ELEMENTARY TREATISE

ON

MECHANICS,

FOR THE USE OF

COLLEGES AND SCHOOLS OF SCIENCE.

BY

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PREFACE.

THE following Treatise was originally prepared to supply a want felt by the compiler, whilst engaged in teaching Natural Philosophy to college classes. It is now proposed to introduce into it, in a simplified form, the results of many years' experience in its use as a text-book. To accomplish this, the entire book has been rewritten, the descriptive matter condensed, the demonstrations simplified, and the practical scope of the work extended; but in no instance has any essential principle been omitted. The most important, if not the only, change in the *plan* of the work is the omission of the Calculus. This change has been made, to cause the work to conform more closely to the original design, which was, to produce a book that should form a suitable connecting link, between purely popular works, on the one hand, and those of the highest grade, on the other. In most of our Colleges, the Calculus is either not taught at all, or else its study is made optional, and pursued without reference to its use as a *tool* for scientific investigation. The change referred to, brings this edition of the work within the range of the College Curriculum, and it is hoped does not impair its value as a text-book for Schools of Science. As modified, it embraces all the elementary propositions of Mechanics, arranged in logical order, rigidly demonstrated and fully illustrated by practical examples; its scope, sufficiently extended to meet the wants of Colleges and Schools of Science; its treatment, so simple that it may be read with profit by those who have not the leisure to make the mathematical sciences a specialty; and its plan, such as to render it a suitable introduction to those higher treatises on Mechanical Philosophy, that all must read who would appreciate and keep pace with the discoveries of modern science.

COLUMBIA COLLEGE,
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CONTENTS.

CHAPTER I.

DEFINITIONS AND INTRODUCTORY REMARKS.

CHAPTER II.

COMPOSITION, RESOLUTION, AND EQUILIBRIUM OF FORCES.

CHAPTER III.

CENTRE OF GRAVITY AND STABILITY.

CHAPTER IV.

ELEMENTARY MACHINES.

CHAPTER V.

RECTILINEAR AND PERIODIC MOTION.

CHAPTER VI.

CURVILINEAR AND ROTARY MOTION.

CHAPTER VII.

MECHANICS OF LIQUIDS.

CHAPTER VIII.

MECHANICS OF GASES AND VAPORS.

CHAPTER IX.

HYDRAULIC AND PNEUMATIC MACHINES.

PRIME MOVERS.

MECHANICS.

CHAPTER I.

DEFINITIONS AND INTRODUCTORY REMARKS.

Definition of a Body.

1. A BODY is a collection of material particles. A body whose dimensions are exceedingly small is called a *material point.*

Rest and Motion.

2. A material point is *at rest,* when it retains a fixed position in space; it is *in motion,* when its position in space is continually changing.

Rest and motion, with respect to surrounding objects, are called *relative rest* and *relative motion,* to distinguish them from *absolute rest* and *absolute motion.*

Rectilinear and Curvilinear Motion.

3. The path traced out by a moving point is called its *trajectory.* When this is a straight line, the motion is *rectilinear;* when it is a curve, the motion is *curvilinear.*

When the motion of a *body* is spoken of as rectilinear or curvilinear, it is understood that some particular *point* *of the body* is referred to, such as the centre of gravity, or the centre of figure.

Motion of Translation and Rotation.

4. A body has a *motion of translation* when all its points move in parallel straight lines; it has a *motion of rotation* when its points move in arcs of circles having their centres in the same line: this line is the *axis of rotation.* All other varieties of motion result from some combination of these two.

Uniform and Varied Motion.

5. The VELOCITY of a point is its rate of motion. When it moves over equal spaces in equal times, the velocity is *constant* and the motion *uniform;* when it moves over unequal spaces in equal times, the velocity is *variable* and the motion *varied.* If the velocity continually increase, the motion is *accelerated;* if it continually decrease, the motion is *retarded.*

In uniform motion, the space passed over in one second is taken as the measure of the velocity. In varied motion, the velocity at any instant is measured by the space that *would* be passed over in a second were the velocity to remain the same as at that instant.

Definition of a Force.

6. A FORCE is anything that tends to change the state of a body with respect to rest or motion. If a body is at rest, anything that tends to put it in motion is a force; if a body is in motion, anything that tends to change either its direction or its rate of motion, is a force.

Classification of Forces.

7. Forces may be divided into two classes, extraneous and molecular: *extraneous forces* act on bodies from without; *molecular forces* are exerted between the neighboring particles of bodies.

Extraneous Forces.

8. EXTRANEOUS FORCES are of two kinds, pressures and moving forces: *pressures* simply *tend* to produce motion; *moving forces* actually produce motion. Thus, if gravity act on a fixed body, it creates pressure; if on a free body, it produces motion.

Moving forces are either impulsive or incessant: *an impulsive force,* or *an impulse,* is one that acts for a moment and then ceases; *an incessant force* is one that acts continuously. We may regard an incessant force as a succession of impulses, imparted at equal, but exceedingly small intervals of time. When the elementary impulses are equal, the force is *constant;* when they are unequal, the force is *variable.* Thus, gravity, at any place, is a constant force; the effort of expanding steam is a variable force.

Molecular Forces.

9. MOLECULAR FORCES are of two kinds, attractive and repellent: *attractive forces* tend to bind the particles of a body together; *repellent forces* tend to thrust them asunder. Both kinds of molecular forces are continually exerted between the molecules of bodies, and on the predominance of one or the other depends the physical state of a body.

Constitution and Classification of Bodies.

10. It is generally believed that matter, in its ultimate form, consists of minute, indivisible, and indestructible parts, called *atoms.* These are grouped in various ways, under the action of molecular forces, to form *molecules,* or *particles;* and these again are united to form larger bodies. The relations that exist between the molecular forces, in different cases, form a basis for the classification of bodies: they are divided into two classes, *solids* and *fluids;* and fluids are again divided into *liquids* and *gases.* In solids, the molecular forces of attraction prevail over those of repulsion; in liquids, they are nearly balanced; in gases, the forces of repulsion prevail over those of attraction. In solids, the particles adhere so as to require some force to separate them; in fluids, the particles move freely amongst each other, yielding to the slightest force. *Solids* tend to preserve both their shape and volume; *liquids* tend to preserve their volume, but take the shape of the containing vessel; *gases* have no tendency to retain either their volume or their shape. Many bodies are capable of existing in different states, according to temperature. Thus, *ice, water,* and *steam* are the same body in different states.

Essential Properties of Bodies.

11. There are certain properties common to all bodies, and without which we could not conceive them to exist: these are *extension, impenetrability,* and *inertia.*

EXTENSION is that property by virtue of which a body occupies a portion of space. Every body has length, breadth, and thickness.

IMPENETRABILITY is that property by virtue of which no two bodies can occupy the same space at the same time. The particles of one body may be thrust aside by those of another, as when a nail is driven into wood; but where one body is, no other body can be.

INERTIA is that property of a body by virtue of which it tends to continue in the state of rest or motion in which it may be placed, until acted on by some force.

Matter has no power to change its state with respect to rest or motion; if at rest, it cannot set itself in motion; or, if moving, it cannot change either the rate or the direction of its motion. If a force act on a body to change its state of rest or motion, it develops a resistance that acts in a contrary direction. This resistance is called the *force of inertia.* The force that a moving body possesses and is capable of giving out, when its motion is opposed, is called *living force.*

Laws of Motion.

12. The laws of motion, commonly known as the Newtonian Laws, depend on the principle of inertia. They may be enunciated as follows:

1*st Law. If a body be at rest, it will remain at rest; or if in motion, it will move uniformly in a straight line, till acted on by some force.*

2*d Law. If a body be acted on by several forces, it will obey each as though the others did not exist, and this whether the body be at rest or in motion.*

3*d Law. If a force act to change the state of a body with respect to rest or motion, the body will offer a resistance equal and directly opposed to the force.*

These laws are deduced from universal experience, and are accepted as axiomatic in treating of the motion of bodies.

Secondary Properties of Bodies.

13. Besides the properties common to all bodies, there are other properties possessed in a greater or less degree by different bodies, that may be called *secondary.* Of these, the most important, from a mechanical point of view, are *porosity, compressibility, dilatability,* and *elasticity,* all of which arise from peculiarity of atomic constitution.

POROSITY is that property by virtue of which the particles of a body are more or less separated. The intermediate spaces are called *pores.* When the pores are small, the body is *dense;* when they are large, it is *rare.* Gold is a dense body, hydrogen a rare one. It is to be observed that the *interatomic* spaces, which are properly called pores, are regularly distributed throughout the body, and should not be confounded with those irregular spaces that may be called *cavities* or *cells,* examples of which may be seen when a loaf of bread is cut across.

COMPRESSIBILITY is that property by virtue of which the particles of a body may be made to approach each other, so as to occupy less space.

DILATABILITY is that property by virtue of which the particles of a body may be separated to a greater distance, so as to occupy more space.

ELASTICITY is that property by virtue of which a body tends to resume its original form, or volume, after compression or extension. The effort that a body exerts to return to its original form or volume after distortion, is called the force of restitution; and when this is very great in comparison with the force of distortion, the body is highly elastic. Ivory is an example of a highly elastic body; clay is very inelastic. Within certain limits most bodies may be considered as elastic,—that is, if they be slightly distorted, they will completely recover their original shape, or volume, on the removal of the force of distortion.

Force of Gravity and Weight.

14. Observation shows that the earth exercises an attractive force on bodies, tending to draw them toward its centre. This force is called the *force of gravity.* It acts on every particle, and if the body be supported, it produces a pressure proportional to the quantity of matter in it; this pressure is called the *weight* of the body.

Newton showed that terrestrial gravity is only a particular manifestation of a general law, which certainly prevails throughout the solar system, and probably throughout the physical universe. This law, sometimes called *the Newtonian law of universal gravitation*, may be enunciated as follows:

Every particle of matter attracts every other particle, with a force that varies directly as the mass of the attracting particle, and inversely as the square of the distance between the particles.

It has also been shown that the attraction of the earth on bodies exterior to it, is very nearly the same as though all its matter were concentrated at its centre. Because the form of the earth is that of an oblate spheroid, having its axis coincident with that of revolution, the force of gravity increases slighty in passing from the equator toward the pole. The weight of a body must therefore increase at the same rate. That this increase of weight may be rendered apparent, the weighing must be performed by a spring balance, or some equivalent method, for, were the ordinary balance used, the increased weight of the body would be accompanied by a like increase in the weight of the counterpoise.

Mass and Density.

15. The MASS of a body is the quantity of matter it contains. We have seen that the weight of a body increases at the same rate as the force of gravity; hence the quotient obtained by dividing the weight at any place by the force of gravity at that place is *constant*. This quotient is always proportional to the quantity of matter in the body, and for this reason is taken as the measure of its mass. Denoting the mass of a body by M, its weight by W, and the force of gravity by g, we have,

$$M = \frac{W}{g}; \text{ whence, } W = Mg \; . \; . \; . \; . \; . \; (1)$$

The DENSITY of a body is the degree of compactness of its particles. It is proportional to the quantity of matter in a given volume. We may take, as the measure of a body's density, the quotient of its mass by its volume; or, denoting the density by D, the volume by V, and the mass by M, we have,

$$D = \frac{M}{V}.$$

Combining this with equation (1), we find,

$$D = \frac{W}{Vg}; \text{ whence, } W = DVg \; . \; . \; . \; . \; . \; (2)$$

Formulas (1) and (2) are of frequent use in Mechanics.

The quantity of matter that weighs one pound is taken as the *unit of mass*. The density of distilled water at 39° Fah. is taken as the *unit of density*.

Momentum or Quantity of Motion.

16. The MOMENTUM, or the QUANTITY OF MOTION of a body, is the product of its mass by its velocity. If a force act to impart motion, it is obvious that the force must in the *first* place be proportional to the mass moved; and in the *second* place, to the velocity it can impart. It is in accordance with this principle that *momentum*, or *quantity of motion*, is used as a measure of force.

Measure of Forces.

17. A force is measured by comparing it with some other force of the same kind taken as a *unit*. There are two kinds of forces—pressures and moving forces; and consequently two kinds of units.

The *unit of pressure* is one pound; when we speak of a pressure of n pounds, we mean a force that would, if directed vertically upward, just sustain a weight of n pounds.

The *unit of an impulsive force* is an impulse capable of imparting a unit of velocity to a unit of mass; that is, an impulse capable of generating a unit of momentum.

An impulsive force is measured by the quantity of motion it can generate. If an impulse f impart a velocity v to a mass m, we have,

$$f = mv \quad . \; . \; . \; . \; . \quad (3)$$

Impulses acting on the same or on equal masses, are proportional to the velocities they impart.

The *unit of a constant force* is a constant force capable of generating a unit of momentum in a unit of time.

A constant force is measured by the quantity of motion it can generate in a unit of time. If a constant force f generate a quantity of motion equal to mv in a unit of time, we have,

$$f = mv \quad . \; . \; . \; . \; . \quad (4)$$

Constant forces acting on equal masses are proportional to the velocities they generate in the same time.

We have seen that an incessant force may be regarded as a succession of impulses, imparted at *equal* intervals of time (Art. 8); hence, constant forces are proportional to their *elementary* impulses.

Variable forces have different values at different times.

The measure of such a force, at any instant, is the quantity of motion it could generate in a unit of time, if its intensity were to remain unchanged for that time. The values of variable forces at different times are proportional to their elementary impulses at those times.

Acceleration due to a Force.

18. The *velocity* that a constant force can generate in a body in a unit of time, is called the *acceleration due to the force.* If we find the value of v, in equation (4) of the last article, we have,

$$v = \frac{f}{m} \quad . \; . \; . \; . \; . \; (5)$$

That is, *the acceleration is equal to the moving force, divided by the mass moved.*

If the acceleration is known, the moving force may be found by multiplying *the acceleration* by *the mass.* In some cases the force acts independently on each particle; the acceleration is then independent of the mass. The force of gravity is an example in which the acceleration is independent of the mass.

Representation of Forces.

19. Forces may be represented geometrically by straight lines, proportional to the forces. A force is given when we know its *intensity*, its *point of application*, and *the direction in which it acts.* When a force is represented by a line, the length of the line represents its intensity; one extremity represents the point of application; and an arrow-head at the other extremity shows the direction of the force. Thus, in Figure 1, OP is the intensity of the force; O, its

Fig. 1.

point of application; and OP, the direction in which it acts. If a force be applied to a solid body, the point of application may be taken anywhere on its line of direction; and it is often found convenient to transfer it from one point of this line to another. The line OP prolonged indefinitely is called the *line of action* of the force OP.

A force may be represented, analytically, by a letter; thus the force OP may be called the force P. In this case we assume the usual algebraic rule for estimating quantities; that is, if a quantity in one sense is *positive*, a quantity in an opposite sense must be *negative*.

Equilibrium.

20. Forces are in *equilibrium* when they balance each other. If a system of forces in equilibrium be applied to a body, they will not change its state with respect to rest or motion: if the body be at rest, it will remain so; if in motion, its motion will remain unchanged, so far as these forces are concerned.

When forces balance each other through the medium of a body at rest, they are said to be in *statical equilibrium;* when they balance each other through the medium of a moving body, they are in *dynamical equilibrium.*

If a body be at rest, or if in uniform motion, we conclude that the forces acting on it are in equilibrium.

Definition of Mechanics.

21. MECHANICS is the science that treats of the action of forces on bodies.

It treats of the laws of equilibrium and motion, and is sometimes divided into two branches, called *Statics* and *Dynamics*. *Statics* treats of pressures; *Dynamics*, of moving forces: when the bodies acted on are liquids, these branches are called *hydrostatics* and *hydrodynamics;* when the bodies acted on are gases, they are called *aerostatics* and *aerodynamics.*

A better division of the subject is into *mechanics of solids* and *mechanics of fluids.*

CHAPTER II.

COMPOSITION, RESOLUTION, AND EQUILIBRIUM OF FORCES.

Definition.

22. *Composition of forces,* is the operation of finding a single force whose effect is the same as that of two or more given forces. The required force is called the *resultant* of the given forces.

Resolution of forces, is the operation of finding two or more forces whose combined effect is equivalent to that of a given force. The required forces are called *components* of the given force.

Composition of Forces whose directions coincide.

23. From the rules laid down for measuring forces, it follows, that the resultant of two forces applied at a point, and acting in the same direction, is equal to the sum of the forces. If two forces act in opposite directions, their resultant is equal to their difference, and it acts in the direction of the greater.

If any number of forces be applied at a point, some in one direction, and others in a contrary direction, their resultant is equal to the sum of those that act in one direction, diminished by the sum of those that act in the opposite direction; or, if we regard the rule for signs, the resultant is equal to the *algebraic sum of the components;* the sign of this sum indicates the direction in which the resultant acts. Thus, if the forces P, P', &c., act on a point, and in a direction that we may assume as *positive*, whilst the forces P'', P''', &c., act on the same point and in the opposite direction, then will their resultant, denoted by R, be given by the equation,

$$R = (P + P' + \&c.,) - (P'' + P''' + \&c.)$$

If the first term of the second member is numerically greater than the second, R is positive, and the resultant acts in the direction that we assumed as positive; if the first term is numerically less than the second, R is negative, and the resultant acts in the opposite direction; if the two terms are equal, the resultant is 0, and the forces are in equilibrium.

All the forces of a system that act in the general direction of any straight line, are called *homologous*, and their *algebraic sum* may be expressed by writing the expression for single force, prefixing the symbol Σ, which indicates the *algebraic sum of homologous* quantities. We might, for example, write the preceding equation under the form,

$$R = \Sigma\,(P) \quad . \; . \; . \; . \; . \quad (6)$$

This equation expresses the fact, that the *resultant of a system of homologous forces, is equal to their algebraic sum.*

Composition of concurrent Forces.

24. Concurrent forces are those whose lines of direction intersect in a common point. The simplest case is that of two forces applied at a common point, but not in the same direction. After this, in order of simplicity, we have the case of several forces applied at a common point and lying in the same plane, and then the case of several forces applied at a common point and not in a single plane.

Parallelogram of Forces.

25. Let O be a material point, and suppose it acted on by two simultaneous impulses, P and Q, represented in direction and intensity by OP and OQ; complete the parallelogram PQ, and draw its diagonal OR.

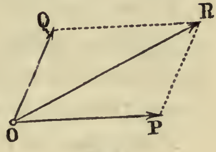


Fig. 2.

If O be taken as the unit of mass, OP and OQ will represent the velocities due to P and Q, (Art. 17), and inasmuch as the point obeys each force, as though the other did not exist, (Art. 12), it will be found at the end of one second somewhere on PR, by virtue of the force P, and somewhere on QR, by virtue of the force Q; it will therefore be at R, and because it moves uniformly in the direction of each force, it must move uniformly in the direction OR. Had O been acted on by an impulse represented by OR, it would in like manner have moved uniformly from O to R in one second. Hence the impulse OR is equivalent in effect to the two impulses OP and OQ; that is,

If two impulsive forces be represented by adjacent sides of a parallelogram, their resultant will be represented by that diagonal of the parallelogram which passes through their common point.

Because constant forces are proportional to their *elementary* impulses, (Art. 17), the above principle holds true for them; and because variable forces are measured by supposing them to become constant for a unit of time, (Art. 17), the principle must hold true for them: it is therefore true for all kinds of moving forces. It is also true for forces of pressure, for if we apply a force equal and directly opposed to the resultant of two moving forces, it will hold them in equilibrium, converting them into forces of pressure, but it will in no manner change the relation between them and their resultant. Hence, the principle is universal; it may be enunciated as follows:

If two forces be represented in direction and intensity by adjacent sides of a parallelogram, their resultant will be represented by that diagonal of the parallelogram which passes through their common point.

This principle is called *the parallelogram of forces.*

Geometrical Applications of the Parallelogram of Forces.

26. 1°. Given two forces; to find their resultant.

Let OP and OQ be the given forces. Complete the parallelogram QP and draw its diagonal OR; this will be the resultant required.

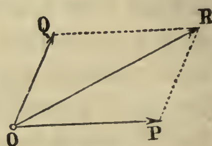


Fig. 3.

2°. Given, a force and one of its components; to find the other.

Let OR be a force and OP one of its components. Draw PR and complete the parallelogram PQ; OQ will be the other component.

3°. Given, a force and the directions of its components; to find the components.

Let OR be a force and OP, OQ, the directions of its components; through R draw RQ and RP parallel to PO and QO; then will OP and OQ be the required components.

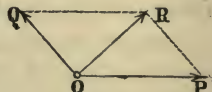


Fig. 4.

4°. Given, a force and the intensities of its components; to find the directions of the components.

Let OR be a force, and let the intensities of its components be represented by lines equal to OP and OQ; with

O as a centre and OP as radius, describe an arc, then with R as a centre and OQ as a radius, describe a second arc, cutting the first at P; draw OP, and RP, and complete the parallelogram PQ; OP and OQ will be the required components.

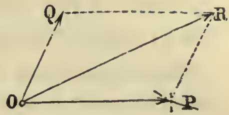


Fig. 5.

Polygon of Forces.

27. Let OQ, OP, OS, and OT, be a system of forces applied at a point, O, and lying in a single plane. To construct their resultant; on OQ and OP construct the parallelogram PQ, and draw its diagonal OR', this will be the resultant of OP and OQ. In like manner construct a parallelogram on OR' and OS; its diagonal OR'', will be the resultant of OP, OQ, and OS. On OR'' and OT construct a parallelogram, and draw its diagonal OR; then will OR be the resultant of all the given forces. This method of construction may be extended to any number of forces whatever.

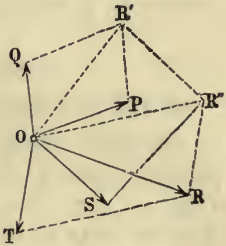


Fig. 6.

If we examine the diagram, we see that QR' is parallel and equal to OP, $R'R''$ is parallel and equal to OS, $R''R$ is parallel and equal to OT, and that OR is drawn from the point of application, O, to the extremity of $R''R$. Hence, we have the following rule for constructing the resultant of several concurrent forces:

Through their common point draw a line parallel and equal to the first force; through the extremity of this draw a line parallel and equal to the second force; and so on, throughout the system; finally, draw a line from the start-

ing point to the extremity of the last line drawn, and this will be the resultant required.

This application of the parallelogram of forces, is called *the polygon of forces.*

The principle holds true, even when the forces are not in one plane. In this case, the lines OQ, QR', $R'R''$, &c., form a *twisted polygon,* that is, a polygon whose sides are not in one plane.

When the point R, in the construction, falls at O, OR reduces to 0, and the forces are in equilibrium.

Parallelopipedon of Forces.

28. Let OP, OQ, and OS, be three concurrent forces not in the same plane. On these, as edges, construct the parallelopipedon OR, and draw OR, OM, and SR. From the principle of Art. 25, OM is the resultant of OP and OQ; and OR is the resultant of OM and OS; hence, OR is the resultant of OP, OQ, and OS; that is, *if three forces be represented by the concurrent edges of a parallelopipedon, their resultant will be represented by the diagonal of the parallelopipedon that passes through their common point.* This principle is called *the parallelopipedon of forces.* It is easily shown that it is a particular case of *the polygon of forces;* for, OP is parallel and equal to the first, PM to the second, MR to the third force, and OR is drawn from the origin, O, to the extremity of MR.

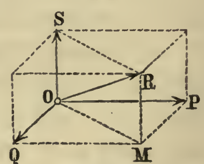


Fig. 7.

Components of a force in the direction of Rectangular Axes.

29. *First.* To find analytical expressions for the components of a force in the direction of two axes.

Let AR be a force in the plane of the rectangular axes OX and OY. On it as a diagonal construct a parallelogram ML, whose sides are parallel to OX and OY. Denote AR by R, AL by X, AM, equal to LR, by Y, and the angle LAR, equal to the angle the force makes with OX, by α. From the figure, we have,

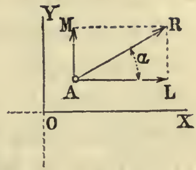


Fig. 8.

$$X = R \cos \alpha, \text{ and } Y = R \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In these expressions the angle α is estimated from the positive direction of the axis of X, around to the force, in accordance with the rule laid down in Trigonometry. The component X will have the same sign as $\cos \alpha$, and the component Y the same sign as $\sin \alpha$.

Secondly. To find the components of a force in the direction of three rectangular axes.

Let OR, denoted by R, be the given force, and OX, OY, and OZ, the given axes. On OR, as a diagonal, construct a parallelopipedon whose edges are parallel to the axes. Then will OL, OM, and ON be the required components. Denote these by X, Y, and Z, and the angles they make with OR by α, β, and γ. Join R with L, M, and N, by straight lines. From the right-angled triangles thus formed, we have,

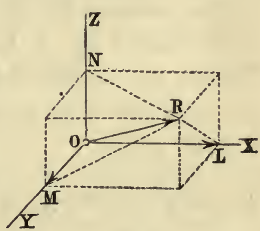


Fig. 9.

$$X = R \cos \alpha, \ Y = R \cos \beta, \text{ and } Z = R \cos \gamma \quad . \quad . \quad . \quad (8)$$

The angles α, β, and γ are estimated from the positive directions of the corresponding axes, as in Trigonometry, and each component has the same sign as the corresponding cosine.

If a force be resolved in the direction of rectangular axes, each component will represent the total effect of the given force in that direction. For this reason such components are called *effective* components. It is plain, that the component in the direction of each axis, is the same as the *projection* of the force on that axis, the projection being made by lines through the extremities of the force, and perpendicular to the axis. Hence, we may find the effective component of a force in the direction of a given line, *geometrically*, by projecting the force on the line, or *analytically*, by multiplying the force into the cosine of its inclination to the line.

Analytical Composition of Rectangular Forces.

30. *First.* When there are but two forces.

Let AL and AM be rectangular forces, denoted by X and Y, and let AR, denoted by R, be their resultant. Denote the angle RAL by α. Then, because $LR = Y$, we have, from the triangle ALR,

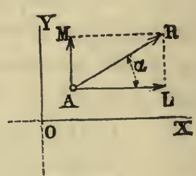


Fig. 10.

$$R = \sqrt{X^2 + Y^2}; \cos \alpha = \frac{X}{R}; \text{ and } \sin \alpha = \frac{Y}{R} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The first of these gives the intensity, the second and third the direction of the resultant.

Secondly. When there are three forces not in one plane. Let OL, OM, and ON, be rectangular forces denoted by X, Y, and Z, and let OR, denoted by R, be their resultant. Denote the angles which R makes with OL, OM, and ON by α, β, and γ. Then, from the figure, we have,

$$R = \sqrt{X^2 + Y^2 + Z^2} \quad . . . (10)$$

$$\cos \alpha = \frac{X}{R}; \ \cos \beta = \frac{Y}{R}; \text{ and,}$$

$$\cos \gamma = \frac{Z}{R} \quad . . . . . (11)$$

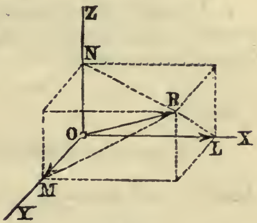


Fig. 11.

The first gives the intensity of the resultant, the others its direction.

EXAMPLES.

1. Two pressures of 9 and 12 pounds act on a point, and at right angles to each other. Required, the resultant pressure.

SOLUTION.

We have,

$X = 9$, and $Y = 12$; $\quad \therefore R = \sqrt{81 + 144} = 15.$

Also, $\cos \alpha = \frac{9}{15} = .6$; $\quad \therefore \alpha = 53° \ 7' \ 47''.$

That is, the resultant pressure is 15 lbs., and it makes an angle of 53° 7′ 47″ with the first force.

2. Two rectangular forces are to each other as 3 to 4, and their resultant is 20 lbs. What are the intensities of the components?

SOLUTION.

We have, $3Y = 4X$, or $Y = \frac{4}{3}X$, and $R = 20$;

$$\therefore 20 = \sqrt{X^2 + \tfrac{16}{9}X^2} = \tfrac{5}{3}X;$$

Hence, $X = 12$, and $Y = 16$.

3. A boat fastened by a rope to a point on shore, is urged by the wind perpendicular to the current, with a force of 18 pounds, and down the current by a force of 22 pounds. What is the tension on the rope, and what angle does it make with the current?

SOLUTION.

We have,

$X = 22$, and $Y = 18$; $\quad \therefore R = \sqrt{808} = 28.425$;

Also, $\cos \alpha = \frac{22}{28.425}$; $\quad \therefore \alpha = 39° \ 17' \ 20''.$

Hence, the tension is 28.425 lbs., and the angle 39° 17′ 20″.

4. Required the intensity and direction of the resultant of three forces at right angles to each other, having the intensities 4, 5, and 6 pounds, respectively.

SOLUTION.

We have,

$$X = 4,\ Y = 5, \text{ and } Z = 6. \qquad \therefore\ R = \sqrt{77} = 8.775.$$

Also, $$\cos\alpha = \frac{4}{8.775},\ \cos\beta = \frac{5}{8.775}, \text{ and } \cos\gamma = \frac{6}{8.775};$$

whence, $$\alpha = 62°\ 52'\ 51'',\ \beta = 55°\ 15'\ 50'', \text{ and } \gamma = 46°\ 51'\ 43''.$$

Hence the resultant pressure is 8.775 lbs., and it makes, with the components taken in order, angles equal to 62° 52′ 51″, 55° 15′ 50″, and 46° 51′ 43″.

5. Three forces at right angles are to each other as 2, 3, and 4, and their resultant is 60 lbs. What are the intensities of the forces?

Ans. 22.284 lbs., 33.426 lbs., and 44.568 lbs.

Application to Groups of Concurrent Forces.

31. The principles explained in the preceding articles, enable us to find the resultant of any number of concurrent forces. Let P, P', P'', &c., be a group of concurrent forces. Call the angles they make with the axis of X, α, α', α'', &c.; the angles they make with the axis of Y, β, β', β'', &c.; and the angles they make with the axis of Z, γ, γ', γ'', &c. Resolve each force into rectangular components parallel to the axes, and denote the resultants of the groups parallel to the axes by X, Y, and Z. We have, (Art. 23),

$$X = \Sigma(P\cos\alpha),\ Y = \Sigma(P\cos\beta),\ Z = \Sigma(P\cos\gamma).$$

If we denote the resultant by R, and the angles it makes with the axes by a, b, and c, we have, as in Article 30,

$$R = \sqrt{X^2 + Y^2 + Z^2}.$$

$$\cos a = \frac{X}{R},\ \cos b = \frac{Y}{R}, \text{ and } \cos c = \frac{Z}{R}.$$

These formulas determine the intensity and direction of the resultant.

When the given forces lie in the plane XY, Z reduces to 0, $\cos\beta$ becomes $\sin\alpha$, $\cos b$ becomes $\sin a$, and the formulas reduce to,

$$X = \Sigma\,(P \cos \alpha), \text{ and } Y = \Sigma\,(P \sin \alpha).$$

$$R = \sqrt{X^2 + Y^2}, \text{ and } \cos a = \frac{X}{R}, \text{ and } \sin a = \frac{Y}{R}.$$

Examples.

1. Three concurrent forces, whose intensities are 50, 40, and 70, lie in the same plane, and make with an axis, angles equal to 15°, 30°, and 45°. Required the resultant.

SOLUTION.

We have,

$$X = 50 \cos 15° + 40 \cos 30° + 70 \cos 45° = 132.435,$$

and

$$Y = 50 \sin 15° + 40 \sin 30° + 70 \sin 45° = 82.44;$$

whence $R = \sqrt{6798 + 17539} = 156;$

and, $\cos a = \dfrac{132.435}{156}; \quad \therefore\ a = 31°\ 54'\ 12''.$

The resultant is 156, and the angle it makes with the axis is 31° 54′ 12″.

2. Three forces 4, 5, and 6, lie in the same plane, and make equal angles with each other. Required the intensity of their resultant and the angle it makes with the least force.

SOLUTION.

Take the least force as the axis of X. Then the angle between it and the second force is 120°, and that between it and the third force is 240°. We have,

$$X = 4 + 5 \cos 120° + 6 \cos 240° = -1.5;$$

$$Y = 5 \sin 120° + 6 \sin 240° = -.866;$$

$$\therefore\ R = \sqrt{3},\ \cos a = -\frac{1.5}{1.732},\ \sin a = -\frac{.866}{1.732}; \quad \therefore\ a = 210°.$$

3. Two forces, one of 5 lbs. and the other of 7 lbs., are applied at the same point, and make with each other an angle of 120°. What is the intensity of their resultant. *Ans.* 6.24 lbs.

Formula for the Resultant of two Forces.

32. Let P and P', be two forces in the same plane, and let the axis of X be taken to coincide with P; α will then be 0, and we shall have $\sin \alpha = 0$, and $\cos \alpha = 1$. The value of X (Art. 31) will be $P + P' \cos \alpha'$, and the value of Y will be $P' \sin \alpha'$. Squaring these values, substituting in Equation (9), and reducing by the relation $\sin^2 \alpha' + \cos^2 \alpha' = 1$, we have,

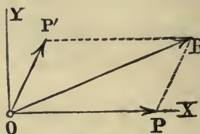


Fig. 12.

$$R = \sqrt{P^2 + P'^2 + 2PP' \cos \alpha'} \quad . \ . \ . \ . \ . \quad (12)$$

The angle α' is the angle between the given forces. Hence,

The resultant of two concurrent forces is equal to the square root of the sum of the squares of the forces, plus twice the product of the forces into the cosine of their included angle.

If α' is greater than 90°, and less than 270°, its cosine is negative, and we have,

$$R = \sqrt{P^2 + P'^2 - 2PP' \cos \alpha'}.$$

If $\alpha' = 0$, its cosine is 1, and we have,

$$R = P + P'.$$

If $\alpha' = 90°$, its cosine is 0, and we have,

$$R = \sqrt{P^2 + P'^2}.$$

If $\alpha' = 180°$, its cosine is -1, and we have,

$$R = P - P'.$$

Examples.

1. Two forces, P and Q, are equal to 24 and 30, and the angle between them is 105°. What is the intensity of their resultant?

$$R = \sqrt{24^2 + 30^2 + 2 \times 24 \times 30 \cos 105°} = 33.21.$$

2. Two forces, P and Q, whose intensities are 5 and 12, have a resultant whose intensity is 13. Required the angle between them.

$$13 = \sqrt{25 + 144 + 2 \times 5 \times 12 \cos \alpha}.$$

$$\therefore \cos \alpha = 0, \text{ or } \alpha = 90^\circ. \quad \textit{Ans.}$$

3. A boat is impelled by the current at the rate of 4 miles per hour, and by the wind at the rate of 7 miles per hour. What will be her rate per hour when the direction of the wind makes an angle of 45° with that of the current?

$$R = \sqrt{16 + 49 + 2 \times 4 \times 7 \cos 45^\circ} = 10.2\text{m}. \quad \textit{Ans.}$$

4. A weight of 50 lbs., suspended by a string, is drawn aside by a horizontal force until the string makes an angle of 30° with the vertical. Required the value of the horizontal force, and the tension of the string. *Ans.* 28.8675 lbs., and 57.735 lbs.

5. Two forces, and their resultant, are all equal. What is the angle between the two forces? *Ans.* 120°.

Relation between two Forces and their Resultant.

33. Let P and Q be two forces, and R their resultant. Then because QP is a parallelogram, the side PR is equal to Q. From the triangle ORP, because the sides are proportional to the sines of the opposite angles, we have,

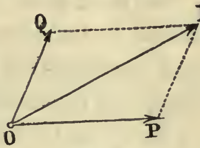


Fig. 13.

$$P : Q : R :: \sin ORP : \sin ROP : \sin OPR.$$

But, $ORP = QOR$, and $OPR = 180^\circ - QOP$; hence, we have,

$$P : Q : R :: \sin QOR : \sin ROP : \sin QOP; \quad . \; . \; . \quad (13)$$

That is, *of two forces and their resultant, each is proportional to the sine of the angle between the other two.*

If we apply a force R' equal and directly opposed to R, the forces P, Q, and R' will be in equilibrium. The angles QOR and QOR' are supplements of each other; hence, $\sin QOR = \sin QOR'$; the angles ROP, and POR', are also supplementary; hence, $\sin ROP = \sin POR'$. We have also $R = R'$. Making these substitutions in the preceding proportion, we have,

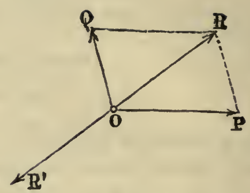


Fig. 14.

$$P : Q : R' :: \sin QOR' : \sin POR' : \sin QOP. \quad . \quad . \quad (14)$$

Hence, *if three forces are in equilibrium, each is proportional to the sine af the angle between the other two.*

Principle of Moments.

34. The moment of a force, with respect to a point, is the product of the intensity of the force, by the perpendicular from the point to the direction of the force.

The fixed point is the *centre of moments;* the perpendicular is the *lever arm of the force;* and the moment itself measures the tendency of the force to produce rotation about the centre of moments.

Let P and Q be two forces, and R their resultant; assume a point C, in their plane, as a centre of moments, and from it, let fall on the forces, perpendiculars, Cp, Cq, and Cr; denote these perpendiculars by p, q, and r. Then will Pp, Qq, and Rr, be the moments of P, Q, and R. Draw CO, and from P let fall the perpendicular PS, on OR. Denote the angle ROP, by α, ROQ, or its equal, ORP, by β, and ROC by φ.

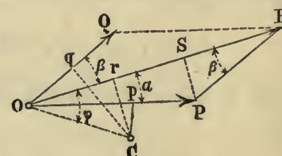


Fig. 15.

Since $PR = Q$, we have from the right-angled triangles OPS and PRS, the equations,

$$R = Q \cos \beta + P \cos \alpha.$$
$$0 = Q \sin \beta - P \sin \alpha.$$

Multiplying both members of the first by $\sin \varphi$, and of the second by $\cos \varphi$, and adding, we find,

$$R \sin \varphi = Q (\sin \varphi \cos \beta + \sin \beta \cos \varphi) + P (\sin \varphi \cos \alpha - \sin \alpha \cos \varphi).$$

Whence, by reduction,

$$R \sin \varphi = Q \sin (\varphi + \beta) + P \sin (\varphi - \alpha).$$

From the figure, we have,

$$\sin \varphi = \frac{r}{OC}, \sin (\varphi - \alpha) = \frac{p}{OC}, \text{ and } \sin (\varphi + \beta) = \frac{q}{OC}.$$

Substituting in the preceding equation, and reducing, we have,

$$Rr = Qq + Pp.$$

When C falls within the angle POR, $\varphi - \alpha$ is negative, and the equation just deduced becomes

$$Rr = Qq - Pp.$$

Hence, in all cases, *the moment of the resultant of two forces is equal to the algebraic sum of the moments of the forces taken separately.*

If we regard the force Q as the resultant of two others, and one of these in turn, as the resultant of two others, and so on, the principle may be extended to any number of concurrent forces in the same plane. This principle may be expressed by the equation,

$$Rr = \Sigma (Pp) \quad . \quad . \quad . \quad . \quad (15)$$

That is, *the moment of the resultant of any number of concurrent forces, in the same plane, is equal to the algebraic sum of the moments of the forces taken separately.*

This is *the principle of moments.*

The moment of the resultant is the *resultant moment;* the moments of the components are *component moments;* and the plane passing through the resultant and centre of moments, is the *plane of moments.*

When a force tends to turn its point of application about the centre of moments, in the direction of the motion of the hands of a watch, its moment is considered *positive;* consequently, when it tends to produce rotation in a contrary direction, its moment must be *negative.* If the resultant moment is negative, the tendency of the system is to produce rotation in a negative direction. If the resultant moment is 0, there is no tendency to rotation. The resultant moment may become 0, in consequence of the lever arm becoming 0, or in consequence of the resultant itself being 0. In the former case, the centre of moments lies on the direction of the resultant, and the sum of the moments of the forces that tend to produce rotation in one direction, is equal to that of those tending to produce rotation in a contrary direction. In the latter case, the system is in equilibrium.

Moment of a Force with respect to an Axis.

35. Let P be a force and OZ any axis. Draw a line, AB, perpendicular to the force and also to the axis. Let A be taken as the point of application of the force, and at this point resolve it into two components P'' and P', the former *parallel*, and the latter *perpendicular* to OZ. The component P'' can

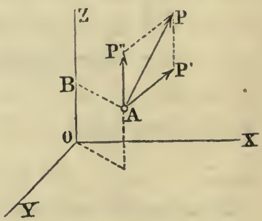


Fig. 16.

have no tendency to produce rotation about the axis;

hence, the moment of P' with respect to B, will be the same as the moment of the given force with respect to the axis. But, P' is the projection of P on a plane perpendicular to the axis, and B is the point in which this plane intersects the axis. Hence, to find the moment of a force with respect to an axis, *project the force on a plane perpendicular to the axis, and find the moment of the projection with respect to the point in which the perpendicular plane cuts the axis.*

The axis is an *axis of rotation*, and any plane perpendicular to it, is a *plane of rotation.*

To find the resultant moment of a system of forces in space, with respect to any line as an axis; assume a plane perpendicular to the given line as a plane of rotation, project the forces on it, and find the moments of the projections with respect to the point in which the plane cuts the axis; these will be the component moments. The resultant moment is the algebraic sum of the component moments.

Principle of Virtual Moments.

36. Let P be a force applied to the material point O; let O be moved by an extraneous force to some position, C, very near to O; project the path OC on the direction of the force: the projection Op, or Op', is called the virtual velocity of the force, and is *positive* when it falls on the direction of the force, as Op, and *negative* when it falls on the prolongation of the force, as Op'. The product of a force by its virtual velocity is called the *virtual moment* of the force.

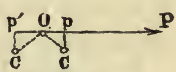


Fig. 17.

Assume the figure of Article 34. Op, Oq, and Or are the virtual velocities of P, Q, and R. Let us denote the virtual velocity of a force by the symbol δ, followed by a small letter of the same name as that which designates the force.

We have from the figure, as in Article 34,

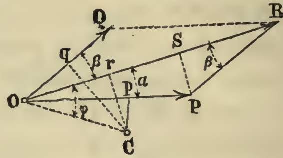


Fig. 18.

$$R = P \cos \alpha + Q \cos \beta.$$
$$0 = P \sin \alpha - Q \sin \beta.$$

Multiplying both members of the first, by $\cos \varphi$, of the second, by $\sin \varphi$, and adding, we have,

$$R \cos \varphi = P (\cos \alpha \cos \varphi + \sin \alpha \sin \varphi) +$$
$$Q (\cos \varphi \cos \beta - \sin \varphi \sin \beta).$$

Or, by reduction,

$$R \cos \varphi = P \cos (\varphi - \alpha) + Q \cos (\varphi + \beta).$$

But, from the right-angled triangles COp, COq, and COr, we have,

$$\cos \varphi = \frac{\delta r}{OC}, \cos (\varphi - \alpha) = \frac{\delta p}{OC}, \text{ and } \cos (\varphi + \beta) = \frac{\delta q}{OC};$$

Substituting in the preceding equation, and reducing, we have,

$$R\delta r = P\delta p + Q\delta q.$$

Hence, *the virtual moment of the resultant of two forces, is equal to the algebraic sum of the virtual moments of the forces taken separately.*

If we regard Q as the resultant of two forces, and one of these as the resultant of two others, and so on, the principle may be extended to any number of forces, applied at the same point. This principle may be expressed by the following equation:

$$R\delta r = \Sigma (P\delta p) ; \quad . \; . \; . \; . \; . \quad (16)$$

Hence, *the virtual moment of the resultant of any number of concurrent forces, is equal to the algebraic sum of the virtual moments of the forces taken separately.*

Resultant of Parallel Forces.

37. Let P and Q be two forces in the same plane, and applied at points invariably connected—for example, at the points M and N of a solid body. Their lines of direction being prolonged, will meet at some point O; and if we suppose the points of application to be transferred to O, their resultant may be determined by the parallelogram of forces.

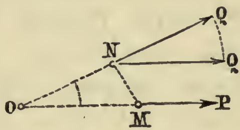


Fig. 19.

Whether the forces be so transferred or not, the direction of the resultant will always pass through O, and will lie between P and Q. Now, supposing the points of application at M and N, let the force Q be turned about N as an axis. As it approaches parallelism with P, O recedes from M and N, and the resultant also approaches parallelism with P. Finally, when Q becomes parallel to P, O is at an infinite distance from M and N, and *the resultant is parallel to P and Q.*

Hence, *if two forces are parallel and act in the same direction, their resultant is parallel to both and lies between them.*

Whatever may be the position of P and Q, the value of the resultant, (Art. 34), will be given by the equation,

$$R = P \cos \alpha + Q \cos \beta.$$

But when the forces are parallel and act in the same direction, we have, $\alpha = 0$, and $\beta = 0$; or, $\cos \alpha = 1$, and $\cos \beta = 1$. Hence,

$$R = P + Q. \quad . \quad . \quad . \quad . \quad (17)$$

That is, *the intensity of the resultant is equal to the sum of the intensities of the two forces.*

Let P and Q be parallel forces acting in the same direction, R their resultant, and S the point in which the direction of R cuts the line joining the points of application of P and Q. Through N draw NL, perpendicular to the forces, and take C, its intersection with R, as a centre of moments. The centre of moments being on the direction of the resultant, the lever arm of the resultant will be 0, and from the principle of moments, (Art. 34), we have,

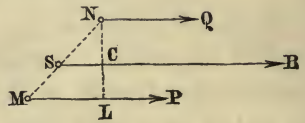


Fig. 20.

$$P \times CL = Q \times CN;$$

or, $$P : Q :: CN : CL.$$

But, from the similar triangles CNS and LNM, we have,

$$CN : CL :: SN : SM.$$

Combining the two proportions, we have,

$$P : Q :: SN : SM. \quad . \quad . \quad . \quad . \quad (18)$$

That is, *the resultant divides the line joining the points of application of the components, inversely as the components.*

If a force R' be applied at S equal and directly opposed to R, it will hold P and Q in equilibrium. The forces R', P, and Q, being in equilibrium, Q must be equal and directly opposed to the resultant of R' and P. But, R' and P are parallel and act in opposite directions, R' being the greater. Hence, the resultant of two parallel forces acting in opposite directions, *is parallel to both, lies without both, on the side and in the direction of the greater, and its* *intensity is equal to the difference of the intensities of the given forces.*

If P and Q, Fig. 21, represent two such forces, and R their resultant; it may be shown, as in the preceding article, that,

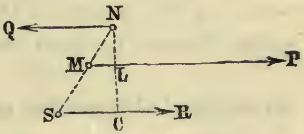


Fig. 21.

$P : Q :: SN : SM$. . . . (19)

By composition, we find,

$$P : Q : P + Q :: SN : SM : SN + SM;$$

and by division,

$$P : Q : P - Q :: SN : SM : SN - SM.$$

When the forces act in the same direction, as in Fig. 20, $P + Q = R$, and $SN + SM = MN$.

When they act in opposite directions, as in Fig. 21, $P - Q = R$, and $SN - SM = MN$.

Substituting in the preceding proportions, for $P + Q$, $P - Q$, $SN + SM$, and $SN - SM$, their values, we have,

$$P : Q : R :: SN : SM : MN. \quad . \quad . \quad . \quad . \quad (20)$$

That is, *of any two parallel forces and their resultant, each is proportional to the distance between the other two.*

We see, from the preceding proportion, that so long as the intensities of P and Q and their points of application remain unchanged, the values of SM and SN also remain unchanged, no matter what direction the forces may have. Hence, if two parallel forces be turned about their points of application, their intensities remaining unchanged, their resultant will turn about a fixed point and continue parallel to the given forces. This fixed point is called the *centre of the parallel forces.*

If P and Q be equal and act in opposite directions, R will be 0, and S will be at an infinite distance. Two such forces constitute *a couple.* The tendency of a couple is to produce rotation; the measure of this tendency, called *the moment of the couple*, is the product of one of the forces, by the distance between the two.

Geometrical Composition and Resolution of Parallel Forces.

38. The preceding principles give the following geometrical constructions.

1. To find the resultant of two parallel forces lying in the same direction:

Let P and Q be the forces, M and N their points of application. Make $MQ' = Q$, and $NP' = P$; draw $P'\,Q'$, cutting MN in S; through S draw SR parallel to MP, and make it equal to $P + Q$; it will be the resultant.

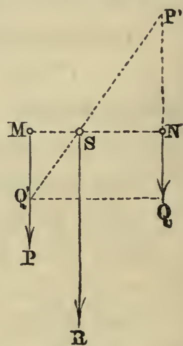


Fig. 22.

For, from the triangles $P'SN$ and $Q'SM$, we have,

$$P'N : Q'M :: SN : SM; \text{ or, } P : Q :: SN : SM.$$

2. To find the resultant of two parallel forces acting in opposite directions:

Let P and Q be the forces, M and N their points of application. Prolong QN till $NA = P$, and make $MB = Q$; draw AB, and produce it till it cuts NM produced in S; draw SR parallel to MP, and equal to BP, it will be the resultant required.

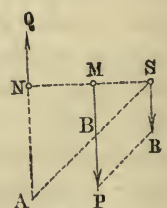


Fig. 23

For, from the triangles SNA and SMB, we have,

$$AN : BM :: SN : SM; \text{ or, } P : Q :: SN : SM.$$

3. To resolve a force into two parallel components in the same direction, and applied at given points:

Let R be the force, M and N the points of application. Through M and N draw lines parallel to R. Make $MA = R$, and draw AN, cutting R in B; make $MP = SB$ and $NQ = BR$; they will be the components.

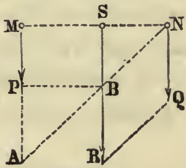


Fig. 24.

For, from the triangles AMN and BSN,

$$BS : AM :: SN : MN;$$

or,

$$BS : R :: SN : MN.$$

But, from proportion (20), we have,

$$P : R :: SN : MN;$$

$$\therefore BS = P, \text{ and } BR = Q.$$

4. To find the resultant of any number of parallel forces.

Let P, P', P'', P''', be parallel forces. Find the resultant of P and P', by the rule already given, it will be $R' = P + P'$; find the resultant of R' and P'', it will be $R'' = P + P' + P''$; find the resultant of R'' and P''', it will be $R = P + P' + P'' + P'''$. If there be a greater number of forces, the operation of composition may be continued; the final result will be the resultant of the system. If some of the forces act in contrary directions, combine all that act in one direction, as just explained, and call their resultant R'; then combine all that act in a contrary direction, and call their resultant

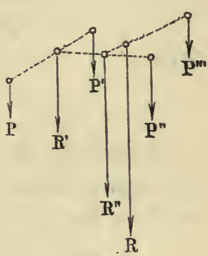


Fig. 25.

R''; finally, combine R' and R''; their resultant, R, will be the resultant of the system.

If the forces P, P', &c., be turned about their points of application, their intensities remaining unchanged, the forces R', R'', R, will also turn about fixed points, continuing parallel to the given forces. The point through which R always passes, is called the *centre of parallel forces.*

Co-ordinates of the Centre of Parallel Forces.

39. Let P, P', P'', &c., be parallel forces, applied at points that maintain fixed positions with respect to a system of rectangular axes, and let R, equal to $\Sigma\,(P)$, be their resultant. Denote the co-ordinates of the points of application of the forces by x, y, z; x', y', z', &c.; and those of R by $x_{,}$, $y_{,}$, $z_{,}$.

Turn the forces about their points of application, till they are parallel to the axis of Y, and in that position find their moments with respect to the axis of Z. In this position the lever arms of the forces are x, x', &c., and the lever arm of R is $x_{,}$. From the principle of moments, (Art. 34), we have

$$Rx_{,} = Px + P''x' +, \&c.$$

or,

$$x_{,} = \frac{\Sigma\,(Px)}{\Sigma(P)} \quad . \; . \; . \; . \; . \; (21)$$

By making the forces in like manner parallel to the axis of Z, and taking their moments with respect to the axis of X, we have,

$$y_{,} = \frac{\Sigma\,(Py)}{\Sigma\,(P)} \quad . \; . \; . \; . \; . \; (22)$$

And in like manner, we find,

$$z_{,} = \frac{\Sigma\,(Pz)}{\Sigma\,(P)} \quad . \; . \; . \; . \; . \; (23)$$

From either of the above expressions we infer that *the lever arm of the resultant of a system of parallel forces with respect to any axis perpendicular to the forces, is equal to the algebraic sum of the moments of the forces with respect to that axis, divided by the algebraic sum of the forces.*

Equations 21, 22, and 23, determine the position of the *centre of parallel of forces.*

Composition of Forces in Space, applied at points invariably connected.

40. Let P, P', P'', &c., be forces in space, applied at points of a solid body. Assume a point O, and through it draw three perpendicular axes. Denote the angles that P, P', P'', &c., make with the axis of X, by α, α', α'', &c.; the angles they make with the axis of Y, by β, β', β'', &c.; the angles they make with the axis of Z, by γ, γ', γ'', &c., and denote the co-ordinates of their points of application by x, y, z; x', y', z'; x'', y'', z'', &c.

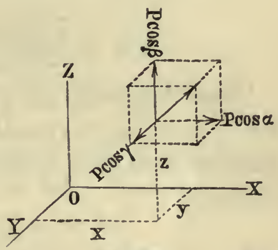


Fig. 26.

Let each force be resolved into components parallel to the axes.

We have for the group parallel to the axis of X,

$$P\cos\alpha,\ P'\cos\alpha',\ P''\cos\alpha'', \text{ \&c.};$$

for the group parallel to the axis of Y,

$$P\cos\beta,\ P'\cos\beta',\ P''\cos\beta'', \text{ \&c.};$$

and, for the group parallel to the axis of Z,

$$P\cos\gamma,\ P'\cos\gamma',\ P''\cos\gamma'', \text{ \&c.}$$

Denoting the resultants of these several groups by X, Y, and Z, we have,

$$X = \Sigma(P\cos\alpha), Y = \Sigma(P\cos\beta), \text{ and } Z = \Sigma(P\cos\gamma). \quad . \quad . \quad . \quad (24)$$

If the given forces have a single resultant, the forces X, Y, and Z, will be applied at a point, the co-ordinates of which are the same as the lever arms of the forces, each taken with respect to the axis whose name comes next in order. Denoting these co-ordinates by x_1, y_1, and z_1, we have, as in Art. 39,

$$\left. \begin{aligned} x_1 &= \frac{\Sigma(P\cos\beta\ x)}{\Sigma(P\cos\beta)} \\ y_1 &= \frac{\Sigma(P\cos\gamma\ y)}{\Sigma(P\cos\gamma)} \\ z_1 &= \frac{\Sigma(P\cos\alpha\ z)}{\Sigma(P\cos\alpha)} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

These determine the point of application of the resultant. Denoting the resultant by R, and the angles it makes with the axes by a, b, and c, we have, from preceding principles,

$$R = \sqrt{X^2 + Y^2 + Z^2} \quad . \quad . \quad . \quad . \quad (26)$$

and

$$\cos a = \frac{X}{R}, \cos b = \frac{Y}{R}, \cos c = \frac{Z}{R} \quad . \quad . \quad . \quad . \quad (27)$$

Hence, the resultant is completely determined.

If the forces are in a plane, that plane may be taken as the plane XY. In this case the formulas for determining the point of application of the resultant become,

$$x_1 = \frac{\Sigma(P\cos\beta\, x)}{\Sigma(P\cos\beta)}, \text{ and } y_1 = \frac{\Sigma(P\cos\alpha\ y)}{\Sigma(P\cos\alpha)}; \quad . \quad . \quad . \quad . \quad (28)$$

and the formulas for finding the intensity and direction of the resultant reduce to,

$$R = \sqrt{X^2 + Y^2}, \cos a = \frac{X}{R}, \cos b = \frac{Y}{R}, \quad . \quad . \quad . \quad (29)$$

in which

$$X = \Sigma\,(P\cos\alpha) \text{ and } Y = \Sigma\,(P\cos\beta) \quad . \quad . \quad . \quad .$$

Conditions of Equilibrium.

41. A system of forces applied at points of a solid body will be in equilibrium when they have no tendency to produce motion, either of translation, or of rotation. We have seen that any system of forces can be resolved into three groups, parallel to three rectangular axes. The tendency of each group is to produce motion parallel to the corresponding axis, and the tendency of the groups taken two and two is to produce rotation about the axis to which they are both perpendicular. In order that there may be no tendency to either kind of motion, we must have the following relations, called *conditions of equilibrium:*

1st. *The algebraic sum of the components of the forces in the direction of any three rectangular axes must be separately equal to* 0.

2d. *The algebraic sum of the moments of the forces, with respect to any three rectangular axes, must be separately equal to* 0.

If the forces lie in a plane, the conditions of equilibrium reduce to these:

1st. *The algebraic sum of the components of the forces, in the direction of any two rectangular axes, separately equal to* 0.

2d. *The algebraic sum of the moments of the forces, with respect to any point in the plane, equal to* 0.

If a body is restrained by a fixed axis, as in case of a pulley, or wheel and axle, the forces will be in equilibrium when *the algebraic sum of the moments of the forces with respect to the axis is equal to* 0.

This case is one frequently met with in discussing machines

CHAPTER III.

CENTRE OF GRAVITY AND STABILITY.

Weight.

42. The force of gravity acts on all the particles of a body, tending to draw them toward the centre of the earth. If this force be resisted it produces a pressure called *weight.* The weight of a body is the resultant of the weights of all its particles. The weights of the particles are sensibly directed toward the centre of the earth, and if the body be small in comparison with the earth, they may be regarded as parallel forces; hence, the weight of a body is parallel to the weights of its particles, and is equal to their sum.

Centre of Gravity.

43. The *centre of gravity* of a body is the point through which the direction of its weight always passes. The weight being the resultant of parallel forces, the centre of gravity is a centre of parallel forces, and so long as the relative position of the particles remains unchanged, this point retains a fixed position in the body. The position of the centre of gravity is entirely independent of the value of the intensity of gravity, provided we regard this force as constant throughout the body, which we may do in most cases. Hence, the centre of gravity is the same for the same body, wherever it may be situated. The determination of the centre of gravity is, then, reduced to the determination of the centre of a system of parallel forces.

In what follows, the lines and surfaces treated of, are regarded as *material,* that is, made up of *material points.* The volumes considered, are supposed to be homogeneous, so that the weights of different parts are proportional to their volumes. This supposition reduces the operation of finding the *centre of gravity,* to the geometric one of finding the centre of figure.

Preliminary Principles.

44. Let there be any number of parallel forces applied at points of a straight line. If we apply the method of finding *the point of application of their resultant,* as explained in Art. 39, it will be seen that *it lies on the given line.* Hence, *the centre of gravity of a material straight line is on that line.*

In like manner it may be shown that *the centre of gravity of a plane curve, or of a plane area, is in that plane.*

Centre of Gravity of a Straight Line.

45. Let M and N be two material points, equal in weight, and firmly connected by a line MN. The resultant of these weights will bisect the line MN in S (Art. 37); hence, S is the centre of gravity of M and N.

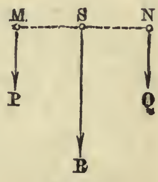


Fig. 27.

Let MN be a material straight line, and S its middle point. We may regard it as composed of material points A, A'; B, B', &c., equal in weight, and symmetrically disposed with respect to S. From what precedes, the centre of gravity of each pair of equidistant points is at S; consequently the centre of gravity of the whole line is at S.

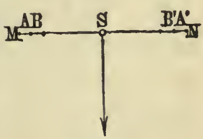


Fig 28.

That is, *the centre of gravity of a straight line is at its middle point.*

Additional Principles.

46. A *line of symmetry* of a plane figure is a straight line that bisects a system of parallel chords of the figure. If the line is perpendicular to the chords it bisects, it is a *line of right symmetry,* otherwise it is a *line of oblique symmetry.* The axes of an ellipse are lines of right symmetry; other diameters are lines of oblique symmetry.

A *plane of symmetry* of a surface, or volume, is a plane that bisects a system of parallel chords of the figure. It may be a plane of *right,* or a plane of *oblique symmetry.*

The intersection of two planes of symmetry is an *axis of symmetry.*

Let $AQBP$ be a curve, and AB a line of symmetry, bisecting the parallel chords PQ. The centre of gravity of each pair of points P, Q, is on AB (Art. 45), hence, the centre of gravity of the entire curve is on AB (Art. 44). Again, the centre of gravity of each chord PQ is on AB, hence the centre of gravity of the entire area is on AB. That is, *if a plane curve, or a plane area, has a line of symmetry, its centre of gravity is on that line.*

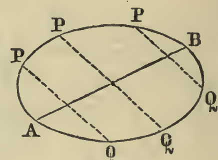


Fig. 29.

In like manner, *if a surface or volume has a plane of symmetry, its centre of gravity is in that plane.*

Two lines of symmetry, or three planes of symmetry intersecting in a point, are sufficient to determine the centre of gravity of the corresponding magnitude. Thus, all diameters of the circle are lines of symmetry, and because they intersect at the centre, it follows that the centre of gravity of both the circumference and area, is at the centre. For a similar reason the centre of gravity of both circumference and area of an ellipse is at its centre.

Any plane through the centre of a sphere, or of an ellipsoid, is a plane of symmetry; hence the centre of gravity of either is at its centre.

The centre of gravity of any surface or volume of revolution is on its axis.

Centre of Gravity of a Triangle.

47. Let ABC be a plane triangle. Join the vertex A with the middle point D, of the opposite side BC; then will AD bisect all lines drawn in the triangle parallel to BC, it is therefore a line of symmetry: hence, the centre of gravity of the triangle is on AD (Art. 46); for a like reason it is on BE, drawn from the vertex B to the middle of the side AC; it is, therefore, at G, their point of intersection.

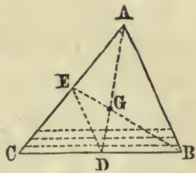


Fig. 30.

Draw ED; then, since ED bisects AC and BC, it is parallel to AB, and the triangles EGD and AGB are similar. The side ED is one-half its homologous side AB, consequently the side GD is one-half its homologous side AG; that is, G is one-third of the distance from D to A.

Hence, *the centre of gravity of a plane triangle is on a line drawn from the vertex to the middle of the base, and at one-third the distance from the base to the vertex.*

Centre of Gravity of a Parallelogram.

48. Let AC be a parallelogram. Draw EF bisecting AB and CD; it will bisect all lines of the parallelogram parallel to these; hence, the centre of gravity is on it; draw also OH bisecting AD and BC; for a similar reason, the centre of gravity is on it; it is, therefore, at G, their point of intersection.

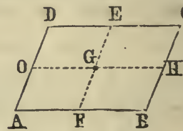


Fig. 31.

Hence, *the centre of gravity of a parallelogram is at the intersection of two straight lines joining the middle points of the opposite sides.*

The diagonals of a parallelogram are also lines of symmetry, each bisecting the chords parallel to the other. Hence, the centre of gravity is at their intersection.

Centre of Gravity of a Trapezoid.

49. Let AC be a trapezoid. Join the middle points, O and P, of the parallel sides, by a straight line; this will bisect all lines parallel to DC; hence, it must contain the centre of gravity. Draw BD, dividing the trapezoid into two triangles. Draw also DO and BP; take $OQ = \frac{1}{3}OD$, and $PR = \frac{1}{3}PB$; then will Q and R be the centres of gravity of these triangles (Art. 47). Join Q and R by a straight line; the centre of gravity of the trapezoid must be on this line (Art. 44). Hence, it is at G where QR cuts OP.

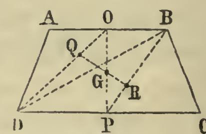


Fig. 32.

Centre of Gravity of a Polygon.

50. Let $ABCDE$ be a polygon, and a, b, c, d, e, the middle points of its sides. The weights of the sides are proportional to their lengths, and may be represented by them.

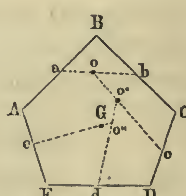


Fig. 33.

Let it be required to find the centre

of gravity of the perimeter; join a and b, and find a point o, such that

$$ao : ob :: BC : BA;$$

then will o be the centre of gravity of AB and BC. Join o and c, and find a point o', such that

$$oo' : o'c :: CD : AB + BC;$$

then will o' be the centre of gravity of the three sides, AB, BC, and CD. Join o' with d, and proceed as before, continuing the operation till the last point, G, is found; this will be the centre of gravity of the perimeter.

To find the centre of gravity of the area, divide it into triangles, and find the centre of gravity of each triangle. The weights of these triangles are proportional to their areas, and may be represented by them. Let O, O', O'', be the centres of gravity of the triangles into which the polygon is divided. Join O and O', and find a point O''', such that

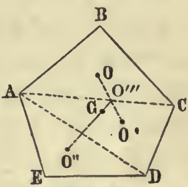


Fig. 34.

$$O'O''' : OO''' :: ABC : ACD;$$

then will O''' be the centre of gravity of the triangles ABC and ACD.

Join O'' and O''', and find a point, G, such that

$$O'''G : O''G :: ADE : ABC + ACD;$$

then will G be the centre of gravity of the polygon.

To find the centre of gravity of a curvilinear area by approximation, we draw a polygon whose perimeter shall nearly coincide with that of the given area, and then find its centre of gravity. The accuracy of this method will depend on the closeness with which the polygon approaches the curvilineal area.

Centre of Gravity of a Pyramid.

51. A pyramid may be regarded as made up of infinitely thin layers, parallel to either of its faces. If a line be drawn from either vertex to the centre of gravity of the opposite face, it will pass through the centres of gravity of all the layers parallel to that face. We may consider the weight of each layer as applied at its centre of gravity, that is, at a point of this line; hence, the centre of gravity of the pyramid is on this line, (Art. 44).

Let $ABCD$ be a triangular pyramid, and K the middle point of DC. Draw KB and KA; lay off $KO = \frac{1}{3}KB$, and $KO' = \frac{1}{3}KA$. Then will O be the centre of gravity of the face DBC, and O' that of the face CAD. Draw AO and BO' intersecting in G. Because the centre of gravity of the pyramid is on both AO and BO', it is at their intersection G. Draw OO'; then KO and KO' being third parts of KB and KA, OO' is parallel to AB, and the triangles OGO' and AGB are similar, consequently their homologous sides are proportional. But OO' is one-third of AB, OG is therefore one-third of GA, or one-fourth of AO.

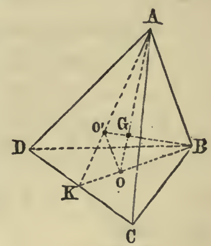


Fig. 35.

Hence, *the centre of gravity of a triangular pyramid is on a line drawn from its vertex to the centre of gravity of its base, and at one-fourth the distance from the base to the vertex.*

Either face of a triangular pyramid may be taken as the base, the opposite vertex being the vertex of the pyramid.

To find the centre of gravity of a polygonal pyramid $A\text{-}BCDEF$, A being the vertex. Conceive it divided into

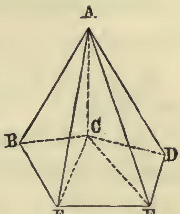


Fig. 36.

triangular pyramids, having a common vertex *A*. If an auxiliary plane be passed parallel to the base, at one-fourth of the distance from the base to the vertex, it follows, from what has just been shown, that the centres of gravity of all the partial pyramids will lie in this plane; hence, the centre of gravity of the entire pyramid must lie in this plane (Art. 44). But it has been shown, that the centre of gravity is somewhere on the line drawn from the vertex to the centre of gravity of the base; it must, therefore, be where this line pierces the auxiliary plane:

Hence, *the centre of gravity of any pyramid is on a line drawn from its vertex to the centre of gravity of its base, and at one-fourth the distance from the base to the vertex.*

A cone is a pyramid having an infinite number of faces:

Hence, *the centre of gravity of a cone is on a line drawn from the vertex to the centre of gravity of the base, and at one-fourth the distance from the base to the vertex.*

Centre of Gravity of a Prism.

52. A prism is made up of layers parallel to the bases, and if a straight line be drawn between the centres of gravity of the bases it will pass through the centres of gravity of all the layers; the centre of gravity of the prism is, therefore, on this line, which we may call the axis of the prism. The prism is also made up of filaments, parallel to the lateral edges, and if a plane be passed parallel to the bases of the prism and midway between them, it will contain the centres of gravity of all the filaments; the centre of gravity of the prism is therefore in this plane. It must then be where this plane cuts the axis.

Hence, *the centre of gravity of a prism is at the middle of its axis.*

A *cylinder*, is a prism whose bases have an infinite number of sides:

Hence, *the centre of gravity of a cylinder whose bases are parallel is at the middle of its axis.*

Centre of Gravity of a Polyhedron.

53. If a point within a polyhedron be joined with each vertex of the polyhedron, we shall form as many pyramids as the solid has faces: the centre of gravity of each pyramid may be found by the rule. If the centres of gravity of the first and second pyramid be joined by a straight line, the common centre of gravity of the two may be found by a process similar to that used in finding the centre of gravity of a polygon, observing that the weights of the pyramids are proportional to their volumes, and may be represented by them. Having compounded the weights of the first and second, and found its point of application, we may, in like manner, compound the weight of these two with that of the third, and so on; the last point of application will be the centre of gravity of the polyhedron.

The centre of gravity of a body bounded by a curved surface may be found by approximation, as follows: Construct a polyhedron whose faces are nearly coincident with the surface of the given body and find its centre of gravity by the method just explained; this will be the point sought.

The accuracy of the method will depend upon the closeness between the given figure and the polyhedron.

The methods of finding the centre of gravity, already given, are sufficient for most purposes. The most general method, however, depends on the Differential and Integral Calculus.

Experimental determination of the Centre of Gravity.

54. The weight of a body always passes through its centre of gravity, no matter what may be the position of the body. If we attach a flexible cord to a body at any point and suspend it freely, it must ultimately come to a state of rest. In this position, the body is acted upon by two forces: its weight, tending to draw it toward the centre of the earth, and the tension of the cord, that resists this force. In order that the body may be in equilibrium, these forces must be equal and directly opposed. But the direction of the weight passes through the centre of gravity of the body; hence, the tension of the string, which acts in the direction of the string, must also pass through the same point. This principle gives rise to the following method of finding the centre of gravity:

Let ABC be a body of any form whatever. Attach a string to any point, C, and suspend it freely; when the body comes to rest, mark the direction of the string; then suspend the body by a second point, B, and when it comes to rest, mark the direction of the string; the point of intersection, G, will be the centre of gravity of the body.

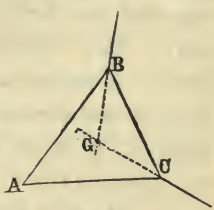


Fig. 37.

Instead of suspending the body by a string, it may be balanced on a point. In this case, the weight acts vertically downward, and is resisted by the reaction of the point; hence, the centre of gravity lies vertically over the point.

If, therefore, a body be balanced at any two points of its surface, and verticals be drawn through the points, in these positions, their intersection will be the centre of gravity of the body.

If a body be suspended by an axis, it can only be at rest when the centre of gravity is in a vertical plane through the axis.

The centre of gravity may be above, below, or on the axis.

In the first case, if the body be slightly deranged, it will continue to revolve till the centre of gravity falls below the axis; in the second case, it will return to its primitive position; in the third case, it will remain in the position in which it is placed.

Centre of Gravity of a System of Bodies.

55. When we have several bodies, and it is required to find their common centre of gravity, it will often be found convenient to employ the principle of moments. To do this, we first find the centre of gravity of each body separately, by rules already given. The weight of each body is then regarded as a force, applied at the centre of gravity of the body. The weights being parallel, we have a system of parallel forces, whose points of application are known. If these points are all in the same plane, we find the lever arms of the resultant of all the weights, with respect to two lines, at right angles to each other in that plane; and these will make known the point of application of the resultant, or, what is the same thing, the centre of gravity of the system. If the points are not in the same plane, the lever arms of the resultant are found, with respect to three axes, at right angles to each other; these make known the point of application of the resultant weight, or the position of the centre of gravity.

EXAMPLES.

1. Required the point of application of the resultant of three equal weights, applied at the vertices of a plane triangle.

SOLUTION.

Let ABC (Fig. 30) represent the triangle. The resultant of the weights at B and C will be applied at D, the middle of BC. The weight acting at D being double that at A, the total resultant will be applied at G, making $GA = 2\,GD$; hence, *the required point is the centre of gravity of the triangle.*

2. Required the point of application of the resultant of a system of equal parallel forces, applied at the vertices of a regular polygon?

Ans. At the centre of the polygon.

3. Parallel forces of 3, 4, 5, and 6 lbs., are applied at the successive vertices of a square, whose side is 12 inches. At what distance from the first vertex is the point of application of their resultant?

SOLUTION.

Take the sides of the square through the first vertex as axes; call the side through the first and second vertex, the axis of X, and that through the first and fourth, the axis of Y. We shall have, from Formulas (21, 22),

$$x_1 = \frac{4 \times 12 + 5 \times 12}{18} = 6;$$

and

$$y_1 = \frac{6 \times 12 + 5 \times 12}{18} = \frac{22}{3}.$$

Denoting the required distance by d, we have,

$$d = \sqrt{x_1^2 + y_1^2} = 9.475 \text{ in.} \quad \textit{Ans.}$$

4. Seven equal forces are applied at seven of the vertices of a cube. What is the distance of the point of application of their resultant from the eighth vertex?

SOLUTION.

Take the eighth vertex as the orgin of co-ordinates, and the three edges passing through it as axes. We shall have, from Equations (21, 22, 23), denoting one edge of the cube by a,

$$x_1 = \tfrac{4}{7}a, \; y_1 = \tfrac{4}{7}a, \text{ and } z_1 = \tfrac{4}{7}a.$$

Denoting the required distance by d, we have,

$$d = \sqrt{x_1^2 + y_1^2 + z_1^2} = \tfrac{4}{7}a\sqrt{3}. \quad \textit{Ans.}$$

5. Two isosceles triangles are constructed on opposite sides of the

base b, having altitudes equal to h and h', h being greater than h'. Where is the centre of gravity of the space within the two triangles?

SOLUTION.

It must lie on the altitude of the greater triangle. Take the common base as an axis of moments; then will the moments of the triangles be $\frac{1}{2}bh \times \frac{1}{3}h$, and $\frac{1}{2}bh' \times \frac{1}{3}h'$; and from Formula (21), we have,

$$x_1 = \frac{\frac{1}{6}b(h^2 - h'^2)}{\frac{1}{2}b(h + h')} = \frac{1}{3}(h - h').$$

That is, the centre of gravity is on the altitude of the greater triangle, at a distance from the base equal to one-third of the difference of the two altitudes.

6. Where is the centre of gravity of the space between two circles tangent to each other internally?

SOLUTION.

Take their common tangent as an axis of moments. The centre of gravity will lie on the common normal, and its distance from the point of contact is given by the equation,

$$x_1 = \frac{\pi r^3 - \pi r'^3}{\pi r^2 - \pi r'^2} = \frac{r^2 + rr' + r'^2}{r + r'}.$$

7. Let there be a square, divided by its diagonals into four equal parts, one of which is removed. Required the distance of the centre of gravity of the remaining figure from the opposite side of the square. *Ans.* $\frac{7}{18}$ of the side of the square.

8. To construct a triangle, having given its base and centre of gravity.

SOLUTION.

Draw through the middle of the base, and the centre of gravity, a straight line; lay off beyond the centre of gravity a distance equal to twice the distance from the middle of the base to the centre of gravity. The point thus found is the vertex.

9. Three men carry a cylindrical bar, one taking hold of one end, and the others at a common point. Required the position of this point, in order that the three may sustain equal portions of the weight.

Ans. At three-fourths the length of the cylinder from the first.

STABILITY AND EQUILIBRIUM.

Stable, Unstable, and Indifferent Equilibrium.

56. A body is in *stable* equilibrium when, on being slightly disturbed from a state of rest, it has a tendency to return to that state. This will be the case when the centre of gravity of the body is at its lowest point. Let *A* be a body suspended from an axis *O*, about which it is free to turn. When the centre of gravity of *A* lies vertically below the axis, it is in equilibrium, for the weight of the body is exactly counterbalanced by the resistance of the axis. Moreover, the equilibrium is *stable;* for if the body be deflected to *A′*, its weight acts with the lever arm *OP* to restore it to its position of rest, *A*.

Fig. 38.

A body is in *unstable* equilibrium when on being slightly disturbed from its state of rest, it tends to depart still farther from it. This will be the case when the centre of gravity of the body occupies its highest position.

Let *A* be a sphere, connected by an inflexible rod with the axis *O*. When the centre of gravity of *A* is vertically above *O*, it is in *unstable* equilibrium; for, if the sphere be deflected to the position *A′*, its weight will act with the lever arm *OP* to increase the deflection. The motion continues till, after a few vibrations, it comes to rest below the axis. In this last position, it is in stable equilibrium.

Fig. 39.

A body is in *indifferent*, or *neutral*, equilibrium when it remains at rest, wherever it may be placed. This is the case when the centre of gravity continues in the same horizontal plane on being slightly disturbed.

Let *A* be a sphere, supported by a horizontal axis *OP* through its centre of gravity. Then, in whatever position it may be placed, it will have no tendency to change this position; it is, therefore, in *indifferent*, or *neutral* equilibrium.

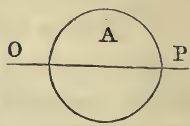


Fig. 40.

In the figure, *A*, *B*, and *C*, represent a cone in positions of *stable*, *unstable*, and *indifferent* equilibrium.

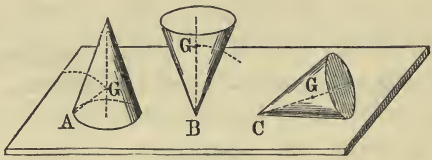


Fig. 41.

If a wheel be mounted on a horizontal axis, about which it is free to turn, the centre of gravity not lying on the axis, it will be in stable equilibrium, when the centre of gravity is directly below the axis; and in unstable equilibrium when it is directly above the axis. When the axis passes through the centre of gravity, it will, in every position, be in neutral equilibrium.

We infer, from the preceding discussion, that when a body at rest is so situated that it cannot be disturbed from its position without raising its centre of gravity, it is in a state of *stable equilibrium;* when a slight disturbance depresses the centre of gravity, it is in a state of *unstable equilibrium;* when the centre of gravity remains constantly in the same horizontal plane, it is in a state of *neutral equilibrium.*

This principle holds true in the combinations of wheels and other pieces used in machinery, and indicates the importance of balancing these elements, so that their centres of gravity may remain in the same horizontal planes.

Stability of Bodies on a Horizontal Plane.

57. A body resting on a horizontal plane may touch it in one, or in more than one point. In the latter case, the salient polygon, formed by joining the extreme points of contact, as *abcd*, is called the *polygon of support.*

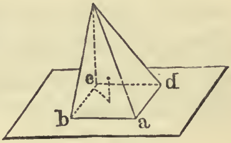


Fig. 42.

When the direction of the weight of the body, that is, the vertical through its centre of gravity, pierces the plane within the polygon of support, the body is *stable,* and will remain in equilibrium, unless acted upon by some other force than the weight of the body. In this case, the body will be most easily overturned about that side of the polygon of support which is nearest to the line of direction of the weight. The moment of the weight, with respect to this side, is called the *moment of stability.* Denoting the weight of the body by W, the distance from its line of direction to the nearest side of the polygon of support by r, and the moment of stability by S, we have,

$$S = Wr.$$

The moment of stability is the moment of the least extraneous force that is capable of overturning the body. The weight of a body remaining the same, its *stability* increases with r. If the polygon of support is a regular polygon, the stability will be greatest, other things being equal, when the direction of the weight passes through its centre. The area of the polygon of support remaining constant, the stability will be greater as the polygon approaches a circle. The polygon of support being regular, but variable in area, the stability will increase as the area increases: low bodies with extended bases, are more stable than high bodies with narrow bases.

When the direction of the weight passes without the polygon of support, the body is *unstable*, and unless supported by some other force than the weight, it will turn about the side nearest the direction of the weight. In this case, the product of the weight into the distance from its direction to the nearest side of the polygon, is called *the moment of instability*. Denoting this moment by I, we have, as before,

$$I = Wr.$$

The moment of instability is equal to the least moment of a force that can prevent the body from overturning.

If the direction of the weight intersect any side of the polygon of support, the body will be *in a state of equilibrium bordering on rotation about that side.*

If the resultant of all the forces acting on a body, including its weight, be oblique to the supporting plane, it may be resolved into two components, one perpendicular to the plane and the other parallel to it. The former is counteracted by the reaction of the plane; the latter tends to make the body to slide along the plane. Hence the importance of making the resultant as nearly normal to the supporting plane as possible.

These principles find application in the arts, more especially in Engineering and Architecture. In structures intended to be stable, the foundation should be as broad as is consistent with the general design of the work, that the polygon of support may be as large as possible. The pieces for transmitting pressures should be so combined that the pressures may be as nearly normal to the bearing surfaces as possible, and their lines of direction should pass as near the centres of the polygons of support as may be. Hence, joints should be made as nearly normal to the pressures as possible.

In the construction of machinery the centres of gravity of rotating pieces should be in their axes, otherwise there will result an irregularity of motion, which, besides making the machine work imperfectly, will ultimately destroy the machine itself.

In loading cars, wagons, &c., we should throw the centre of gravity of the load as near the track as possible. This is partially effected by placing the heavier articles at the bottom of the load.

Pressure of one body on another.

58. Let A be a movable body pressed against a fixed body B, and touching it at a single point. In order that A may be in equilibrium, the resultant of all the forces acting on it, including its weight, must pass through the point of contact, P'; otherwise there would be a tendency to rotation about P', which would be measured by the moment of the resultant with respect to this point. Furthermore,

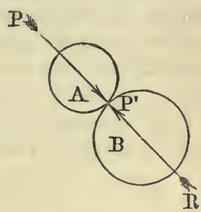


Fig. 43.

the direction of the resultant must be normal to the surface of B at the point P', else the body A would have a tendency to slide along the body B, which tendency would be measured by the tangential component. The pressure on B develops a force of reaction, which is equal and directly opposed to it. The resultant of all the forces including the reaction must be equal to zero (Art. 41). That is, *when a body, resting on another and acted upon by any number of forces, is in equilibrium, the resultant of all the forces called into play is equal to* 0.

If all the forces called into play are taken into account, *the algebraic sums of their moments with respect to any three rectangular axes will be separately equal to* 0.

If the bodies A and B touch in more than one point, the polygonal figure formed by uniting the extreme points of contact may be called the polygon of contact. In this case, the resultant of all the forces must pass within the polygon of contact.

Practical Problems.

1. A horizontal beam AB, which sustains a load, is supported on a pivot at A, and by a cord DE, the point E being vertically over A. Required the tension of DE, and the vertical pressure on A.

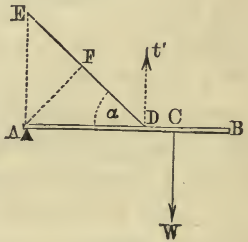


Fig. 44.

SOLUTION.

Denote the weight of the beam and load by W, and suppose its point of application to be C. Denote CA by p, the perpendicular distance, AF, from A to DE, by p', and the tension of the cord by t. If we take A as a centre of moments, we have, in case of equilibrium,

$$Wp = tp'; \qquad \therefore\ t = W\frac{p}{p'}.$$

Or, denoting the angle EDA by α, and the distance AD by b, we have,

$$p' = b \sin \alpha; \qquad \therefore\ t = W\frac{p}{b \sin\alpha}.$$

To find the vertical pressure on A, resolve t into components, parallel and perpendicular to AB. We have for the latter component, denoted by t',

$$t = t \sin \alpha = W\frac{p}{b}.$$

The vertical pressure on A, plus the weight W, must be equal to t'. Denoting the vertical pressure by P, we have,

$$P + W = W\frac{p}{b}; \text{ or, } P = W\left(\frac{p}{b} - 1\right) = W\left(\frac{p-b}{b}\right);$$

or,

$$P = W\frac{DC}{AD}.$$

When $DC=0$; or, when D and C coincide, the vertical pressure is 0.

2. A rope, AD, supports a pole, DO, one end of which rests on a horizontal plane, and from the other is suspended a weight W. Required the tension of the rope, and the thrust, or pressure, on the pole, its weight being neglected.

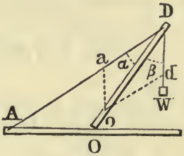


Fig. 45.

SOLUTION.

Denote the tension of the rope by t, the pressure on the pole by p, the angle ADO by α, and the angle ODW by β.

There are three forces acting at D, which hold each other in equilibrium; the weight W, acting downward, the tension of the rope, acting from D, toward A, and the reaction of the pole, acting from O toward D. Lay off Dd, to represent the weight, and complete the parallelogram $doaD$; then will Da represent the tension of the rope, and Do the thrust on the pole.

From Art. 33, we have,

$$t : W : : \sin\beta : \sin\alpha; \qquad \therefore\ t = W\frac{\sin\beta}{\sin\alpha}.$$

We have, also, from the same principle,

$$p : W : : \sin(\alpha+\beta) : \sin\alpha; \qquad \therefore\ p = W\frac{\sin(\alpha+\beta)}{\sin\alpha}.$$

If the rope is horizontal, we have $\alpha = 90° - \beta$, which gives,

$$t = W\tan\beta, \text{ and } p = \frac{W}{\cos\beta}.$$

3. A beam FB, is suspended by ropes attached at its extremities, and fastened to pins A and H. Required the tensions of the ropes.

SOLUTION.

Denote the weight of the beam and its load by W, and let c be its point of application. Denote the tension of the rope BH, by t, and that of FA by t'. The forces in equilibrium, are W, t, and t'. The plane of these forces must be vertical, and further, the directions of the forces must intersect in a point. Produce AF, and BH, till they intersect in K, and draw Kc; take Kc, to represent the weight of the beam and its load, and complete the parallelogram $KbCf$; then

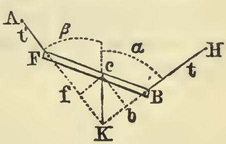


Fig. 46.

will Kb represent t, and Kf will represent t'. Denote the angle cKB by α, and cKF by β. We shall have, as in the last problem,

$$W : t :: \sin(\alpha + \beta) : \sin\beta; \qquad \therefore\ t = W\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

And,

$$W : t' :: \sin(\alpha + \beta) : \sin\alpha; \qquad \therefore\ t' = W\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

4. A gate AH, is supported at O on a pivot, and at A by a hinge, attached to a post AB. Required the pressure on the pivot, and the tension of the hinge.

SOLUTION.

Denote the weight of the gate and its load, by W, and let C be its point of application. Produce the vertical through C, till it intersects the horizontal through A in D, and draw DO. Then will AD and DO be the directions of the required components of W. Lay off Dc, to represent W, and complete the parallelogram, $Dcoa$; then will Do represent the pressure on O, and aD the tension on the hinge, A. Denoting the angle oDc by α, the pressure on the pivot by p, and on the hinge by p', we have,

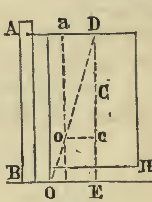


Fig. 47.

$$p = \frac{W}{\cos\alpha}, \text{ and } p' = p\sin\alpha.$$

If we denote OE by b, and DE by h, we shall have,

$$\cos\alpha = \frac{h}{\sqrt{b^2+h^2}}, \text{ and } \sin\alpha = \frac{b}{\sqrt{b^2+h^2}}.$$

Hence,

$$p = \frac{W\sqrt{b^2+h^2}}{h}, \text{ and } p' = \frac{pb}{\sqrt{b^2+h^2}}.$$

5. Having two rafters, AC and BC, abutting in notches of a tie-beam AB, it is required to find the pressure, or *thrust*, on the rafters, and the direction and intensity of the pressure on the joints at the tie-beam.

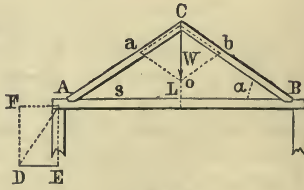


Fig. 48.

SOLUTION.

Denote the weight of the rafters and their load by $2w$; we may regard this weight as made up of three parts—a weight w, applied at

C, and two equal weights $\frac{1}{2}w$, applied at A and B respectively. Denote the half span AL by s, the rise CL by h, and the length of the rafter AC or CB by l. Denote, also, the angle CBL by α, the thrust on each rafter by t, and the resultant pressure at each of the joints A and B by p.

Lay off Co to represent the weight w, and complete the parallelogram $Cboa$; then will Ca and Cb represent the thrust on the rafters; and, since $Cboa$ is a rhombus, we have,

$$t \sin \alpha = \tfrac{1}{2}w \qquad \therefore\ t = \frac{w}{2 \sin\alpha} = \frac{wl}{2h}.$$

Conceive t to be applied at A, and there resolve it into components parallel to CL and LA; we have, for these components,

$$t \sin \alpha = \tfrac{1}{2}w, \text{ and } t \cos \alpha = \frac{ws}{2h}.$$

The latter component gives the strain on the tie-beam, AB.

To find the pressure on the joint, we have, acting downward, the forces $\frac{1}{2}w$ and $\frac{1}{2}w$, or the single force w, and, acting from L toward A, the force $\frac{ws}{2h}$; hence,

$$p = \sqrt{w^2 + \frac{w^2 s^2}{4h^2}} = \frac{w}{2h}\sqrt{4h^2 + s^2}$$

If we denote the angle DAE by β, we have from the right-angled triangle DAE,

$$\tan \beta = \frac{DE}{AE} = \frac{ws}{2h} \div w = \frac{s}{2h}.$$

The joint should be perpendicular to the force p, that is, it should make with the horizon an angle whose tangent is $\frac{s}{2h}$.

6. In the last problem suppose the rafters to abut against the wall. Required the least thickness that must be given to it to prevent it from being overturned.

SOLUTION.

Denote the weight thrown on the wall by w, the length of wall that sustains the pressure p by l', its height by h', its thickness by x, and the weight of each cubic foot of the wall by w'; then will the weight of this part be $w'h'l'x$.

The force $\frac{ws}{2h}$ acts with an arm of lever h' to overturn the wall about its lower and outer edge; this force is resisted by the weight

$w + w'h'l'x$, acting through the centre of gravity of the wall with a lever arm equal to $\frac{1}{2}x$. If there be an equilibrium, the moments of these two forces are equal, that is,

$$\frac{ws}{2h} \times h' = (w + w'h'l'x)\frac{x}{2}, \text{ or } \frac{wsh'}{h} = wx + w'h'l'x^2.$$

Reducing, we have, $x^2 + \frac{w}{w'h'l'}x = \frac{ws}{w'l'h}$;

$$\text{or, } x = -\frac{w}{2w'h'l'} \pm \sqrt{\frac{ws}{w'hl'} + \frac{w^2}{4w'^2h'^2l'^2}}.$$

7. A sustaining wall has a cross section in the form of a trapezoid, the face on which the pressure is thrown being vertical, and the opposite face having a slope of *six* perpendicular to *one* horizontal. Required the least thickness that must be given to the wall at top, that it may not be overturned by a horizontal pressure, whose point of application is at a distance from the bottom of the wall equal to one-third its height.

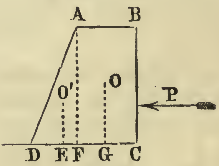


Fig. 49.

SOLUTION.

Pass a plane through the edge A parallel to BC, and consider a portion of the wall whose length is one foot. Denote the pressure on this by P, the height of the wall by $6h$, its thickness at top by x, and the weight of a cubic foot by w. Let fall from the centres of gravity O and O' of the two portions, perpendiculars OG and $O'E$, and take the edge D as an axis of moments. The weight of the portion $ABCF$ is equal to $6whx$, and its lever arm, DG, is equal to $h + \frac{1}{2}x$. The weight of the portion ADF is $3wh^2$, and its lever arm, DE, is $\frac{2}{3}h$. In case of equilibrium, the sum of the moments of their weights must be equal to the moment of P, whose lever arm is $2h$. Hence,

$$6whx(h + \tfrac{1}{2}x) + 3wh^2 \times \tfrac{2}{3}h = P \times 2h;$$

or,
$$6whx + 3wx^2 + 2wh^2 = 2P.$$

Whence,
$$x^2 + 2hx = \frac{2(P - wh^2)}{3w};$$

$$\therefore\ x = -h \pm \sqrt{\frac{2(P - wh^2)}{3w} + h^2}.$$

8. Required the conditions of stability of a square pillar acted on by a force oblique to the axis, and applied at the centre of gravity of the upper base.

SOLUTION.

Denote the intensity of the force by P, its inclination to the vertical by α, the breadth of the pillar by $2a$, its height by x, and its weight by W. Through the centre of gravity of the pillar draw a vertical AC, and lay off AC equal to W; prolong PA and lay off AB equal to P; complete the parallelogram $ABDC$, and prolong the diagonal till it intersects HG at F. If F is between H and G, the pillar will be stable; if at H, it will be indifferent; if without H, it will be unstable. To find an expression for FG, draw DE perpendicular to AG. From the similar triangles ADE and AFG, we have,

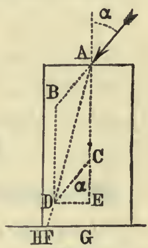


Fig. 50.

$$AE : AG :: DE : FG; \qquad \therefore FG = \frac{AG \times DE}{AE}.$$

But $AG = x$, $DE = P\sin\alpha$, and $AE = W + P\cos\alpha$, hence, we have,

$$FG = \frac{Px\sin\alpha}{W + P\cos\alpha};$$

And, since HG equals a, we have the following conditions for stability, indifference, and instability, respectively:

$$a > \frac{Px\sin\alpha}{W + P\cos\alpha};$$

$$a = \frac{Px\sin\alpha}{W + P\cos\alpha};$$

$$a < \frac{Px\sin\alpha}{W + P\cos\alpha}.$$

CHAPTER IV.

ELEMENTARY MACHINES.

Definitions and General Principles.

59. A MACHINE is a contrivance by means of which a force applied at one point is made to produce an effect at some other point.

The applied force is called the *power*, and the force to be overcome the *resistance;* the source of the power is called the *motor*.

Some of the more common motors are *muscular effort*, as exhibited by man and beast in various kinds of work; the *weight and living force of water*, as shown in the various kinds of water-mills; the *expansive force of vapors and gases*, as displayed in steam and caloric engines; the *force of air in motion*, as exhibited in the windmill, and in the propulsion of sailing vessels; the *force of magnetic attraction and repulsion*, as shown in the magnetic telegraph and various magnetic machines; the *elastic force of springs*, as shown in watches and various other machines. Of these the most important are *steam* and *water power*.

Work.

60. Work is the effect produced by a force in overcoming a resistance; it implies the simultaneous existence of both pressure and motion.

The measure of the work done by a force, is the product of the effective pressure, by the distance through which it is exerted.

Machines simply transmit and modify the action of forces. They add nothing to the work of the motor; on the contrary, they absorb and render inefficient much of that which is impressed on them. For example, in a water-mill, only a portion of the *work* expended by the motor is transmitted to the machine, on account of the imperfect manner of applying it, and of this portion a large part is absorbed and rendered practically useless by resistances, so that only a small portion of the work expended by the motor becomes *effective.*

Of the *applied work*, a part is expended in overcoming *friction, stiffness of cords, bands,* or *chains, resistance of the air, adhesion of the parts,* &c. This goes to *wear out* the machine. A second portion is expended in overcoming shocks, arising from the nature of the work to be accomplished, as well as from imperfect connection of the parts, and from want of hardness and elasticity in the connecting pieces. This also goes to *strain and wear out* the machine, and to increase the waste already mentioned. There is often a waste of work arising from a greater supply of motive power than is required to attain the desired result. Thus, in the movement of a train of cars, the excess of work of the steam, above what is necessary to bring the train to the station, is wasted, being consumed by the application of brakes, an operation that not only wears out the brakes, but also, by creating shocks, ultimately destroys the cars themselves.

Such are some of the sources of loss of work. A part of these may, by judicious combinations, be greatly diminished; but, under the most favorable circumstances, there is a continued loss of work, which requires a continued supply of power.

In a machine, the quotient obtained by dividing the quantity of *useful*, or *effective work*, by the quantity of *applied work*, is called the *modulus* of the machine. As the resistances are diminished, the modulus increases, and the machine becomes more perfect. Could the modulus become equal to 1, the machine would be *perfect*. Once set in motion, it would continue to move forever, realizing the idea of *perpetual motion*. It is needless to say that, until the laws of nature are changed, no such realization can be looked for.

Trains of Mechanism.

61. A machine usually consists of an assemblage of moving pieces called *elements*, kept in position by a connected system called a *frame*. Of the moving pieces, that which receives the power is called the *recipient*, that which performs the work, is called the *operator* or *tool*, and the connecting pieces constitute what is called a *train of mechanism*. Of two consecutive elements, that which imparts motion is called a *driver*, and that which receives *motion* is called a *follower*. Each piece, except the extremes, is a *follower*, with respect to that which precedes, and a *driver*, with respect to that which follows.

In studying a train of mechanism we find the relation between the power and resistance for each element neglecting hurtful resistances. We then modify these results so as to take account of all these resistances, such as friction, adhesion, stiffness of cords, &c. Having found the relation between the power and resistance for each piece, we begin at one extreme and combine them, recollecting, that the *resistance* for each driver is equal to the *power* for its follower.

We might also find the modulus of each element, and take the product of these partial moduli as the modulus of the machine.

We shall first show the relations between the power and resistance in the different elements on the supposition that there are no hurtful resistances.

The Mechanical Powers.

62. The *elements* to which all machines can be reduced, are sometimes called *mechanical powers.* They are seven in number—viz., the *cord,* the *lever,* the *inclined plane,* the *pulley,* the *wheel and axle,* the *screw,* and the *wedge.* The first three are *simple* elements; the pulley, and the wheel and axle are combinations of the cord and lever; the screw is a combination of two inclined planes twisted round an axle; and the wedge is a simple combination of two inclined planes.

The Cord.

63. Let AB be a cord solicited by two forces, P and R, applied at its extremities, A and B. In order that the cord may be in equilibrium, it is evident, in the first place, that the forces must act in the direction of the cord, and in such manner as to stretch it, otherwise the cord would bend; and in the second place, the forces must be equal, otherwise the greater would prevail, and motion would ensue. Hence, if two forces applied at the extremities of a cord are in equilibrium, *the forces are equal and directly opposed.*

Fig. 51.

The tension of a cord is the force by which any two of its adjacent particles are urged to separate. If a cord be solicited in opposite directions by equal forces, its tension is measured by either force. If the forces are unequal, the tension is measured by the less.

Let AB be a cord solicited by groups of forces applied at its extremities. In order that these forces may be in equilibrium, the resultants of the groups at A and B must be *equal and directly opposed.* Hence, if we suppose the forces at each point resolved into com-

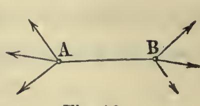


Fig. 52.

ponents coinciding with, and at right angles to, AB, *the normal components at each point must be in equilibrium, and the resultants of the remaining components at A and B must be equal and directly opposed.*

Let $ABCD$ be a cord, at the points A, B, C, D, of which groups of forces are applied. If these forces are in equilibrium through the intervention of the cord, there must necessarily be an equilibrium at each point, and this whatever may be the lengths of AB, BC, and CD.

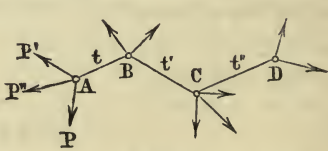


Fig. 53.

If we make these infinitely small, the equilibrium will still subsist. But in that case the points A, B, C, and D, will coincide, and all the forces will be applied at a single point. Hence, we conclude, *that a system of forces applied in any manner at points of a cord will be in equilibrium, when, if applied at a single point without change of intensity or direction, they will maintain each other in equilibrium.*

Hence, cords in machinery simply transmit the action of forces, without modifying their effects in any other manner.

The Lever.

64. A *lever* is an inflexible bar, free to turn about an axis, called the *fulcrum.*

Levers are divided into three classes, according to the relative positions of the points of application of the power and resistance.

In the *first* class, the fulcrum is between the power and resistance. The ordinary balance is an example of this class of levers. The substance to be weighed is the resistance; the counterpoising weight is the power, and the axis of suspension is the fulcrum.

1st Class.

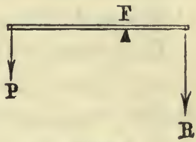


Fig. 54.

In the *second* class, the resistance is between the power and the fulcrum. The ordinary nut-cracker is an example of this class. The nut is the resistance; the power is applied at the ends of the blades, and the fulcrum is at the hinge.

2d Class.

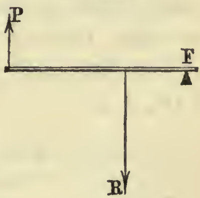


Fig. 55.

In the *third* class, the power is between the fulcrum and the resistance. A pair of tongs furnishes an example of this class. The resistance is the substance seized between the blades; the power is applied at the middle of the blades; and the fulcrum is at the hinge.

3d Class.

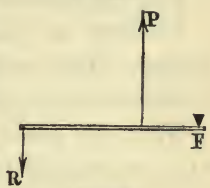


Fig. 56.

Levers may be *curved*, or *straight;* and the power and resistance may be either parallel or oblique to each other. We shall suppose the power and resistance to be perpendicular to the fulcrum; for, if not so situated, we might conceive each to be resolved into two components—one perpendicular, and the other parallel, to the axis. The latter would bend the lever laterally, or make it slide along the axis, developing hurtful resistance, while the former alone would tend to turn the lever about the fulcrum.

The perpendicular distances from the fulcrum to the lines of direction of the power and resistance, are called *lever arms* of these forces. In the bent lever MFN, the perpendicular distances FA, and FB, are the lever arms of P and R.

To determine the conditions of equilibrium of the lever, let us denote the power by P, the resistance by R, and their lever arms by p and r. We have the case of a body restrained by an axis, and if we take this as the axis of moments, we shall have for the condition of equilibrium (Art. 41),

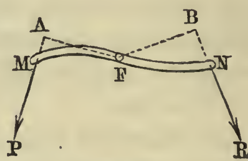


Fig. 57.

$$Pp = Rr; \text{ or, } P : R :: r : p \quad \ldots\ldots (30)$$

That is, *the power is to the resistance, as the lever arm of the resistance, to the lever arm of the power.*

This relation holds good for every kind of lever.

The ratio of the power to the resistance when in equilibrium, either statical or dynamical, is called the *leverage*, or *mechanical advantage*.

When the power is less than the resistance, there is said to be *a gain of power, but a loss of velocity;* the space passed over by the power, in performing any work, is as many times greater than that passed over by the resistance, as the resistance is greater than the power. When the power is greater than the resistance, there is said to be *a loss of power, but a gain of velocity.* When the power and resistance are equal, there is neither gain nor loss of power, but simply a change of direction.

In levers of the first class, there may be either gain or loss of power; in those of the second class, there is always gain of power; in those of the third class, there is always loss of power. A gain of power is always attended with a corresponding loss of velocity, and the reverse.

If several forces act on a lever at different points, all being perpendicular to the direction of the fulcrum, they will be in equilibrium, when *the algebraic sum of their moments, with respect to the fulcrum, is equal to* 0.

Among the forces must be included the weight of the lever, which is to be regarded as vertical force, applied at the centre of gravity.

The pressure on the fulcrum is the resultant of all the forces, including the weight of the lever.

The Compound Lever.

65. A compound lever is a combination of simple levers AB, BC, CD, so arranged that the resistance in one acts as a power in the next, throughout the combination. Thus, a power P produces at B a resistance R', which, in turn, produces at C a resistance R'', and so on. Let us assume the notation of the figure. From the principle of the simple lever, we have the relations,

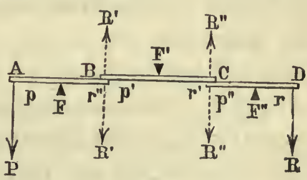


Fig. 58.

$$Pp = R'r'',\ R'p' = R''r',\ R''p'' = Rr.$$

Multiplying these equations, member by member, and striking out common factors, we have,

$$Ppp'p'' = Rrr'r''; \text{ or, } P : R :: rr'r'' : pp'p''. \quad . \quad . \quad (31)$$

And similarly for any number of levers.

Hence, in the compound lever, *the power is to the resist-*

ance as the continued product of the alternate arms of lever, commencing at the resistance, is to the continued product of the alternate arms of lever, commencing at the power.

By suitably adjusting the simple levers, any amount of mechanical advantage may be obtained.

The Elbow-joint Press.

66. Let CA, BD, and DE represent bars, with hinge-joints at B and D. The bar CA has its fulcrum at C, and the bar DE works through a guide between D and E. When A is depressed, DE is forced against the upright F, so as to compress a body placed between E and F. This machine is called the *elbow-joint press*, and is used in printing, in moulding bullets, in striking coins and medals, in punching holes, &c.

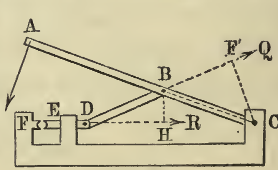


Fig. 59.

Let P denote the power applied at A, perpendicular to AC, Q the resistance in the direction DB, and R the component of Q, in the direction ED. Let C be taken as an axis of moments, and then, because P and Q are in equilibrium, we have,

$$P \times AC = Q \times F'C, \text{ or, } Q = P \times \frac{AC}{F'C}.$$

But, we have,

$$R = Q \cos BDH.$$

Substituting and reducing, we have,

$$\frac{R}{P} = \frac{AC \cos BDH}{F'C} \quad \ldots\ldots \quad (32)$$

When B is depressed, $\cos BDH$ approaches 1, and $F'C$ continually diminishes, that is, the mechanical advantage increases; and finally, when B reaches ER, it becomes infinite. There is no limit to the amount of compression that can be obtained, except that fixed by the strength of the material. It is to be observed that the space through which the pressure is exerted varies inversely as the mechanical advantage.

Weighing Machines.

67. Nearly all weighing machines depend on the principle of the lever; the resistance is the weight to be determined, and the power is a counterpoising weight of known value.

There are two principal classes of weighing machines: in the *first*, the lever arm of the power is constant, and the power varies; in the *second*, the power is constant, and its lever arm varies. The ordinary balance is an example of the first class, and the steelyard of the second.

The Common Balance.

68. The common balance consists of a lever, AB, called the beam, having a knife-edge fulcrum, F, and two scale-pans, D and E, suspended from its extremities by means of knife-edge joints at A and B. The beam is supported by a standard, FK, resting on a foot-plate, L. The standard is made vertical by levelling screws passing through the foot-plate. The knife-edges and their supports are of hardened steel; and to prevent unnecessary wear, an arrangement is made for throwing them from their bearings when not in use. A needle, N, playing in front of a graduated scale, GH, shows the amount of deflection of the beam.

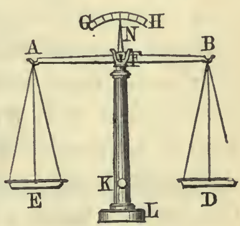


Fig. 60.

In the finer balances employed in scientific investigation, many additional contrivances are introduced, to render the machine more perfect. For a complete description of these balances the reader is referred to more extended treatises.

A good balance should fulfil the following conditions: 1°, it should be *true;* 2°, it should be *stable*—that is, when the beam is deflected it should tend to return to a horizontal position; 3°, it should be *sensitive*—that is, it should be deflected from the horizontal by a small force.

In order that a balance may be *true,* its lever arms must be equal in length, and both the beam and scale-pans must be symmetrical with respect to two planes through the centre of gravity of the beam, the first plane being perpendicular to the beam, and the second perpendicular to the fulcrum.

In order that it may be *stable,* the centre of gravity of the beam must be below the fulcrum, and the line joining the points of suspension of the scale-pans must not pass above the fulcrum.

In order that it may be *sensitive,* the line joining the points of suspension must not pass below the fulcrum, the lever arms must be as long, and the beam as light as is consistent with strength and stiffness, the knife-edges must be horizontal and parallel to each other, and the friction at the joints must be as small as possible. The sensitiveness of a balance diminishes as the load increases.

The true weight of a body may be found by a balance whose lever arms are not equal, by means of the principle demonstrated below.

Denote the length of the lever arms, by r and r', and the weight of the body, by W. When the weight W is applied at the extremity of the arm r, denote the counterpoising weight by W'; and when it is at the extremity of the arm r', denote the counterpoising weight by W''. We shall have, from the principle of the lever,

$$Wr = W'r', \text{ and } Wr' = W''r.$$

Multiplying these equations, member by member, we have,

$$W^2rr' = W''W'rr'; \quad \therefore \quad W = \sqrt{W' W''};$$

that is, *the true weight is equal to the square root of the product of the apparent weights.*

A still better method, and one that is more free from the effect of errors in construction, is to place the body to be weighed in one scale, and put weights in the other, till the beam is horizontal; then remove the body to be weighed, and replace it by known weights, till the beam is again horizontal; the sum of the replacing weights will be the weight required. If, in changing the load, the positions of the knife-edges be not changed, this method is almost perfect; but this is a condition difficult to fulfil.

The Steelyard.

69. The steelyard is an instrument for weighing bodies. It consists of a lever, AB, called the beam; a fulcrum, F; a scale-pan, D, attached at the extremity of one arm; and a known weight, E, movable along the other arm. We shall suppose the weight of E to be 1 lb. This instrument is sometimes more convenient than the balance, but it is not so accurate. The conditions of sensibility are essentially the same as for the balance. To graduate the instrument, place a pound-weight in the pan, D, and move

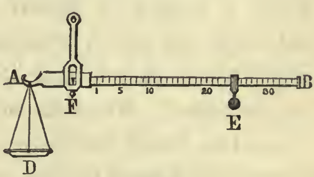


Fig. 61.

the counterpoise E till the beam rests horizontal—let that point be marked 1; next place a 10 lb. weight in the pan, and move the counterpoise E till the beam is again horizontal, and let that point be marked 10; divide the intermediate space into nine equal parts, and mark the points of division as shown in the figure. These spaces may be subdivided at pleasure, and the scale extended to any desirable limits. We have supposed the centre of gravity to coincide with the fulcrum; when this is not the case, the weight of the instrument must be taken into account as a force applied at its centre of gravity. We may then graduate the beam by experiment, or we may compute the lever arms, corresponding to different weights, by the principle of moments.

To weigh a body with the steelyard, place it in the scale-pan, and move the counterpoise E along the beam till an equilibrium is established; the mark on the beam will indicate the weight.

The bent Lever Balance.

70. This balance consists of a bent lever, ACB; a fulcrum, C; a scale-pan, D; and a graduated arc, EF, whose centre is the centre of motion, C. When a weight is placed in the scale-pan, the pan is depressed, the weight B is raised, and its lever arm increased. When the moments of the two forces become equal, the instrument comes to rest, and the weight is indicated by a needle projecting from B, and playing in front of the arc FE. The zero of the arc FE is at the point indicated by the needle when there is no load in the pan D.

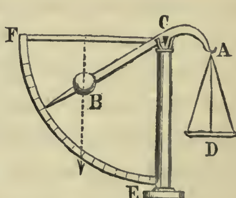


Fig. 62.

The instrument may be graduated experimentally by placing weights of 1, 2, 3, &c., pounds in the pan, and marking the points at which the needle comes to rest; or it may be graduated by the principle of moments.

To weigh a body with the bent lever balance, place it in the scale-pan, and note the point at which the needle comes to rest; the reading will give the weight sought.

Compound Balances.

71. Compound balances are used in weighing heavy articles, as merchandise, coal, freight for shipping, &c. A great variety of combinations have been employed, one of which is shown in the figure.

AB is a platform on which the object to be weighed is placed; *BC* is a guard firmly attached to the platform; the platform is supported on the knife-edge fulcrum *E*, and the piece *D*, through the medium of a brace *CD*; *GF* is a lever turning about the fulcrum *F*, and suspended by a rod from the point *L*; *LN* is a lever having its fulcrum at *M*, and sustaining the piece *D* by a rod *KH*; *O* is a scale-pan suspended from the end *N* of the lever *LN*. The instrument is so constructed, that

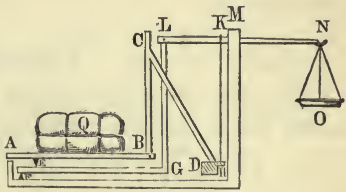


Fig. 63.

$$EF : GF :: KM : LM;$$

and *KM* is generally made equal to $\frac{1}{10}$ of *MN*. The parts are so arranged that the beam *LN* shall rest horizontally when no weight is placed on the platform.

If, now, a body *Q* be placed on the platform, a part of its weight will be thrown on the piece *D*, and, acting downward, will produce an equal pressure at K. The remaining part will be thrown on E, and, acting on the lever FG, will produce a downward pressure at G, which will be transmitted to L; but, on account of the relation given by the above proportion, the effect of this pressure on the lever LN will be the same as though the pressure thrown on E had been applied directly at K. The final effect is, therefore, the same as though the weight of Q had been applied at K, and, to counterbalance it, a weight equal to $\frac{1}{10}$ of Q must be placed in the scale-pan O.

To weigh a body, place it on the platform, and add weights to the scale-pan till LN is horizontal, then 10 times the sum of the weights added will be the weight required. By applying the principle of the steelyard to this balance, objects may be weighed by using a constant counterpoise.

Examples.

1. In a lever of the first class, the lever arm of the resistance is $2\frac{2}{3}$ inches, that of the power, $33\frac{1}{3}$, and the resistance 100 lbs. What power is necessary to hold the resistance in equilibrium? *Ans.* 8 lbs.

2. Four weights of 1, 3, 5, and 7 lbs., are suspended from points of a straight lever, eight inches apart. How far from the point of application of the first weight must the fulcrum be situated, that the weights may be in equilibrium?

SOLUTION.

Let x denote the required distance. Then, from Art. (34)

$$1 \times x + 3(x - 8) + 5(x - 16) + 7(x - 24) = 0;$$

$$\therefore\ x = 17 \text{ in.} \quad \textit{Ans.}$$

3. A lever, of uniform thickness, and 12 feet long, is kept horizontal by a weight of 100 lbs. applied at one extremity, and a force P applied at the other extremity, so as to make an angle of 30° with the horizon. The fulcrum is 20 inches from the point of application of the weight, and the weight of the lever is 10 lbs. What is the value of P, and what is the pressure on the fulcrum?

SOLUTION.

The lever arm of P is equal to 124 in. $\times$ sin 30° $=$ 62 in., and the lever arm of the weight of the lever is 52 in. Hence,

$$20 \times 100 = 10 \times 52 + P \times 62; \qquad \therefore\ P = 24 \text{ lbs. nearly.}$$

We have, also,

$$R = \sqrt{X^2 + Y^2} = \sqrt{(110 + 24 \sin 30^\circ)^2 + (24 \cos 30^\circ)^2}.$$

$$\therefore\ R = 123.8 \text{ lbs.};$$

and, $$\cos a = \frac{X}{R} = \frac{20.785}{123.8} = .16789;$$

$$\therefore\ a = 80^\circ\ 20'\ 02''.$$

4. A heavy lever rests on a fulcrum 2 feet from one end, 8 feet from the other, and is kept horizontal by a weight of 100 lbs., applied at the first end, and a weight of 18 lbs., applied at the other end. What is the weight of the lever, supposed of uniform thickness throughout?

SOLUTION.

Denote the required weight by x; its arm of lever is 3 feet. We have, from the principle of the lever,

$$100 \times 2 = x \times 3 + 18 \times 8; \qquad \therefore\ x = 18\tfrac{2}{3} \text{ lbs.} \quad \textit{Ans.}$$

5. Two weights keep a horizontal lever at rest; the pressure on the fulcrum is 10 lbs., the difference of the weights is 4 lbs., and the difference of lever arms is 9 inches. What are the weights, and their lever arms?

Ans. The weights are 7 lbs. and 3 lbs.; their lever arms are $15\frac{3}{4}$ in., and $6\frac{3}{4}$ in.

6. The apparent weight of a body weighed in one pan of a false balance is $5\frac{1}{2}$ lbs., and in the other pan it is $6\frac{6}{11}$ lbs. What is the true weight?

$$W = \sqrt{\tfrac{11}{2} \times \tfrac{72}{11}} = 6 \text{ lbs.} \quad \textit{Ans.}$$

The Inclined Plane.

72. An inclined plane is one that is inclined to the horizon.

In this machine, the power may be a force applied to a body either to prevent motion down the plane, or to produce motion up the plane, and the resistance, the weight of the body acting vertically downward. The power may be applied in any direction whatever; but we shall suppose it to be in a vertical plane, perpendicular to the inclined plane.

Let AB be an inclined plane, O a body on it, R its weight, and P the force necessary to hold it in equilibrium. In order that these two forces may keep the body at rest, their resultant must be perpendicular to AB (Art. 58).

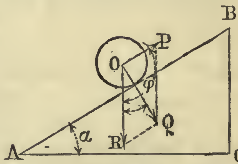


Fig. 64.

If the direction of P is given, its intensity may be found as follows: draw OR to represent the weight, and OQ perpendicular to AB; through R draw RQ parallel to OP, and through Q draw QP parallel to OR; then will OP represent the required intensity, and OQ the pressure on the plane.

If the intensity of P is given, its direction may be found as follows: draw OR and OQ as before; with R as a centre, and the given intensity as a radius, describe an arc cutting OQ in Q; draw RQ, and through O draw OP parallel, and equal to RQ; it will represent the direction of the force P.

If we denote the angle between P and R by φ, and the inclination of the plane by α, we have the angle ROQ equal to α, since OQ is perpendicular to AB, and OR to AC, and, consequently, $QOP = \varphi - \alpha$. From Art. 33 we have,

$$P : R :: \sin\alpha : \sin(\varphi - \alpha) \ldots (33)$$

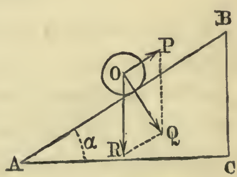


Fig. 65.

From which, if either P or φ be given, the other can be found.

When the power is parallel to the plane, we have,

$$\varphi - \alpha = 90°,$$

or,
$$\sin(\varphi - \alpha) = 1;$$

also,
$$\sin \alpha = \frac{BC}{AB}.$$

Substituting these in the preceding proportion, and reducing, we have,

$$P : R :: BC : AB \quad . \; . \; . \; . \quad (34)$$

That is, *the power is to the resistance, as the height of the plane is to its length.* When power is parallel to the base of the plane, we have, $\varphi - \alpha = 90° - \alpha$; whence,

$$\sin(\varphi - \alpha) = \cos \alpha = \frac{AC}{AB};$$

also,
$$\sin \alpha = \frac{BC}{AB};$$

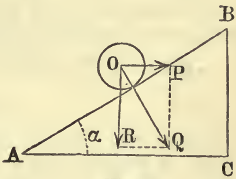


Fig. 66.

Substituting in (33), and reducing, we have,

$$P : R :: BC : AC \quad . \; . \; . \; . \quad (35)$$

That is, *the power is to the resistance, as the height of the plane is to its base.*

From the last proportion we have,

$$P = R\frac{BC}{AC} = R\tan\alpha.$$

If α increase, the value of P will increase, and when α becomes 90°, P becomes infinite; that is, no finite horizontal force can sustain a body against a vertical wall, without the aid of friction.

Examples.

1. A power of 1 lb., acting parallel to an inclined plane, supports a weight of 2 lbs. What is the inclination of the plane? *Ans.* 30°

2. The power, resistance, and normal pressure, in the case of an inclined plane, are, respectively, 9, 13, and 6 lbs. What is the inclination of the plane, and what angle does the power make with the plane?

SOLUTION.

If we denote the angle between the power and resistance by φ, and the inclination of the plane by α, we have, from (Art. 32),

$$6 = \sqrt{13^2 + 9^2 + 2 \times 9 \times 13 \cos \varphi};$$

$$\therefore \ \varphi = 156^\circ \ 8' \ 20''.$$

Also, from (Art. 33), for the inclination of the plane,

$$6 : 9 :: \sin 156^\circ \ 8' \ 20'' : \sin \alpha; \qquad \therefore \ \alpha = 37^\circ \ 21' \ 26''.$$

Inclination of power to plane $= \varphi - 90^\circ - \alpha = 28^\circ \ 46' \ 54''$. *Ans.*

3. A body is supported on an inclined plane by a force of 10 lbs., acting parallel to the plane; but it requires a force of 12 lbs. to support it when the force acts parallel to the base. What is the weight of the body, and the inclination of the plane?

Ans. The weight is 18.09 lbs., and the inclination 33° 33′ 25″.

The Pulley.

73. A pulley is a wheel having a groove around its circumference to receive a cord; the wheel turns on an axis at right angles to its plane, and this axis is supported by a frame called a *block*. The pulley is said to be *fixed*, when the block is fixed, and *movable*, when the block is movable. Pulleys are used singly, or in combinations.

Single Fixed Pulley.

74. In this machine the block is fixed. Denote the power by P, the resistance by R, and the radius of the pulley by r. It is plain that both the power and resistance should be at right angles to the axis. Hence, if we take the axis of the pulley as an axis of moments, we have, (Art. 41), in case of equilibrium,

$$Pr = Rr; \text{ or, } P = R.$$

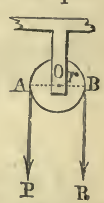


Fig. 67.

That is, *the power is equal to the resistance.*

The effect of this pulley is simply to change the direction of a force.

Single Movable Pulley.

75. In this pulley the block is movable. The resistance is applied by means of a hook attached to the block; one end of a rope, enveloping the lower part of the pulley, is attached at a fixed point, C, and the power is applied at its other extremity. We shall suppose, in the first place, that the two branches of the rope are parallel.

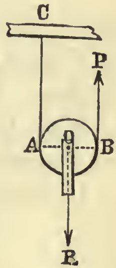


Fig. 68.

Adopting the notation of the preceding article, and taking A as a centre of movements, we have, in case of equilibrium (Art. 41),

$$P \times 2r = Rr; \quad \therefore \quad P = \tfrac{1}{2}R.$$

That is, when the power and resistance are parallel, *the power is one-half the resistance.* The tension of the cord CA is the same as that of BP. It is, therefore, equal to one-half the resistance. If the resistance of the point C be replaced by a force equal to P, the equilibrium will be undisturbed.

Let the two branches of the enveloping cord be oblique to each other. Suppose the resistance C to be replaced by a force equal to P, and denote the angle between the two branches of the rope by 2φ. If there is an equilibrium between P, P, and R, we must have

$$2\,P\cos\varphi = R.$$

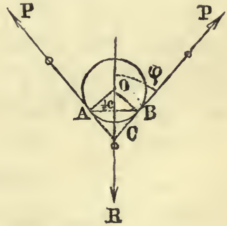


Fig. 69.

Draw the chord AB, and denote its length by c; draw, also, the radius OB. Then, because

OR is perpendicular to AB and BP to OB, the angle ABO is one-half ACB, or equal to φ. Hence,

$$\cos\varphi = \tfrac{1}{2}c \div r = \frac{c}{2r}.$$

Substituting in the preceding equation, and reducing, we have,

$$Pc = Rr; \qquad \therefore \quad P : R :: r : c \;.\;.\;.\;.\; (36)$$

That is, *the power is to the resistance, as the radius of the pulley, is to the chord of the arc enveloped by the rope.*

When the chord is greater than the radius, there is a gain of *mechanical advantage;* when less, there is a loss.

If the chord is equal to the diameter, we have, as before,

$$P = \tfrac{1}{2}R.$$

Combination of Movable Pulleys.

76. The figure represents a combination of movable pulleys, in which there are as many cords as pulleys; one end of each cord is attached at a fixed point, the other end being fastened to the hook of the next pulley in order, up to the last cord, at the second extremity of which the power is applied.

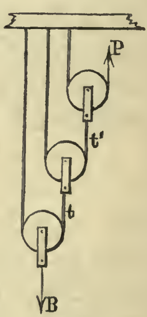


Fig. 70.

Denote the tension of the cord between the first and second pulley by t, that of the cord between the second and third pulley by t'. From the preceding Article, we have,

$$t = \tfrac{1}{2}R; \;\; t' = \tfrac{1}{2}t; \;\; P = \tfrac{1}{2}t'.$$

Multiplying these equations together, member by member, and reducing, we have,

$$P = (\tfrac{1}{2})^3 R.$$

Had there been n pulleys in the combination, we should have obtained, in a similar manner,

$$P = (\tfrac{1}{2})^n.R; \quad \therefore P : R :: 1 : 2^n \quad . \; . \; . \; . \; . \; (37)$$

That is, *the power is to the resistance, as* 1 *is to* 2^n, n *denoting the number of pulleys.*

Combinations of Pulleys in Blocks.

77. These combinations are effected in various ways. In most cases, but one rope is employed, which, being attached to a hook of one block, passes round a pulley in the other block, then round one in the first, and so on, from block to block, till it has passed round each pulley in the system. The power is applied at the free end of the rope. Sometimes the pulleys in each block are placed side by side, sometimes one above another, as in the figure, in which case the inner ones are made smaller than the outer ones. The conditions of equilibrium are the same in both cases. To deduce the conditions of equilibrium in the case represented, denote the power by P, and the resistance by R. When there is an equilibrium, the tension of each branch of the rope that aids in supporting the resistance must be equal to P; but, since the last pulley simply serves to change the direction of the force P, there will be four such branches in the case considered; hence, we shall have,

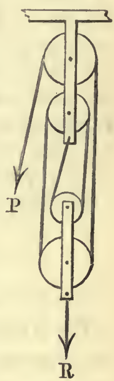


Fig. 71.

$$4P = R, \text{ or, } P = \tfrac{1}{4}R.$$

Had there been n pulleys in the combination, there would have been n supporting branches, and we should have had,

$$nP = R, \text{ or, } P : R :: 1 : n \quad . \; . \; . \; . \; . \; (38)$$

That is, *the power is to the resistance, as* 1 *is to the number of branches of the rope that support the resistance.*

The principles already considered are sufficient to determine the relation between the power and resistance in any combination whatever.

EXAMPLES.

1. In a system of six movable pulleys, of the kind described in Art. 76, what weight can be sustained by a power of 12 lbs.? *Ans.* 768 lbs.

2. In a combination of pulleys in two blocks, when there are six pulleys in each block, what weight can a power of 12 lbs. sustain in equilibrium? *Ans.* 144 lbs.

3. In a combination of separate movable pulleys, the resistance is 576 lbs., and the power that keeps it in equilibrium is 9 lbs. How many pulleys in the combination? *Ans.* 6.

4. In a combination of pulleys in two blocks, with a single rope, the power is 62 lbs., and the resistance 496 lbs. How many pulleys in each block? *Ans.* 4.

5. In a combination of two movable pulleys, the inclinations of the ropes at each pulley is 60°. What is the power required to support a weight of 27 lbs.? *Ans.* 9 lbs.

The Wheel and Axle.

78. The wheel and axle consists of a wheel, *A*, mounted on an axle, *B*. The power is applied at one extremity of a rope wrapped around the wheel, and the resistance at one extremity of a second rope, wrapped around the axle in a contrary direction. The whole is supported by pivots projecting from the ends of the axle. In deducing the conditions of equilibrium, we shall suppose the power and resistance to be at right angles to the axis.

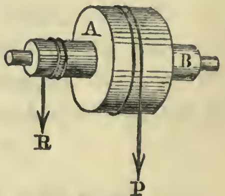


Fig. 72.

Denote the power by P, the resistance by R, the radius of the wheel by r, and the radius of the axle by r'. We shall have, in case of equilibrium (Art. 41),

$$Pr = Rr', \text{ or, } P : R :: r' : r \quad . \quad . \quad . \quad . \quad (39)$$

That is, *the power is to the resistance, as the radius of the axle, to the radius of the wheel.*

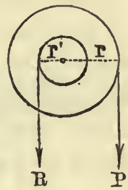


Fig. 73.

By suitably varying the dimensions of the wheel and axle, any amount of *mechanical advantage* may be obtained.

If we draw a line from the point of contact of the first rope with the wheel, to the point of contact of the second rope with the axle, the power and resistance being parallel, it will cut the axis of revolution at the point that divides the line through the points of contact into parts, inversely proportional to the power and resistance. Hence, this is the point of application of the resultant of these forces. The resultant is equal to the sum of the forces, and by the principle of moments, the pressure on each pivot may be computed. When the weight of the machine is taken into account, we regard it as a vertical force applied at the centre of gravity of the wheel and axle. The pressures on each pivot due to this weight may be computed separately, and the results combined with those already found.

Combinations of Wheels and Axles.

79. If the rope of the first axle be passed around a second wheel, and the rope of the second axle around a third wheel, and so on, a combination will result, capable of affording great mechanical advantage. The figure represents a combination of two wheels and axles. To deduce the conditions of equilibrium, denote the power by

P, the resistance by R, the radius of the first wheel by r, that of the first axle by r', that of the second wheel by r'', and that of the second axle by r'''. If we denote the tension of the connecting rope by t, this may be regarded as a power applied to the second wheel. From what was demonstrated for the wheel and axle, we shall have,

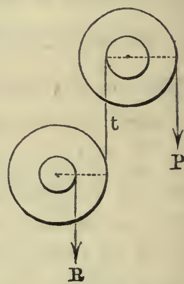


Fig. 74.

$$Pr = tr', \text{ and } tr'' = Rr'''.$$

Multiplying these equations member by member, and reducing, we have,

$$Prr'' = Rr'r'''; \text{ or, } P : R :: r'r''' : rr''.$$

In like manner, were there any number of wheels and axles in the combination, we might deduce the relation,

$$Prr''r^{\text{iv}} \ldots = Rr'r'''r^{\text{v}} \ldots;$$

or, $$P : R :: r'r'''r^{\text{v}} \ldots\ldots : rr''r^{\text{iv}} \ldots\ldots \quad (40)$$

That is, *the power is to the resistance, as the continued product of the radii of the axles, to the continued product of the radii of the wheels.*

The principle just explained, is applicable to machinery in which motion is transmitted from wheel to wheel by bands, or belts. An endless band, called the driving belt, passes around one drum mounted on the axle of the driving wheel, and around another on that of the driven wheel.

The Crank and Axle, or Windlass.

80. This machine consists of an axle, AB, and a crank, BCD. The power is applied to the crank-handle, DC, and the resistance to a rope wrapped around the axle. The distance, BC, from the handle to the axis, is the crank-arm.

The relation between the power and resistance is the same as in the wheel and axle, except that we substitute the crank-arm for the radius of the wheel.

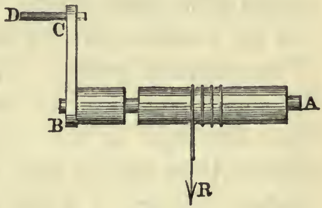


Fig. 75.

Hence, *the power is to the resistance, as the radius of the axle, to the crank-arm.*

This machine is used in drawing water from wells, in raising ore from mines, and the like. It is also used in combination with other machines. Instead of the crank, as shown in the figure, two holes are sometimes bored at right angles to each other and to the axis, and levers inserted, at the extremities of which the power is applied. The condition of equilibrium remains unchanged, provided we substitute for the crank-arm, the distance from the point of application of the power to the axis.

The Capstan.

81. The *Capstan* differs in no material respect from the windlass, except in having its axis vertical. The capstan consists of a vertical axle passing through guides, and having holes at its upper end for the insertion of levers. It is used on shipboard for raising anchors. The conditions of equilibrium are the same as in the windlass.

The Differential Windlass.

82. This differs from the common windlass in having its axle formed of two cylinders, *A* and *B*, of different diameters. A rope is attached to the larger cylinder, and wrapped several times around it, after which it passes round the movable pulley, *C*, and, returning, is wrapped in a contrary direction about the smaller cylinder, to which the second end of the rope is made fast. The power is applied at the crank-handle, FE, and the resistance to the hook of the movable pulley. When the crank is turned so as to wind the rope on the larger cylinder, it unwinds it from the smaller one, but in a less degree, and the total effect is to raise the resistance, R. To deduce the conditions of equilibrium, denote the power by P,

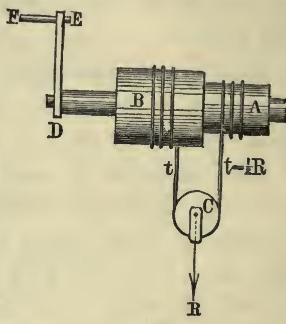


Fig. 76.

the resistance by R, the crank-arm by c, the radius of the larger cylinder by r, and that of the smaller cylinder by r'. The resistance acts equally on the two branches of the rope from which it is suspended, hence the tension of each branch may be represented by $\frac{1}{2}R$. Suppose the power acts to wind the rope on the larger cylinder. The moment of the power will be Pc; the moment of the tension of the branch A will be $\frac{1}{2}Rr'$, this acts to assist the power; the moment of the tension of the branch B will be $\frac{1}{2}Rr$, this acts to oppose the power. From the principle of moments, we have,

$$Pc + \tfrac{1}{2}Rr' = \tfrac{1}{2}Rr, \text{ or, } Pc = \tfrac{1}{2}R(r - r');$$

whence,

$$P : R :: r - r' : 2c \; . \; . \; . \; . \; . \; (41)$$

That is, *the power is to the resistance, as the difference of the radii of the cylinders, to twice the crank-arm.*

By increasing the crank-arm and diminishing the difference between the radii of the cylinders, any amount of mechanical advantage may be obtained; but the amount of rope required for a single turn is so great as to render the contrivance in the form described of little practical value. This difficulty is avoided in a machine known as WESTON'S pulley-block. In this combination, there are two pulleys nearly equal in size, and turning together as one in the upper block. An endless chain takes the place of the rope, and is prevented from slipping by projecting pins. The power is applied at the portion of the chain that leaves the larger pulley, and the chain continues to run till the weight is raised. To trace the course of the chain, let us commence at the point where it leaves the lower pulley: from this it ascends, passing around the larger pulley in the upper block; descending so as to leave a sufficient amount of slack, it again rises to the upper block, passes around the smaller pulley, and returns to the place of beginning.

Wheel-work.

83. The principle employed in finding the relation between the power and resistance in a train of wheel-work is the same as that used in discussing the wheel and axle and its modifications. To illustrate, we have taken a case in which the power is applied to a crank-handle that is attached to the axis of a toothed wheel, *A;* the teeth of this wheel work into the spaces of the toothed wheel, *B*, and the resistance is attached to a rope wound round the arbor of the last wheel. In order that *A* may communicate motion to *B*, the number of teeth in their circumferences should be proportional to their radii, and the spaces between the teeth in one wheel should be

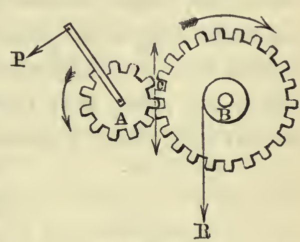


Fig. 77.

large enough to receive the teeth of the other, but not large enough to allow much play. The teeth should always come in contact at the same distances from the centres of the wheels, and those distances are taken as the radii of the wheels.

Denote the power by P, the resistance by R, the crank-arm by c, the radius of the wheel A by r, that of B by r', that of the arbor by r'', and suppose the power and resistance in equilibrium. The power tends to turn the wheels in the direction of the arrow-heads. This tendency is counteracted by the resistance which tends to produce motion in a contrary direction. If we denote the pressure at C by R', we have, from what has preceded,

$$Pc = R'r \text{ and } R'r' = Rr'';$$

whence, by multiplication and reduction,

$$Pcr' = Rrr'', \text{ or, } P : R :: rr'' : cr' \quad . \; . \; . \; . \quad (42)$$

That is, *the power is to the resistance, as the continued product of the alternate arms of lever, beginning at the resistance, to the continued product of the alternate arms of lever beginning at the power.*

Had there been any number of wheels in the train between the power and resistance, we should have found similar conditions of equilibrium.

Examples.

1. A power of 5 lbs., acting at the circumference of a wheel whose radius is 5 feet, supports a resistance of 200 lbs., applied at the circumference of the axle. What is the radius of the axle?

Ans. $1\frac{1}{2}$ inches.

2. The radius of the axle of a windlass is 3 inches, and the crank-arm 15 inches. What power must be applied to the crank-handle, to support a resistance of 180 lbs., applied at the circumference of the axle? *Ans.* 36 lbs

3. A power, P, acts on a rope 2 inches in diameter, passing over a wheel whose radius is 3 feet, and supports a resistance of 320 lbs., applied by a rope of the same diameter, passing over an angle whose radius is 4 inches. What is the value of P, the thickness of the rope being taken into account? *Ans.* $43\frac{9}{37}$ lbs.

The Screw.

84. The screw is a combination of two inclined planes twisted round an axis. It consists of a solid cylinder, enveloped by a spiral projection called the *thread.* The thread may be generated as follows: let an isosceles triangle be placed so that its base shall coincide with an element of the cylinder, and its plane pass through the axis. Let the triangle be revolved uniformly about the axis, and at the same time moved uniformly in the direction of the axis, at such a rate that it shall pass over a distance equal to the base of the triangle in one revolution. The solid generated by the triangle is the thread of the screw. The sides of the triangle generate helicoidal surfaces, which constitute the upper and lower surfaces of the thread. Each point of these lines generates a curve called a helix, which is similar to an inclined plane bent round a cylinder. The vertex generates the *outer helix*, and the angular points of the base trace out the *inner helix.* The screw just described has a triangular thread. Had we used a rectangle, instead of a triangle, and imposed the condition, that the motion in the direction of the axis during one revolution, should be twice its base, we should have had a screw with a rectangular thread, as in the figure.

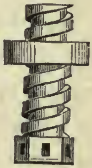


Fig. 78.

The screw works into a piece called a *nut*, generated in a manner analogous to that just described, except that what is solid in the screw is hollow in the nut; it is, therefore, exactly adapted to receive the thread of the screw. Sometimes, the screw is fast, and the nut turns on it; in this case, the nut has a motion of revolution, combined with a longitudinal motion. Sometimes, the nut is fast, and the screw turns within it; in this case, the screw has a motion in the direction of its axis, in connection with a motion of rotation. The conditions of equilibrium are the same for each. In both cases, the power is applied at the extremity of a lever. We shall suppose the nut to remain fast, and the screw to be movable, and that the resistance is parallel to the axis of the screw. If the axis is vertical, and the resistance a weight, we may regard that weight as resting on one of the helices, and sustained in equilibrium by a horizontal force. If the supporting helix be developed on a vertical plane, by unrolling the surface of the cylinder on which it lies, it will form an inclined plane, whose base is equal to the base of the cylinder on which it lies, and whose altitude is the distance between the threads of the screw.

Let AB be the development of the helix, and F the force applied parallel to the base, and *immediately* to the weight R, to sustain it on the plane. We have, (Art. 72),

$$F : R :: BC : AC.$$

But the power is actually applied through the medium of a lever. Denoting the radius, OG, of the cylinder of the supporting helix, by r, and the arm of lever of the power, P, by p, we have, from the principle of the lever,

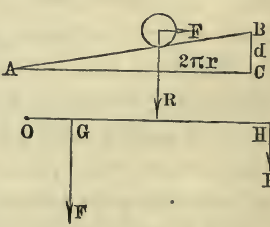


Fig. 79.

$$P : F :: r : p;$$

or,

$$P : F :: 2\pi r : 2\pi p.$$

Combining this proportion with the preceding one, and recollecting that $AC = 2\pi r$, we deduce the proportion,

$$P : R :: BC : 2\pi p \quad . \; . \; . \; . \; . \; (43)$$

That is, *the power is to the resistance, as the distance between the threads, to the circumference described by the point of application of the power.*

By diminishing the distance between the threads, other things being equal, any amount of *mechanical advantage* may be obtained.

The screw is used for producing great pressures through small distances, as in pressing books for the binder, packing merchandise, expressing oils, and the like. On account of the great amount of friction, and other hurtful resistances developed, the modulus of the machine is small.

The Differential Screw.

85. The differential screw consists of an ordinary screw, into the end of which works a smaller screw, having its axis coincident with the first. The distance between the threads of the second screw is less than that of the first, and this difference may be made as small as desirable. The second screw is so arranged that it admits of longitudinal motion, but not of rotation. By the action of the differential screw, the weight is raised vertically through a distance equal to the difference of the distances between the threads on the two screws, for each revolution of the point of application of the power.

Hence, *the power is to the resistance, as the difference of the distances between the threads of the two screws to the circumference described by the point of application of the power.*

The Endless Screw.

86. The endless screw is a screw secured by shoulders, so that it cannot be moved longitudinally, and working into a toothed wheel. The distance between the teeth is nearly the same as the distance between the threads of the screw. When the screw is turned, it imparts a rotary motion to the wheel, which may be utilized by any mechanical device. The conditions of equilibrium are the same as for the screw, the resistance in this case being offered by the wheel, in the direction of its circumference.

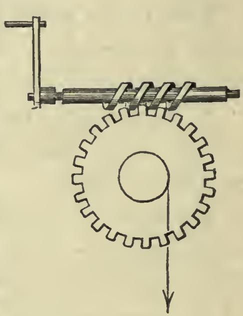


Fig. 80.

Machines of this kind are used for counting the number of revolutions of an axis. An endless screw is arranged to turn as many times as the axis, and being connected with a train of light wheel-work, the last piece of which bears an index, the number of revolutions can be ascertained at any instant. For example, suppose the first wheel to have 100 teeth, and to bear on its arbor a pinion having 10 teeth; suppose this to engage with another wheel having 100 teeth, and so on. When the endless screw has made 10,000 revolutions, the first wheel will have made 100 revolutions, the second will have made 10 revolutions, and the third 1 revolution. By a suitable arrangement of indices and dials, the exact number of revolutions, at any instant, may be read off.

Examples.

1. What must be the distance between the threads of a screw, that a power of 28 lbs., acting at the extremity of a lever 25 inches long, may sustain a weight of 10,000 lbs.? *Ans.* .4396 inches.

2. The distance between the threads of a screw is $\frac{1}{3}$ of an inch. What resistance can be supported by a power of 60 lbs., acting at the extremity of a lever 15 inches long? *Ans.* 16,964 lbs.

3. The distance from the axis of the trunions of a gun weighing 2,016 lbs. to the elevating screw is 3 feet, and the distance of the centre of gravity of the gun from the same axis is 4 inches. If the distance between the threads of the screw be $\frac{2}{3}$ of an inch, and the length of the lever 5 inches, what power must be applied to sustain the gun in a horizontal position? *Ans.* 4.754 lbs.

The Wedge.

87. The wedge is a combination of two inclined planes. It is bounded by a rectangle, BD, called the *back;* two rectangles, AF and DF, called *faces;* and two isosceles triangles, called *ends.* The line, EF, in which the faces meet, is the *edge.*

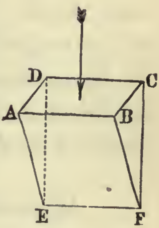


Fig. 81.

The power is applied at the back, to which it should be normal, and the resistance is applied to the faces, and normal to them. One half the resistance is applied to one face, and the other half to the other face. Let ABC be a section of a wedge by a plane at right angles to the edge. Denote the power by P, the resistance opposed to each face by $\frac{1}{2}R$, and the angle BAC by 2φ. Produce the directions of the resistances till they intersect in O. This point will be on the line of the direction of the power. Because the three forces P, $\frac{1}{2}R$, and $\frac{1}{2}R$ are in equilibrium, we have, (Art. 33),

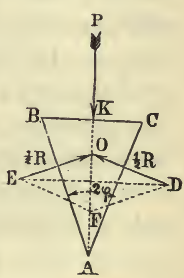


Fig. 82.

$$P : \tfrac{1}{2}R :: \sin EOD : \sin POD \quad . \ . \ . \ (44)$$

But, DO and FO are perpendicular to AC and AB; hence,

$$\sin EOD = \sin 2\varphi = 2\sin\varphi\cos\varphi.$$

In like manner, PO and DO are perpendicular to KC and AC; hence,

$$\sin POD = \sin ACK = \cos\varphi.$$

Substituting, and reducing, we have,

$$P : \tfrac{1}{2}R :: 2\sin\varphi : 1,$$

or, $$P : R :: KC : AC \ldots\ldots (45)$$

That is, *the power is to the resistance, as half the breadth of the back, is to the length of the face.*

The mechanical advantage of the wedge may be increased by diminishing the breadth of the back, or, in other words, by making the edge sharper. The principle of the wedge finds an application in cutting instruments. By diminishing the thickness of the back, the instrument is weakened; hence the necessity of forming cutting instruments of hard and tenacious material.

Application of the Principle of Virtual Moments.

88. The preceding conditions of equilibrium might have been deduced from the principle of virtual moments. To illustrate the mode of proceeding, let us take the case of a single movable pulley, and suppose P and R to be in equilibrium. Let the machine be set in motion until P has acted through a very small distance, FG, in its own direction; the force, R, will have acted in the same time through some distance, DE, *contrary* to its own direction. From the principle of virtual moments, we have,

Fig. 83.

$$P \times FG - R \times DE = 0.$$

In order that R may act through a distance, DE, each branch of the rope must be shortened by an equal amount; in other words, the force, P, must act through twice the distance, DE. Making $FG = 2DE$, and reducing, we have,

$$P = \frac{1}{2}R,$$

as already shown. In like manner, the conditions of equilibrium for other machines may be deduced.

Hurtful Resistances.

89. The principal hurtful resistances that must be taken into account in modifying the relations between the power and resistance, are *friction*, *adhesion*, *stiffness of cords*, and *atmospheric resistance*.

Friction.

90. Friction is the resistance one body experiences in moving on another, the two being pressed together by some force. This resistance arises from inequalities in the surfaces, the projections of one sinking into the depressions of the other. In order to overcome this resistance, sufficient force must be applied to break off, or bend down, the projecting points, or else to lift the moving body clear of them. The force thus applied, is equal, and directly opposed to the force of friction, which is tangential to the two surfaces. The force that presses the surfaces together, is normal to both at the point of contact.

Between certain bodies, friction is somewhat different when motion is just beginning, from what it is when motion has been established. The friction developed when a body is passing from a state of rest to a state of motion, is called *friction of quiescence;* that between bodies in motion, is called *friction of motion.*

The following *laws of friction* have been established by experiment, viz.:

First, *friction of quiescence between the same bodies, is proportional to the normal pressure, and independent of the extent of the surfaces in contact.*

Secondly, *friction of motion between the same bodies, is proportional to the normal pressure, and independent, both of the extent of surface of contact, and of the velocity of the moving body.*

Thirdly, *for compressible bodies, friction of quiescence is greater than friction of motion: for bodies which are incompressible, the difference is scarcely appreciable.*

Friction may be diminished by the interposition of unguents, which fill up the cavities, and so diminish the roughness of the rubbing surfaces. For slow motions and great pressures, the more substantial unguents are used, such as lard, tallow, and certain mixtures; for rapid motions, and light pressures, oils are generally employed.

Methods of finding the Coefficient of Friction.

91. The quotient obtained by dividing the force of friction by the normal pressure, is called the *coefficient of friction;* its value for any two substances, may be determined as follows:

Let AB be a horizontal plane formed of one of the substances, and O a cubical block of the other. Attach a string, OC, to the block, so that its direction shall pass through the centre of gravity, and be parallel to AB; let the string pass over a fixed pulley, C, and let a weight, F, be attached to its extremity.

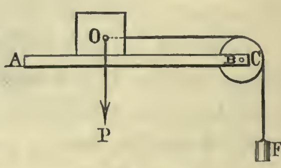


Fig. 84.

Increase F till O just begins to slide along the plane, then will F be the force of friction. Denote the normal pressure, by P, and the coefficient of friction, by f. From the definition, we have,

$$f = \frac{F}{P}.$$

In this manner, values for f may be found for different substances, and arranged in tables.

The value of f, for any substance, is the *unit,* or *coefficient* of friction. Hence, we may define the unit, or coefficient of friction, to be *the friction due to a normal pressure of one pound.*

Having the normal pressure in pounds, and the coefficient of friction, the entire friction may be found by multiplying these quantities together.

There is a second method of finding the value of f, as follows:

Let AB be an inclined plane, formed of one of the substances, and O a block, of the other. Elevate the plane till the block just begins to slide down by its own weight. Denote the inclination, at this instant, by α, and the weight of O, by W. Resolve W into two components, one normal to the plane, and the other parallel to it.

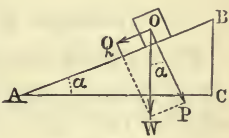


Fig. 85.

Denote the former by P, and the latter by Q. Since OW is perpendicular to AC, and OP to AB, the angle, WOP, is equal to α. Hence,

$$P = W\cos\alpha, \text{ and } Q = W\sin\alpha.$$

The normal pressure being equal to $W\cos\alpha$, and the force of friction being $W\sin\alpha$, we shall have, from the principle already explained,

$$f = \frac{W\sin\alpha}{W\cos\alpha}, \text{ or, } f = \tan\alpha.$$

The angle α is called the angle of friction.

The values of f, in some of the more common cases, are given in the following

TABLE.

Bodies between which friction takes place.	*Coefficient of friction.*
Iron on oak	.62
Cast-iron on oak	.49
Oak on oak, fibres parallel	.48
Do., do., greased	.10
Cast-iron on cast-iron	.15
Wrought-iron on wrought-iron	.14
Brass on iron	.16
Brass on brass	.20
Wrought-iron on cast-iron	.19
Cast-iron on elm	.19
Soft limestone on the same	.64
Hard limestone on the same	.38

Influence of Friction on an Inclined Plane.

92. To show the manner of taking account of friction, let us consider the case of a body sliding on an inclined plane. Let AB be the plane, O the body, P the power, situated in a plane perpendicular both to the horizon and to the given plane, and suppose the body on the eve of motion up the plane. Denote the weight of the body by R, the inclination of the plane by α,

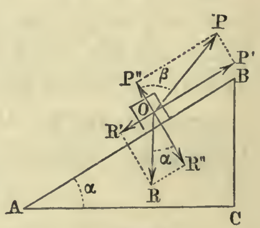


Fig. 86.

and the angle between the power and the normal to the plane by β. Let P and R be resolved into components parallel and perpendicular to the plane. We have, for the parallel components, $R\sin\alpha$ and $P\sin\beta$, and for the perpendicular components, $R\cos\alpha$ and $P\cos\beta$. The resultant of the normal components is $R\cos\alpha - P\cos\beta$; and the force of friction (Art. 91) is equal to

$$f(R\cos\alpha - P\cos\beta).$$

Because the body is on the eve of motion up the plane, the component $P\sin\beta$ must be equal and directly opposed to the resultant of the force of friction and the component $R\sin\alpha$; hence, we must have,

$$P\sin\beta = R\sin\alpha + f(R\cos\alpha - P\cos\beta).$$

Performing the multiplications indicated, and reducing, we have,

$$P = R \cdot \frac{\sin\alpha + f\cos\alpha}{\sin\beta + f\cos\beta} \quad \ldots\ldots \quad (46)$$

If an equilibrium exist, the body being on the eve of motion down the plane, we have,

$$P\sin\beta + f(R\cos\alpha - P\cos\beta) = R\sin\alpha.$$

Whence, by reduction,

$$P = R \left\{ \frac{\sin\alpha - f\cos\alpha}{\sin\beta - f\cos\beta} \right\} \quad \ldots\ldots \quad (47)$$

When α, β, and f, are given, P may be found in terms of R.

Example.

Let the plane be of oak, the sliding body of cast-iron, the inclination of the plane to the horizon 20°, and the angle between the power and a normal to the given plane 64°. Required the relation between P and R, when the body is on the eve of motion.

We have, $f = .49$; $\sin\,\alpha = .34$; $\cos\,\alpha = .94$; $\sin\,\beta = 90°$; and $\cos\,\beta = .44$. Substituting, in (46) and (47), and reducing, we have, in the former, $P = .71\,R$, and in the latter, $P = .38\,R$.

Limiting Angle of Resistance.

93. Let AB be a plane, and O a body resting on it. Let R be the resultant of all the forces acting on it, including its weight. Denote the angle between R and the normal to AB, by α, and suppose R to be resolved into two components, P and Q, the former parallel to AB, and the latter perpendicular to it; we have,

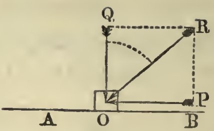


Fig. 87.

$$P = R\sin\alpha, \text{ and } Q = R\cos\alpha.$$

The friction due to the normal pressure is equal to $fR\cos\alpha$. When the tangential component $R\sin\alpha$ is less than $fR\cos\alpha$, the body will remain at rest; when it is greater than $fR\cos\alpha$, the body will slide along the plane; and when the two are equal, the body will be in a state bordering on motion along the plane. Placing the two equal, we have,

$$fR\cos\alpha = R\sin\alpha; \quad \therefore \quad \tan\alpha = f.$$

This value of α is called the limiting angle of resistance, and is equal to the inclination of the plane, when the body is about to slide down by its own weight. If OR be revolved about the normal, it will generate a conical surface, called the *limiting cone* of resistance. If the resultant of *all* the forces acting on O, lie within this cone, the body will remain at rest; if it lie without, the body will move along the plane in the direction determined by a plane through the force and the normal; if it lie on the surface of the cone, the body will be on the eve of motion along the plane in a direction determined as before. The last principle is applicable in many cases, and may be enunciated as follows: *When one body is on the eve of sliding along another, the resultant of all the forces acting on the former, including its weight, makes an angle with the normal to the surfaces*

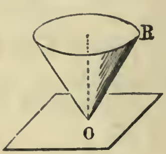


Fig. 88.

at their point of contact equal to the angle of friction of the two bodies.

Friction on an Axle.

94. The principle demonstrated in the last article enables us to determine the position of equilibrium of a horizontal axle revolving in a cylindrical box.

Let O' be the centre of the cross section of the axle, and O that of the box, and let N be their point of contact when the power is on the point of overcoming friction. At N let NT be drawn tangent to both circles. The axle may now be regarded as a body resting on the inclined plane, NT, and on the eve of sliding along it. Hence, the resultant of all the forces acting on the axle, except friction, must pass through N, and make an angle with NO equal to the angle of friction between the axle and box. If the axle be rolled further up the side of the box, it will *slide* back to N; if it be thrust down the box, it will *roll* back to N. If all the forces acting on the axle, except friction, are vertical, NT will make with the horizon an angle equal to that of friction. In this case the relation between the power and resistance may be found, as in Art. 92.

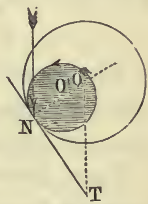


Fig. 89.

Line of Least Traction.

95. The force employed to draw a body uniformly along an inclined plane, is called the *force of traction;* and the direction of this force is the line of traction. In equation (46), P is the force of traction, and β is the angle the line of traction makes with the normal. When β varies, other things being equal, the value of P also varies; there is evidently some value of β that will render P least possible;

the direction of P, in this case, is the line of least traction; it is along this line that a force can be applied with greatest advantage, to draw a body up an inclined plane. If we examine the expression for P, in equation (46), we see that the numerator is constant; therefore, the expression for P will be least possible when the denominator is greatest possible. By a simple process of the Differential Calculus, it may be shown that the denominator will be greatest possible, or a maximum, when,

$$f = \cot\beta, \text{ or, } f = \tan(90° - \beta).$$

That is, the power will be applied most advantageously, when it makes an angle with the inclined plane equal to the angle of friction.

From the second value of P, it may be shown, in like manner, that a force will be most advantageously applied, to prevent a body from sliding down a plane, when its direction makes an angle with the plane equal to the supplement of the angle of friction, the angle being estimated, as before, from that part of the plane lying above the body.

Resistance to Rolling.

96. Resistance to rolling, sometimes called *rolling friction*, is the resistance experienced when one body rolls on another, the two being pressed together by some force. It arises from inequalities in the two surfaces, and also from distortion caused by the force that presses the bodies together. The *coefficient* is the quotient obtained by dividing the entire *resistance* by the normal pressure.

The following laws have been established, when a cylindrical body rolls on a plane:

First, *the friction is proportional to the normal pressure.*

Secondly, *it is inversely proportional to the diameter of the cylinder or wheel.*

Thirdly, *it increases as the surface of contact and velocity increase.*

In many cases there is a combination of both sliding and rolling friction in the same machine. Thus, in a car on a railroad track, the friction at the axle is sliding, and that between the wheel and track is rolling.

Work of Friction.

97. The work of friction is equal to the work of the force necessary to overcome it. It is therefore measured by the product of the force of friction into the path through which it is exerted. In case of an axle revolving in a box, the work during one revolution is equal to the force of friction multiplied by the circumference of the axle.

Adhesion.

98. Adhesion is the resistance one body experiences in moving on another in consequence of cohesion between the molecules of the surfaces in contact. This resistance increases when the surfaces are allowed to remain in contact for some time, but is very slight when motion has been established. Both theory and experiment show that adhesion between the same surfaces, is proportional to the extent of the surface of contact.

The *coefficient of adhesion* is obtained by dividing the entire adhesion by the area of the surface of contact. Denoting the entire adhesion by A, the area of the surface of contact by S, and the *coefficient of adhesion* by a, we have,

$$a = \frac{A}{S}, \text{ or, } A = aS.$$

To find the entire adhesion, multiply the unit of adhesion by the area of the surface of contact.

Stiffness of Cords.

99. Let O be a pulley, with a cord, AB, wrapped round its circumference; and suppose a force, P, applied at B, to overcome a resistance, R. As the rope winds on the pulley, at C, its rigidity acts to increase the arm of lever of R, and to overcome this rigidity an additional force is required. This additional force may be represented by the expression,

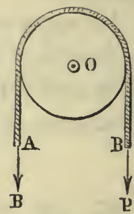


Fig. 90.

$$d\left(\frac{a + bR}{D}\right);$$

in which d depends on the character and size of the rope, a on its natural rigidity, bR on the rigidity due to the load, and D is the diameter of the wheel. The values of d, a, and b have been found by experiment for different kinds of rope, and tabulated.

Atmospheric Resistance.

100. The atmosphere offers a resistance to bodies moving through it, in consequence of the inertia of its particles. For the same extent of surface the resistance varies as the square of the velocity. For, if the velocity be doubled, twice as many particles will be met with in a given time, and each particle will be impinged against by the moving body with twice the force; hence, the resistance will be quadrupled. In a similar manner it may be shown that if the velocity be tripled, the resistance will be nine times as great, and so on. If, therefore, the resistance on a square foot of surface be determined for a given velocity, the resistance offered to any surface, and for any volocity, may be computed.

For the detailed methods of taking hurtful resistances into account, the reader is referred to more extended treatises on practical mechanics.

CHAPTER V.

RECTILINEAR AND PERIODIC MOTION.

Motion.

101. A point is in motion when it continually changes its position in space. When the path of the moving point is a straight line, the motion is *rectilinear;* when it is a curved line, the motion is *curvilinear.* When the motion is curvilinear, we may regard the path as made up of infinitely short straight lines; that is, we may consider it as a polygon, whose sides are infinitely small. If any side of this polygon be prolonged in the direction of the motion, it will be tangent to the curve. Hence, we say, that *a point moves in the direction of a tangent to its path.*

Uniform Motion.

102. Uniform motion is that in which the moving point describes equal spaces in any equal portions of time. If we denote the space passed over in one second by v, and in t seconds by s, we have, from the definition,

$$s = vt; \quad \therefore \; v = \frac{s}{t}$$

From the first of these equations, we see that *the space described in any time is equal to the product of the velocity and time;* from the second, we see that *the velocity is equal to the space described in any time, divided by that time.*

If the moving point had passed over a space s', at the beginning of the time t, the relation between the spaces and times would be given by the equation,

$$s = s' + vt \quad . \quad . \quad . \quad . \quad . \quad (48)$$

In this equation, s' is called the *initial* space.

Uniformly Varied Motion.

103. *Uniformly varied motion,* is that in which the velocity increases or diminishes uniformly. In the former case, the motion is *accelerated,* in the latter, *retarded.* In both the moving force is *constant.*

To find the relation between the spaces passed over, and the velocities generated, in any time, let the acceleration due to the moving force, (Art. 18), be denoted by f, and the velocity generated in t seconds by v. The acceleration is the velocity generated in one second, and because the velocity generated is proportional to the time, we have, from the definition,

$$v = ft \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Because the velocity increases uniformly, the space described in any time is the same as though the body had moved uniformly during that time, with its *mean,* or *average* velocity. At the beginning of the time t, the velocity is 0, at the end of that time it is ft; hence, the average velocity during the time t is $\frac{1}{2}ft$; multiplying this by the time t, we have, for the space described, $\frac{1}{2}ft \times t$, or, denoting the space by s, we have,

$$s = \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Equations (49) and (50) express the circumstances of motion of a body moving from a state of rest, under the action of a constant force: from the former we see that *the velocities are proportional to the times,* and from the latter we see that *the spaces are proportional to the squares of the times.*

If in equation (50) we make $t = 1$, we find,

$$s = \frac{1}{2}f; \text{ or, } f = 2s;$$

That is, *if a body move from rest, under the action of a constant force, the acceleration is measured by twice the space passed over in the first second.*

It follows, from the principle of inertia, that the velocity generated in any time is entirely independent of the state of the body at the beginning of that time. If, therefore, the body has a velocity v' at the beginning of the time t, equation (49) will become

$$v = v' + ft \quad . \quad . \quad . \quad . \quad . \quad (51)$$

In this equation v' is called the *initial velocity.*

If we suppose the body to have passed over a space s', called the *initial space*, before the beginning of t, the final space will be made up of three parts; *first*, the initial space, s'; *second*, the space due to the initial velocity v', and equal to $v't$; *third*, the space due to the action of the constant force f during the time t, equal to $\frac{1}{2}ft^2$. Hence,

$$s = s' + v't + \frac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Equations (51), and (52), may be made to conform to any case of uniformly varied motion, by giving suitable values to s', v', and f; it is to be observed, that any one of these quantities may be either *plus* or *minus.* When f is essentially positive the motion is *accelerated,* when f is essentially negative the motion is *retarded.*

Application to Falling Bodies.

104. The force of gravity is the force exerted by the earth on all bodies exterior to it. It is found, by observation, that *this force is directed toward the centre of the earth*, and that *its intensity varies inversely, as the square of the distance from the centre.*

Because the centre of the earth is so distant from the surface, the variation in intensity for small elevations above the surface is inappreciable. Hence, we may regard the force of gravity at any place on, or near, the earth's surface, as constant; in which case, the equations of the preceding article are applicable. The force of gravity acts equally on all the particles of a body, and were there no resistance offered, it would impart the same velocity, in the same time, to any two bodies whatever. The atmosphere, however, offers a resistance, which tends to retard the motion of bodies falling through it; and of two bodies of equal mass, it retards that one most, which presents the greatest surface to the direction of the motion. In discussing the laws of falling bodies, it will be found convenient to regard them as being in a vacuum, and in this case the equations of the preceding article are immediately applicable. The effects of atmospheric resistance may be taken into account, as corrections, or in certain cases the motions may be made so slow that their effects may be neglected.

If we denote the acceleration due to gravity by g, and the space fallen through by h, both being regarded as positive downward, we have, from (49) and (50),

$$v = gt \quad \ldots\ldots \quad (53)$$

$$h = \tfrac{1}{2}gt^2 \quad \ldots\ldots \quad (54)$$

That is, *the velocities are proportional to the times, and the spaces to the squares of the times.*

The value of g in the latitude of New York is not far from $32\frac{1}{6}$ feet; making $g = 32\frac{1}{6}$ feet, and giving to t the values 1^s, 2^s, 3^s, &c., in equations (53) and (54), we have the results given in the following

TABLE.

TIME ELAPSED.	VELOCITIES ACQUIRED.	SPACES DESCRIBED.
Seconds.	*Feet.*	*Feet.*
1	$32\frac{1}{6}$	$16\frac{1}{12}$
2	$64\frac{1}{3}$	$64\frac{1}{3}$
3	$96\frac{1}{2}$	$144\frac{3}{4}$
4	$128\frac{2}{3}$	$257\frac{1}{3}$
5	$160\frac{5}{6}$	$402\frac{1}{12}$
&c.	&c.	&c.

Solving equation (54) with respect to t, we have,

$$t = \sqrt{\frac{2h}{g}} \quad . \; . \; . \; . \; . \quad (55)$$

That is, *the number of seconds required for a body to fall through any height is equal to the square root of the quotient obtained by dividing twice the height in feet by* $32\frac{1}{6}$.

Substituting this value of t, in equation (53), we have,

$$v = g\sqrt{\frac{2h}{g}}, \text{ or } v^2 = 2gh;$$

whence, by solving with respect to v, and h,

$$v = \sqrt{2gh}, \quad \text{and} \quad h = \frac{v^2}{2g} \quad . \; . \; . \; . \; . \quad (56)$$

In these equations, v, is called the *velocity due to the height* h, and h, *the height due to the velocity* v.

If the body be projected downward with a velocity v', the circumstances of motion will be made known by the equations,

$$v = v' + gt,$$
$$h = v't + \tfrac{1}{2}gt^2.$$

In these equations, the origin of spaces is at the point from which the body is projected downward.

Motion of Bodies projected vertically upward.

105. Suppose a body projected vertically upward from the origin of spaces, with a velocity v', and afterward acted on by the force of gravity. In this case, the force of gravity acts to retard the motion. Making, in (51) and (52), $s' = o$, $f = -g$, and $s = h$, they become,

$$v = v' - gt \quad \ldots\ldots \quad (57)$$

$$h = v't - \tfrac{1}{2}gt^2 \quad \ldots\ldots \quad (58)$$

In these equations h is positive upward, and negative downward.

From equation (57), we see that the velocity diminishes as the time increases. The velocity is 0, when,

$$v' - gt = 0, \text{ or, when } t = \frac{v'}{g}.$$

When t is greater than $\frac{v'}{g}$, v is negative, and the body retraces its path: hence, *the time required for the body to reach its highest elevation, is equal to the initial velocity, divided by the force of gravity.*

Eliminating t, from (57) and (58), we have,

$$h = \frac{v'^2 - v^2}{2g} \quad \ldots\ldots \quad (59)$$

Making $v = 0$, in the last equation, we have,

$$h = \frac{v'^2}{2g} \quad \ldots\ldots \quad (60)$$

Hence, *the greatest height to which the body will ascend, is equal to the square of the initial velocity, divided by twice the force of gravity.*

This height is that due to the initial velocity, (Art. 104).

If, in (57), we make $t = \frac{v'}{g} - t'$, we find,

$$v = gt' \quad . \; . \; . \; . \; . \quad (61)$$

If, in the same equation, we make $t = \frac{v'}{g} + t'$, we find,

$$v = -gt' \quad . \; . \; . \; . \; . \quad (62)$$

Hence, *the velocities at equal times before and after reaching the highest points are equal.*

The difference of signs shows that the body is moving in opposite directions at the times considered.

If we substitute these values of v successively, in (59), we find in both cases,

$$h = \frac{v'^2 - g^2t'^2}{2g};$$

hence, the points at which the velocities are equal, in ascending and descending, are equally distant from the highest point; that is, they are coincident. Hence, *if a body be projected vertically upward, it will ascend to a certain point, and then return upon its path, in such manner, that the velocities in ascending and descending are equal at the same points.*

EXAMPLES.

1. Through what distance will a body fall from rest in a vacuum, in 10 seconds, and through what space will it fall during the last second? *Ans.* $1608\frac{1}{3}$ ft., and $305\frac{1}{2}$ ft.

2. In what time will a body fall from rest through 1200 feet? *Ans.* 8.63 sec.

3. A body was observed to fall through a height of 100 feet in the last second. How long was the body falling, and through what distance did it descend?

SOLUTION.

If we denote the distance by h, and the time by t, we have,

$$h = \tfrac{1}{2}gt^2, \text{ and } h - 100 = \tfrac{1}{2}g(t-1)^2;$$

$$\therefore\ t = 3.6 \text{ sec., and } h = 208.44 \text{ ft. } Ans.$$

4. A body falls through 300 feet. Through what distance does it fall in the last two seconds?

The entire time occupied, is 4.32 seconds. The distance fallen through in 2.32 sec., is 86.57 ft. Hence, the distance required is 300 ft. − 86.57 ft. = 213.43 ft. *Ans.*

5. A body is projected upward, with a velocity of 60 feet. To what height will it rise? *Ans.* 55.9 ft.

6. A body is projected upward, with a velocity of 483 ft. In what time will it rise 1610 feet?

We have, from equation (58),

$$1610 = 483t - 16\tfrac{1}{12}t^2; \qquad \therefore\ t = \tfrac{2898}{193} \pm \tfrac{2161}{193};$$

$$\text{or, } t = 26.2 \text{ sec., and } t = 3.82 \text{ sec.}$$

The smaller value of t gives the time required; the larger value gives the time occupied in rising to its greatest height, and returning to the point 1610 feet from the starting point.

7. A body is projected upward, with a velocity of 161 feet, from a point 214⅔ feet above the earth. In what time will it reach the earth, and with what velocity will it strike?

SOLUTION.

The body will rise 402.9 ft. The time of rising will be 5 sec.; the time of falling to the earth will be 6.2 sec. Hence, the required time is 11.2 sec. The required velocity is 199 ft.

8. Suppose a body to have fallen through 50 feet, when a second begins to fall just 100 feet below it. How far will the latter body fall before it is overtaken by the former? *Ans.* 50 feet.

Restrained Vertical Motion.

106. We have seen, (Art. 18), that *the acceleration due to a moving force is equal to the moving force divided by the mass moved.* Hence, in the case of a body falling freely, the moving force varies directly as the mass moved, and the acceleration is constant. If, however, we increase the mass moved, without changing the moving force, we shall correspondingly diminish the acceleration; and in this manner we may render it as small as possible. This result may be attained by the combination represented in the figure. In it, A is a fixed pulley, mounted on a horizontal axis, W and W' are unequal weights attached to the extremities of a flexible cord passing over the pulley. If the weight, W, be greater than W', the former will descend, and draw the latter up.

Fig. 31.

In this case, the moving force is the difference of the weights, W and W'; the mass moved is the sum of the masses of W and W', together with that of the pulley and connecting cord. The different parts of the pulley move with different velocities, but the effect of its mass may be replaced by that of some other mass at the circumference of the pulley. Denoting this mass, together with the mass of the cord, by m'', and the masses of W and W' by m and m', we have—to represent the entire mass moved—the expression $m + m' + m''$, and for the moving force we shall have $(m - m')g$; hence, by the rule, the acceleration, denoted by g', is equal to,

$$\frac{m - m'}{m + m' + m''}g.$$

This force being constant, the motion produced by it is *uniformly varied*, and the circumstances of that motion will be made known by substituting the above expression for f, in equation (49) and (50).

Atwood's Machine.

107. Atwood's machine is a contrivance to illustrate the laws of falling bodies. It consists of a vertical post, *AB*, about 12 feet in height, supporting, at its upper extremity, a fixed pulley, *A*. To obviate, as much as possible, the resistance of friction, the axle is made to turn on friction rollers. A silk string passes over the pulley, and at its extremities are fastened two equal weights, *C* and *D*. In order to impart motion to the weights, a small weight, *G*, in the form of a bar, is laid on *C*, and by diminishing its mass, the acceleration may be rendered as small as desirable. The rod, *AB*, graduated to feet and decimals, is provided with sliding stages, *E* and *F*; the upper one is in the form of a ring, which will permit *C* to pass, but not *G*; the lower one is in the form of a plate, which is intended to intercept the weight *C*. Connected with the instrument is a seconds pendulum for measuring time.

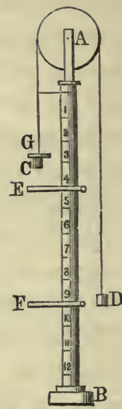


Fig. 92.

Suppose the weights, *C* and *D*, each equal to 150 grains, the weight of the bar 24 grains, and let a weight of 62 grains, placed at the circumference of the pulley, produce the same resistance by its inertia as that actually produced by the pulley and cord. Then will the fraction $\frac{m - m'}{m + m' + m''}$ become equal to $\frac{24}{386}$; and this, multiplied by $32\frac{1}{6}$, gives $g' = 2$. This value, substituted for g, in (53) and (54), gives,

$$v = 2t, \text{ and } h = t^2.$$

If, in these equations, we make $t = 1$ sec., we have $h = 1$, and $v = 2$. If we make $t = 2$ sec., we, in like manner, have $h = 4$, and $v = 4$. If we make $t = 3$ sec., we have $h = 9$, and $v = 6$, and so on. To verify these results experimentally, commencing with the first:—The weight, *C*, is drawn up till it comes opposite the 0 of the graduated scale, and the bar, *G*, is placed on it. The weight thus set is held in its place by a spring. The ring, *E*, is set at 1 foot, and the stage, *F*, at 3 feet from the 0. When the pendulum reaches one of its extreme limits, the spring is pressed back, the weight, *CG*, descends, and as the pendulum completes its vibration, the bar, *G*, strikes the ring, and is retained. The acceleration then becomes 0, and *C* moves on uniformly, with the velocity acquired, in the first second; and it will be observed that *C* strikes the second stage just as the pendulum completes its second vibration. Had *F* been set at 5 feet from the 0, *C* would have reached it at the end of the third vibration of the pendulum. Had it been 7 feet from the 0, it would have reached it at the end of the fourth vibration, and so on.

To verify the next result, we set the ring, *E*, at four feet from the 0, and the stage, *F*, at 8 feet from the 0, and proceed as before. The ring will intercept the bar at the end of the second vibration, and the weight will strike the stage at the end of the third vibration, and so on.

By making the weight of the bar less than 24 grains, the acceleration is diminished, and, consequently, the spaces and velocities, correspondingly diminished. The results may be verified as before.

Motion of Bodies on Inclined Planes.

108. If a body be placed on an inclined plane, and abandoned to the action of its own weight, it will either slide or roll down the plane, provided there be no friction between it and the plane. If the body is spherical, it will roll, and in this case friction may be disregarded. Let the weight of the body be resolved into two components, one perpendicular to the plane, and the other parallel to it: the plane of these components will be vertical, and also perpendicular to the given plane. The effect of the first component will be counteracted by the resistance of the plane, whilst the second will act as a constant force, urging the body down the plane. The force being constant, the body will have a uniformly varied motion, and equations (53) and (54) will be applicable. The acceleration may be found by projecting the acceleration due to gravity on the inclined plane.

Let AB represent the inclined plane, and P the centre of gravity of a body resting on it. Let PQ represent the force of gravity, denoted by g, and PR, its component, parallel to AB, PS being the normal component.

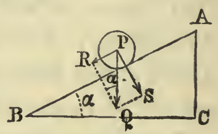


Fig. 93.

Denote PR by g', and the angle ABC by α. Then, since PQ is perpendicular to BC, and QR to AB, the angle, RQP, is equal to ABC, or to α. From the right-angled triangle, PQR, we have,

$$g' = g\sin\alpha.$$

But the triangle, ABC, is right-angled, and, if we denote its height, AC, by h, and its length, AB, by l, we shall have $\sin\alpha = \frac{h}{l}$, which, being substituted above, gives,

$$g' = \frac{gh}{l} \quad . \; . \; . \; . \; . \; (63)$$

This value of g' is the acceleration due to the moving force. Substituting it for f, in equations (51) and (52), we have,

$$v = v' + \frac{gh}{l}t,$$

$$s = s' + v't + \frac{gh}{2l}t^2.$$

If the body start from rest at A, taken as the origin of spaces, then will $v' = 0$, and $s' = 0$, giving,

$$v = \frac{gh}{l}t \quad . \; . \; . \; . \; . \quad (64)$$

$$s = \frac{gh}{2l}t^2 \quad . \; . \; . \; . \; . \quad (65)$$

To find the time required for a body to move from the top to the bottom of the plane, make $s = l$, in (65); there will result,

$$l = \frac{gh}{2l}t^2; \quad \therefore \quad t = l\sqrt{\frac{2}{gh}} \quad . \; . \; . \; . \; . \quad (66)$$

Hence, *the time varies directly as the length, and inversely as the square root of the height.*

For planes having the same height, but different lengths, the radical factor of the value of t remains constant. Hence, *the times required for a body to move down planes having the same height, are to each other as their lengths.*

To determine the velocity with which a body reaches the bottom of the plane, substitute for t, in equation (64), *its value taken from equation* (66). We have, after reduction,

$$v = \sqrt{2gh}.$$

But this is the velocity due to the height h, (Art. 104). Hence, *the velocity generated in a body whilst moving down an inclined plane, is equal to that generated in falling through the height of the plane.*

Examples.

1. An inclined plane is 10 feet long and 1 foot high. How long will it take for a body to move from top to bottom, and what velocity will it acquire?

SOLUTION.

We have, from equation (66),

$$t = l\sqrt{\frac{2}{gh}};$$

substituting for l, 10, and for h, 1, we have,

$$t = 2\tfrac{1}{2} \text{ seconds, nearly.}$$

From the formula, $v = \sqrt{2gh}$, we have, by making $h = 1$,

$$v = \sqrt{64.33} = 8.02 \text{ ft.}$$

2. How far will a body descend from rest in 4 seconds, on an inclined plane whose length is 400 feet, and whose height is 300 feet?

Ans. 193 ft.

3. How long will it take a body to descend 100 feet on a plane whose length is 150 feet, and whose height is 60? *Ans.* 3.9 sec.

4. There is a track, $2\frac{1}{2}$ miles in length, whose inclination is 1 in 35. What velocity will a car attain, in running the length of the road, by its own weight, hurtful resistances being neglected?

Ans. 155.75 ft., or, 106.2 m. per hour.

5. A railway train, having a velocity of 45 miles per hour, is detached from the locomotive on an ascending grade of 1 in 200. How far, and for what time, will the train continue to ascend the inclined plane?

SOLUTION.

We find the velocity 66 ft. Hence, $66 = \sqrt{2gh}$; or, $h = 67.7$ ft. for the vertical height. Hence, $67.7 \times 200 = 13{,}540$ ft., or, 2.5644 m., the distance the train will proceed. We have,

$$t = l\sqrt{\frac{2}{gh}} = 410.3 \text{ sec., or, 6 min. 50.3 sec.,}$$

the time required to come to rest.

6. A body weighing 5 lbs. descends vertically, and draws a weight of 6 lbs. up an inclined plane of 45°. How far will the first body descend in 10 seconds?

SOLUTION.

The moving force is $5\,lbs. - 6\,lbs. \times \sin 45°$; consequently the acceleration is, (Art. 106),

$$g' = \frac{5 - 6 \sin 45°}{11} g = 2.213\,ft.$$

$$\therefore s = \tfrac{1}{2} g't^2 = 111\,ft., \text{ nearly.}$$

Motion of a Body down a succession of Inclined Planes.

169. If a body start from the top of an inclined plane, with an initial velocity, v', it will reach the bottom with a velocity equal to the initial velocity, plus that acquired whilst on the plane. This velocity, called the *terminal velocity*, will be equal to that which the body would acquire in falling through the height due to the initial velocity, *plus* the height of the plane. Hence, if a body start from rest at A, and, after having passed over one plane, AB, enter on a second, BC, without loss of velocity, it will reach the bottom of the second plane with the same velocity that it would have acquired by falling through DC, the sum of the heights of the two planes. Were there a succession of inclined planes, so arranged that there would be no loss of velocity in passing from one to another, it might be shown, by similar reasoning, that the terminal velocity would be that due to the vertical distance of the terminal point below the point of starting.

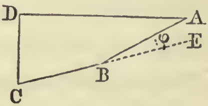


Fig. 94.

By a course of reasoning analogous to that employed in discussing the motion of bodies projected vertically upward, it might be shown that, if a body were projected upward, in the direction of the lower plane, with the terminal velocity, it would ascend along the several planes to the top of the highest one, where the velocity would be 0. The body would then, under the action of its own weight, retrace its path in such manner that the velocity at every point in descending would be the same as in ascending, but in a contrary direction. The time occupied in passing over any part of the path in descending, would be equal to that occupied in passing over the same portion in ascending.

In the preceding discussion, we have supposed that there is no loss of velocity in passing from one plane to another. To ascertain under what circumstances this condition will be fulfilled, let us take two planes, AB and BC. Prolong CB upward, and denote the angle, ABE, by φ. Denote the velocity of the body on reaching B, by v'. Let v' be resolved into two components, one in the direction of BC, and the other at right angles to it. The effect of the latter is destroyed by the resistance of the plane, and the former is the effective velocity in the direction of the plane, BC. From the rule for resolution of velocities, we have, the effective component of v' equal to $v' \cos\varphi$. Hence, the loss of velocity is $v' - v' \cos\varphi$, or, $v'(1 - \cos\varphi)$. But when φ is infinitely small, its cosine is 1, and there is no loss of velocity. Hence, the loss of velocity due to change of direction will be 0, when the path is a curved line. The principle is general, and may be enunciated as follows:

When a body is constrained to describe a curvilinear path, there is no loss of velocity due to change of direction of the body's motion.

Periodic Motion.

110. *Periodic motion*, is a species of variable motion, in which the spaces described in *certain* equal periods, are equal. An example of this kind of motion is found in *curvilinear vibration*. Let ABC be a vertical curve, symmetrical with respect to DB. Let AC be horizontal, and denote EB by h. If a body were placed at A, and abandoned to the action of its own weight, being constrained to remain on the curve, it would, in accordance with the principles of

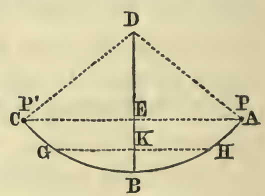


Fig. 95.

the last article, move toward B with an accelerated motion, and, on arriving at B, would possess a velocity due to the height h. By virtue of its inertia, it would ascend the branch, BC, with retarded motion, and would finally reach C, where its velocity would be 0. The body would then be in the same condition that it was at A, and would, consequently, descend to B, and again ascend to A, whence it would again descend, and so on. Were there no retarding causes, the motion would continue forever. From what precedes, it follows that the time occupied by the body in passing from A to B is equal to that in passing from B to C, and also the time in passing from C to B is equal to that in passing from B to A. Further, the velocities of the body when at G and H, two points on the same horizontal, are equal, either one, being that due to the height EK. These principles are used in discussing the pendulum.

Angular Velocity, and Angular Acceleration.

111. When a body revolves about an axis, its points being at different distances from the axis, will have different velocities. The *angular velocity* is the velocity of a point whose distance from the axis is 1. To obtain the velocity of any other point, we multiply its distance from the axis by the angular velocity. The force of gravity acts uniformly on the different points of a body, and the *impressed* acceleration is the same for all the particles. If the body is constrained to turn about a horizontal axis, the effective acceleration of different particles will depend on their distance from the axis. The effective acceleration of a point, at a unit's distance from the axis, is called the *angular acceleration* of the body.

The Simple Pendulum.

112. A pendulum is a heavy body suspended from a horizontal axis, about which it is free to vibrate.

To investigate the circumstances of vibration, let us first consider the hypothetical case of a material point, vibrating about an axis to which it is attached by a rod destitute of weight. Such a pendulum is called a SIMPLE PENDULUM. The laws of vibration in this case will be identical with those explained in Art. 110, the arc, ABC, being an arc of a circle.

Let ABC be the arc through which vibration takes place, and denote its radius, DA, by l. The angle, ADC, is called the *amplitude* of vibration; half of this angle, ADB, is called the *angle of deviation*.

If the point start from rest at A, it will, on reaching any point, H, have a velocity v, due to the height, EK, denoted by h, (Art. 104). Hence,

$$v = \sqrt{2gh} \quad . \; . \; . \; . \; . \quad (67)$$

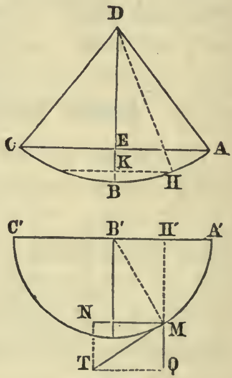


Fig. 96.

Let the angle of deviation be so small, that the chords of the arcs, AB and HB, may be considered equal to the arcs themselves. We shall have (Legendre, Bk. IV., Prop. XXIII., Cor.),

$$\overline{AB}^2 = 2l \times EB, \text{ and } \overline{HB}^2 = 2l \times KB,$$

whence, by subtraction,

$$\overline{AB}^2 - \overline{HB}^2 = 2l(EB - KB) = 2l \times h.$$

Denoting AB by a, HB by x, and solving the last equation, we have,

$$h = \frac{a^2 - x^2}{2l}.$$

Substituting this value of h, in (67), it becomes,

$$v = \sqrt{\frac{g}{l}(a^2 - x^2)} \quad . \; . \; . \; . \; . \; (68)$$

Now let us develop the arc, ABC, into a straight line, $A'B'C''$, and suppose a point to start from A' at the same time that the pendulum starts from A, and to vibrate back and forth upon $A'B'C''$ with the same velocities as the pendulum; then, when the pendulum is at any point, H, this point will be at a corresponding point, H', and the times of vibration of the two will be the same.

To find the time of vibration along $A'B'C'$, describe on it a semi-circle, $A'MC''$, and suppose a third point to start from A' at the same time as the second, and to move uniformly around the arc with a velocity equal to $a\sqrt{\frac{g}{l}}$. Then will the time required for this particle to reach C' be equal to the space divided by the velocity, (Art. 102). Denoting this time by t, and remembering that $A'B' = a$, we have,

$$t = \frac{\pi a}{a\sqrt{\frac{g}{l}}} = \pi\sqrt{\frac{l}{g}}.$$

Make $H'B' = x$, and draw $H'M$ perpendicular to $A'C''$, and at M decompose the velocity of the third particle, MT, into two components, MN and MQ, parallel and perpendicular to $A'C'$.

We have, for the horizontal component,

$$MN = MT \cos TMN \quad . \; . \; . \; . \; . \; (69)$$

But, $MT = a\sqrt{\frac{g}{l}}$, and because MT and MN are perpendicular to $B'M$ and $H'M$, we have, $\cos TMN = \cos B'MH' = \frac{H'M}{B'M}$. But $B'M = a$, and $H'M = \sqrt{a^2 - x^2}$; hence, $\cos TMN = \frac{\sqrt{a^2 - x^2}}{a}$. Substituting these values in equation (69), we have, for the horizontal velocity,

$$MN = \sqrt{\frac{g}{l}(a^2 - x^2)},$$

which is the same value as that obtained for v, in equation (68). Hence, we infer that the horizontal velocity of the third point is always equal to that of the second point, consequently the times required to pass from A' to C' must be equal; that is, the time of vibration of the second point, and consequently of the pendulum, must be $\pi\sqrt{\frac{l}{g}}$. Denoting this time by t, we have,

$$t = \pi\sqrt{\frac{l}{g}} \quad . \; . \; . \; . \; . \quad (70)$$

Hence, *the time of vibration of a simple pendulum is equal to the number* 3.1416, *multiplied into the square root of the quotient obtained by dividing the length of the pendulum by the force of gravity.*

For a pendulum, whose length is l', we shall have,

$$t' = \pi\sqrt{\frac{l'}{g}} \quad . \; . \; . \; . \; . \quad (71)$$

From equations (70) and (71), we have, by division,

$$\frac{t}{t'} = \frac{\sqrt{l}}{\sqrt{l'}}; \text{ or, } t : t' :: \sqrt{l} : \sqrt{l'} \quad . \; . \; . \; . \; . \quad (72)$$

That is, *the times of vibration of simple pendulums, are to each other as the square roots of their lengths.*

If we suppose the lengths of two pendulums to be the same, but the force of gravity to vary, as it does in different latitudes, and at different elevations, we shall have,

$$t = \pi \sqrt{\frac{l}{g}}, \text{ and } t'' = \pi \sqrt{\frac{l}{g''}}.$$

Whence, by division,

$$\frac{t}{t''} = \pi \sqrt{\frac{g''}{g}}, \quad \text{or,} \quad t : t'' :: \sqrt{g''} : \sqrt{g} \ . \ . \ . \ (73)$$

That is, *the times of vibration of the same pendulum, at different places, are to each other inversely as the square roots of the forces of gravity at the places.*

If we suppose the times of vibration to be the same, and the force of gravity to vary, the lengths will vary also, and we shall have,

$$t = \pi \sqrt{\frac{l}{g}}, \quad \text{and} \quad t =. \pi \sqrt{\frac{l'}{g'}}.$$

Equating these values, and squaring, we have,

$$\frac{l}{g} = \frac{l'}{g'}; \quad \text{or,} \quad l : l' :: g : g' \ . \ . \ . \ . \ . \ (74)$$

That is, *the lengths of pendulums that vibrate in equal times at different places, are to each other as the forces of gravity at those places.*

Vibrations of equal duration are called *isochronal.*

De l'Ambert's Principle.

113. When several bodies are rigidly connected, it often happens that they are constrained to move in a different manner from what they would, if free. Some move *faster* and some *slower* than they would, were it not for the connection. In the former case there is a *gain,* and in the latter a *loss,* of moving force, in consequence of the connection. It is obvious that the resultant of all the impressed forces is equal to that of all the effective forces, for if the latter were reversed, they would hold the former in equilibrium. Hence, *all the moving forces lost and gained in consequence of the connection are in equilibrium.*

This is *de l'Ambert's principle.*

The Compound Pendulum.

114. A compound pendulum is a body free to vibrate about a horizontal axis, called the *axis of suspension.* The straight line drawn from the centre of gravity of the pendulum perpendicular to the axis of suspension is called the *axis of the pendulum.*

In practical applications, the pendulum is so shaped that the plane through the axis of suspension and the centre of gravity divides it symmetrically.

Were the particles of the pendulum entirely disconnected, but constrained to remain at invariable distances from the axis of suspension, we should have a collection of simple pendulums. Those at equal distances from the axis would vibrate in equal times, and those unequally distant would vibrate in unequal times. The particles nearest the axis would vibrate more rapidly than the compound pendulum, and those most remote would vibrate slower; hence, there must be intermediate points that would vibrate in the same time as the pendulum. These points lie on the surface of a circular cylinder whose axis is that of suspension; the point in which this cylinder cuts the axis of the pendulum is called the *centre of oscillation.* If the entire mass of the pendulum were concentrated at this point, the time of its vibration would be unchanged. Hence, the *centre of oscillation of a pendulum* is a point of its axis, at which, if the mass of the pendulum were concentrated, its time of vibration would be unchanged. A line drawn through this point, parallel to the axis of suspension, is called the *axis of oscillation.* The distance between the axis of oscillation and the axis of suspension is the length of an *equivalent simple pendulum*—that is, of a simple pendulum, whose time of vibration is the same as that of the compound pendulum.

Angular Acceleration of a Compound Pendulum.

115. Let CK be a compound pendulum, C its axis of suspension, G its centre of gravity, and suppose the plane of the paper to pass through the centre of gravity, G, and perpendicular to the axis, C. We may regard the pendulum as made up of infinitely small filaments, parallel to the axis of suspension, and consequently perpendicular to the paper. The circumstances of vibration will be unchanged if we suppose each element to be concentrated in the point where it meets the plane of the paper. Denote the mass of any element, as S, by m, its distance from C, by r, and the mass of the pendulum by M.

Through C draw a horizontal line, CB, and draw SH, GA, and OB perpendicular to it. On HS prolonged, take SE to represent the moving force *impressed* on S. Then will SE be equal to mg, (Art. 18), and its moment with respect to C will be $mg \times HC$. Denote the *angular acceleration* by ω; then will the actual acceleration of S, in the direction perpendicular to SC, be equal to $r\omega$, and the *effective*

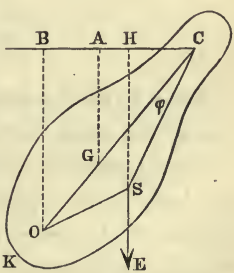


Fig. 97.

moving force to $mr\omega$; because this force acts at right angles to SC, its moment is equal to $mr^2\omega$. Because mg is the moving force *impressed* on S, and $mr\omega$ the *effective* moving force, the expression, $mg - mr\omega$, will be the moving force lost or gained by S in consequence of its connection with the other particles. There will be a loss when mg is *greater* than $mr\omega$, and a gain when mg is *less* than $mr\omega$. The moment of this force with respect to C is equal to $mg \times CH - mr^2\omega$. Similar expressions may be found for each of the elementary particles of the pendulum.

By de l'Ambert's principle, the moving forces lost and gained, in consequence of the connection of the parts, are in equilibrium; hence, the algebraic sum of their moments with respect to an axis, C, is equal to 0—that is,

$$\Sigma(mg \times CH) - \Sigma(mr^2\omega) = 0.$$

But ω and g are the same for each particle; hence,

$$\omega = \frac{\Sigma(m \times CH)}{\Sigma(mr^2)}g.$$

From the principle of moments, we have,

$$\Sigma(m \times CH) = M \times CA.$$

Substituting above, we have, finally,

$$\omega = \frac{M \times CA}{\Sigma(mr^2)}g \quad . \; . \; . \; . \; . \quad (75)$$

That is, *the angular acceleration varies as, CA, the lever arm of the weight of the pendulum.*

The expression $\Sigma(mr^2)$ is called the *moment of inertia* of the body with respect to the axis of suspension, Mg is the weight of the body, and $Mg \times CA$ is the moment of the weight, with respect to the same axis.

Hence, *the angular acceleration is equal to the moment* *of the weight, divided by the moment of inertia, both with respect to the axis of suspension.*

Length of an Equivalent Simple Pendulum.

116. To find the length of a simple pendulum that will vibrate in the same time as the given compound pendulum, let O be the centre of oscillation, and draw OB perpendicular to CB. Denote CO by l, and CG by k. Were the entire mass concentrated at O, we should have, for its moment of inertia, Ml^2, for the moment of the mass, $M \times CB$, and for the angular acceleration,

$$\omega = \frac{M \times CB}{Ml^2} g.$$

But the pendulum is to vibrate in the same time, whether it exist as a compound pendulum, or as a simple pendulum, its mass being concentrated at its centre of oscillation; the value of ω must, therefore, be the same in both cases. Placing the value just deduced equal to that in equation (75), we have,

$$\frac{M \times CB}{Ml^2} g = \frac{M \times CA}{\Sigma(mr^2)} g;$$

whence, by reduction,

$$Ml^2 = \Sigma(mr^2) \times \frac{CB}{CA}.$$

From the similar triangles, CGA and COB, we have,

$$\frac{CB}{CA} = \frac{l}{k}.$$

Substituting, and reducing, we have,

$$l = \frac{\Sigma(mr^2)}{Mk} \quad \ldots\ldots \quad (76)$$

Reciprocity of Axes of Suspension and Oscillation.

117. Let C be the axis of suspension, O the centre of oscillation, and let a line be drawn through O parallel to the axis of suspension. This line is called *the axis of oscillation.* Let the plane of the paper be taken as before, and suppose the elements projected on it, as in the last article.

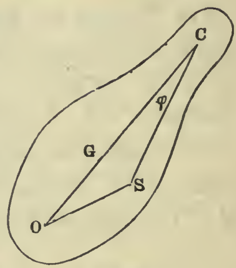


Fig. 98.

Let S be any element, and denote its distance from the axis of suspension by r, and from the axis of oscillation by t; denote OC by l, and the angle OCS by φ.

If the axis of oscillation be taken as an axis of suspension, and the length of the corresponding simple pendulum denoted by l', we have, from the preceding article,

$$l' = \frac{\Sigma(mt^2)}{M(l-k)} \quad \ldots\ldots \quad (77)$$

In the triangle, OSC, we have,

$$t^2 = r^2 + l^2 - 2rl\cos\varphi;$$

hence,

$$\Sigma(mt^2) = \Sigma(mr^2) + \Sigma(ml^2) - 2\Sigma(mr\cos\varphi)l.$$

But, from equation (76), we have,

$$\Sigma(mr^2) = Mkl;$$

and because l is invariable, we have,

$$\Sigma(ml^2) = \Sigma(m)l^2 = Ml^2;$$

if we suppose CO horizontal, $r\cos\varphi$, the projection of r on CO, will be the lever arm of m, and the expression, $\Sigma(mr\cos\varphi)$, will be the algebraic sum of the moments of the elementary masses with respect to C; hence, we shall have,

$$\Sigma(mr\cos\varphi)l = Mkl.$$

Substituting for these expressions their values above, and putting the value of $\Sigma(mt^2)$, thus found, in (77), we have,

$$l' = \frac{Mkl + Ml^2 - 2Mkl}{M(l-k)} = \frac{M(l^2 - kl)}{M(l-k)};$$

or,

$$l' = l.$$

Hence, *the axes of suspension and oscillation are convertible;* that is, *if either be taken as an axis of suspension, the other will be the axis of oscillation.*

This property of the compound pendulum is employed to determine the length of the seconds' pendulum, and the value of the force of gravity at different places on the surface of the earth.

A straight bar of iron, CD, is provided with knife-edge axes, A and B, of hardened steel, at right angles to the axis of the bar, and having their edges turned toward each other. These axes are so placed that they are not symmetrical with respect to the bar. The pendulum thus constructed is suspended on horizontal plates of polished agate, and allowed to vibrate about each axis in turn till, by filing away one end of the bar, the times of vibration about the axes are made equal. The distance, AB, is then the length of a simple pendulum that will vibrate in the same time as the bar, about either axis. The adjustment may also be made by using a sliding piece, that can be made fast to the bar, by a clamp-screw.

Fig. 99.

To employ the pendulum thus adjusted to find the length of a seconds' pendulum at any place, the pendulum is suspended, and allowed to vibrate through a small angle, the number of vibrations counted, and the time noted by a chronometer. The time divided by the number of vibrations, gives the time of a single vibration. The distance between the axes, measured by a scale of equal parts, gives the length of the corresponding simple pendulum. To find the length of the simple seconds' pendulum, we make use of proportion (72), substituting in it for t' and l' the values just found, and for t, 1 second; the remaining quantity l, may be found by solving the proportion. This value is the length of the seconds' pendulum at the place where the observation is made. In making the observations, a variety of precautions must be taken, and several corrections applied, the explanation of which does not fall within the scope of this treatise. By a series of carefully conducted experiments, it was found that the length of a seconds' pendulum in the Tower of London is 3.2616 ft., or 39.13921 in. By a similar course of proceeding, the length of the seconds' pendulum has been determined for a great number of places on the earth's surface, at different latitudes, and from these the corresponding values of the force of gravity at those points have been determined, according to the following principle:

From the equation, $t = \pi\sqrt{\frac{l}{g}}$, we find, by solving with respect to g, and making $t = 1$,

$$g = \pi^2 l.$$

From this equation the value of g may be found at different places, by substituting for l the length of the seconds' pendulum at those places. By comparing the values of g, it is found that they are everywhere the same on the same parallel of latitude, and that they vary in passing from latitude to latitude.

The following formula for determining the value of g, at different places, is given by Prof. Airy. In it G represents the value of gravity at the equator, g its value in any latitude, l.

$$g = G(1 + .005133 \sin^2 l) \quad . \; . \; . \; . \; . \quad (78)$$

The value of G is 32.088 ft.; this gives for gravity at the pole, 32.2527 ft.

Practical Application of the Pendulum.

118. One of the most important uses of the pendulum is to regulate the motion of clocks. A clock consists of a train of wheelwork, the last wheel of the train connecting with a pendulum-rod by a piece of mechanism called, an *escapement.* The wheelwork is kept in motion by a descending weight, or by the elastic force of a spring, and the wheels are so arranged that one tooth of the last wheel in the train escapes from the pendulum-rod at each vibration of the pendulum, or at each *beat.* The number of beats is rendered visible on a dial-plate by indices, called *hands.*

On account of expansion and contraction, the length of the pendulum is liable to variation, which gives rise to irregularity in the times of vibration. To obviate this, and to render the times of vibration uniform, several devices have been resorted to, giving rise to what are called *compensating pendulums.* We shall indicate two of the most important of these, observing that the remaining ones are nearly the same in principle, differing only in mode of application.

Graham's Mercurial Pendulum.

119. Graham's mercurial pendulum consists of a rod of steel about 42 inches long, branched toward its lower end, to embrace a cylindrical glass vessel 7 or 8 inches deep, and having between 6 and 7 inches of this depth filled with mercury. The exact quantity of mercury, being dependent on the weight and expansibility of the other parts of the pendulum, may be determined by experiment in each case. When the temperature increases, the steel rod is lengthened, and, at the same time, the mercury rises in the cylinder. When the temperature decreases, the steel bar is shortened, and the mercury falls in the cylinder. By a proper adjustment of the quantity of mercury, the effect of the lengthening, or shortening of the rod is exactly counterbalanced by the rising or falling of the centre of gravity of the mercury, and the axis of oscillation is kept at an invariable distance from the axis of suspension.

Harrison's Gridiron Pendulum.

120. Harrison's gridiron pendulum consists of five rods of steel and four of brass, placed alternately with each other, the middle rod, or that from which the *bob* is suspended, being of steel. These rods are connected by cross-pieces in such a manner that, whilst the expansion of the steel rods tends to elongate the pendulum, or lower the bob, the expansion of the brass rods tends to shorten the pendulum, or raise the bob. By duly proportioning the sizes and lengths of the bars, the axis of oscillation may be maintained at an invariable distance from the axis of suspension. From what has preceded, it follows that whenever the distance from the axis of oscillation to the axis of suspension remains invariable, the times of vibration must be absolutely equal at the same place. The pendulums just described are principally used for astronomical clocks,

Fig. 100.

where great accuracy and uniformity in the measure of time are indispensable.

Basis of a System of Weights and Measures.

121. The pendulum is of further importance, in a practical point of view, in furnishing the standard that has been made the basis of the English system of weights and measures.

It was enacted by Parliament, in 1824, that the distance between the centres of two gold studs in a certain described brass bar, the bar being at a temperature of 62° F., should be an "imperial standard yard." To be able to restore it in case of its destruction, it was enacted that the yard should be considered as bearing to the length of the seconds' pendulum in the latitude of London, in vacuum, and at the level of the sea, the ratio of 36 to 39.1393. From the yard, every other unit of linear measure may be derived, and thence all measures of area and volume.

It was also enacted that a certain described brass weight, made in 1758, and called 2 lbs. Troy, should be regarded as authentic, and that a weight equal to one-half that should be "the imperial standard Troy pound." The $\frac{1}{5760}$th part of the Troy pound was called a *grain*, of which 7000 constituted a pound avoirdupois. To provide for the contingency of a loss of the standard, it was connected with the system of measures, by enacting, that if lost, it should be restored by allowing 252.724 grains for the weight of a cubic inch of distilled water, at 62° F., the water being weighed in vacuum and by brass weights. From the grain thus established, all other units of weight may be derived.

Our own system of weights and measures is the same as the English.

EXAMPLES.

1. The length of a seconds' pendulum is 39.13921 in. If it be shortened 0.130464 in., how many vibrations will be gained in a day of 24 hours?

SOLUTION.

The times of vibration of pendulums at the same place, are as the square roots of their lengths. Hence, the number of vibrations in any given time, are inversely as the square roots of their lengths. If, therefore, we denote the number of vibrations gained in 24 hours, or 86400 seconds, by x, we have,

$$86400 : 86400 + x :: \sqrt{39.008747} : \sqrt{39.13921}.$$

Whence, $x = 144$, nearly. *Ans.*

2. A seconds' pendulum on being carried to the top of a mountain, was observed to lose 5 vibrations per day of 86400 seconds. Required the height of the mountain, reckoning the radius of the earth at 4000 miles.

SOLUTION.

The squares of the times of vibration, at two points, are inversely as the forces of gravity at those points. But the forces of gravity at the points are inversely as the squares of their distances from the centre of the earth. Hence, the times of vibration are proportional to the distances of the points from the centre of the earth; and, consequently, the number of vibrations in any given time, as 24 hours, for example, will be inversely as those distances. If, therefore, we denote the height of the mountain in miles by x, we have,

$$86400 : 86405 :: 4000 : 4000 + x.$$

Whence, $x = \frac{20000}{86400} = 0.2315$ miles, or, 1222 feet. *Ans.*

3. What is the time of vibration of a pendulum whose length is 60 inches, when the force of gravity is $32\frac{1}{6}$ ft.? *Ans.* 1.2387 sec.

4. How many vibrations will a pendulum 36 inches in length make in one minute, the force of gravity being the same as before? *Ans.* 62.53.

5. A pendulum makes 43170 vibrations in 12 hours. How much must it be shortened that it may beat seconds?

SOLUTION.

We shall have, as in example 1st,

$$43170 : 43200 :: \sqrt{39.13921} : \sqrt{39.13921 + x}.$$

Whence, $x = 0.0544$ in. *Ans.*

6. In a certain latitude, the length of a pendulum vibrating seconds is 39 inches. What is the length of a pendulum vibrating seconds, in the same latitude, at the height of 21000 feet above the first station, the radius of the earth being 3960 miles? *Ans.* 38.9218 in.

7. If a pendulum make 40000 vibrations in 6 hours, at the level of the sea, how many vibrations will it make in the same time, at an elevation of 10560 feet, the radius of the earth being 3960 miles? *Ans.* 39979.8.

Centre of Percussion.

122. The centre of percussion of a suspended body, is the point at which an impulse may be applied, perpendicular to the plane through it and the axis, without shock to the axis. This point is identical with the centre of oscillation. For, suppose, whilst the body is vibrating about the axis, an impulse to be applied at the centre of oscillation, capable of generating a quantity of motion equal and directly opposed to the resultant of the quantities of motion of all the particles; the direction of this impulse will be opposite to the motion of the centre of oscillation, that is, perpendicular to the plane through it and the axis, and it is obvious, from the property of the centre of oscillation, that it will bring the body to rest without shock to the axis. Were the same impulse applied to the body, at rest, it would generate a quantity of motion equal to that destroyed, but in a contrary direction, and without shock on the axis. The direction of the impulse remaining the same, no matter what may be its intensity, there will be no shock. It is a matter of common observation, that if a rod held in the hand be struck at a certain point, the hand will not feel the blow, but if struck at any other point, a shock will be felt, the intensity of which depends on the intensity of the blow, and on its point of application.

Moment of Inertia.

123. The moment of inertia of a body with respect to an axis, is the *algebraic sum* of the products obtained by multiplying the mass of each elementary particle by the square of its distance from the axis. Denoting the moment of inertia with respect to any axis, by K, the mass of any element of the body, by m, and its distance from the axis, by r, we have, from the definition,

$$K = \Sigma(mr^2) \quad . \; . \; . \; . \; . \quad (79)$$

The moment of inertia varies, in the same body, according to the position of the axis. To investigate the law of variation, let AB represent a section of the body by a plane perpendicular to the axis; C, the point in which this plane cuts the axis; and G, the point in which it cuts a parallel axis through the centre of gravity. Let P be any element of the body, whose mass is m, and denote PC by r, PG by s, and CG by k.

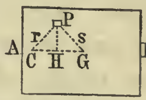


Fig. 101.

From the triangle CPG, according to a principle of trigonometry, we have,

$$r^2 = s^2 + k^2 - 2sk\cos CGP.$$

Substituting, in (79), and separating the terms, we have,

$$K = \Sigma(ms^2) + \Sigma(mk^2) - 2\Sigma(msk\cos CGP).$$

Or, since k is constant, and $\Sigma(m) = M$, the mass of the body, we have,

$$K = \Sigma(ms^2) + Mk^2 - 2k\Sigma(ms\cos CGP).$$

But $s\cos CGP = GH$, the lever arm of the mass m, with respect to the axis through the centre of gravity. Hence, $\Sigma(ms\cos CGP)$, is the algebraic sum of the moments of all the particles of the body with respect to the axis through the centre of gravity; but from the principle of moments, this is 0. Hence,

$$K = \Sigma(ms^2) + Mk^2 \quad . \quad . \quad . \quad . \quad . \quad (80)$$

The first term of the second member is the moment of inertia, with respect to the axis through the centre of gravity.

Hence, *the moment of inertia of a body with respect to any axis, is equal to the moment of inertia with respect to a parallel axis through the centre of gravity,* plus *the mass of the body into the square of the distance between the two axes.*

The moment of inertia is least possible when the axis passes through the centre of gravity. If any number of parallel axes be taken at equal distances from the centre of gravity, the moment of inertia with respect to each, will be the same.

The moment of inertia with respect to any axis, may be determined experimentally as follows. Make the axis horizontal, and allow the body to vibrate about it, as a compound pendulum. Find the time of a single vibration, and denote it by t. This value of t, in equation (70), makes known the value of l. Determine the centre of gravity, and denote its distance from the axis, by k. Find the mass of the body, and denote by M.

We have, from equation (76),

$$Mkl = \Sigma(mr^2) = K.$$

Substitute for M, l, and k, the values already found, and the value of K will be the moment of inertia, with respect to the assumed axis. Subtract from this the value of Mk^2, and the remainder will be the moment of inertia with respect to a parallel axis through the centre of gravity.

The moment of inertia of a homogeneous body of regular figure, is most readily found by means of the calculus.

The results thus determined, in a few of the more common cases of practical mechanics, are appended.

1. The moment of inertia of a rod, or bar, of uniform thickness, with respect to an axis perpendicular to the length of the rod, is given by the formula,

$$K = M\left(\frac{l^2}{3} + d^2\right) \quad \ldots\ldots \quad (81)$$

in which, K is the moment of inertia, M the mass of the rod, $2l$ its length, and d the distance of the centre of gravity from the axis.

2. The moment of inertia of a thin circular plate about a line in its own plane, is given by the formula,

$$K = M\left(\frac{r^2}{4} + d^2\right) \quad \ldots\ldots \quad (82)$$

in which, K, M, and d, are the same as before, and r is the radius of the circular plate.

3. The moment of inertia of a circular plate, with reference to an axis perpendicular to its plane, is given by the formula,

$$K = M\left(\frac{r^2}{2} + d^2\right) \quad \ldots\ldots \quad (83)$$

in which, the quantities are the same as before.

4. The moment of inertia of a circular ring, with reference to an axis perpendicular to its plane, is given by the formula,

$$K = M\left(\frac{r^2 + r'^2}{2} + d^2\right) \quad \ldots\ldots \quad (84)$$

in which, r and r' are the exterior and interior radii of the ring.

5. The moment of inertia of a right cylinder with respect to an axis perpendicular to the axis of the cylinder, is given by the formula,

$$K = M\left(\frac{r^2}{4} + \frac{l^2}{3} + d^2\right) \quad . \; . \; . \; . \; . \quad (85)$$

in which, r is the radius of the base, and $2l$ the length of the cylinder.

By making $d = o$ in any of the above formulas, we find the corresponding moment of inertia for a parallel axis through the centre of gravity.

Centre and Radius of Gyration.

124. The *centre of gyration* with respect to an axis, is a point at which, if the entire mass of a body be concentrated, its moment of inertia will remain unchanged. The distance of this point from the axis is the *radius of gyration.*

Let M denote the mass of a body, and k' its radius of gyration; then will the moment of inertia of the concentrated mass with respect to the axis, be equal to Mk'^2; but this must, by definition, be equal to the moment of inertia with respect to the same axis, or $\Sigma(mr^2)$; hence,

$$Mk'^2 = \Sigma(mr^2); \text{ or, } k' = \sqrt{\frac{\Sigma(mr^2)}{M}} \quad . \; . \; . \; . \; . \quad (86)$$

That is, *the radius of gyration is equal to the square root of the quotient obtained by dividing the moment of inertia by the mass.*

Since M is constant for the same body, the radius of gyration will be least possible when the moment of inertia is least possible, that is, when the axis passes through the centre of gravity. This minimum radius is called the *principal radius of gyration.* If we denote the principal radius of gyration by k, we shall have, from the examples of article (123), the following results:

Example 1... $k' = \sqrt{\frac{l^2}{3} + d^2}$; $\quad k = l\sqrt{\frac{1}{3}}$.

Example 2... $k' = \sqrt{\frac{r^2}{4} + d^2}$; $\quad k = \frac{r}{2}$.

Example 3... $k' = \sqrt{\frac{r^2}{2} + d^2}$; $\quad k = r\sqrt{\frac{1}{2}}$.

Example 4... $k' = \sqrt{\frac{r'^2 + r^2}{2} + d^2}$; $\quad k = \sqrt{\frac{r'^2 + r^2}{2}}$

Example 5... $k' = \sqrt{\frac{r^2}{4} + \frac{l^2}{3} + d^2}$; $\quad k = \sqrt{\frac{r^2}{4} + \frac{l^2}{3}}$.

To find the relation between the length of an equivalent simple pendulum and the principal radius of gyration of a suspended body, let us replace the expression $\Sigma(mr^2)$, in equation (76), by its value Mk'^2 and reduce. We find,

$$l = \frac{k'^2}{k} \quad \therefore \quad k' = \sqrt{kl};$$

that is, the centre of gravity, the centre of oscillation, and the centre of gyration, are on a common perpendicular to the axis of suspension, and so situated that the distance of the last from an axis is a mean proportional between the distances of the other two from the same axis.

CHAPTER VI.

CURVILINEAR AND ROTARY MOTION.

Motion of Projectiles.

125. If a body be projected obliquely upward in a vacuum, and then abandoned to the force of gravity, it will be continually deflected from a rectilinear path, and, after describing a curvilinear trajectory, will finally reach the horizontal plane from which it started.

The starting-point is the *point of projection;* the distance from the point of projection to the point at which the projectile again reaches the same horizontal plane is the *range,* and the time occupied is *the time of flight.* The only forces to be considered, are the *initial impulse* and *the force of gravity.* Hence, the trajectory will lie in a vertical plane through the direction of the initial impulse. Let CAB be this plane, A the point of projection, AB the range, and AC a vertical through A.

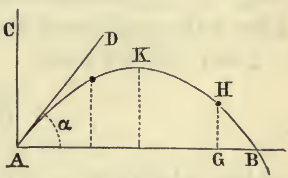


Fig. 102.

Take AB and AC as co-ordinate axes; denote the angle of projection, DAB, by α, and the velocity due to the initial impulse by v. Resolve v into two components, one in the direction AC, and the other in the direction AB. We have, for the former, $v \sin\alpha$, and, for the latter, $v \cos\alpha$.

The velocities, and, consequently, the spaces described in the direction of the co-ordinate axes, will (Art. 12) be entirely independent of each other. Denote the space described in the direction AC, in any time t, by y. The circumstances of motion in this direction, are those of a body projected vertically upward with an initial velocity, $v \sin\alpha$, and then continually acted on by the force of gravity. Hence, equation (58) is applicable. Making, in that equation, $h = y$, and $v' = v \sin\alpha$, we have,

$$y = v \sin\alpha\, t - \tfrac{1}{2}gt^2 \quad . \; . \; . \; . \; . \quad (87)$$

Denote the space described in the direction of the axis, AB, in the time t, by x. The only force acting in this direction is the component of the initial impulse. Hence, the motion will be uniform, and the first equation of Art. 102, is applicable. Making $s = x$, and $v = v \cos\alpha$, we have,

$$x = v \cos\alpha\, t \quad . \; . \; . \; . \; . \quad (88)$$

If we suppose t to be the same in equations (87) and (88), they will be simultaneous, and, taken together, will make known the position of the projectile at any instant.

From (88), we have,

$$t = \frac{x}{v \cos\alpha},$$

which, substituted in (87), gives,

$$y = \frac{\sin\alpha}{\cos\alpha}x - \frac{gx^2}{2v^2\cos^2\alpha} \quad . \; . \; . \; . \; . \quad (89)$$

an equation which is independent of t. It, therefore, expresses the relation between x and y for any value of t, and is, consequently, the equation of the trajectory. But, equation (89) is the equation of a parabola whose axis is vertical. Hence, the trajectory is a parabola.

To find the range, make $y = 0$, in (89), and deduce the corresponding value of x. Placing the value of y equal to 0, we have,

$$\frac{\sin\alpha}{\cos\alpha}x - \frac{gx^2}{2v^2\cos^2\alpha} = 0;$$

$$\therefore\ x = 0, \text{ and } x = \frac{2v^2\sin\alpha\cos\alpha}{g}.$$

The first value of x corresponds to the point of projection, and the second is the value of the *range*, AB.

From trigonometry, we have,

$$2\sin\alpha\cos\alpha = \sin 2\alpha.$$

If we denote the height due to the initial velocity, by h, we have,

$$v^2 = 2gh.$$

Substituting these in the second value of x, and denoting the range by r, we have,

$$r = 2h\sin 2\alpha \quad . \quad . \quad . \quad . \quad . \quad (90)$$

The greatest value of r will correspond to $\alpha = 45°$, in which case, $2\alpha = 90°$, and $\sin 2\alpha = 1$. Hence, we have, for the greatest range,

$$r = 2h.$$

That is, it is equal to *twice the height due to the initial velocity.*

If, in (90), we replace α by $90° - \alpha$, we have,

$$r = 2h\sin(180° - 2\alpha) = 2h\sin 2\alpha,$$

the same value as before. Hence, there are two angles of projection, complements of each other, that give the same range. The trajectories in the two cases are not the same, as may be shown by substituting the values of α, and $90° - \alpha$, in equation (89). The greater angle of projection gives a higher elevation, and, consequently, the projectile descends more vertically. It is for this reason that the gunner selects the greater of the two, when he desires to crush an object, and the less when he desires to batter. or overturn the object. If $\alpha = 90°$, the value of r is 0. That is, if a body be projected vertically upward, it will return to the point of projection.

To find *the time of flight*, make $x = r$, in (88), and deduce the corresponding value of t. This gives,

$$t = \frac{r}{v \cos\alpha} \quad . \ . \ . \ . \ . \ (91)$$

The range remaining the same, the time of flight will be greatest when α is greatest. Equation (88) also gives the time required for the body to describe any distance in the direction of the horizontal line, AB.

In equation (91) there are four quantities, t, r, v, and α, and from it, if any three are given, the remaining one may be determined.

As an application of the principles just deduced, let it be required to determine the angle of projection, that the projectile may strike a point, H, at a horizontal distance, $AG = x'$ from the point of projection, and at a height, $GH = y'$, above it.

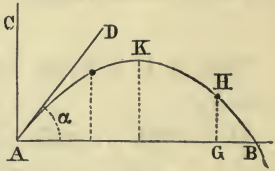


Fig. 102

Since H lies on the trajectory, its co-ordinates must satisfy the equation of the curve, giving,

$$y' = x' \tan\alpha - \frac{gx'^2}{2v^2\cos^2\alpha}.$$

From trigonometry, we have,

$$\cos^2\alpha = \frac{1}{\sec^2\alpha} = \frac{1}{1 + \tan^2\alpha}.$$

Substituting this in the preceding equation, we have, after clearing of fractions,

$$2v^2y' = 2v^2x'\tan\alpha - gx'^2(1 + \tan^2\alpha);$$

or, transposing and reducing,

$$\tan^2\alpha - \frac{2v^2}{gx'}\tan\alpha = -\frac{2v^2y' + gx'^2}{gx'^2}.$$

Hence,

$$\tan\alpha = \frac{v^2}{gx'} \pm \sqrt{\frac{v^4}{g^2x'^2} - \frac{2v^2y' + gx'^2}{gx'^2}};$$

or, making $v^2 = 2gh$,

$$\tan\alpha = \frac{2h}{x'} \pm \sqrt{\frac{4h^2}{x'^2} - \frac{4hy' + x'^2}{x'^2}} = \frac{2h \pm \sqrt{4h^2 - 4hy' - x'^2}}{x'}.$$

This shows that there are two angles of projection, under either of which, the point may be struck.

If we suppose

$$x'^2 = 4h^2 - 4hy' \quad . \; . \; . \; . \; . \quad (92)$$

the quantity under the radical sign will be 0, and the two angles of projection will become one.

If x' and y' be regarded as variables, equation (92) represents a parabola whose axis is a vertical, through the point of projection. Its vertex is at a distance, h, above the point, A, its focus is at A, and its parameter is $4h$, or twice the range.

If we suppose

$$x'^2 < 4h^2 - 4hy',$$

the point (x', y'), will lie within the parabola just described, the quantity under the radical sign will be positive, and there will be two real values of $\tan\alpha$, and, consequently, two angles of projection, under either of which the point may be struck.

If we suppose

$$x'^2 > 4h^2 - 4hy',$$

the point (x', y'), will be without this parabola, the values of $\tan\alpha$ will both be imaginary, and there will be no angle under which the point can be struck.

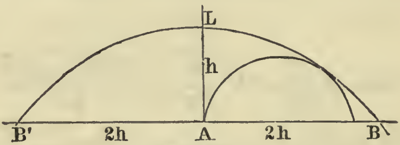


Fig. 103.

Let the parabola $B'LB$ represent the curve whose equation is

$$x'^2 = 4h^2 - 4hy'.$$

Conceive it to be revolved about AL, as an axis, generating a paraboloid of revolution. Then, from what precedes, we conclude, *first*, that every point within the surface may be reached from A, under two different angles of projection; *secondly*, that every point on the surface can be reached, but only by a single angle of projection; *thirdly*, that no point without the surface can be reached at all.

If a body be projected horizontally from an elevated point, A, its trajectory will be made known by equation (89), simply making $\alpha = 0$; whence, $\sin\alpha = 0$, and $\cos\alpha = 1$. Substituting and reducing, we have,

$$y = -\frac{gx^2}{2v^2} \quad \ldots\ldots \quad (93)$$

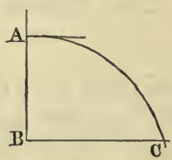


Fig. 104.

For every value of x, y is negative, which shows that the trajectory lies below the horizontal through the point of projection. If we suppose ordinates to be positive downward, we have,

$$y = \frac{gx^2}{2v^2} \quad . \quad . \quad . \quad . \quad . \quad (94)$$

To find the point at which the trajectory will reach any horizontal plane, BC, whose distance below A is h', make $y = h'$, in (94), whence,

$$x = BC = v\sqrt{\frac{2h'}{g}} \quad . \quad . \quad . \quad . \quad . \quad (95)$$

On account of the resistance of the air, the results of the preceding discussion must be greatly modified. They approach more nearly to the observed phenomena, as the velocity is diminished and the density of the projectile increased. The atmospheric resistance increases as the square of the velocity, and as the cross section of the projectile exposed to the action of the resistance. In the air, it is found, under ordinary circumstances, that the maximum range is obtained by an angle of projection, not far from 34°.

Examples.

1. What is the time of flight of a projectile in vacuum, when the angle of projection is 45°, and the range 6000 feet?

SOLUTION.

When the angle of projection is 45°, the range is equal to twice the height due to the velocity of projection. Denoting this velocity by v, we have,

$$v^2 = 2gh = 2 \times 32\tfrac{1}{6} \times 3000 = 193000.$$

Whence,

$$v = 439.3 \text{ ft.}$$

From equation (91), we have,

$$t = \frac{r}{v\cos\alpha} = \frac{6000}{439.3 \cos 45^\circ} = 19.3 \text{ sec. } \textit{Ans.}$$

2. What is the range of a projectile, when the angle of projection is 30°, and the initial velocity 200 feet? *Ans.* 1076.9 ft.

3. The angle of projection under which a shell is thrown is 32°, and the range 3250 feet. What is the time of flight?

Ans. 11.25 sec., nearly.

Centripetal and Centrifugal Forces.

126. Curvilinear motion can only result from the action of an incessant force, whose direction differs from that of the original impulse. This force may arise from one or more active forces, or it may result from the resistance offered by a rigid body, as when a ball is compelled to run in a groove. Whatever may be the nature of the forces, we can always conceive them to be replaced by a single incessant force acting transversely to the path of the body. Let this force be resolved into two components, one normal to the path of the body, and the other tangential to it. The latter force may act to accelerate, or to retard the motion of the body, according to the direction of the resultant force; the former alone is effective in changing the direction of motion. The normal component is always directed toward the concave side of the curve, and is called the *centripetal force.* The body resists this force, by virtue of its inertia, and, from the law of inertia, this resistance must be *equal and directly opposed* to the centripetal force. This resistance is called *the centrifugal force.* Hence, we may define centrifugal force to be *the resistance a body offers to a force that tends to deflect it from a rectilineal path.* The centripetal and centrifugal forces together, are called *central* forces.

Measure of the Centrifugal Force.

127. To deduce an expression for the measure of the centrifugal force, let us first consider the case of a material point, constrained to move in a circular path, by a force constantly directed toward the centre, as when a body is confined by a string and whirled around a fixed point. In this case, the tangential component of the deflecting force is 0; there is no loss of velocity in consequence of a change of direction in the motion, (Art. 109); hence, the motion of the point is uniform.

Let ABD be the path of the body, and V its centre. Suppose the circumference of the circle to be a regular polygon, having an infinite number of sides, of which AB is one; and denote each side by s. When the body reaches A, it tends, by virtue of its inertia, to move in the direction of the tangent, AT; but, in consequence of the action of the centripetal force directed toward V, it is constrained to describe the side s in the time t. If we draw BC parallel to AT, it will be perpendicular to the diameter AD, and AC will represent the space through which the body has been drawn from the tangent, in the time t. If we denote the acceleration due to the centripetal force by f, and suppose it to be constant during the time t, we have, from Art. 103,

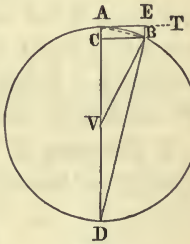


Fig. 105.

$$AC = \tfrac{1}{2}ft^2 \quad . \; . \; . \; . \; . \quad (96)$$

From the right-angled triangle, ABD, we have, since $AB = s$,

$$s^2 = AC \times AD; \text{ or, } s^2 = AC \times 2r.$$

Whence,

$$AC = \frac{s^2}{2r}.$$

Substituting this value of AC, in (96), and solving with respect to f,

$$f = \frac{s^2}{t^2} \times \frac{1}{r}.$$

But $\frac{s^2}{t^2} = v^2$ (Art. 102), in which v is the velocity of the moving point. Substituting in the preceding equation, we have,

$$f = \frac{v^2}{r} \quad \ldots\ldots \quad (97)$$

Here f is the acceleration due to the centripetal force, but this is equal to the centrifugal force, hence, *the acceleration due to the centrifugal force, is equal to the square of the velocity, divided by the radius of the circle.*

If the mass of the body be denoted by M, and the *entire centrifugal* force by F, we have, (Art. 18),

$$F = \frac{Mv^2}{r} \quad \ldots\ldots \quad (98)$$

If we suppose the body moving on any curve, we may, whilst it is passing over any two consecutive elements, regard it as moving on the arc of the osculatory circle to the curve; and, further, we may regard the velocity as uniform during the infinitely small time required to describe these elements. The direction of the centrifugal force being normal to the curve, must pass through the centre of the osculatory circle. Hence, all the circumstances of motion are the same as before, and equations (97) and (98) will be applicable, provided r be taken as the radius of the curvature. Hence, we may enunciate the law of centrifugal force as follows:

The acceleration due to the centrifugal force is equal to the square of the velocity of the body divided by the radius of curvature.

The entire centrifugal force is equal to the acceleration, multiplied by the mass of the body.

In the case of a body whirled around a centre, and restrained by a string, the tension of the string is measured by the centrifugal force. The radius remaining constant, the tension increases as the square of the velocity.

Centrifugal Force at points of the Earth's Surface.

128. Let it be required to determine the centrifugal force at different points of the earth's surface, due to rotation on its axis.

Suppose the earth spherical. Let A be a point on the surface, PQP' a meridian section through A, PP' the axis, RQ the equator, and AB, perpendicular to PP', the radius of the parallel of latitude through A. Denote the radius of the earth by r, the radius of the parallel through A by r', and the latitude of A, or the angle ACQ, by l. The time of revolution being the same for every point on the earth's surface, the velocities of Q and A will be to each other as their distances from the axis. Denoting these velocities by v and v', we have,

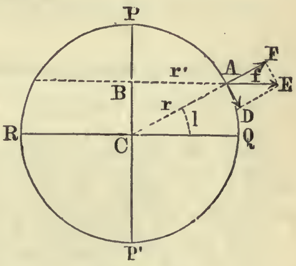


Fig. 106.

$$v : v' :: r : r',$$

whence,

$$v' = \frac{vr'}{r}.$$

But from the right-angled triangle, CAB, since the angle at A is equal to l, we have,

$$r' = r \cos l.$$

Substituting this value of r' in the value of v', and reducing, we have,

$$v' = v \cos l.$$

If we denote the centrifugal force at the equator by f, we have,

$$f = \frac{v^2}{r} \quad . \; . \; . \; . \; . \quad (99)$$

In like manner, if we denote the centrifugal force at A, by f', we have,

$$f' = \frac{v'^2}{r'}.$$

Substituting for v' and r' their values, previously deduced, we get,

$$f' = \frac{v^2 \cos l}{r} \quad . \; . \; . \; . \; . \quad (100)$$

Combining equations (99) and (100), we find,

$$f : f' :: 1 : \cos l, \qquad \therefore \; f' = f \cos l \quad . \; . \; . \; . \; . \quad (101)$$

That is, *the centrifugal force at any point on the earth's surface, is equal to the centrifugal force at the equator, multiplied by the cosine of the latitude.*

Let AE, perpendicular to PP', represent f', and resolve it into two components, one tangential, and the other normal to the meridian section. Prolong CA, and draw AD perpendicular to it at A. Complete the rectangle, FD on AE, as a diagonal. Then will AD be the tangential, and AF the normal component. In the right-angled triangle, AFE, the angle at A is equal to l. Hence,

$$FE = AD = f' \sin l = f \cos l \sin l = \frac{f \sin 2l}{2} \quad . \; . \; . \; . \; . \quad (102)$$

$$AF = f' \cos l = f \cos^2 l \quad . \; . \; . \; . \; . \quad (103)$$

From (102), we see that the tangential component is 0 at the equator, goes on increasing till $l = 45°$, where it is a maximum, and then goes on decreasing till the latitude is 90°, when it again becomes 0.

The effect of the tangential component is to heap up the particles of the earth about the equator, and, were the earth in a fluid state, this process would go on till the effect of the tangential component was counterbalanced by the component of gravity acting down the inclined plane thusformed, when the particles would be in equilibrium.

The higher analysis shows that the form of equilibrium is that of an oblate spheroid, differing but slightly from that which our globe is found to possess by actual measurement.

From equation (103), we see that the normal component of the centrifugal force varies as the square of the cosine of the latitude.

This component is directly opposed to gravity, and, consequently, tends to diminish the apparent weight of all bodies on the surface of the earth. The value of this component is greatest at the equator, and diminishes toward the poles, where it is 0. From the action of the normal component of the centrifugal force, and because the flattened form of the earth due to the tangential component brings the polar regions nearer the centre of the earth, the measured force of gravity ought to increase in passing from the equator toward the poles. This is found to be the case.

The radius of the earth at the equator is about 3962.8 miles, which, multiplied by 2π, will give the entire circumference of the equator. If this be divided by the number of seconds in a day, 86400, we find the value of v. Substituting this value of v and that of r just given, in equation (99), we find,

$$f = 0.1112 \text{ ft.},$$

for the centrifugal force at the equator. If this be multiplied by the square of the cosine of the latitude of any place, we have the value of the normal component of the centrifugal force at that place.

Centrifugal Force of Extended Masses.

129. We have supposed, in what precedes, the dimensions of the body under consideration to be extremely small; let us next examine the case of a body, of any dimensions whatever, constrained to revolve about a fixed axis. If the body be divided into infinitely small elements, whose directions are parallel to the axis, the centrifugal force of each element will be equal to the mass of the element into the square of its velocity, divided by its distance from the axis. If a plane be passed through the centre of gravity of the body, perpendicular to the axis, we may, without impairing the generality of the result, suppose the mass of each element concentrated at the point in which this plane cuts the line of direction of the element.

Let XCY be the plane through the centre of gravity perpendicular to the axis of revolution, AB the projection of the body on the plane, and C the point in which it cuts the axis. Take C as the origin of a system of rectangular co-ordinates; let CX be the axis of X, CY the axis of Y, and m be the point at which the mass of one filament is concentrated, and denote that mass by m. Denote the co-ordinates of m by x and y, its distance from C by r, and its velocity by v. The centrifugal force of the mass, m, is equal to

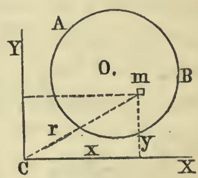


Fig. 107.

$$\frac{mv^2}{r}.$$

If we denote the angular velocity of the body by V', the velocity of m will be rV', which, in the expression just deduced, gives,

$$mrV'^2.$$

Let this force be resolved into components parallel to CX and CY. We have, for these components,

$$mrV'^2 \cos mCX, \quad \text{and,} \quad mrV'^2 \sin mCX.$$

But, from the figure,

$$\cos mCX = \frac{x}{r}, \quad \text{and,} \quad \sin mCX = \frac{y}{r}.$$

Substituting these in the preceding expressions, and reducing, we have, for the components,

$$mxV'^2, \quad \text{and,} \quad myV'^2.$$

Similar expressions may be deduced for each of the other filaments. If we denote the resultant of the components parallel to CX by X, and of those parallel to CY by Y, we have,

$$X = \Sigma(mx)V'^2, \quad \text{and,} \quad Y = \Sigma(my)V'^2.$$

If we denote the mass of the body by M, and suppose it concentrated at its centre of gravity, O, whose co-ordinates are x_1, and y_1, and whose distance from C is r_1, we shall have, from the principle of the centre of gravity, (Art. 55),

$$\Sigma(mx) = Mx_1, \text{ and } \Sigma(my) = My_1.$$

Substituting above, we have,

$$X = MV'^2x_1, \quad \text{and,} \quad Y = MV'^2y_1.$$

If we denote the resultant centrifugal force by R, we have,

$$R = \sqrt{X^2 + Y^2} = MV'^2\sqrt{x_1^2 + y_1^2} = MV'^2r_1.$$

But if the velocity of the centre of gravity be denoted by V, we have,

$$V = V' r_1 ; \text{ or, } V'^2 = \frac{V^2}{r_1^2} ;$$

which, in the preceding result, gives,

$$R = \frac{MV^2}{r_1} \quad . \; . \; . \; . \; . \; (104)$$

The direction of R is given by the equations,

$$\cos a = \frac{X}{R} = \frac{x_1}{r_1}, \quad \text{and} \quad \cos b = \frac{Y}{R} = \frac{y_1}{r_1};$$

hence, it passes through the centre of gravity, O; that is, *the centrifugal force of an extended mass, constrained to revolve about a fixed axis, is the same as though the mass were concentrated at its centre of gravity.*

Principal Axes.

130. Suppose the axis about which a body is revolving to be *free*, so that the body can move in any manner. If the body is homogeneous and the axis not one of symmetry, the centrifugal forces of the elements of the body will not balance each other, and unequal pressures will be exerted on different parts of the axis. This inequality of pressure will change the position of the axis of revolution at each instant, and the change will go on, till an axis is reached, that is pressed equally in all directions by the centrifugal forces of the elements. Such an axis is called a *principal axis.* It may be shown, by the higher analysis, that a body has at least three principal axes, which pass through its centre of gravity, and are at right angles to each other. It may also be shown that the moment of inertia with respect to one of these axes is *greater*, and with respect to another *less*, than with reference to any other line through the centre of gravity. When the body is revolving about the former, its rotation is *stable;* when about the latter, it is *unstable.* The former may be called an *axis of stability*, and the latter an *axis of instability.* In the case of certain regular bodies, there may be an infinite number of either kind. Thus, in an oblate spheroid, the polar axis is an axis of stability, and the only one, whilst any diameter of an equatorial section is an axis of instability. In a prolate spheroid, the polar axis is an axis of instability, and the only one, whilst any diameter of the equatorial section is an axis of stability. In a right cone with a circular base, the axis of the cone is an axis of instability; but any line through the centre of gravity, and perpendicular to the axis, is an axis of stability.

Experimental Illustrations.

131. The principles relating to centrifugal force admit of experimental illustration. The instrument represented in the figure may be employed to show the value of the centrifugal force. *A* is a vertical axle, on which is mounted a wheel, *F*, communicating with a train of wheel-work, by means of which the axle may be made to revolve with any angular velocity. At the upper end of the axle is a forked branch, *BC*, sustaining a stretched wire. *D* and *E* are balls pierced by the wire, and free to move along it. Between *B* and *E* is a spiral spring, whose axis coincides with the wire.

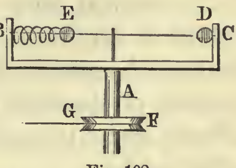


Fig. 108.

Immediately below the spring, on the horizontal part of the fork, is a scale for determining the distance of the ball, *E*, from the axis, and for measuring the degree of compression of the spring. Before using the instrument, the force required to produce any degree of compression of the spring is determined experimentally, and marked on the scale.

If a motion of rotation be communicated to the axis, the ball D will at once recede to C, but the ball E will be restrained by the spring. As the velocity of rotation increases, the spring is compressed more and more, and the ball E approaches B. By a suitable arrangement of wheelwork, the angular velocity of the axis corresponding to any compression may be ascertained. We have, therefore, all the data necessary to verify the law of centrifugal force.

If a vessel of water be made to revolve about a vertical axis, the inner particles recede from the axis on account of the centrifugal force, and are heaped up about the sides of the vessel, imparting a concave form to the upper surface. The concavity becomes greater as the angular velocity is increased.

If a circular hoop of flexible material be mounted on one of its diameters, its lower point being fastened to the horizontal beam, and a motion of rotation imparted, the portions of the hoop farthest from the axis will be most affected by centrifugal force, and the hoop will assume an elliptical form.

If a sponge, filled with water, be attached to one of the arms of a whirling machine, and motion of rotation imparted, the water will be thrown from the sponge. This principle has been used for drying clothes. An annular trough of copper is mounted on an axis by radial arms, and the axis connected with a train of wheelwork, by means of which it may be put in motion. The outer wall of the trough is pierced with holes for the escape of water, and a lid confines the articles to be dried. To use this instrument, the linen, after being washed, is placed in the annular space, and a rapid rotation imparted to the machine. The linen is thrown against theouter wall of the instrument, and the water, urged by the centrifugal force, escapes through the holes. Sometimes as many as 1,500 revolutions per minute are given to the drying machine, in which case, the drying process is very rapid and very perfect.

If a body revolve with sufficient velocity, it may happen that the centrifugal force generated will be greater than the force of cohesion that binds the particles together, and the body be torn asunder. It is a common occurrence for large grindstones, when put into rapid rotation, to burst, the fragments being thrown away from the axis, and often producing much destruction.

When a wagon, or carriage, is driven round a corner, or is forced to run on a circular track, the centrifugal force is often sufficient to throw loose articles from the vehicle, and even to overthrow the vehicle itself. When a car on a railroad track is forced to turn a sharp curve, the centrifugal force throws the cars against the rail, producing a great amount of friction. To obviate this difficulty, it is customary to raise the outer rail, so that the resultant of the centrifugal force, and the force of gravity, shall be perpendicular to the plane of the rails.

Elevation of the Outer Rail of a Curved Track.

132. To find the elevation of the outer rail, so that the resultant of the weight and centrifugal force shall be perpendicular to the line joining the rails, assume a cross section through the centre of gravity, *G*. Take the horizontal, GA, to represent the centrifugal force, and GB to represent gravity. Construct their resultant, GC. Then must DE be perpendicular to GC

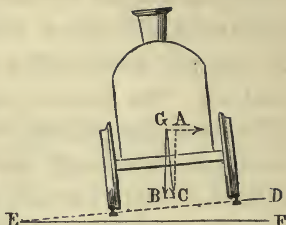


Fig. 109.

Denote the velocity of the car by v, the radius of the curved track by r, the force of gravity by g, and the angle, DEF, or its equal, BGC, by α. From the right-angled triangle, GBC, we have,

$$\tan\alpha = \frac{BC}{GB}.$$

But, BC, or its equivalent, GA, is equal to $\frac{v^2}{r}$, and GB is equal to g; hence,

$$\tan\alpha = \frac{v^2}{gr}.$$

Denoting the distance between the rails, by d, and the elevation of the outer rail above the inner one, by h, we have,

$$\tan\alpha = \frac{h}{d}, \text{ very nearly.}$$

Equating the two values of $\tan\alpha$, we have,

$$\frac{h}{d} = \frac{v^2}{gr}, \quad \therefore \quad h = \frac{dv^2}{gr}.$$

Hence, the elevation of the outer rail varies as the square of the velocity directly, and as the radius of the curve inversely.

It is obvious that the elevation ought to be different for different velocities, which, from the nature of the case, is impossible. The correction is, therefore, made for some assumed velocity, and then such a form is given to the tire of the wheels as will complete the correction for other velocities.

The Conical Pendulum.

133. The conical pendulum consists of a ball attached to one end of a rod, the other end of which is connected, by a hinge-joint, with a vertical axle. When the axle is put in motion, the centrifugal force causes the ball to recede from the axis, until an equilibrium is established between the weight of the ball, the centrifugal force, and the resistance of the connecting rod. When the velocity is constant, the centrifugal force is constant, and the centre of the ball describes a horizontal circle, whose radius depends on the velocity. To determine the time of revolution:

Let BD be the axis, A the ball, B the hinge-joint, and AB the connecting rod, whose mass is so small, that it may be neglected, in comparison with that of the ball.

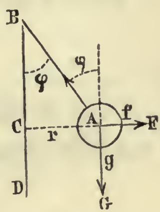


Fig. 110.

Denote the time of revolution, by t, the length of the arm, by l, the centrifugal force, by f, and the angle, ABC, by φ. Draw AC perpendicular to BD, and denote AC, by r, and BC, by h.

From the triangle, ABC, we have, $r = h\tan\varphi$; and since r is the radius of the circle described by A, the distance passed over by A, in the time t, is equal to $2\pi r$, or, $2\pi h\tan\varphi$. Denoting the velocity of A, by v, we have, from article (102),

$$v = \frac{2\pi h\tan\varphi}{t}.$$

But the centrifugal force is equal to the square of the velocity, divided by the radius; hence,

$$f = \frac{4\pi^2 h\tan\varphi}{t^2} \quad . \; . \; . \; . \; . \quad (105)$$

The forces that act on A, are the centrifugal force, in the direction AF, the force of gravity, in the direction AG, and the resistance of the connecting rod, in the direction AB. In order that the ball may remain at an invariable distance from the axis, these must be in equilibrium. Hence, (Art. 33),

$$g : f :: \sin BAF :: \sin BAG;$$

but, $$\sin BAF = \sin(90^\circ + \varphi) = \cos\varphi;$$

and, $$\sin BAG = \sin(180^\circ - \varphi) = \sin\varphi.$$

We have, therefore,

$$g : f :: \cos\varphi : \sin\varphi, \quad \therefore\ f = g\tan\varphi.$$

Equating these values of f, we have,

$$\frac{4\pi^2 h \tan\varphi}{t^2} = g\tan\varphi.$$

Solving with respect to t,

$$t = 2\pi\sqrt{\frac{h}{g}} \quad . \; . \; . \; . \; . \quad (106)$$

That is, *the time of revolution, is equal to the time of a double vibration of a pendulum whose length is h.*

The Governor.

134. The principle of the conical pendulum is employed in the *governor*, a machine attached to engines, to regulate the supply of motive force.

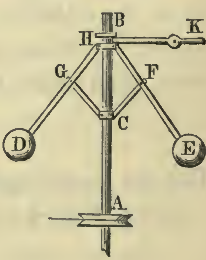


Fig. 111.

AB is a vertical axis connected with the machine near its working-point, and revolving with a velocity proportional to that of the working-point; FE and GD are arms turning about AB, and bearing heavy balls, D and E, at their extremities; these bars are united by hinge-joints with two other bars at G and F, and also to a ring at H, that is free to slide up and down the shaft.

The ring, H, is connected with a lever, HK, that acts on the valve in the pipe that admits steam to the cylinder.

When the shaft revolves, the centrifugal force causes the balls to recede from the axis, and the ring, H, is depressed; and when the velocity has become sufficiently great, the lever closes the valve. If the velocity slackens, the balls approach the axis, and the ring, H, ascends, opening the valve. In any given case, if we know the velocity required at the working-point, we can compute the required angular velocity of the shaft, and, consequently, the value of t. This value of t, substituted in equation (106), gives the value of h. We may, therefore, prouerly adjust the ring, and the lever, HK.

EXAMPLES.

1. A ball weighing 10 lbs. is whirled round in a circle whose radius is 10 feet, with a velocity of 30 feet. What is the acceleration due to centrifugal force? *Ans.* 90 ft.

2. In the preceding example, what is the tension on the cord that restrains the ball?

SOLUTION.

Denote the tension, in pounds, by t; then, since the pressures produced are proportional to the accelerations, we have,

$$10 : t :: g : 90, \qquad \therefore\ t = 28 \text{ lbs., nearly. } Ans.$$

3. A body is whirled round in a circular path whose radius is 5 feet, and the centrifugal force is equal to the weight of the body. What is the velocity of the moving body?

SOLUTION.

Denoting the velocity by v, we have the centrifugal force equal to $\frac{v^2}{5}$; but, by the conditions of the problem, this is equal to gravity;

hence, $$\frac{v^2}{5} = 32\tfrac{1}{6}; \text{ or, } v = 12.7 \text{ ft. } Ans.$$

4. In how many seconds must the earth revolve that the centrifugal force at the equator may counterbalance the force of gravity, the radius of the equator being 3962.8 miles?

SOLUTION.

Reducing miles to feet, and denoting the required velocity, by v, we have,

$$\frac{v^2}{20923584} = 32\tfrac{1}{6}, \qquad \therefore\ v = \sqrt{32\tfrac{1}{6} \times 20923584},$$

But the time of revolution is equal to the circumference of the equator, divided by the velocity. Denoting the time by t, we have,

$$t = \frac{2\pi \times 20923584}{v};$$

and, substituting for v, its value, taken from the preceding equation, we have,

$$t = \frac{2\pi\sqrt{20923584}}{\sqrt{32\tfrac{1}{6}}} = \frac{2\pi \times 4574}{5.67} = 5068 \text{ secs. } \textit{Ans.}$$

5. A body is placed on a horizontal plane, which is made to revolve about a vertical axis, with an angular velocity of 2 feet. How far must the body be situated from the axis that it may be on the point of sliding outward, the coefficient of friction between the body and plane being equal to .6?

SOLUTION.

Denote the required distance by r; then will the velocity of the body be $2r$, and the centrifugal force $4r$. But the acceleration due to the force of friction is equal to $0.6 \times g = 19.3$ ft. From the conditions of the problem, these are equal, hence,

$$4r = 19.3 \text{ ft.}, \qquad \therefore\ r = 4.825 \text{ ft. } \textit{Ans.}$$

6. What must be the elevation of the outer rail of a track, the radius being 3960 ft., the distance between the rails 5 feet, and the velocity of the car 30 miles per hour, that there may be no lateral thrust? *Ans.* 0.076 ft., or 0.9 in., nearly.

7. The distance between the rails is 5 feet, the radius of the curve 600 feet, and the height of the centre of gravity of the car 5 feet. What velocity must the car have that it may be on the point of being overturned by the centrifugal force, the rails being on the same level?

We have,

$$v = \sqrt{\frac{5 \times 32\tfrac{1}{6} \times 600}{2 \times 5}} = 98 \text{ ft., or } 66\tfrac{3}{4} \text{ m., per hour. } \textit{Ans.}$$

Definition and Measure of Work.

135. By the term *work*, in mechanics, is meant the effect produced by a force in overcoming a resistance. It implies that a force is exerted through a certain space; thus, a force exerted to raise a weight is said to *work*, and the *quantity of work* performed depends, *first* on the weight raised, and *secondly* on the height through which it is raised. Because other kinds of work may be assimilated to that of raising a weight, it is customary to assume the work necessary to raise a given weight, to a given height, as a standard to which all kinds of work may be referred.

In this country, and in Great Britain, the unit generally adopted is the work required to raise a weight of *one pound* through a height of *one foot*. This unit is called a *foot-pound*. In France, the assumed unit is the work required to raise a *kilogramme* through a *metre*; it is called a *kilogrammetre*.

If we denote the force exerted by P, the space through which it is exerted by p, and the quantity of work performed by Q, we shall have,

$$Q = Pp.$$

If the force is variable, we may conceive the path divided into equal parts, so small that, for each part, the pressure may be regarded as constant. If we denote the length of one of these parts by p, and the force exerted whilst describing it by P, we shall have, for the corresponding quantity of work, Pp, and for the entire quantity of work, denoted by Q, we shall have the sum of the elementary quantities of work; or, since p is the same for each,

$$Q = p\,\Sigma(P).$$

The quotient obtained by dividing the entire quantity of work by the entire path, is called the *mean pressure*, or the *mean resistance*, and is evidently the force which, acting uniformly through the same path, would accomplish the same work.

In estimating work performed by engines and other machines, a unit is adopted that involves the additional idea of time. This unit is called a *horse power*. A horse power is a power capable of raising 33,000 lbs. through a height of 1 foot in 1 minute. When we say that a machine is one of 10 horse power, we mean that it is capable of performing 330,000 units of work in a minute.

Work, when the Power acts Obliquely.

136. Let PD be a force, and AB the path that the body D is constrained to follow. Denote the angle PDs by α, and suppose P to be resolved into two components, one perpendicular, and the other parallel to AB. We have, for the former, $P\sin\alpha$, and, for the latter, $P\cos\alpha$.

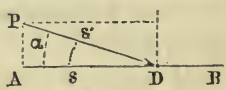


Fig. 112.

The former can produce no work, since, from the nature of the case, the point cannot move in the direction of the normal; hence, the latter is the only component that works. Let sD be the space through which the body is moved, and denote the quantity of work, by Q; we have,

$$Q = P\cos\alpha \times sD.$$

Let fall the perpendicular ss' from s, on the direction of the force, P. From the right-angled triangle, Dss', we have,

$$sD \times \cos\alpha = s'D.$$

Substituting this in the preceding equation, we get,

$$Q = P \times s'D.$$

That is, the quantity of work of a force acting obliquely to the path along which the point of application is constrained to move, is equal to the intensity of the force multiplied by the projection of the path on the direction of the force.

If we take sD, infinitely small, $s'D$ will be the virtual velocity of D, and the expression for the quantity of work of P will be its virtual moment, (Art. 36). Hence, we say that the elementary quantity of work of a force is equal to its virtual moment, and, from the principle of virtual moments, we conclude that the algebraic sum of the elementary quantities of work of any number of forces applied at the same point, is equal to the elementary quantity of work of their resultant. What is true for the elementary quantities of work at any instant, must be equally true at any other instant. Hence, the algebraic sum of all the elementary quantities of work of the components is equal to the algebraic sum of the elementary quantities of work of their resultant; that is, the work of the components is equal to the work of their resultant.

This principle hardly seems to require demonstration, for, from the definition of a resultant, it would seem to be true of necessity. If the forces be in equilibrium, the entire quantity of work is equal to 0.

This principle is used in computing the quantity of work required to raise material for a wall or building; for raising material from a shaft; for raising water from one reservoir to another; and for a great variety of similar operations. In this connection, the principle may be enunciated as follows:

The algebraic sum of the quantities of work required to raise the parts of a system through any vertical spaces, is equal to the quantity of work required to move the whole *system through the height described by the centre of gravity of the system.*

It also follows, that, *if all the pieces of a machine, which moves without friction, be in equilibrium in all positions, the centre of gravity of the system will neither ascend nor descend whilst the machine is in motion.*

Rotation.

137. When a body, restrained by an axis, is acted on by a force not passing through the axis, it takes up a motion of rotation. In this variety of motion, each point describes a circle whose plane is perpendicular to, and whose centre is in, the axis. The velocity of any particle is equal to its distance from the axis multiplied by the angular velocity of the body. The time of revolution of all the particles being the same, the velocities of different particles are proportional to their distances from the axis.

Quantity of Work of a Force producing Rotation.

138. If a force be applied obliquely to the axis of rotation, we may conceive it to be resolved into two components, one parallel, and the other perpendicular to the axis. The effect of the former will be counteracted by the resistance of the axis; the effect of the latter will be exactly the same as that of the applied force. We need, therefore, consider those components only, whose directions are perpendicular to the axis.

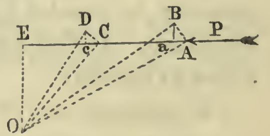


Fig. 113.

Let P be a force whose direction is perpendicular to the axis, but does not intersect it. Let O be the point in which a plane through P, perpendicular to the axis, intersects it. Let A and C

be any two points on the direction of P. Suppose P to turn the system through an infinitely small angle, and let B and D be the new positions of A and C. Draw OE, Ba, and Dc perpendicular to PE; draw also, AO, BO, CO, and DO. Denote OA by r, OC by r', OE by p, and the path described by a point at a unit's distance from O, by θ'. Since the angles AOB, and COD are equal, from the nature of the motion, we have, $AB = r\theta'$, and $CD = r'\theta'$; and since the angular displacement is infinitely small, these may be regarded as straight lines perpendicular to OA and OC. From the right-angled triangles ABa and CDc, we have,

$$Aa = r\theta' \cos BAa, \text{ and } Cc = r'\theta' \cos DCc.$$

In the right-angled triangles ABa, and OAE, AB is perpendicular to OA, and Aa to OE; hence, BAa, and AOE, are equal; hence,

$$\cos BAa = \cos AOE = \frac{p}{r}.$$

In like manner,

$$\cos DCc = \cos COE = \frac{p}{r'}.$$

Substituting in the preceding equations, we have,

$$Aa = p\theta', \text{ and } Cc = p\theta'; \therefore Aa = Cc;$$

whence,

$$P \,.\, Aa = P \,.\, Cc = Pp\theta'.$$

The first member of the equation is this quantity of work of P, when its point of application is A; the second is its quantity of work, when the point of application is at C. Hence, we conclude, that *the elementary quantity of work of a rotating force is always the same, wherever its point of application may be taken, provided its line of direction remain unchanged.*

We conclude, also, that the elementary quantity of work is equal to the intensity of the force multiplied by its lever arm into the elementary space described by a point at a unit's distance from the axis.

If we suppose the force to act for a unit of time, the intensity and lever arm remaining the same, and denote the *angular velocity*, by θ, we shall have,

$$Q' = Pp\theta.$$

For any number of forces similarly applied, we shall have,

$$Q = \Sigma(Pp)\theta \quad . \quad . \quad . \quad . \quad . \quad (107)$$

If the forces are in equilibrium, we have, (Art. 34), $\Sigma(Pp) = 0$; consequently, $Q = 0$.

Hence, if any number of forces tending to produce rotation are in equilibrium, the entire quantity of work of the forces is equal to 0.

Accumulation of Work.

139. When a force acts on a body, to impart motion, it expends a certain quantity of work in overcoming the body's inertia. This work is said to be *stored up* in the body; and if a resistance be offered to its motion, the entire quantity of work will be given out, and expended on the resistance. A body in motion may, therefore, be regarded as the representative of a quantity of work which, under certain circumstances, is capable of being utilized. The work stored up, or *accumulated*, depends on the mass of the moving body, and also on the velocity with which it moves. To find an expression for it, let us denote the weight of the body by W, its velocity by v, and the quantity of accumulated work by Q. If we suppose it to be projected vertically upward, with the velocity, v, it will rise to *the height due to that velocity*, that is, the work stored up in the body is sufficient to raise the weight, W, through a height, h. Hence,

$$Q = Wh.$$

But, $h = \frac{1}{2}\frac{v^2}{g}$, (Art. 105). Substituting this value of h, we have,

$$Q = \frac{1}{2}\frac{W}{g}v^2.$$

Denoting the mass of the body by M, we have, (Art. 15), $\frac{W}{g} = M$, and this, in the preceding equation, gives,

$$Q = \frac{1}{2}Mv^2.$$

Hence, *the accumulated work in a moving body is equal to one-half the body's mass into the square of its velocity.*

The expression $\frac{1}{2}Mv^2$ is called the *living force* of the body. Hence, *the living force of a body is equal to half its mass, multiplied by the square of its velocity.* The living force of a body is the measure of the quantity of work expended in producing the velocity, or, of the quantity of work the body is capable of giving out.

When forces tend to increase the velocity, their work is positive; when they tend to diminish it, their work is negative. It is the aggregate of all the work expended, both positive and negative, that is measured by the quantity, $\frac{1}{2}mv^2$.

If, at any instant, a body whose mass is m, has a velocity v, and, at a subsequent instant, its velocity has become v', we have for the accumulated work at these two instants

$$Q = \frac{1}{2}mv^2, \quad Q' = \frac{1}{2}mv'^2;$$

and, for the aggregate quantity of work expended in the interval,

$$Q'' = \frac{1}{2}m(v'^2 - v^2) \quad . \quad . \quad . \quad . \quad . \quad (108)$$

When the motive forces, during the interval, perform more work than the resistances, v' is greater than v, and there is an accumulation of work. When the work of the resistances exceeds that of the motive forces, v exceeds v', Q'', is negative, and there is a loss of work which is expended on the resistances.

Living Force of Revolving Bodies.

140. Denote the angular velocity of a revolving body by θ, the masses of its elementary particles by m, m', &c., and their distances from the axis of rotation, by r, r', &c. Their velocities will be $r\theta$, $r'\theta$, &c., and their living forces, $\frac{1}{2}mr^2\theta^2$, $\frac{1}{2}m'r'^2\theta^2$, &c. Denoting the entire living force of the body by L, we have, by summation, recollecting that θ is the same for all the terms,

$$L = \tfrac{1}{2}\Sigma(mr^2)\theta^2 \quad . \; . \; . \; . \; . \quad (109)$$

But $\Sigma(mr^2)$ is the moment of inertia of the body with respect to the axis of rotation. Denoting the entire mass by M, and its radius of gyration, with respect to the axis of rotation, by k, we have,

$$\Sigma(mr^2) = Mk^2; \quad \therefore \quad L = \tfrac{1}{2}Mk^2\theta^2 \quad . \; . \; . \; . \; . \quad (110)$$

If, at any subsequent instant, the angular velocity has become θ', we have,

$$L' = \tfrac{1}{2}Mk^2\theta'^2;$$

and, for the gain or loss of living force in the interval,

$$L'' = \tfrac{1}{2}Mk^2(\theta'^2 - \theta^2) \quad . \; . \; . \; . \; . \quad (111)$$

If, in equation (110), we make $\theta = 1$, we have,

$$L''' = \tfrac{1}{2}\Sigma(mr^2); \text{ or, } \Sigma(mr^2) = 2L.$$

That is, the moment of inertia of a body, with respect to an axis, is equal to twice its living force when the angular velocity is equal to 1, or, to twice the quantity of work that must be expended to generate a unit of angular velocity.

The principle of living force is applied in discussing the motion of machines. When the power performs more work than is necessary to overcome the resistances, the velocities of the parts increase, and a quantity of work is stored up, to be given out again when the resistances require more work to overcome them than is furnished by the motor.

In many machines, pieces are introduced to equalize the motion; this is particularly the case when either the power or the resistance is variable. Such pieces are called *fly-wheels*.

Fly-Wheels.

141. A fly-wheel is a heavy wheel mounted on an axis, near the point of application of the force it is designed to regulate. It is generally composed of a rim, connected with the axis by radial arms. Sometimes it consists of radial bars, carrying spheres of metal at their outer extremities. Let us denote the mass of the wheel by M, its radius of gyration by k, the quantity of work stored up in any time by Q, and the initial and terminal angular velocities by θ' and θ''. We shall have, from equation (111),

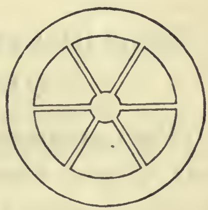


Fig. 114.

$$Q = \tfrac{1}{2}Mk^2(\theta''^2 - \theta'^2) \quad . \quad . \quad . \quad . \quad . \quad (112)$$

If $\theta'' > \theta'$, Q is positive and work is stored up; if, $\theta'' < \theta'$, Q is negative, and the wheel gives out work.

If the angular velocity increase from θ' to θ'', and then decrease to θ', and so on, alternately, the work accumulated during the first part of each cycle is given out during the second part, and any device that will make θ' and θ'' more nearly equal, will contribute toward equalizing the motion of the wheel. By suitably increasing the mass and radius of gyration, their difference may be made as small as desirable. Let the half-sum of the greatest and least angular velocities be called the mean angular velocity, and denote it by θ'''. We shall have $\frac{\theta'' + \theta'}{2} = \theta'''$, and by factoring the second member of (112), we have,

$$Q = \tfrac{1}{2}Mk^2(\theta'' + \theta')\ (\theta'' - \theta');$$

whence, by substituting the value of $\theta'' + \theta'$,

$$Q = Mk^2(\theta'' - \theta')\theta''' \quad . \ . \ . \ . \ . \quad (113)$$

Let us suppose the difference between the greatest and least velocity, equal to the n^{th} part of their mean, that is, that

$$\theta'' - \theta' = \frac{\theta'''}{n}.$$

This, in (113), gives

$$Q = \frac{Mk^2\theta'''^2}{n}; \text{ or, } Mk^2 = \frac{nQ}{\theta'''^2}.$$

From this equation the moment of inertia of the wheel may be found, when we know n, Q, and θ'''. The value of n may be assumed; for most kinds of work a value of from 6 to 10 will be found to give sufficient uniformity; the value of θ''' depends on the character of the work to be performed, and Q is made known by the character of the motion to be regulated.

Composition of Rotations.

142. Let a body, $ACBD$, be acted on by an impulse that would cause it to revolve about AB with an angular velocity v, and at the same instant let it be acted on by a second impulse that would cause it to revolve about DC with an angular velocity v'. Suppose the axes to intersect at O, and from any assumed point in their plane, draw perpendiculars to AB and DC, denoting the former by x and the latter by y. Then will the velocity of the assumed point due to the first force be vx, and its velocity due to the second force will be $v'y$. Now, the point can always be taken, so that rotation about the first axis shall tend to *depress* the point below the plane, and about the second axis to *elevate* it above the plane. In this case the effective velocity of the point is $v'y - vx$.

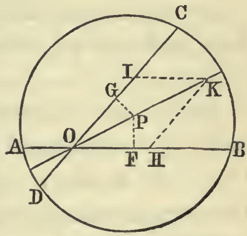


Fig. 115.

If this velocity is equal to 0, the assumed point remains fast, and, we have,

$$vx = v'y\,;\ \text{or,}\ x : y :: v' : v \quad . \ . \ . \ . \ . \ (114)$$

To find the position of the point, in the case supposed, lay off OH equal to v, and OI equal to v', and on these as sides, construct the parallelogram IH, and draw its diagonal OK. Then will any point, P, of this diagonal satisfy proportion (114). For, let OH and OI for a moment be regarded as forces, and OK their resultant, and suppose PF and PG to be perpendicular to OH and OI. Then if P be taken as a centre of moments, we have, (Art. 34),

$$OH \times PF = OI \times PG\,;\ \text{or,}\ v \times PF = v' \times PG.$$

From which we find,

$$PF : PG :: v' : v\,;\ \text{or,}\ PF : PG :: x : y.$$

Hence, every point of OK remains at rest; it is consequently *the resultant axis of rotation.* We have, therefore, the following principles:

If a body be acted on simultaneously by two impulses, each *tending to impart rotation about a separate axis, the resultant motion will be one of rotation about a third axis lying in the plane of the other two, and passing through their point of intersection.*

The direction of the resultant axis coincides with the diagonal of a parallelogram, whose sides are the component axes, and whose lengths are proportional to the angular velocities.

Let OH and OI represent the angular velocities v and v', and OK the diagonal of the parallelogram constructed on these lines as sides. Take any point, I, on the second axis, and let fall perpendiculars on OH and OK; denote the former by r, and the latter by r''; denote, also, the angular velocity about OK, by v''. Since the space passed over by I, in any time, t, depends only on the first force, it will be the same whether we regard the revolution as taking place about OH or OK. If we suppose the rotation to take place about OH, the space passed over in the time, t, will be rvt; if we suppose the rotation to take place about OK, the space passed over in the same time will be $r''v''t$. Placing these equal, we have, after reduction,

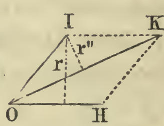


Fig. 116

$$v'' = \frac{r}{r''}v \quad . \; . \; . \; . \; . \quad (115)$$

If we suppose, as before, that OH and OI are forces, and OK their resultant, and take I as a centre of moments, we have,

$$OK \times r'' = vr; \text{ or, } OK = \frac{r}{r''}v.$$

By comparing this with equation (115), we have,

$$v'' = OK.$$

Hence, *the resultant angular velocity is equal to the* *diagonal of the parallelogram described on the component angular velocities as sides.*

By a course of reasoning similar to that employed in demonstrating the parallelopipedon of forces, we might show, that,

If a body be acted on by three simultaneous impulses, each tending to produce rotation about axes intersecting, the resultant motion will be one of rotation about the diagonal of the parallelopipedon whose adjacent edges are the component angular velocities, and the resultant angular velocity will be equal to the length of this diagonal.

The principles just deduced are called *the parallelogram* and *the parallelopipedon of rotations.*

Application to the Gyroscope.

143. The gyroscope is an instrument that may be used to illustrate the laws of rotary motion. It consists of a heavy wheel, *A*, mounted on an axle, *BC*. This axle is attached, by pivots, to the inner edge of a circular hoop, *DE*, within which the wheel, *A*, can turn freely. On one side of the hoop, and in the prolongation of the axle, *BC*, is a bar, *EF*, having a conical hole drilled on its lower face to receive the point of a vertical standard, *G*. If a string be wrapped round the axle, *BC*, and then rapidly unwound, so as to impart a motion of rotation to the wheel, *A*, in the direction indicated by the arrow-head, it is observed that the machine, instead of sinking downward under the action of gravity, takes up a retrograde orbital motion about *G*, as indicated by the arrow-head, *H*. For a time, the orbital motion increases, and, under certain

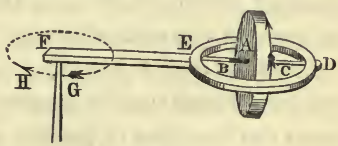


Fig. 117.

circumstances, the bar, EF, is observed to rise upward in a retrograde spiral direction; and, if the cavity for receiving the pivot is pretty shallow, the bar may even be thrown off the standard. Instead of a bar, EF, the instrument may simply have an ear at E, and be suspended by a string. The phenomena are the same as before.

Before explaining the phenomena, it will be necessary to assume conventional rules for giving signs to the different rotations.

Let OX, OY, and OZ, be three rectangular axes. It has been agreed to call all distances, estimated from O, toward either X, Y, or Z, *positive;* consequently, all distances estimated in contrary directions must be negative. If a body revolve about either axis, in such a manner as to appear to an eye on the positive portion of the axis, and looking toward the origin, to move in the same direction as the hands of a watch, that rotation is called *positive.* If rotation take place in an opposite direction, it is *negative.* The arrow-head, A, indicates the direction of positive rotation about the axis of X, the arrow-head, B, the direction of positive rotation about the axis of Y, and the arrow-head, C, the direction of positive rotation about the axis of Z.

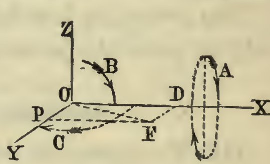


Fig. 118.

Suppose the axis of the wheel of the gyroscope to coincide with the axis of X, taken horizontal; let the standard coincide with the axis of Z, the axis of Y being perpendicular to both. Let positive rotation be communicated to the wheel by a string, and then let the instrument be abandoned to the action of gravity. During the first instant, the force of gravity will impart to it a positive rotation about the axis of Y. Denote the angular velocity about the axis of X, by v, and about the axis of Y, by v'; lay off in a positive direction on the axis of X, OD equal to v, and on the positive direction of the axis of Y, OP equal to v', and complete the parallelogram, OF. Then will OF represent the resultant axis of revolution, and the angular velocity. In moving from OD to OF, the axis has a positive, or *retrograde orbital motion* about the axis of Z. To construct the resultant axis for the second instant, we must compound three angular velocities. Lay off on a perpendicular to OF and OZ, the angular velocity due to gravity, and on OZ the angular velocity in the orbit; construct a parallelopipedon on the three velocities, and draw its diagonal through O. This diagonal will coincide with the axis for the second instant, and will represent the resultant angular velocity. For the next instant, we proceed as before, and so on continually. Since, in each case, the diagonal is greater than either edge of the parallelopipedon, it follows that the angular velocity will continually increase, and, were there no hurtful resistance, this increase would go on indefinitely. The effect of gravity is continually exerted to depress the centre of gravity of the instrument, whilst the effect of the orbital rotation is to elevate it. When the latter prevails, the axis of the gyroscope rises; when the former prevails, the gyroscope descends. Whether one or the other of these conditions is fulfilled, depends on the angular velocity of the wheel, and the position of the centre of gravity of the instrument. Were the instrument counterpoised so as to place the centre of gravity exactly over the pivot, there would be no orbital motion, neither would the instrument rise or fall. Were the centre of gravity thrown on the opposite side of the pivot, the rotation due to gravity would be negative, and the orbital motion would be *direct*.

CHAPTER VII.

MECHANICS OF LIQUIDS.

Classification of Fluids.

144. A FLUID is a body whose particles move freely amongst each other, each particle yielding to the slightest force.

Fluids are of two classes: *liquids,* of which water is a type, and *gases,* or *vapors,* of which air and steam are types. The distinctive property of the first class is, that they are almost *incompressible;* thus, water, on being pressed by a force of 15 lbs. on each square inch of surface, suffers a diminution of not more than the $\frac{1}{2000000}$th of its bulk. Bodies of the second class are readily compressible; thus, air and steam are easily compressed into smaller volumes, and when the pressure is removed, they expand and occupy larger volumes.

Most liquids are imperfect; that is, there is more or less adherence between their particles, giving rise to viscosity. In what follows, they will be regarded as destitute of *viscosity,* and *homogeneous.* In certain cases fluids may also be regarded as destitute of weight, without impairing the validity of the conclusions.

Principle of Equal Pressures.

145. From the nature of a fluid, each of its particles is perfectly movable in all directions. From this we deduce the following fundamental law, viz.: *If a fluid be in equilibrium, under the action of any forces whatever, each par-*

ticle of the mass is equally pressed in all directions; for, if any particle were more strongly pressed in one direction than in the others, it would yield in that direction, and motion would ensue, which is contrary to the hypothesis.

This is called the *principle of equal pressures.*

It follows from the principle of equal pressures, that if a fluid, confined in a vessel, be pressed at any part of its surface, the pressure will be transmitted without change of intensity to every part of the inner surface of the vessel.

This may be illustrated as follows: let a vessel, AB, be filled with water, and let two pistons, C and D, be fitted to corresponding openings in the side of the vessel, and suppose the fluid to be in equilibrium. If any extraneous force be applied to either piston, a second force must be applied to the other to hold the first in equilibrium, and it will be found that these forces are proportional to the areas of the pistons to which they are applied. This relation holds true, no matter what the areas of the pistons, or at what portion of the vessel they may be applied.

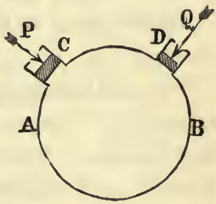


Fig. 119.

A pressure transmitted through a fluid in equilibrium, to the surface of a containing vessel, is normal to that surface; for if it were not, we might resolve it into two components, one normal to the surface, and the other tangential; the effect of the former would be destroyed by the resistance of the vessel, whilst the latter would impart motion to the fluid, which is contrary to the supposition of equilibrium. In like manner, it may be shown, that the resultant of all the pressures, acting at any point of the free surface of a fluid, is normal to the surface at that point.

When the only force acting is gravity, the surface is level. For small areas, a level surface coincides sensibly with a horizontal plane. For larger areas, as lakes and oceans, a level surface coincides with the general surface of the earth. Were the earth at rest, the level surface of lakes and oceans would be spherical; but, on account of the centrifugal force arising from the rotation of the earth, it is that of an ellipsoid, whose axis of revolution is the axis of the earth.

Pressure due to Weight.

146. If an incompressible fluid be in equilibrium, the pressure at any point arising from the weight of the fluid, is proportional to the depth of that point below the free surface.

Take an infinitely small surface, supposed horizontal, and conceive it to be the base of a vertical prism whose altitude is its distance from the free surface. Let this filament be divided, by horizontal planes, into infinitely small, or elementary prisms. From *the principle of equal pressures,* the pressure on the lower face of any one of these prisms is greater than that on its upper face, by the weight of the prism, whilst the lateral pressures counteract each other. Hence, the pressure on the lower face of the first prism from the top, is equal to its weight; that on the lower face of the second is equal to the weight of the first, *plus* the weight of the second, and so on to the bottom. Hence, the pressure on the assumed surface is equal to the weight of the entire column of fluid above it. Had the assumed elementary surface been oblique to the horizon, or perpendicular to it, and at the same depth as before, the pressure on it would have been the same, but its direction would have been normal to the surface. We have, therefore, the following law:

The pressure on an elementary portion of the surface of a vessel containing a heavy fluid, is equal to the weight of a prism of the fluid, whose base is the surface pressed, and whose altitude is its depth below the free surface of the fluid.

Denoting the area of the elementary surface, by s, its depth below the free surface, by z, the weight of a unit of volume of the fluid, by w, and the pressure, by p, we shall have,

$$p = wzs \quad . \; . \; . \; . \; . \quad (116)$$

We have seen that the pressure on any element of a surface is normal to the surface. Denote the angle this normal makes with the vertical, estimated from above downward, by φ, and resolve the pressure into two components, one vertical and the other horizontal; denoting the vertical component by p', we have,

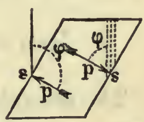


Fig. 120.

$$p' = wzs\cos\varphi \quad . \; . \; . \; . \; . \quad (117)$$

But $s\cos\varphi$ is equal to the horizontal projection of the element s, in other words, it is a horizontal section of a vertical prism, of which that surface is the base.

Hence, *the vertical component of the pressure on any element of the surface is equal to the weight of a column of the fluid, whose base is the horizontal projection of the element, and whose altitude is the distance of the element from the free surface of the fluid.*

The distance, z, has been taken as *positive* from the surface of the fluid downward. If $\varphi < 90°$, we have $\cos\varphi$ positive; hence p', will be *positive*, which shows that the vertical pressure is exerted downward. If $\varphi > 90°$, we have $\cos\varphi$ *negative*, hence p' is negative, which shows that the vertical pressure is exerted upward (see Fig. 120).

Suppose the interior surface of a vessel containing a heavy fluid to be divided into elementary portions, whose areas are denoted by s, s', s'', &c.; denote the distances of these elements from the free surface, by z, z', z'', &c. From the principle just demonstrated, the pressures on these surfaces will be wsz, $ws'z'$, $ws''z''$, &c., and the entire pressure on the interior of the vessel will be equal to,

$$w(sz + s'z' + s''z'' + \&c.); \text{ or, } w \times \Sigma(sz).$$

Let Z denote the depth of a column of fluid, whose base is the surface pressed, and whose weight is equal to the entire pressure, then will this pressure be $w(s + s' + s'' + \&c.)Z$; or, $wZ \,.\, \Sigma s$. Equating these values, we have,

$$w \,.\, \Sigma(sz) = wZ \,.\, \Sigma(s), \quad \therefore \quad Z = \frac{\Sigma(sz)}{\Sigma(s)} \quad \ldots\ldots \quad (118)$$

The second member of (118) is the distance of the centre of gravity of the surface pressed, from the free surface of the fluid. Hence,

The pressure of a heavy fluid on the interior of a vessel is equal to the weight of a cylinder of the fluid, whose base is the area pressed, and whose altitude is the distance of its centre of gravity from the free surface of the fluid.

Examples.

1. A hollow sphere is filled with a liquid. How does the pressure, on the interior surface, compare with the weight of the liquid?

SOLUTION.

Denote the radius of the surface, by r, and the weight of a unit of the liquid, by w. The surface pressed is $4\pi r^2$; and, its centre of gravity is at a distance r from the free surface of the liquid; thence the pressure on the interior surface is equal to,

$$w \times 4\pi r^2 \times r = 4\pi w r^3.$$

But the weight of the liquid is equal to

$$\tfrac{4}{3}\pi w r^3.$$

That is, *the entire pressure is three times the weight of the liquid.*

2. A hollow cylinder, with a circular base, is filled with a liquid. How does the pressure on the interior surface compare with the weight of the liquid?

SOLUTION.

Denote the radius of the base, by r, and the altitude, by h. The centre of gravity of the lateral surface is at a distance from the upper surface of the fluid equal to $\frac{1}{2}h$. If we denote the weight of the unit of volume of the liquid, by w, we have, for the pressure on the interior surface,

$$wh\pi r^2 + 2w\pi r \,.\, \tfrac{1}{2}h^2 = w\pi rh(r + h).$$

But the weight of the liquid is equal to

$$w\pi r^2 h.$$

Hence, *the total pressure is* $\frac{r+h}{r}$ *times the weight of the liquid.*

If $h = r$, the pressure is twice the weight.

If $r = 2h$, the pressure is $\frac{3}{2}$ of the weight.

If $h = 2r$, the pressure is three times the weight, and so on.

In all cases, the pressure exceeds the weight of the liquid.

3. A right cone, with a circular base, stands on its base, and is filled with a liquid. How does the pressure on the internal surface compare with the weight of the liquid?

SOLUTION.

Denote the radius of the base, by r, and the altitude, by h, then will the slant height be equal to

$$\sqrt{h^2 + r^2}.$$

The distance of the centre of gravity of the lateral surface, below the free surface of the liquid, is $\frac{2}{3}h$. If we denote the weight of a unit of volume of the liquid, by w, we have, for the total pressure on the interior surface,

$$w\pi r^2 h + \tfrac{2}{3}w\pi rh\sqrt{h^2 + r^2} = w\pi rh(r + \tfrac{2}{3}\sqrt{h^2 + r^2}).$$

But the weight of the liquid is

$$\tfrac{1}{3}w\pi r^2 h = w\pi rh \times \tfrac{1}{3}r.$$

Hence, *the total pressure is equal to* $\frac{3r + 2\sqrt{h^2 + r^2}}{r}$ *times the weight.*

4. Required the relation between the pressure and the weight in the preceding case, when the cone stands on its vertex.

SOLUTION.

The total pressure is

$$\tfrac{1}{3}w\pi rh\sqrt{h^2 + r^2};$$

and, consequently, *it is equal to* $\dfrac{\sqrt{h^2 + r^2}}{r}$ *times the weight of the liquid.*

5. What is the pressure on the lateral faces of a cubical vessel filled with water, the edge of the cube being 4 feet, and the weight of the water $62\frac{1}{2}$ lbs. per cubic foot? *Ans.* 8000 lbs.

6. A cylindrical vessel is filled with water. The height of the vessel is 4 feet, and the radius of the base 6 feet. What is the pressure on the lateral surface? *Ans* 18850 lbs., nearly.

Centre of Pressure on a Plane Surface.

147. The centre of pressure on a surface, is the point at which the resultant pressure intersects the surface.

Let $ABCD$ be a plane, pressed by a fluid on its upper surface, AB its intersection with the free surface of the fluid, G its centre of gravity, O the centre of pressure, and s the area of an element of the surface at S. Denote the inclination of the plane to the horizontal, by α, the distances from O to AB, by x, from G to AB, by p, and from S to AB, by r. Denote, also, the area AC, by A, and the weight of a unit of volume of the fluid, by w. The distance from G to the free surface of the fluid, is $p\sin\alpha$, and that of any element of the plane, is $r\sin\alpha$.

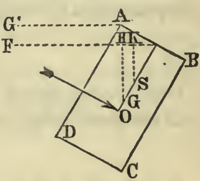


Fig. 121.

From the preceding article, we see that the entire pressure is $wAp\sin\alpha$, and its moment, with respect to AB, is,

$$wAp\sin\alpha \times x.$$

The elementary pressure on s, in like manner, is $wsr\sin\alpha$, its moment, with respect to AB, is $wsr^2 \sin\alpha$, and the sum of all the elementary moments is,

$$w \sin\alpha\, \Sigma(sr^2).$$

But the resultant moment is equal to the algebraical sum of the elementary moments. Hence,

$$wAp \sin\alpha \times x = w \sin\alpha\, \Sigma(sr^2);$$

and, by reduction,

$$x = \frac{\Sigma(sr^2)}{Ap} \quad . \; . \; . \; . \; . \quad (119)$$

The numerator, is the moment of inertia of $ABCD$, with respect to AB, and the denominator is the moment of the area with respect to the same line. Hence, *the distance from the centre of pressure to the intersection of the plane with the surface, is equal to the moment of inertia of the plane, divided by the moment of the plane.*

If we take AD, perpendicular to AB, as an axis of moments, denoting the distance of O from it, by y, and of S from it, by l, we have,

$$wAp \sin\alpha y = w \sin\alpha \Sigma(srl);$$

and, by reduction,

$$y = \frac{\Sigma(srl)}{Ap} \quad . \; . \; . \; . \; . \quad (120)$$

The values of x and y determine the centre of pressure.

It may be observed that x is the distance from AB to the centre of percussion of the plane, and y is the distance from AD to the centre of gravity of the plane. Hence, the centre of pressure is the same as the centre of percussion.

EXAMPLES.

1. Where is the centre of pressure on a rectangular flood-gate, the upper line of the gate coinciding with the surface of the water?

SOLUTION.

It will be on the line joining the middle points of the upper and lower edges of the gate. Denote its distance from the upper edge, by z, the depth of the gate, by $2l$, and its mass, by M. The distance of the centre of gravity from the upper edge is l.

From example 1, (Art. 123), we have, for the moment of inertia of the rectangle,

$$M\left(\frac{l^2}{3}+l^2\right)=M\tfrac{4}{3}l^2.$$

But the moment of the rectangle is,

$$Ml;$$

hence, by division, we have,

$$z=\tfrac{4}{3}l=\tfrac{2}{3}(2l).$$

That is, the centre of pressure is two-thirds of the distance from the upper to the lower edge of the gate.

2. Let it be required to find the pressure on a submerged gate, $ABCD$, the plane of the gate being vertical; also, the distance of the centre of pressure below the surface of the water.

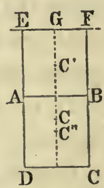


Fig. 122.

SOLUTION.

Let EF be the intersection of the plane with the surface of the water, and suppose the rectangle, AC, to be prolonged till it reaches EF. Let C, C', and C'', be the centres of pressure of the rectangles EC, EB, and AC respectively. Denote GC'', by z, ED, by a, and EA, by a'. Denote the breadth of the gate, by b, and the weight of a unit of volume of the water, by w.

The pressure on EC will be $\frac{1}{2}a^2bw$, and the pressure on EB will be $\frac{1}{2}a'^2bw$; hence, the pressure on AC will be,

$$\tfrac{1}{2}bw(a^2-a'^2);$$

which is the pressure required; from the principle of moments, the moment of the pressure on AC, is equal to the moment of the pressure on EC, *minus* the moment of the pressure on EB. Hence,

$$\tfrac{1}{2}bw(a^2-a'^2)\times z=\tfrac{1}{2}bwa^2\times\tfrac{2}{3}a-\tfrac{1}{2}bwa'^2\times\tfrac{2}{3}a',$$

$$\therefore\ z=\tfrac{2}{3}\,\frac{a^3-a'^3}{a^2-a'^2},$$

which is the required distance from the surface of the water.

3. To find the pressure on a gate, when both sides are pressed, the water being at different levels on the sides. Also, to find the centre of pressure.

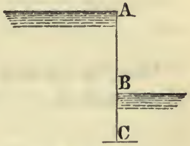


Fig. 123.

SOLUTION.

Denote the depth of water on one side, by a, and on the other, by a', the other elements being the same as before.

The total pressure will be,

$$\tfrac{1}{2}bw(a^2 - a'^2).$$

Estimating z from C upward,

$$z = \tfrac{1}{3}\frac{a^3 - a'^3}{a^2 - a'^2}. \quad \textit{Ans.}$$

4. A sluice-gate, 10 feet square, is placed vertically, its upper edge coinciding with the surface of the water. What is the pressure on the upper and lower halves of the gate, respectively, the weight of a cubic foot of water being $62\tfrac{1}{2}$ lbs.?

Ans. 7812.5 lbs., and 23437.5 lbs.

5. What must be the thickness of a rectangular dam of granite, that it may neither rotate about its outer edge, nor slide along its base, the weight of a cubic foot of granite being 160 lbs., and the coefficient of friction between it and the soil being .6?

SOLUTION.

First, to prevent rotation. Denote the height of the wall, by h, and suppose the water to extend from bottom to top. Denote the thickness, by t, and the length, by l. The weight in pounds, will be,

$$lht \times 160;$$

and this being exerted through its centre of gravity, the moment of the weight with respect to the outer edge, is,

$$\tfrac{1}{2}t^2lh \times 160 = 80lht^2.$$

The pressure of the water against the inner face, in pounds, is equal to

$$\tfrac{1}{2}lh^2 \times 62.5 = lh^2 \times 31.25.$$

This pressure is applied at the centre of pressure, which is (example 1) at a distance from the bottom of the wall equal to $\tfrac{1}{3}h$; hence, its moment with respect to the outer edge of the wall, is equal to

$$lh^3 \times 10.4166.$$

The pressure of the water tends to produce rotation outward, and the weight of the wall acts to prevent it. In order that these forces may be in equilibrium, their moments must be equal; or,

$$80lht^2 = lh^3 \times 10.4166.$$

Whence,

$$t = h\sqrt{.1302} = .36 \times h.$$

Next, to prevent sliding. The force of friction due to the weight of the wall, is,

$$160lht \times .6 = 96lht;$$

and that the wall may not slide, this must be equal to the pressure exerted horizontally against the wall. Hence,

$$96lht = 31.25lh^2.$$

Whence,

$$t = .325h.$$

If the wall is thick enough to prevent rotation, it is secure against sliding.

6. What must be the thickness of a rectangular dam 15 feet high, the weight of the material being 140 lbs. to the cubic foot, when the water rises to the top, that the structure may be just on the point of overturning? *Ans.* 5.7 ft.

7. The staves of a cylindrical cistern filled with water, are held together by a single hoop. Where should the hoop be situated?

Ans. At a distance from the bottom equal to one-third of the height of the cistern.

8. Required the pressure of the sea on the cork of an empty bottle, when sunk to the depth of 600 feet, the diameter of the cork being $\frac{3}{4}$ of an inch, and a cubic foot of sea-water weighing 64 lbs.?

Ans. 134 lbs.

Buoyant Effort of Fluids.

148. Let A be a solid, suspended in a fluid. Conceive it divided into vertical prisms, whose horizontal sections are infinitely small. Each prism is pressed down by a force equal to the weight of a column of fluid, whose base, (Art. 146), is the horizontal section of the filament, and whose altitude is the distance of its upper surface from the surface of the fluid; it is pressed up by a force equal to the weight of a column of fluid having the same base, and an altitude equal to the

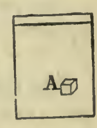


Fig. 124.

distance of the lower base of the filament from the surface of the fluid. The resultant of these pressures is exerted vertically upward, and is equal to the weight of a column of the fluid, equal in bulk to that of the filament, and having its point of application at the centre of gravity of the filament; the lateral pressures destroy each other's effects; hence, the resultant pressure on the body, is a vertical force exerted upward, whose intensity is equal to the weight of the displaced fluid, and whose point of application is the centre of gravity of the displaced fluid. This upward pressure is the *buoyant effort* of the fluid, and its point of application is *the centre of buoyancy.* The direction of the buoyant effort, in any position of the body, is *a line of support.* That line of support which passes through the centre of gravity, of the body is *the line of rest.*

Floating Bodies.

149. A body immersed in a fluid, is urged downward by its weight applied at its centre of gravity, and upward, by the buoyant effort of the fluid applied at the centre of buoyancy.

The body can only be in equilibrium when the line through the centre of gravity of the body, and the centre of buoyancy, is vertical; in other words, when the line of rest is vertical. When the weight of the body exceeds the buoyant effort, the body sinks to the bottom; when they are equal, it remains in equilibrium, wherever placed. When the buoyant effort is greater than the weight, it rises to the surface, and, after a few oscillations, comes to rest, in such a position, that the weight of the displaced fluid is equal to that of the body, when it is said to *float.* The upper surface of the fluid is then called *the plane of flota-*

tion, and its intersection with the surface of the body, *the line of flotation.*

If a floating body be slightly disturbed from its position of equilibrium, the centres of gravity and buoyancy are no longer in the same vertical. Let *DE* represent the plane of flotation, *G* the centre of gravity of the body, (Fig. 126), *GH* its line of rest, and *C* the centre of buoyancy.

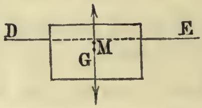


Fig. 125.

If the line of support, *CB*, intersect the line of rest in *M*, above *G*, as in Fig. 126, the buoyant effort and the weight conspire to restore the body to equilibrium; in this case, the equilibrium is *stable.*

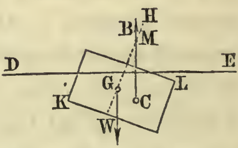


Fig. 126.

If *M* is below *G*, as in Fig. 127, the buoyant effort and the weight conspire to overturn the body; in this case the body, before being disturbed, must have been in *unstable* equilibrium.

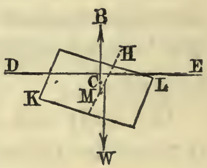


Fig. 127.

If the centres of buoyancy and gravity are always on the same vertical, *M* coincides with *G* (Fig. 128), and the body is in *indifferent* equilibrium. The limiting position of *M*, obtained by deflecting the body through an infinitely small angle, is the *metacentre* of the body. Hence,

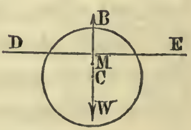


Fig. 128.

If the metacentre is above the centre of gravity, the body is in stable equilibrium; if below the centre of gravity, the body is in unstable equilibrium; if the points coincide, the body is in indifferent equilibrium.

The stability of the floating body is greater, as the metacentre is higher above the centre of gravity. This condition is fulfilled in loading ships, by stowing the heavier objects near the bottom of the vessel.

Specific Gravity.

150. The specific gravity of a body is its *relative weight*, that is, it is the number of times the body is heavier than an equivalent volume of some other body, taken as a standard.

The specific gravity of a body is obtained by dividing the weight of any volume of the body, by that of an equivalent volume of the standard.

For solids and liquids, distilled water is taken as a standard. Because this liquid is of different densities at different temperatures, it becomes necessary to assume a standard temperature for it: for a like reason, a standard temperature must be taken for the body whose specific gravity is to be found. Different standards of temperature have been assumed by different writers; we shall adopt those assumed by JAMIN, who takes for the standard temperature of water, 4° C., or about 39° F., and for the standard temperature of the body, 0° C., or 32° F. The former is the temperature at which water has a maximum density, and the latter is that of melting ice.

In finding the specific gravity of a body, we first determine it with respect to water at any temperature; this we may call the *observed specific gravity*. We then correct the result for the temperature of the water, by means of a table of densities of water at different temperatures, that at 39° being 1; this result we call the *apparent specific gravity*. Finally, we correct this for the temperature of the body, and thus find the *true specific gravity*.

1st. Let d be the density of water at the temperature, t, its density at 39° being 1; let s be the *observed* specific gravity of a body referred to water at the temperature, t, and let s' be its specific gravity referred to water at 39°.

Because the specific gravity of a body varies inversely as the density of the water to which it is referred, we have,

$$s : s' :: 1 : d; \quad \therefore \quad s' = ds.$$

That is, *to find the apparent specific gravity of a body, multiply its observed specific gravity, at the temperature, t, by the corresponding tabular density of water.*

2dly. Suppose the body to have the same temperature, t, as the water to which it is referred. Denote the volume of the body at the temperature, t, by v', and at 32°, by v; denote the corresponding specific gravities by s' and s.

Because the specific gravity varies inversely as its volume, we have,

$$s : s' :: v' : v; \quad \therefore \quad s = s'\frac{v'}{v}.$$

That is, *to find the true specific gravity of a body, multiply its apparent specific gravity by the quotient of its volume at the temperature, t, by its volume at* 32°.

This quotient may be found from the body's known rate of expansion.

It is only in nice determinations that it is necessary to take account of the latter correction.

Gases are usually referred to air as a standard; but as air is easily referred to water, we may take distilled water at 39° F. as a standard for all bodies.

Sometimes it is convenient to find the specific gravity of a body with respect to some other body whose specific gravity is already known. In this case the required specific gravity is equal to the product of that which is found, by that which is already known. Thus, if A is m times as heavy as B, and if B is n times as heavy as C, then will A be mn times as heavy as C.

Methods of finding Specific Gravity.

151. There are two principal methods of finding the specific gravity of a body; *first*, by means of the balance, and *secondly*, by means of the hydrometer. The former alone can be used for nice determinations, such as are needed in the operations of analytical chemistry; the latter is of easier application, and is sufficiently accurate for most practical purposes.

Hydrostatic Balance.

152. This balance is similar to that described in Article 68; the scale-pans, however, are provided with hooks for suspending bodies, as shown in the figure.

Fig. 129.

In balances of modern construction the vessel containing water is placed on a movable bench or shelf, that strides one of the scale-pans, without interfering with its movements, and the body is then suspended from the beam by a thread or wire. In both cases a body attached to the string may be weighed either in the air or in the water, at pleasure.

Specific Gravity of an Insoluble Body.

153. Fasten the suspending wire to one scale-pan, or to one extremity of the beam, as the case may be, and counterpoise it by weights in the opposite pan. Then attach the body to the wire and counterpoise it by weights in the other pan; *these give the weight of the body in air:* next immerse the body in water, so as not to touch the containing vessel; the buoyant effort of the water will thrust the body up with a force equal to the weight of the displaced water: restore the equilibrium by weights placed in the first pan; *these will give the weight of the displaced water:* divide the weight of the body in air by the weight of the displaced water, and the quotient will be the observed specific gravity.

Thus, if a piece of copper weigh 2047 grains in air, and lose 230 grains when weighed in water, its specific gravity is $\frac{2047}{230}$, or 8.9.

If the body will not sink in water, determine its weight in air, as before; then attach to it a body so heavy that the combination will sink; find the weight of the water displaced by the combination, and also the weight displaced by the heavy body, take their difference, and the result will be the weight of the water displaced by the body in question; then proceed as before.

Thus, a body weighs 600 grains in air; when attached to a piece of copper, the combination weighs 2647 grains in air, and suffers a loss of 834 grains in water, the copper alone losing 230 grains. The buoyant effort of the fluid exerted on the body is therefore 604 grains, and the specific gravity of the body is $\frac{600}{604}$, or 0.993.

Specific Gravity of a Soluble Body.

154. Find its specific gravity with respect to some liquid in which it is not soluble; find also the specific gravity of this liquid with respect to water; take the product of these, and it will be the specific gravity sought, (Art. 150).

Thus, if the specific gravity of a body with respect to oil be 3.7, and the specific gravity of the oil with respect to water be 0.9, the specific gravity of the body is 3.7×0.9, or 3.33.

It is often convenient to use a saturated solution of the substance in question as the auxiliary liquid.

Specific Gravity of Liquids.

155. 1*st Method.*—The most convenient method is by the specific gravity bottle. This is a bottle constructed to hold exactly 1000 grains of distilled water. Accompanying it is a brass weight, just equal to the empty bottle. To use it, let it be filled with the liquid in question, and placed in one scale-pan; in the other pan place the brass counterpoise, and weights enough to balance the liquid; divide the number of grains in the weight of the liquid by 1000, and the quotient will be the specific gravity.

Thus, if the bottle filled with a liquid weighs 945 grains, beside the counterpoise, its specific gravity is 0.945.

2*d Method.*—Take a body, that will sink both in the liquid and in water, and which is not acted upon by either; determine its loss of weight, first in the liquid, then in water; divide the former by the latter, and the quotient will be the specific gravity sought. The reason is evident.

Thus, if a glass ball lose 30 grains when weighed in water, and 24 in alcohol, the specific gravity of the alcohol is $\frac{24}{30}$, or 0.8.

3*d Method.*—Let *AB* and *CD* be graduated glass tubes, half an inch in diameter, open at both ends. Let their upper ends communicate with the receiver of an air-pump, and their lower ends dip into two vessels, one containing distilled water, and the other the liquid whose specific gravity is to be determined. Let the air be partially exhausted from the receiver by an air-pump; the liquid will rise in the tubes

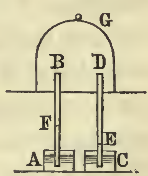


Fig. 130.

to heights inversely as the specific gravities of the liquids. If we divide the height of the column of water by that of the other liquid, the quotient will be the specific gravity sought. By producing different degrees of rarefaction, the columns will rise to different heights, but their ratios ought to be the same. We are thus enabled to make a series of observations, each corresponding to a different degree of rarefaction, from which a more accurate result can be had, than from a single observation.

Specific Gravity of Air.

156. Take a globe, fitted with a stop-cock, and, by means of an air-pump, or condensing syringe, force in as much air as is convenient, close the stop-cock, and weigh the globe thus filled. Provide a glass tube, graduated to cubic inches and decimals of a cubic inch, and, having filled it with mercury, invert it over a mercury bath. Open the stop-cock, and allow the compressed air to escape into the tube, taking care to place the tube in such a position that the mercury without the tube is at the same level as within. The reading on the tube gives the volume of escaped air. Weigh the globe again, and subtract the weight thus found from the first weight; this difference is the weight of the escaped air. Having reduced the measured volume of air to what it would have occupied at a standard temperature and pressure, by rules yet to be deduced, compute the weight of an equivalent volume of water; divide the weight of the corrected volume of air by that of an equivalent volume of distilled water, and the quotient will be the specific gravity sought.

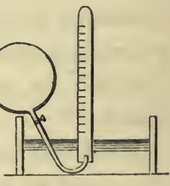


Fig. 131.

Hydrometers.

157. A hydrometer is a floating body, used in finding specific gravities. Its construction depends on the principle of flotation. Hydrometers are of two kinds. 1°. Those in which the submerged volume is constant. 2°. Those in which the weight of the instrument is constant.

Nicholson's Hydrometer.

158. This instrument consists of a hollow cylinder, *A*, at the lower extremity of which is a basket, *B*, and at the upper extremity a wire, bearing a scale-pan, *C*. At the bottom of the basket is a ball, *E*, containing mercury, to cause the instrument to float in an upright position. By means of this ballast, the instrument is adjusted so that a given weight, say 500 grains, placed in the pan, *C*, will sink it in distilled water to a notch, *D*, filed in the neck.

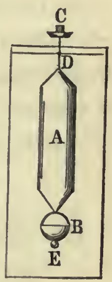


Fig. 132.

This instrument is in reality a weighing-machine, and as such can be used for determining the approximate weights of bodies within certain limits; in the instrument described, no body can be weighed whose weight exceeds 500 grains.

To find the specific gravity of a solid, place it in the pan, *C*, and add weights till the instrument sinks, in distilled water, to the notch, *D*. The added weights, subtracted from 500 grains, give the weight of the body in air. Place the body in the basket, *B*, which generally has a reticulated cover, to prevent the body from floating away, and add other weights to the pan, until the instrument again sinks to the notch, *D*. The weights last added give the weight of water displaced by the body. Divide the first of these by the second, and the quotient will be the specific gravity required.

To find the specific gravity of a liquid. Having weighed the instrument, place it in the liquid, and add weights to the scale-pan, till it sinks to *D*. The weight of the instrument, plus the weights added, will be the weight of the liquid displaced by the instrument. The weight of the instrument added to 500 grains gives the weight of an equal volume of distilled water. The quotient of the first by the second is the specific gravity required.

A modification of this instrument, in which the basket, *B*, is omitted, is sometimes used for determining specific gravities of liquids only. This kind of hydrometer, known as Fahrenheit's hydrometer, is generally made of glass, that it may not be acted on chemically by the liquids into which it is plunged.

Scale Areometer.

159. The scale areometer is a hydrometer whose weight is constant; the specific gravity of a liquid is made known by the depth to which it sinks in it. The instrument consists of a glass cylinder, *A*, with a stem, *C*, of uniform diameter. At the bottom of the cylinder is a bulb, *B*, containing mercury, to make the instrument float upright. By introducing a suitable quantity of mercury, the instrument may be adjusted so as to float at any desired point of the stem.

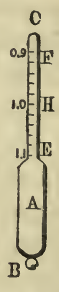


Fig. 133.

When it is designed to determine the specific gravity of liquids, both lighter and heavier than distilled water, it is called a *universal hydrometer*, and is so ballasted as to float in distilled water at the middle of the stem. This point is marked on the stem with a file, and is numbered 1 on the scale. A liquid is then formed, by dissolving salt in water, whose specific gravity is 1.1, and the instrument is allowed to float freely in it; the point, *E*, to which it sinks, is marked on the stem, and the intermediate part of the scale, *HE*, is divided into 10 equal parts. In like manner a mixture of alcohol and water is formed, whose specific gravity is 0.9, the corresponding position of the plane of flotation is marked on the stem, and the space between it and the division 1 is divided into 10 equal parts. The graduation is continued, both up and down, through the whole length of the stem. The graduation is marked on a piece of paper within the stem. To use this hydrometer, we put it into the liquid and allow it to come to rest; the division of the scale that corresponds to the surface of flotation shows the specific gravity of the liquid. The hypothesis on which this instrument is graduated, is, that the increments of specific gravity are proportional to the increments of the submerged portion of the stem. This hypothesis is only approximately true, but it approaches more nearly to the truth as the diameter of the stem diminishes.

When it is only desired to use the instrument for liquids heavier than water, the instrument is ballasted so that the division 1 shall be near the top of the stem. If it is to be used for liquids lighter than water, it is ballasted so that the division 1 shall be near the bottom of the stem. In this case we determine the point 0.9 by using a mixture of alcohol and water, the *principle* of graduation being the same as in the first instance.

Volumeter.

160. The volumeter is a modification of the scale areometer, differing from it only in graduation. The graduation is effected as follows: The instrument is placed in distilled water, and allowed to come to rest, and the point of the stem where the surface cuts it, is marked with a file. The submerged volume is then accurately determined, and the stem is graduated in such manner that each division indicates a volume equal to a hundredth part of the volume originally submerged. The divisions are then numbered from the first mark in both directions, as indicated in the figure. To use the instrument, place it in the liquid, and note the division to which it sinks; divide 100 by the number indicated, and the quotient will be the specific gravity sought. The principle employed is, that the specific gravities of liquids are inversely as the volumes of equal weights. Suppose that the instrument indicates x parts; then the weight of the instrument displaces x parts of the liquid, whilst it displaces 100 parts of water. Denoting the specific gravity of the liquid by S, and that of water by 1, we have,

Fig. 134

$$S : 1 :: 100 : x, \quad \therefore S = \frac{100}{x}.$$

A table may be computed to save performing the division.

Densimeter.

161. The densimeter admits of use when only a small portion of the liquid can be had. Its construction differs from that of the volumeter, in having a small cup at the upper extremity of the stem, to receive the fluid whose specific gravity is to be determined.

The instrument is so ballasted that when the cup is empty, the densimeter sinks in distilled water to a point, *B*, near the bottom of the stem. This point is the 0 of the instrument. The cup is then filled with distilled water, and the point, *C*, to which it sinks, is marked; the space, *BC*, is divided into any number of equal parts, say 10, and the graduation is continued to the top of the tube.

Fig. 135.

To use the instrument, place it in distilled water, and fill the cup with the liquid in question, and note the division to which it sinks. Divide the number of this division by 10, and the quotient will be the specific gravity required. The principle of the densimeter is, that the specific gravity of a body of a constant volume is proportional to the volume of water it causes the instrument to displace.

Centesimal Alcoometer of Gay Lussac.

162. This instrument is similar in construction to the scale areometer; the graduation, however, is made on a different principle. Its object is, to determine the percentage of alcohol in a mixture of alcohol and water. The graduation is made as follows: the instrument is first placed in absolute alcohol, and ballasted so that it will sink nearly to the top of the stem. This point is marked 100. Next, a mixture of 95 parts of alcohol and 5 of water, is made, and the point to which the instrument sinks, is marked 95. The intermediate space is divided into 5 equal parts. Next, a mixture of 90 parts of alcohol and 10 of water is made; the point to which the instrument sinks, is marked 90, and the space between this and 95, is divided into 5 equal parts. In this manner, the entire stem is graduated by successive operations. The spaces on the scale are not equal at different points, but, for a space of five parts, they may be so regarded, without sensible error.

To use the instrument, place it in the mixture of alcohol and water, and read the division to which it sinks; this will indicate the percentage of alcohol in the mixture.

In all the instruments, the temperature has to be taken into account; this is effected by tables that accompany the different instruments.

On the principle of the alcoometer, a great variety of areometers are constructed, for determining the strength of wines, syrups, and other liquids employed in the arts.

In some nicely constructed hydrometers, the mercury used as ballast serves also to fill the bulb of a delicate thermometer, whose stem rises into the cylinder of the instrument, and thus enables us to note the temperature of the fluid in which it is immersed.

EXAMPLES.

1. A cubic foot of water weighs 1000 ounces. Required the weight of a cubical block of stone, whose edge is 4 feet, its specific gravity being 2.5. *Ans.* 10000 lbs.

2. Required the number of cubic feet in a body whose weight is 1000 lbs., its specific gravity being 1.25. *Ans.* 12.8.

3. Two lumps of metal weigh 3 lbs., and 1 lb., and their specific gravities are 5 and 9. What will be the specific gravity of an alloy formed by melting them together, supposing no contraction of volume to take place? *Ans.* 5.625.

4. A body weighing 20 grains has a specific gravity of 2.5. Required its loss of weight in water. *Ans.* 8 grains.

5. A body weighs 25 grains in water, and 40 grains in a liquid whose specific gravity is .7. What is the weight of the body in vacuum? *Ans.* 75 grains.

6. A NICHOLSON's hydrometer weighs 250 grains, and it requires an additional weight of 336 grains to sink it to the notch in the stem, in a mixture of alcohol and water. What is the specific gravity of the mixture? *Ans.* .781.

7. A block of wood sinks in distilled water till $\frac{7}{8}$ of its volume is submerged. What is its specific gravity? *Ans.* .875.

8. The weight of a piece of cork in air, is $\frac{3}{4}$ oz.; the weight of a piece of lead in water, is $6\frac{4}{9}$ oz.; the weight of the cork and lead together in water, is $4\frac{7}{100}$ oz. What is the specific gravity of the cork? *Ans.* 0.24.

9. A solid, whose weight is 250 grains, weighs in water, 147 grains, and, in another fluid, 120 grains. What is the specific gravity of the latter fluid? *Ans.* 1.262.

10. A solid weighs 60 grains in air, 40 in water, and 30 in an acid. What is the specific gravity of the acid? *Ans.* 1.5.

The following table is compiled from the *Ordnance Manual.*

TABLE OF SPECIFIC GRAVITIES OF SOLIDS AND LIQUIDS.

SOLIDS.	SPEC. GRAV.	SOLIDS.	SPEC. GRAV.
Antimony, cast....	6.712	Limestone.........	3.180
Brass, cast.......	8.396	Marble, common...	2.686
Copper, cast.......	8.788	Salt, common......	2.130
Gold, hammered...	19.361	Sand..............	1.800
Iron, bar..........	7.788	Slate..............	2.612
Iron, cast.........	7.207	Stone, common....	2.520
Lead, cast.........	11.352	Tallow............	0.945
Mercury at 32° F..	13.598	Boxwood.........	0.912
" at 60°....	13.580	Cedar......... ...	0.596
Platina, rolled.....	22.069	Cherry............	0.715
" cast.......	20.337	Lignum vitæ......	1.333
Silver, hammered..	10 511	Mahogany.........	0.854
Tin, cast..........	7.291	Oak, heart........	1.170
Zinc, cast..........	6.861	Pine, yellow.......	0.660
Bricks.............	1.900	Nitric acid........	1.217
Chalk...	2.784	Sulphuric acid.....	1.841
Coal, bituminous..	1.270	Alcohol, absolute..	0.792
Diamond..........	3.521	Ether, sulphuric...	0.715
Earth, common ...	1.500	Sea water.........	1.026
Gypsum...........	2.168	Olive oil..........	0.915
Ivory.............	1.822	Oil of Turpentine..	0.870

Thermometer.

163. A THERMOMETER, is an instrument for measuring the temperatures of bodies. All bodies expand when heated, and contract when cooled, and, other things being equal, always occupy the same volumes at the same temperatures. Different bodies expand and contract in different ratios for equal increments of temperature. As a general rule, liquids expand more rapidly than solids, and gases more rapidly than liquids. The construction of the thermometer depends on this principle of unequal expansibility of bodies. A great variety of forms have been used, only one of which will be described.

The mercurial thermometer consists of a bulb, A, at the upper extremity of which is a tube of uniform bore, hermetically sealed at its upper end. The bulb and tube are nearly filled with mercury, and to the whole is attached a frame, on which is a scale for temperature.

Fig. 136.

A thermometer may be constructed as follows: A tube of uniform bore is selected, and on one extremity a bulb is blown, which may be cylindrical, or spherical; the former shape is, on many accounts, the preferable one. At the other extremity, a conical-shaped funnel is blown, open at top. The funnel is filled with mercury, which should be of the purest quality, and the whole being held vertical, the heat of a spirit-lamp is applied to the bulb, which expanding the air contained in it, forces a portion in bubbles up through the mercury in the funnel. The instrument is next allowed to cool, when a portion of mercury is forced down the tube into the bulb. By a repetition of this process, the entire bulb may be filled with mercury, as well as the tube itself. Heat is then applied to the bulb, until the mercury is made to boil; and, on being cooled down to a little above the highest temperature that it is desired to measure, the top of the tube is melted off by a jet of flame, urged by a blow-pipe, and the whole hermetically sealed. The instrument, thus prepared, is attached to a frame, and graduated as follows:

The instrument is plunged into a bath of melting ice, and, after remaining a sufficient time for the instrument to take the temperature of the ice, the height of the mercury in the tube is marked on the scale. This gives the *freezing point.* The instrument is next plunged into a bath of boiling water, and allowed to remain long enough to acquire the temperature of the water and steam. The height of the mercury is then marked on the scale. This gives the *boiling point.* The freezing and boiling points having been determined, the intermediate space is divided into a certain number of equal parts, according to the scale adopted, and the graduation is continued, both up and down, to any desired extent.

Three principal scales are used. FAHRENHEIT'S *scale,* in which the space between the freezing and boiling point is divided into 180 equal parts, called degrees, the freezing point being marked 32°, and the boiling point 212°. In this scale, the 0 point is 32 degrees below the freezing point. *The Centigrade scale,* in which the space between the fixed points is divided into 100 equal parts, called degrees. The 0 of this scale is at the freezing point. REAUMUR'S *scale,* in which the same space is divided into 80 equal parts, called degrees. The 0 of this scale also is at the freezing point.

If we denote the number of degrees on the Fahrenheit, Centigrade, and Reaumur scales, by F, C, and R respectively, the following formula will enable us to pass from any one of these scales to any other:

$$\tfrac{1}{9}(F^\circ - 32^\circ) = \tfrac{1}{5}C^\circ = \tfrac{1}{4}R^\circ$$

The scale most in use in this country is FAHRENHEIT'S. The other two are used in Europe, particularly the Centigrade scale.

Velocity of a Liquid through a small Orifice.

164. Let ABD be a vessel, having a small orifice at its bottom, and filled with a liquid.

Denote the cross section of the orifice, by a, and its depth below the upper surface, by h. Let D be an infinitely small layer of the liquid at the orifice, and denote its height, by h'. This layer is (Art. 146) urged downward by a force equal to the weight of a column of the liquid whose base is the orifice, and whose height is h; denoting this pressure, by p, and the weight of a unit of volume of the liquid, by w, we have,

Fig. 137.

$$p = wah.$$

Were the element pressed downward by its own weight alone, the pressure being denoted by p', we should have,

$$p' = wah'.$$

Dividing the former by the latter, we have,

$$\frac{p}{p'} = \frac{h}{h'};$$

that is, *the pressures are as the heights h and h'.*

Let us suppose, the element falls through the height, h', first under the action of the force, p, and then under the action of the force, p'. Denoting the velocities generated, by v and v', we have, (Art. 104),

$$v = \sqrt{2ph'}, \text{ and, } v' = \sqrt{2p'h'};$$

whence, by reduction,

$$v : v' :: \sqrt{p} : \sqrt{p'}, \therefore v = v'\sqrt{\frac{p}{p'}}.$$

But, when the element falls under the action of v', or its own weight, we have,

$$v' = \sqrt{2gh'}.$$

Substituting this volume, v', and replacing $\frac{p}{p'}$, by its value, $\frac{h}{h'}$, we have, after reduction,

$$v = \sqrt{2gh}.$$

Hence, *a liquid issues from an orifice in the bottom of a vessel, with a velocity equal to that acquired by a body in falling through a height equal to the distance of the orifice below the free surface.*

We have seen that the pressure due to the weight of a fluid on any point of the surface of a vessel, is normal to the surface, and is proportional to the depth of the point below the free surface. Hence, if an orifice be made at any point, the liquid will flow out in a jet, normal to the surface at that point, and with a velocity due to the distance of the orifice from the free surface of the fluid.

If the orifice is on a vertical side of a vessel, the initial direction of the jet will be horizontal; if it be at a point where the tangent plane is oblique to the horizon, the initial direction of the jet will be oblique; if the opening is on the upper side of a portion of a vessel where the tangent is horizontal, the jet will be directed upward, and will rise to a height due to the velocity; that is, to the height of the upper surface of the fluid.

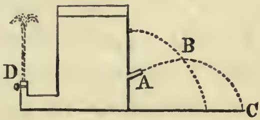


Fig. 138.

Modification due to Extraneous Pressure.

165. If the upper surface of the liquid, in any of the preceding cases, be pressed by a force, as when it is urged downward by a piston, we may denote the height of a column of the fluid whose weight is equal to the extraneous pressure, by h'. The velocity of efflux will then be given by the equation,

$$v = \sqrt{2g(h + h')}.$$

The pressure of the atmosphere acts equally on the upper surface and the opening; hence, in ordinary cases, it may be neglected; but were the liquid to flow into a vacuum, or into rarefied air, the pressure must be taken into account, and this may be done by means of the formula just given.

Should the flow take place into condensed air, or into any medium which opposes a greater resistance than the atmospheric pressure, the extraneous pressure would act upward, h' would be negative, and the preceding formula would become,

$$v = \sqrt{2g(h - h')}.$$

Spouting of Liquids on a Horizontal Plane.

166. Let KL be a vessel filled with water, D an orifice in its vertical side, and DE the path of the spouting fluid. We may regard each drop as a projectile shot forth horizontally, and then acted on by gravity. Its path is, therefore, a parabola, and the circumstances of its motion are made known by equations (89) and (94).

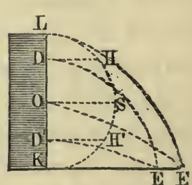


Fig. 139.

Denote DK, by h', and DL, by h. We have, from equation (94), by making y equal to h', and $x = KE$,

$$KE = \sqrt{\frac{2v^2h'}{g}}$$

But we have found $v = \sqrt{2gh}$; hence, by substitution,

$$KE = 2\sqrt{hh'}.$$

If we describe a semicircle on KL, and through D draw an ordinate, DH, we have, from a property of the circle,

$$DH = \sqrt{DK \cdot DL} = \sqrt{hh'}.$$

Hence we have, by substitution,

$$KE = 2\overline{DH}.$$

Since there are two points on KL at which the ordinates are equal, there must be two orifices through which the fluid will spout to the same distance on the horizontal plane; one of these is as far above the centre, O, as the other is below it.

If the orifice be at O, midway between K and L, the ordinate, OS, will be greatest possible, and the range, KE', will be a maximum. The range in this case will be equal to the diameter of the circle, LHK, or to the distance from the surface of the water in the vessel to the horizontal plane.

If the jet is directed obliquely upward by a short pipe, A, (Fig. 138), the path described by each particle will still be the arc of a parabola, ABC. Since each particle of the liquid may be regarded as a body projected obliquely upward, the nature of the path and the circumstances of the motion will be given by equation (89).

If a semi-parabola, LE', is described, having its axis vertical, its vertex at L, and focus at K, then may every point, P, within the curve, be reached by two separate jets issuing from the side of the vessel; every point on the curve can be reached by one, and only one; points lying without the curve cannot be reached by any jet whatever.

In like manner, the same equation will make known the nature of the path and the circumstances of motion, when the jet is directed obliquely downward by a short tube.

Coefficients of Efflux and Velocity.

167. When a vessel empties itself by a small orifice at the bottom, it is observed that the particles of fluid near the top descend in vertical lines; when they approach the bottom they incline toward the orifice, the converging lines of particles tending to cross each other as they emerge from the vessel. The result is, the stream grows narrower, after leaving the vessel, until it reaches a point at a distance from the vessel equal to about the radius of the orifice, when the contraction becomes a minimum, and below that point the vein again spreads out. This phenomenon is called, *contraction of the vein.* The cross section at the most contracted part is not far from $\frac{64}{100}$ of the area of the orifice, when the vessel is very thin. If we denote the area of the orifice, by a, and the area of the least cross section of the vein, by a', we have,

$$a' = ka,$$

in which k is a number to be determined by experiment. This number is called the *coefficient of contraction.*

To find the quantity of water discharged through an orifice at the bottom of the containing vessel, in one second, we multiply the smallest section of the vein by the velocity. Denoting the quantity discharged in one second, by Q', we have,

$$Q' = ka\sqrt{2gh}.$$

This formula is only true on the supposition that the actual velocity is the same as the theoretical velocity, which is not the case, as has been shown by experiment. The theoretical velocity is equal to $\sqrt{2gh}$, and if we denote the actual velocity, by v', we have,

$$v' = l\sqrt{2gh},$$

in which l is to be determined by experiment; this value of l is slightly less than 1, and is called the *coefficient of velocity*. In order to get the actual discharge, we must replace $\sqrt{2gh}$ by $l\sqrt{2gh}$, in the preceding equation. Doing so, and denoting the actual discharge per second, by Q, we have,

$$Q = kla\sqrt{2gh}.$$

The product, kl, is called the *coefficient of efflux*. It has been shown by experiment, that this coefficient for orifices in thin plates, is not quite constant. It decreases slightly, as the area of the orifice and the velocity are increased; and it is further found to be greater for circular orifices than for those of any other shape.

If we denote the coefficient of efflux, by m, we have,

$$Q = ma\sqrt{2gh}.$$

In this equation, h is called *head of water*. Hence, we may define the *head of water* to be the distance from the orifice to the plane of the upper surface of the fluid.

The mean value of m corresponding to orifices of from $\frac{1}{2}$ to 6 inches in diameter, with from 4 to 20 feet head of water, has been found to be about .615. If we take $k = .64$, we have,

$$l = \frac{m}{k} = \frac{.615}{.640} = .96.$$

That is, the actual velocity is only $\frac{96}{100}$ of the theoretical velocity. This diminution is due to friction, viscosity, &c.

Efflux through Short Tubes.

168. It is found that the discharge from a given orifice increases, when the thickness of the plate through which the flow takes place increases; also, when a short tube is introduced.

When a tube, AB, is employed not more than four times as long as the diameter of the orifice, the value of m becomes, on an average, equal to .813; that is, the discharge per second is 1.325 times as great, when the tube is used, as without it. In using the cylindrical tube, the contraction takes place at the outlet of the vessel, and not at the outlet of the tube.

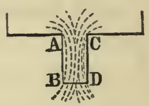


Fig. 140.

Compound mouth-pieces are sometimes formed of two conic frustums, as shown in the figure, having the form of the vein. It has been shown by Etelwein, that the most effective tubes of this form should have the diameter, CD, equal to .833 of AB. The angle made by the sides, CF and DE, should be about 5°, and the length of this portion should be three times that of the other.

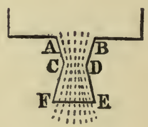


Fig. 141.

Examples.

1. With what theoretical velocity will water issue from a small orifice $16\frac{1}{12}$ feet below the surface of the fluid? *Ans.* $32\frac{1}{6}$ ft.

2. If the area of the orifice, in the last example, is $\frac{1}{10}$ of a square foot, and the coefficient of efflux .615, how many cubic feet of water will be discharged per minute? *Ans.* 118.695 ft.

3. A vessel, constantly filled with water, is 4 feet high, with a cross section of one square foot; an orifice in the bottom has an area of one square inch. In what time will three-fourths of the water be drawn off, the coefficient of efflux being .6?

Ans. $\frac{3}{4}$ minute, nearly.

4. A vessel is kept constantly full of water. How many cubic feet

will be discharged per minute from an orifice 9 feet below the upper surface, having an area of one square inch, the coefficient of efflux being .6? *Ans.* 6 cubic feet, about.

5. In the last example, what will be the discharge per minute, if we suppose each square foot of the upper surface to be pressed by a force of 645 lbs.? *Ans.* $8\frac{3}{4}$ cubic feet, about.

6. The head of water is 16 feet, and the orifice is $\frac{1}{100}$ of a square foot. What quantity of water will be discharged per second, when the orifice is through a thin plate?

SOLUTION.

In this case, we have,

$$Q = .615 \times .01\sqrt{2 \times 32\frac{1}{6} \times 16} = .197 \text{ cubic feet.}$$

When a short cylindrical tube is used, we have,

$$Q = .197 \times 1.325 = .261 \text{ cubic feet.}$$

Capillary Phenomena.

169. When a liquid is in equilibrium, under the action of its own weight, it has been shown that its upper surface is level. It is observed, however, in the neighborhood of solid bodies, such as the walls of a vessel, that the surface is sometimes elevated, and sometimes depressed, according to the nature of the liquid and solid in contact. These elevations and depressions result from the action of molecular forces, exerted between the particles of the liquid and solid in contact; from the fact that they are more apparent in small tubes, of the diameter of a hair, they have been called *capillary phenomena*, and the forces giving rise to them, *capillary forces*.

The following are some of the observed effects of capillary action: When a solid is plunged into a liquid capable of moistening it, as when glass is plunged into water, the surface of the liquid is heaped up about the solid, taking a concave form, as shown in Fig. 142.

When a solid is plunged into a liquid not capable of moistening it, as when glass is plunged into mercury, the surface of the liquid is depressed about the solid, taking a convex form, as shown in Fig. 143.

Fig. 142.

The surface of the liquid in the neighborhood of the surfaces of the containing vessel takes the concave or convex form according as the material of the vessel is capable of being moistened, or not, by the liquid.

Fig. 143

These phenomena become more apparent when we plunge a tube into a liquid; according as the tube is, or is not, capable of being moistened by the liquid, the liquid will rise in the tube, or be depressed in it. When the liquid rises in the tube, its upper surface takes a concave shape; when it is depressed, it takes a convex form. The elevations, or depressions, increase as the diameter of the tube diminishes.

Elevation and Depression between Plates.

170. If two plates of any substance be placed parallel to each other, it is found that the laws of ascent and descent of the liquid into which they are plunged, are the same as for tubes. For example: if two plates of glass parallel to each other, and pretty close together, are plunged into water, it is found that the water will rise between them to a height, inversely proportional to their distance apart; and further, that this height is equal to about one-half the height to which water would rise in a glass tube whose internal diameter is equal to the distance between the plates.

If the same plates be plunged into mercury, there will be a depression according to a corresponding law.

If two plates of glass, AB and AC, inclined to each other, as shown in Fig. 144, be plunged into a liquid that will moisten them, the liquid will rise between them. It will rise higher near the junction, the surface taking a curved form, such that any section made by a plane through AD, will be an equilateral hyperbola.

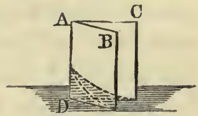


Fig. 144.

If the line of junction of the two plates is horizontal, a small quantity of a liquid that will moisten them, assumes the shape shown at A; if it do not moisten them, it takes the form shown at B.

Fig. 145.

Attraction and Repulsion of Floating Bodies.

171. If two small balls of wood, both of which can be moistened by water, or two small balls of wax, that cannot be moistened, be placed in a vessel of water, and brought so near each other that the surfaces of capillary elevation, or depression interfere, the balls will attract each other and come together. If one ball of wood and one of wax be brought so near that the surfaces of capillary elevation and depression interfere, the bodies will repel each other, and separate. If two needles be carefully oiled and laid on the surface of water, they will repel the water from their neighborhood, and float. If, whilst floating, they are brought sufficiently near to each other to permit the surfaces of capillary depression to interfere, the needles will immediately rush together. The reason of the needles floating is, that they repel the water, heaping it up on each side, thus forming a cavity in the surface; the needle is buoyed up by a force equal to the weight of the displaced fluid, and, when this exceeds the weight of the needle, it floats. On this principle certain insects move freely over a sheet of water; their feet are lubricated with an oily substance which repels the water, producing a hollow around each foot, and gives rise to a buoyant effort greater than the weight of the insect.

The principle of mutual attraction between bodies, both of which repel water, or both of which attract it, accounts for the fact that small floating bodies have a tendency to collect in groups about the borders of the containing vessel. When the material of which the vessel is made, exercises a different capillary action from that of the floating particles, they will aggregate themselves at a distance from the surface of the vessel.

Applications of the Principles of Capillarity.

172. It is a consequence of capillary action that water rises to fill the pores of a sponge, or lump of sugar. The same principle causes oil to rise in the wick of a lamp, which is but a bundle of fibres very nearly in contact, leaving capillary interstices between them.

The siphon filter is the same, in principle, as the wick of a lamp. It consists of a bundle of fibres like a lamp-wick, one end of which dips into the liquid to be filtered, whilst the other hangs over the edge of the vessel. The liquid ascends the fibrous mass by capillary attraction, and continues to advance till it reaches the overhanging end, when, if this is lower than the upper surface of the liquid, it will fall by drops from the end of the wick, the impurities being left behind.

The principle of capillary attraction is used for splitting rocks and raising weights. To employ this principle in cleaving mill-stones, as is done in France, the stone is first dressed to the form of a cylinder of the required diameter.

Grooves are then cut around it where the divisions are to take place, and into these grooves thoroughly dried wedges of willow-wood are driven. On being exposed to the action of moisture, the cells of the wood absorb water, expand, and finally split the rock.

To raise a weight, a thoroughly dry cord is fastened to the weight, and then stretched to a point above. If the cord be moistened, the fibres absorb moisture, expand laterally, the rope is diminished in length, and the weight raised.

The principle of capillary action is also employed in metallurgy, in purifying metals, by cupellation.

Endosmose and Exosmose.

173. The names *endosmose* and *exosmose* have been given to currents, flowing in contrary directions between two liquids, when separated by a porous partition, either organic or inorganic. The discovery of this phenomena is due to M. DUTROCHET, who called the flowing in, endosmose, and the flowing out, exosmose. The existence of the currents was established by an instrument, to which he gave the name *endosmometer.* This instrument consists of a tube of glass, at one end of which is attached a membranous sack, secured by a ligature. If the sack be filled with gum water, a solution of sugar, albumen, or almost any solution denser than water, and then plunged into water, it is observed, after a time, that the fluid rises in the stem, and is depressed in the vessel, showing that water has entered the sack by passing through the pores. By applying suitable tests, it is also found that a portion of the liquid in the sack has passed through the pores into the vessel.

Two currents are thus established. If the operation be reversed, and the bladder and tube be filled with pure water, the liquid in the vessel will rise, whilst that in the tube falls. The phenomena of endosmose and exosmose are extremely various, and serve to explain a great variety of interesting facts in animal and vegetable physiology. The cause of the currents, is the action of molecular forces between the particles of the bodies employed.

CHAPTER VIII.

MECHANICS OF GASES AND VAPORS.

Gases and Vapors.

174. Gases and vapors are distinguished from other fluids, by their great compressibility, and correspondingly great elasticity. These fluids continually tend to occupy a greater space; this expansion goes on till counteracted by some extraneous force, as that of gravity, or the resistance offered by a containing vessel.

The force of expansion, common to gases and vapors, is called their *tension*, or *elastic force*. We shall take for the unit of this force, at any point, the pressure that would be exerted on a square inch, were the pressure the same at every point of the square inch as at the point in question. If we denote this unit, by p, the area pressed, by a, and the entire pressure, by P, we have,

$$P = ap \quad . \ . \ . \ . \ . \quad (121)$$

Most of the principles demonstrated for liquids hold good for gases and vapors, but there are certain properties arising from elasticity that are peculiar to aeriform fluids, some of which it is now proposed to investigate.

Atmospheric Air.

175. The gaseous fluid that envelops our globe, and extends on all sides to a distance of many miles, is called the *atmosphere*. It consists principally of nitrogen and oxygen, together with small but variable portions of watery vapor and carbonic acid, all in a state of mixture. On an average, it is found that 1000 parts by volume of atmospheric air, taken near the surface of the earth, consist of about,

788 parts of nitrogen,
197 parts of oxygen,
14 parts of watery vapor,
1 part of carbonic acid.

The atmosphere may be taken as a type of gases, for it is found that the laws regulating density, expansibility, and elasticity, are the same for all gases and vapors, so long as they maintain a purely gaseous form. It is found, however, in the case of vapors, and of those gases which have been reduced to a liquid form, that the law changes just before actual liquefaction.

This change appears somewhat analogous to that observed when water passes from the liquid to the solid form. Although water does not actually freeze till reduced to a temperature of 32° Fah., it is found that it reaches its maximum density at about 39°, at which temperature the particles seem to commence arranging themselves according to new laws, preparatory to taking the solid form.

Atmospheric Pressure.

Fig. 146.

176. If a tube, 35 or 36 inches long, open at one end and closed at the other, be filled with pure mercury, and inverted in a basin of the same, the mercury will fall in the tube until the vertical distance from the surface of the mercury in the tube to that in the basin is about 30 inches. This column of mercury is sustained by the pressure of the atmosphere exerted on the surface of the mercury in the basin, and transmitted through the fluid, according to the general law of *transmission of pressures.* The column of mercury sustained by the elasticity of the atmosphere is called the *barometric column,* because it is generally measured by an instrument called a barometer. In fact, the instrument just described, when provided with a suitable scale for measuring the height of the column, is a complete barometer. The height of the barometric column fluctuates somewhat, even at the same place, on account of changes of temperature, and other causes yet to be considered.

Observation has shown, that the average height of the barometric column at the level of the sea, is a trifle less than 30 inches.

The weight of a column of mercury 30 inches high, having a cross section of one inch, is nearly 15 pounds. Hence, the unit of atmospheric pressure is 15 pounds.

This unit is called an *atmosphere,* and is often employed in estimating the pressure of elastic fluids, particularly steam. Hence, to say that the pressure of steam in a boiler is two atmospheres, is equivalent to saying, that there is a pressure of 30 pounds on each square inch of the interior of the boiler. In general, when we say that the tension of a gas or vapor is n atmospheres, we mean that each square inch is pressed by a force of n times 15 pounds.

Mariotte's Law.

177. When a given mass of gas or vapor is compressed so as to occupy a smaller space, its elastic force is increased; if its volume is increased, its elastic force is diminished.

The law of increase and diminution of elastic force, discovered by MARIOTTE, and bearing his name, may be enunciated as follows:

The elastic force of a given mass of gas, whose temperature remains the same, varies inversely as the volume it occupies.

As long as the mass remains the same, its density varies inversely as its volume. Hence,

The elastic force of a gas, whose temperature remains the same, varies as its density; and conversely, *its density varies as its elastic force.*

MARIOTTE'S law may be verified for atmospheric air, by an instrument called MARIOTTE'S Tube. This is a tube, $ABCD$, of uniform bore, bent so that its two branches are parallel to each other. The shorter branch, AB, is closed at its upper extremity, whilst the longer one is open. Between the two branches, and attached to the frame, is a scale of equal parts.

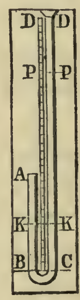


Fig. 147.

To use the instrument, place it in a vertical position, and pour mercury into the tube, until it just cuts off communication between the two branches. The mercury will then stand at the same level, BC, in both branches, and the tension of the air in AB, will be exactly equal to that of the external atmosphere. If an additional quantity of mercury be poured into the longer branch, the air in the shorter branch will be compressed, and the mercury will rise in both branches, but higher in the longer, than in the shorter one. Suppose the mercury to have risen in the shorter branch, to K, and in the longer one, to P. There will be an equilibrium in the mercury lying below the horizontal plane, KK; there will also be an equilibrium between the tension of the air in AK, and the forces which give rise to that tension. These forces are, the pressure of the external atmosphere, transmitted through the mercury, and the weight of a column of mercury whose base is the cross section of the tube, and whose altitude is PK. If we denote the height of the column of mercury sustained by the pressure of the external atmosphere, by h, the tension of the air in AK, will be measured by the weight of a column of mercury, whose base is the cross section of the tube, and whose height is $h + PK$. Since the weight is proportional to the height, the tension of the confined air is proportional to $h + PK$.

Now, whatever may be the value of PK, it is found that,

$$AK : AB :: h : h + PK;$$

whence,

$$AK = \frac{AB \cdot h}{h + PK}.$$

If $PK = h$, we have, $AK = \frac{1}{2}AB$; if $PK = 2h$, we have, $AK = \frac{1}{3}AB$; if $PK = nh$, n being any positive number, entire or fractional, we have, $AK = \frac{AB}{n+1}$. This formula, deduced from MARIOTTE'S law, was verified by DULONG and ARAGO for all values of n, up to $n = 27$. The law may also be verified when the pressure is less than an atmosphere, by the following apparatus: AK is a tube of uniform bore, closed at its upper and open at its lower extremity; CD is a deep cistern of mercury. The tube, AK, is either graduated into equal parts, commencing at A, or has attached to it a scale of brass or ivory.

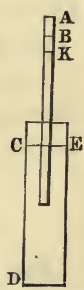


Fig. 148.

To use the instrument, pour mercury into the tube till it is nearly full; place the finger over the open end, invert it in the cistern, and depress it till the mercury stands at the same level without and within the tube, and suppose the surface of the mercury in this case to be at B. Then will the tension of the air in AB, be equal to that of the external atmosphere. If the tube be raised vertically, the air in AB will expand, its tension will diminish, and the mercury will fall in the tube, to maintain the equilibrium. Suppose the level of the mercury in the tube, to have reached K. In this position of the instrument the tension of the air in AK, added to the weight of the column of mercury, KE, will be equal to the tension of the external air.

Now, it is found, whatever may be the value of KE, that

$$AK : AB :: h : h - EK;$$

whence,

$$AK = \frac{AB \cdot h}{h - EK}.$$

If $EK = \frac{1}{2}h$, we have, $AK = 2AB$; if $EK = \frac{2}{3}h$, we have, $AK = 3AB$; in general, if $EK = \frac{n}{n+1}h$, we have, $AK = AB(n + 1)$.

This formula has been verified, for all values of n, up to $n = 111$.

It is a law of Physics that, when a gas is suddenly compressed, heat is evolved, and when a gas is suddenly expanded, heat is absorbed; hence, in making the experiment, care must be taken that the temperature be kept uniform.

More recent experiments have shown that Mariotte's law is not *strictly true*, especially for high tensions, yet its variation is so small that the error committed in regarding it as true is not appreciable in any practical case.

Gay Lussac's Law.

178. If the volume of any gas or vapor remain the same, and its temperature be increased, its tension is increased also. If the pressure remain the same, the volume of the gas increases as the temperature is raised. The law of increase and diminution, as deduced by GAY LUSSAC, whose name it bears, may be enunciated as follows:

In a given mass of gas, or vapor, if the volume remain the same, the tension varies as the temperature; if the tension remain the same, the volume varies as the temperature.

According to REGNAULT, if a given mass of air be heated from 32° Fahrenheit to 212°, the tension remaining constant, its volume will be increased by the .3665th part of its volume at 32°. Hence, the increase for each degree of temperature is the .00204th part of its volume at 32°. If we denote the volume at 32°, by v, and the volume at the temperature, t', by v', we have,

$$v' = v[1 + .00204(t' - 32)] \quad . \; . \; . \; . \; . \quad (122)$$

Solving with reference to v, we have,

$$v = \frac{v'}{1 + .00204(t' - 32)} \quad . \; . \; . \; . \; . \quad (123)$$

Formula (123) enables us to compute the volume of a mass of air at 32°, when we know its volume at the temperature, t', the pressure remaining constant.

To find the volume at the temperature, t'', we have simply to substitute t'' for t' in (122). Denoting this volume by v'', we have,

$$v'' = v[1 + .00204(t'' - 32)].$$

Substituting for v its value, from (123), we get,

$$v'' = v' \frac{1 + .00204(t'' - 32)}{1 + .00204(t' - 32)} \quad . \; . \; . \; . \; . \quad (124)$$

This formula enables us to compute the volume of a mass of air, at a temperature, t'', when we know its volume at the temperature, t'; and, since the density varies inversely as the volume, we may also, by means of the same formula, find the density of any mass of air, at the temperature, t'', when we have given its density at the temperature, t'.

Manometers.

179. A manometer is an instrument for measuring the tension of gases and vapors, particularly of steam. Two principal varieties of manometers are used for measuring the tension of steam, the *open*, and the *closed manometer*.

The open Manometer.

180. The open manometer consists of an open glass tube, AB, terminating near the bottom of a cistern, EF. The cistern is of wrought-iron, steam tight, and filled with mercury. Its dimensions are such, that the upper surface of the mercury will not be materially lowered, when a portion of the mercury is forced up the tube. ED is a tube, by means of which, steam may be admitted from the boiler to the surface of the mercury in the cistern. This tube is sometimes filled with water, through which the pressure of the steam is transmitted to the mercury.

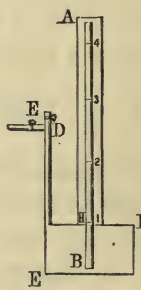


Fig. 149.

To graduate the instrument. All communication with the boiler is cut off, by closing the stop-cock, E, and communication with the external air is made by opening the stop-cock, D. The point of the tube, AB, to which the mercury rises, is noted, and a distance is laid off, upward, from this point, equal to what the barometric column wants of 30 inches, and the point, H, thus determined, is marked 1. This point will be very near the surface of the mercury in the cistern. From the point, H, distances of 30, 60, 90, &c., inches are laid off upward, and the corresponding points numbered 2, 3, 4, &c. These divisions correspond to atmospheres, and may be subdivided into tenths and hundredths.

To use the instrument, the stop-cock, D, is closed, and communication made with the boiler, by opening the stop-cock, E. The height to which the mercury rises in the tube indicates the tension of the steam in the boiler, which may be read from the scale in terms of atmospheres and decimals of an atmosphere. If the pressure in pounds is wished, it may at once be found by multiplying the reading of the instrument by 15.

The principal objection to this kind of manometer is its want of portability, and the great length of tube required, when high tensions are to be measured.

The closed Manometer.

181. The general construction of the closed manometer is the same as that of the open manometer, except that the tube, AB, is closed at the top. The air confined in the tube, is compressed in the same way as in MARIOTTE'S tube.

To graduate this instrument. We determine the division, H, as before. The remaining divisions are found by applying MARIOTTE'S law.

Denote the distance in inches, from H to the top of the tube, by l; the pressure on the mercury, in atmospheres, by n, and the distance in inches, from H to the upper surface of the mercury in the tube, by x.

The tension of the air in the tube is equal to that on the mercury in the cistern, diminished by the weight of a column of mercury whose altitude is x. Hence, in atmospheres, it is

$$n - \frac{x}{30}.$$

The bore of the tube being uniform, the volume of the compressed air is proportional to its height. When the pressure is 1 atmosphere, the height is l; when the pressure is $n - \frac{x}{30}$ atmospheres, the height is $l - x$. Hence, from MARIOTTE'S law,

$$1 : n - \frac{x}{30} :: l - x : l.$$

Whence, by reduction,

$$x^2 - (30n + l)x = - 30l(n - 1).$$

Solving, with respect to x, we have,

$$x = \frac{30n + l}{2} \pm \sqrt{- 30l(n - 1) + \left(\frac{30n + l}{2}\right)^2}.$$

The upper sign of the radical is not used, as it would give a value for x, greater than l. Taking the lower sign, and assuming $l = 30$ in., we have,

$$x = 15n + 15 - \sqrt{- 900(n - 1) + (15n + 15)^2}.$$

Making $n = 2, 3, 4$, &c., in succession, we find for x, the values, 11.46 in., 17.58 in., 20.92 in., &c. These distances being set off from H, upward, and marked 2, 3, 4, &c., indicate atmospheres. The intermediate spaces may be subdivided by the same formula.

In making the graduation, we have supposed the temperature to remain the same. If, however, it does not remain the same, the reading of the instrument must be corrected by a table computed for the purpose.

The instrument is used in the same manner as that already described. Neither can be used for measuring tensions less than 1 atmosphere.

The Siphon Gauge.

182. The siphon gauge is used to measure tensions of gases and vapors, less than an atmosphere. It consists of a tube, ABC, bent so that its two branches are parallel. The branch, BC, is closed at the top, and filled with mercury, which is retained by the pressure of the atmosphere; the branch, AB, is open at the top. If the air be rarefied in any manner, or, if the mouth of the tube be exposed to the action of a gas whose tension is sufficiently small, the mercury will no longer be supported in BC, but will fall in it and rise in BA. The distance between the surfaces of the mercury in the two branches, given by a scale between them, indicates the tension of the gas. If this distance is expressed in inches, the tension can be found, in atmospheres, by dividing by 30, or, in pounds, by dividing by 2.

Fig. 150.

The Diving-Bell.

183. The diving-bell is a bell-shaped vessel, open at the bottom, used for descending into the water. The bell is placed with its mouth horizontal, and let down by a rope, AB, the whole apparatus being sunk by weights properly adjusted. The air contained in the bell is compressed by the pressure of the water, but its increased elasticity prevents the water from rising to the top of the bell, which is provided with seats for the accommodation of those within the bell. The air, constantly contaminated by

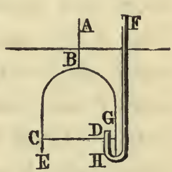


Fig. 151.

breathing, is continually replaced by fresh air, pumped in through a tube, *FG*. Were there no additional air introduced, the volume of the compressed air, at any depth, might be computed by MARIOTTE'S law. The unit of the compressing force, in this case, is the weight of a column of water whose cross section is a square inch, and whose height is the distance from *DC* to the surface of the water.

The Barometer.

184. The barometer is an instrument for measuring the pressure of the atmosphere. It consists of a glass tube, hermetically sealed at one extremity, filled with mercury, and inverted in a basin of that fluid. The pressure of the air is indicated by the height of mercury it supports.

A variety of forms of mercurial barometers have been devised, all involving the same mechanical principle. The most important of these are the *siphon* and the *cistern barometers*.

The Siphon Barometer.

185. The siphon barometer consists of a tube, *CDE*, bent so that its two branches, *CD* and *DE*, are parallel to each other. A scale is placed between them, and attached to the same frame with the tube. The longer branch, *CD*, is hermetically sealed at the top, and filled with mercury; the shorter one is open to the air. When the instrument is placed vertically, the mercury sinks in the longer branch and rises in the shorter one. The distance between the surface of the mercury in the two branches indicates the pressure of the atmosphere.

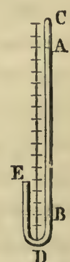


Fig. 152.

The Cistern Barometer.

186. The cistern barometer consists of a glass tube, filled and inverted in a cistern of mercury. The tube is surrounded by a frame of metal, attached to the cistern. Two longitudinal openings, near the upper part of the frame, permit the upper surface of the mercury to be seen. A slide, moved up and down by a rack and pinion, may be brought exactly to the upper level of the mercury. The height of the column is then read from a scale, whose 0 is at the surface of the mercury in the cistern. The scale is graduated to inches and tenths, and the smaller divisions are read by a vernier.

ig. 153.

The figure shows the parts of a complete cistern barometer; *KK* represents the frame; *HH*, the cistern, of glass, at the upper part, that the mercury in the cistern may be seen through it; *L*, a thermometer, to show the temperature of the mercury; *N*, the sliding-ring bearing the vernier, and moved up and down by the screw, *M*.

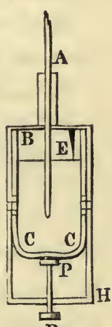


Fig. 154.

The cistern is shown on an enlarged scale in Fig. 154; *A* is the barometer tube, terminating in a small opening, to prevent sudden shocks when the instrument is moved from place to place; *H*, the frame of the cistern; *B*, the upper portion of the cistern, made of glass, that the mercury may be seen; *E*, a piece of ivory, projecting from the upper surface of the cistern, whose point corresponds to the 0 of the scale; *CC*, the lower part of the cistern, made of leather, or other flexible material, and attached to the glass part; *D*, a screw, working through the frame, and against the bottom of the bag, *CC*, by means of a plate, *P*. The screw, *D*, serves to bring the surface of the mercury to the point of ivory, *E*, and also to force the mercury to the top of the tube, when it is desired to transport the barometer from place to place.

To use this barometer, it is suspended vertically, and the level of the mercury in the cistern brought to the point of ivory, *E*, by the screw, *D*; a smart rap on the frame will detach the mercury from the glass, to which it tends to adhere. The ring, *N*, is run up or down till its lower edge appears tangent to the surface of the mercury in the tube, and the altitude is read from the scale. The height of the attached thermometer should also be noted.

The requirements of a good barometer are, sufficient width of tube, perfect purity of mercury, and a scale with an accurately graduated vernier.

The bore of the tube should be as large as practicable, to diminish the effect of capillary action. On account of the repulsion between the glass and mercury, the latter is depressed in the tube, and this depression increases as the diameter of the tube diminishes.

In all cases, this depression should be allowed for, and the reading corrected by a table computed for the purpose.

To secure purity of the mercury, it should be carefully distilled, and after the tube is filled, it should be boiled to drive off any bubbles of air that might adhere to the tube.

Uses of the Barometer.

187. The primary object of the barometer is, to measure the pressure of the atmosphere. It is used by mariners as a weather-glass. It is also employed for determining the heights of points on the earth's surface, above the level of the ocean. The principle on which it is employed for the latter purpose is, that the pressure of the atmosphere at any place depends on the weight of a column of air reaching from the place to the upper limit of the atmosphere. As we ascend above the level of the ocean, the weight of the column diminishes; consequently, the pressure becomes less, a fact that is shown by the mercury falling in the tube.

Difference of Level.

188. Let aB represent a vertical prism of air, whose cross section is one square inch. Denote the pressure at B, by p, and at aa', by p'; denote the density of the air at B, by d, and at aa' by d', and suppose the temperature throughout the column to be 32° Fah.

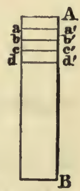


Fig. 155.

Pass a horizontal plane, bb', infinitely near to aa', and denote the weight of the air in ab, by w. Conceive the entire column to be divided by horizontal planes into prisms, whose weights are each equal to w, and denote their heights, beginning at a, by s, s', s'', &c.

From MARIOTTE'S law, we have,

$$\frac{p}{p'} = \frac{d}{d'}; \qquad \therefore \quad \frac{1}{d'} = \frac{p}{p'd}.$$

The air throughout each elementary prism may be regarded as homogeneous; the density of the air in ab is therefore equal to its weight, divided by its volume into gravity (Art. 15). But its volume is equal to $1 \times 1 \times s = s$; hence,

$$d' = \frac{w}{gs}; \qquad \therefore \quad s = \frac{w}{g} \times \frac{1}{d'}.$$

Substituting for $\frac{1}{d'}$, its value in the preceding equation, we have,

$$s = \frac{p}{dg} \times \frac{w}{p'} \quad . \; . \; . \; . \; . \; (125)$$

From the formula for log. $(1 + y)$, deduced in algebra, we have, by substituting for y the fraction $\frac{w}{p'}$, the equation,

$$l\left(1 + \frac{w}{p'}\right) = \frac{w}{p'} - \frac{w^2}{2p'^2} + \frac{w^3}{3p'^3} - , \text{ \&c.}$$

But $\frac{w}{p'}$ is infinitely small; hence, all the terms in the second member, after the first, may be neglected, giving,

$$\frac{w}{p'} = l\left(1 + \frac{w}{p'}\right); \quad \text{or,} \quad \frac{w}{p'} = l\left(\frac{p' + w}{p'}\right);$$

or, finally,

$$\frac{w}{p'} = l(p' + w) - lp',$$

in which l denotes the Naperian logarithm.

In this equation, p' is the pressure on the prism, ab; hence, $p' + w$ is the pressure on the next prism below, that is, on bc.

If we substitute the value of $\frac{w}{p'}$ in equation (125), we have, for the height of ab,

$$s = \frac{p}{dg}[l(p' + w) - lp'].$$

Substituting in succession for p', the values, $p' + w$, $p' + 2w$, $p' + 3w$, &c., we find the heights of bc, cd, &c., to the nth at the base, B, as follows:

$$s = \frac{p}{dg}[l(p' + w) - lp'],$$

$$s' = \frac{p}{dg}[l(p' + 2w) - l(p' + w)],$$

$$s'' = \frac{p}{dg}[l(p' + 3w) - l(p' + 2w)],$$

. . . .

$$s^{(n-1)'} = \frac{p}{dg}[l(p' + nw) - l(p' + (n-1)w)].$$

Adding the equations member to member, and denoting the sum of the first members, which will be equal to AB, by z, we have,

$$z = \frac{p}{dg}[l(p' + nw) - lp'] = \frac{p}{dg} \times l\left(\frac{p' + nw}{p'}\right).$$

But nw is the weight of the air in aB; hence, we have, $p' + nw = p$, or,

$$z = \frac{p}{dg} l \frac{p}{p'} \quad . \; . \; . \; . \; . \quad (126)$$

Denoting the modulus of the common system of logarithms by M, and designating common logarithms by the symbol *log*, we have,

$$z = \frac{p}{Mdg} \log \frac{p}{p'}.$$

The pressures, p and p', are measured by the heights of the columns of mercury which they will support; denoting these heights by H and H', we have,

$$\frac{p}{p'} = \frac{H}{H'}.$$

whencc, by substitution,

$$z = \frac{p}{Mdg} \log \frac{H}{H'} \quad . \; . \; . \; . \; . \quad (127)$$

We supposed the temperature, of both air and mercury, to be 32°. To make the formula general, let T be the temperature of the mercury at B, T' its temperature at a. and denote the corresponding heights of the barometric column, by h and h'; also, let t be the temperature of the air at B, and t' its temperature at a.

The quantity $\frac{p}{d}$ is the ratio of the density of the air at B, to the corresponding pressure, the temperature being 32°. According to MARIOTTE'S law, this ratio remains constant, whatever may be the altitude of B above the level of the ocean.

If we denote the latitude of the place, by l, we have, (Art. 117),

$$g = G(1 + .005133\sin^2 l).$$

It has been shown, by experiment, that, a column of mercury when heated, increases in length at the rate of $\frac{1}{9990}$th of its length at 32°, for each degree. Hence,

$$h = H\left(1 + \frac{T - 32}{9990}\right) = H\frac{9990 + T - 32}{9990};$$

$$h' = H'\left(1 + \frac{T' - 32}{9990}\right) = H'\frac{9990 + T' - 32}{9990}.$$

Dividing the first equation by the second, member by member, we have,

$$\frac{h}{h'} = \frac{H}{H'} \cdot \frac{9990 + T - 32}{9990 + T' - 32}.$$

Dividing both terms of the coefficient of $\frac{H}{H'}$ by the denominator, and neglecting $T' - 32$, in comparison with 9990, we have,

$$\frac{h}{h'} = \frac{H}{H'}\left(1 + \frac{T - T'}{9990}\right) = \frac{H}{H'}\,[1 + .0001\,(T - T')]$$

Whence, by reduction,

$$\frac{H}{H'} = \frac{h}{h'} \cdot \frac{1}{1 + .0001(T - T')}.$$

The quantity z denotes, not only the height, but also the volume of the column of air, aB, at 32°. When the temperature is changed from 32°, the pressures remaining the same, this volume will vary, according to the law of GAY LUSSAC.

If we suppose the temperature of the entire column to be a mean between the temperatures at B and a, which we may do without sensible error, the height of the column will become, equation (122),

$$z\left[1 + .00204\left(\frac{t + t'}{2} - 32\right)\right] = z[1 + .00102(t + t' - 64)].$$

Hence, to adapt equation (127) to the conditions proposed, we must multiply the value of z by the factor,

$$1 + .00102(t + t' - 64).$$

Substituting in equation (127), for $\frac{H}{H'}$ and g, the values given above, and multiplying the resulting value of z, by the factor $1 + .00102(t + t' - 64)$, we have,

$$z = \frac{p}{Md} \cdot \frac{1 + .00102(t + t' - 64)}{G(1 + .005133\sin^2 l)} \log \frac{h}{h'[1 + .0001(T - T')]} \quad \ldots\ldots (128)$$

The factor $\frac{p}{MdG}$ is constant, and may be determined as follows: Select two points, one considerably higher than the other, and determine, by trigonometrical measurement, their difference of level. At the lower point, take the reading of the barometer, of its attached thermometer, and of a detached thermometer exposed to the air. Make similar observations at the upper station. These observations, together with the latitude of the place, will give all the quantities entering equation (128), except the factor in question. Hence, this factor may be deduced. It is found to be 60345.51 ft. Hence, we have, finally, the barometric formula,

$$z = 60345.51 \text{ ft} \times$$

$$\frac{1 + .00102(t + t' - 64)}{1 + .005133\sin^2 l} \log \frac{h}{h'[1 + .0001(T - T')]} \quad . . (129)$$

To use this formula for determining the difference of level between two stations, observe, simultaneously, if possible, the heights of the barometer, and of the attached, and detached thermometers, at the two stations. Substitute these results for the corresponding quantities in the formula; also substitute for l the latitude of the place, and the resulting value of z will be the difference of level required.

If the observations cannot be made simultaneously at the two stations, make a set of observations at the lower station; after a certain interval, make a set at the upper station; then, after an equal interval, make another set at the lower station. Take a mean of the results of observation at the lower station, as a single set, and proceed as before.

For the more convenient application of the formula, tables have been computed, by which the arithmetical operations are much facilitated.

Steam.

189. If water be exposed to the atmosphere, at ordinary temperatures, a portion is converted into vapor, mixes with the atmosphere, and constitutes one of the elements of the aerial ocean. The tension of watery vapor thus formed, is very slight, and the atmosphere soon ceases to absorb any more. If the temperature of the water be raised, an additional amount of vapor is evolved, and of greater tension.

When the temperature is raised to a point at which the tension of the vapor is equal to that of the atmosphere, ebullition commences, and vaporization goes on with great rapidity. If heat be added beyond the point of ebullition, neither the water, nor the vapor will increase in temperature till all the water is converted into steam. When the barometer stands at 30 inches, the boiling-point of pure water is 212° Fah.

If we take the quantity of heat that is necessary to raise *one pound* of water from the temperature 32° F. to the temperature 33° F., as a *unit of heat*, the total amount of heat necessary to raise a pound of water from 32° F. to 212° F. will be 180 *units*, and Regnault has shown that the additional amount of heat necessary to convert the entire pound of water into steam of the temperature 212° F. is equal to 966.6 units. Hence, we say that it requires 180 + 966.6 or 1146.6 units of heat to convert a pound of water at 32° F. into a pound of steam at 212° F. Of this amount 966.6 units are said to become latent, that is, this amount of heat is employed in converting the water at 212° into steam of the same temperature. From this we see that the amount of heat that becomes latent in converting a quantity of water at 212° F. into steam at the same temperature, is nearly $5\frac{1}{3}$ times as much as is required to raise it from the temperature 32° F. to the boiling point.

If steam is generated under a pressure greater or less than *one atmosphere*, the boiling point of the water will be either greater or less than 212° F. In this case, Regnault has shown by experiment that the total number of units of heat required to convert a pound of water at 32° F. into steam, will be given by the formula,

$$Q = 1091.7 + .305\,(t - 32°),$$

in which t is the boiling point of water under the given pressure expressed in degrees of Fahrenheit's scale.

Thus, to convert 1 *lb.* of water at 32° F. into steam of the temperature 250° F., would require 1158.2 units of heat.

When water is converted into steam under a pressure of one atmosphere, each cubic inch produces about 1700 cubic inches of steam, of the temperature of 212°; or, since a cubic foot contains 1728 cubic inches, we may say, in round numbers, that *a cubic inch of water gives a cubic foot of steam.*

If water be converted into steam under greater or less pressure than an atmosphere, the density is increased or diminished, and, consequently, the volume is diminished, or increased. The temperature being also increased or diminished, the increase of density, or decrease of volume will not be exactly proportional to the increase of pressure; but, for purposes of approximation, we may consider the densities as directly, and the volumes as inversely proportional to the pressures under which the steam is generated. On this hypothesis, if a cubic inch of water be evaporated under a pressure of half an atmosphere, it will afford two cubic feet of steam; if generated under a pressure of two atmospheres, it will only afford half a cubic foot of steam.

Work of Steam.

190. When water is converted into steam, a certain amount of work is generated, and, from what has been shown, this work is very nearly the same, whatever may be the temperature at which the water is evaporated.

Suppose a cylinder, whose cross section is one square inch, to contain a cubic inch of water, above which is an air-tight piston, that may be loaded with weights at pleasure. In the first place, if the piston is pressed down by a weight of 15 pounds, and the inch of water converted into steam, the weight will be raised to the height of 1728 inches, or 144 feet. Hence, the quantity of work is 144 × 15, or, 2160 units. Again, if the piston be loaded with a weight of 30 pounds, the conversion of water into steam will give but 864 cubic inches, and the weight will be raised through 72 feet. In this case, the quantity of work will be 72 × 30, or, 2160 units, as before. We conclude, therefore, that the quantity of work is the same, or nearly so, whatever may be the pressure under which steam is generated. We also conclude, that the quantity of work is nearly proportional to the amount of fuel consumed.

Besides the quantity of work developed by simply converting water into steam, a further quantity of work is developed by allowing the steam to expand after entering the cylinder. This principle is used in steam-engines working expansively.

To find the quantity of work developed by steam acting expansively: Let AB represent a cylinder, closed at A, and having an air-tight piston, B. Suppose the steam to enter at the bottom of the cylinder, and to push the piston upward to C, and then suppose the opening at which the steam enters, to be closed: if the piston is not too heavily loaded, the steam will continue to expand, and the piston will be raised to some position, B. The expansive force of the steam will obey MARIOTTE's law, and the quantity of work due to expansion may be computed by the formula in the next article.

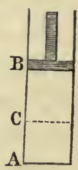


Fig. 156.

Work due to the Expansion of a Gas or Vapor.

191. Let the gas, or vapor, be confined in a cylinder closed at its lower end, and having a piston working air-tight. When the gas occupies a portion of the cylinder whose height is h, denote the pressure on each square inch of the piston, by p; when the gas expands, so that the altitude of the column becomes x, denote the pressure on a square inch, by y.

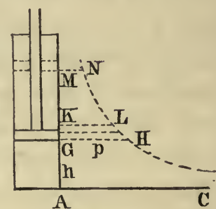


Fig. 157.

Since the volumes of the gas, under these suppositions, are proportional to their altitudes we shall have, from MARIOTTE's law,

$$p : y :: x : h;$$

whence,

$$xy = ph.$$

If p and h are constant, and x and y vary, the above equation will be that of an equilateral hyperbola referred to its asymptotes.

Draw AC perpendicular to AM, and on these lines, as asymptotes, construct the curve, NLH, from the equation, $xy = ph$. Make $AG = h$, and draw GH parallel to AC; it will represent the pressure, p. Make $AM = x$, and draw MN parallel to AC; it will represent the pressure, y. In like manner, the pressure at any height of the piston may be constructed.

Let KL be drawn infinitely near to GH, and parallel with it. The elementary area, $GKLH$, will not differ sensibly from a rectangle whose base is p, and altitude is GK. Hence, its area may be taken as the measure of the work whilst the piston is rising through the infinitely small space, *GK*. In like manner, the area of any infinitely small element, bounded by lines parallel to *AC*, may be taken to represent the work whilst the piston is rising through the height of the element. If we take the sum of all the elements between *GH* and *MN*, this sum, or the area, *GMNH*, will represent the work of the force of expansion whilst the piston is rising from *G* to *M*. But the area included between an equilateral hyperbola and one of its asymptotes, limited by lines parallel to the other asymptote, is equal to the product of the co-ordinates of any point, multiplied by the Naperian logarithm of the quotient obtained by dividing one of the limiting ordinates by the other; or, in this particular case, it is equal to $ph \times l\left(\frac{p}{y}\right)$. Hence, if we designate the quantity of work performed by the expansive force whilst the piston is moving over *GM*, by q, we shall have,

$$q = ph \times l\left(\frac{p}{y}\right).$$

This is the work exerted on each square inch of the piston; if we denote the area of the piston, by A, and the total quantity of work, by Q, we shall have,

$$Q = Aph \times l\left(\frac{p}{y}\right) = Aph \times l\left(\frac{x}{h}\right) \quad \ldots\ldots \quad (130)$$

CHAPTER IX.

HYDRAULIC AND PNEUMATIC MACHINES.

Definitions.

192. HYDRAULIC MACHINES are machines for raising and distributing water, as *pumps, siphons, hydraulic rams, &c.* The name is also applied to machines in which water power is the motor, or in which water is employed to transmit pressures, as *water-wheels, hydraulic presses, &c.*

PNEUMATIC MACHINES are machines to rarefy and condense air, or to impart motion to air, as *air-pumps, ventilating-blowers, &c.* The name is also applied to those machines in which the living force of air is the motive power, such as windmills, &c.

Water Pumps.

193. A *water pump* is a machine for raising water from a lower to a higher level, by the aid of atmospheric pressure. Three separate principles are employed in pumps: the *sucking*, the *lifting*, and the *forcing* principle. Pumps are named according to the principle employed.

Sucking and Lifting Pump.

194. This pump consists of a barrel, *A*, to the lower extremity of which is attached a sucking-pipe, *B*, leading to a reservoir. An air-tight piston, *C*, is worked up and down in the barrel by a lever, *E*, attached to a piston-rod, *D*; *P* is a valve opening upward, which, when the pump is at rest, closes by its own weight. This valve is called the *piston-valve.* A second valve, *G,* also opening upward, is placed at the junction of the pipe with the barrel; this is called the *sleeping-valve.* The space, *LM,* through which the piston moves up and down, is *the play of the piston.*

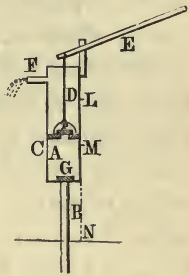


Fig. 158.

To explain the action of the pump; suppose the piston to be in its lowest position, and everything in equilibrium. If the extremity of the lever, *E,* be depressed, and the piston raised, the air in the lower part of the barrel is rarefied, and that in the pipe, *B,* by virtue of its greater tension, opens the valve, and a portion escapes into the barrel. The air in the pipe, thus rarefied, exerts less pressure on the water in the reservoir than the external air, and, consequently, the water rises in the pipe, until the tension of the internal air, plus the weight of the column of water raised, is equal to the tension of the external air; the valve, *G,* then closes by its own weight.

If the piston be again depressed to the lowest limit, the air in the lower part of the barrel is compressed, its tension becomes greater than that of the external air, the valve, *P* is forced open, and a portion of the air escapes. If the piston be raised once more, the water, for the same reason as before, rises still higher in the pipe, and after a few double strokes of the piston, the air is completely exhausted from beneath the piston, the water passes through the piston-valve, and finally escapes at the spout, *F.*

The water is raised to the piston by the pressure of the air on the surface of the water in the reservoir; hence, the piston should not be placed at a greater distance above the water in the reservoir, than the height at which the pressure of the air will sustain a column of water. In fact, it should be placed a little lower than this limit. The specific gravity of mercury being about 13.5, the height of a column of water that will counterbalance the pressure of the atmosphere may be found by multiplying the height of the barometric column by $13\frac{1}{2}$.

At the level of the sea the average height of the barometric column is $2\frac{1}{2}$ feet; hence, the theoretical height to which water can be raised by the principle of suction alone, is a little less than 34 feet.

The water having passed through the piston-valve, may be raised to any height by the lifting principle, the only limitation being the strength of the pump.

There are certain relations that must exist between the play of the piston and its height above the water in the reservoir, in order that the water may be raised to the piston; if the play is too small, it will happen after a few strokes of the piston, that the air in the barrel is not sufficiently compressed to open the piston-valve; when this state of affairs takes place, the water ceases to rise.

To investigate the relation that should exist between the play and the height of the piston above the water: Denote the play of the piston, by p, the distance from the surface of the water in the reservoir to the highest position of the piston, by a, and the height at which the water ceases to rise, by x. The distance from the water in the pump to the highest position of the piston will be $a - x$, and the distance to the lowest position of the piston, $a - p - x$. Denote the height at which the atmospheric pressure sustains a column of water in vacuum, by h, and the weight of a column of water, whose base is the cross section of the pump, and altitude is 1, by w; then will wh denote the pressure of the atmosphere exerted upward through the water in the reservoir and pump.

When the piston is at its lowest position, the pressure of the confined air must be equal to that of the external atmosphere; that is, to wh. When the piston is at its highest position, the confined air will be rarefied, the volume occupied being proportional to its height. Denoting the pressure of the rarefied air by wh', we shall have, from MARIOTTE'S law,

$$wh : wh' :: a - x : a - p - x.$$

$$\therefore \quad h' = h\frac{a - p - x}{a - x}.$$

If the water does not rise when the piston is in its highest position, the pressure of the rarefied air, *plus* the weight of the column already raised, will be equal to the pressure of the external atmosphere; or,

$$wh\frac{a - p - x}{a - x} + wx = wh.$$

Solving this with respect to x, we have,

$$x = \frac{a \pm \sqrt{a^2 - 4ph}}{2}.$$

If, $$4ph > a^2; \quad \text{or,} \quad p > \frac{a^2}{4h},$$

the value of x is imaginary, and there is no point at which the water ceases to rise. Hence, the above inequality expresses the relation that must exist, in order that the pump may be effective. This condition, expressed in words, gives the following rule:

The play of the piston must be greater than the square of the distance from the water in the reservoir, to the highest position of the piston, divided by four times the height at *which the atmosphere will support a column of water in a vacuum.*

Let it be required to find the least play of the piston, when its highest position is 16 feet above the water in the reservoir, and the barometer at 28 inches.

In this case,

$$a = 16 \text{ ft.}, \quad \text{and } h = 28 \text{ in.} \times 13\tfrac{1}{2} = 378 \text{ in.} = 31\tfrac{1}{2} \text{ ft.}$$

Hence, $$p > \tfrac{256}{126} \text{ ft.}; \quad \text{or,} \quad p > 2\tfrac{2}{63} \text{ ft.}$$

To find the quantity of work required to make a double stroke of the piston, after the water reaches the spout.

In depressing the piston, no force is required, except that necessary to overcome inertia and friction. Neglecting these for the present, the quantity of work in the downward stroke, may be regarded as 0. In raising the piston, its upper surface is pressed downward, by the pressure of the atmosphere, wh, *plus* the weight of the column of water from the piston to the spout; and it is pressed upward, by the pressure of the atmosphere, transmitted through the pump, *minus* the weight of a column of water, whose cross section is that of the barrel, and whose altitude is the distance from the piston, to the water in the reservoir. If we subtract the latter from the former, the difference will be the downward pressure. This difference is equal to the weight of a column of water, whose base is the cross section of the barrel, and whose height is the distance of the spout above the reservoir. Denoting this height by H, the pressure is equal to wH. The path through which the pressure is exerted during the ascent of the piston, is the play of the piston, or p. Denoting the quantity of work required, by Q, we shall have,

$$Q = wpH.$$

But wp is the weight of a volume of water, whose base is the cross section of the barrel, and whose altitude is the play of the piston. Hence, Q is equal to the quantity of work necessary to raise this volume of water from the level of the reservoir to the spout. This volume is evidently equal to that actually delivered at each double stroke of the piston. Hence, the quantity of work expended in pumping, with the sucking and lifting pump, hurtful resistances being neglected, is equal to the quantity of work necessary to lift the amount of water, actually delivered, from the level of the reservoir to the spout. In addition to this, a sufficient amount of power must be exerted to overcome hurtful resistances. The disadvantage of this pump, is the irregularity with which the force acts, being 0 in depressing the piston, and a maximum in raising it. This is an important objection when machinery is employed in pumping; but it may be partially overcome, by using two pumps, so arranged, that one piston ascends as the other descends. Another objection to the use of this pump, is the irregularity of flow, the inertia of the column of water having to be overcome at each upward stroke.

Sucking and Forcing Pump.

195. This pump consists of a barrel, *A*, with a sucking pipe, *B*, and a sleeping-valve, *G*, as in the pump just discussed. The piston, *C*, is solid, and is worked up and down by a lever, *E*, and a piston-rod, *D*. At the bottom of the barrel, a pipe leads to an air-vessel, *K*, through a second sleeping-valve, *F*, which opens upward, and closes by its own weight. A delivery-pipe, *H*, enters the air-vessel at the top, and terminates near the bottom.

To explain the action of this pump, suppose the piston, *C*, to be in its lowest position. If the piston be raised to its highest position, the air in the barrel is rarefied, its tension is diminished, the air in the tube, *B*, thrusts open the valve, and a portion escapes into the barrel. The pressure of the external air then forces water up the pipe, *B*, until the tension of the rarefied air, *plus* the weight of the water raised, is equal to the tension of the external air. An equilibrium being produced, the valve, *G*, closes by its own weight. If the piston be depressed, the air in the barrel is condensed, its tension increases till it becomes greater than that of the external air, when the valve, *F*, is thrust open, and a portion escapes through the delivery-pipe, *H*. After a few double strokes of the piston, the water rises through the valve, *G*, and, as the piston descends, is forced into the air-vessel, the air is condensed in the upper part of the vessel, and, acting by its elastic force, forces a portion of the water up the delivery-pipe and out at the spout, *P*. The object of the air-vessel is, to keep up a continued stream through the pipe, *H*, otherwise it would be necessary to overcome the inertia of the entire column of water in the pipe at every double stroke. The flow having commenced, a volume of water is delivered from the spout, at each double stroke, equal to that of a cylinder whose base is the area of the piston, and whose altitude is the play of the piston.

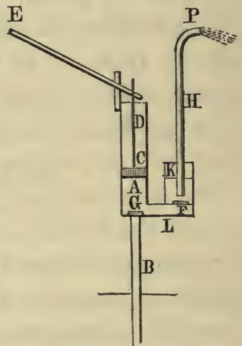


Fig. 159.

The same relation between the parts should exist as in the sucking and lifting pump.

To find the quantity of work consumed at each double stroke, after the flow has become regular, hurtful resistance being neglected:

When the piston is descending, it is pressed downward by the tension of the air on its upper surface, and upward by the tension of the atmosphere, transmitted through the delivery-pipe, *plus* the weight of a column of water whose base is the area of the piston, and whose altitude is the distance of the spout above the piston. This distance is variable during the stroke, but its mean value is the distance of the middle of the play below the spout. The difference between these pressures is exerted upward, and is equal to the weight of a column of water whose base is the area of the piston, and whose altitude is the distance from the middle of the play to the spout. The distance through which the force is exerted, is the play of the piston. Denoting the quantity of work during the descending stroke, by Q', the weight of a column of water, whose base is the area of the piston, and altitude is 1, by w, and the height of the spout above the middle of the play, by h', we have,

$$Q' = wh' \times p.$$

When the piston is ascending, it is pressed downward by the tension of the atmosphere on its upper surface, and upward by the tension of the atmosphere, transmitted through the water in the reservoir and pump, *minus* the weight of a column of water whose base is the area of the piston, and whose altitude is the height of the piston above the reservoir. This height is variable, but its mean value is the height of the middle of the play above the reservoir. The distance through which this force is exerted, is the play of the piston. Denoting the quantity of work during the ascending stroke, by Q'', and the height of the middle of the play above the reservoir, by h'', we have,

$$Q'' = wh'' \times p.$$

Denoting the entire quantity of work during a double stroke, by Q, we have,

$$Q = Q' + Q'' = wp(h' + h'').$$

But wp is the weight of a volume of water, whose base is the piston, and whose altitude is the play; that is, it is the weight of the volume delivered at each double stroke.

The quantity, $h' + h''$, is the height of the spout above the reservoir. Hence, the work expended, is equal to that required to raise the volume delivered from the level of the reservoir to the spout. To this must be added the work necessary to overcome hurtful resistances, as friction, &c.

If $h' = h''$, we have, $Q' = Q''$; that is, the quantity of work during the ascending stroke, equal to that during the descending stroke. Hence, the work of the motor is more nearly uniform, when the middle of the play is at equal distances from the reservoir and spout.

Fire-Engine.

196. The fire-engine is a double sucking and forcing pump, the piston-rods being so connected, that when one piston ascends, the other descends. The sucking and delivery pipes are made of leather, and attached to the machine by metallic screw-joints.

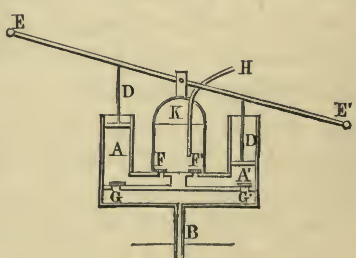


Fig. 160.

The figure exhibits a cross section of the essential part of a fire-engine.

A, A', are the barrels, the pistons are connected by rods, D, D, with the lever, E, E'; B is the sucking-pipe, terminating in a box from which the water may enter either barrel through the valves, *G*, *G′*; *K* is the air-vessel, common to both pumps, and communicating with them by valves, *F*, *F′*; *H* is the delivery-pipe.

It is mounted on wheels for convenience of locomotion. The lever, *E*, *E′*, is worked by rods at right angles to the lever, so arranged that several men can apply their strength at once. The action of the pump differs in no respect from that of the forcing pump; but when the instrument is worked vigorously, a large quantity of water is forced into the air vessel, the tension of the air is much augmented, and its elastic force, thus brought into play, propels the water to a considerable distance from the delivery-pipe. It is this capacity of throwing a jet of water to a great distance, that gives to the engine its value in extinguishing fires.

A pump similar to the fire-engine, is often used, under the name of *the double-action forcing pump*, for other purposes.

The Rotary Pump.

197. The rotary pump is a modification of the sucking and forcing pump. Its construction will be understood from the drawing, which represents a section through the axis of the sucking-pipe, at right angles to the axis of the rotating portion.

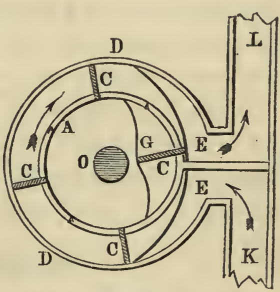


Fig. 161.

A is a ring of metal, revolving about an axis; *D*, *D*, is a second ring of metal, concentric with the first, and forming with it an intermediate annular space. This space communicates with the sucking-pipe, *K*, and the delivery-pipe, *L*. Four radial paddles, *C*, are so disposed as to slide backward and forward through suitable openings in the ring, *A*, and are moved around with it. *G* is a guide, fastened to the end of the cylinder enclosing the revolving apparatus, and cut as represented in the figure; *E*, *E*, are springs, attached to the ring, *D*, and acting, by their elastic force, to press the paddles firmly against the guide. These springs do not impede the flow of water *from* the pipe, *K*, and *into* the pipe, *L*.

When the axis, *O*, revolves, each paddle, as it passes the partition, is pressed against the guide, but is forced out again, by the form of the guide, against the wall, *D*. Each paddle drives the air in front of it in the direction of the arrow-head, and finally expels it through the pipe, *L*. The air behind the paddle is rarefied, and the pressure of the external air forces a column of water up the pipe. After a few revolutions, the air is entirely exhausted from the pipe, *K*. The water enters the channel, *C*, *C*, and is forced up the pipe, *L*, from which it escapes by a spout. The work expended in raising a volume of water to the spout, by this pump, is equal to that required to lift it from the level of the cistern to the spout. This may be shown in the same manner as was explained under the head of the sucking and forcing pump. To this quantity of work, must be added the work necessary to overcome hurtful resistances.

A machine, similar to the rotary pump, is constructed for exhausting foul air from a mine; or, by reversing the direction of rotation, to force fresh air to the bottom of the mine.

The Hydrostatic Press.

198. The hydrostatic press is a machine for exerting a great pressure through a small space. It is used in compressing seeds to obtain oil, in packing hay and other goods, also in raising great weights. Its construction, though requiring the use of a sucking pump, depends upon the principle of equal pressures, (Art. 145).

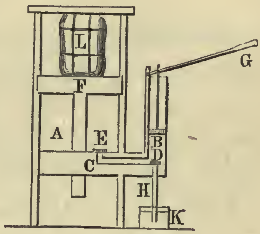


Fig. 162.

It consists of two cylinders, A and B, each provided with a solid piston. The cylinders communicate by a pipe, C, whose entrance to the larger cylinder is closed by a sleeping-valve, E. The smaller cylinder communicates with the reservoir, K, by a sucking-pipe, H, whose upper extremity is closed by a sleeping-valve, D. The piston, B, is worked by the lever, G. By raising and depressing the lever, G, the water is raised from the reservoir and forced into the cylinder, A; and when the space below the piston, F, is filled, a force is exerted upward, as many times greater than that applied to B, as the area of F is greater than B, (Art. 145). This force may be utilized in compressing a body, L, between the piston and the frame of the press.

Denote the area of the larger piston, by P, of the smaller, by p, the pressure applied to B, by f, and that exerted at F, by F; we shall have,

$$F : f :: P : p, \quad \therefore F = \frac{fP}{p}.$$

If we denote the longer arm of the lever, G, by L, the shorter arm, by l, and the force applied at the extremity of the longer arm, by K, we have, from the principle of the lever, (Art. 64),

$$K : f :: l : L, \quad \therefore f = \frac{KL}{l}.$$

Substituting above, we have,

$$F = \frac{PKL}{pl}.$$

To illustrate, let the area of the larger piston be 100 square inches, that of the smaller piston 1 square inch, the longer arm of the lever 30 inches, the shorter arm 2 inches, and let a force of 100 pounds be applied at the end of the longer arm of the lever; to find the pressure on F.

From the conditions,

$$P = 100, \ K = 100, \ L = 30, \ p = 1, \text{ and } l = 2.$$

Hence,

$$F = \frac{100 \times 100 \times 30}{2} = 150{,}000 \text{ lbs.}$$

We have not taken into account the hurtful resistances, hence the pressure of 150,000 pounds must be somewhat diminished.

The volume of water forced from the smaller to the larger cylinder, during a single descent of the piston, B, will occupy, in the two cylinders, spaces whose heights are inversely as the areas of the pistons. Hence, the path, over which f is exerted, is to the path over which F is exerted, as P is to p. Or, denoting these paths by s and S, we have,

$$s : S :: P : p;$$

or, since $P : p :: F : f$, we shall have,

$$s : S :: F : f, \quad \therefore fs = FS.$$

That is, *the work of the power and resistance are equal*, a principle that holds good in all machines.

Examples.

1. The cross section of a sucking and forcing pump is 6 square feet, the play of the piston 3 feet, and the height of the spout, above the reservoir, 50 feet. What must be the effective horse-power of an

engine to impart 30 double strokes per minute, hurtful resistances being neglected?

SOLUTION.

The number of units of work required to be performed each minute, is equal to

$$6 \times 3 \times 50 \times 62\tfrac{1}{2} \times 30 = 1687500 \text{ lb. ft.}$$

Hence,

$$n = \tfrac{1687500}{33000} = 51\tfrac{4}{33}. \quad Ans.$$

2. In a hydrostatic press, the areas of the pistons are, 2 and 400 square inches, and the arms of the lever are, 1 and 20 inches. Required the pressure on the larger piston for each pound of pressure on the longer arm of lever? *Ans.* 4000 lbs.

3. The areas of the pistons of a hydrostatic press are, 3 and 300 square inches, and the shorter arm of the lever is one inch. What must be the length of the longer arm, that a force of 1 lb. may produce a pressure of 1000 lbs.? *Ans.* 10 inches.

The Siphon.

199. The siphon is a bent tube, for transferring a liquid from a higher to a lower level, over an intermediate elevation. The siphon consists of two branches, AB and BC, of which the outer one is the longer. To use the instrument, the tube is filled with the liquid, the end of the longer branch being stopped with the finger, or a stop-cock; in which case, the pressure of the atmosphere prevents the liquid from escaping at the other end. The instrument is then inverted, the end, C, being submerged in the liquid, and the stop removed from A. The liquid will flow through the tube, and the flow will continue till the level of the liquid in the reservoir reaches the mouth of the tube, C.

Fig. 163.

To find the velocity with which water will issue from the siphon, let us consider an infinitely small layer at the orifice, A. This layer is pressed downward, by the tension of the atmosphere exerted on the surface of the reservoir, *minus* the weight of the water in the branch, BD, *plus* the weight of the water in the branch, BA. It is pressed upward by the tension of the atmosphere. The difference of these forces, is the weight of the water in DA, and the velocity of the stratum will be due to that depth. Denoting the vertical height of DA, by h, we shall have, for the velocity,

$$v = \sqrt{2gh}.$$

This is the theoretical velocity, but it is never quite realized in practice, on account of resistances, that have been neglected in the preceding investigation.

Fig. 164.

The siphon may be filled by applying the mouth to the end, A, and exhausting the air by suction. The tension of the atmosphere, on the upper surface of the reservoir, presses the water up the tube, and fills it, after which the flow goes on as before. Sometimes, a sucking-tube, AD, is inserted near the opening, A, rising nearly to the bend of the siphon. In this case, the opening, A, is closed, and the air exhausted through the sucking-tube, AD, after which the flow goes on as before.

The Wurtemburg Siphon.

200. In the Wurtemburg siphon, the ends of the tube are bent twice, at right angles, as shown in the figure. The advantage of this is, that the tube, once filled, remains so, as long as the plane of its axis is kept vertical. The siphon may be lifted out and replaced at pleasure, thereby stopping and reproducing the flow at will.

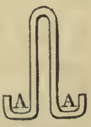


Fig. 165.

It is to be observed that the siphon is only effective when the distance from the highest point of the tube to the level of the water in the reservoir is less than the height at which the atmospheric pressure sustains a column of water in a vacuum. This will, generally, be less than 34 feet.

The Intermitting Siphon

201. The intermitting siphon is represented in the figure. *AB* is a curved tube issuing from the bottom of a reservoir. The reservoir is supplied with water by a tube, *E*, having a smaller bore than the siphon.

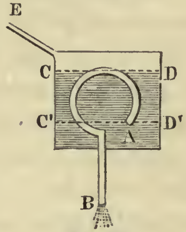


Fig. 166.

To explain its action, suppose the reservoir to be empty, and the tube, *E*, to be open; as soon as the reservoir is filled to the level, *CD*, the water begins to flow from the opening, *B*, and the flow once commenced, continues till the level of the reservoir is reduced to *C'D'*, through the opening, *A*. The flow then ceases till the cistern is again filled to *CD*, and so on as before.

Intermitting Springs.

202. Let *A* represent a subterranean cavity, communicating with the surface of the earth by a channel, *ABC*, bent like a siphon. Suppose the reservoir to be fed by percolation through the crevices, or by a small channel, *D*. When the water in the reservoir rises to the horizontal plane, *BD*, the flow commences at *C*, and, if the channel is sufficiently large, the flow continues till the water is reduced to the level plane through *C*. An intermission then occurs till the reservoir is again filled; and so on, intermittingly.

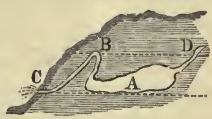


Fig. 167.

Siphon of Constant Flow.

203. We have seen that the velocity of efflux depends on the height of water in the reservoir above the external opening of the siphon. When the water is drawn from the reservoir, the surface sinks, this height diminishes, and, consequently, the velocity continually diminishes.

If, however, the shorter branch, *CD*, be passed through a cork large enough to float the siphon, the instrument will sink as the upper surface is depressed, the height of *DA* will remain constant, and, consequently, the flow will be uniform till the siphon comes in contact with the upper edge of the reservoir. By suitably adjusting the siphon in the cork, the velocity of efflux can be increased or decreased within certain limits. In this manner, any desired quantity of the fluid can be drawn off in a given time.

The siphon is used in the arts, for decanting liquids. It is also employed to draw a portion of a liquid from the interior of a vessel when that liquid is overlaid by one of less specific gravity.

The Hydraulic Ram.

204. The hydraulic ram is a machine for raising water by means of shocks caused by the sudden stoppage of a stream of water.

It consists of a reservoir, *B*, supplied by an inclined pipe, *A;* at the upper surface of the reservoir, is an orifice closed by a valve, *D;* this valve is kept in place by a metallic framework immediately below the orifice; *G* is an air-vessel communicating with the reservoir by an opening, *F*, with a spherical valve, *E;* this valve closes the orifice, *F*, except when forced upward, in which case its motion is restrained by a framework or cage; *H* is a delivery-pipe entering the

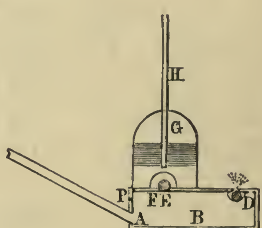


Fig. 168.

air-vessel at its upper part, and terminating near the bottom. At *P* is a small valve, to supply the loss of air in the air-vessel, arising from absorption.

To explain the action of the instrument, suppose it empty, and the parts in equilibrium. If a current of water be admitted to the reservoir, through the pipe, *A*, the reservoir is soon filled, and the water commences rushing out at *D;* the impulse of the water forces the valve, *D*, upward, and closes the opening; the velocity of the water in the reservoir is checked; the reaction forces open the valve, *E*, and a portion of the water enters the air-chamber, *G;* the force of the shock having been expended, the valves both fall by their own weight; a second shock takes place, as before; an additional quantity of water is forced into the air-vessel, and so on continuously. As the water is forced into the air-vessel, the air becomes compressed; and acting by its elastic force, urges a stream of water up the pipe, *H.* The shocks occur in rapid succession, and thus a constant stream is kept up.

To explain the use of the valve, *P*, it may be remarked that water absorbs more air under a greater, than under a smaller pressure. Hence, as it passes through the air-chamber, a portion of the contained air is taken up by the water and carried out through the pipe, *H*. But each time that the valve, *D*, falls, there is a tendency to a vacuum in the upper part of the reservoir, in consequence of the rush of the fluid to escape through the opening. The pressure of the external air then forces the valve, *P*, open, a portion of air enters, and is afterward forced up with the water into the vessel, *G*, to keep up the supply.

The hydraulic ram is only used to raise small quantities of water, as for the supply of a house, or garden. Only a small fraction of the fluid that enters the supply-pipe actually passes out through the delivery-pipe; but if the head of water is pretty large, a column may be raised to a great height. Water is often raised, in this manner, to the highest parts of lofty buildings.

Sometimes, an additional air-vessel is introduced over the valve, *E*, to deaden the shock of the valve in its play.

Archimedes' Screw.

205. This is a machine for raising water through small heights, and, in its simplest form, it consists of a tube wound spirally around a cylinder. The cylinder is mounted so that its axis is oblique to the horizon, the lower end dipping into the reservoir. When the cylinder is turned on its axis, the lower end of the tube describes the circumference of a circle, whose plane is perpendicular to the axis. When the mouth of the tube comes to the level of the axis and begins to ascend, there is a certain quantity of water in the tube, which continues to occupy the lowest part of the spire; and, if the cylinder is properly inclined to the horizon, this flow is toward the upper end of the tube. At each revolution, a quantity of water enters the tube, and that already in the tube is raised, higher and higher, till, at last, it flows from the upper end of the tube.

The Chain Pump.

206. The chain pump is an instrument for raising water through small elevations.

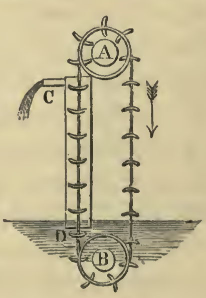


Fig. 169.

It consists of an endless chain passing over wheels, *A* and *B*, having their axes horizontal, one below the surface of the water, and the other above the spout of the pump. Attached to this chain, and at right angles to it, are circular disks, fitting the tube, *CD*. If the cylinder, *A*, be turned in the direction of the arrow-head, the buckets or disks rise through the tube, *CD*, driving the water before them, until it reaches the spout, *C*, and escapes. One great objection to this machine is, the difficulty of making the disks fit the tube. Hence, there is a constant leakage, requiring great additional expenditure of force.

Sometimes the body of the pump is inclined, in which case it does not differ much in principle from a wheel with flat buckets, that has also been used for raising water.

The Air Pump.

207. The air pump is a machine for rarefying air.

It consists of a barrel, *A*, in which a piston, *B*, is worked up and down by a lever, *C*, attached to a piston-rod, *D*. The barrel communicates with a vessel, *E*, called a receiver, by a narrow pipe. The receiver is usually of glass, ground to fit air-tight on a smooth bed-plate, *KK*. The joint between the receiver

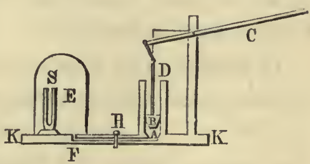


Fig. 170.

and plate may be rendered more perfectly air-tight by interposing a layer of tallow. A stop-cock, *H*, permits communication to be made at pleasure between the barrel and receiver, or between the barrel and external air. When the stop-cock is turned in a particular direction, the barrel and receiver communicate; but on turning it through 90 degrees, the communication with the receiver is cut off, and a communication is opened between the barrel and external air. Instead of the stop-cock, valves are often used, that are opened and closed by the elastic force of the air, or by the force that works the pump. The communicating pipe should be exceedingly small, and the piston, *B*, when at its lowest point, should fit accurately to the bottom of the barrel.

To explain the action of the air pump, suppose the piston to be at its lowest position. The stop-cock, *H*, is turned so as to open a communication between the barrel and receiver, and the piston is raised to its highest point by a force applied to the lever, *C*. The air, which before occupied the receiver and pipe, expands so as to fill the barrel, receiver, and pipe. The stop-cock is then turned to cut off communication between the barrel and receiver, and open the barrel to the external air, and the piston again depressed to its lowest position. The air in the barrel is expelled by the depression of the piston. The air in the receiver is now more rare than at the beginning, and by a continued repetition of the process, any degree of rarefaction may be attained.

To measure the rarefaction of the air in the receiver, a siphon-gauge may be used, or a glass tube, 30 inches long, may be made to communicate at its upper extremity with the receiver, whilst its lower extremity dips into a cistern of mercury. As the air is rarefied in the receiver, the pressure on the mercury in the tube becomes less than on that in the cistern, and the mercury rises in the tube. The tension of the air in the receiver is indicated by the difference between the height of the barometric column and that of the mercury in the tube.

To investigate a formula for the tension of the air in the receiver, after any number of double strokes, let us denote the capacity of the receiver, by r, that of the connecting-pipe, by p, and that of the space between the bottom of the barrel and the highest position of the piston, by b.

Denote the original tension of the air, by t; its tension after the first upward stroke of the piston, by t'; after the second, third, ... n^{th}, upward strokes, by t'', t''', ... $t^{n'}$.

The air which occupied the receiver and pipe, after the first upward stroke, fills the receiver, pipe, and barrel: according to MARIOTTE'S law, its tension in the two cases varies inversely as the volumes occupied; hence,

$$t : t' :: p + r + b : p + r, \quad \therefore\ t' = t\frac{p + r}{p + r + b}.$$

In like manner, we shall have, after the second upward stroke,

$$t' : t'' :: p + r + b : p + r, \quad \therefore\ t'' = t'\frac{p + r}{p + b + r}.$$

Substituting for t' its value, deduced from the preceding equation, we have,

$$t'' = t\left(\frac{p + r}{p + b + r}\right)^2.$$

In like manner, we find,

$$t''' = t\left(\frac{p + r}{p + b + r}\right)^3;$$

and, in general, after the n^{th} stroke,

$$t^{n'} = t\left(\frac{p + r}{p + b + r}\right)^n.$$

If the pipe is exceedingly small, its capacity may be neglected in comparison with that of the receiver, and we then have,

$$t^{n'} = t\left(\frac{r}{b + r}\right)^n.$$

Let it be required, for example, to determine the tension of the air after 5 upward strokes, when the capacity of the barrel is one-third that of the receiver.

In this case, $\frac{r}{b+r} = \frac{3}{4}$, and $n = 5$, whence,

$$t^{v} = t\tfrac{243}{1024}.$$

Hence, the tension is less than a fourth part of that of the external air.

Instead of the receiver, the pipe may be connected by a screw-joint with any closed vessel, as a hollow globe, or glass flask. In this case, by reversing the direction of the stop-cock, in the up and down motion of the piston, the instrument may be used as a condenser. When so used, the tension, after n downward strokes of the piston, is given by the formula,

$$t^{n'} = t\left(\frac{r + nb}{r}\right).$$

Taking the same case as that before considered, with the exception that the instrument is used as a condenser instead of a rarefier, we have, after 5 downward strokes,

$$t^{v} = \tfrac{8}{3}t.$$

That is, the tension is eight-thirds that of the external air. external air.

Artificial Fountains.

208. An artificial fountain is an instrument by which a liquid is forced upward in the form of a jet, by the tension of condensed air. The simplest form of artificial fountain is called HERO'S ball.

Hero's Ball.

209. This consists of a globe, *A*, into the top of which is inserted a tube, *B*, reaching nearly to the bottom of the globe. This tube is provided with a stop-cock, *C*, by which it may be closed, or opened at pleasure. A second tube, *D*,

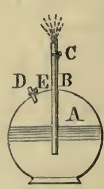


Fig. 171.

enters the globe near the top, which is also provided with a stop-cock, *E*.

To use the instrument, close the stop-cock, *C*, and fill the lower portion of the globe with water through *D*; then connect *D* with a condenser, and pump air into the upper part of the globe, and confine it there by closing the stop-cock, *E*. If, now, the stop-cock, *C*, be opened, the pressure of the confined air on the surface of the water in the globe forces a jet through the tube, *B*. This jet rises to a greater or less height, according to the greater or less quantity of air that was forced into the globe. The water will continue to flow through the tube as long as the tension of the confined air is greater than that of the external atmosphere, or till the level of the water in the globe reaches the lower end of the tube.

Instead of using the condenser, air may be introduced by blowing with the mouth through the tube, *D*, and confined by turning the stop-cock, *E*.

The principle of HERO's ball is the same as that of the air-chamber in the forcing-pump and fire-engine, already explained.

Hero's Fountain.

210. HERO'S fountain is constructed on the same principle as HERO's ball, except that the compression of the air is effected by the weight of a column of water, instead of by a condenser.

A is a cistern, similar to HERO's ball, with a tube, *B*, extending nearly to the bottom of the cistern. *C* is a second cistern placed at some distance below *A*. This cistern is connected with a basin, *D*, by a bent tube, *E*, and also with the upper part of the cistern, *A*, by

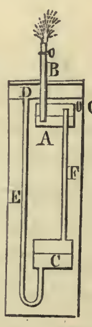


Fig. 172.

a tube, *F*. When the fountain is to be used, *A* is nearly filled with water, *C* being empty. A quantity of water is then poured into the basin, *D*, which, acting by its weight, sinks into *C*, compressing the air in the upper portion of it into a smaller space, thus increasing its tension. This increase of tension acting on the surface of the water in *A*, forces a jet through *B*, which rises to a greater or less height according to the greater or less tension. The flow will continue till the level of the water in *A* reaches the bottom of the tube, *B*. The measure of the compressing force on a unit of surface of the water in *C*, is the weight of a column of water, whose base is that unit, and whose altitude is the difference of level between the water in *D* and in *C*.

If HERO'S ball be partially filled with water and placed under the receiver of an air-pump, the water will be observed to rise in the tube, forming a fountain, as the air in the receiver is exhausted. The principle is the same as before; the flow is due to an excess of pressure on the water within the globe over that without. In both cases, the flow is resisted by the tension of the air without, and is urged on by the tension within.

Wine-Taster and Dropping-Bottle.

211. The wine-taster is used to bring up a small portion of wine or other liquid from a cask. It consists of a tube, open at the top, and terminating below in a narrow tube, also open. When it is to be used, it is inserted to any depth in the liquid, which rises in the tube to the level of the liquid without. The finger is then placed so as to close the upper end of the tube, and the instrument raised out of the cask. The fluid escapes from the lower end, until

Fig. 173.

the pressure of the rarefied air in the tube, *plus* the weight of a column of liquid, whose cross section is that of the tube, and whose altitude is that of the fluid retained, is equal to the pressure of the external air. If the tube be placed over a tumbler, and the finger removed from the upper orifice, the fluid brought up flows into the tumbler.

If the lower orifice is very small, a few drops may be allowed to escape, by taking off the finger and immediately replacing it. The instrument then constitutes the dropping-bottle.

The Atmospheric Inkstand.

212. The atmospheric inkstand consists of a cylinder, *A*, which communicates by a tube with a second cylinder, *B*. A piston, *C*, is moved up and down in *A*, by means of a screw, *D*. Suppose the spaces, *A* and *B*, to be filled with ink. If the piston, *C*, be raised, the pressure of the external air forces the ink to follow it, and the part, *B*, is emptied. If the operation be reversed, and the piston, *C*, depressed, the ink is again forced into the space, *B*. This operation may be repeated at pleasure.

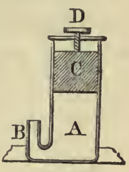


Fig.174.

Prime Movers.

Definition of a Prime Mover.

213. A PRIME MOVER is a contrivance, by means of which the power furnished by a motor is made to impart motion to a train of mechanism. The principal motors are, *water-power*, *wind-power*, and *steam*. The corresponding prime movers are, *water-wheels*, *windmills*, and *steam-engines*.

Water-Wheels.

214. A WATER-WHEEL is a wheel set in motion by the action of water. Water-wheels are divided into two classes —*vertical* and *horizontal.*

There are three principal varieties of vertical wheels:— *overshot, undershot,* and *breast wheels.* The most important horizontal wheel is the *turbine.*

The *overshot wheel* consists of a cylindrical drum, *A*, terminated at its ends by projecting rings, *B*, called *crowns.* The space between the crowns is divided into cells, by curved or angular partitions; these cells, called *buckets,* are constructed so as to retain the water as long as possible. The water is delivered by a sluice-way, *C*, near the top of the wheel, and, acting by its weight, it imparts motion to the wheel, which is communicated to the train by suitable transmitting pieces. This wheel is employed where there is but a small volume of water, with considerable fall.

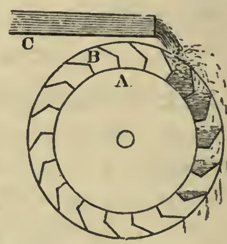


Fig. 175.

The *undershot wheel* is similar, in its general construction, to the overshot wheel; the partitions between the cells, which may be either plane or curved, are called floats. The water is delivered at the bottom of the wheel, and impinging against the floats, acts by its living force to set the wheel in motion. The

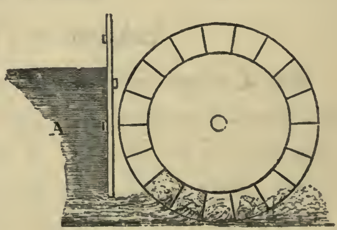


Fig. 176

velocity of the water depends on its *head*, that is, its height in the reservoir, *A*.

The *breast wheel* differs from the undershot wheel in having the water delivered at a higher level, and also in having a casing, or trough, *A*, called a *breast*, which nearly fits the periphery of the wheel that revolves within it. In this wheel, the water acts partly by its weight and partly by its living force.

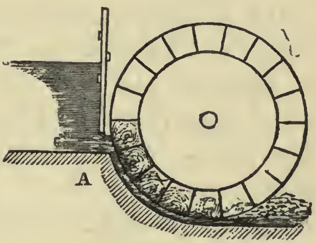


Fig. 177.

The *turbine* turns on a vertical axis, and its floats radiate from it, being curved somewhat like the blades of a screw propeller. The water enters at the centre of the wheel, flows downward and outward, and acts, both by its weight and living force, to impart motion of rotation to the wheel.

Windmills.

215. A WINDMILL is a wheel set in motion by the living force of a current of air. It consists of a horizontal axle, always parallel to the direction of the wind, with projecting arms carrying sails, set obliquely to the axis, somewhat like the blades of a screw propeller. The force of the wind, which acts on each sail at its centre of pressure, may be resolved into two components, one perpendicular, and the other parallel to the sail. The former alone is *effective;* this may be further resolved into two components, one perpendicular, and the other parallel to the axis of rotation. The first of these alone is concerned in producing rotation, and the measure of its effect is the product of its intensity by its lever arm—that is, its distance from the axis.

The Steam-Engine.

216. A STEAM-ENGINE is a contrivance for utilizing the expansive force of steam. The term is generally employed to designate not only the engine proper, but also the various appendages for generating and condensing steam. The relation between the heat applied and the amount of steam generated, as also its general mode of action, have been explained in a previous chapter.

Varieties of Steam Engines.

217. Steam-engines may be *condensing*, or *non-condensing*. In the *former*, the steam, after having acted on the piston, is condensed, and the warm water returned to the boiler; in the *latter*, the steam is not condensed, but having acted on the piston, is blown off, into the air. In a condensing engine, steam may be used of a lower tension than 15 lbs. to the inch; in which case it is called a *low-pressure* engine. In a non-condensing engine, the steam must be of a greater tension than 15 lbs. to the inch, in order that it may be blown off into the air. An engine in which steam is used of a higher tension than 15 lbs., is called a high-pressure engine. A condensing engine may be either *high* or *low pressure*. A *non*-condensing engine must be high pressure.

Condensing engines are more economical of fuel, but they are heavier and more complex in construction; for this reason they are necessarily *stationary*. Non-condensing engines are used for locomotives; where fuel is abundant they are sometimes used as stationary engines.

The Boiler and its Appendages.

218. The BOILER is a shell of metal, generally of wrought iron, in which steam is generated. Boilers are of various forms. One of the simplest is cylindrical, with hemispherical ends. Sometimes two smaller cylinders, called heaters, are placed below the main boiler, and connected with it by suitable pipes. In the Cornish boiler, the cylindrical shell has a large flue, and sometimes two flues, passing through it, from end to end. The tubular boiler has a great many small tubes, or flues, passing through it for the transmission of flame and heated gases. The object in all cases is to generate steam rapidly and economically. To accomplish this, the boiler is set in the furnace so as to give as large a heating surface, in proportion to its capacity, as possible, and the flues and heat passages are constructed to keep the currents of hot air and gas in contact with the heating surface, as long as is compatible with free combustion.

The following are some of the principal appendages to the boiler: 1°, the *furnace*, or *fireplace*, with its flues and dampers, for regulating the draft and keeping up combustion; 2°, the *feed apparatus*, for furnishing water, either from the condenser or from a reservoir, to supply the place of that converted into steam; 3°, the *safety-valve*, a valve opening into the boiler and secured in position by a spring or weighted lever, until the tension of the steam reaches the limit of safety; 4°, the *gauge*, to indicate the height of the water in the boiler; 5°, the *manometer*, for showing the actual tension of the steam in the boiler; 6°, the *blow-off apparatus*, consisting of a cock near the bottom of the boiler, which, when opened, permits the pressure of the steam to force out the sediment and impurities that collect there; and, 7°, the *steam-pipe*, that conducts the steam from the boiler to the engine proper.

The Engine proper.

219. The essential parts of the engine proper are shown in the cut. As the figure is only intended to illustrate the general principles of the engine, the parts are arranged in such manner as to exhibit them best at a single view.

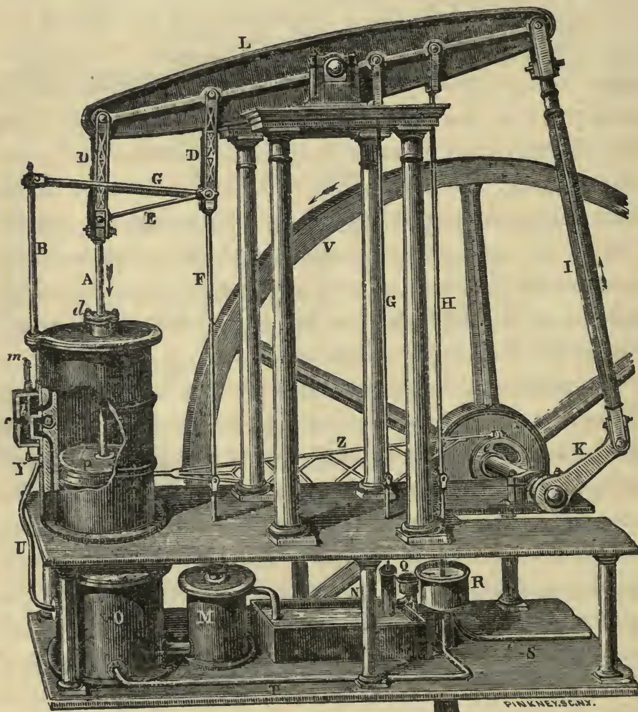


Fig. 178.

The *cylinder* is shown on the left, with a portion broken away. Its interior surface is smooth, and of uniform bore throughout.

The *piston, P,* receives the pressure of the steam, alternately on its upper and lower faces, and is thus made to move up and down in the cylinder, the joint between them being made steam-tight by a suitable packing.

The *piston-rod, A,* working through a *stuffing-box, d,* and kept parallel to the axis by the *parallel motion, D, D, E,* acts on one end of the *working-beam, L,* and imparts to it an oscillatory motion.

The *connecting-rod, I,* transmits the oscillatory motion to the *crank, K,* by means of which it is transformed into rotary motion about the *shaft* of the engine.

The *steam-chest, b,* receives the steam from the boiler through the *steam-pipe, c,* and by means of the *sliding-valve* connected with the rod, *m,* is permitted to pass through the proper channels, or *steam-ports,* alternately to the upper and lower faces of the piston. In the position of the engine shown in the figure, the steam from the boiler passes into the *upper steam-passage,* rises to the top of the cylinder, enters it there, and acts to force the piston *down;* the steam below the piston passes up through the *lower steam-passage,* is prevented from entering the steam-chest by the sliding-valve, passes out at the opening, *a,* and is thence conveyed by the *eduction pipe, U,* to the *condenser, O;* when the piston reaches the bottom of the cylinder the motion imparted to the shaft operates on the eccentric, *e,* to move the *eccentric rod, Z,* which, in turn, through the *bent lever, m,* draws the sliding-valve *up,* so as to cover the upper, and uncover the lower steam-passages; the opening, *a,* of the eduction pipe is then in communication with the upper end of the cylinder, through the upper passage and the sliding-valve. In this state of affairs, the steam from the boiler enters the cylinder by the lower passage, the piston is forced up, the steam above the piston is driven into the eduction pipe, *U*, and thence to the condenser; when the piston reaches the upper limit of its play, the position of the sliding-valve is again reversed, and so on continually.

The *cold-water pump*, *R*, worked by the rod, *H*, draws cold water from a reservoir, and forces it through a pipe, *T*, into the condenser. This pipe, terminating in a rose, delivers the water in the form of a cold shower, which acts to condense the steam that is continually forced into it.

The *air-pump*, *M*, worked by a rod, *F*, draws the hot water, and the air that is mixed with it, from the condenser, and forces it into the *hot-well*, *N*.

The *feed-pump*, *Q*, worked by the rod, *G*, draws the water from the hot-well and forces it back to the boiler.

The Locomotive.

220. The *Locomotive* is represented in section by the accompanying figure. The essential parts are the following:

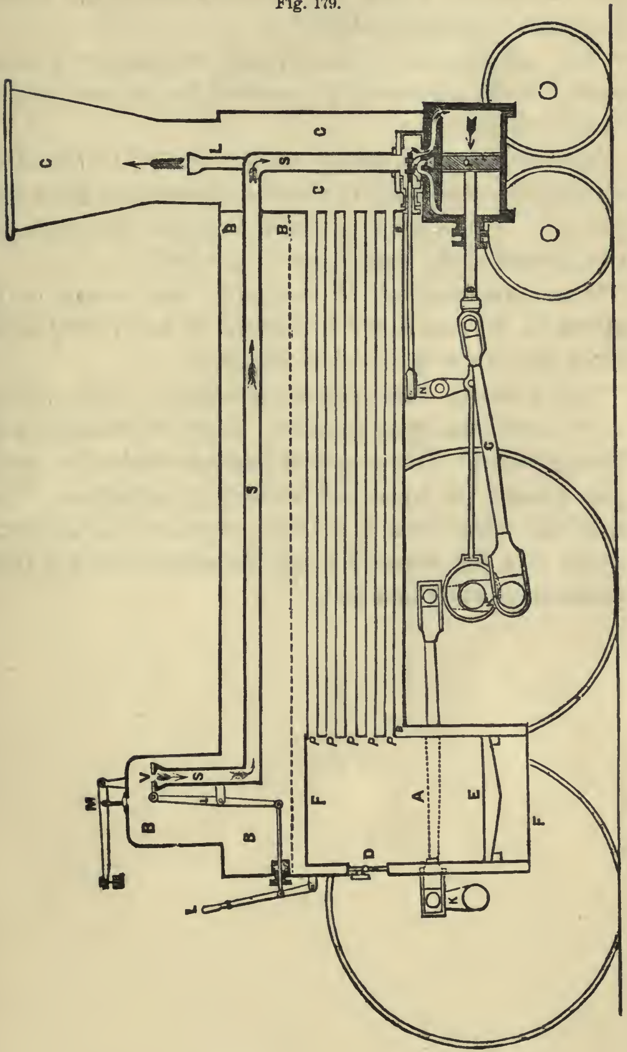
The *boiler*, *B*, *B*, with its *flues*, *p*, *p*, and its *safety-valve*, *M*. The dotted line shows the height of the water in the boiler.

The *fire-box*, *A*, communicating with the *smoke-box*, *C*, by means of the flues, *p*, *p*. The fire-box has a double wall, the interval being filled with water, communicating with that in the boiler. Fuel is supplied by the door, *D*, and air enters the fire-box from below, through the *grate*, *E*.

The *steam-dome*, *B*, is an elevated portion of the boiler, whose object is to permit steam to enter the steam-pipe, without any admixture of water, as might happen if it were taken from a lower level.

The *steam-pipe*, *S*, *S*, conveys steam from the dome to

Fig. 179.



the *steam-chest*, where it is distributed in the manner described in the last article.

The *cylinder*, the *piston*, *P*, and the *piston-rod*, *R*, are similar to the corresponding parts of the engine described in the last article.

The steam, after acting on the piston, is blown off through the *blast-pipe*, *L*, which terminates in the smoke-box, *C*. The current produced increases the draft, and thus promotes the combustion of the fuel.

The *connecting-rod*, *G*, transmits the motion of the piston to the *crank*, which converts it into *rotary motion* about the axis of the driving-wheels, *F*.

The alternate back and forth motion of the sliding-valve is effected by an eccentric, placed on the axle of the driving-wheel. The supply of water is obtained by pumps placed under the frame and worked by eccentrics. These suck the water from a reservoir, mounted on the *tender*, which is a car attached to the locomotive for the transportation of water and fuel.