadratic by the preceding rules.

Ex. 18. Let $x^6-2x^3=48.$

Complete the square, $x^6-2x^3+1=49;$

Extract the square root, $x^3-1=\pm 7;$

$$\therefore x^3=8 \text{ or } -6;$$

$$\text{and } \therefore x=2 \text{ or } \sqrt[3]{-6}.$$

Ex. 19. $2x-7\sqrt{x}=99.$

$$x-\frac{7}{2}\sqrt{x}=\frac{99}{2},$$

$$x-\frac{7}{2}\sqrt{x}+\left(\frac{7}{4}\right)^2=\frac{99}{2}+\frac{49}{16}=\frac{841}{16}.$$

$$\sqrt{x}-\frac{7}{4}=\pm\frac{29}{4},$$

$$\sqrt{x}=\frac{7}{4}\pm\frac{29}{4}=9 \text{ or } -\frac{11}{2}.$$

$\therefore$ by squaring both sides, $x=81$ or $\frac{121}{4}.$

Ex. 20. $x^4+4x^2=12$. Ans. $x=\pm\sqrt{2}$ or $\pm\sqrt{-6}$.

Ex. 21. $x^6-8x^3=513$. ... $x=3$ or $\sqrt[3]{-19}$.

Rule II.

Let $ax^2\pm bx=c$,

Multiply each side of the equation by $4a$, then $4a^2x^2\pm 4abx=4ac$.

Add b^2 to each side, we have $4a^2x^2\pm 4abx+b^2=4ac+b^2$.

Extract the sq. root as before, $2ax\pm b=\pm\sqrt{4ac+b^2}$

$$\therefore 2ax=\pm\sqrt{4ac+b^2}\mp b$$

$$\text{and } x=\frac{\pm\sqrt{4ac+b^2}\mp b}{2a}.$$

From which we infer, that "if each side of the equation be multiplied by *four times the coefficient of* x^2, and to each side there be added *the square of the coefficient of* x, the quantity on the left-hand side of the equation will be the square of $2ax\pm b$. Extract the square root of each side of the equation, and there arises a *simple equation*, from which the value of x may be determined."*

If $a=1$, the equation is reduced to the form $x^2\pm px=q$; in this case, therefore, the Rule may be applied, by "multiplying each side of the equation by 4, and adding the square of the coefficient of x."

From the form in which the value of x is exhibited in this Rule, it is evident that it will have *two* values; one corresponding to the sign $+$, and the other to the sign $-$, of the radical quantity.

Ex. 1.

Let $3x^2+5x=42$.

Multiply each side of the equation by $(4a)$ 12; then $36x^2+60x=504$.

* The principle of this Rule will be found in the *Bija Ganita*, a *Hindoo* Treatise on the Elements of Algebra. For a full account of that work, as translated by Mr. Strachey, see Dr. Hutton's *Tracts*, vol. ii. Tract 33.

Add (b^2) 25 to each side of the equation, we have $\left.\right\} 36x^2+60x+25=504+25=529.$

Extract the square root of each side of the equation, which gives ..... $6x+5=\pm\sqrt{529}=\pm 23$;

$$\therefore 6x=\pm 23-5=18 \text{ or } -28;$$

$$\text{and } x=\frac{18}{6}=3,$$

$$\text{or } x=\frac{18}{6}=-\frac{14}{3}=-4\tfrac{2}{3}.$$

Ex. 2.

Let $x^2-15x=-54.$

Multiply by 4, then $4x^2-60x=-216.$

Add (b^2) 225 to each side $\left.\right\}$ and $4x^2-60x+225=225-216=9.$

Extract the square root $2x-15=\pm\sqrt{9}=\pm 3$;

$$\therefore 2x=15\pm 3=18 \text{ or } 12,$$

$$\text{and } x=\frac{18}{2} \text{ or } \frac{12}{2}=9 \text{ or } 6.$$

ON THE SOLUTION OF PROBLEMS PRODUCING QUADRATIC EQUATIONS.

69. In the solution of problems which involve quadratic equations, sometimes *both*, and sometimes only *one* of the values of the unknown quantity, will answer the conditions required. This is a circumstance which may always be very readily determined by the nature of the problem itself.

Problem 1.

To divide the number 56 into two such parts, that their product shall be 640.

Let $x=$ *one* part,

then $56-x=$ the *other* part,

and $x\,(56-x)=$ *product* of the two parts.

Hence, by the question, $x\,(56-x)=640,$

or $56x-x^2=640.$

By transposition, $x^2-56x=-640$.

By completing the square, (RULE I.) $\left.\right\}$ $x^2-56x+784=784-640=144$;

$$\therefore x-28=\pm\sqrt{144}=\pm 12,$$

$$\text{and } x=28\pm 12=40 \text{ or } 16.$$

In this case it appears that the *two* values of the unknown quantity are the *two* parts into which the given number was required to be divided.

PROB. 2. There are two numbers whose difference is 7, and half their product *plus* 30 is equal to the square of the *less* number. What are the numbers?

Let $x=$ the *less* number,

then $x+7=$ the *greater* number,

and $\dfrac{x\times(x+7)}{2}+30=$ half their product *plus* 30.

Hence, by the question, $\dfrac{x\times(x+7)}{2}+30=x^2$ (square of *less*),

$$\text{or } \frac{x^2+7x}{2}+30=x^2.$$

Multiply by 2 ..... $x^2+7x+60=2x^2$.

By transposition ... $x^2-7x=60$.

Multiply by 4, and add 49 (RULE II.) $\left.\right\}$ $4x^2-28x+49=240+49=289$,

$$\therefore 2x-7=\sqrt{289}=17$$

$$2x=17+7=24, \text{ or } x=12 \textit{ less} \text{ number};$$

hence $x+7=12+7=19$ *greater* number.

PROB. 3. To divide the number 30 into two such parts, that their product may be equal to *eight* times their difference.

Let $x=$ the *less* part,

then $30-x=$ the *greater* part,

and $30-x-x$ or $30-2x=$ their *difference*.

Hence, by the question, $x\times(30-x)=8\times(30-2x)$,

$$\text{or } 30x-x^2=240-16x.$$

By transposition, $x^2-46x=-240$.

Complete the square, (RULE I.) $\Big\}\ x^2-46x+529=529-240=289$;

$$\therefore x-23=\pm\sqrt{289}=\pm 17,$$

and $x=23\pm 17=40$ or $6=$ *less* part;

$30-x=30-6=24=$ *greater* part.

In this case, the solution of the equation gives 40 and 6 for the *less* part. Now as 40 cannot possibly be a *part* of 30, we take 6 for the *less* part, which gives 24 for the *greater* part; and the two numbers, 24 and 6, answer the conditions required.

PROB. 4. A person bought cloth for £33 15*s.*, which he sold again at £2. 8*s.* per piece, and gained by the bargain as much as one piece cost him. Required the number of pieces.

Let $x=$ the number of pieces.

Then $\frac{675}{x}=$ the number of *shillings* each piece *cost*,

and $48x=$ the number of *shillings* he *sold* the whole for;

$\therefore 48x-675=$ what he gained by the bargain.

Hence, by the problem, $48x-675=\frac{675}{x}$.

By transposition and division, $\Big\}\ x^2-\frac{225}{16}x=\frac{225}{16}$.

Complete the sq. (RULE I.) $\Big\}\ x^2-\frac{225}{16}x+\frac{50625}{1024}=\frac{225}{16}+\frac{50625}{1024}=\frac{65025}{1024}$.

$$\therefore x-\frac{225}{32}=\sqrt{\frac{65025}{1024}}=\frac{255}{32},$$

and $x=\frac{255+225}{32}=15$.

PROB. 5. A and B set off at the *same time* to a place at the distance of 150 miles. A travels 3 miles an hour *faster* than B, and arrives at his journey's end 8 hours and 20 minutes *before* him. At what rate did each person travel per hour?

Let $x =$ rate per hour at which B travels.

Then $x + 3 =$ ..... A .....

And $\frac{150}{x} =$ number of hours for which B travels.

$\frac{150}{x+3} =$ ..... A .....

But A is 8 hours 20 minutes ($8\frac{1}{3}$ hours) *sooner* at his journey's end than B;

$$\text{Hence } \frac{150}{x+3} + 8\tfrac{1}{3} = \frac{150}{x},$$

$$\text{or } \frac{150}{x+3} + \frac{25}{3} = \frac{150}{x}.$$

$$\text{By reduction, } x^2 + 3x = 54.$$

Complete the square, $x^2 + 3x + \frac{9}{4} = 54 + \frac{9}{4} = \frac{225}{4}$ (RULE I.):

$$\therefore x + \frac{3}{2} = \sqrt{\frac{225}{4}} = \frac{15}{2};$$

$$\text{and } x = \frac{15-3}{2} = 6 \text{ miles an hour for B,}$$

$$x + 3 = 9 \text{ ..... A.}$$

PROB. 6. Some bees had alighted upon a tree; at one flight the square root of half of them went away; at another $\frac{8}{9}$ths of them; two bees then remained. How many alighted on the tree?*

Let $2x^2 =$ the No. of bees.

$$\text{then } x + \frac{16x^2}{9} + 2 = 2x^2,$$

$$\text{or } 9x + 16x^2 + 18 = 18x^2$$

$$\therefore 18x^2 - 16x^2 - 9x = 18,$$

$$\text{or } 2x^2 - 9x = 18.$$

(RULE II.) Multiply by 8,

$$16x^2 - 72x = 144.$$

* This question is taken from Mr. Strachey's translation of the *Bija Ganita;* and the several steps of the operation will, upon comparison, be found to accord with the *Hindoo* method of solution, as it stands in that translation, p. 62.

Add 81; then $16x^2-72x+81=225$,

$$\text{or } 4x-9=15;$$

$$\therefore 4x=15+9=24,$$

$$\text{and } x=\frac{24}{4}=6;$$

$$\therefore 2x^2=72, \text{ No. of bees.}$$

PROB. 7. To divide the number 33 into two such parts that their product shall be 162. Ans. 27 and 6.

PROB. 8. What two numbers are those whose sum is 29, and product 100? Ans. 25 and 4.

PROB. 9. The difference of two numbers is 5, and $\frac{1}{4}$th part of their product is 26. What are the numbers? Ans. 13 and 8.

PROB. 10. The difference of two numbers is 6; and if 47 be added to *twice the square of the less*, it will be equal to the *square of the greater*. What are the numbers? Ans. 17 and 11.

PROB. 11. There are two numbers whose sum is 30; and $\frac{1}{3}$d of their product *plus* 18 is equal to the square of the *less* number. What are the numbers? Ans. 21 and 9.

PROB. 12. There are two numbers whose product is 120. If 2 be added to the less, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? Ans. 15 and 8.

PROB. 13. A and B distribute £1200 each among a certain number of persons: A relieves 40 persons more than B, and B gives £5 a-piece to each person *more* than A. How many persons were relieved by A and B respectively? Ans. 120 by A, 80 by B.

PROB. 14. A person bought a certain number of sheep for £120. If there had been 8 more, each sheep would have cost him 10 shillings less. How many sheep were there? Ans. 40.

PROB. 15. A person bought a certain number of sheep for £57. Having lost 8 of them, and sold the remainder at 8 shillings a-head profit, he is no loser by the bargain. How many sheep did he buy? Ans. 38.

PROB. 16. A and B set off at the same time to a place at the distance of 300 miles. A travels at the rate of one mile an hour faster than B, and arrives at his journey's end 10 hours before him. At what rate did each person travel per *hour?* Ans. A travelled 6 miles per hour.
B ..... 5 .....

PROB. 17. To divide the number 16 into two such parts, that their product shall be equal to 70.

Let $x =$ one part,
then $16-x =$ the other part.

Hence $x\,(16-x)$ or $16x-x^2=70$.

Transpose, and $x^2-16x=-70$.

Complete the square,

$$x^2-16x+64=-70+64=-6,$$

$$\therefore\ x-8=\pm\sqrt{-6},\ \text{or}\ x=8\pm\sqrt{-6}.*$$

PROB. 18. To divide the number 20 into two such parts, that their product shall be 105 ..... $x=10\pm\sqrt{-5}$.

PROB. 19. To resolve the number a into two such factors, that the sum of their nth powers shall be equal to b.

Let $x=$ one factor,†

then $\frac{a}{x}=$ the other factor.

Hence $x^n+\frac{a^n}{x^n}=b$,

or $x^{2n}+a^n=bx^n$;

$\therefore\ x^{2n}-bx^n=-a^n$.

* It is very well known that the *greatest* product which can arise from the multiplication of the two parts into which any given number may be divided, is when these two parts are *equal;* the greatest product therefore, which could arise from the division of the number 16 into two parts, is when each of them is 8; hence, in requiring "to divide the number 16 into two such parts that their product should be 70," the solution of the question is *impossible.*

† By *factors* are here meant the two numbers which being multiplied together shall generate the given number; if therefore $x=$ one factor, $\frac{a}{x}$ must be the other factor, for $x\times\frac{a}{x}=a$.

By Rule II.

$$4x^{2n}-4bx^n+b^2=b^2-4a^n,$$

$$\text{and } 2x^n-b=\sqrt{b^2-4a^n},$$

$$\text{or } 2x^n=b+\sqrt{b^2-4a^n}, \text{ and } x^n=\frac{b+\sqrt{b^2-4a^n}}{2};$$

$$\therefore x=\sqrt[n]{\frac{b+\sqrt{b^2-4a^n}}{2}}.$$

Prob. 20. To resolve the number 18 into two such factors, that the sum of their *cubes* shall be 243.

Ans. 6 and 3.

ON THE SOLUTION OF QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

The solution of equations with *two* unknown quantities, in which one or both these quantities are found in a quadratic form, can only, in *particular cases*,* be effected by means of the preceding Rules. Of these cases, the two following are very well known.

Case I.

70. "When one of the equations by which the values of the unknown quantities are to be determined, is a *simple* equation;" in which case, the Rule is, "to find a value of one of the unknown quantities from that simple equation, and then substitute for it the value so found, in the other equation; the resulting equation will be a quadratic, which may be solved by the ordinary Rules."

* The most complete form under which quadratic equations containing two unknown quantities could be expressed, is this:

$$ax^2+by^2+cxy+dx+ey=m$$

$$a'x^2+b'y^2+c'xy+d'x+e'y=m';$$ but the general solution of these equations can only be effected bv means of equations of higher dimensions than quadratics.

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70. There are two well-known cases, which admit of solution by the preceding rules; state them, and the rules employed for reducing the two equations to one quadratic of the usual form.

Ex. 1.

$$\left.\begin{array}{l}\text{Let } x+2y=7,\\ \text{and } x^2+3xy-y^2=23\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

From 1st equation, $x=7-2y$, $\therefore x^2=49-28y+4y^2$;

Substitute these values for x and x^2 in the 2d equation,

then $49-28y+4y^2+21y-6y^2-y^2=23$,

or $3y^2+7y=49-23=26$.

By Rule II. $36y^2+84y+49=312+49=361$,

$\therefore 6y+7=19$

$6y=19-7=12$, or $y=2$

$x=7-2y=7-4=3$.

Ex. 2.

$$\left.\begin{array}{l}\text{Let } \dfrac{2x+y}{3}=9\\ \text{and } 3xy=210\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

From 1st equation, $2x+y=27$;

$\therefore 2x=27-y$.

and $x=\dfrac{27-y}{2}$

Hence, $3xy=3\times\dfrac{27-y}{2}\times y=210$,

or $3\times(27-y)\times y=420$

$81y-3y^2=420$

$27y-y^2=140$;

or $y^2-27y=-140$.

By Rule II., $4y^2-108y+729=729-560=169$;

$\therefore 2y-27=13$, or $y=\dfrac{27+13}{2}=20$,

and $x=\dfrac{27-20}{2}=3\frac{1}{2}$.

Ex. 3.

There is a certain number consisting of two digits. The left-hand digit is equal to 3 times the right-hand digit; and if 12 be subtracted from the number itself, the remainder

will be equal to the square of the left-hand digit. What is the number?

Let x be the left-hand digit, and y the other; then, by Art. 61, $10x+y$ is the number.

$$\left.\begin{array}{r}\text{Hence, } x=3y\\ \text{and } 10x+y-12=x^2\end{array}\right\}\text{by the question;}$$

$$\therefore \text{ by substitution } \left\{ 30y+y-12=9y^2, \text{ (for } 10x=30y, \text{ and } x^2=9y^2)\right.;$$

$$9y^2-31y=-12;$$

$$\therefore y^2-\frac{31}{9}y=-\frac{12}{9}.$$

By Rule I., $y^2-\frac{31}{9}y+\frac{961}{324}=\frac{961}{324}-\frac{12}{9}=\frac{961-432}{324}=\frac{529}{324}$.

$$\text{Hence, } y-\frac{31}{18}=\frac{23}{18}; \text{ or } y=\frac{54}{18}=3,$$

$$x=3y=9;$$

and consequently the number is 93.

Ex. 4. $\left.\begin{array}{r}\text{Let } 2x-3y=1\\ 2x^2+xy-5y^2=20\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=3$.

Ex. 5. There are two numbers, such, that if the less be taken from three times the greater, the remainder will be 35; and if four times the greater be divided by three times the less *plus* one, the quotient will be equal to the less number. What are the numbers? Ans. 13 and 4.

Ex. 6 What number is that, the *sum* of whose digits is 15, and if 31 be added to their *product*, the digits will be inverted? Ans. 78.

Case II.

71. When x^2, y^2, or xy, is found in every term of the two equations, they assume the form of

$$ax^2+b\,xy+c\,y^2=d,$$

$$a'x^2+b'xy+c'y^2=d';$$ and their solution may be effected:—as in the following Examples:

Ex. 1.

Let $2x^2+3xy+y^2=20$
$5x^2+4y^2=41$;

Assume $x=vy$, then $2v^2y^2+3vy^2+y^2=20$, or $y^2=\frac{20}{2v^2+3v+1}$,

and $5v^2y^2+4y^2=41$, or $y^2=\frac{41}{5v^2+4}$;

Hence $\frac{20}{2v^2+3v+1}=\frac{41}{5v^2+4}$,

which reduced is, $6v^2-41v=-13$;

$\therefore v^2-\frac{41v}{6}=-\frac{13}{6}$.

By Rule I., $v^2-\frac{41v}{6}+\frac{1681}{144}=\frac{1369}{144}$;

$\therefore v-\frac{41}{12}=\frac{\pm 37}{12}$; or $v=\frac{41\pm 37}{12}=\frac{13}{2}$ or $\frac{1}{3}$.

Let $v=\frac{1}{3}$, then $y^2=\frac{41}{5v^2+4}=\frac{41}{\frac{5}{9}+4}=\frac{369}{41}=9$, or $y=3$,

$x=vy=\frac{1}{3}\times 3=1$.

Ex. 2.

What two numbers are those, whose sum multiplied by the greater is 77? and whose difference multiplied by the less is equal to 12?

Let $x=$ greater number,
$y=$ less.

Then $(x+y)\times x=x^2+xy=77$,
and $(x-y)\times y=xy-y^2=12$.
Assume $x=vy$;

$$\text{Then } \left.\begin{array}{l} v^2y^2+vy^2=77, \\ \text{and } vy^2-y^2=12 \end{array}\right\} \begin{array}{l} \text{or } y^2=\frac{77}{v^2+v}; \\ \text{or } y^2=\frac{12}{v-1}. \end{array}$$

Hence, $\frac{12}{v-1}=\frac{77}{v^2+v}$,

$$\text{or } 12v^2+12v=77v-77;$$

$$\text{which gives } v^2-\frac{65}{12}v=-\frac{77}{12}$$

$$\text{and } v^2-\frac{65}{12}v+\frac{4225}{576}=\frac{529}{576};$$

$$\therefore v=\frac{65\pm23}{24}=\frac{88 \text{ or } 42}{24}=\frac{11}{3} \text{ or } \frac{7}{4}.$$

Either value of v will answer the conditions of the question; but take $v=\frac{7}{4}$; then $y^2=\frac{12}{v-1}=\frac{12}{\frac{7}{4}-1}=\frac{48}{7-4}=\frac{48}{3}=16$,

$$\text{and } y=4,$$

$$x=vy=\frac{7}{4}\times4=7.$$

Hence, the numbers are 4 and 7.

Ex. 3. Find two numbers, such, that the square of the greater *minus* the square of the less may be 56; and the square of the less *plus* $\frac{1}{3}$rd their product may be 40.

Ans. 9 and 5.

Ex. 4. There are two numbers, such, that 3 times the square of the greater *plus* twice the square of the less is 110; and half their product *plus* the square of the less is 4. What are the numbers?* Ans. 6 and 1.

CHAPTER VI.

ON ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSIONS.

72. If a series of quantities increase or decrease by the continual *addition* or *subtraction* of the same quantity, then those quantities are said to be in *Arithmetical* Progression.

* For a great variety of questions relating to quadratic equations which contain two unknown quantities, see Bland's *Algebraical Problems*.

Thus the numbers, 1, 2, 3, 4, 5, 6, &c. (which *increase* by the *addition* of 1 to each successive term), and the numbers 21, 19, 17, 15, 13, 11, &c. (which *decrease* by the *subtraction* of 2 from each successive term), are in arithmetical progression.

73. Ingeneral, if a represents the *first* term of any arithmetical progression, and d the *common difference*, then may the series itself be expressed by a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c., which will evidently be an *increasing* or a *decreasing* one, according as d is *positive* or *negative*.

In the foregoing series, the *coefficient* of d in the *second* term is *one;* in the *third* term it is *two;* in the *fourth* it is *three*, &c., i.e. the coefficient of d in any term is always *less by unity* than the number which denotes *the place of that term in the series.* Hence, if the number of terms in the series be denoted by (n), the nth or *last* term in the progression will be $a+(n-1)\,d$: and, if the nth term be represented by l; then

$$l=a+(n-1)\,d.$$

Ex. 1. Find the 50th term of the series, 1, 3, 5, 7, &c.

$$\text{Here } \left.\begin{aligned} a&=1\\ d&=2\\ n&=50 \end{aligned}\right\} \therefore \begin{aligned} l&=1+(50-1)\,2\\ &=1+49\times 2\\ &=99. \end{aligned}$$

Ex. 2. Find the 12th term of the series 50, 47, 44, &c.

$$\text{Here } \left.\begin{aligned} a&=50\\ d&=-3\\ n&=12 \end{aligned}\right\} \therefore \begin{aligned} l&=50+(12-1)\times -3.\\ &=50-11\times 3\\ &=17. \end{aligned}$$

Ex. 3. Find the 25th term of the series, 5, 8, 11, &c.
Ans. 77.

Ex. 4. ..... 12th ..... 15, 12, 9, &c.
Ans. -18.

Ex. 5. Find 6 arithmetic *means* (or intermediate terms) between 1 and 43.

Here the number of terms is 8, namely, the 6 terms to be inserted, and the 2 given terms, and consequently

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73. What is an *arithmetic progression?* Give an example of a series of quantities in arithmetical progression.

$$\left.\begin{array}{l} a = 1 \\ l = 43 \\ n = 8 \end{array}\right\} \text{But } a + (n-1)\,d = l \quad \therefore\ 1 + 7d = 43;\quad \therefore\ d = 6.$$

And the means required are 7, 13, 19, 25, 31, 37.

Ex. 6. Find 7 arithmetic means between 3 and 59.
Ans. 10, 17, 24, 31, 38, 45, 52.

Ex. 7. Find 8 arithmetic means between 4 and 67.

Ex. 8. Insert 9 arithmetic means between 9 and 109.

74. Let a be the *first term* of a series of quantities in arithmetic progression, d the *common difference*, n the *number of terms*, l the *last term*, and S the *sum of the series:* Then

$$S = a + (a+d) + (a+2d) + \ldots\ldots\ldots + l$$

and, writing this series in a reverse order,

$$S = l + (l-d) + (l-2d) + \ldots\ldots\ldots + a.$$

These two equations being added together, there results

$$2\,S = (a+l) + (a+l) + (a+l) + \ldots\ldots\ldots + (a+l)$$
$$= (a+l) \times n, \text{ since there are } n \text{ terms};$$

$$\therefore\ S = (a+l)\frac{n}{2} \ \ldots\ldots\ldots\ (1).$$

Hence it appears that *the sum of the series is equal to the sum of the first and last terms multiplied by half the number of terms:*

And since $l = a + (n-1)\,d$;

$$\therefore\ S = \left\{2a + (n-1)\,d\right\}\frac{n}{2} \ \ldots\ldots\ldots\ (2).$$

From this equation, any three of the four quantities a, d, n, s, being given, the fourth can be found.

Ex. 1. Find the sum of the series 1, 3, 5, 7, 9, 11, &c. continued to 120 terms.

$$\text{Here } \left.\begin{array}{l} a = 1 \\ d = 2 \\ n = 120 \end{array}\right\} \therefore\ S = \left\{2a + (n-1)d\right\} \times \frac{n}{2}.$$
$$= \left\{2 \times 1 + (120-1)\,2\right\} \times \frac{120}{2}.$$
$$= (2 + 119 \times 2) \times 60 = 240 \times 60 = 14400.$$

Ex. 2. Find the sum of the series 15, 11, 7, 3, -1, -5, &c. to 20 terms.

$$\left.\begin{array}{l}\text{Here } a = 15\\ d = -4\\ n = 20\end{array}\right\} \therefore S = \left\{2a + (n-1)\,d\right\} \times \frac{n}{2}$$
$$= \left\{2 \times 15 + (20-1) \times -4\right\} \times \frac{20}{2}$$
$$= (30 - 19 \times 4) \times 10$$
$$= (30 - 76) \times 10$$
$$= -46 \times 10 = -460.$$

Ex. 3. Find the sum of 25 terms of the series 2, 5, 8, 11, 14, &c. Ans. 950.

Ex. 4. Find the sum of 36 terms of the series 40, 38, 36, 34, &c. Ans. 180.

Ex. 5. Find the sum of 150 terms of the series $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, $\frac{7}{3}$, &c.

$$\left.\begin{array}{l}\text{Here } a = \frac{1}{3}\\ d = \frac{1}{3}\\ n = 150\end{array}\right\} \therefore S = \left\{2a + (n-1)\,d\right\}\frac{n}{2}$$
$$= \left\{2 \times \tfrac{1}{3} + (150-1) \times \tfrac{1}{3}\right\}\frac{150}{2}$$
$$= \left(\frac{2}{3} + \frac{149}{3}\right)75 = \frac{151}{3} \times 75 = 3775.$$

Ex. 6. Find the sum of 32 terms of the series, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, &c. Ans. 280.

PROBLEMS.

Prob. 1. The *sum* of an arithmetic series is 1240, *common difference* -4, and *number of terms* 20. What is the *first term?*

$$\left.\begin{array}{l}\text{Here } S = 1240\\ d = -4\\ n = 20\end{array}\right\} \therefore \quad S = \left\{2a + (n-1)\,d\right\}\frac{n}{2}$$
$$1240 = \left\{2a + (20-1) \times -4\right\}\frac{20}{2}$$
$$= (2a - 19 \times 4)\,10$$
$$124 = 2a - 76;$$
$$\therefore 2a = 124 + 76 = 200,$$
$$\text{and } \therefore a = 100.$$

Hence the series is 100, 96, 92, &c.

PROB. 2. The *sum* of an arithmetic series is 1455, the *first term* 5, and the *number of terms* 30. What is the *common difference?*

$$\left.\begin{array}{l}\text{Here } S=1455\\ a=\quad 5\\ n=\quad 30\end{array}\right\} \therefore \left.\begin{array}{l}\left\{2a+(n-1)d\right\}\frac{n}{2}=S\\ \left\{2\times5+(30-1)\,d\right\}\frac{30}{2}=1455\\ (10+29d)\,15=1455.\end{array}\right.$$

Dividing both sides by 15, $10+29d=97$,

$$29d=87;$$

$$\therefore d=3.$$

Hence the series is 5, 8, 11, 14, &c.

PROB. 3. The *sum* of an arithmetic series is 567, the *first term* 7, the *common difference* 2. Find the *number of terms.*

$$\left.\begin{array}{l}\text{Here } S=567\\ a=\quad 7\\ d=\quad 2\end{array}\right\} \therefore \text{since} \left.\begin{array}{l}\left\{2a+(n-1)\,d\right\}\frac{n}{2}=S\\ \left\{2\times7+(n-1)\,2\right\}\frac{n}{2}=567\\ n^2+6n=567.\end{array}\right.$$

Completing the square, $n^2+6n+9=576$,

Extracting the square root, $x+3=\pm24$;

$$\therefore x=21 \text{ or} -27.$$

PROB. 4. The *sum* of an arithmetic series is 950, the *common difference* 3, and *number of terms* 25. What is the *first term?* Ans. 2.

PROB. 5. The *sum* of an arithmetic series is 165, the *first term* 3, and the *number of terms* 10. What is the *common difference?* Ans. 3.

PROB. 6. The *sum* of an arithmetic series is 440, *first term* 3, and *common difference* 2. What is the *number of terms?* Ans. 20.

PROB. 7. The *sum* of an arithmetic series is 54, the *first term* 14, and *common difference* -2. What is the *number of terms?* Ans. 9 or 6.

PROB. 8. A traveller, bound to a place at the distance of 198 miles, goes 30 miles the *first* day, 28 the *second*, 26 the *third*, and so on. In how many days will he arrive at his journey's end?

Here is given $\left.\begin{array}{l} a = 30 \\ d = -2 \\ S = 198 \end{array}\right\}$ to find the *number of terms*.

$$\text{Now} \left\{2a + (n-1)\,d\right\}\frac{n}{2} = S,$$

$$\therefore \left\{2\times 30 + (n-1)\times -2\right\}\frac{n}{2} = 198,$$

$$(31-n)\,n = 198,$$

$$\text{or, } n^2 - 31n = -198,$$

$$n^2 - 31n + \left(\frac{31}{2}\right)^2 = \frac{961}{4} - 198 = \frac{169}{4}$$

$$n - \frac{31}{2} = \pm\frac{13}{2};$$

$$\therefore n = \frac{31}{2} \pm \frac{13}{2} = 22 \text{ or } 9.$$

To explain the apparent difficulty arising from the two positive values of n, which gives us *two different periods* of the traveller's arrival at his journey's end, we must observe, that if the proposed series 30, 28, 26, &c., be carried to 22 terms, the 16th term will be *nothing*, and the remaining six terms will be *negative;* by which is indicated the *rest* of the traveller on the 16th day, and his *return in the opposite direction* during the six days following; and this will bring him *again*, at the end of the 22d day, to the same point at which he was at the end of the 9th, viz. 198 miles from the place whence he set out.

PROB. 9. How much ground does a person pass over in gathering up 200 stones placed in a straight line, at intervals of 2 feet from each other; supposing that he brings each stone *singly* to a basket standing at the distance of 20 yards from the first stone, and that he starts from the spot where the basket stands?

It is evident that the space passed over by this person will be *twice* the sum of an arithmetic series, whose *first term* is

20 *yards* (i.e. 60 *feet*), *common difference* 2 feet, and *number of terms* 200.

$$\text{Here } \left.\begin{array}{l} a = 60 \\ d = 2 \\ n = 200 \end{array}\right\} \therefore S = \left\{ 2a + (n-1)d \right\} \times \frac{n}{2}.$$
$$= (120 + 398) \times 100.$$
$$= 518 \times 100 = 51800 \text{ feet.}$$

Hence the distance required $=$ 103,600 feet $=$ 19 miles, 4 furlongs, 640 feet.

PROB. 10. A person bought 47 sheep, and gave 1 shilling for the *first* sheep, 3 for the *second*, 5 for the *third*, and so on. What did *all* the sheep cost him? Ans. £110. 9*s*.

PROB. 11. A gentleman began the year by giving away a *farthing* the *first* day, a *halfpenny* the *second*, *three farthings* the *third*, and so on. What money had he disposed of in charity at the end of the year?

Ans. £69. 11*s*. $6\frac{3}{4}$*d*.

PROB. 12. A travels *uniformly* at the rate of 6 miles an hour, and sets off upon his journey 3 hours and 20 minutes before B; B follows him at the rate of 5 miles the *first* hour, 6 the *second*, 7 the *third*, and so on. In how many hours will B overtake A? Ans. In 8 hours.

PROB. 13. There is a certain number of quantities in arithmetic progression, whose *common difference* is 2, and whose *sum* is equal to eight times their *number*; moreover, if 13 be added to the *second* term, and this sum be divided by the *number of terms*, the quotient will be equal to the *first term*. What are the numbers?

$$\left.\begin{array}{l}\text{Let the } \textit{first term} = x, \\ \text{and } \textit{No. of terms} = y;\end{array}\right\} \text{then the } \textit{second} \text{ term will be } x + 2.$$

In the expression $\overline{2a + (n-1)d} \times \frac{n}{2}$, substitute x for a, 2 for b, and y for n, and it becomes $\overline{2x + (y-1)2} \times \frac{y}{2} (= xy + y^2 - y)$, for the *sum* of the series.

By the problem, $xy + y^2 - y = 8y$, or $y = 9 - x$,

$$\text{and } \frac{x + 2 + 13}{y} = x.$$

$$\text{Hence, } \frac{x+2+13}{9-x}=x, \text{ or } x^2-8x=-15;$$

$$\therefore x^2-8x+16=16-15=1,$$

$$\text{and } x-4=\pm 1; \therefore x=5 \text{ or } 3,$$

$$y=9-x \quad =4 \text{ or } 6.$$

From which it appears that there are *two* sets of numbers which will answer the conditions required; viz. 5, 7, 9, 11, or 3, 5, 7, 9, 11, 13.

PROB. 14. There is a certain number of quantities in arithmetic progression, whose *first term* is 2, and whose *sum* is equal to 8 times their number; if 7 be added to the *third* term, and that sum be divided by the number of terms, the quotient will be equal to the *common difference*. What are the numbers? Ans. 2, 5, 8, 11, 14.

ON GEOMETRIC PROGRESSION.

75. If a series of quantities increase or decrease by the continual *multiplication* or *division* by the same quantity, then those quantities are said to be in *Geometrical* Progression. Thus the numbers, 1, 2, 4, 8, 16, &c. (which *increase* by the continual *multiplication* by 2), and the numbers 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. (which *decrease* by the continued *division* by 3, or *multiplication* by $\frac{1}{3}$), are in Geometrical Progression.

76. In general, if a represents the *first term* of such a series, and r the *common multiple* or *ratio*, then may the series itself be represented by a, ar, ar^2, ar^3, ar^4, &c. which will evidently be an *increasing* or *decreasing* series, according as r is a *whole number* or a *proper fraction*. In the foregoing series, the *index of* r in any term is *less by unity* than the number which denotes *the place of that term in the series*. Hence, if the number of terms in the series be denoted by (n), the *last term* will be ar^{n-1}.

77. From the series given in the two preceding articles it is evident by mere inspection, that the *common ratio* can be found by *dividing* the *second* term by the *first*, or *by dividing any term by that which precedes it*.

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75. Define a *geometrical progression*, and give an example.—77. How is the *common ratio* of a series of numbers in geometrical progression found?

Ex. 1. Find the *common ratio* of the geometrical progression 1, 2, 4, 8, &c.

Here the common ratio $=\frac{2}{1}=2$.

Ex. 2. Find the *common ratio* of the series $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}$, &c.

In this example the common ratio $=\frac{4}{9}\div\frac{2}{3}=\frac{2}{3}$.

Ex. 3. Find the *common ratio* in the series $\frac{5}{3}, 1, \frac{3}{5}, \frac{9}{25}$, &c.

Ans. $\frac{3}{5}$.

78. Let S be the sum of the series a, ar^2, ar^3, &c., then

$$a+ar+ar^2+ar^3+ \text{ \&c. } \ldots ar^{n-2}+ar^{n-1} \ldots = S.$$

Multiply the equation by r, and it becomes

$$ar+ar^2+ar^3+ \text{ \&c. } \ldots ar^{n-2}+ar^{n-1}+ar^n=rS.$$

Subtract the *upper* equation from the *lower*, and we have,

$$ar^n-a=rS-S, \text{ or } (r-1)\,S=ar^n-a;$$

$$\text{and therefore, } S=\frac{ar^n-a}{r-1}.$$

If r is a *proper fraction*, then r and *its powers* are less than 1.

For the convenience of calculation, therefore, it is better in this case to transpose the equation into $S=\frac{a-ar^n}{1-r}$, by multiplying the numerator and denominator of the fraction $\frac{ar^n-a}{r-1}$ by -1.

79. If l be the last term of a series of this kind, then $l=ar^{n-1}$, $\therefore rl=ar^n$; hence $S=\left(\frac{ar-a}{r-1}\right)=\frac{rl-a}{r-1}$. From this equation, therefore, if any three of the four quantities S, a, r, l, be given, the fourth may be found.

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78. What is the expression for the sum of n terms of a series of numbers in geometrical progression?

Ex. 1.

Find the sum of the series 1, 3, 9, 27, &c. to 12 terms.

$$\text{Here } \left.\begin{array}{l} a = 1 \\ r = 3 \\ n = 12 \end{array}\right\} \therefore S = \frac{ar^n - a}{r-1} = \frac{1 \times 3^{12} - 1}{3-1}.$$

$$= \frac{81^3 - 1}{2}.$$

$$= \frac{531441 - 1}{2} = \frac{531440}{2} = 265720.$$

Ex. 2.

Find the sum of ten terms of the series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27}$, &c.

$$\left.\begin{array}{l} a = 1 \\ r = \frac{2}{3} \\ n = 10 \end{array}\right\} \therefore S = \frac{a - ar^n}{1-r} = \frac{1 - \left(\frac{2}{3}\right)^{10}}{1 - \frac{2}{3}} = \frac{1 - \left(\frac{2}{3}\right)^{10} \times 3}{3-2} = 1 - \left(\frac{2}{3}\right)^{10} \times 3.$$

$$\text{Now } \left(\frac{2}{3}\right)^{10} = \frac{2^{10}}{3^{10}} = \frac{1024}{59049};$$

$$\therefore 1 - \left(\frac{2}{3}\right)^{10} = 1 - \frac{1024}{59049} = \frac{58025}{59049},$$

$$\text{and } S = \frac{3 \times 58025}{59049} = \frac{174075}{59049}.$$

Ex. 3. Find the sum of 7 terms of the series, 1, 3, 9, 27, 81, &c. Ans. 1093.

Ex. 4. Find the sum of 1, 2, 4, 8, 16, &c. to 14 terms. Ans. 16383.

Ex. 5. Find the sum of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. to 8 terms.

Ans. $\frac{3280}{2187}$.

Ex. 6. Find three geometric means between 2 and 32.

$$\text{Here } \left.\begin{array}{l} a = .2 \\ l = 32 \\ n = 5 \end{array}\right\} \begin{array}{l} \text{And } ar^{n-1} = l \\ \therefore 2r^4 = 32, \\ r^4 = 16, \\ \therefore r = 2. \end{array}$$

And the means required are 4, 8, 16.

Ex. 7. Find two geometric means between 4 and 256.
Ans. 16 and 64.

Ex. 8. Find three geometric means between $\frac{1}{9}$ and 9.
Ans. $\frac{1}{3}$, 1, 3.

Ex. 9. Find a geometric mean between a and l.

Let x be the geometric mean required;

Then a, x, l, are three terms in geometric progression,

$$\text{and } \frac{x}{a}=\frac{l}{x}$$

$$\text{or } x^2=al$$

$$\therefore x=\sqrt{al}.$$

Ex. 10. What is the geometric mean between 16 and 64?
Ans. 32.

Ex. 11. Insert four geometric means between $\frac{1}{3}$ and 81.
Ans. 1, 3, 9, 27.

PROBLEMS.

PROB. 1. Find three numbers in geometric progression, such that their *sum* shall be equal to 7; and the *sum of their squares* to 21.

Let $\frac{x}{2}$, x, xy, be the numbers. Then by the problem,

$$\frac{x}{2}+x+xy \quad =7 \quad \ldots (1)$$

$$\frac{x^2}{y^2}+x^2+x^2y^2=21 \quad \ldots (2)$$

From equation (1), $\quad x\left(\frac{1}{y}+1+y\right) \quad =7$

$\therefore$ by squaring, $\quad x^2\left(\frac{1}{y^2}+\frac{2}{y}+3+2y+y^2\right)=49$

From (2) $\quad x^2\left(\frac{1}{y^2} \quad +1 \quad +y^2\right)=21$

$\therefore$ by subtraction, $\quad x^2\left(\frac{2}{y}+2+2y\right) \quad =28,$

$$\text{or } 14x=28;$$

$$\therefore x=2.$$

This value of x being inserted in (1),

$$\frac{1}{y}+1+y=\frac{7}{2}$$

$$\therefore y^2-\frac{5}{2}y+\left(\frac{5}{2}\right)^2=\frac{25}{16}-1=\frac{9}{16}$$

$$\therefore y=\frac{5\pm3}{4}=2 \text{ or } \frac{1}{2}$$

Hence, the numbers are 1, 2, 4; or 4, 2, 1.

PROB. 2. There are three numbers in geometric progression whose *product* is 64, and *sum* 14. What are the numbers? Ans. 2, 4, 8; or 8, 4, 2.

PROB. 3. There are three numbers in geometric progression whose *sum* is 21, and the *sum of their squares* 189. What are the numbers? Ans. 3, 6, 12.

PROB. 4. There are three numbers in geometric progression; the sum of the *first* and *last* is 52, and the *square of the mean* is 100. What are the numbers?

Ans. 2, 10, 50.

PROB. 5. There are three numbers in geometric progression, whose sum is 31, and the sum of the *first* and *last* is 26. What are the numbers? Ans. 1, 5, 25.

ON THE SUMMATION OF AN INFINITE SERIES OF FRACTIONS IN GEOMETRIC PROGRESSION; AND ON THE METHOD OF FINDING THE VALUE OF CIRCULATING DECIMALS.

79. The general expression for the sum of a geometric series whose common ratio (r) is a *fraction*, is (Art. 78) $S=\frac{a-ar^n}{1-r}$. Suppose now n to be indefinitely *great*, then r^n (r being a proper fraction) will be indefinitely *small*,* so

* When r is a proper fraction, it is evident that r^n *decreases* as n *increases*; let $r=\frac{1}{10}$ for instance, then $r^2=\frac{1}{100}$, $r^3=\frac{1}{1000}$, $r^4=\frac{1}{10000}$, &c., and when n is indefinitely great, the *denominator* of the fraction becomes so large with respect to the *numerator*, that the value of the fraction itself becomes less than any assignable quantity.

that ar^n may be considered as *nothing* with respect to a in the numerator $a-ar^n$ of the fraction expressing the value of S; the *limit*, therefore, to which this value of S approaches, when the number of terms is infinite, is $\frac{a}{1-r}$.

Ex. 1.

Find the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, &c. *ad infinitum.*

$$\left.\begin{array}{l}\text{Here } a=1\\ r=\frac{1}{2}\end{array}\right\} \therefore S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=\frac{2}{2-1}=2.$$

Ex. 2.

Find the value of $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+$ &c. *ad infinitum.*

$$\left.\begin{array}{l}\text{Here } a=\frac{1}{5}\\ r=\frac{1}{5}\end{array}\right\} \therefore S=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{5-1}=\frac{1}{4}.$$

Ex. 3. Find the value of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ &c. *ad infinitum.*

Ans. $\frac{3}{2}$.

Ex. 4. ..... $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+$ &c. *ad infinitum.*

Ans. 4.

Ex. 5. ..... $\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+$ &c. *ad infinitum.*

Ans. $\frac{2}{3}$.

80. These operations furnish us with an expeditious method of finding the value of *circulating decimals*, the numbers composing which are geometric progressions, whose

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80. What is the expression for the sum of a geometric series, when the number of terms is infinite?

common ratios are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. according to the number of factors contained in the *repeating* decimal.

Ex. 1.

Find the value of the circulating decimal .33333, &c. This decimal is represented by the geometric series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ &c. whose *first term* is $\frac{3}{10}$, and *common ratio* $\frac{1}{10}$

$$\left.\begin{array}{l}\text{Hence } a=\frac{3}{10},\\ \qquad\quad r=\frac{1}{10},\end{array}\right\} \therefore S=\frac{a}{1-r}=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3}.$$

Ex. 2. Find the value of .32323232, &c. *ad infinitum.*

$$\left.\begin{array}{l}\text{Here } a=\frac{32}{100}\\ \qquad\quad r=\frac{1}{100}\end{array}\right\} \begin{array}{l}\frac{3}{10}+\frac{2}{100}+\frac{3}{1000}+ \text{ &c.} .32\\ \therefore S=\frac{a}{1-r}=\frac{\frac{32}{100}}{1-\frac{1}{100}}=\frac{32}{100-1}=\frac{32}{99}.\end{array}$$

Ex. 3. Find the value of .713333, &c. *ad infinitum.*

The series of fractions representing the value of this decimal are $\frac{71}{100}+$ (geometric series) $\frac{3}{1000}+\frac{3}{10000}+$ &c. $=\frac{71}{100}+S$. or $\frac{7}{10}+\frac{1}{100}+\frac{3}{1000}+\cancel{\frac{3}{10000}}+$ &c.

$$\left.\begin{array}{l}\text{Here } a=\frac{3}{1000}\\ \qquad\quad r=\frac{1}{10}\end{array}\right\} \therefore S=\frac{\frac{3}{1000}}{1-\frac{1}{10}}=\frac{3}{1000-100}=\frac{3}{900}=\frac{1}{300}.$$

Hence the value of the decimal $=\left(\frac{71}{100}+S\right)\frac{71}{100}+\frac{1}{300}=\frac{214}{300}=$ $=\frac{107}{150}$.

$$=\left(\frac{71}{100}+S\right)=\frac{71}{100}+\frac{1}{300}=\frac{213}{300}+\frac{1}{300}=\frac{214}{300}=\frac{107}{150}$$

$\frac{8}{10}+\frac{1}{100}+\frac{3}{1000}+\frac{4}{10000}+\frac{3}{100000}+\frac{4}{1000000}+\&c.$

Ex. 4. Find the value of .81343434, &c. *ad infinitum.*

$$\text{Here } \left.\begin{array}{l} a=\frac{34}{10000} \\ r=\frac{1}{100} \end{array}\right\} \therefore S=\frac{a}{1-r}=\frac{\frac{34}{10000}}{1-\frac{1}{100}}=\frac{34}{10000-100}=\frac{34}{9900}.$$

$$\text{And value of the decimal}=\frac{81}{100}+S=\frac{81}{100}+\frac{34}{9900}=\frac{8053}{9900}.$$

Ex. 5. Find the value of .77777, &c. *ad infinitum.*

$\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{7}{10000}+\&c.$ Ans. $\frac{7}{9}$.

Ex. 6. ..... .232323, &c. *ad infinitum.*

Ans. $\frac{23}{99}$.

Ex. 7. ..... .83333, &c. *ad infinitum.*

Ans. $\frac{5}{6}$.

Ex. 8. ..... .7141414, &c. *ad infinitum.*

Ans. $\frac{707}{990}$.

Ex. 9. ..... .956666, &c. *ad infinitum.*

Ans. $\frac{287}{300}$.

Here all the worked examples of these

The value of a *circulating decimal* may also be found as follows:—In Ex. 4 above,

$$\begin{array}{rl} \text{Let } S= & .813434\ldots\ldots \\ \therefore 10000\,S= & 8134.3434\ldots\ldots \\ \text{and } 100\,S= & 81.3434\ldots\ldots \\ \hline \therefore 9900\,S= & 8053 \end{array}$$

$$\therefore S=\frac{8053}{9900}, \text{ as before.}$$

Prob. 1. A body in motion moves over 1 mile the *first* second, but being acted upon by some retarding cause, it only moves over $\frac{1}{2}$ a mile the *second* second, $\frac{1}{4}$ the *third*,

and so on. Show that, according to this law of motion, the body, though it move on to *all eternity*, will never pass over a space greater than 2 miles.

ON HARMONIC PROGRESSION.

81. A series of quantities, whose *reciprocals* are in arithmetic progression, are said to be in *Harmonic Progression*. Thus the numbers 2, 3, 6, are in *harmonic progression* since their reciprocals $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, are in *arithmetic* progression ($-\frac{1}{3}$ being the *common difference*).

Ex. 1. Find a harmonic mean between 1 and $\frac{2}{3}$.

Let x be the *mean* required:

Then $1, \frac{1}{x}, \frac{3}{2}$, are in arithmetic progression,

$$\text{And } \frac{1}{x}-1=\frac{3}{2}-\frac{1}{x}$$

$$\therefore \frac{2}{x}=1+\frac{3}{2}$$

$$=\frac{5}{2}$$

$$\therefore x=\frac{4}{5}.$$

Obs. To find the com. diff. (or) in any Arith. Series, we must subtract any term from the term following, as let the Series 2.5.8. 11.14. 17. 20. 23. be in AP then $5-2=11-8$ & $17-14=23-2$ or $23-20=5-2$ & $11-8=17-14$ either of which will give the value of (d). $\therefore \frac{1}{x}-1=\frac{3}{2}-\frac{1}{x}$.

Ex. 2. Find a third number to be in harmonic progression with 6 and 4.

Let x be the number required:

Then $\frac{1}{6}, \frac{1}{4}, \frac{1}{x}$, are in arithmetic progression.

$$\text{And } \frac{1}{4}-\frac{1}{6}=\frac{1}{x}-\frac{1}{4}$$

$$\therefore \frac{1}{x}=\frac{1}{2}-\frac{1}{6}$$

$$=\frac{3}{6}-\frac{1}{6}$$

$$=\frac{1}{3}$$

$$\therefore x=3$$

either which will give the value of (d).

Ex. 3. Insert three harmonic *means* between 9 and 3.

The *reciprocals* of 9 and 3 are $\frac{1}{9}$ and $\frac{1}{3}$, which are the *first* and last term of an *arithmetic* progression, between which 3 *arithmetic means* are to be inserted. We have therefore, according to Art. 73—

$$\left.\begin{array}{l} a=\frac{1}{9} \\ l=\frac{1}{3} \\ x=5 \end{array}\right\} \text{And } a+(n-1)\,d=l,$$

$$\therefore \frac{1}{9}+(5-1)\,d=\frac{1}{3}$$

$$4d=\frac{1}{3}-\frac{1}{9}$$

$$=\frac{3}{9}-\frac{1}{9}$$

$$=\frac{2}{9}$$

$$\therefore d=\frac{1}{18}.$$

Hence, $\frac{1}{6}, \frac{2}{9}, \frac{5}{18}$, are the arithmetic *means* to be inserted between $\frac{1}{9}$ and $\frac{1}{3}$, and therefore *their reciprocals* 6, $\frac{9}{2}, \frac{18}{5}$, are the three harmonic *means* required.

Ex. 4. Find a harmonic mean between 12 and 6.
Ans. 8.

Ex. 5. The numbers 4 and 6 are two terms of a harmonic progression; find a third term. Ans. 12.

Ex. 6. Find two harmonic means between 84 and 56.
Ans. 72 and 63.

Ex. 7. Insert three harmonic means between 15 and 3.
Ans. $\frac{15}{2}$, 5, $\frac{15}{4}$.

Read this particularly

82. Let a, b, c, d, e, &c. be a series of quantities in *harmonic* progression; then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c. are in *arith-*

metic progression, and according to the definition of an *arithmetic* progression (Art. 72), we have

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b} \quad \ldots\ldots (1)$$

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{d}-\frac{1}{c} \quad \ldots\ldots (2)$$

$$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d} \quad \ldots\ldots (3)$$

$$\&c. = \&c.$$

From (1) $$\frac{a-b}{ab}=\frac{b-c}{bc}$$

$$\frac{a-b}{a}=\frac{b-c}{c}$$ or $c(a-b)=a(b-c)$

hence $\frac{a-b}{b-c}=\frac{a}{c}$

$$\therefore \quad \frac{a}{c}=\frac{a-b}{b-c}.$$

or, converting this equation into a proportion,

$$a : c :: a-b : b-c$$

Similarly from (2) $\quad b : d :: b-c : c-d$

..... (3) $\quad c : e :: c-d : d-e$ get this up by hea

and so on for any number of quantities.

this These proportions are frequently assumed as the *definition* to quantities in *harmonic* progression, and may be thus expressed in words:—if any *three* quantities in *harmonic* progression be taken, *the first is to the third as the difference between the first and second is to the difference between the second and third.*

PROB. 1. Given $a^x=b^y=c^z$, where a, b, c, are in geometric progression. Prove that x, y, z, are in harmonic progression.

$$a^x=b^y; \quad \therefore a=b^{\frac{y}{x}} \quad \ldots\ldots\ldots (1).$$

$$c^z=b^y; \quad \therefore c=b^{\frac{y}{z}} \quad \ldots\ldots\ldots (2).$$

By multiplying (1) by (2) $ac=b^{\frac{y}{x}+\frac{y}{z}}$

But by geometric progression $ac = b^2$

$$\therefore b^2 = b^{\frac{y}{x}+\frac{y}{z}};$$

$$\text{and } \therefore 2 = \frac{y}{x} + \frac{y}{z},$$

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$$\text{or } \frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y}.$$

CHAPTER VII.

ON PERMUTATIONS AND COMBINATIONS.

83. By *Permutations* are meant the number of *changes* which any quantities a, b, c, d, e, &c., can undergo with respect to their order, when taken *two and two* together, *three and three*, &c. &c. Thus ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc, are the different permutations of the *four* quantities a, b, c, d, when taken *two and two* together; abc, acb, bac, bca, cab, cba, of the *three* quantities a, b, c, when taken *three and three* together, &c. &c.

84. Let there be n quantities, a, b, c, d, e, &c.: then, by Art. 83, it appears that there will be $(n-1)$ permutations in which a stands first; for the same reason there will be $(n-1)$ permutations in which b stands first; and so of c, d, e, &c. Hence there will be n times $(n-1)$ permutations of the form ab, ac, ad, ae, &c.; ba, bc, bd, be, &c.; ca, cb, cd, ce, &c.; i.e. "*the number of permutations of* n *things taken two and two is* $n\,(n-1)$."

85. If these n quantities be taken *three and three* together, then there will be $n\,(n-1)\times(n-2)$ *permutations*. For if $(n-1)$ be substituted for n in the last article, then the number of permutations of $n-1$ things taken *two and two* together will be $(n-1)\times(n-2)$; hence the number of permutations of b, c, d, e, &c. taken *two and two* together,

are $(n-1)\times(n-2)$, and consequently there are $(n-1)\times(n-2)$ permutations of the quantities a, b, c, d, e, &c. taken *three and three* together, in which a may stand first; for the same reason there are $(n-1)(n-2)$ permutations in which b may stand first; and so of c, d, e, &c. The numbers of the permutations of this kind will therefore amount to $n\ (n-1)\ (n-2)$.

86. *To find the number of permutations of* n *things taken* r *together.*

By Arts. 85 and 86—

The No. taken *two* together $= n\ (n-1)$

..... *three* ..... $= n\ (n-1)\times(n-2)$

Similarly..... *four* ..... $= n\ (n-1)\times(n-2)\times(n-3)$

If the law, which is observed in these particular cases, be supposed to hold generally, that is, if the number of permutations of n things a, b, c, d, &c. taken $r-1$ together, be

$$n\ (n-1)\ (n-2)\ \ldots\ldots\ (n-r+2)$$

Then, by omitting a, it is equally true that the number of permutations of $n-1$ things b, c, d, &c. taken $r-1$ together, will be by putting $n-1$ for n in this last expression

$$(n-1)\ (n-2)\ \ldots\ldots\ (n-r+1)$$

Now, if a be placed before each of these permutations, there will be

$$(n-1)\ (n-2)\ \ldots\ldots\ (n-r+1)$$

permutations of things taken r together, in which a stands *first*. It is clear that there will in like manner be the same number of permutations of things taken r together, in which each of the other things b, c, d, &c. stand *respectively first;* and as there are n things the entire number of permutations of n things taken r together, will be the sum of the permutations taken r together in which the n things a, b, c, d, &c. respectively stand first, that is, n times $(n-1)$ $(n-2)$ ..... $(n-r+1)$,

$$\text{or } n\ (n-1)\ (n-2)\ \ldots\ldots\ (n-r+1)$$

It has thus been proved, that if the law by which the expression for the number of permutations of n things taken $r-1$ together is found, be true, it is also true for the next superior number, or when n things are taken r together; but the law of the expression has been found to hold for

the number of permutations of n things taken *two* together, and for the number of permutations of n things taken *three* together; it is therefore true by the theorem just demonstrated when the n things are taken *four* together, and if true when taken *four* together, it is true also when taken *five* together, and so on for any number not greater than n, which may be taken together.

This proof affords an excellent example of *demonstrative induction*, a method of reasoning of great importance in the mathematical sciences.

87. If $r=n$, i.e. if the permutations respect all the quantities at once, then (since $n-r=0$) the *number* of them will be $n(n-1)(n-2)$ &c.....2.1. Thus, the number of permutations which might be formed from the letters composing the word "*virtue*" are $6\times5\times4\times3\times2\times1=720$.

88. But if the *same* letter should occur any number of times, then it is evident that we must *divide* the whole number of permutations by the number of permutations which would have arisen if *different* letters had occurred instead of the repetition of the same letter. Thus if the same letter should occur *twice*, then we must divide by 2×1; if it should occur *thrice*, we must divide by $3\times2\times1$; if p times, by $1.2.3...p$; and so for any other letter which may occur more than once. Hence the general expression for the number of permutations of n things, of which there are p of *one* kind, r of *another*, q of *another*, &c., &c. is $\frac{n(n-1)(n-2)(n-3)....2.1}{1.2.3..p\times1.2.3..r\times1.2.3..q}$. Thus the permutations which may be formed by the letters composing the word "*easiness*" (since s occurs *thrice*, e *twice*) are $\frac{8.7.6.5.4.3.2.1}{1.2.3.\times1.2}=3360$.

Ex. 1. What is the number of different arrangements which can be made of 6 persons at a dinner table?

$$\text{The number}=1\times2\times3\times4\times5\times6=720.$$

Ex. 2. Required the number of changes which can be rung upon 8 bells.

$$\text{The No. of changes}=1\times2\times3\times4\times5\times6\times7\times8=40320.$$

Ex. 3. With 5 flags of different colours, how many *signals* can be made?

The number of signals, when the *flags* are taken,

Singly, are	=	5
Two together ... = 5 . 4	=	20
Three ..... = 5 . 4 . 3	=	60
Four ..... = 5 . 4 . 3 . 2	=	120
Five ..... = 5 . 4 . 3 . 2 . 1 ...	=	120
∴ the total number of signals	=	325

Ex. 4. How many permutations can be formed out of 10 letters, taken 5 at a time? Ans. 30240.

Ex. 5. How many permutations can be formed out of the words *Algebra* and *Missippi* respectively, all the letters being taken at once? Ans. 2520 and 1680.

ON COMBINATIONS.

89. By *Combinations* are meant the number of *collections* which can be formed out of the quantities, a, b, c, d, e, &c., taken *two and two* together, *three and three* together, &c. &c., without having regard to the *order* in which the quantities are arranged in each collection. Thus ab, ac, ad, bc, bd, cd, are the *combinations* which can be formed out of the *four* quantities a, b, c, d, taken *two and two* together; abc, abd, acd, bcd, the combinations which may be formed out of the same quantities, when taken *three and three* together; &c. &c.

90. From the expression (in Art. 86) for finding the number of *permutations* of n things taken r *and* r together, we immediately deduce the theorem for finding the number of *combinations* of n things taken in the same manner. For the permutations of n things taken *two and two* together being $n(n-1)$, and as each *combination* admits of as many *permutations* as may be made by *two* things (which is 2×1), the number of *combinations* must be equal to the number of *permutations* divided by 2; i.e. the number of *combinations* of n things taken *two and two* together is $\frac{n(n-1)}{2}$. For

the same reason, the *combinations* of n things, taken *three and three* together, must be equal to $\frac{n(n-1)(n-2)}{1.2.3}$; and in general, the combinations of n things taken r *and* r together must be equal to $\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$.

Ex. 1. Find the number of combinations which can be formed out of 8 letters, when taken 5 at a time.

$$\text{The number} = \frac{8\times7\times6\times5\times4}{1\times2\times3\times4\times5} = 56.$$

Ex. 2. What is the total number of combinations which can be formed out of 6 colours taken in every possible way?

No. of combinations when the colours are taken—

1 at a time ..... = 6

2 ..... $= \frac{6.5}{1.2}$ ..... = 15

3 ..... $= \frac{6.5.4}{1.2.3}$ ..... = 20

4 ..... $= \frac{6.5.4.3}{1.2.3.4}$ ... = 15

5 ..... $= \frac{6.5.4.3.2}{1.2.3.4.5}$ ... = 6

6 ..... $= \frac{6.5.4.3.2.1}{1.2.3.4.5.6}$ = 1

Hence the total number = 63

Ex. 3. Find how many different combinations of 8 letters, taken in every possible way, can be made.

Ans. 255.

*** Several other useful and interesting subjects of an elementary character yet remain to be treated of. The Editor is preparing for publication a Second Part, which will embrace these subjects.

THE END.

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