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F. Cajori

AN

ELEMENTARY TREATISE

ON

PLANE & SPHERICAL TRIGONOMETRY,

WITH THEIR APPLICATIONS TO

NAVIGATION, SURVEYING, HEIGHTS & DISTANCES, AND SPHERICAL ASTRONOMY,

AND PARTICULARLY ADAPTED TO EXPLAINING THE

CONSTRUCTION OF BOWDITCH'S NAVIGATOR, AND THE NAUTICAL ALMANAC.

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PLANE TRIGONOMETRY.

PLANE TRIGONOMETRY.

CHAPTER I.

GENERAL PRINCIPLES OF PLANE TRIGONOMETRY.

1. *Trigonometry* is the science which treats of angles and triangles.

2. *Plane Trigonometry* treats of plane triangles. [B. p. 36.]*

3. *To solve a Triangle* is to calculate certain of its sides and angles when the others are known.

It has been proved in Geometry that, when three of the six parts of a triangle are given, the triangle can be constructed, provided one at least of the given parts is a side. In these cases, then, the unknown parts of the triangle can be determined geometrically, and it may readily be inferred that they can also be determined algebraically.

But a great difficulty is met with on the very threshold of the attempt to apply the calculus to triangles. It arises from the circumstance, that two kinds of quantities are to be intro-

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* References between brackets, preceded by the letter B., refer to the pages in the stereotype edition of Bowditch's Navigator.

Solution of all triangles reduced to that of right triangles.

duced into the same formulas, sides, and angles. These quantities are not only of an entirely different species, but the law of their relative increase and decrease is so complicated, that they cannot be determined from each other by any of the common operations of Algebra.

4. To diminish the difficulty of solving triangles as much as possible, every method has been taken to compare triangles with each other, and the solution of all triangles has been reduced to that of a *Limited Series of Right Triangles.*

a. It is a well known proposition of Geometry, that, in all triangles, which are equiangular with respect to each other, the ratios of the homologous sides are also equal. [B. p. 12.] If, then, a series of dissimilar triangles were constructed containing every possible variety of angles: and, if the angles and the ratios of the sides were all known, we should find it easy to calculate every case of triangles. Suppose, for instance, that in the triangle ABC (fig. 1.), the sides of which we shall denote by the small letters a, b, c, respectively opposite to the angles A, B, C, there are given the two sides b and c and the included angle A, to find the side a and the angles B and C. We are to look through the series of calculated triangles, till we find one which has an angle equal to A, and the ratio of the including sides equal to that of b and c. As this triangle is similar to ABC, its angles and the ratio of its sides must also be those of the triangle ABC, which is therefore completely determined. For, to find the side a, we have only to multiply the ratio which we have found of b to a, that is, the fraction $\frac{a}{b}$ by the side b or the ratio $\frac{a}{c}$ by the side c.

Solution of all triangles reduced to that of right triangles.

b. A series of calculated triangles is not, however, needed for any other than right triangles. For every oblique triangle is either the sum or the difference of two right triangles; and the sides and angles of the oblique triangle are the same with those of the right triangles, or may be obtained from them by addition or by subtraction. Thus the triangle ABC is the sum (fig. 2.) or the difference (fig. 3.) of the two right triangles ABP and BPC. In both figures the sides AB, BC, and the angle A belong at once to the oblique and the right triangles, and so does the angle BCA (fig. 2.) or its supplement (fig. 3.); while the angle ABC is the sum (fig. 2.), or, the difference (fig. 3.) of ABP and PBC; and the side AC is the sum (fig. 2.), or the difference (fig. 3.) of AP and PC.

c. But, as even a series of right triangles, which should contain every variety of angle, would be unlimited, it could never be constructed or calculated. Fortunately, such a series is not required; and it is sufficient for all practical purposes to calculate a series in which the successive angles differ only by a minute, or, at the least, b ya second. The other triangles can be obtained, when needed, by that simple principle of interpolation made use of to obtain the intermediate logarithms from those given in the tables.

CHAPTER II.

SINES, TANGENTS, AND SECANTS.

5. Confining ourselves, for the present, to right triangles, we now proceed to introduce some terms, for the purpose of giving simplicity and brevity to our language.

The *Sine* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle by the hypothenuse.

Thus, if we denote (fig. 4.) the legs BC and AC by the letters a and b, and the hypothenuse AB by the letter h, we have.

$$\text{sin. } A = \frac{a}{h}, \quad \text{sin. } B = \frac{b}{h}. \qquad (1)$$

6. The *Tangent* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle, by the adjacent leg.

Thus, (fig. 4.),

$$\text{tang. } A = \frac{a}{b}, \quad \text{tang. } B = \frac{b}{a}. \qquad (2)$$

7. The *Secant* of an angle is the quotient obtained by dividing the hypothenuse by the leg adjacent to the angle.

Cosine, cotangent, cosecant.

Thus, (fig. 4.)

$$\text{sec. } A = \frac{h}{b}, \quad \text{sec. } B = \frac{h}{a}. \tag{3}$$

8. The *Cosine, Cotangent*, and *Cosecant* of an angle are respectively the sine, tangent, and secant of its complement.

9. *Corollary.* Since the two acute angles of a right triangle are complements of each other, the sine, tangent, and secant of the one must be the cosine, cotangent, and cosecant of the other.

Thus, (fig. 4.),

$$\left.\begin{aligned}
\text{sin. } A &= \text{cos. } B = \frac{a}{h} \\
\text{cos. } A &= \text{sin. } B = \frac{b}{h} \\
\text{tang. } A &= \text{cotan. } B = \frac{a}{b} \\
\text{cotan. } A &= \text{tang. } B = \frac{b}{a} \\
\text{sec. } A &= \text{cosec. } B = \frac{h}{b} \\
\text{cosec. } A &= \text{sec. } B = \frac{h}{a}
\end{aligned}\right\} \tag{4}$$

10. *Corollary.* By inspecting the preceding equations (4), we perceive that the sine and cosecant of an angle are reciprocals of each other; as are also the cosine and secant, and also the tangent and cotangent.

So that

$$\left.\begin{aligned} \text{cosec.}\ A \times \text{sin.}\ A &= \frac{h}{a} \times \frac{a}{h} = \frac{ah}{ah} = 1 \\ \text{sec.}\ A \times \text{cos.}\ A &= \frac{h}{b} \times \frac{b}{h} = \frac{bh}{bh} = 1 \\ \text{tang.}\ A \times \text{cotan.}\ A &= \frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1 \end{aligned}\right\} \quad (5)$$

whence

$$\left.\begin{aligned} \text{cosec.}\ A &= \frac{1}{\text{sin.}\ A}, \text{ or sin. } A = \frac{1}{\text{cosec.}\ A} \\ \text{sec.}\ A &= \frac{1}{\text{cos.}\ A}, \text{ or cos. } A = \frac{1}{\text{sec.}\ A} \\ \text{cotan.}\ A &= \frac{1}{\text{tan.}\ A}, \text{ or tang. } A = \frac{1}{\text{cotan.}\ A} \end{aligned}\right\} \quad (6)$$

As soon, then, as the sine, cosine, and tangent of an angle are known, their reciprocals the cosecant, secant, and cotangent may easily be obtained.

11. *Problem. To find the tangent when the sine and cosine of an angle are known.*

Solution. The quotient of sin. A divided by cos. A is, by equations (4),

$$\frac{\text{sin.}\ A}{\text{cos.}\ A} = \frac{a}{h} \div \frac{b}{h} = \frac{ah}{bh} = \frac{a}{b}.$$

But by (4)

$$\text{tang.}\ A = \frac{a}{b};$$

hence

$$\text{tang.}\ A = \frac{\text{sin.}\ A}{\text{cos.}\ A}. \quad (7)$$

12. *Corollary.* Since the cotangent is the reciprocal of the tangent, we have

$$\text{cotan. } A = \frac{\text{cos. } A}{\text{sin. } A}. \qquad (8)$$

13. *Problem. To find the cosine of an angle when its sine is known.*

Solution. We have, by the Pythagorean proposition, in the right triangle ABC (fig. 4.)

$$a^2 + b^2 = h^2.$$

But by (4)

$$(\text{sin } A.)^2 + (\text{cos. } A)^2 = \frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2 + b^2}{h^2} = \frac{h^2}{h^2} = 1,$$

or $$(\text{sin. } A)^2 + (\text{cos. } A)^2 = 1\,; \qquad (9)$$

that is, *the sum of the squares of the sine and cosine is equal to unity.*

Hence $$(\text{cos. } A)^2 = 1 - (\text{sin. } A)^2,$$

$$\text{cos. } A = \sqrt{1 - (\text{sin. } A)^2}. \qquad (10)$$

14. *Corollary.* Since

$$h^2 - a^2 = b^2,$$

we have by (4)

$$(\text{sec. } A)^2 - (\text{tang. } A)^2 = \frac{h^2}{b^2} - \frac{a^2}{b^2} = \frac{h^2 - a^2}{b^2} = \frac{b^2}{b^2} = 1,$$

or $$(\text{sec. } A)^2 - (\text{tang. } A)^2 = 1\,; \qquad (11)$$

whence $$(\text{sec. } A)^2 = 1 + (\text{tang. } A)^2.$$

15. *Corollary.* Since

$$h^2 - b^2 = a^2$$

we have by (4)

$$(\text{cosec. } A)^2 - (\text{cotan. } A)^2 = \frac{h^2}{a^2} - \frac{b^2}{a^2} = \frac{h^2 - b^2}{a^2} = \frac{a^2}{a^2} = 1,$$

or $(\text{cosec. } A)^2 - (\text{cotan. } A)^2 = 1;$ (12)

whence $(\text{cosec. } A)^2 = 1 + (\text{cotan. } A)^2.$

16. *Scholium.* The whole difficulty of calculating the trigonometric tables of sines and cosines, tangents and cotangents, secants and cosecants is, by the preceding propositions, reduced to that of calculating the sines alone.

17. Examples.

1. Given the sine of the angle A, equal to 0.4568, calculate its cosine, tangent, cotangent, secant, and cosecant.

Solution. By equation (10)

$$\text{cos. } A = \sqrt{1 - (\text{sin. } A)^2} = \sqrt{(1 + \text{sin. } A)(1 - \text{sin. } A)}.$$

$1 + \text{sin. } A = 1.4568$	0.16340
$1 - \text{sin. } A = 0.5432$	9.73496
$(\text{cos. } A)^2$	2|9.89836
$\text{cos. } A = 0.8896$	9.94918.

By (7) and (8)

$$\text{tang. } A = \frac{\text{sin. } A}{\text{cos. } A} \qquad \text{cotan. } A = \frac{\text{cos. } A}{\text{sin. } A}.$$

Calculation of cosine, &c.

sin. $A = 0.4568$	9.65973	(ar. co.)	10.34027
cos. $A = 0.8896$ (ar. co.)	10.05082		9.94918
tang. $A = 0.5135$	9.71055	(ar. co.)	10.28945

cotan. $A = 1.9474$.

By (6)

$$\text{sec. } A = \frac{1}{\text{cos. } A}, \qquad \text{cosec. } A = \frac{1}{\text{sin. } A}.$$

$$\text{log. sec. } A = -\text{log. cos. } A = 0.05082,$$

$$\text{sec. } A = 1.1241.$$

$$\text{log. cosec. } A = -\text{log. sin. } A = 0.34027,$$

$$\text{cosec. } A = 2.1891.$$

2. Given sin. $A = 0.1111$; find the cosine, tangent, cotangent, secant, and cosecant of A.

Ans. cos. $A = 0.9938$
tang. $A = 0.1118$
cotan. $A = 8.9452$
sec. $A = 1.0062$
cosec. $A = 9.0010$.

3. Given sin. $A = 0.9891$; find the cosine, tangent, cotangent, secant, and cosecant of A,

Ans. cos. $A = 0.1472$
tang. $A = 6.7173$
cotan. $A = 0.1489$
sec. $A = 6.7914$
cosec. $A = 1.0110$.

Sine, &c. in the circle whose radius is unity.

18. *Theorem. The sine of an angle is equal to the perpendicular let fall from one extremity of the arc, which measures it in the circle, whose radius is unity, upon the radius passing through the other extremity.*

Proof. Let BCA (fig. 5.) be the angle, and let the radius of the circle $ABA'A$ be

$$AC = \text{unity} = 1.$$

Let fall, on the radius AC, the perpendicular BP, and we have by § 5, in the right triangle BCP,

$$\sin. BCP = \frac{BP}{BC} = \frac{BP}{1} = BP.$$

19. *Theorem. In the circle of which the radius is unity, the cosine of an angle is equal to the portion of the radius, which is drawn perpendicular to the sine, included between the sine and the centre.*

Proof. For if BCA (fig. 5.) is the angle, we have, by § 9,

$$\cos. BCA = \frac{CP}{BC} = \frac{CP}{1} = CP.$$

20. *Theorem. In the circle of which the radius is unity, the secant is equal to the length of the radius drawn through one extremity of the arc which measures the angle, and produced till it meets the tangent drawn through the other extremity.*

The trigonometric tangent is equal to that portion of the tangent, drawn through one extremity of the arc, which is intercepted between the two radii which terminate the arc.

Proof. If CB (fig. 5.) is produced to meet the tangent A at T, we have, by (2) and (3), in the right triangle BCT,

$$\text{sec. } BCA = \frac{CT}{AC} = \frac{CT}{1} = CT$$

$$\text{tang. } BCA = \frac{AT}{AC} = \frac{AT}{1} = AT.$$

21. *Scholium.* The preceding theorems (18 – 20), have been adopted by most writers upon trigonometry, as the definitions of sine, cosine, tangent, and secant, except that the radius of the circle has not been limited to unity. [B. p. 6.]

By not limiting the radius to unity the sines, &c. have not been fixed values, but have varied with the length of the radius; whereas their values, in the system here adopted, are the fixed ratios of their values as ordinarily given to the radius of the circle in which they are measured. Thus, if R is the radius, we have

sin., cos., &c. in the common system $= R \times$ sin., cos., &c. in this system.

22. *Corollary.* If the angle is very small, as C (fig. 6.), the arc AB will be sensibly a straight line, perpendicular to the two radii CA and CB, drawn to its extremities, and will sensibly coincide with the sine and tangent; while the cosine will sensibly coincide with the radius CA, and the secant with the radius CB.

Hence, the sine and tangent of a very small angle are nearly equal to the arc which measures the angle, *in the circle the radius of which is unity; and its cosine and secant are nearly equal to unity.*

23. *Problem. To find the sine of a very small angle.*

Solution. Let the angle C (fig. 6.) be the given angle, and suppose it to be exactly one minute. The arc AB must in this case be $\frac{1}{10800}$ of the semicircumference, of which unity or CA is radius. But the value of the semicircumference, of which unity is radius, has been found in Geometry to be 3.1415926. Therefore, by §22,

$$\text{sin. } 1' = AB = \frac{3.1415926}{10800} = 0.00029. \qquad (12)$$

In the same way we might find the sine of any other small angle, or we might, in preference, find it by the following proposition.

24. *Theorem. The sines of very small angles are proportional to the angles themselves.*

Proof. Let there be the two small angles, BCA and $B'CA$ (fig. 7.) Draw the arc ABB' with the centre C, and the radius unity. Then, as angles are proportional to the arcs which measure them,

$$BCA : B'CA = BA : B'A.$$

But, by § 22,

$$\text{sin. } BCA = BA, \text{ sin. } B'CA = B'A;$$

whence

$$BCA : B'CA = \text{sin. } BCA : \text{sin. } B'CA.$$

a. This proposition is limited to angles so small, that their arcs may be considered as straight lines. It is found in practice, that the angles may be as large as two degrees, provided the approximations are not carried beyond five places of decimals. The investigation of the sines of larger angles requires the introduction of some new formulas.

25. Examples.

1. Find the sine of $12' \; 13''$, knowing that

$$\sin. 1' = 0.00029.$$

Solution. By (28)

$$1' : 12' \; 13'' : : \sin. 1' : \sin. 12' \; 13'',$$

or

$$60'' : 733'' : : 0.00029 : \sin. 12' \; 13''.$$

Hence

$$\sin. 12' \; 13'' = \frac{733 \times 0.00029}{60} = 0.00354. \quad \textit{Ans}$$

2. Find the sine of $7' \; 15''$, knowing that

$$\sin. 1' = 0.00029.$$

Ans. $\sin. 7' \; 15'' = 0.00210.$

3. Find the sine of $2' \; 31''$, knowing that

$$\sin. 1' = 0.00029.$$

Ans. $\sin. 2' \; 31'' = 0.00073.$

26. *Problem. Given the sine of any angle, to find the sine of another angle which exceeds it by a very small quantity.*

Sine of an angle differing very little from a given angle.

Solution. Let the given angle be BCA (fig. 8), which we will denote by the letter M; and let the angle whose sine is required be $B'CA$, exceeding the former by the small angle $B'CB$, which we will denote by the letter m; so that

$$M = BCA, \qquad m = B'CB,$$

$$M + m = B'CA.$$

From the vertex C as a centre, with the radius unity, describe the arc ABB'. From the points B and B' let fall BP and $B'P'$ perpendicular to AC.

We have, by § 18 and 19,

$$\text{Sine. } M = BP$$

$$\text{sin. } B'CA = \text{sin. } (M + m) = B'P'$$

$$\text{cos. } M = PC;$$

Draw BR perpendicular to $B'P'$, and

$$B'P' = BP + B'R,$$

or

$$\text{sin. } (M + m) = \text{sin. } M + B'R.$$

The triangles BCP and $BB'R$, having their sides perpendicular each to each, are similar, and give the proportion

$$BC : BB' = CP : B'R.$$

But, by § 22,

$$BB' = \text{sin. } m.$$

Hence

$$1 : \text{sin. } m = \text{cos. } M : B'R;$$

and $$B'R = \text{sin. } m. \text{ cos. } M,$$

which gives, by substitution,

Cosine of an angle differing very little from a given angle.

$$\sin. (M + m) = \sin. M + \sin. m. \cos. M. \qquad (13)$$

27. *Corollary.* If m were $1'$, (13) would become

$$\sin. (M + 1') = \sin. M + \sin. 1'. \cos. M,$$
$$= \sin. M + 0.00029 \cos. M. \qquad (14)$$

We may, by this formula, find the sine of $2'$ from that of $1'$, thence that of $3'$, then of $4'$, of $5'$, &c., to the sine of angle of any number of degrees and minutes.

28. *Corollary.* We can, in a similar way, deduce the value of $\cos. (M + m)$.

For, by § 19,

$$\cos. (M + m) = P'C = PC - PP',$$
$$= \cos. M - BR.$$

But the similar triangles

$BB'R$ and BCP give the proportion

$$BC : BB' = BP : BR,$$

or

$$1 : \sin. m = \sin. M : BR.$$

Hence

$$BR = \sin. m. \sin. M,$$

whence

$$\cos. (M + m) = \cos. M - \sin. m. \sin. M, \qquad (15)$$

and, if we make $m = 1'$, this equation becomes

$$\cos. (M + 1') = \cos. M - \sin. 1'. \sin M,$$
$$= \cos. M - 0.00029 \sin. M. \qquad (16)$$

29. Examples.

1. Given the sine of $23° 28'$ equal to 0.39822, to find the sine of $23° 29'$.

Solution. We find the cosine of $23° 28'$ by (10) to be

$$\cos. 23° 28' = 0.91729.$$

Hence, by (14), making $M = 23° 28'$

$$\begin{aligned} \sin. 23° 29' &= \sin. 23° 28' + 0.00029 \cos. 23° 28', \\ &= 0.39822 + 0.00026, \\ &= 0.39848. \end{aligned}$$

Ans. $\sin. 23° 29' = 0.39848.$

2. Given the sine and cosine of $46° 58'$ as follows,

$$\sin. 46° 58' = 0.73096, \cos. 46° 58' = 0.68242,$$

find the sine of $46° 59'$.

Ans. $\sin. 46° 59' = 0.73116.$

3. Given the sine and cosine of $11° 10'$ as follows,

$$\sin. 11° 10' = 0.19366, \cos. 11° 10' = 0.98107,$$

find the cosine of $11° 11'$.

Ans. $\cos. 11° 11' = 0.98101.$

30. By the formulas here given a complete table of sines and cosines might be calculated. Such tables have been actually calculated; and table XXIV. of the Navigator is such a table; their logarithms are given in table XXVII. of the Navigator.

Natural and artificial sines. Radius of table.

The sines, cosines, &c. of table XXIV. are called *natural*, to distinguish them from their logarithms, which are sometimes called their *artificial* sines, cosines, &c.

The radius of table XXIV. is

$$10^5 = 100000,$$

so that this table is, by § 21, reduced to the present system by dividing each number by 100000, that is, by prefixing the decimal point to each of the numbers of the table.

The radius of table XXVII. is

$$10^{10} = 10000000000,$$

so that this table is reduced to the present system by subtracting from each number the logarithm of this radius, which is 10, that is by subtracting 10 from each characteristic.

The method of using these two tables is fully explained in pp. 33 – 35, and p. 390, of the Navigator.

CHAPTER III.

RIGHT TRIANGLES.

31. *Problem. To solve a right triangle, when the hypothenuse and one of the angles are known.* [B. p. 38.]

Solution. Given (fig. 4) the hypothenuse h and the angle A, to solve the triangle.

First. To find the other acute angle B, subtract the given angle from 90°.

Secondly. To find the opposite side a, we have by (1)

$$\sin. A = \frac{a}{h},$$

which, multiplied by h, gives

$$a = h \sin. A; \qquad (17)$$

or, by logarithms,

$$\log. a = \log. h + \log. \sin. A.$$

Thirdly. To find the side b, we have by (4)

$$\cos. A = \frac{b}{h},$$

which, multiplied by h, gives

$$b = h \cos. A; \qquad (18)$$

or, by logarithms,

$$\log. b = \log. h + \log. \cos. A.$$

Leg and an angle given.

32. *Problem. To solve a right triangle, when a leg and the opposite angle are known.* [B. p. 39.]

Solution. Given (fig. 4.) the leg a, and the opposite angle A, to solve the triangle.

First. The angle B is the complement of A.

Secondly. To find the hypothenuse h, we have by (17)

$$a = h \sin. A,$$

which, divided by sin. A, gives by (6)

$$h = \frac{a}{\sin. A} = a \operatorname{cosec.} A; \qquad (19)$$

or, by logarithms,

$$\log. h = \log. a + (\text{ar. co.}) \log. \sin. A$$

$$= \log. a + \log. \operatorname{cosec.} A.$$

Thirdly. To find the other leg b, we have by (4)

$$\operatorname{cotan.} A = \frac{b}{a},$$

which, multiplied by a, gives

$$b = a \operatorname{cotan.} A; \qquad (20)$$

or, by logarithms,

$$\log. b = \log. a + \log. \operatorname{cotan.} A.$$

33. *Problem. To solve a right triangle, when a leg and the adjacent angle are known.* [B. p. 39.]

Solution. Given (fig. 4.) the leg a and the angle B, to solve the triangle.

First. The angle A is the complement of B.

Secondly. The other parts may be found by (19) and (20), or from the following equations, which are readily deduced from equations (4) and (6),

$$h = \frac{a}{\cos.\ B} = a \sec.\ B, \qquad (21)$$

$$b = a \text{ tang. } B; \qquad (22)$$

or, by logarithms,

$$\log.\ h = \log.\ a + \log.\ \sec.\ B,$$

$$\log.\ b = \log.\ a + \log.\ \text{tang. } B.$$

36. *Problem. To solve a right triangle, when the hypothenuse and a leg are known.* [B. p. 40.]

Solution. Given (fig. 4.) the hypothenuse h and the leg a, to solve the triangle.

First. The angles A and B are obtained from equation (4),

$$\sin.\ A = \cos.\ B = \frac{a}{h}; \qquad (23)$$

or, by logarithms,

$$\log.\ \sin.\ A = \log.\ \cos.\ B = \log.\ a + (\text{ar. co.}) \log.\ h.$$

Secondly. The leg b is deduced from the Pythagorean property of the right triangle, which gives

$$a^2 + b^2 = h^2, \qquad (24)$$

whence

$$b^2 = h^2 - a^2 = (h + a)(h - a),$$

$$b = \surd (h^2 - a^2) = \surd [(h + a)(h - a)]; \qquad (25)$$

by logarithms,

$$\log.\ b. = \tfrac{1}{2} \log.\ (h^2 - a^2) = \tfrac{1}{2} [\log.\ (h + a) + \log.\ (h - a)].$$

The legs given.

35. *Problem. To solve a right triangle, when the two legs are known.* [B. p. 40.]

Solution. Given (fig. 4.) the legs a and b, to solve the triangle.

First. The angles are obtained from (4)

$$\text{tang. } A = \text{cotan. } B = \frac{a}{b};$$

or, by logarithms,

$$\text{log. tang. } A = \text{log. cotan. } B = \text{log. } a + (\text{ar. co.}) \text{ log. } b.$$

Secondly. To find the hypothenuse, we have by (24)

$$h = \sqrt{(a^2 + b^2)}.$$

Thirdly. An easier way of finding the hypothenuse is to make use of (19) or (20)

$$h = a \text{ cosec. } A = a \text{ sec. } B;$$

or, by logarithms,

$$\text{log. } h = \text{log. } a. + \text{log. cosec. } A = \text{log. } a + \text{log. sec. } B.$$

36. EXAMPLES.

1. Given the hypothenuse of a right triangle equal to 49.58, and one of the acute angles equal to 54° 44′; to solve the triangle.

Solution. The other angle = 90° — 54° 44′ = 35° 16′. Then making $h = 49.58$, and $A = 54° 44′$; we have, by (17) and (18), Examples of right triangles.

$h = 49.58$	1.69531		1.69531
$A = 54° 44'$	*sin. 9.91194		cos. 9.76146
$a = 40.481$	1.60725;	$b = 28.627$	1.45677.

Ans. The other angle $= 35° 16'$;

The legs $= \begin{cases} 40.481, \\ 28.627. \end{cases}$

2. Given the hypothenuse of a right triangle equal to 54.571, and one of the legs equal to 23.479; to solve the triangle.

Solution. Making $h = 54.571$, $a = 23.479$;

we have, by (23),

$a = 23.479$		1.37068
$h = 54.571$	(ar. co.)	8.26304
$A = 25° 29'$ sin.		
$B = 64° 31'$ cos.	}	9.63372.

By (25),

$h + a = 78.050$		1.89237
$h - a = 31.092$		1.49265
b^2	2	3.38502
$b = 49.262$		1.69251

Ans. The other leg $= 49.262$

The angles $= \begin{cases} 25° 29' \\ 64° 31' \end{cases}$

3. Given the two legs of a right triangle equal to 44.375, and 22.165; to solve the triangle.

* To avoid negative characteristics the logarithms are retained as in the tables, according to the usual practice, with the logarithms of decimals, as in B. p. 29.

Examples of right triangles.

Solution. Making $a = 44.375$, $b = 22.165$; we have,

$a = 44.375$		1.64714	1.64714
$b = 22.165$	(ar. co.)	8.65433	
$A = 63^\circ\ 27'\ 28''$ tang. / $B = 26^\circ\ 32'\ 32''$ cotan.		10.30147;	cosec. / sec. 10.04837
		$h = 49.603$	1.69551

Ans. The hypothenuse $= 49.603$

The angles $= \begin{cases} 63^\circ\ 27'\ 28'' \\ 26^\circ\ 32'\ 32'' \end{cases}$

4. Given the hypothenuse of a right triangle equal to 37.364, and one of the acute angles equal to $12^\circ\ 30'$; to solve the triangle.

Ans. The other angle $= 77^\circ\ 30'$

The legs $= \begin{cases} 8.087 \\ 36.478 \end{cases}$

5. Given one of the legs of a right triangle equal to 14.548, and the opposite angle equal to $54^\circ\ 24'$; to solve the triangle.

Ans. The hypothenuse $= 17.892$

The other leg $= 10.415$

The other angle $= 35^\circ\ 36'$.

6. Given one of the legs of a right triangle equal to 11.111, and the adjacent angle equal to $11^\circ\ 11'$, to solve the triangle.

Ans. The hypothenuse $= 11.326$

The other leg $= 2.197$

The other angle $= 78^\circ\ 49'$.

7. Given the hypothenuse of a right triangle equal to 100, and one of the legs equal to 1, to solve the triangle.

Ans. The other leg = 99.995

$$\text{The angles} = \begin{cases} 0^\circ\ 34'\ 23'' \\ 89^\circ\ 25'\ 37'' \end{cases}$$

8. Given the two legs of a right triangle equal to 8.148, and 10.864, to solve the triangle.

Ans. The hypothenuse = 13.58

$$\text{The angles} = \begin{cases} 36^\circ\ 52'\ 11'' \\ 53^\circ\ \ 7'\ 49'' \end{cases}$$

CHAPTER IV.

GENERAL FORMULAS.

37. The solution of oblique triangles requires the introduction of several trigonometrical formulas, which it is convenient to bring together and investigate all at once.

38. *Problem. To find the sine of the sum of two angles.*

Solution. Let the two angles be BAC and $B'AC$ (fig. 9), represented by the letters M and N. At any point C, in the line AC, erect the perpendicular BB'. From B let fall on AB' the perpendicular BP. Then represent the several lines, as follows,

$$a = BC, a' = B'C, b = AC$$

$$h = AB, h' = AB', x = BP$$

$$M = BAC, \; N = B'AC.$$

Then, by (4),

$$\sin. BAC = \sin. M = \frac{a}{h}, \qquad \sin. N = \frac{a'}{h'}$$

$$\cos. M = \frac{b}{h}, \qquad \cos. N = \frac{b}{h'}$$

$$\sin. BAP = \sin. (M + N) = \frac{BP}{AB} = \frac{x}{h}.$$

Now the triangles BPB' and $B'AC$, being right-angled, and having the angle B' common, are equiangular and similar.

Whence we derive the proportion

$$AB' : BB' = AC : BP,$$

or

$$h' : a + a' = b : x;$$

whence

$$x = \frac{a\,b + a'\,b}{h'},$$

and

$$\sin.\ (M + N) = \frac{x}{h} = \frac{a\,b + a'\,b}{h\,h'}.$$

The second member of this equation may be separated into factors, as follows,

$$\sin.\ (M + N) = \frac{a\,b}{hh'} + \frac{ba'}{hh'}$$

$$= \frac{a}{h} \cdot \frac{b}{h'} + \frac{b}{h} \cdot \frac{a'}{h'};$$

whence, by substitution, we obtain

$$\sin.\ (M + N) = \sin.\ M\ \cos.\ N + \cos.\ M\ \sin.\ N. \qquad (26)$$

39. *Problem. To find the sine of the difference of two angles.*

Solution. Let the two angles be BAC and $B'AC$ (fig. 10), represented by M and N. At any point C in the line AC erect the perpendicular $BB'C$. From B let fall on AB' the perpendicular BP. Then, using the notation of § 38, we have

$$\sin.\ BAP = \sin.\ (M - N) = \frac{B\,P}{A\,B} = \frac{x}{h}.$$

Cosine of the sum of two angles.

The triangles $B'AC$ and $BB'P$ are similar, because they are right-angled, and the angles at B' are vertical and equal.

Whence

$$AB' : BB' = AC : BP,$$

or

$$h' : a - a' = b : x;$$

whence

$$x = \frac{ab - a'b}{h'},$$

and

$$\sin.(M - N) = \frac{x}{h} = \frac{ab - ba'}{hh'}$$

$$= \frac{ab}{hh'} - \frac{ba'}{hh'}$$

$$= \frac{a}{h} \cdot \frac{b}{h'} - \frac{b}{h} \cdot \frac{a'}{h'};$$

and by substitution,

$$\sin.(M - N) = \sin. M \cos. N - \cos. M \sin. N. \qquad (27)$$

40. *Problem. To find the cosine of the sum of two angles.*

Solution. Making use of (fig. 9), with the notation of § 38, and also the following

$$y = AP, z = PB';$$

we have

$$\cos.(M + N) = \frac{AP}{AB} = \frac{y}{h}.$$

But

$$y = AB' - PB' = h' - z.$$

Cosine of the sum of two angles.

The similar triangles BPB' and $B'AC$, give the proportion

$$AB' : BB' = B'C \,.\, B'P,$$

or

$$h' : a + a' = a' : z;$$

whence

$$z = \frac{a\,a' + a'^{\,2}}{h'},$$

and

$$y = h' - z = h' - \frac{a\,a' + a'^2}{h'}$$

$$= \frac{h'^2 - a'^2 - a\,a'}{h'}.$$

But, from the right triangle $AB'C$,

$$h'^2 - a'^2 = (AB')^{\,2} - (B'C)^2 = (AC)^2 = b^2\,;$$

whence

$$y = \frac{b^2 - a\,a'}{h'}$$

and

$$\cos.\,(M + N) = \frac{y}{h} = \frac{b^2 - a\,a'}{hh'}$$

$$= \frac{b^2}{hh'} - \frac{aa'}{hh'},$$

$$= \frac{b}{h} \cdot \frac{b}{h'} - \frac{a}{h} \cdot \frac{a'}{h'}\,;$$

whence by substitution,

$$\cos.\,(M + N) = \cos.\,M.\,\cos.\,N - \sin.\,M.\,\sin.\,N. \qquad (28)$$

41. *Problem. To find the cosine of the difference of two angles.*

Cosine of the difference of two angles.

Solution. Making use of (fig. 10.) with the notation of the preceding section, we have

$$\cos. BAB' = \cos. (M - N) = \frac{A\,P}{A\,B} = \frac{y}{h}.$$

But $$y = AB' + PB' = h' + z.$$

The similar triangles $BB'P$ and $B'AC$ give the proportion

$$AB' : BB' = B'C : B'P,$$

or $$h' : a - a' = a' : z;$$

whence $$z = \frac{a\,a' - a'^2}{h'},$$

and $$y = h' + z = h' + \frac{a\,a' - a'^2}{h'}$$

$$= \frac{h'^2 - a'^2 + a\,a'}{h'}.$$

But $$h'^2 - a'^2 = b^2.$$

Hence $$y = \frac{b^2 + a\,a'}{h'},$$

and $$\cos. (M - N) = \frac{y}{h} = \frac{b^2 + a\,a'}{h\,h'}$$

$$= \frac{b^2}{hh'} + \frac{aa'}{hh'}$$

$$= \frac{b}{h} \cdot \frac{b}{h'} + \frac{a}{h} \cdot \frac{a'}{h'};$$

or, by substitution,

$$\cos. (M - N) = \cos. M \cos. N + \sin. M \sin. N. \qquad (29)$$

42. *Corollary.* The similarity, in all but the signs, of the formulas (26) and (27) is such, that they may both be written in the same form, as follows,

$$\sin. (M \pm N) = \sin. M \cos. N \pm \cos. M \sin. N. \quad (30)$$

in which the upper signs correspond with each other, and also the lower ones.

In the same way, by the comparison of (28) and (29), we are led to the form

$$\cos. (M \pm N) = \cos. M. \cos. N \mp \sin. M. \sin. N, \quad (31)$$

in which the upper signs correspond with each other, and also the lower ones.

43. *Corollary.* The sum of the equations (26) and (27) is

$$\sin. (M + N) + \sin. (M - N) = 2 \sin. M \cos. N. \quad (32)$$

Their difference is

$$\sin. (M + N) - \sin. (M - N) = 2 \cos. M \sin. N. \quad (33)$$

44. *Corollary.* The sum of (28) and (29) is

$$\cos. (M + N) + \cos. (M - N) = 2 \cos. M \cos. N. \quad (34)$$

Their difference is

$$\cos. (M - N) - \cos. (M + N) = 2 \sin. M \sin. N. \quad (35)$$

45. *Corollary.* If, in (32–25), we make

$$M + N = A, \text{ and } M - N = B;$$

that is,

$$M = \tfrac{1}{2}(A + B), \qquad N = \tfrac{1}{2}(A - B);$$

they become, as follows,

Sum and difference of sines and cosines.

$$\sin. A + \sin. B = 2 \sin. \tfrac{1}{2} (A + B) \cos. \tfrac{1}{2} (A - B) \quad (36)$$

$$\sin. A - \sin. B = 2 \cos. \tfrac{1}{2} (A + B) \sin. \tfrac{1}{2} (A - B) \quad (37)$$

$$\cos. A + \cos. B = 2 \cos. \tfrac{1}{2} (A + B) \cos. \tfrac{1}{2} (A - B) \quad (38)$$

$$\cos. B - \cos. A = 2 \sin. \tfrac{1}{2} (A + B) \sin. \tfrac{1}{2} (A - B). \quad (39)$$

46. *Corollary.* The quotient, obtained by dividing (36) by (37), is

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin. \frac{1}{2} (A + B) \cos. \frac{1}{2} (A - B)}{\cos. \frac{1}{2} (A + B) \sin. \frac{1}{2} (A - B)}.$$

Reducing the second member by means of equations (6), (7), (8), we have

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \text{tang.} \tfrac{1}{2} (A + B) \text{ cotan.} \tfrac{1}{2} (A - B)$$

$$= \frac{\text{tang.} \frac{1}{2} (A + B)}{\text{tang.} \frac{1}{2} (A - B)} = \frac{\text{cotan.} \frac{1}{2} (A - B)}{\text{cotan.} \frac{1}{2} (A + B)}. \quad (40)$$

47. *Corollary.* The quotient of (39) divided by (38) is, by reduction,

$$\frac{\cos. B - \cos. A}{\cos. B + \cos. A} = \text{tang.} \tfrac{1}{2}(A+B) \text{ tang.} \tfrac{1}{2}(A-B)$$

$$= \frac{\text{tang.} \frac{1}{2}(A+B)}{\text{cotan.} \frac{1}{2}(A-B)} = \frac{\text{tang.} \frac{1}{2}(A-B}{\text{cotan.} \frac{1}{2}(A+B}. \quad (41)$$

48. *Corollary.* Putting in (26) and (28), M and N both equal to A, we obtain

$$\sin. 2 A = \sin. A \cos. A + \sin. A \cos. A = 2 \sin. A \cos. A \quad (42)$$

$$\cos. 2 A = \cos. A \cos. A - \sin. A \sin. A$$
$$= (\cos. A)^2 - (\sin. A)^2. \quad (43)$$

49. *Corollary.* The sum of (43), and of the following equation, which is the same as (9),

$$1 = (\cos. A)^2 + (\sin. A)^2,$$

is

$$1 + \cos. 2 A = 2 (\cos. A)^2. \tag{44}$$

Their difference is

$$1 - \cos. 2 A = 2 (\sin. A)^2. \tag{45}$$

50. *Corollary.* Making $2 A = C$, or $C = \frac{1}{2} A$, in (42–49), we obtain

$$\sin. C = 2 \sin \tfrac{1}{2} C \cos. \tfrac{1}{2} C \tag{46}$$

$$\cos. C = (\cos. \tfrac{1}{2} C)^2 - (\sin. \tfrac{1}{2} C)^2 \tag{47}$$

$$1 + \cos. C = 2 (\cos. \tfrac{1}{2} C)^2 \tag{48}$$

$$1 - \cos. C = 2 (\sin. \tfrac{1}{2} C)^2. \tag{49}$$

The equations (48) and (49) give

$$\cos. \tfrac{1}{2} C = \sqrt{[\tfrac{1}{2}(1 + \cos. C)]} \tag{50}$$

$$\sin. \tfrac{1}{2} C = \sqrt{[\tfrac{1}{2}(1 - \cos. C)]} \tag{51}$$

$$\text{tang. } \tfrac{1}{2} C = \sqrt{\left(\frac{1 - \cos. C}{1 + \cos. C}\right)}. \tag{52}$$

51. *Problem. To find the tangent of the sum and of the difference of two angles.*

Solution. First. To find the tangent of the sum of two angles, which we will suppose to be M and N, we have from (7),

$$\text{tang. } (M + N) = \frac{\sin. (M + N)}{\cos. (M + N)}.$$

Tangent of sum and difference of angles.

Substituting (26) and (28),

$$\text{tang. } (M+N) = \frac{\text{sin. } M \text{ cos. } N + \text{cos. } M \text{ sin. } N}{\text{cos. } M \text{ cos. } N - \text{sin. } M \text{ sin. } N}.$$

Divide every term of both numerator and denominator of the second member by cos. M cos. N;

$$\text{tang. } (M+N) = \frac{\dfrac{\text{sin. } M \text{ cos. } N}{\text{cos. } M \text{ cos. } N} + \dfrac{\text{cos. } M \text{ sin. } N}{\text{cos. } M \text{ cos. } N}}{\dfrac{\text{cos. } M \text{ cos. } N}{\text{cos. } M \text{ cos. } N} - \dfrac{\text{sin. } M \text{ sin. } N}{\text{cos. } M \text{ cos. } N}},$$

$$= \frac{\dfrac{\text{sin. } M}{\text{cos. } M} + \dfrac{\text{sin. } N}{\text{cos. } N}}{1 - \dfrac{\text{sin. } M}{\text{cos. } M} \cdot \dfrac{\text{sin. } N}{\text{cos. } N}},$$

which, reduced by means of (7), becomes

$$\text{tang. } (M+N) = \frac{\text{tang. } M + \text{tang. } N}{1 - \text{tang. } M \text{ tang. } N}. \qquad (53)$$

Secondly. To find the tangent of the difference of M and N, since by (7)

$$\text{tang. } (M-N) = \frac{\text{sin. } (M-N)}{\text{cos. } (M-N)},$$

a bare inspection of (30) and (31) shows that we have only to change the signs, which connect the terms in the value of tang. $(M+N)$ to obtain that of tang. $(M-N)$. This change, being made in (53), produces

$$\text{tang. } (M-N) = \frac{\text{tang. } M - \text{tang. } N}{1 + \text{tang. } M \text{ tang. } N}. \qquad (54)$$

Tangent and cotangent of double an angle.

52. *Corollary.* As the cotangent is merely the reciprocal of the tangent, we have, by inverting the fractions, from (53) and (54),

$$\text{cotan.}\,(M+N) = \frac{1 - \text{tang.}\,M\,\text{tang.}\,N}{\text{tang.}\,M + \text{tang.}\,N}, \qquad (55)$$

$$\text{cotan.}\,(M-N) = \frac{1 + \text{tang.}\,M\,\text{tang.}\,N}{\text{tang.}\,M - \text{tang.}\,N}. \qquad (56)$$

53. *Corollary.* Make $M = N = A$, in (53) and (55). They become

$$\text{tang.}\,2\,A = \frac{2\,\text{tang.}\,A}{1 - (\text{tang.}\,A)^2}, \qquad (57)$$

$$\text{cotan.}\,2\,A = \frac{1 - (\text{tang.}\,A)^2}{2\,\text{tang.}\,A}. \qquad (58)$$

CHAPTER V.

VALUES OF THE SINES, COSINES, TANGENTS, COTANGENTS, SECANTS, AND COSECANTS OF CERTAIN ANGLES.

54. *Problem. To find the sine, &c. of* 0° *and* 90°.

Solution. Supposing $M = N$, in (27) and (29), we have

$$\sin.(M - M) = \sin. 0° = \sin. M \cos. M - \cos. M \sin. M$$

$$\cos.(M - M) = \cos. 0° = (\cos. M)^2 + (\sin. M)^2;$$

whence, by (9), and the consideration that 0° and 90° are complements of each other,

$$\sin. 0° = \cos. 90° = 0 \qquad (59)$$

$$\cos. 0° = \sin. 90° = 1. \qquad (60)$$

From (6) and (7), we have

$$\text{tang. } 0° = \text{cotan. } 90° = \frac{\sin. 0°}{\cos. 0°} = \tfrac{0}{1} = 0, \qquad (61)$$

$$\text{cotan. } 0° = \text{tang. } 90° = \frac{1}{\text{tang. } 0°} = \tfrac{1}{0} = \infty \qquad (62)$$

$$\text{sec. } 0° = \text{cosec. } 90° = \frac{1}{\cos. 0°} = \tfrac{1}{1} = 1 \qquad (63)$$

$$\text{cosec. } 0° = \text{sec. } 90° = \frac{1}{\sin. 0°} = \tfrac{1}{0} = \infty. \qquad (64)$$

Sine, &c. of 180° and 270°.

55. *Problem. To find the sine, &c. of* 180°.

Solution. Make $A = 90°$, in (42) and (43), they become, by means of (59) and (60),

$$\sin. 180° = 2 \sin. 90° \cos. 90° = 0 \tag{65}$$

$$\cos. 180° = (\cos. 90°)^2 - (\sin. 90°)^2 = -1. \tag{66}$$

Hence from (6) and (7),

$$\text{tang. } 180° = \frac{\sin. 180°}{\cos. 180°} = \frac{0}{-1} = 0 \tag{67}$$

$$\text{cotan. } 180° = \frac{\cos. 180°}{\sin. 180°} = \frac{-1}{0} = -\infty \tag{68}$$

$$\sec. 180° = \frac{1}{\cos. 180°} = \frac{1}{-1} = -1 \tag{69}$$

$$\text{cosec. } 180° = \frac{\sin. 180°}{1} = \tfrac{1}{0} = \infty. \tag{70}$$

56. *Problem. To find the sine, &c. of* 270°.

Solution. Make $M = 180°$ and $N = 90°$ in (26) and (28). They become, by means of (59, 60, 65, 66),

$$\sin. 270° = \sin. 180° \cos. 90° + \cos. 180° \sin. 90° = -1 \tag{71}$$

$$\cos. 270° = \cos. 180° \cos. 90° - \sin. 180° \sin. 90° = 0. \tag{72}$$

Hence, from (6) and (7),

$$\text{tang. } 270° = \frac{\sin. 270°}{\cos. 270°} = \frac{-1}{0} = -\infty \tag{73}$$

$$\text{cotan. } 270° = \frac{\cos. 270°}{\sin. 270°} = \frac{0}{-1} = 0 \tag{74}$$

$$\text{sec. } 270° = \frac{1}{\text{cos. } 270°} = \tfrac{1}{0} = \infty \qquad (75)$$

$$\text{cosec. } 270° = \frac{1}{\text{sin. } 270°} = \frac{1}{-1} = -1. \qquad (76)$$

57. *Problem. To find the sine, &c. of* 360°.

Solution. Make $A = 180°$ in (42) and (43); and they become by (65, 66, 59, 60)

$$\text{sin. } 360° = 0 = \text{sin. } 0° \qquad (77)$$

$$\text{cos. } 360° = 1 = \text{cos. } 0°. \qquad (78)$$

Hence the sine, &c. of 360° *are the same as those of* 0°.

58. *Problem. To find the sine, &c. of* 45°.

Solution. Make $C = 90°$ in (50) and (51). They become, by means of (59),

$$\text{cos. } 45° = \sqrt{[\tfrac{1}{2}(1 + \text{cos. } 90°)]} = \sqrt{\tfrac{1}{2}} \qquad (79)$$

$$\text{sin. } 45° = \sqrt{[\tfrac{1}{2}(1 - \text{cos. } 90°)]} = \sqrt{\tfrac{1}{2}} = \text{cos. } 45°. \qquad (80)$$

Hence, from (6) and (7),

$$\text{tang. } 45° = \frac{\text{sin. } 45°}{\text{cos. } 45°} = 1 \qquad (81)$$

$$\text{cotan. } 45° = \frac{1}{\text{tang. } 45°} = 1 = \text{tang. } 45° \qquad (82)$$

$$\text{sec. } 45° = \frac{1}{\text{cos. } 45°} = \frac{1}{\sqrt{\tfrac{1}{2}}} = \sqrt{2} \qquad (83)$$

$$\text{cosec. } 45° = \frac{1}{\text{sin. } 45°} = \frac{1}{\sqrt{\tfrac{1}{2}}} = \sqrt{2} = \text{sec. } 45°. \qquad (84)$$

59. *Problem. To find the sine, &c. of* 30° *and* 60°.

Solution. Make $A = 30°$ in (42). It becomes, from the consideration that 30° and 60° are complements of each other,

$$\sin. 60° = \cos. 30° = 2 \sin. 30° \cos. 30°.$$

Dividing by cos. 30°, we have

$$1 = 2 \sin. 30°,$$

or
$$\sin. 30° = \tfrac{1}{2} = \cos. 60° \qquad (85)$$

whence, from (6), (7), and (10),

$$\cos. 30° = \sin. 60° = \sqrt{(1 - \tfrac{1}{4})} = \tfrac{1}{2}\sqrt{3} \qquad (86)$$

$$\text{tang. } 30° = \text{cotan. } 60° = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \sqrt{\tfrac{1}{3}} \qquad (87)$$

$$\text{cotan. } 30° = \text{tang. } 60° = \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{3} \qquad (88)$$

$$\text{sec. } 30° = \text{cosec. } 60° = \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}} \qquad (89)$$

$$\text{cosec. } 30° = \text{sec. } 60° = \frac{1}{\frac{1}{2}} = 2. \qquad (90)$$

60. *Problem. To find the sine, &c. of the supplement of an angle.*

Solution. Make $M = 180°$ in (27) and (29). They become, by means of (65) and (66),

$$\sin. (180° - N) = \sin. 180° \cos. N - \cos. 180° \sin. N$$
$$= \sin. N \qquad (91)$$

$$\cos. (180° - N) = \cos. 180° \cos. N + \sin. 180° \sin. N$$
$$= -\cos. N \qquad (92)$$

whence, from (6) and (7),

$$\text{tang.}\ (180° - N) = - \text{tang.}\ N \qquad (93)$$

$$\text{cotan.}\ (180° - N) = - \text{cotan.}\ N \qquad (94)$$

$$\text{sec.}\ (180° - N) = - \text{sec.}\ N \qquad (95)$$

$$\text{cosec.}\ (180° - N) = \text{cosec.}\ N; \qquad (96)$$

that is, *the sine and cosecant of the supplement of an angle are the same with those of the angle itself and the cosine, tangent, cotangent, and secant of the supplement are the negative of those of the angle.*

61. *Corollary.* Since, when an angle is acute its supplement is obtuse, it follows from the preceding proposition, that *the sine and cosecant of an obtuse angle are positive, while its cosine, tangent, cotangent, and secant are negative.*

This proposition must be carefully borne in mind in using the trigonometric tables, as it affords the means of discriminating between the two angles which are given in B. Table XXVII, and of deciding which of these two angles is the required one.

62. *Corollary.* The preceding corollary might also have been obtained from (26) and (28). For by making $M = 90°$, we have by (59) and (60)

$$\text{sin.}\ (90° + N) = \text{cos.}\ N \qquad (97)$$

$$\text{cos.}\ (90° + N) = - \text{sin.}\ N; \qquad (98)$$

whence, by (6) and (7),

$$\text{tang.}\ (90° + N) = - \text{cotan.}\ N \qquad (99)$$

$$\text{cotan.}\,(90^\circ + N) = -\,\text{tang.}\,N \qquad (100)$$

$$\text{sec.}\,(90^\circ + N) = -\,\text{cosec.}\,N \qquad (101)$$

$$\text{cosec.}\,(90^\circ + N) = \text{sec.}\,N; \qquad (102)$$

that is, *the sine and cosecant of an angle, which exceeds* 90°, *are equal to the cosine and secant of its excess above* 90°, *while its cosine, tangent, cotangent, and secant are equal to the negative of the sine, cotangent, tangent, and cosecant of this excess.*

63. *Problem. To find the sine, &c. of a negative angle.*

Solution. Make $N = 0^\circ$ in (27) and (29). They become, by means of (59) and (60),

$$\text{sin.}\,(-N) = -\,\text{sin.}\,N \qquad (103)$$

$$\text{cos.}\,(-N) = \text{cos.}\,N \qquad (104)$$

whence, from (6) and (7),

$$\text{tang.}\,(-N) = -\,\text{tang.}\,N \qquad (105)$$

$$\text{cotan.}\,(-N) = -\,\text{cotan.}\,N \qquad (106)$$

$$\text{sec.}\,(-N) = \text{sec.}\,N \qquad (107)$$

$$\text{cosec.}\,(-N) = -\,\text{cosec.}\,N; \qquad (108)$$

so that *the cosine and secant of the negative of an angle are the same with those of the angle itself; and the sine, tangent, cotangent, and cosecant of the negative of the angle are the negative of those of the angle.*

Sine, &c. of an angle greater than 180°.

64. *Problem. To find the sine, &c. of an angle which exceeds* 180°.

Solution. Make $M = 180°$ in (26) and (28). They become, by means of (65) and (66),

$$\sin. (180° + N) = - \sin. N \qquad (109)$$

$$\cos. (180° + N) = - \cos. N \qquad (110)$$

whence, from (6) and (7),

$$\text{tang.} (180° + N) = \text{tang.} N \qquad (111)$$

$$\text{cotan.} (180° + N) = \text{cotan.} N \qquad (112)$$

$$\sec. (180° + N) = - \sec. N \qquad (113)$$

$$\text{cosec.} (180° + N) = - \text{cosec.} N; \qquad (114)$$

that is, *the tangent and cotangent of an angle, which exceeds* 180°, *are equal to those of its excess above* 180°; *and the sine, cosine, secant, and cosecant of this angle are the negative of those of its excess.*

65. *Corollary.* If the excess of the angle above 180° is less than 90°, the angle is contained between 180° and 270°; so that *the tangent and cotangent of an angle which exceeds* 180°, *and is less than* 270°, *are positive; while its sine, cosine, secant, and cosecant are negative.*

66. *Corollary.* If the excess of the angle above 180° is greater than 90° and less than 180°, the angle is contained between 270° and 360°; so that, by § 64 and 61, *the cosine and secant of an angle, which exceeds*

270° *and is less than* 360°, *is positive; while its sine, tangent, cotangent, and cosecant are negative.*

67. *Corollary.* The results of the two preceding corollaries might have been obtained from (27) and (29). For by making $M = 360°$, we have, by § 57,

$$\sin.(360° - N) = -\sin. N \qquad (115)$$

$$\cos.(360° - N) = \cos. N \qquad (116)$$

whence, by (6) and (7),

$$\text{tang.}(360° - N) = -\text{tang.} N \qquad (117)$$

$$\text{cotan.}(360° - N) = -\text{cotan.} N \qquad (118)$$

$$\sec.(360° - N) = \sec. N \qquad (119)$$

$$\text{cosec.}(360° - N) = -\text{cosec.} N \qquad (120)$$

that is, *the cosine and secant of an angle are the same with those of the remainder after subtracting the angle from* 360°; *while its sine, tangent, cotangent, and cosecant are the negative of those of this remainder.*

68. *Problem. To find the sine, &c. of an angle which exceeds* 360°.

Solution. Make $M = 360°$ in (26) and (28). They become, by means of (77) and (78),

$$\sin.(360° + N) = \sin. N \qquad (121)$$

$$\cos.(360° + N) = \cos. N \qquad (122)$$

that is, *the sine, &c. of an angle which exceeds* 360° *are equal to those of its excess above* 360°.

69. *Theorem. The sine, tangent, and secant of an acute angle increase with the increase of the angle; the cosine, cotangent, and cosecant decrease.*

Proof. I. The excess of the sine of $M + m$ over the sine of M is, by (13), equal to $\sin. m \cos. M$, which is a positive quantity when M is acute; and, therefore, the sine of the acute angle increases with the increase of the angle.

II. The excess of $\cos. M$ over $\cos. (M + m)$ is, by (15), equal to $\sin. m \sin. M$, which is a positive quantity; and, therefore, the cosine of the acute angle decreases with the increase of the angle.

III. The tangent of an angle is, by (7), the quotient of its sine divided by its cosine. It is, therefore, a fraction whose numerator increases with the increase of the angle, while its denominator decreases. Either of these changes in the terms of the fraction would increase its value; and, therefore, the tangent of an acute angle increases with the increase of the angle.

IV. The cosecant, secant, and cotangent of an angle are, by (6), the respective reciprocals of the sine, cosine, and tangent. But the reciprocal of a quantity increases with the decrease of the quantity, and the reverse. It follows, then, from the preceding demonstrations, that its secant increases with the increase of the acute angle, while its cosecant and cotangent decrease.

70. *Theorem. The absolute values (neglecting their signs) of the sine, tangent, and secant of an obtuse*

angle decrease with the increase of the angle; while those of the cosine, cotangent, and cosecant increase.

Proof. The supplement of an obtuse angle is an acute angle, of which the absolute values of the sine, &c. are, by § 60, the same as those of the angle itself. But this acute angle decreases with the increase of the obtuse angle, and at the same time its sine, tangent, and secant decrease, while its cosine, cotangent, and cosecant increase.

CHAPTER VI.

OBLIQUE TRIANGLES.

71. *Theorem. The sides of a triangle are directly proportional to the sines of the opposite angles.* [B. p. 13]

Proof. In the triangle ABC (figs. 2 and 3), denote the sides opposite the angles A, B, C, respectively, by the letters a, b, c. We are to prove that

$$\sin. A : \sin. B : \sin. C = a : b : c. \qquad (123)$$

From the vertex B, let fall on the opposite side the perpendicular BP, which we will denote by the letter p. Then, in the triangle BAP, we have by (1)

$$\sin. A = \frac{BP}{AB} = \frac{p}{c},$$

or

$$p = c \sin. A. \qquad (124)$$

Also, in the triangle BPC, we have, by (1) and (91), and from the consideration that BCP is the angle C (fig. 2.), and its supplement (fig. 3.),

$$\sin. BCP = \sin. C = \frac{BP}{BC} = \frac{p}{a},$$

or

$$p = a \sin. C. \qquad (125)$$

Comparing (124) and (125), we have

$$c \sin. A = a \sin. C,$$

which may be converted into the following proportion

$$\sin. A : \sin. C = a : c.$$

In the same way, it may be proved that

$$\sin. A : \sin. B = a : b;$$

and these two proportions may be written in one as in (123).

72. *Problem. To solve a triangle when one of its sides and two of its angles are known.* [B. p. 41.]

Solution. First. The third angle may be found by subtracting the sum of the two given angles from 180°.

Secondly. To find either of the other sides, we have only to make use of a proportion, derived from § 71. As the sine of the angle opposite the given side is to the sine of the angle opposite the required side, so is the given side to the required side. Thus, if a (fig. 1.) were the given and b the required side, we should have the proportion

$$\sin. A : \sin. B = a : b;$$

whence by (6)

$$b = \frac{a \sin. B}{\sin. A} = a \sin. B \text{ cosec. } A. \qquad (126)$$

73. Examples.

1. Given one side of a triangle equal to 22.791, and the adjacent angles equal to 32° 41′ and 47° 54′; to solve the triangle.

Solution. The other angle $= 180° - (32° \ 41' + 47° \ 54')$ $= 99° \ 25'$.

Given two sides and an angle opposite one of them.

By (126)

99° 25′ cosec.	10.00589		10.00589
32° 41′ sin.	9.73239	47° 54′ sin.	9.87039
22.791	1.35776		1.35776
12.475	*1.09604 ;	17.141	*1.23404.

Ans. The other angle = 99° 25′

The other sides = $\begin{cases} 12.475 \\ 17.141 \end{cases}$

2. Given one side of a triangle equal to 327.06, and the adjacent angles equal to 154° 22′ and 17° 35′; to solve the triangle.

Ans. The other angle = 8° 3′

The other sides = $\begin{cases} 1010.4 \\ 705.5 \end{cases}$

74. *Problem. To solve a triangle when two of its sides and an angle opposite one of the given sides are known.* [B. p. 42.]

Solution. First. The angle opposite the other given side is found by the proportion of § 71. As the side opposite the given angle is to the other given side, so is the sine of the given angle to the sine of the required angle. Thus, if (fig. 1.) a and b are the given sides and A the given angle, the angle B is found, by the proportion

* 20 is subtracted from each of these characteristics, because the two sines and cosecant were taken from the tables without any diminution, as required by § 30.

Given two sides and an angle opposite one of them.

$$a : b = \sin. A : \sin. B;$$

whence

$$\sin. B = \frac{b \sin. A}{a}. \qquad (127)$$

Secondly. The third angle is found by subtracting the sum of the two known angles from 180°.

Thirdly. The third side is found by the proportion. As the sine of the given angle is to the sine of the angle opposite the required side, so is the side opposite the given angle to the required side. That is, in the present case,

$$\sin. A : \sin. C = a : c;$$

whence

$$c = \frac{a \sin. C}{\sin. A}, = a \sin. c \text{ cosec. } A. \qquad (128)$$

75. *Scholium.* Two angles are given in the tables corresponding to the same sine, which are supplements of each other, one being acute and the other obtuse. Two values of B (127) are then given in the tables, and both these values may be possible, when the given value of b is greater than that of a, and the given value of A is less than 90°; for, in this case, there may be two triangles, ABC (fig. 11.) and $AB'C$, which satisfy the data.

76. *Scholium.* The problem is impossible, when the given value of b is greater than that of a, and the given value of A is obtuse. For the greatest side of an obtuse-angled triangle must always be opposite the obtuse angle.

77. *Scholium.* The problem is impossible, when the given value of b is so much greater than that of a, that we have

Given two sides and an angte opposite one of them.

$$b \text{ sin. } A > a;$$

for, in this case, the given value of a is less than that of the perpendicular CP (fig. 11.), from C upon AP.

78. *Scholium.* The obtuse value of B does not satisfy the problem, when b is less than a; for the obtuse angle of a triangle cannot be opposite a smaller side. In this case, therefore, the problem admits of only one solution.

79. Examples.

1. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the first side equal to 55° 28′ 12″, to solve the triangle.

Solution. Making

$$b = 92{\cdot}341,\ a = 77{\cdot}245,\ A = 55^\circ\ 28'\ 12'',$$

we have, by (127),

$a = 77{\cdot}245$		(ar. co.)	8·11213
$b = 92{\cdot}341$			1·96540
$A = 55^\circ\ 28'\ 12''$		sin.	9·91584
$B = 80'\ 1'$	or $= 99^\circ\ 59'$	sin.	9·99337
$A + B = 135^\circ\ 29'\ 12''$	or $= 155^\circ\ 27'\ 12''$		
$C = 44^\circ\ 30'\ 48''$	or $= 24^\circ\ 32'\ 48''$		

Then, by (128),

Given two sides and an angle opposite one of them.

$a = 77{\cdot}245$		1·88787		1·88787
$C = 44° \, 30' \, 48''$	sin.	9·84576	or $= 24° \, 32' \, 48''$ sin.	9·61850
$A = 55° \, 28' \, 12''$	cosec.	10·08416		10·08416
$C = 65{\cdot}734$		1·81779	or $= 38{\cdot}952$	1·59053

Ans. The third side = 65·734 or = 38·952

$$\text{The other angles} = \begin{cases} 80° \, 1' \\ 44° \, 30' \, 48'' \end{cases} \text{or} = \begin{cases} 99° \, 59' \\ 24° \, 32' \, 48'' \end{cases}$$

2. Given two sides of a triangle equal to 77·245 and 92·341, and the angle opposite the second side equal to 55° 28′ 12″; to solve the triangle.

Ans. The third side = 110·7

$$\text{The other angles} = \begin{cases} 43° \, 33' \, 44'' \\ 80° \, 58' \, 4'' \end{cases}$$

3. Given two sides of a triangle equal to 40 and 50, and the angle opposite the first side equal to 45°, to solve the triangle.

Ans. The third side = 54.061 or = 16·65

$$\text{The other angles} = \begin{cases} 62° \, 7' \\ 72° \, 53' \end{cases} \text{or} = \begin{cases} 117° \, 53' \\ 17° \, 7' \end{cases}$$

4. Given two sides of a triangle equal to 77·245 and 92·341, and the angle opposite the second side equal to 124° 31′ 48″, to solve the triangle.

Ans. The third side = 23·129

$$\text{The other angles} = \begin{cases} 43° \, 33' \, 44'' \\ 11° \, 54' \, 28'' \end{cases}$$

5. Given two sides of a triangle equal to 77·245 and 92·341, and the angle opposite the first side equal to 124° 31′ 48″, to solve the triangle.

Ans. The question is impossible.

6. Given two sides of a triangle equal to 75·486 and 92·341, and the angle opposite the first side equal to 55° 28′ 12″, to solve the triangle.

Ans. The question is impossible.

80. *Theorem. The sum of two sides of a triangle is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.* [B. p. 13.]

Proof. We have (fig. 1.)

$$a : b = \text{sin. } A : \text{sin. } B\,;$$

whence, by the theory of proportions,

$$a + b : a - b = \text{sin. } A + \text{sin. } B : \text{sin. } A - \text{sin. } B,$$

which, expressed fractionally, is

$$\frac{a+b}{a-b} = \frac{\text{sin. } A + \text{sin. } B}{\text{sin. } A - \text{sin. } B}.$$

But, by (40),

$$\frac{\text{sin. } A + \text{sin. } B}{\text{sin. } A - \text{sin. } B} = \frac{\text{tang. } \frac{1}{2}\,(A+B)}{\text{tang. } \frac{1}{2}\,(A-B)}\,;$$

whence

$$\frac{a+b}{a-b} = \frac{\text{tang. } \frac{1}{2}\,(A+B)}{\text{tang. } \frac{1}{2}\,(A-B)} \qquad (129)$$

or

$$a + b : a - b = \text{tang. } \tfrac{1}{2}\,(A + B) : \text{tang. } \tfrac{1}{2}\,(A - B).$$

81. *Problem. To solve a triangle when two of its sides and the included angle are given.* [B. p. 43.]

Solution. Let the two sides a and b (fig. 1.) be given, and the included angle C, to solve the triangle.

First. To find the other two angles. Subtract the given angle C from 180°, and the remainder is the sum of A and B, for the sum of the three angles of a triangle is 180°, that is,

$$A + B = 180° - C,$$

and

$$\tfrac{1}{2}\,(A + B) = 90° - \tfrac{1}{2}\,C = \text{complement of } \tfrac{1}{2}\,C.$$

The difference of A and B is then found by (129)

$$a + b : a - b = \text{tang. } \tfrac{1}{2}\,(A + B) : \text{tang. } \tfrac{1}{2}\,(A - B).$$

But we have

$$\text{tang. } \tfrac{1}{2}\,(A + B) = \text{cotan. } \tfrac{1}{2}\,C;$$

whence

$$\text{tang. } \tfrac{1}{2}\,(A - B) = \frac{a-b}{a+b}\text{tang.}\tfrac{1}{2}(A+B) = \frac{a-b}{a+b}\text{cotan.}\tfrac{1}{2}C. \quad (130)$$

The greater angle, which must be opposite the greater side, is then found by adding their half sum to their half difference; and the smaller angle by subtracting the half difference from the half sum.

Secondly. The third side is found by the proportion

$$\sin.\ A : \sin.\ C = a : c;$$

whence

$$c = \frac{a \sin. C}{\sin. A}.$$

82. Examples.

1. Given two sides of a triangle equal to 99·341 and 1.234, and their included angle equal to 169° 58′, to solve the triangle.

Solution. Making $a = 99.341$, $b = 1.234$; and

$$C = 169° \ 58', \ \tfrac{1}{2} C = 84° \ 59';$$

we have, by (130),

$a + b =$ 100.575	(ar. co.)	7.99751
$a - b =$ 98.107		1.99170
$\frac{1}{2}(A + B) =$ 5° 1′	tang.	8.94340
$\frac{1}{2}(A - B) =$ 4° 53′ 39″	tang.	8.93261
$A =$ 9° 54′ 39″		
$B =$ 0° 7′ 21′		
$a =$ 99.341		1.99712
$C =$ 169° 58′	sin.	9.24110
$A =$ 9° 54′ 39″	cosec.	10.76416
$c =$ 100.55		2.00238

Ans. The third side = 100.55

The other angles = { 9° 54′ 39′ ; 0° 7′ 21″ }

Segments of base made by perpendicular from opposite vertex.

2. Given two sides of a triangle equal to 0.121 and 5.421 and the included angle equal to $1° 2'$; to solve the triangle.

Ans. The other side $= 5.294$

The other angles $= \begin{cases} 178° \ 56' \ 35'' \\ 0° \ \ 1' \ 25'' \end{cases}$

83. *Theorem. One side of a triangle is to the sum of the other two, as their difference is to the difference of the segments of the first side made by a perpendicular from the opposite vertex, if the perpendicular fall within the triangle; or to the sum of the distances of the extremities of the base from the foot of the perpendicular, if it fall without the triangle.* [B. p. 14.]

Proof. Let AB (figs. 12 and 13) be the side of triangle ABC on which the perpendicular is let fall, and BP the perpendicular.

From B as a centre with a radius equal to BC, the shorter of the other two sides, describe the circumference $CC'E'E$. Produce AB to E' and AC to C', if necessary.

Then, since AC and AB are secants, we have,

$$AC : AE' = AE : AC'.$$

But

$$AE' = AB + BE' = AB + BC$$

$$AE = AB - BE = AB - BC$$

and

$$\text{(fig. 12.) } AC' = AP - PC' = AP - PC$$

$$\text{(fig. 13.) } AC' = AP + PC' = AP + PC$$

whence

$$\text{(fig. 12.) } AC : AB + BC = AB - BC : AP - PC$$

$$\text{(fig. 13.) } AC : AB + BC = AB - BC : AP + PC$$

84. *Problem. To solve a triangle when its three sides are given.* [B. p. 43.]

Solution. On the side b (figs. 2 and 3.) let fall the perpendicular BP.

Then, by § 83,

$$\text{(fig. 2.) } b : c + a = c - a : PA - PC$$

$$\text{(fig. 3.) } b : c + a = c - a : PA + PC.$$

These proportions give the difference of the segments (fig. 2.), or their sum (fig. 3.). Then, adding the half difference to the half sum, we obtain the larger segment corresponding to the larger of the two sides a and c. And, subtracting the half difference from the half sum, we obtain the smaller segment.

Then, in triangles BCP and ABP, we have, by (4) and (92),

$$\cos. A = \frac{AP}{c};$$

and

$$\text{(fig. 2.) } \cos. C = \frac{PC}{a},$$

$$\text{(fig. 3.) } \cos. C = -\cos. BCP = -\frac{PC}{a}.$$

The third angle B is found by subtracting the sum of A and C from 180°.

85. *Corollary.* From the preceding section, we have

Given the three sides.

$$\text{(fig. 2.) } PA - PC = \frac{(c + a)(c - a)}{b} = \frac{c^2 - a^2}{b}$$

$$\text{(fig. 3.) } PA + PC = \frac{(c + a)(c - a)}{b} = \frac{c^2 - a^2}{b}$$

which, added to

$$\text{(fig. 2.) } PA + PC = AC = b$$
$$\text{(fig. 3.) } PA - PC = AC = b$$

gives

$$2\,P\,A = \frac{c^2 - a^2}{b} + b = \frac{b^2 + c^2 - a^2}{b}$$

Hence

$$P\,A = \frac{b^2 + c^2 - a^2}{2\,b}$$

and

$$\cos. A = \frac{PA}{c} = \frac{b^2 + c^2 - a^2}{2\,b\,c} \qquad (131)$$

86. *Corollary.* If (131) is cleared from fractions it becomes by transposition

$$a^2 = b^2 + c^2 - 2\,bc \cos. A. \qquad (132)$$

87. *Corollary.* Add unity to both sides of (131) and we have

$$1 + \cos. A = \frac{b^2 + c^2 - a^2}{2\,b\,c} + 1 = \frac{b^2 + 2\,b\,c + c^2 - a^2}{2\,b\,c}$$

$$= \frac{(b + c)^2 - a^2}{2\,b\,c} \qquad (133)$$

Since the numerator of (133) is the difference of two squares, it may be separated into two factors, and we have

$$1 + \cos. A = \frac{(b + c + a)(b + c - a)}{2\,b\,c}.$$

Now, representing half the sum of the three sides of a triangle by s, we have

$$2\,s = a + b + c, \qquad (134)$$

and

$$2\,s - 2\,a = 2\,(s - a) = a + b + c - 2\,a = b + c - a. \qquad (135)$$

If we substitute these values in the above equation, it becomes

$$1 + \cos. A = \frac{4\,s\,(s - a)}{2\,b\,c} = \frac{2s\,(s - a,}{b\,c}. \qquad (136)$$

But, by (48),

$$1 + \cos. A = 2\,(\cos. \tfrac{1}{2} A)^2.$$

Hence

$$2\,(\cos. \tfrac{1}{2} A)^2 = \frac{2\,s\,(s - a)}{b\,c}$$

or

$$(\cos. \tfrac{1}{2} A)^2 = \frac{s\,(s - a)}{b\,c} \qquad (137)$$

$$\cos. \tfrac{1}{2} A = \sqrt{\left(\frac{s\,(s - a)}{b\,c}\right)}, \qquad (138)$$

which corresponds to proposition LXI. of B. p. 14.

In the same way, we have

$$\cos. \tfrac{1}{2} B = \sqrt{\left(\frac{s\,(s - b)}{a\,c}\right)} \qquad (139)$$

$$\cos. \tfrac{1}{2} C = \sqrt{\left(\frac{s\,(s - c)}{a\,b}\right)}. \qquad (140)$$

Given the three sides.

88. *Corollary.* Subtract both sides of (131) from unity, and we have

$$1 - \cos. A = 1 - \frac{b^2 + c^2 - a^2}{2\,b\,c} = \frac{a^2 + 2\,b\,c - b^2 - c^2}{2\,b\,c}$$

$$= \frac{a^2 - (b - c)^2}{2\,b\,c}. \qquad (141)$$

Since the numerator of (141) is the difference of two squares, it may be separated into two factors, as follows,

$$1 - \cos. A = \frac{(a - b + c)\,(a + b - c)}{2\,b\,c}.$$

But from (134)

$$2\,s - 2\,b = 2\,(s - b) = a + b + c - 2\,b = a - b + c \quad (142)$$

$$2\,s - 2\,c = 2\,(s - c) = a + b + c - 2\,c = a + b - c. \quad (143)$$

If we substitute these values in the above equations, it becomes

$$1 - \cos. A = \frac{4\,(s - b)\,(s - c)}{2\,b\,c} = \frac{2\,(s - b)\,(s - c)}{b\,c}. \quad (144)$$

But, by (49),

$$1 - \cos. A = 2\,(\sin. \tfrac{1}{2}\,A)^2.$$

Hence, by reduction,

$$\sin. \tfrac{1}{2}\,A = \sqrt{\left(\frac{(s - b)\,(s - c)}{b\,c}\right)}. \quad (145)$$

In the same way, we have

$$\sin. \tfrac{1}{2}\,B = \sqrt{\left(\frac{(s - a)\,(s - c)}{a\,c}\right)} \quad (146)$$

$$\sin. \tfrac{1}{2}\,C = \sqrt{\left(\frac{(s - a)\,(s - b)}{a\,b}\right)}. \quad (147)$$

Given the three sides.

89. *Corollary.* The quotients of (145, 146, and 147), divided by (138, 139, and 140), are by (7)

$$\text{tang. } \tfrac{1}{2} A = \sqrt{\left(\frac{(s-b)\,(s-c)}{s\,(s-a)}\right)} \tag{148}$$

$$\text{tang. } \tfrac{1}{2} B = \sqrt{\left(\frac{(s-a)\,(s-c)}{s\,(s-b)}\right)} \tag{149}$$

$$\text{tang. } \tfrac{1}{2} C = \sqrt{\left(\frac{(s-a)\,(s-b)}{s\,(s-c)}\right)}. \tag{150}$$

90. The product of (136) by (144) is

$$1 - (\text{cos. } A)^2 = \frac{4\,s\,(s-a)\,(s-b)\,(s-c)}{b^2\,c^2}.$$

But from (10)

$$1 - (\text{cos. } A)^2 = (\text{sin. } A)^2.$$

Hence

$$(\text{sin. } A)^2 = \frac{4\,s\,(s-a)\,(s-b)\,(s-c)}{b^2\,c^2},$$

or

$$\text{sin. } A = \frac{2\sqrt{[s\,(s-a)\,(s-b)\,(s-c)]}}{b\,c}. \tag{151}$$

91. *Scholium.* The problem would be impossible, if the given value of either side exceeded the sum of the other two.

92. Examples.

1. Given the three sides of a triangle equal to 12.348, 13.561, 14.091; to solve the triangle

Solution. First Method.

Make (fig. 2.) $a = 12.348$ $b = 13.561$
$c = 14.091.$

Then by § 84

$b = 13.561$	(ar. co.) 8.86771	
$c + a = 26.439$	1.42224	
$c - a = 1.743$	0.24130	
$PA - PC = 3.3982$	0.53125	
$PA = 8.4796$	0.92838	
$PC = 5.0814$		0.70598
$c = 14.091$	(ar. co.) 8.85106	
$a = 12.348$		(ar. co.) 8.90840
$A = 53°\ 0'$	cos. 9.77944	
$C = 65°\ 42'$		cos. 9.61438

$$B = 180° - (A + C)$$

$$= 180° - 118°\ 42' = 61°\ 18'$$

Given the three sides.

Second Method.

By (138, 139, and 140),

$a = 12.348$		(ar. co.) 8.90840	(ar. co.) 8.90840
$b = 13.561$	(ar. co.) 8.86771		(ar. co.) 8.86771
$c = 14.091$	(ar. co.) 8.85106	(ar. co.) 8.85106	
$s = 20.000$	1.30103	1.30103	1.30103
$s-a=7.652$	0.88377		
$s-b=6.439$		0.80882	
$s-c=5.909$			0.77151
	2 | 19.90357	2 | 19.86931	2 | 19.84865
cos.	9.95179	9.93466	9.92433

$$\tfrac{1}{2} A = 26^\circ\ 30', \quad \tfrac{1}{2} B = 30^\circ\ 39', \quad \tfrac{1}{2} C = 32^\circ\ 51'$$

$$A = 53^\circ\ 0', \quad B = 61^\circ\ 18', \quad C = 65^\circ\ 42'.$$

Ans. The angles $= \begin{cases} 53^\circ\ 0' \\ 61^\circ\ 18' \\ 65^\circ\ 42'. \end{cases}$

In the same way equations (145 – 147) would furnish a third method, (148 – 150) a fourth method, and (151) a fifth method.

2. Given the three sides of a triangle equal to 17.856, 13.349, and 11.111; to solve the triangle.

Ans. The angles $= \begin{cases} 93^\circ\ 19'\ 16'' \\ 48^\circ\ 16'\ 24'' \\ 38^\circ\ 24'\ 20''. \end{cases}$

NAVIGATION AND SURVEYING.

NAVIGATION AND SURVEYING.

CHAPTER I.

PLANE SAILING.

1. The daily revolution of the earth is performed around a straight line, passing through its centre, which is called the *earth's axis*.

The extremities of this axis on the surface of the earth are the *terrestrial poles*, one being the *north pole*, and the other the *south pole*.

The section of the earth, made by a plane passing through its centre and perpendicular to its axis, is the *terrestrial equator*. [B. p. 48.]

2. *Parallels of latitude* are the circumferences of small circles, the planes of which are parallel to the equator.

3. *Meridians* are the semicircumferences of great circles, which pass from one pole to the other.

The *first meridian* is one arbitrarily assumed, to which all others are referred. In most countries, that has been taken as the first meridian which passes through the capital of the country. But, in the United States, we have usually adhered to the English custom, and we consider the meridian, which passes through the Observatory of Greenwich, as the first meridian. [B. p. 48.]

4. The *latitude* of a place is its angular distance from the equator, the vertex of the angle being at the centre of the earth; or, it is the arc of the meridian, passing through the place, which is comprehended between the place and the equator. [B. p. 48.]

Latitude is reckoned north and south of the equator from 0° to 90°.

5. The *difference of latitude* of two places is the angular distance between the parallels of latitude in which they are respectively situated, the vertex of the angle being at the centre of the earth; or it is the arc of a meridian which is comprehended between the parallels of latitude. [B. p. 52.]

The difference of latitude of two places is equal to the difference of their latitudes, if they are on the same side of the equator; and to the sum of their latitudes, if they are on opposite sides of the equator. [B. p. 50.]

6. The *longitude* of a place is the angle made by the plane of the first meridian with the plane of the meridian passing through the place; or it is the arc of the equator comprehended between these two meridians. [B. p. 48.]

Longitude is reckoned East and West of the first meridian from 0° to 180°.

7. The *difference of longitude* of two places is the angle contained between the planes of the meridians passing through the two places; or it is the arc of the equator comprehended between these two meridians.

The difference of longitude of two places is equal to the difference of their longitudes, if they are on the same side of the first meridian; and to the sum of their longitudes, if they are on opposite sides of the first meridian, unless their sum be greater than 180°; *in which case the sum must be subtracted from* 360° *to give the difference of longitude.* [B. p. 50.]

8. The *distance* between two places in Navigation is the portion of a curve which would be described by a ship sailing from one place to the other in a path, which crosses every meridian at the same angle. [B. p. 52.]

9. The *course* of the ship, or the *bearing* of the two places from each other, is the angle which the ship's path makes with the meridian. [B. p. 52.]

10. The *departure* of two places is the distance of either from the meridian of the other, when they are so near each other that the earth's surface may be considered as plane and its curvature neglected. But, if the two places are at a great distance from each other,

the distance is to be divided into small portions, and the *departure* of the two places is the sum of the departures corresponding to all these portions.

11. Instead of dividing the quadrant into 90 degrees, navigators are in the habit of dividing it into eight equal parts called *points;* and of subdividing the points into halves and quarters. A point, therefore, is equal to one eighth of 90°, or to 11° 15′. [B. p. 52.]

Names are given to the directions determined by the different points, as in the diagram (fig .14), which represents the face of the card of the *Mariner's Compass.*

The *Mariner's Compass* consists of this card, attached to a magnetic needle, which has the property of constantly pointing toward the north and thereby determining the ship's course.

On page 53 of the Navigator a table is given of the angles which every *point* of the compass makes with the meridian, and on page 169, table XXV. the log., sines, &c. are given.

12. The object of *Plane Sailing* is to calculate the Distance, Course or Bearing, Difference of Latitude and Departure, when either two of them are known. [B. p. 52.]

13. *Problem. To find the difference of latitude and departure, when the distance and course are known.* [B. p. 54.]

Solution. First. When the distance is so small that the curvature of the earth's surface may be neglected. Let *A B* (fig. 15.) be the distance. Draw through A the meridian AC, and let fall on it the perpendicular BC. The angle A is the course, AC is the difference of latitude, and BC is the departure. Then, by (17, and 18.)

$$\text{Diff. of lat.} = \text{dist.} \times \text{cos. course.} \qquad (152)$$

$$\text{Departure} = \text{dist.} \times \text{sin. course.} \qquad (153)$$

Secondly. When the distance is great, as AB (fig. 16), then divide it into smaller portions, as Aa, ab, bc, &c. Through the points of division, draw the meridians AN, an, bp, &c. Let fall the perpendiculars am, bn, cp, &c. Then, as the course is every where the same, each of the angles mAa, nab, pbc, &c. is equal to the angle A, or the course. Moreover, the distances, Am, an, bp, &c. are the differences of latitude respectively of A and a, a and b, b and c, &c. Also am, bn, cp, &c. are the departures of the points A and a, a and b, b and c, &c. Therefore, as the difference of latitude of A and B is evidently equal to the sum of these partial differences of latitude; and as the departure of A and B is by § 10 equal to the sum of the partial departures, we have

$$\text{Diff. of lat.} = Am + an + bp + \text{\&c.}$$

$$\text{Departure} = am + bn + cp + \text{\&c.}$$

But the right triangles mAa, nab, pbc, &c. give by (152, and 153.)

$$Am = Aa \times \text{cos. course}, \quad am = Aa \times \text{sin. course};$$

$$an = ab \times \text{cos. course}, \quad bn = ab \times \text{sin. course};$$

$$bp = bc \times \text{cos. course}, \quad cp = bc \times \text{sin. course}.$$

&c. &c.

Given course and departure.

The sums of these equations give

$$\text{Diff. of lat.} = A\,m + a\,n + b\,c + \&c.$$
$$= (A\,a + a\,b + b\,c + \&c.) \times \text{cos. course},$$
$$\text{Departure} = a\,m + b\,n + c\,p + \&c.$$
$$= (A\,a + a\,b + b\,c + \&c.) \times \text{sin. course}.$$

But

$$A\,a + a\,b + b\,c + \&c. = AB = \text{distance}.$$

Hence,

$$\text{Diff. of lat.} = \text{dist.} \times \text{cos. course},$$
$$\text{Departure} = \text{dist.} \times \text{sin. course};$$

precisely the same with (152) and (153).

This shows that the method of calculating the difference of latitude and departure is the same for all distances, and that *all the problems of Plane Sailing may be solved by the right triangle* (fig. 15.) [B. p. 52.]

A table of difference and latitude and departure are given in pages 1 – 6, Tables I. and II. of the Navigator. which might be calculated by (152 and 153.)

14. *Problem. To find the distance and difference of latitude, when the course and departure are known.* [B. p. 55.]

Solution. There are given (fig. 15.) the angle A and the side $B\,C$. Hence, by (19, and 20),

$$\text{Distance} = \text{departure} \times \text{cosec. course}. \qquad (154)$$
$$\text{Diff. of lat.} = \text{departure} \times \text{cotan. course}. \qquad (155)$$

15. *Problem. To find the distance and departure, when the course and difference of latitude are known.* [B. p. 55.]

Solution. There are given (fig. 15.) the angle A and the side AC. Then, by (21, and 22).

$$\text{Distance} = \text{diff. of lat.} \times \text{sec. course.} \qquad (156)$$

$$\text{Departure} = \text{diff. of lat.} \times \text{tang. course.} \qquad (157)$$

16. *Problem. To find the course and difference of latitude, when the distance and departure are known.* [B. p. 57.]

Solution. There are given (fig. 15.) the hypothenuse AB and the side BC. Then, by (23, and 25),

$$\text{sin. course} = \frac{\text{departure}}{\text{distance}}, \qquad (158)$$

$$\text{Diff. of lat.} = \sqrt{[(\text{dist.}^2) - (\text{departure})^2]}. \qquad (159)$$

17. *Problem. To find the course and departure, when the distance and difference of latitude are known.* [B. p. 56.]

Solution. There are given (fig. 15.) the hypothenuse AB and the leg AC. Then, by (23 and 25),

$$\text{cos. course} = \frac{\text{diff. of lat.}}{\text{distance.}}, \qquad (160)$$

$$\text{Departure} = \sqrt{[(\text{dist.})^2 - (\text{diff. of lat.})^2]}. \qquad (161)$$

18. *Problem. To find the course and distance, when the departure and difference of latitude are known.* [B. p. 57.]

Examples.

Solution. There are given (fig. 15.) the legs AC and BC. Then,

$$\text{tang. course} = \frac{\text{departure}}{\text{diff. of. lat.}} \qquad (162)$$

$$\text{Dist.} = \text{diff. of lat.} \times \text{sec. course.} \qquad (163)$$

19. Examples.

1. A ship sails from latitude 3° 45′ S., upon a course N. by E., a distance of 2345 miles; to find the latitude at which it arrives, and the departure which it makes.

Ans. Latitude = 34° 35′ N.
Departure = 458 miles.

2. A ship sails from latitude 62° 19′ N., upon a course W. N. W., till it makes a departure of 1000 miles; to find the latitude at which it arrives, and the distance sailed.

Ans. Latitude = 69° 13′ N.
Distance = 1082 miles.

3. The bearing of Paris from Athens is N. 54° 56′ W.; find the distance and departure of these two places from each other.

Ans. Distance = 1135 miles.
Departure = 929 miles.

4. A ship sails from latitude 72° 3′ S. a distance of 2000 miles, upon a course between the north and the west, that is, *northwesterly*, until it makes a departure of 1000 miles; find the latitude at which it arrives, and the course.

Examples.

Ans. Latitude = 43° 11′ S.
Course = N. 30° W.

5. The distance from New Orleans to Portland is 958 miles; find the bearing and departure.

Ans. Bearing = N. 49° 24′ E.
Departure = 1257 miles.

6. The departure of Boston from Canton is 8790 miles; find the bearing and distance.

Ans. Bearing = N. 82° 31′ E.
Distance = 8865 miles.

CHAPTER II.

TRAVERSE SAILING.

20. A traverse is an irregular track made by a ship when sailing on several different courses.

The object of *Traverse Sailing* is to reduce a traverse to a single course, where the distances sailed are so small that the earth's surface may be considered as a plane. [B. p. 59.]

21. *Problem. To reduce several successive tracks of a ship to one; that is, to find the single track, leading to the place, which the ship has actually reached, by sailing on a traverse.* [B. p. 59.]

Solution. Suppose the ship, to start from the point A (fig. 17.) and to sail, first from A to B, then from B to C, then from C to E, and lastly from E to F; to find the bearing and distance of F from A. Call the differences of latitude, corresponding to the 1st, 2d, 3d, and 4th tracks, the 1st, 2d, 3d, and 4th differences of latitude; and call the corresponding departures the 1st, 2d, 3d, and 4th departures. Then we need no demonstration to prove that,

Diff. of lat. of A and F = 1st diff. of lat. — 2d diff. of lat. + 3d diff. of lat. — 4th diff. &c.;

or that *the difference of latitude* of A and F *is found* *by taking the sum of the differences of latitude corresponding to the northerly courses, and also the sum of those corresponding to the southerly courses, and the difference of these sums is the required difference of latitude.*

By neglecting the earth's curvature, we also have,

Dep. of A and $F =$ 1st dep. — 2d dep. — 3d dep. + 4th dep.

or *the departure* of A and F *is found by taking the sum of the departures corresponding to the easterly courses, also the sum of those corresponding to the westerly courses; and the difference of these sums is the required departure.*

Having thus found the difference of latitude and departure of A and F, their distance and bearing are found by § 18.

22. The calculations of traverse sailing are usually put into a tabular form, as in the following example. In the *first* column of the table are the numbers of the courses; in the *second* and *third* columns are the courses and distances; in the *fourth* and *fifth* columns are the differences of latitude, the column, headed N, corresponding to the northerly courses, and that headed S, to the southerly courses; in the *sixth* and *seventh* columns are the departures, the column, headed E, corresponding to the easterly courses, and that, headed W, to the westerly courses. [B. p. 59.]

23. EXAMPLES.

1. A ship sails on several successive tracks, in the order and with the courses and distances of the first three columns of the following table; find the bearing and distance of the place at which the ship arrives, from that from which it started.

No.	Course.	Dist.	N.	S.	E.	W.
1	N. N. E.	30	27.7		11.5	
2	N. W.	80	56.6			56.6
3	West.	60				60.0
4	S. E. by S.	55		45.7	30.6	
5	North.	43	43.0			
6	S. by W.	152		149.1		29.7
	Sum of columns,		127.3	194.8	42.1	146.3
				127.3		42.1

Diff. lat. = 67.5 S. dep.=104.2W.

Dep. = 104.2	2.01787		
Diff. of lat. = 67.5 (ar. co.)	8.17070		1.82930
Bearing = 57° 4′ tang.	0.18857	sec.	0.26467
Dist. = 124.1			2.09397

Ans. Bearing = S. 57° 4′ W.

Distance = 124.1 miles.

2. A ship sails on the following successive tracks, South 10 miles, W. S. W. 25 miles, S. W. 30 miles, and West 20 miles; it is bound to a port which is at a distance of 100 miles from the place of starting, and its bearing is W. by N.

Examples.

Required the bearing and distance of the port to which the ship is bound, from the place at which it has arrived.

Ans. Bearing = S. 51° 47′ W.
Distance = 239 miles.

CHAPTER III.

PARALLEL SAILING.

24. *Parallel Sailing* considers only the case where the ship sails exactly east or west, and therefore remains constantly on the same parallel of latitude. Its object is to find the change in longitude corresponding to the ship's track. [B. p. 63.]

25. *Problem. To find the difference of longitude in parallel sailing.* [B. p. 65.]

Solution. Let AB (fig. 18.) be the distance sailed by the ship on the parallel of latitude AB. As the course is exactly east or west, the distance sailed must be itself equal to its departure.

The latitude of the parallel is ADA' or AA'. The angle $AEB = A'DB'$, or the arc $A'B'$, is the difference of longitude. Denote the radius of the earth $A'D = B'D = AD$ by R, and the radius of the parallel $AE = BE$ by r; also the circumference of the earth by C, and that of the parallel by c.

Since AB and $A'B'$ correspond to the equal angles AEB and $A'DB'$, they must be similar arcs, and give the proportion,

$$AB : A'B' = c : C,$$

or

$$\text{Dep.} : \text{diff. long.} = c : C.$$

But, as circumferences are proportional to their radii,

$$c : C = r : R.$$

Hence, leaving out the common ratio,

$$\text{Dep.} : \text{diff. long.} = r : R.$$

Putting the product of the extremes equal to that of the means,

$$r.\ \text{diff. of long.} = R.\ \text{departure}.$$

But, in the triangle $A\,D\,E$, since

$$D\,A\,E = A\,D\,A' = \text{latitude},$$

we have, from (17),

$$r = R \times \text{cos. lat.}$$

which, substituted in the above equation, gives, if the result is divided by R,

$$\text{Diff. long.} \times \text{cos. lat.} = \text{departure}. \qquad (164)$$

Hence, by (8),

$$\text{Diff. long.} = \frac{\text{departure}}{\text{cos. lat.}} = \text{dep.} \times \text{sec. lat.} \qquad (165)$$

26. *Corollary.* Since the distance is the same as the departure in parallel sailing. The word *distance* may be substituted for *departure* in (164) and (165).

27. *Corollary.* It appears, from (164) and (165), that if a right triangle (fig. 18.) is constructed, the hypothenuse of which is the difference of longitude, and one of the acute angles the latitude, the leg adjacent to this angle is the departure. *All the cases of parallel sailing may, then, be reduced to the solution of this triangle.*

28. *Problem. To find the distance between two places which are upon the same parallel of latitude.*

Solution. This problem is solved at once by (164).

29. The Table, p. 64, of the Navigator, which "shows for every degree of latitude how many miles distant two meridians are, whose difference of longitude is one degree," is readily calculated by this problem.

30. Examples.

1. A ship sails from Boston 1000 miles exactly east; find the longitude at which it arrives.

Ans. Longitude sought = 51° 48′ W.

2. Find the distance of Barcelona (Spain) from Nantucket (Massachusetts).

Ans. Distance = 3252 miles.

3. Find the distance between two meridians, whose difference of longitude is one degree in the latitude of 45°.

Ans. Distance = 42.43 miles.

CHAPTER IV.

MIDDLE LATITUDE SAILING.

31. The object of *Middle Latitude Sailing* is to give an approximative method of calculating the difference of longitude, when the difference of latitude is small. [B. p. 66.]

32. *Problem. To find the difference of longitude by Middle Latitude Sailing, when the distance and course are known, and also the latitude of either extremity of the ship's track.* [B. p. 71.]

Solution. The difference of latitude and departure are found by (152) and (153),

$$\text{Diff. lat.} = \text{dist.} \times \text{cos. course}$$

$$\text{Departure} = \text{dist.} \times \text{sin. course.}$$

The difference of longitude may then be found by means of (165). But there is a difficulty with regard to the latitude to be used in (165); for, of the two extremities of the ship's track, the latitude of one is smaller, while the latitude of the other extremity is larger than the latitude of the rest of the track. Navigators have evaded this difficulty by using the *Middle Latitude* between the two, as sufficiently accurate, when the difference of latitude is small. Now the middle latitude is the arithmetical mean between the latitudes of the extremities, so that we have, To find the difference of longitude.

Middle lat. $=\frac{1}{2}$ sum of the lats. of the extremities of the track ; (166)

and, by (165),

$$\text{Diff. long.} = \frac{\text{departure}}{\text{cos. mid. lat.}} = \text{dep.} \times \text{sec. mid. lat.} \quad (167)$$

or, by substituting (153),

$$\text{Diff. long.} = \text{dist.} \times \text{sin. course} \times \text{sec. mid. lat.} \quad (168)$$

" This method of calculating the difference of longitude may be rendered perfectly accurate by applying to the middle latitude a correction," which is given in the Navigator, and the method of computing, which will be explained in the succeeding chapter. [B. p. 76.]

33. By combining the triangle (fig. 16.) of Plane sailing with that (fig. 18.) of Parallel sailing *a triangle* (fig. 19.) *is obtained, by which all the cases of Middle Latitude sailing may be solved.*

34. *Problem. To find the distance and bearing of two places from each other, when their latitudes and longitudes are known.* [B. p. 68.]

Solution. From (fig. 19) we have

$$\text{Departure} = \text{diff. long.} \times \text{cos. mid. lat.} \quad (169)$$

$$\text{tang. bearing} = \frac{\text{departure}}{\text{diff. lat.}} \quad (170)$$

$$\text{dist.} = \text{diff. lat.} \times \text{sec. bearing.} \quad (171)$$

Cases of middle latitude sailing.

35. *Problem. To find the course, distance, and difference of longitude, when both latitudes and the departure are given.* [B. p. 70.]

Solution. The difference of longitude is found by (167), the course by (170), and the distance by (171).

36. *Problem. To find the departure, distance, and difference of longitude, when both latitudes and the course are given.* [B. p. 72.]

Solution. The departure is found by the formula

$$\text{departure} = \text{diff. lat.} \times \text{tang. course}; \tag{172}$$

the distance by (171); and the difference of longitude may be found by (167), or by substituting (172) in (167)

$$\text{diff. long.} = \text{diff. lat.} \times \text{tang. course} \times \text{sec. mid. lat.} \tag{173}$$

37. *Problem. To find the course, departure, and difference of longitude, when both latitudes and the distance are given.* [B. p. 73.]

Solution. The course is found by the formula

$$\text{cos. course} = \frac{\text{diff. lat.}}{\text{dist.}}; \tag{174}$$

the departure by

$$\text{departure} = \text{dist.} \times \text{sin. course}; \tag{175}$$

and the difference of longitude by (167).

38. *Problem. To find the difference of latitude, dis-*

tance, and difference of longitude, when one latitude, course, and departure are given. [B. p. 74.]

Solution. The difference of latitude is found by the formula

$$\text{diff. lat.} = \text{dep.} \times \text{cotan. course}; \qquad (176)$$

the distance by the formula

$$\text{dist.} = \text{dep.} \times \text{cosec. course}; \qquad (177)$$

and the difference of longitude by (167).

39. *Problem. To find the course, difference of latitude, and difference of longitude, when one latitude, the distance, and departure are given.*] B. p. 75.]

Solution. The course is found by the formula

$$\text{sin. course} = \frac{\text{dep.}}{\text{dist.}}; \qquad (178)$$

the difference of latitude by the formula

$$\text{diff. lat.} = \text{dist.} \times \text{cos. course}; \qquad (179)$$

and the difference of longitude by (167).

40. Examples.

1. A ship sailed from Halifax (Nova Scotia) a distance of 2509 miles, upon a course S. 79° 34′ E.; find the place at which it arrived.

Solution. By § 32,

Examples.

dist. = 2509		3·39950	3.39950
course = 79° 34′		cos. 9.25790	sin. 9.99276
diff. lat. = 454′ = 7° 34′ S.		2.65740	
given lat. = 44° 36′ N.	mid. lat. = 40° 49′		
required lat. = 37° 2′ N.	cor. = 7′		
	cor. mid. lat. = 40° 56′		sin. 10.12178
diff. long. = 3266′ = 54° 26′ E.			3.51404
given long. = 63° 28′ W.			
required long. = 9° 2′ W.			

Ans. The place arrived at is one mile south of Cape St. Vincent in Portugal.

2. Find the bearing and distance of Canton from Washington.

Solution. By § 34,

lat. of Washington = 38° 53′ N.	long. = 77° 3′ W.		
lat. of Canton = 23° 7′ N.	long. = 113° 14′ E.		
diff. lat. = 946′ = 15° 46′,	sum of longs. = 190° 17′		
mid. lat. = 31° 0′	diff. long. = 169° 43′ = 10183′		
cor. = 31′			
cor. mid. lat. = 31° 31′		cos. 9.93069	
diff. long. = 10183′		4.00788	
diff. lat. = 946′		ar. co. 7.02411	2.97589
bearing = S. 83° 47′ W.		tang. 10.96268	sec. 10.96526
dist. = 8733 miles			3.94115

Examples.

3. A ship sails from New York a distance of $675\frac{1}{2}$ miles, upon a course S. E. $\frac{1}{4}$ S.; find the place at which it arrives.

Ans. Three miles to the west of Georgetown in Bermuda.

4. Find the bearing and distance of Portland (Maine) from New Orleans.

Ans. The bearing = N. 49° 25′ E.
The distance = 1257 miles.

5. A ship from the Cape of Good Hope sails northwesterly until its latitude is 22° 3′ S., and its departure 3110 miles; find its course, distance sailed, longitude, and its distance from Cape St. Thomas (Brazil).

Ans. Course = N. 76° 38′ W.
Distance = 3197 miles.
Longitude = 18° W.
Distance to the Cape St. Thomas = 22 miles.

6. A ship sails from Boston upon a course E. by N. until it arrives in latitude 45° 21′ N.; find the distance, its longitude, and its distance and bearing from Liverpool.

Ans. Distance sailed = 920 miles.
Longitude = 50° 10′ W.
Distance from Liverpool = 1905 miles
Bearing from Liverpool = S. 75° 22′ W.

7. A ship sails southwesterly from Gibraltar a distance of 1500 miles, when it is in latitude 14° 44′ N.; find its course and longitude and distance from Cape Verde.

Examples.

Ans. Course = S. 37° 21′ W.

Longitude = 18° 3′ W.

Dist. from Cape Verde = 339 miles.

8. A ship sails from Nantucket upon a course S. 62° 11′ E., until its departure is 2274 miles; find the distance sailed, and the place arrived at.

Ans. Distance = 2571 miles.

The place arrived at is 261 miles north of Santa Cruz.

9. A ship sails southwesterly from Land's End (England), a distance of 3461 miles, when its departure is 3300 miles; find the course and the place arrived at.

Ans. The course = S. 72° 27′ W.

The place arrived at is Charleston (South Carolina) Light House.

CHAPTER V.

MERCATOR'S SAILING.

41. The object of *Mercator's Sailing* is to give an accurate method of calculating the difference of longitude. [B. p. 78.]

42. *Problem. To find the difference of longitude, when the distance, the course, and one latitude are known.*

Solution. Let $A\,B$ (fig. 16) be the ship's track. Divide it into the small portions $A\,a$, $a\,b$, $b\,c$, &c., which are such that the difference of longitude is the same for each of them, and let

$d =$ this small difference in longitude.

Let also

$L =$ the latitude of A,

$L' =$ the latitude of B,

$l =$ the latitude of one of the points of division as b,

$l' =$ the latitude of the next point c,

$C =$ the course.

The distance $b\,c$ may then be supposed so small, that the formulas of middle latitude sailing may be applied to it; and (173) gives

Difference of longitude.

$$d = (l' - l) \times \text{tang. } C \times \text{sec. } \tfrac{1}{2}(l' + l), \qquad (180)$$

or

$$\tfrac{1}{2} d \text{ cotan. } C = \frac{\tfrac{1}{2}(l' - l)}{\text{cos. } \tfrac{1}{2}(l' + l)}. \qquad (181)$$

If, now, the mile is adopted as the unit of length, and if

$$R = \text{the earth's radius in miles}, \qquad (182)$$

$\frac{\tfrac{1}{2}(l' - l)}{R}$ is the leugth of the arc $\tfrac{1}{2}(l' - l)$,

expressed in terms of the radius as unity; and since this arc is very small, its length is by § 22 equal to its sine or

$$\frac{\tfrac{1}{2}(l' - l)}{R} = \text{sin. } \tfrac{1}{2}(l' - l); \qquad (183)$$

which substituted in (181) gives

$$\frac{d \text{ cotan. } C}{2R} = \frac{\text{sin. } \tfrac{1}{2}(l' - l)}{\text{cos. } \tfrac{1}{2}(l' + l)}. \qquad (184)$$

Let now

$$m = \frac{d \text{ cotan. } C}{2R} = \frac{\text{sin. } \tfrac{1}{2}(l' - l)}{\text{cos. } \tfrac{1}{2}(l' + l)}; \qquad (185)$$

and (185) may be written in the usual form of a proportion

$$\text{sin. } \tfrac{1}{2}(l' - l) : \text{cos. } \tfrac{1}{2}(l' + l) = m : 1; \qquad (186)$$

whence, by the theory of proportions

$$\frac{\text{cos. } \tfrac{1}{2}(l' + l) + \text{sin. } \tfrac{1}{2}(l' - l)}{\text{cos. } \tfrac{1}{2}(l' + l) - \text{sin. } \tfrac{1}{2}(l' - l)} = \frac{1 + m}{1 - m}. \qquad (187)$$

But if in (40) we put

$$A = 90° - \tfrac{1}{2}(l' + l),\ B = \tfrac{1}{2}(l' - l); \qquad (188)$$

we have

Difference of longitude.

$$A + B = 90° - l,\ A - B = 90° - l', \tag{189}$$

and (46) becomes

$$\frac{\text{cos.}\ \frac{1}{2}(l' + l) + \text{sin.}\ \frac{1}{2}(l' - l)}{\text{cos.}\ \frac{1}{2}(l' + l) - \text{sin.}\ \frac{1}{2}(l' - l)} = \frac{\text{cotan.}\ (45° - \frac{1}{2}l')}{\text{cotan.}\ (45° - \frac{1}{2}l)}; \tag{190}$$

whence, if we put

$$M = \frac{1 + m}{1 - m}, \tag{191}$$

$$\frac{\text{cotan.}\ (45° - \frac{1}{2}l')}{\text{cotan.}\ (45° - \frac{1}{2}l)} = M. \tag{192}$$

Hence, the successive values of cotan. $(45° - \frac{1}{2}l)$ at the points A, a, b, &c., form a geometric progression; and if

D = the difference of longitude of A and B,

n = the number of portions of AB;

we have by (185)

$$n = \frac{D}{d} = \frac{D}{2\,R\,m\ \text{tang.}\ C}, \tag{193}$$

and by the theory of geometric progression

$$\text{cotan.}\ (45° - \tfrac{1}{2}L') = \text{cotan.}\ (45° - \tfrac{1}{2}L)\ M^n, \tag{194}$$

and by logarithms

$$\text{log. cotan.}\ (45° - \tfrac{1}{2}L') - \text{log. cotan.}\ (45° - \tfrac{1}{2}L) = \text{log.}\ M^n. \tag{195}$$

If, lastly, we put

$$e = M^{\frac{1}{2m}}; \tag{196}$$

we have

$$M^n = e^{\frac{D}{R\ \text{tang.}\ C}}. \tag{197}$$

$$\log. M^n = \frac{D}{R \text{ tang. } C} \log. e; \qquad (198)$$

which substituted in (195) gives by a simple reduction

$$\left[\frac{R}{\log. e} \log. \text{cotan.} (45° - \tfrac{1}{2} L') - \frac{R}{\log. e} \log. \text{cotan.} (45° - \tfrac{1}{2} L)\right] \times \text{tang. } C = D \qquad (199)$$

Now the value of $\frac{R}{\log. e}$ log. cotan. $(45° - \frac{1}{2} L)$ has been calculated for every mile of latitude, and inserted in tables. [B. Table III.] It is called the *Meridional Parts of the Latitude*, and the method of computing it is given in the following section.

The difference between the meridional parts of the two latitudes, when the latitudes are both north or both south, is called the Meridional Difference of Latitude; but when one of the latitudes is north and the other south, the sum of the meridional parts is the meridional difference of latitude.

Hence (199) gives

$$D = \text{diff. long.} = \text{mer. diff. lat.} \times \text{tang course.} \qquad (200)$$

43. *Corollary.* The difference of longitude is as in (fig. 20.) the leg DE of a right triangle, of which AD is the meridional difference of latitude, and the angle A the course; and by combining this triangle with the triangle ABC of plane sailing, *all the cases of Mercator's Sailing are reduced to the solution of these two similar right triangles.*

44. *Problem. To calculate the Table of Meridional Parts.*

Solution. I. The value of R is found from the consideration that if

$$\pi = \text{the ratio of a circumference to its diameter} = 3.1416, \tag{201}$$

we have,

$$2\pi R = \text{the circumference of the earth} = 360^\circ = 21600'$$

and

$$R = \frac{10800}{3.1416} = 3437.7. \tag{202}$$

II. In finding the value of e, the portions of the distance are supposed to be infinitely small, hence m is by (185) also infinitely small, and its reciprocal is infinitely great.

If $1 + m$ is divided by $1 - m$, as follows,

$$\begin{array}{l} 1 - m\ )\ 1 + m\ (\ 1 + 2m + 2m^2 + \&c. \\ \quad\quad\quad\ \underline{1 - m} \\ \quad\quad\quad\quad + 2m \\ \quad\quad\quad\quad\ \underline{2m - 2m^2} \\ \quad\quad\quad\quad\quad\quad + 2m^2 \\ \quad\quad\quad\quad\quad\quad\ \underline{2m^2 - 2m^3} \\ \quad\quad\quad\quad\quad\quad\quad\quad + 2m^3 \end{array}$$

we have by (191)

$$M = 1 + 2m + 2m^2 + \&c. \tag{203}$$

But since m is infinitely small, m^2, m^3, &c. are infinitely smaller, and the error of rejecting them in (203) is less than any assignable quantity; whence

$$M = 1 + 2\,m \tag{204}$$

and by the binomial theorem, and (196),

$$e = (1 + 2\,m)^{\frac{1}{2\,m}}$$

$$= 1 + \frac{1}{2\,m} \cdot 2\,m + \frac{1}{2\,m}\left(\frac{1}{2\,m} - 1\right)\frac{(2\,m)^2}{1 \,.\, 2}$$

$$+ \frac{1}{2\,m}\left(\frac{1}{2\,m} - 1\right)\left(\frac{1}{2\,m} - 2\right)\frac{(2\,m)^3}{1 \,.\, 2 \,.\, 3} + \&c. \tag{205}$$

But $\frac{1}{2\,m}$ is infinite and gives therefore, without any assignable error,

$$\frac{1}{2\,m} - 1 = \frac{1}{2\,m},\; \frac{1}{2\,m} - 2 = \frac{1}{2\,m},\; \&c. \tag{206}$$

which, substituted in (205), gives

$$e = 1 + \frac{1}{2\,m} \cdot 2\,m + \frac{1}{(2\,m)^2}\,\frac{(2\,m)^2}{1 \,.\, 2} + \frac{1}{(2\,m)^3}\,\frac{(2\,m)^3}{1 \,.\, 2 \,.\, 3} + \&c.$$

$$= 1 + 1 + \frac{1}{1 \,.\, 2} + \frac{1}{1 \,.\, 2 \,.\, 3} + \frac{1}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \&c. \tag{207}$$

so that e is the sum of a series of terms, *the first of which is unity, and each succeeding term is obtained by dividing the preceding term by the place of this preceding term.*

The value of e is thus computed.

$$
\begin{array}{rl}
1) & 1.000000 \\
2) & 1.000000 \\
3) & .500000 \\
4) & .166667 \\
5) & .041667 \\
6) & .008333 \\
7) & .001389 \\
8) & .000198 \\
9) & .000025 \\
 & .000003 \\
\hline
\end{array}
$$

$$e = 2.71828 \tag{208}$$

The sixth place of the value of e is neglected as inaccurate.

This value of e is remarkable, as being *the base of the system of logarithms invented by Neper*, and which are called the *Neperian* or *hyperbolic logarithms.*

III. The values of R (202) and e (208) give

$$\frac{R}{\text{log. } e} = \frac{3437.7}{\text{log. } (2.71828)} = \frac{3437.7}{0.43429} = 7915.7, \tag{209}$$

so that we have by (199)

$$\text{Mer. parts of } L = 7915.7 \text{ log. cotan. } (45° - \tfrac{1}{2} L), \tag{210}$$

which agrees with the explanation of Table III. given in the Preface to the Navigator.

45. Examples.

1. Calculate the meridional parts for latitude 45° 48′.

Solution. 2)45° 48′

$45° - \frac{1}{2} L = 45° - 22° 54' = 22° 6'$

22° 6′	log. cotan.	0.39141	log	9.59263
		7915.7		3.89849
mer. parts of 45° 48′ =		3098		3.49112

2. Calculate the meridional parts of latitude 28° 14′.

Ans. 1767.

3. Calculate the meridional parts of latitude 83° 59′.

Ans. 10127.

46. *Problem. To calculate the correction for middle latitude sailing.*

Solution. If the angle DBC (fig. 19.) were exactly what it should be in order that the hypothenuse BD should be the difference of longitude, and the leg BC the departure, it would be the corrected middle latitude, and we should have

diff. long. = sec. cor. mid. lat. × departure

= sec. cor. mid. lat. × diff. lat. × tang. course, (211)

which, compared with (200) gives, by dividing by tang. course,

mer. diff. lat. = sec. cor. mid. lat. × diff. lat. (212)

whence $\quad \text{sec. cor. mid. lat.} = \dfrac{\text{mer. diff. lat.}}{\text{diff. lat.}}$. (213)

If, from the corrected middle latitude, calculated by this formula, the actual middle latitude is subtracted, the correction of the middle latitude is obtained, as in the table on p. 76 of the Navigator. The meridional difference of latitude should be obtained for these calculations, not from the tables of meridional parts, but directly from the tables of logarithmic sines, &c. by means of (209); and when the difference of latitude is less than 14°, tables should be used in which the logarithms are given to seven places of decimals.

The following examples are, for the convenience of the learner, limited to cases for which the common tables are sufficiently accurate.

47. Examples.

1. Find the correction for middle latitude sailing, when the middle latitude is 35°, and the difference of latitude 14°.

Solution. Greater lat. = 35° + 7° = 42°

Less lat. = 35° — 7° = 28°

45° — ½ gr. lat. = 24° cotan.	0.35142		
45° — ½ less lat. = 31° cotan.	0.22123		
	0.13019	log.	9.11458
	7915.7		3.89849
diff. lat. = 840′		ar. co.	7.07572
corrected mid. lat. = 35° 24′		sec.	10.08879

correction = 35° 24′ — 35° = 24′.

Cases of Mercator's sailing.

2. Find the correction for middle latitude sailing, when the middle latitude is 60°, and the difference of latitude 16°.

Ans. 46′.

3. Find the correction for middle latitude sailing, when the middle latitude is 72°, and the difference of latitude 20°.

Ans. 124′.

48. *Problem. To find the bearing and distance of two given places.* [B. p. 79.]

Solution. We have by (fig. 20.) for the bearing,

$$\text{tang. bearing} = \frac{\text{diff. long.}}{\text{mer. diff. lat.}}, \qquad (214)$$

and the distance is found by (171).

49. *Problem. To find the course, distance, and difference of longitude, when both latitudes and the departure are given.* [B. p. 80.]

Solution. The course is found by (170), the difference of longitude by (200), and the distance by (171).

50. *Problem. To find the distance and difference of longitude, when both latitudes and the course are given.* [B. p. 82.]

Solution. The distance is found by (171), and the difference of longitude by (200).

51. *Problem. To find the course and difference of longitude, when both latitudes and the distance are given.* [B. p. 83.]

Solution. The course is found by (174), and the difference of longitude by (200).

Examples.

52. *Problem. To find the distance, the difference of latitude, and the difference of longitude, when one latitude, the course, and departure are given.* [B. p. 84.]

Solution The distance is found by (177), the difference of latitude by (176), and the difference of longitude by (200).

53. *Problem. To find the course, the difference of latitude, and the difference of longitude, when one latitude, the distance, and the departure are given.* [B. p. 85.]

Solution. The course is found by (178), the difference of latitude by (179), and the difference of longitude by (200), or by the following proportion deduced from the similar triangles of (fig. 20.)

Diff. lat. : dep. = mer. diff. lat. : diff. long. (215)

54. Examples.

1. A ship sails from Boston a distance of 6747 miles, upon a course S. 46° 59½′ E.; to find the place at which it arrives.

Solution.

Dist. = 6747		3.82911		
Course = 46° 59½′	cos.	9.83385		tang. 10.03022
Diff. lat. = 76° 42′ S. = 4602′,		3.66296	mer. d. l. = 5007,	3.69958
Lat. left = 42° 21′ N.	mer. p.	2810		
Lat. in = 34° 21′ S.		2197	diff. long. = 5368′	3.72980
			= 89° 28′ E.	
	mer. diff. lat. =	5007	long. left = 71° 5′ W.	
			long. in 1829′ W.	

Examples.

Ans. The place arrived at is the Cape of Good Hope.

2. Find the bearing and distance from Moscow to St. Helena.

Solution.

Moscow,	lat. 55° 46′ N.	mer. parts 4049	long. 37.° 33′ E.
St. Helena,	lat. 15° 55′ S.	mer. parts 968	long. 5° 36′ W.
Diff. lat.	= 71° 41′	mer. diff. lat. 5017	diff. l. = 43° 9′
	= 4301′		= 2589′

Mer. diff. lat. = 5017	(ar. co.) 6.29956		
diff. long. = 2589	3.41313		
course = S. 27° 18′ W.	tang. 9.71269	sec.	10.05127
	diff. lat. = 4301		3.63357
	dist. = 4840 miles		3.68484

3. A ship sails from a position 200 miles to the east of Cape Horn a distance of 3635 miles, upon a course N. N. E.; find the position at which it has arrived.

Ans. It has arrived at the equator in the longitude of 16° 6′ W.

4. Required the bearing and distance of Botany Bay from London.

Ans. The Bearing = S. 57° 31′ E.
Distance = 9551 miles.

5. A ship sails northwesterly from Lima until it arrives in the latitude 23° 7′ N., and has made a departure of 9983 miles; find the place at which it has arrived.

Ans. Canton.

Examples.

6. A ship sails from Disappointment Island in the North Pacific Ocean, upon a course S. 61° 41′ E., until it has arrived in latitude 14° 7′ S.; find the place at which it has arrived.

Ans. The Disappointment Islands in the South Pacific Ocean.

7. A ship sails from Icy Cape (North West Coast of America) a distance of 9138 miles southeasterly, when it has arrived in latitude 62° 30′ S.; find the place at which it has arrived.

Ans. Yankee Straits in New South Shetland.

8. A ship sails from Java Head, upon a course S. 68° 53′ W., until it has made a departure of 4749 miles; find the position at which it has arrived.

Ans. It has arrived at a position 180 miles south of the Cape of Good Hope.

9. A ship sails southeasterly from the South Point of the Great Bank of Newfoundland a distance of 2821 miles, when it has made a departure of 910 miles; find the position at which it has arrived.

Ans. Its position is 208 miles north of Cape St. Roque.

CHAPTER VI.

SURVEYING.

55. The object of *Surveying* is to determine the dimensions and areas of portions of the earth's surface. In the application of Plane Trigonometry, the portions of the earth are supposed to be so small that the curvature of the earth is neglected. They are, in this case, nothing more than common fields bounded by lines either straight or curved.

56. *Problem. To find the area of a triangular field, when its angles and one of its sides are known.*

Solution. Let ABC (fig. 2.) be the triangle to be measured, and c the given side. The area of the triangle is equal to half the product of its base by its altitude, or

$$\text{area of } ABC = \tfrac{1}{2}\, b\, p. \qquad (216)$$

But, by (123),

$$\sin. b : \sin. B : : c : b,$$

whence

$$b = \frac{c \sin. B}{\sin. C};$$

and, by (124),

$$p = c \sin. A.$$

Substituting (216), we have

$$\text{area of } ABC = \frac{c^2 \sin. A \sin. B}{2 \sin. C}. \qquad (217)$$

57. *Problem. To find the area of a triangular field, when two of its sides and the included angle are known.*

Solution. Let ABC (fig. 2.) be the triangle to be measured, b and c the given sides, and A the given angle. Then, by (216),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, p,$$

and, by (124),

$$p = c \sin. A.$$

Hence

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin. A. \qquad (218)$$

or, *the area of a triangle is equal to half the continued product of two of its sides and the sine of the included angle.*

58. *Problem. To find the area of a triangular field, when its three sides are known.*

Solution. Let ABC (fig. 1.) be the given triangle. Then, by (218),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin. A;$$

but, by (151),

$$\sin. A = \frac{2\sqrt{[s(s-a)(s-b)(s-c)]}}{b\, c},$$

in which s denotes the half sum of the three sides of the triangle.

Area of triangle.

Hence

$$b\,c \sin. A = 2\sqrt{[s(s-a)(s-b)(s-c)]};$$

and

$$\text{area of } ABC = \sqrt{[s(s-a)(s-b)(s-c)]}; \quad (219)$$

or, *to find the area of a triangular field, subtract each side separately from the half sum of the sides; and the square root of the continued product of the half sum and the three remainders is the required area.*

59. Examples.

1. Given the three sides of a triangular field, equal to 45.56 ch., 52.98 ch., and 61.22 ch.; to find its area.

Solution. In (fig. 1.) let $a = 45.56$ ch., $b = 52.98$ ch., $c = 61.22$ ch.

$2s =$	159.76 ch.	
$s =$	79.88 ch.	1.90244
$s - a =$	34.32 ch.	1.53555
$s - b =$	26.90 ch.	1.42975
$s - c =$	18.66 ch.	1.27091
		2 | 6.13865
Area of $ABC =$	1173. 1 sq. ch.	3.06932

Ans. The area = 117 A. 1 R. 9 r.

2. Given the three sides of a triangular field equal to 32.56 ch., 57.84 ch., and 44.44 ch.; to find its area.

Ans. The area = 71 A. 3 R. 29 r.

3. Given one side of a triangular field equal to 17.35 ch., and the adjacent angles equal to 100° and 70°; to find its area.

Ans. The area = 85 A. 3 R. 16 r.

4. Given two sides of a triangular field equal to 12.34 ch. and 17.97 ch., and the included angle equal to 44° 56′; to find its area.

Ans. The area = 7 A. 3 R. 13 r.

60. *Problem. To find the area of an irregular field bounded by straight lines.*

First Method of Solution. Divide the field into triangles in any manner best suited to the nature of the ground. Measure all those sides and angles which can be measured conveniently, remembering that three parts of each triangle, one of which is a side, must be known to determine it.

But it is desirable to measure more than three parts of each triangle, when it can be done; because the comparison of them with each other will often serve to correct the errors of observation. Thus, if the three angles were measured, and their sum found to differ from 180°, it would show there was an error; and the error, if small, might be divided between the angles; but if the error was large, it would show the observations were inaccurate, and must be taken again.

The area of each triangle is to be calculated by one of the preceding formulas, and the sum of the areas of the triangles is the area of the whole field.

This method of solution is general, and may be applied to surfaces of any extent, provided each triangle is so small as not to be affected by the earth's curvature.

Second Method of Solution. Let $ABCEFH$ (fig. 21.) be the field to be measured. Starting from its most easterly or its most westerly point, the point A for instance, measure successively round the field the bearings and lengths of all its sides.

Through A draw the meridian NS, on which let fall the perpendiculars BB', CC', EE', FF', and HH'. Also draw $CB''E''$, EF'', and HF''' parallel to NS.

Then the area of the required field is

$$ABCEFH = AC'CEFF' - [AC'CB + AHFF'].$$

But

$$AC'CEFF' = C'CEE' + E'EFF';$$

and

$$AC'CB + AHFF' = C'CBB' + B'BA + AHH' + H'HFF'.$$

Hence

$$ABCEFH = [C'CEE' + E'EFF'] - [C'CBB' + B'BA + AHH' + H'HFF'];$$

or doubling and changing a very little the order of the terms,

$$\left.\begin{array}{l} 2\,ABCEFH = [2\,C'CEE' + 2\,E'EFF'] - \\ {[2\,B'BA + 2\,CC'BB' + 2\,H'HFF' + 2\,AHH']}. \end{array}\right\} \quad (220)$$

Again,

$$\left.\begin{array}{lll} 2\,B'BA & = BB' & \times AB' \\ 2\,CC'BB' & = (BB' + CC') & \times B'C' \\ 2\,C'CEE' & = (EE' + CC') & \times E'C' \\ 2\,E'EFF' & = (EE' + FF') & \times E'F' \\ 2\,H'HFF' & = (HH' + FF') & \times H'F' \\ 2\,AHH' & = HH' & \times AH'. \end{array}\right\} \quad (221)$$

So that the determination of the required area is now reduced to the calculation of the several lines in the second members of (221.) But the rest of the solution may be more easily comprehended by means of the following table, which is precisely similar in its arrangement to the table actually used by surveyors, when calculating areas by this process.

Sides	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
AB	AB′		BB′		BB′	BB′	BB′A	
BC	B′C′			BB″	CC′	BB′ + CC′	CC′BB′	
CE		C′E′	EE″		EE′	CC′ + EE′		C′CEE′
EF		E′F′	FF″		FF′	EE′ + FF′		E′EFF′
FH	F′H′			FF‴	HH′	FF′ + HH′	H′HFF′	
HA	H′A			HH′	O	HH′	AHH′	

In the *first* column of the table are the successive sides of the field.

In the *second* and *third* columns are the differences of latitude of the several sides, the column headed N, corresponding to the sides running in a northerly direction, and that headed S, corresponding to those running in a southerly direction.

These two columns are calculated by the formula

$$\text{Diff. lat.} = \text{dist.} \times \cos. \text{bearing}.$$

In the *fourth* and *fifth* columns are the departures of the several sides; the column headed E, corresponding to the sides running in an easterly direction, and that headed W, to those running in a westerly direction.

These two columns are calculated by the formula

$$\text{Departure} = \text{dist.} \times \text{sin. bearing.}$$

In the *sixth* column, headed *Departure*, are the departures of the several vertices, which end each side of the field from the vertex *A*. This column is calculated from the two columns E and W, in the following manner. *The first number in column Departure is the same as the first in the two columns E and W; and every other number in column Departure is obtained by adding the corresponding number in columns E and W, if it is of the same column with the first number in those two columns, to the previous number in column Departure; and by subtracting it, if it is of a different column.*

Thus,

$$BB' = BB'$$
$$CC' = B'B'' = BB' - BB''$$
$$EE' = E'E'' + EE'' = CC' + EE''$$
$$FF' = F'F'' + FF'' = EE' + FF''$$
$$HH' = F'F''' = FF' - FF'''$$
$$O = HH' - HH'.$$

In the *seventh* column, headed *Sum*, are the first factors of the second members of (221). This column is calculated from column Departure in the following manner. *The first number in column Sum is the same as the first in column Departure; every other number in column Sum is the sum of the corresponding num-*

ber in column Departure added to the previous number in column Departure, as is evident from simple inspection.

In the *eighth* and *ninth* columns are the values of the areas, which compose the first members of (221). *These columns are calculated by multiplying the members in column Sum by the corresponding numbers in columns N and S*, which contain the second factors of the second members of (221). *The products are written in the column of North Areas, when the second factors are taken from column N, and in that of South Areas, when the second factors are taken from column S.*

If we compare the columns of North and South Areas with (220), we find that all those areas, which are preceded by the negative sign, are the same with those in the column of North Areas; while all those, which are connected with the positive sign, belong to the column of South Areas. *To obtain, therefore, the value* of the second member of (220), that is, *of double the required area, we have only to find the difference between the sums of the columns of North and South Areas.* [B. p. 107.]

61. *Corollary.* The columns N, S, E, and W, are those which would be calculated in Traverse Sailing, if a ship was supposed to start from the point A, and proceed round the sides of the field till it returned to the point A. The difference of the sums of columns N and S is, then, by traverse

Correction of errors.

sailing, the difference of latitude between the point from which the ship starts, and the point at which it arrives; and the difference of columns E and W is the departure of the same two points. But as both the points are here the same, their difference of latitude and their departure must be nothing, or

Sum of column N = sum of column S;

Sum of column E = sum of column W.

But when, as is almost always the case, the sums of these columns differ from each other, the difference must arise from errors of observation. If the error is great, new observations must be taken; but if it is small, it may be divided among the sides by the following proportion.

The sum of the sides : each side = whole error : error corresponding to each side.

The errors corresponding to the sides are then to be subtracted from the differences of latitude, or the departures which are in the larger column, and added to to those which are in the smaller column.

62. Examples.

1. Given the bearings and lengths of the sides of a field, as in the three first columns of the following table; to find its area.

Solution. The table is computed by § 60.

No.	Bearing.	Dist.	N.	S.	E.	W.	Cor. N.	Cor. E.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
1	N. 45° W.	21 ch.	14.85			14.85	.02	.05	14.83			14.90	14.90	14.90	220.9670	
2	N. 24° E.	32 ch.	29.24		13.01		.03	.08	29.21		12.93		1.97	16.87	492.7727	
3	S. 86° W.	54 ch.		3.76		53.87	.05	.14		3.81		54.01	55.98	57.95		214.9945
4	South.	10 ch.		10.00			.01	.03		10.01		.03	56.01	111.99		1121.0199
5	S. 70° W.	11 ch.		3.76		10.34	.01	.03		3.77		10.37	66.38	122.39		451.4103
6	S. 20° E.	99 ch.		93.03	33.86		.10	.26		93.13	33.60		32.78	99.16		9234.7708
7	East.	6 ch.			6.00		.00	.02			5.98		26.80	59.58		
8	N. 22° E.	72 ch.	66.76		26.98		.08	.18	66.68		26.80		0.00	26.80	1787.0240	
			110.85 110.55	110.55	79.85 79.06	79.06	.30	.79	110.72	110.72	79.31	79.31			2500.7637	11022.1955 2500.7637
			.30 N.		.79 E.											2) 8521.4318
																10) 4260.7159
																426.0716 4
																0.2864 40
																11.4560

Ans. The Area = 426 A. 0 R. 11 r.

Area of irregular field.

2. Given the lengths and bearings of the sides of a field, as in the following table; to find its area.

No.	Bearings.	Dist.
1	N. 17° E.	25 ch.
2	East.	28 ch.
3	South.	54 ch.
4	S. 4° W.	22 ch.
5	N. 33° W.	62 ch.

Ans. The area = 173 A. 0 R. 36 r.

63. *Problem. To find the area of a field bounded by sides, irregularly curved.*

Solution. Let *ABCEFHIKL* (fig. 22.) be the field to be measured, the boundary *ABCEFHIKL* being irregularly curved. Take any points *C* and *F*, so that by joining *AC*, *CF*, and *FL*, the field *ACFL*, bounded by straight lines, may not differ much from the given field.

Find the area of *ACFL*, by either of the preceding methods, and then measure the parts included between the curved and the straight sides by the following method of *offsetts*.

Take the points *a*, *b*, *c*, *d*, so that the lines *A a*, *a b*, *b c*, *c d*, *d C* may be sensibly straight. Let fall on *AC* the perpendiculars $a a'$, $b b'$, $c c'$, $d d'$. Measure these perpendiculars, and also the distances $A a'$, $b'c'$, $b'c'$, $c'd'$, $d'C$.

The triangles Aaa', Cdd', and the trapeziums $aba'b'$, $bcb'c'$, $cdc'd$ are then easily calculated, and their sum is the area of ABC.

In the same way may the areas of CEF, FHI, and IKL be calculated; and then the required area is found by the equation

$$ABCEFHIKL = ACFL - ABC + CEF + FHI - IKL.$$

EXAMPLE.

Given (fig. 22.) $Aa' = 5$ ch., $a'b' = 2$ ch., $b'c' = 6$ ch., $c'd' = 1$ ch., $d'C = 4$ ch.; also $aa' = 3$ ch., $bb' = 2$ ch., $cc' = 25$ ch., $dd' = 1$ ch.; to find the area of ABC.

Ans. Required area = 2 A. 3 R. 36 r.

CHAPTER VII.

HEIGHTS AND DISTANCES.

64. The plane of the *sensible horizon* at any place, is the tangent plane to the earth's surface at that place. [B. p. 48.]

The horizontal plane coincides with that of the surface of tranquil waters, when this surface is so small that its curvature may be neglected; and it is perpendicular to the *plumb line.*

65. The *angle of elevation* of an object is the angle which the line drawn to it makes with the horizontal plane, when the object is above the horizon; the *angle of depression* is the same angle when the object is below the horizon.

66. The *bearing of one object from another* is the angle included by the two lines which are drawn from the observer to these two objects.

67. *Problem. To determine the height of a vertical tower, situated on a horizontal plane.* [B. p. 94.]

Solution. Observation. Let *AB* (fig. 23.) be the tower, whose height is to be determined. Measure off the distance *BC* on the horizontal plane of any convenient length. At the point *C* observe the angle of elevation *ACB*.

Calculation. We have, then, given in the right triangle *ACB* the angle *C* and the base *BC*, as in problem, § 33 of Pl. Trig., and the leg *AB* is found by (22).

EXAMPLE.

At the distance of 95 feet from a tower, the angle of elevation of the tower is found to be 48° 19′. Required the height of the tower.

Ans. 106.69 feet.

68. *Problem. To find the height of a vertical tower situated on an inclined plane.*

Solution. Observation. Let *AB* (fig. 24.) be the tower situated on the inclined plane *BC*. Observe the angle *B*, which the tower makes with the plane. Measure off the distance *BC* of any convenient length. Observe the angle *C*, made by a line drawn to the top of the tower with *BC*.

Calculation. In the oblique triangle *ABC*, there are given the side *BC* and the two adjacent angles *B* and *C*, as in § 72 of Pl. Trig.

EXAMPLE.

Given (fig. 24.) $BC = 89$ feet, $B = 113° 12′$, $C = 23° 27′$; to find *AB*.

Ans. $AB = 51.595$ feet.

69. *Problem. To find the distance of an inaccessible object.* [B. p. 89 and 95.]

Solution. Observation. Let *B* (fig. 2.) be the point, the distance of which is to be determined, and *A* the place of the observer. Measure off the distance AC of any convenient length, and observe the angles A and C.

Calculation. AB and BC are found by § 72 of Pl. Trig.

70. *Corollary.* The perpendicular distance BP of the point B from the line AC, and the distances AP and PC are found in the triangle ABP and BPC, by § 31 of Pl. Trig.

71. *Corollary.* Instead of directly observing the angles A and C, the bearings of the lines AB, AC, and BC may be observed, when the plane ABC is horizontal, and the angles A and C are easily determined.

72. Examples.

1. An observer sees a cape, which bears N. by E.; after sailing 30 miles N. W. he sees the same cape bearing east; find the distance of the cape from the the two points of observation.

Ans. The first distance = 21.63 miles.

The second dist. = 25.43 miles.

2. Two observers stationed on opposite sides of a cloud observe the angles of elevation to be 44° 56′, and 36° 4′, their distance apart being 700 feet; find the distance of the cloud from each observer and its perpendicular altitude.

Ans. Distances from observers = 417.2 feet, and = 500.6 ft.

Height = 294.7 feet.

3. The angle of elevation of the top of a tower at one station is observed to be 68° 19′, and at another station 546 feet

farther from the tower, the angle of elevation is 32° 34′; find the height and distance of the tower, the two points of observation being supposed to be in the same horizontal plane with the foot of the tower.

Ans. The height . . . . . = 234.28 ft.

The dist. from the nearest point of observ. = 135.86 ft.

73. *Problem. To find the distance of an object from the foot of a tower of known height, the observer being at the top of the tower.*

Solution. Observation. Let the tower be *AB* (fig. 23.), and the object *C*. Measure the angle of depression *HAC*.

Calculation. Since

$$ACB = HAC,$$

we know in the triangle *ACB* the leg *AB* and the opposite angle *C*, as in § 32 of Pl. Trig.

EXAMPLE.

Given the height of the tower = 150 feet, and the angle of depression = 17° 25′; to find the distance from the foot of the tower.

Ans. 478.16 feet.

74. *Problem. To find the height of an inaccessible object above a horizontal plane, and its distance from the observer.* [B. p. 96.]

Solution. Observation. Let *A* (fig. 25.) be the object. At two different stations, *B* and *C*, whose distance apart and bearing from each other are known, observe the bearings of the object, and also the angle of elevation at one of the stations, as *B*.

Calculation. In the triangle *BCD*, the side *BC* and its adjacent angles are known, so that *BD* is found by § 72 of Pl. Trig. In the right triangle *ABD*, the height *AD* is, then, computed by § 33 of Pl. Trig.

EXAMPLE.

At one station the bearing of a cloud is N. N. W., and its angle of elevation 50° 35′. At a second station, whose bearing from the first station is N. by E., and distance 5000 feet, the bearing of the cloud is W. by N.; find the height of the cloud.

Ans. 7316.5 feet.

75. *Problem. To find the distance of two objects, whose relative position is known.* [B. p. 90.]

Solution. Observation. Let *B* and *C* (fig. 1.) be the two known objects, and *A* the position of the observer. Observe the bearings of *B* and *C* from *A*.

Calculation. In the triangle *ABC*, the side *BC* and the two angles are known. The sides of *AB* and *AC* are found by § 72 of Pl. Trig.

EXAMPLE.

The bearings of the two objects are, of the first N. E. by E., and of the second E. by S.; the known distance of the first object from the second is 23.25 miles, and the bearing N. W.; find their distance from the observer.

Ans. The distance of the first object is = 18.27 miles.
That of the second object = 32.25 miles.

76. *Problem. To find the distance apart of two objects separated by an impassable barrier.* [B. p. 91.]

Solution. Observation. Let A and B (fig. 1.) be the objects; the distance of which from each other is sought. Measure the distances and bearings from any point C to both A and B.

Calculation. In the triangle ABC the two sides AC and BC and the included angle C are known. The sides AB and BC may be found by § 81 of Pl. Trig.

Example.

Two ships sail from the same port, the one N. 10° E. a distance of 200 miles, the second N. 70° E. a distance of 150 miles; find their bearing and distance.

Ans. The distance . . . . = 180.3 miles.
The bearing of the first ship from the second = N. 36° 6′ W.

77. *Problem. To find the distance apart of two inaccessible objects situated in the same plane with the observer, and their bearing from each other.* [B. p. 92.]

Solution. Observation. Let A and B (fig. 26.) be the two inaccessible objects. At two stations, C and E, observe the bearings of A and B and the bearing and distance of C from E.

Calculation. In the triangle AEC we have the side CE, and the angles ACE and AEC, so that AC is found by § 72 of Pl. Trig.

Distance apart of two objects.

In the same way $B\,C$ is calculated from the triangle BCE.

Lastly, in triangle ABC, we know the two sides AC and BC, and the included angle for

$$ACB = ACE - BCE.$$

Hence AB and the angles BAC and CBA are found by § 81.

EXAMPLE.

An observer from a ship saw two headlands; the first bore E. N. E., and the second N. W. by N. After he had sailed N. by W. 16.25 miles, the first headland bore E. and the second N. W. by W.; find the bearing and distance of the first headland from the second.

Ans. Distance = 55.9 miles.

Bearing = N. 65° 33′ W.

78. *Problem. To find the distance of an object of known height, which is just seen in the horizon.*

Solution. I. *If light moved in a straight line,* and if A (fig. 27.) were the eye of the observer, and B the object, the the straight line APB would be that of the visual ray. The point P, at which the ray touches the curved surface CPD of the earth, is the point of the visible horizon at which the object is seen. The distances PA and PB may be calculated separately, when the heights AC and BD are known. For this purpose, let O be the earth's centre, let BD be produced to E, and let

$$h = AC, \quad H = BD,$$

$$l = PA, \quad L = PB,$$

$$R = \text{the earth's radius}$$

Distance of an object seen in the horizon.

Since BP is a tangent, and BOE a secant to the earth, we have

$$BE : BP = BP : BD;$$

and BD is so small in comparison with the radius, that we may take

$$BE = DE = 2\,R,$$

and the above proportion becomes

$$2\,R : L = L : H;$$

whence

$$L^2 = 2\,RH, \quad L = \sqrt{(2\,RH)}, \tag{222}$$

$$H = \frac{L^2}{2\,R}; \tag{223}$$

and in the same way

$$l^2 = 2\,R\,h, \quad l = \sqrt{(2\,R\,h)}, \tag{224}$$

$$h = \frac{l^2}{2\,R}. \tag{225}$$

II. Light does not, however, move in a straight line near the earth's surface, but *in a line curved towards the other centre, which is nearly an arc of a circle, whose radius is seven times the earth's radius;* so that for the point of contact P and the distances l and L, the positions of the eye and of the object are A' and B'. Now if we put

$$BB' = H', \; B'D = H_1 = H - H'$$

$$A'C = h_1,$$

we can find the value of H' with sufficient accuracy by changing in (223) R into $7\,R$, which gives

Distance of an object seen in the horizon.

$$H' = \frac{L^2}{14R} = \tfrac{1}{7}H$$

$$H_1 = H - H' = \tfrac{6}{7}H = \frac{3L^2}{7R}, \qquad (226)$$

whence $$L = \sqrt{(\tfrac{7}{3}RH_1)}. \qquad (227)$$

III. In calculating the value of L by (227), it is usually desired in statute miles, while the height H_1 is given in feet. Now we have in the Preface to the Navigator, page v,

$$R = 20911790 \text{ feet}, \qquad (228)$$

whence $$\tfrac{7}{3}R = 48794177 \text{ feet},$$

$$\log. \sqrt{(\tfrac{7}{3}R)} = \tfrac{1}{2}\log. \tfrac{7}{3}R = 3.84418,$$

and

$$\log. (L \text{ in feet}) = 3.84418 + \tfrac{1}{2}\log. (H_1 \text{ in feet}).$$

But $$L \text{ in miles} = \frac{L \text{ in feet}}{5280},$$

so that

$$\log. L \text{ in miles} = \log. L \text{ in feet} - 3.72263$$

$$= 0.12155 + \tfrac{1}{2}\log. H_1 \text{ in feet}, \qquad (229)$$

which agrees with the formula given in the preface to the Navigator for calculating table X.

IV. The Table may be used for finding L or l, when H_1 and h_1 are given, and then the required distance is the sum of L and l.

79. *Corollary.* Table X gives the correction for the error which is committed in § 67, by neglecting the earth's curva-

ture, for it is evident that to the height BP (fig. 28.) of the object above the visible level must be added the height PC of the level above the curved surface of the earth, as in B. p. 95.

80. Examples.

1. Calculate the distance in table X at which an object can be seen from the surface of the earth, when its height is 5000 feet.

Solution.

$\frac{1}{2}$ log. 5000	$=$ $\frac{1}{2}$ (3.69897)	$=$ 1.84948
	constant log.	$=$ 0.12155
dist. $=$ 93.5 (as in table X)		1.97103

2. Being on a hill 200 feet above the sea, I see just appearing in the horizon the top of a mast, which I know to be 150 feet above the water; how far distant is it?

Solution. By table X,

200 feet corresponds to 18.71 miles.

150 feet corresponds to 16.20

The distance is 34.91 miles.

3. At the distance of $7\frac{1}{2}$ statute miles from a hill the angle of elevation of its top is $2^\circ\ 13'$; find its height in feet, the observer being 20 feet above the sea.

Distance of an object seen in the horizon.

Solution.

2° 13	tang.	8.58779	
$7\frac{1}{2}$ miles = 39600		4.59770	By table X.
1533 feet		3.18549	7.50
1 foot correction,		height 20	5.12
height = 1534 feet,		height 1	1.58

4. Calculate the distance in table X, when the height is 450 feet.

Ans. 28.06 miles.

5. Upon a height of 5000 feet, the top of a hill, one mile high, is just visible in the horizon; how far distant is the hill?

Ans. 189.6 miles.

6. At the distance of 25 miles from a mountain the angle of elevation of its top is 3°; find its height, the observer being 60 feet above the intervening sea.

Ans. 7033 feet.

SPHERICAL TRIGONOMETRY.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS.

1. *Spherical Trigonometry* treats of the solution of *spherical triangles*.

A *Spherical Triangle* is a portion of the surface of a sphere included between three arcs of great circles.

In the present treatise those spherical triangles only are treated of, in which the sides and angles are less than 180°.

2. *The angle*, formed by two sides of a spherical triangle, is the same as the angle formed by their planes.

3. Besides the usual method of denoting sides and angles by degrees, minutes, &c.; another method of denoting them is so often used in Spherical Astronomy, that it will be found convenient to explain it here.

The circumference is supposed to be divided into 24 equal arcs, called *hours;* each hour is divided into 60 *minutes of time*, each minute into 60 *seconds of time*, and so on.

Hours, minutes, seconds, &c. of time are denoted by h, m, s, &c.

Arcs expressed in time.

4. *Problem. To convert degrees, minutes, &c. into hours, minutes, &c. of time.*

Solution. Since

$$360^\circ = 24^h$$

we have $15^\circ = 1^h$, and $1^\circ = \frac{1}{15}{}^h = 4^m$,

and $15' = 1^m$, and $1' = 4^s$,

$15'' = 1^s$, and $1'' = 4^t$.

Hence $a^\circ = 4\,a^m$, $a' = 4\,a^s$, $a'' = 4\,a^t$;

so that *to convert degrees, minutes, &c. into time, multiply by* 4, *and change the marks* $^\circ\ '\ ''$ *respectively, into* $^m\ ^s\ ^t$.

5. *Corollary. To convert time into degrees, minutes, &c., multiply the the hours by* 15 *for degrees, and divide the minutes, seconds, &c. of time by* 4, *changing the marks* $^m\ ^s\ ^t$ *into* $^\circ\ '\ ''$.

The turning of degrees, minutes, &c. into time, and the reverse, may be at once performed by table XXI of the Navigator.

6. Examples.

1. Convert 225° 47′ 38″ into time.

Solution. By § 4. By Table XXI.

By § 4.			By Table XXI.
$225^\circ = 900^m =$		15^h	15^h
$47' = 188^s =$		$3^m\ 8^s$	$3^m\ 8^s$
$38'' = 152^t =$		$2^s\ 32^t$	$2^s\ 32^t$
$225^\circ\ 47'\ 38''$	$=$	$15^h\ 3^m\ 10^s\ 32^t$	$15^h\ 3^m\ 10^s\ 32^t$

Arcs expressed in time.

2. Convert $17^h\ 19^m\ 13^s$ into degrees, minutes, &c.

Solution. By § 5.

$$17^h = 255°$$
$$19^m\ 13^s = 4°\ 48'\ 15''$$
$$17^h\ 19^m\ 13^s = 259°\ 48'\ 15''$$

By Table XXI.

$$17^h\ 16^m = 259°$$
$$3^m\ 12^s = 48'$$
$$1^s = 15''$$
$$17^h\ 19^m\ 13^s = 259°\ 48'\ 15''$$

3. Convert $12°\ 34'\ 56''$ into time.

Ans. $50^m\ 19^s\ 44^t$.

4. Convert $99°\ 59'\ 59''$ into time.

Ans. $6^h\ 39^m\ 59^s\ 56^t$.

5. Convert $3^h\ 2^m\ 12^s$ into degrees, minutes, &c.

Ans. $45°\ 33'$.

6. Convert $11^h\ 59^m\ 59^s$ into degrees, minutes, &c.

Ans. $179°\ 59'\ 45''$.

7. When an arc is given in time, its log., sine, &c. can be found directly from table XXVII, by means of the column headed *Hour P. M.*, in which twice the time is given, so that the double of the angle must be found in this column.

The use of the table of proportional parts for these columns is explained upon page 35 of the Navigator. When the time exceeds 6^h, the difference between it and 12^h or 24^h must be used.

Arcs expressed in time.

EXAMPLES.

1. Find the log. cosine of $19^h\ 33^m\ 11^s$.

Solution.

$$24^h - 19^h\ 33^m\ 11^s = 4^h\ 26^m\ 49^s$$

$$2 \times (4^h\ 26^m\ 49^s) = 8^h\ 53^m\ 38^s$$

$8^h\ 53^m\ 36^s$ P. M.	cos.	9.59720
prop. parts of 2^s		7
$8^h\ 53^m\ 38^s$ P. M.	cos.	9.59713

2. Find the angle in time of which the log. tang. is 10.12049.

$7^h\ 2^m\ 40^s$ P. M.	tang.	10.12026
7^s prop. parts		23
2) $7^h\ 2^m\ 47^s$ P. M.		10.12049

Ans. $3^h\ 31^m\ 23\frac{1}{2}^s$

3. Find the log. sine of $3^h\ 12^m\ 2^s$. *Ans.* 9.87113.

4. Find the log. cosine of $11^h\ 3^m\ 13^s$. *Ans.* 9.98653.

5. Find the log. tang. of $15^h\ 0^m\ 9^s$. *Ans.* 10.00057.

6. Find the log. cotan. of $22^h\ 59^m\ 59^s$. *Ans.* 10.57183.

7. Find the angle in time whose log. secant is 10.23456.
Ans. $3^h\ 37^m\ 26^s$.

8. Find the angle in time whose log. cosecant is 10.12346.
Ans. $3^h\ 15^m\ 15^s$.

8. An *isosceles* spherical triangle is one, which has two of its sides equal.

An *equilateral* spherical triangle is one, which has all its sides equal.

9. A spherical *right* triangle is one, which has a right angle; all other spherical triangles are called *oblique*.

We shall in spherical trigonometry, as we did in plane trigonometry, attend first to the solution of right triangles.

CHAPTER II.

SPHERICAL RIGHT TRIANGLES.

10. *Problem. To investigate some relations between the sides and angles of a spherical right triangle.*

Solution. The importance of this problem is obvious; for, unless some relations were known between the sides and the angles, they could not be determined from each other, and there could be no such thing as the solution of a spherical triangle.

Let, then, ABC (fig. 29.) be a spherical right triangle, right-angled at C. Call the hypothenuse AB, h; and call the legs BC and AC, opposite the angles A and B, respectively a and b.

Let O be the centre of the sphere. Join OA, OB, OC.

The angle A is, by art. 2, equal to the angle of the planes BOA and COA. The angle B is equal to the angle of the planes BOC and BOA. The angle of the planes BOC and AOC is equal to the angle C, that is, to a right angle; these two planes are, therefore, perpendicular to each other.

Moreover, the angle BOA, measured by BA, is equal to BA or h; BOC is equal to its measure BC or a, and AOC is equal to its measure AC or b.

Through any point A' of the line OA, suppose a plane to pass perpendicular to OA. Its intersections $A'C'$ and $A'B'$

with the planes COA and BOA must be perpendicular to OA', because they are drawn through the foot of this perpendicular.

As the plane $B'A'C'$ is perpendicular to OA, it must be perpendicular to AOC; and its intersection $B'C'$ with the plane BOC, which is also perpendicular to AOC, must likewise be perpendicular to AOC. Hence $B'C'$ must be perpendicular to $A'C'$ and OC', which pass through its foot in the plane AOC.

All the triangles $A'OB'$, $A'OC'$, $B'OC'$, and $A'B'C'$ are then right-angled; and the comparison of them leads to the desired equations, as follows:

First. We have from triangle $A'OB'$ by (4)

$$\cos.\ A'OB' = \cos.\ h = \frac{OA'}{OB'};$$

and from triangles $A'OC'$ and $B'OC'$

$$\cos.\ A'OC' = \cos.\ b = \frac{OA'}{OC'},$$

$$\cos.\ B'OC' = \cos.\ a = \frac{OC'}{OB'}.$$

The product of the two last equations is

$$\cos.\ a \cos.\ b = \frac{OA'}{OC'} \times \frac{OC'}{OB'} = \frac{OA'}{OB'};$$

hence, from the equality of the second members of these equations,

$$\cos.\ h = \cos.\ a \cos.\ b. \qquad (230)$$

Investigation of Neper's Rules.

Secondly. From triangle $A'BC'$ we have by (4), and the fact that the angle $B'A'C'$ is equal to the inclination of the two planes BOC and BOA,

$$\cos. B'A'C' = \cos. A = \frac{A'C'}{A'B'};$$

and, from triangles $A'OC'$ and $A'OB'$, by (4),

$$\text{tang. } C'OA' = \text{tang. } b = \frac{A'C'}{A'O},$$

$$\text{cotan. } B'OA' = \text{cotan. } h = \frac{A'O}{A'B'}.$$

The product of these equations is

$$\text{tang. } b \text{ cotan. } h = \frac{A'C'}{A'O} \times \frac{A'O}{A'B'} = \frac{A'C'}{A'B'};$$

hence $$\cos. A = \text{tang. } b \text{ cotan. } h. \qquad (231)$$

Thirdly. Corresponding to the preceding equation between the hypothenuse h, the angle A, and the adjacent side b, there must be a precisely similar equation between the hypothenuse h, the angle B, and the adjacent side a; which is

$$\cos. B = \text{tang. } a \text{ cotan. } h. \qquad (232)$$

Fourthly. From triangles $B'OC'$, $B'OA'$, and $B'A'C'$, by (4),

$$\sin. B'OC' = \sin. a = \frac{B'C'}{OB'},$$

$$\sin. B'OA' = \sin. h = \frac{B'A'}{OB'},$$

$$\sin. B'A'C' = \sin. A = \frac{B'C'}{B'A'}.$$

The product of these last two equations is

$$\sin. h \sin. A = \frac{B'A'}{OB'} \times \frac{B'C'}{B'A'} = \frac{B'C'}{OB'};$$

hence $$\sin. a = \sin. h \sin. A. \tag{233}$$

Fifthly. The preceding equation between h, the angle A, and the opposite side a, leads to the following corresponding one between h, the angle B, and the opposite side b;

$$\sin. b = \sin. h \sin. B. \tag{234}$$

Sixthly. From triangles $C'OA'$, $B'A'C'$, and $B'OC'$, by (4),

$$\sin. C'OA' = \sin. b = \frac{A'C'}{OC'},$$

$$\text{cotan.} B'A'C' = \text{cotan.} A = \frac{A'C'}{B'C'},$$

$$\text{tang.} B'OC' = \text{tang.} a = \frac{B'C'}{OC'}.$$

The product of these last two equations is

$$\text{cotan.} A \text{ tang.} a = \frac{A'C'}{B'C'} \times \frac{B'C'}{OC'} = \frac{A'C'}{OC'};$$

hence $$\sin. b = \text{cotan.} A \text{ tang.} a. \tag{235}$$

Seventhly. The preceding equation between the angle A, the opposite side a, and the adjacent side b, leads to the following corresponding one between the angle B, the opposite side b, and the adjacent side a;

$$\sin. a = \text{cotan.} B \text{ tang.} b. \tag{236}$$

Eighthly. From (7)

$$\text{tang.}\, a = \frac{\sin.\, a}{\cos.\, a},$$

$$\text{tang.}\, b = \frac{\sin.\, b}{\cos.\, b};$$

which, substituted in (235) and (236), give

$$\sin.\, a = \frac{\text{cotan.}\, B \sin.\, b}{\cos.\, b},$$

$$\sin.\, b = \frac{\text{cotan.}\, A \sin.\, a}{\cos.\, a}.$$

Multiplying the first of these equations by cos. b and the second by cos. a, we have

$$\sin.\, a \cos.\, b = \text{cotan.}\, B \sin.\, b,$$

$$\sin.\, b \cos.\, a = \text{cotan}\, A \sin.\, a.$$

The product of these equations is

$$\sin.\, a \sin.\, b \cos.\, a \cos.\, b = \text{cotan.}\, A\ \text{cotan.}\, B \sin.\, a \sin.\, b;$$

which, divided by sin. a sin. b, becomes

$$\cos.\, a \cos.\, b = \text{cotan.}\, A\ \text{cotan.}\, B.$$

But, by (230),

$$\cos.\, h = \cos.\, a \cos.\, b;$$

hence $$\cos.\, h = \text{cotan.}\, A\ \text{cotan.}\, B. \quad (237)$$

Ninthly. We have, by (230) and (234),

$$\cos.\, a = \frac{\cos.\, h}{\cos.\, b},$$

$$\sin.\, B = \frac{\sin.\, b}{\sin.\, h},$$

the product of which is, by (7) and (8),

$$\cos. a \sin. B = \frac{\sin. b \cos. h}{\cos. b \sin. h} = \frac{\sin. b}{\cos. b} \cdot \frac{\cos. h}{\sin. h}$$

$$= \text{tang.}\, b \,\text{cotan.}\, h.$$

But, by (231),

$$\cos. A = \text{tang.}\, b \,\text{cotan.}\, h;$$

hence $$\cos. A = \cos. a \sin. B. \quad (238)$$

Tenthly. The preceding equation between the side a, the opposite angle A, and the adjacent angle B, leads to the following similar one between the side b, the opposite angle B, and the adjacent angle A;

$$\cos. B = \cos. b \cdot \sin. A. \quad (239)$$

11. *Corollary.* The ten equations, [230 – 239], have, by a most happy artifice, been reduced to two very simple theorems, called, from their celebrated inventor, *Neper's Rules.*

In these rules, the complements of the hypothenuse and the angles are used instead of the hypothenuse and the angles themselves, and the right angle is neglected.

Of the five parts, then, the legs, the complement of the hypothenuse, and the complements of the angles; either part may be called the *middle part.* The two parts, including the middle part on each side, are called the *adjacent parts*; and the other two parts are called the *opposite parts.* The two theorems are as follows.

I. *The sine of the middle part is equal to the product of the tangents of the two adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the two opposite parts.* [B. p. 436.]

Proof. To demonstrate the preceding rules, it is only necessary to compare all the equations which can be deduced from them, with those previously obtained. [230 – 239.]

Let there be the spherical right triangle ABC (fig. 30.) right-angled at C.

First. If co. h were made the middle part, then, by the above rule, co. A and co. B would be adjacent parts, and a and b opposite parts; and we should have

$$\sin. (\text{co. } h) = \text{tang. } (\text{co. } A) \text{ tang. } (\text{co. } B)$$

$$\sin. (\text{co. } h) = \cos. a \cos. b;$$

or

$$\cos. h = \text{cotan. } A \text{ cotan. } B,$$

$$\cos. h = \cos. a \cos. b;$$

which are the same as (237) and (230).

Secondly. If co. A were made the middle part; then co. h and b would be adjacent parts, and co. B and a opposite parts; and we should have

$$\sin. (\text{co. } A) = \text{tang. } (\text{co. } h) \text{ tang. } b,$$

$$\sin. (\text{co. } A) = \cos. (\text{co. } B) \cos. a;$$

or

$$\cos. A = \text{cotan. } h \text{ tang. } b,$$

$$\cos. A = \sin. B \cos. a;$$

which are the same as (231) and (238).

In like manner, if co. B were made the middle part, we should have

$$\cos. B = \text{cotan.}\, h \text{ tang.}\, a,$$

$$\cos. B = \sin. A \cos. b;$$

which are the same as (232) and (239).

Thirdly. If a were made the middle part, then co. B and b would be the adjacent parts, and co. A and co. h the opposite parts; and we should have

$$\sin. a = \text{tang.}\, (\text{co.}\, B) \text{ tang}\, b,$$

$$\sin. a = \cos. (\text{co.}\, A) \cos. (\text{co.}\, h);$$

or

$$\sin. a = \text{cotan.}\, B \text{ tang.}\, b,$$

$$\sin. a = \sin. A \sin. h;$$

which are the same as (236) and (233).

In like manner, if b were made the middle part, we should have

$$\sin. b = \text{cotan.}\, A \text{ tang.}\, a,$$

$$\sin. b = \sin. B \sin. h;$$

which are the same as (235) and (234).

Having thus made each part successively the middle part, the ten equations, which we have obtained, must be all the equations included in Neper's Rules; and we perceive that they are identical with the ten equations [230 – 239].

12. *Theorem. The three sides of a spherical right triangle are either all less than* 90° *; or else, one is less while the other two are greater than* 90°, unless one of them is equal to 90°, as in § 16.

Proof. When h is less than 90°, the first member of (230) is positive; and therefore the factors of its second member must either be both positive or both negative; that is, the two legs a and b must, by Pl. Trig. § 61, be both acute or both obtuse.

But when h is obtuse, the first member of (230) is negative; and therefore one of the factors of the second member must be positive, while the other negative; that is, of the two legs a and b, one must be acute, while the other is obtuse.

13. *Theorem. The hypothenuse differs less from* 90° *than does either of the legs*, the case of either side equal to 90° being excepted.

Proof. The factors cos. a and cos. b of the second member of the equation (230) are, by (4), fractions whose numerators are less than their denominators. Their product, neglecting the signs, must then be less than either of them, as cos a for instance, or

$$\cos. h < \cos. a;$$

and therefore, by Pl. Trig. § 69 and 70, h must differ less from 90° than a does.

14. *Theorem. An angle and its opposite leg in a spherical right triangle must be both acute, or both obtuse, or, by* § 16, *both equal to* 90°.

Proof. When A is acute, the first member of (238) is positive, and therefore the factor cos. a of the second member, being multiplied by the positive factor sin. B must be positive; that is, a must be acute. But if A is obtuse, the first member of (238) is negative, and therefore the factor cos. a of the second member must be negative; that is, a must be obtuse.

One side equal to 90°.

15. *Theorem. An angle differs less from* 90° *than its opposite leg*, the case of either side, equal to 90°, being excepted.

Proof. Since the second member of (238) is the product of the two fractions cos. a and sin. B, the first member must be less than either of them. Thus, neglecting the sines,

$$\cos. A < \cos. a;$$

hence A differs less from 90° than a does.

16. *Theorem. When in a spherical right triangle either side is equal to* 90°, *one of the other two sides is also equal to* 90°; *and each side is equal to its opposite angle.*

Proof. First. If either of the legs is equal to 90°, the corresponding factor of the second member of (230) is, by (59), equal to zero; which gives

$$\cos. h = 0,$$

or, by (59),

$$h = 90°.$$

Again, if we have

$$h = 90°,$$

it follows, from (59) and (230), that

$$0 = \cos. a \cos. b,$$

and therefore either cos. a or cos. b must be zero; that is, either a or b must be equal to 90°.

Secondly. When either side is equal to 90°, it follows, from the preceding proof, that

$$h = 90° ;$$

which substituted in (233) produces, by (60),

$$\sin. a = \sin. A ;$$

which gives

$$a = A ;$$

because, from § 14, a could not be equal to the supplement of A.

17. *Corollary.* When both the legs of a spherical right triangle are equal to 90°, all the sides and angles are equal to 90°.

18. *Theorem. When two of the angles of a spherical triangle are equal to* 90°, *the opposite sides are also equal to* 90°.

Proof. For in this case, one of the factors of the second member of the equation (237) must, by (61), be equal to zero, since either A or B is 90° ; hence

$$\cos. h = 0 ;$$

or, by (59),

$$h = 90° ;$$

and the remainder of the proposition follows from § 16.

19. *Corollary.* When all the angles of a spherical right triangle are equal to 90°, all the sides are also equal to 90°.

20. *Theorem. The sum of the angles of a spherical right triangle is greater than* 180°, *and less than* 360°; *and each angle is less than the sum of the other two.*

Proof. I. It is proved in Geometry, that the sum of the angles of any spherical triangle is greater than 180°.

II. It is proved in Geometry, that each angle of any spherical triangle is greater than the difference between two right angles and the sum of the other two angles. Hence if the sum of the two angles A and B is greater than 180°, we have

$$90° > A + B - 180°,$$

or

$$A + B < 270°,$$

or

$$A + B + 90° < 360°;$$

that is, the sum of the three angles is less than 360°; for in case the sum of the angles A and B is less than 180°, the sum of the three angles is obviously less than 360°.

III. When the right angle is greatest of the three angles, we have

$$90° + A + B > 180°,$$

or

$$A + B > 90°;$$

that is, the greater angle is in this case less than the sum of the other two.

But if one of the other angles A is the greatest of the three angles, we have, by the proposition of Geometry last referred to,

$$B > 90° + A - 180°,$$

or

$$B > A - 90°,$$

or

$$A < B + 90°;$$

so that in every case one angle is less than the sum of the other two.

21. To solve a spherical right triangle, two parts must be known in addition to the right angle. From the two known parts, the other three parts are to be determined, separately, by equations derived from Neper's Rules. The two given parts, with the one to be determined are, in each case, to enter into the same equation *These three parts are either all adjacent to each other, in which case the middle one is taken as the* MIDDLE PART, *and the other two are, by* § 11, ADJACENT PARTS ; *or one is separated from the other two, and then the part, which stands by itself, is the* MIDDLE PART, *and the other two are, by* § 11, OPPOTITE PARTS.

22. *Problem. To solv3 a spherical right triangle, when the hypothenuse and one of the angles are given.*

Solution. Let ABC (fig. 30.) be the right triangle, right-angled at C; and let the sides be denoted as in § 10. Let h and A be given; to solve the triangle.

First. To find the other angle B. The three parts, which are to enter into the same equation, are co. h, co. A, and co. B; and, by § 21, as they are all adjacent to each other, co. h is the middle part, and co. A and co. B are adjacent parts. Hence, by Neper's Rules,

$$\sin. (\text{co. } h) = \text{tang. } (\text{co. } A) \text{ tang. } (\text{co. } B),$$

or

$$\cos. h = \text{cotan. } A \text{ cotan. } B;$$

and, by (6),

Hypothenuse and an angle given.

$$\text{cotan. } B = \frac{\text{cos. } h}{\text{cotan. } A} = \text{cos. } h \text{ tang. } A$$

Secondly. To find the opposite leg a. The three parts are co. A, co. h, and a; of which, by § 21, a is the middle part, and co. h and co. A are the opposite parts. Hence, by Neper's Rules,

$$\text{sin. } a = \text{cos. (co. } h) \text{ cos. (co. } A),$$

or

$$\text{sin. } a = \text{sin. } h \text{ sin. } A.$$

Thirdly. To find the adjacent leg b. The three parts are co. A, co. h, and b; of which co. A is the middle part, and co. h and b are the adjacent parts. Hence, by Neper's Rules,

$$\text{sin. (co. } A) = \text{tang. (co. } h) \text{ tang. } b,$$

or

$$\text{cos. } A = \text{cotan. } h \text{ tang. } b;$$

and, by (6),

$$\text{tang. } b = \frac{\text{cos. } A}{\text{cotan. } h} = \text{tang. } h \text{ cos. } A.$$

23. *Scholium.* The tables always give two angles, which are supplements of each other, corresponding to each sine, cosine, &c. But it is easy to choose the proper angle for the particular case, by referring to § 12 and 14; or by having regard to the signs of the different terms of the equation, as determined by Pl. Trig. § 61.

24. *Scholium.* When h and A are both equal to 90°, the values of cotan. B and tang. b are indeterminate; for the numerators and denominators of the fractional values are, by (59) and (61), equal to zero; and in this case there are an

infinite number of triangles which satisfy the given values of h and A.

The problem is impossible by § 18, if the given value of h differs from 90°, while that of A is equal to 90°.

25. Examples.

1. Given in the spherical right triangle (fig. 30.), $h = 145°$ and $A = 23° 28'$; to solve the triangle.

Solution.

h, cos.	9.91336 *n,	sin.	9.75859,	tang.	9.84523 n.
A, tang.	9.63761,	sin.	9.60012,	cos.	9.96251
B, cotan.	9.55097 n;	a sin.	9.35871;	b tang.	9.80774 n.

Ans. $B = 109° 34' 33''$, $a = 13° 12' 12''$, $b = 147° 17' 15''$.

2. Given in the spherical right triangle, (fig. 30.), $h = 32° 34'$, and $A = 44° 44'$; to solve the triangle.

Ans. $B = 50° \; 8' 21''$,
$a = 22° 15' 43''$,
$b = 24° 24' 19''$.

26. ***Problem.*** *To solve a spherical right triangle, when its hypothenuse and one of its legs are given.*

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* The letter n placed after a logarithm indicates it to be the logarithm of a negative quantity, and it is plain that, when the number of such logarithms to be added together is even, the sum is the logarithm of a positive quantity; but if odd, the sum is the logarithm of a negative quantity.

Hypothenuse and a leg given.

Solution. Let ABC (fig. 30.) be the triangle; h the given hypothenuse, and a the given leg.

First. To find the opposite angle A; a is the middle part, and co. A and co. h are the opposite parts. Hence

$$\sin. a = \cos. (\text{co. } h) \cos. (\text{co. } A);$$

or

$$\sin. a = \sin. h \sin. A;$$

and, by (6),

$$\sin. A = \frac{\sin. a}{\sin. h} = \sin. a \text{ cosec. } h.$$

Secondly. To find the adjacent angle B; co. B is the middle part, and co. h and a are the adjacent parts. Hence

$$\sin. (\text{co. } B) = \text{tang. } a \text{ tang. } (\text{co. } h),$$

or

$$\cos. B = \text{tang. } a \text{ cotan. } h.$$

Thirdly. To find the other leg b; co. h is the middle part, and a and b are the opposite parts. Hence

$$\cos. h = \cos. a \cos. b;$$

and, by (6),

$$\cos. b = \frac{\cos. h}{\cos. a} = \sec. a \cos. h.$$

27. *Scholium.* The question is impossible by § 13, when the given value of the hypothenuse differs more from 90° than that of the leg.

28. *Solution.* When h and a are both equal to 90°, it may be shown, as in § 24, that the values of B and b are indeterminate.

29. Example.

Given in the spherical right triangle (fig. 30.), $a = 141° \ 11'$, and $h = 127° \ 12'$; to solve the triangle.

$$\text{Ans.} \quad A = 128° \ \ 5' \ 54'', \quad B = 52° \ 22' \ 24'', \quad b = 39° \ \ 6' \ 23''.$$

30. *Problem. To solve a spherical right triangle, when one of its legs and the opposite angle are given.*

Solution. Let ABC (fig. 30.) be the triangle, a the given leg, and A the given angle.

First. To find the hypothenuse h; a is the middle part, and co. h and co. A are the opposite parts. Hence

$$\sin. a = \sin. h \sin. A;$$

and, by (6),

$$\sin. h = \frac{\sin. a}{\sin. A} = \sin. a \operatorname{cosec.} A.$$

Secondly. Te find the other angle B; co. A is the middle part, and a and co. B are the opposite parts. Hence

$$\cos. A = \cos. a \sin. B;$$

and, by (6),

$$\sin. B = \frac{\cos. A}{\cos. a} = \sec. a \cos. A.$$

Thirdly. To find the other leg. b; b is the middle part, and a and co. A are the adjacent parts. Hence

$$\sin. b = \operatorname{tang.} a \operatorname{cotan.} A.$$

A leg and the opposite angle given.

31. *Scholium.* There are two triangles ABC and $A'BC$ (fig. 31.) formed by producing the sides AB and AC, to the point of meeting A', both of which satisfy the conditions of the problem. For the side BC or a, and the angle A, or by § 2 its equal A', belong to both the triangles.

Now ABA' and ACA' are semicircumferences. Hence h', the hypothenuse of $A'BC$, is the supplement of h; b' is the supplement of b; and $A'BC$ is the supplement of ABC. One set of values, then, of the unknown quantities, given by the tables, as in § 23, corresponds to the triangle ABC, and the other set to $A'BC$.

32. *Corollary.* When the given values of a and A are equal, the values of h, B, and b become

$$\sin. h = 1, \quad \sin. B = 1, \quad \sin. b = 1;$$

or, by (60),

$$h = 90°, \qquad B = 90°, \qquad b = 90°;$$

as in § 16.

33. *Corollary.* When a and A are equal to 90°, the values of b and B are indeterminate, as in § 24.

34. *Scholium.* The problem is, by § 14, impossible, when the given values of the leg and its opposite angle are such, that one is obtuse, while the other is acute, or that one is equal to 90°, while the other differs from 90°; and, by § 15, it is impossible, when the given value of the angle differs more from 90° than that of the leg.

35. Example.

Given in the spherical right triangle, (fig. 30.), $a = 35° 44'$, and $A = 37° 28'$; to solve the triangle.

$$Ans. \quad \left.\begin{array}{l} h = 73° 45' 15'' \\ B = 77° 54' \\ b = 69° 50' 24'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} h = 106° 14' 45'' \\ B = 102° \ 6' \\ b = 110° \ 9' 36''. \end{array}\right.$$

36. *Problem. To solve a spherical right triangle, when one of its legs and the adjacent angle are given.*

Solution. Let ABC (fig. 30.) be the triangle, a the given leg, and B the given angle.

First. To find the hypothenuse h; co. B is the middle part, and co. h and a are adjacent parts. Hence

$$\cos. B = \text{tang. } a \text{ cotan. } h;$$

and, by (6),

$$\text{cotan. } h = \frac{\cos. B}{\text{tang. } a} = \text{cotan. } a \cos. B.$$

Secondly. To find the other angle A; co. A is the middle part, and co. B and a are opposite parts. Hence

$$\cos. A = \cos. a \sin. B.$$

Thirdly. To find the other leg b; a is the middle part, and co. B and b are adjacent parts. Hence

$$\sin. a = \text{tang. } b \text{ cotan. } B;$$

and, by (6),

$$\text{tang. } b = \frac{\sin. a}{\text{cotan. } B} = \sin. a \text{ tang. } B.$$

37. Example.

Given in the spherical right triangle, (fig. 30.), $a = 118° 54'$, and $B = 12° 19'$; to solve the triangle.

$$Ans.\quad h = 118° 20' 20'',$$
$$A = 95° 55' 2'',$$
$$b = 10° 49' 17''.$$

38. *Problem. To solve a spherical right triangle, when its two legs are given.*

Solution. Let ABC (fig. 30.) be the triangle, a and b the given legs.

First. To find the hypothenuse h; co. h is the middle part, a and b are opposite parts. Hence

$$\cos. h = \cos. a \cos. b.$$

Secondly. To find one of the angles, as A; b is the middle part, and co. A and a are adjacent parts. Hence

$$\sin. b = \text{tang.}\, a \,\text{cotan.}\, A;$$

and, by (6),

$$\text{cotan.}\, A = \frac{\sin.}{\text{tang.}\, a} = \text{cotan.}\, a \sin. b.$$

In the same way,

$$\text{cotan.}\, B = \text{cotan.}\, b \sin. a.$$

39. Example.

Given in the spherical right triangle, (fig. 30.), $a = 1°$, and $b = 100°$; to solve the triangle.

Ans. $h = 99° \; 59' \; 52''$,

$A = 1° \; 0' \; 56''$,

$B = 90° \; 11' \; 24''$.

40. *Problem. To solve a spherical right triangle, when the two angles are given.*

Solution. Let ABC (fig. 30.) be the triangle, A and B the given angles.

First. To find the hypothenuse h; co. h is the middle part, and co. A and co. B are adjacent parts. Hence

$$\cos. h = \text{cotan } A \text{ cotan. } B.$$

Secondly. To find one of the legs, as a; co. A is the middle part, and co. B and a are the opposite parts. Hence

$$\cos. A = \cos. a \sin. B;$$

and, by (6),

$$\cos. a = \frac{\cos. A}{\sin. B} = \cos. A \text{ cosec. } B.$$

In the same way,

$$\cos. b = \text{cosec. } A \cos. B.$$

41. *Scholium.* The problem is, by § 20, impossible, when the sum of the given values of A and B is less than 90°, or

The angles given.

greater than 270°, or when their difference is greater than 90.°

42. Example.

Given in the spherical right triangle, (fig. 30.), $A = 91° 11'$, and $B = 111° 11'$, to solve the triangle.

Ans. $h = 89° 32' 28''$,

$a = 91° 16' 8''$,

~~$b = 109° 52' 16''$.~~

b = 111°–11′–16″.

CHAPTER III.

SPHERICAL OBLIQUE TRIANGLES.

42. *Theorem. The sines of the sides in any spherical triangle are proportional to the sines of the opposite angles.* [B. p. 437.]

Proof. Let ABC (figs. 32 and 33.) be the given triangle. Denote by a, b, c, the sides respectively opposite to the angles A, B, C. From either of the vertices let fall the perpendicular BP upon the opposite side AC. Then, in the right triangle ABP, making BP the middle part, co. c and co. BAP are the opposite parts. Hence, by Neper's Rules,

$$\sin. BP = \sin. c \sin. BAP = \sin. c \sin. A.$$

For BAP is either the same as A, or it is its supplement, and in either case has the same sine, by (91).

Again, in triangle BPC, making BP the middle part, co. a and co. C are the opposite parts. Hence, by Neper's Rules,

$$\sin. BP = \sin. a \sin. C;$$

and, from the two preceding equations,

$$\sin. c \sin. A = \sin. a \sin. C,$$

which may be written as a proportion, as follows;

$$\sin. a : \sin. A = \sin c : \sin. C.$$

In the same way,

$$\sin. a : \sin. A = \sin. b \,.\, \sin. B.$$

43. *Theorem. Bowditch's Rules for Oblique Triangles.* If, in a spherical triangle, two right triangles are formed by a perpendicular let fall from one of its verticles upon the opposite side; and if, in the two right triangles, the middle parts are so taken that the perpendicular is an adjacent part in both of them; then

The sines of the middle parts in the two triangles are proportional to the tangents of the adjacent parts.

But, if the perpendicular is an opposite part in both the triangles, then

The sines of the middle parts are proportional to the cosines of the opposite parts. [B. p. 437.]

Proof. Let M denote the middle part in one of the right triangles, A an adjacent part, and O an opposite part. Also let m denote the middle part in the other right triangle, a an adjacent part, and o an opposite part; and let p denote the perpendicular.

First. If the perpendicular is an adjacent part in both triangles, we have, by Neper's Rules,

$$\text{sin. } M = \text{tang. } A \text{ tang. } p,$$

$$\text{sin. } m = \text{tang. } a \text{ tang. } p;$$

whence

$$\frac{\text{sin. } M}{\text{sin. } m} = \frac{\text{tang. } A \text{ tang. } p}{\text{tang. } a \text{ tang. } p} = \frac{\text{tang. } A}{\text{tang. } a},$$

or $$\text{sin. } M : \text{sin. } m = \text{tang. } A : \text{tang. } a.$$

Secondly. If the perpendicular is an opposite part in both the triangles, we have, by Neper's Rules,

$$\sin. M = \cos. O \cos. p,$$

$$\sin. m = \cos. o \cos. p;$$

whence

$$\frac{\sin. M}{\sin. m} = \frac{\cos. O \cos. p}{\cos. o \cos. p} = \frac{\cos. O}{\cos. o},$$

or $$\sin. M : \sin. m = \cos. O : \cos\ o.$$

44. *Problem. To solve a spherical triangle, when two of its sides and the included angle are given.* [B. p. 438.]

Solution. Let ABC (figs. 32 and 33.) be the triangle, a and b the given sides, and C the given angle. From B let fall on AC the perpendicular BP.

First. To find PC, we know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by means of Neper's Rules,

$$\text{tang. } PC = \cos. C \text{ tang. } a. \tag{239}$$

Secondly. AP is the difference between AC and PC, that is,

$$\text{(fig. 32.) } AP = b - PC \text{ or (fig 33.) } AP = PC - b. \tag{240}$$

Thirdly. To find the side c. If, in the triangle BPC, co. a is the middle part, PC and PB are opposite parts; and if, in the triangle ABP, co. c is the middle part, BP and AP are the opposite parts. Hence, by Bowditch's Rules,

$$\cos. PC : \cos. AP = \sin. (\text{co. } a) : \sin. (\text{co. } c),$$

or $$\cos. PC : \cos. AP = \cos. a : \cos. c. \tag{241}$$

Rules for acute or obtuse angles and sides.

Fourthly. To find the angle A. If, in the triangle BPC, PC is the middle part, co. C and BP are adjacent parts; and if, in the triangle ABP, AP is the middle part, co. BAP and BP are adjacent parts. Hence, by Bowditch's Rules,

$$\sin. PC : \sin. PA = \text{cotan.}\, C : \text{cotan.}\, BAP; \qquad (242)$$

and BAP is the angle A (fig. 32.), when the perpendicular falls within the triangle; or it is the supplement of A (fig. 33.), when the perpendicular falls without the triangle.

Fifthly. B is found by means of (344),

$$\sin. c : \sin. C = \sin. b : \sin. B. \qquad (243)$$

45. *Scholium.* In determining PC, c, and BAP, by (239), (241), and (242), the signs of the several terms must be carefully attended to; by means of Pl. Trig. § 61.

But to determine which value of B, determined by (243), is the true value, regard must be had to the following rules, which are proved in Geometry.

I. *The greater side of a spherical triangle is always opposite to the greater angle.*

II. *Each side is less than the sum of the other two.*

III. *The sum of the sides is less than* 360°.

IV. *Each angle is greater than the difference between* 180°, *and the sum of the other two angles.*

There are, however, cases in which these conditions are all satisfied by each of the values of B. In any such case this angle can be determined in the same way in which the angle A was determined, by letting fall a perpendicular from the vertex A on the side BC. But this difficulty can always be avoided, by letting fall the perpendicular upon that of the two given sides which differs the most from 90°.

46. *Corollary.* By (240), (104), and (29), we have

$$\cos. AP = \cos. (b - PC) = \cos. (PC - b)$$
$$= \cos. b \cos. PC + \sin. b \sin. PC, \qquad (244)$$

which, substituted in (241), gives

$$\cos. PC : \cos. b \cos. PC + \sin. b \sin. PC = \cos. a : \cos. c.$$

Dividing the two terms of the first ratio by cos. PC, we have by (7),

$$1 : \cos. b + \sin. b \,\text{tang}.\, PC = \cos. a : \cos. c. \qquad (245)$$

The product of the means being equal to that of the extremes, we have

$$\cos. c = \cos. a \cos. b + \sin. b \cos. a \,\text{tang}.\, PC. \qquad (246)$$

But by (239)

$$\text{tang}.\, PC = \cos. C \,\text{tang}.\, a = \frac{\cos. C \sin. a}{\cos. a},$$

or $$\cos. a \,\text{tang}.\, PC = \cos. C \sin. a; \qquad (247)$$

which, substituted in (246), gives

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C, \qquad (248)$$

which is *one of the fundamental equations of Spherical Trigonometry.*

47. *Corollary.* We have, by (48),

$$\cos. C = 1 + 2 (\cos. \tfrac{1}{2} C)^2,$$

$$= 2(\cos \tfrac{C}{2})^2 - 1$$

which, substituted in (248), gives, by (28),

$$\cos. c = \cos. (a + b) + 2 \sin. a \sin. b (\cos. \tfrac{1}{2} C)^2, \quad (249)$$

from which the value of the side c can readily be found by using the table of Natural Sines.

48. *Corollary.* We have, by (49),

$$\cos. C = 1 - 2 (\sin. \tfrac{1}{2} C)^2,$$

which, substituted in (248), gives, by (29),

$$\cos. c = \cos. (a - b) - 2 \sin. a \sin. b (\sin. \tfrac{1}{2} C)^2, \quad (250)$$

which can be used like formula (249).

49. *Corollary.* The use of formula (250) is much facilitated by means of the column of Rising in Table XXIII of the Navigator. This column contains the values of

$$\log. 2 (\sin. \tfrac{1}{2} C)^2 = 2 \log. \sin. \tfrac{1}{2} C + \log. 2$$
$$= 2 \log. \sin. \tfrac{1}{2} C + 0.30103. \quad (251)$$

But the decimal point is supposed to be changed so as to correspond to the table of Natural Sines, that is, 5 is added to the logarithm; and 20 is to be subtracted from the value of $2 \log. \sin. \frac{1}{2} C$, which is given by table XXVII, as is evident from Pl. Trig. § 30. So that the column Rising of Table XXIII is constructed by the formula

$$\log. \text{Ris. } C = 2 \log. \sin. \tfrac{1}{2} C + 5.30102 - 20$$
$$= 2 \log. \sin. \tfrac{1}{2} C - 14.69897, \quad (252)$$

which agrees with the explanation in the Preface to the Navigator.

Two sides and the included angle given.

50. By using table XXIII, the following rule is obtained for finding the third side, when two sides and the included angle are given.

Add together the log. Rising of the given angle, and the log. cosines of the two given sides. The sum is the logarithm of a number, which is to be subtracted from the natural cosine of the difference of two given sides (regard being had to the sign of this cosine). The difference is the natural cosine of the required side.

51. Examples.

1. Calculate the value of log. Ris. of $4^h\ 28^m$.

Solution.

$\frac{1}{2}(4^h\ 28^m) = 2^h\ 14^m$	sin.	9.74189
		2
		19.48378
	—	14.69897
log. Ris. $4^h\ 28^m$	=	4.78481

2. Given in the spherical triangle two sides equal to 45° 54′, and 138° 32′, and the included angle 98° 44′; to solve the triangle.

Solution. I. by (239),

$C =$ 98° 44′	cos.	9.18137 *n.*
$a =$ 45° 54′	tang.	0.01365
$PC =$ 171° 6′ 16″	tang.	9.19502 *n.*

Two sides and the included angle given.

By (240),

$$AP = 171° \;\; 6' \, 16'' - 138° \, 32' = 32° \, 34' \, 16''.$$

By (241),

$PC =$	$171° \;\; 6' \, 16''$	cos. (ar. co.)	10.00525 *n.*
$AP =$	$32° \, 34' \, 16''$	cos.	9.92569
$a =$	$45° \, 54'$	cos.	9.84255
$c =$	$126° \, 24' \, 45''$	cos.	9.77349 *n.*

By (242),

$PC =$	$171° \;\; 6' \, 16''$	sin. (ar. co.)	10.81071
$AP =$	$32° \, 34' \, 16''$	sin.	9.73106
$C =$	$98° \, 44'$	cotan.	9.18644 *n.*
$BAP =$	$118° \;\; 8' \, 19''$	cotan.	9.72821 *n.*

$$A = 180° - 118° \, 8' \, 19'' = 61° \, 51' \, 41''.$$

By (243),

$c =$	$126° \, 24' \, 45''$	sin. (ar. co.)	10.09433
$C =$	$90° \, 44'$	sin.	9.99494
$b =$	$138° \, 32'$	sin.	9.82098
$B =$	$125° \, 34' \, 48''$	sin.	9.91025

Ans. $c = 126° \, 24' \, 45''$

$A = 61° \, 51' \, 41''$

$B = 125° \, 34' \, 48''.$

Two sides and the included angle given.

II. The third side is thus caloulated by means of (249),

	2	log.	0.30103
	$45° 54'$	sin.	9.85620
	$138° 32'$	sin.	9.82098
$\frac{1}{2}(98° 44') =$	$49° 22'$	2 cos.	19.62744
0.40332			9.60565

$-0.99683 = \text{Nat. cos.}(138° 32' + 45° 54') = \text{N. cos. } 184' 26$

$-0.59351 = \text{Nat. cos } 126° 24' 23'' = c.$

III. The third side is thus calculated by § 41.

$98° 44' = 6^h 34^m 56^s$			log. Ris.	5.06139
$45° 54'$			sin.	9.85620
$138° 32'$			sin.	9.82098
		54774		4.73857
$92° 38'$	N. cos.	— 4594		
$c = 126° 25' 8''$	N. cos.	— 59368		

3. Calculate the log. Ris. of $11^h 12^m 20^s$.

Ans. 5.29632.

4. Given in a spherical triangle two sides equal to 100°, and 125°, and the included angle equal to 45°; to solve the triangle.

Ans. The third side $= 47° 55' 52''$

The other two angles $= 69° 43' 48''$, and $= 128° 42' 48''$.

52. *Problem. To solve a spherical triangle, when one of its sides and the two adjacent angles are given.* [B. p. 438.]

Solution. Let ABC (figs. 32 and 33.) be the triangle; a the given side, and B and C the given angles. From B let fall on AC the perpendicular BP.

First. To find PBC, we know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by Neper's Rules,

$$\text{cotan. } PBC = \text{cos. } a \text{ tang. } C. \qquad (253)$$

Secondly. ABP is the difference between ABC and PBC, that is,

$$\text{(fig. 32.) } ABP = B - PBC,$$

$$\text{or} \qquad \text{(fig. 33.) } ABP = PBC - B. \qquad (254)$$

Thirdly. To find the angle A. If, in the triangle PBC, co. C is the middle part, PB and co. PBC are the opposite parts; and if, in the triangle ABP, co. BAP is the middle part, PB and co. ABP are the opposite parts. Hence, by Bowditch's Rules,

$$\text{cos. (co. } PBC) : \text{cos. (co. } ABP) = \text{sin. (co. } C) : \text{sin. (co. } BAP),$$

$$\text{or} \qquad \text{sin. } PBC : \text{sin. } ABP = \text{cos. } C : \text{cos. } BAP; \qquad (255)$$

and BAP is either the angle A or its supplement.

Fourthly. To find the side c. If, in the triangle PBC, co PBC is the middle part, PB and co. a are the adjacent parts; and if, in the triangle ABP, co. ABP is the middle part, PB and co. c are the adjacent parts. Hence, by Bowditch's Rules,

$$\text{cos. } PBC : \text{cos. } ABP = \text{cotan. } a : \text{cotan. } c. \qquad (256)$$

A side and the two adjacent angles given.

Fifthly. b is found by the proportion

$$\sin. C : \sin. c = \sin. B : \sin. b. \quad (257)$$

53. *Scholium.* In determining PBC, BAP, and c by (253), (255), and (256), the signs of the several terms must be carefully attended to, by means of Pl. Trig. § 61.

To determine which value of b, obtained from (257), is the true value, regard must be had to the rules of § 45. But if all these conditions are satisfied by both values of b, then b may be calculated by letting fall a perpendicular from C on the side c in the same way in which c has been obtained in the preceding solution. But this case can be avoided by letting fall the perpendicular from the vertex of that one of the two given angles, which differs the most from 90°.

54. *Corollary.* Since $180° - a$, $180° - b$, and $180° - c$ are the angles of the polar triangle, and $180° - A$, $180° - B$, and $180° - C$ are its sides; we have given in the polar triangle the two sides $180° - B$, and $180° - C$, and the included $180° - a$; so that the polar triangle might be solved by § 44.

55. *Corollary.* If formula (248) is applied to the polar triangle of the preceding section, it becomes by Pl. Trig. § 60,

$$-\cos. A = \cos. B \cos. C - \sin. B \sin. C \cos. a,$$

or

$$\cos. A = -\cos. B \cos. C + \sin. B \sin. C \cos. a. \quad (258)$$

56. *Corollary.* In the same way (249) becomes by (92) and (116),

$$\cos. A = -\cos. (B + C) - 2 \sin. B \sin. C (\sin. \tfrac{1}{2} a)^2, \quad (259)$$

from which the value of the third angle may be found by means of table XXIII.

57. *Corollary.* In the same way (250) becomes by (92),

$$\cos. A = -\cos. (B - C) + 2 \sin. B \sin. C (\cos. \tfrac{1}{2} a)^2, \quad (260)$$

from which the value of the third side may be found.

58. Examples.

1. Given in a spherical triangle one side equal to 175° 27′, and the two adjacent angles equal to 126° 12′, and 109° 16′; to solve the triangle.

Solution. I. By (253),

$a = 175° \ 27'$	cos.	9.99863 *n.*
$C = 109° \ 16'$	tang.	0.45650 *n.*
$PBC = 19° \ 19' \ 24''$	cotan.	0.45513

By (254),

$$ABP = 126° \ 12' - 19° \ 19' \ 24'' = 106° \ 52' \ 36''.$$

By (255),

$PBC = 19° \ 19' \ 24''$	sin. (ar. co.)	10.48031
$ABP = 106° \ 52' \ 36''$	sin.	9.98088
$C = 109° \ 16'$	cos.	9.51847 *n.*
$BAP = 162° \ 36'$	cos.	9.97966 *n.*

A side and the two adjacent angles given.

By (256),

$PBC = 19° 19' 24''$	cos. (ar. co.)	10.02518	
$ABP = 106° 52' 36''$	cos.	9.46288	*n.*
$a = 175° 27'$	cotan.	1.09920	*n.*
$c = 14° 30' 9''$	cotan.	0.58726	

$$A = BAP = 162° 36'.$$

By (257),

$C = 109° 16'$	sin.	(ar. co.)	10.02503
$c = 14° 30' 9''$	sin.		9.39867
$B = 126° 12'$	sin.		9.90685
$b = 167° 38' 21''$	sin.		9.33055

Ans. $A = 162° 36'$
$b = 167° 38' 21''$
$c = 14° 30' 9''$.

II. The third angle is thus calculated by means of (259).

$175° 27' = 11^h 41^m 48^s$		log. Ris.	5.30035
$126° 12'$		sin.	9.90685
$109° 16'$		sin.	9.97497
		-152115	5.18217
$235° 28'$	$-$ N. cos.	56689	
$A = 162° 36' 12''$	N. cos.	$-$ 95426	

Two sides and an opposite angle given.

III. The third angle is thus calculated by means of (260),

		2 log.	0.30103
$\frac{1}{2}(175° 27') = 87° 43' 30''$		2 cos.	17.19750
126° 12′		sin.	9.90685
109° 16′		sin.	9.97497
	0.00240		7.38035
16° 56′	— N. cos. — 0.95664		
$A = 162° 36'$	N. cos. — 0.95424		

2. Given in a spherical triangle one side $= 45° 54'$, and the two adjacent angles $= 125° 37'$, and $= 98° 44'$; to solve the triangle.

Ans. The third angle $= 61° 55' 2''$,

The other two sides $= 138° 34' 22''$, and $= 126° 26' 11''$.

59. *Problem. To solve a spherical triangle, when two sides and an angle opposite one of them are given.* [B. p. 437.]

Solution. Let ABC (figs. 32 and 33.) be the triangle, a and c the given sides, and C the given angle. From B let fall on AC the perpendicular BP.

First. To find PC. We know, in the right triangle PBC, the side a and the angle C. Hence, by Neper's Rules,

$$\text{tang. } PC = \cos. C \text{ tang. } a. \tag{261}$$

Secondly. To find AP. If, in the triangle PBC, co. a is the middle part, CP and PB are the opposite parts; and if,

Two sides and an opposite angle given.

in the triangle *ABP*, co. c is the middle part, *AP* and *PB* are the opposite parts. Hence, by Bowditch's Rules,

$$\cos. a : \cos. c = \cos. PC : \cos. AP. \quad (262)$$

Thirdly. To find b. There are, in general, two triangles which resolve the problem, in one of which (fig. 32.)

$$b = PC + AP, \quad (263)$$

and in the other (fig. 33.)

$$b = PC - AP. \quad (264)$$

But, if *AP* is greater than *PC*, there is but one triangle, as in (fig. 32.), and b is obtained by (263); or, if the sum of *AP* and *PC* is greater than 180°, there is but one triangle, as in (fig. 33.), and b is obtained by (264).

Fourthly. A and B are found by the proportion

$$\sin. c : \sin. C = \sin. a : \sin. A \quad (265)$$

$$\sin. c : \sin. C = \sin. b : \sin. B. \quad (266)$$

60. *Scholium.* In determining *PC* and *AP* by (261) and (262), the signs of the several terms must be carefully attended to by means of Pl. Trig. § 61.

The two values of A, given by (265), correspond respectively to the two triangles which satisfy the problem; and the one, which belongs to each triangle, is to be selected, so that the angle *BAP*, which is the same as A in (fig. 32.), and the supplement of A in (fig. 33.), may be obtuse if C is obtuse, and acute if C is acute. For *BP* is the side opposite *BAP* in the right triangle *ABP*, and the side opposite C in the triangle *BCP*; and therefore, by § 14, *BP*, *BAP*, and C are all at the same time acute, or all obtuse.

Of the two values of B, given by (266), the one which belongs to each triangle is to be determined by means of the rules of § 45.

61. *Scholium.* The problem is, by a proposition of Geometry, impossible, when the given value of c differs more from 90° than that of a; if, at the same time, the value of one of the two quantities, c and C, is acute, while that of the other is obtuse. And in this case we should find that AP was larger than PC, and at the same time that the sum of AP and PC was more than 180°.

62. Examples.

1. Given in the spherical triangle, one side = 35°, a second side = 142°, the angle opposite the second side = 176°; to solve the triangle.

Solution. By (261),

$C = 176°$	cos.	9.99894 *n.*
$a = 35°$	tang.	9.84523
$PC = 145°\ 3'\ 56''$	tang.	9.84417 *n.*

By (262),

$a = 35°$	cos. (ar. co.)	10.08664
$PC = 145°\ 3'\ 56''$	cos.	9.91371 *n.*
$c = 142°$	cos.	9.89653 *n.*
$AP = 37°\ 56'\ 30''$	cos.	9.89688

Two sides and an opposite angle given.

By (264),

$$b = 145° \; 3' \; 56'' - 37° \; 56' \; 30'' = 107° \; 7' \; 26''.$$

By (265),

$c = 142°$		sin. (ar. co.)	10.21066
$C = 176°$		sin.	8.84358
$a = 35°$		sin.	9.75859
$A =$	$3° \; 43' \; 34''$	sin.	8.81283

By (266),

$c = 142°$		sin. (ar. co.)	10.21066
$C = 176°$		sin.	8.84358
$b = 107° \; 7' \; 26''$		sin.	9.98030
$B =$	$6° \; 12' \; 58''$	sin.	9.03454

Ans. $b = 107° \; 7' \; 26''$
$A = 3° \; 43' \; 34''$
$B = 6° \; 12' \; 58''$.

2. Given in a spherical triangle, one side $= 54°$, a second side $= 22°$, the angle opposite the second side $= 12°$; to solve the triangle.

Ans. The third side $= 73° \; 14' \; 29''$, or $= 33° \; 32' \; 59''$.
One angle $= 26° \; 40' \; 49''$, or $= 153° \; 19' \; 11''$.
The third angle $= 147° \; 53' \; 51''$, or $= 17° \; 51' \; 43''$.

63. *Problem. To solve a spherical triangle, when two angles and a side opposite one of them are given.* [B. p. 438.]

Two angles and an opposite side given.

Solution. Let ABC (figs. 32 and 33.) be the triangle, A and C the given angles, and a the given side.

From B let fall on AC the perpendicular BP. This perpendicular must fall within the triangle if A and C are either both obtuse or both acute; but it falls without, if one is obtuse and the other acute.

First. PC may be found by (261).

Secondly. To find AP. If, in the triangle PBC, PC is the middle part, co. C and PB are the adjacent parts; and if, in the triangle ABP, AP is the middle part, co. BAP and BP are the adjacent parts. Hence, by Bowditch's Rules,

$$\text{cotan. } C : \text{cotan. } BAP = \text{sin. } PC : \text{sin. } AP. \qquad (267)$$

Thirdly. To find b. We have

$$\text{(fig. 32.)} \qquad b = PC + AP, \qquad (268)$$

$$\text{(fig. 33.)} \qquad b = PC - AP. \qquad (269)$$

Fourthly. c and B are found by the proportion

$$\text{sin. } A : \text{sin. } a = \text{sin. } C : \text{sin. } c, \qquad (270)$$

$$\text{sin. } a : \text{sin. } A = \text{sin. } b : \text{sin. } B. \qquad (271)$$

64. *Scholium.* Either value of AP, given by (267), may be used, and there will be two different triangles solving the problem, except when $AP + PC$ (fig. 32.) is greater than 180°, or PC (fig. 33.) is less than AP. It may be that both values of AP satisfy the conditions of the problem, or that only one value satisfies them, or that neither value does; in which last case the problem is impossible.

Of the values of c, determined by (270), the true value must be ascertained from the right triangle ABP by § 12; or since PB and C are both acute or both obtuse at the same time; it follows, from § 12, that when C and AP are both acute or both obtuse, that c is acute; but when one of them is obtuse and the other acute, c is obtuse.

From the two values of B (271), the true value must be selected by means of the rules of § 45.

65. *Scholium.* The problem is impossible, by Geometry, when A differs more from 90° than does C, and when at the same time one of the two quantities a and A is acute, while the other is obtuse. This case is precisely the same as the impossible case of § 61.

66. Examples.

1. Given in a spherical triangle, one angle = 95°, a second angle = 104°, and the side opposite the first angle = 138°; to solve the triangle.

Solution. By (261),

$C =$	104°	cos.	9.38368 n.
$a =$	138°	tang.	9.95444 n.
$PC =$	12° 17′ 20″	tang.	9.33812

By (267),

$C =$	104°	cotan. (ar. co.)	0.60323 n.
$PC =$	12° 17′ 20″	sin.	9.32802
$BAP =$	95°	cotan.	8.94195 n.
$AP =$	4° 16′ 59″	sin.	8.87320

Two angles and an opposite side given.

By (268),

$$b = 12° \ 17' \ 20'' + 4° \ 16' \ 59'' = 16° \ 34' \ 19''.$$

By (270),

$A =$	$95°$	sin. (ar. co.)	10.00166
$a =$	$138°$	sin.	9.82551
$C =$	$104°$	sin.	9.98690
$c =$	$139° \ 19' \ 40''$	sin.	9.81407

By (271),

$a =$	$138°$	sin. (ar. co.)	10.17449
$A =$	$95°$	sin.	9.99834
$b =$	$16° \ 34' \ 19''$	sin.	9.45518
$B =$	$25° \ 7' \ 38''$	sin.	9.62801

Again, by (269),

$$b = 12° \ 17' \ 20'' - 4° \ 16' \ 59'' = 7° \ 0' \ 21''$$

$$c = 180° - 139° \ 19' \ 40'' = 40° \ 40' \ 20''.$$

By (271),

$a =$	$138°$	sin. (ar. co.)	10.17449
$A =$	$95°$	sin.	9.99834
$b =$	$7° \ 0' \ 21''$	sin.	9.08623
$B =$	$10° \ 27' \ 42''$	sin.	9.25906

Ans. $b = 16° \ 34' \ 19''$ or $= 7° \ 0' \ 21''$
$c = 139° \ 19' \ 40''$ or $= 40° \ 40' \ 20''$
$B = 25° \ 7' \ 38''$ or $= 10° \ 27' \ 42''$.

2. Given in a spherical triangle, one angle $= 104°$, a second angle $= 95°$, and the side opposite the first angle $= 138°$; to solve the triangle.

Ans. The two sides are $17° 22' 13''$, and $136° 36' 27''$.
The other angle is $25° 39' 9''$.

67. *Problem. To solve a spherical triangle, when its three sides are given.*

Solution. Equation (248) gives, by transposition and division,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}, \qquad (272)$$

whence the value of the angle C may be calculated, and in the same way either of the other angles.

68. *Corollary.* An equation, more easy for calculation by logarithms, may be obtained from (249), which gives, by transposition and division,

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{\cos. c - \cos. (a + b)}{\sin. a \sin\ b}. \qquad (273)$$

Now, letting s denote half the sum of the sides, or

$$s = \tfrac{1}{2} (a + b + c); \qquad (274)$$

if we make in (35)

$$M = \tfrac{1}{2} (a + b + c) = s,$$
$$N = \tfrac{1}{2} (a + b - c) = s - c;$$

we have

$$M + N = a + b,$$
$$M - N = c;$$

and (35) becomes

$$\cos. c - \cos. (a + b) = 2 \sin. s \sin. (s - c);$$

which, substituted in (273), gives

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{2 \sin. s \sin. (s - c)}{\sin. a \sin. b}, \qquad (275)$$

and

$$\cos. \tfrac{1}{2} C = \sqrt{\left(\frac{\sin. s \sin. (s - c)}{\sin. a \sin. b}\right)}. \qquad (276)$$

69. *Corollary.* The angles A and B may be found by the two following equations, which are easily deduced from (276),

$$\cos. \tfrac{1}{2} A = \sqrt{\left(\frac{\sin. s \sin. (s - a)}{\sin. b \sin. c}\right)}, \qquad (277)$$

$$\cos. \tfrac{1}{2} B = \sqrt{\left(\frac{\sin. s \sin. (s - b)}{\sin. a \sin. c}\right)}. \qquad (278)$$

70. *Corollary.* Another equation, equally simple in calculation, can be obtained from (250), which gives, by transposition and division,

$$\cos. (a - b) = \cos. a \cos. b + \sin. a \sin b,$$

which, substituted in the numerator of (250), gives

$$2 (\sin. \tfrac{1}{2} C)^2 = \frac{\cos. (a - b) - \cos. c}{\sin. a \sin. b}, \qquad (279)$$

whence C can be found by Table XXIII.

71. *Corollary.* If, in (35), we make

$$M = \tfrac{1}{2} (a - b + c) = s - b,$$

$$N = \tfrac{1}{2} (-a + b + c) = s - a,$$

The three sides given.

we have

$$M + N = c,$$
$$M - N = a - b,$$

and (35) becomes

$$\cos.\,(a - b) - \cos.\,c = 2 \sin.\,(s - a) \sin.\,(s - b);$$

which, substituted in (279), gives

$$2\,(\sin.\,\tfrac{1}{2}\,C)^2 = \frac{2 \sin.\,(s - a) \sin.\,(s - b)}{\sin.\,a \sin.\,b}, \qquad (280)$$

and

$$\sin.\,\tfrac{1}{2}\,C = \sqrt{\left(\frac{\sin.\,(s - a) \sin.\,(s - b)}{\sin.\,a \sin.\,b}\right)}. \qquad (281)$$

72. *Corollary.* In the same way we might deduce the following equations;

$$\sin.\,\tfrac{1}{2}\,A = \sqrt{\left(\frac{\sin.\,(s - b) \sin.\,(s - c)}{\sin.\,b \sin.\,c}\right)}, \qquad (282)$$

$$\sin.\,\tfrac{1}{2}\,B = \sqrt{\left(\frac{\sin.\,(s - a) \sin.\,(s - c)}{\sin.\,a \sin.\,c}\right)}. \qquad (283)$$

73. *Corollary.* The quotient of (282), divided by (277), is by (7),

$$\text{tang.}\,\tfrac{1}{2}A = \frac{\sin.\,\tfrac{1}{2}\,A}{\cos.\,\tfrac{1}{2}\,A} = \sqrt{\left(\frac{\sin.\,(s - b) \sin.\,(s - c)}{\sin.\,s \sin.\,(s - a)}\right)}. \quad (284)$$

In the same way,

$$\text{tang.}\,\tfrac{1}{2}\,B = \sqrt{\left(\frac{\sin.\,(s - a) \sin.\,(s - c)}{\sin.\,s \sin.\,(s - b)}\right)}, \qquad (285)$$

$$\text{tang.}\,\tfrac{1}{2}\,C = \sqrt{\left(\frac{\sin.\,(s - a) \sin.\,(s - b)}{\sin.\,s \sin.\,(s - c)}\right)}. \qquad (286)$$

The three sides given.

74. Examples.

I. Given in the spherical triangle ABC the three sides equal to 46°, 72°, and 68°; to solve the triangle.

Solution. I.	By (277),	by (278),	by (279),
$a=46°$ sin.		(ar.co.) 10.14307	(ar.co.) 10.14307
$b=72°$ sin.	(ar.co.) 10.02179		(ar.co.) 10.02179
$c=68°$ sin.	(ar.co.) 10.03283	(ar.co.) 10.03283	
$s=93°$ sin.	9.99940	9.99940	9.99940
$s-a=47°$ sin.	9.86413		
$s-b=21°$ sin.		9.55433	
$s-c=25°$ sin.			9.62595
	2)19.91815	2)19.72963	2)19.79021
cos.	9.95908	9.86482	9.89511

$\frac{1}{2}A = 24° \, 29'$, $\frac{1}{2}B = 42° \, 54'$, $\frac{1}{2}C = 38° \, 14' \, 18''$;

Ans. $A = 48° \, 58'$, $B = 85° \, 48'$, $C = 76° \, 28' \, 56''$.

II. By Table XXIII and equation (279),

$a - b = 26°$	N. cos.	89879		
$c = 68°$	N. cos.	37461		
		52418	log.	4.71948
$a = 46°$			sin. (ar. co.)	0.14307
$b = 72°$			sin. (ar. co.)	0.02179
$C = 5^h \, 5^m \, 55^s = 76° \, 28' \, 45''$			log. Ris.	4.88434

2. Given in a spherical triangle the three sides equal to 3°, 4°, and 5°; to solve the triangle.

Ans. The three angles are 36° 54′, 53° 10′, and 90° 2′.

75. Neper obtained two theorems for the solution of a spherical triangle, when a side and the two adjacent angles are given, by which the two sides can be calculated without the necessity of calculating the third angle. These theorems, which are given in § 78 and 79, can be obtained from equations (284–286) by the assistance of the following lemmas.

76. *Lemma.* If we have the equation

$$\frac{\text{tang. } M}{\text{tang. } N} = \frac{x}{y}, \tag{287}$$

we can deduce from it the following equation,

$$\frac{\text{sin. } (M + N)}{\text{sin. } (M - N)} = \frac{x + y}{x - y}. \tag{288}$$

Proof. We have from (7)

$$\text{tang.} M = \frac{\text{sin. } M}{\text{cos.} M}, \text{ and tang. } N = \frac{\text{sin. } N}{\text{cos.} N};$$

which, substituted in (287), give

$$\frac{\text{sin. } M \text{cos. } N}{\text{cos. } M \cdot \text{sin. } N} = \frac{x}{y}.$$

This equation is the same as the proportion

$$\text{sin. } M \text{ cos. } N : \text{cos. } M \text{ sin. } N = x : y;$$

hence, by the theory of proportions,

$$\sin. M \cos. N + \cos. M \sin. N : \sin. M \cos. N - \cos. M \sin. N = x + y : x - y,$$

or, by (26) and (27),

$$\sin. (M + N) : \sin. (M - N) = x + y : x - y;$$

which may be written in the form of an equation, as in (288).

77. *Lemma.* If we have the equation

$$\text{tang.}\, M \,\text{tang.}\, N = \frac{x}{y}; \tag{289}$$

we can deduce from it the equation

$$\frac{\cos. (M + N)}{\cos. (M - N)} = \frac{y - x}{y + x}. \tag{290}$$

Proof. We have, by (289) and (7),

$$\frac{\sin. M \sin. N}{\cos. M \cos. N} = \frac{x}{y}.$$

This equation is the same as the proportion

$$\cos. M \cos. N : \sin. M \sin. N = y : x;$$

hence, by the theory of proportions,

$$\cos. M \cos. N - \sin. M \sin. N : \cos. M \cos. N + \sin. M \sin. N = y - x : x + y,$$

or, by (28) and (29),

$$\cos. (M + N) : \cos. (M - N) = y - x : y + x;$$

which may be written as in (290).

78. *Theorem. The sine of half the sum of two angles of a spherical triangle is to the sine of half their difference, as the tangent of half the side to which they are*

both adjacent is to the tangent of half the difference of the other two sides; that is, in the spherical triangle ABC (figs. 32 and 33.),

$$\sin.\tfrac{1}{2}(A+C) : \sin.\tfrac{1}{2}(A-C) = \text{tang}.\tfrac{1}{2}b : \text{tang}.\tfrac{1}{2}(a-c). \quad (291)$$

Proof. The quotient of (284), divided by (286) is, by an easy reduction,

$$\frac{\text{tang}.\tfrac{1}{2}A}{\text{tang}.\tfrac{1}{2}C} = \frac{\sin.(s-c)}{\sin.(s-a)}. \quad (292)$$

Hence, by § 76,

$$\frac{\sin.\tfrac{1}{2}(A+C)}{\sin.\tfrac{1}{2}(A-C)} = \frac{\sin.(s-c)+\sin.(s-a)}{\sin.(s-c)-\sin.(s-a)}. \quad 93)$$

If we make in equation (40)

$$A = s - c = \tfrac{1}{2}(a+b-c),$$

$$B = s - a = \tfrac{1}{2}(-a+b+c);$$

we have

$$A + B = b,$$

$$A - B = a - c;$$

and (40) becomes

$$\frac{\sin.(s-c)+\sin.(s-a)}{\sin.(s-c)-\sin.(s-a)} = \frac{\text{tang}.\tfrac{1}{2}b}{\text{tang}.\tfrac{1}{2}(a-c)}.$$

This equation, substituted in the second member of (293), gives

$$\frac{\sin.\tfrac{1}{2}(A+C)}{\sin.\tfrac{1}{2}(A-C)} = \frac{\text{tang}.\tfrac{1}{2}b}{\text{tang}.\tfrac{1}{2}(a-c)}; \quad (294)$$

which is the same as (291).

79. *Theorem. The cosine of half the sum of two angles of a spherical triangle is to the cosine of half*

their difference, as the tangent of half the side to which they are both adjacent is to the tangent of half the sum of the other two sides; that is, in the spherical triangle ABC (figs. 32 and 33.),

$$\cos.\tfrac{1}{2}(A+C):\cos.\tfrac{1}{2}(A-C)=\text{tang.}\tfrac{1}{2}b:\text{tang.}\tfrac{1}{2}(a+c). \quad (295)$$

Proof. The product of (284) and (286) is, by a simple reduction,

$$\text{tang.}\tfrac{1}{2}A\,\text{tang.}\tfrac{1}{2}C=\frac{\sin.(s-b)}{\sin.s};$$

hence, by § 77,

$$\frac{\cos.\frac{1}{2}(A+C)}{\cos.\frac{1}{2}(A-C)}=\frac{\sin.s-\sin.(s-b)}{\sin.s+\sin.(s-b)}. \quad (296)$$

If in equation (40) inverted we make

$$A=s=\tfrac{1}{2}(a+b+c),$$
$$B=s-b=\tfrac{1}{2}(a-b+c);$$

we have

$$A+B=a+c,$$
$$A-B=b;$$

and (40) becomes

$$\frac{\sin.s-\sin.(s-b)}{\sin.s+\sin.(s-b)}=\frac{\text{tang.}\frac{1}{2}b}{\text{tang.}\frac{1}{2}(a+c)}.$$

This equation, substituted in (296), gives

$$\frac{\cos.\frac{1}{2}(A+C)}{\cos.\frac{1}{2}(A-C)}=\frac{\text{tang.}\frac{1}{2}b}{\text{tang.}\frac{1}{2}(a+c)}, \quad (297)$$

which is the same as (295).

Neper's Analogies.

80. *Scholium.* In using (291) and (295), the signs of the terms must be attended to by means of Pl. Trig. § 61.

81. Examples.

1. Given in a spherical triangle two angles = 158°, and = 98°, and the included side = 144°; to find the other sides.

Solution. By (291),

$\frac{1}{2}(A+C)$	$= 128°$	sin. (ar. co.)	10.10347
$\frac{1}{2}(A-C)$	$= 30°$	sin.	9.69897
$\frac{1}{2}\,b$	$= 72°$	tang.	0.48822
$\frac{1}{2}(a-c)$	$= 62°\ 53'\ 1''$	tang.	0.29066

By (295),

$\frac{1}{2}(A+C)$	$= 128°$	cos. (ar. co.)	10.21066 *n.*
$\frac{1}{2}(A-C)$	$= 30°$	cos.	9.93753
$\frac{1}{2}\,b$	$= 72°$	tang.	0.48822
$\frac{1}{2}(a+c)$	$= 103°\ 0'\ 25''$	tang.	0.63641 *n.*

Ans. $a = 165°\ 53'\ 26''$,

$c = 40°\ 7'\ 24''$.

2. Given in a spherical triangle two angles = 170°, and = 2°, and the included side = 92°; to find the other sides.

Ans. $a = 103°\ 6'\ 30''$,

$c = 11°\ 17'\ 30''$.

Three angles given.

82. *Problem. To solve a spherical triangle, when its three angles are given.*

Solution. If A, B, C are the angles of the given triangle, and a, b, c its sides $180° - A$, $180° - B$, $180° - C$ are the sides of the polar triangle, and $180° - a$, $180° - b$, $180° - c$ the angles of the polar triangle, the sides are then given in the polar triangle; to find the angles. For this purpose we may use the formulas of the preceding problem.

83. *Corollary.* Applying (272) to the polar triangle gives

$$\cos. c = -\frac{\cos. C + \cos. A \text{ and } B}{\sin. A \sin. B}. \quad (298)$$

84. *Corollary.* Equations (276–278) give, for the polar triangle, if we put

$$S = \tfrac{1}{2}(A + B + C), \quad (299)$$

if we use (71 and 72),

$$\sin. \tfrac{1}{2} a = \sqrt{\left(\frac{-\cos. S \cos. (S - A)}{\sin. B \sin. C}\right)}, \quad (300)$$

$$\sin. \tfrac{1}{2} b = \sqrt{\left(\frac{-\cos. S \cos. (S - B)}{\sin. A \sin. C}\right)}, \quad (301)$$

$$\sin. \tfrac{1}{2} c = \sqrt{\left(\frac{-\cos. S \cos. (S - C)}{\sin. A \sin. B}\right)}. \quad (302)$$

85. *Corollary.* Equations (281–283), applied to the polar triangle, give

$$\cos. \tfrac{1}{2} a = \sqrt{\left(\frac{\cos. (S - B) \cos. (S - C)}{\sin. B \sin. C}\right)}, \quad (303)$$

$$\cos. \tfrac{1}{2} b = \sqrt{\left(\frac{\cos. (S - A) \cos. (S - C)}{\sin. A \sin. C}\right)}, \quad (304)$$

Three angles given.

$$\cos. \tfrac{1}{2} c = \sqrt{\left(\frac{\cos. (S - A) \cos. (S - B)}{\sin. A \sin. B}\right)}. \quad (305)$$

86. *Corollary.* Equations (284–286), applied to the polar triangle, give

$$\text{tang.} \tfrac{1}{2} a = \sqrt{\left(\frac{-\cos. S \cos. (S - A)}{\cos. (S - B) \cos. (S - C)}\right)}, \quad (306)$$

$$\text{tang.} \tfrac{1}{2} b = \sqrt{\left(\frac{-\cos. S \cos. (S - B)}{\cos. (S - A) \cos. (S - C)}\right)}, \quad (307)$$

$$\text{tang.} \tfrac{1}{2} c = \sqrt{\left(\frac{-\cos. S \cos. (S - C)}{\cos. (S - A) \cos. (S - B)}\right)}. \quad (308)$$

87. *Corollary.* Equation (273), applied to the polar triangle, is

$$2 (\sin. \tfrac{1}{2} c)^2 = \frac{-\cos. C - \cos. (A + B)}{\sin. A \sin. B}, \quad (309)$$

which may be used like equation (279).

88. Example.

Given in the spherical triangle ABC, the three angles equal to 89°, 5°, and 88°; to solve the triangle.

Ans. The three sides are 53° 10′, 4°, and 53° 8′.

89. *Theorem. The sine of half the sum of two sides of a spherical triangle is to the tangent of half their difference, as the cotangent of half the included angle is to the tangent of half the difference of the other two angles*, that is, in ABC (figs. 32 and 33.),

$$\sin. \tfrac{1}{2}(a + c) : \sin. \tfrac{1}{2}(a - c) = \text{cotan.} \tfrac{1}{2} B : \text{tang.} \tfrac{1}{2}(A - C) \quad (310)$$

Proof. This theorem is at once obtained by applying § 78 to the polar triangle.

90. *Theorem. The cosine of half the sum of two sides of a triangle is to the cosine of half their difference, as the cotangent of half the included angle is to the tangent of half the sum of the other two angles*, or in (figs. 32 and 33.),

$$\cos.\tfrac{1}{2}(a+c(: \cos.\tfrac{1}{2}(a-c) = \text{cotan}.\tfrac{1}{2}B : \text{tang}.\tfrac{1}{2}(A+C). \quad (311)$$

Proof. This theorem is at once obtained by applying § 79 to the polar triangle

91. *Corollary.* These two theorems, similar to § 78 and 79, were given by Neper for the solution of the case, in which two sides and the included angle are given. By means of them the other two angles can be found without the necessity of calculating the third side. In using them regard must be had to the signs of the terms by means of Pl. Trig. § 61.

92. Examples.

1. Given in a spherical triangle two sides = 149°, and = 49°, and the included angle = 88°; to find the other angles.

Solution. By § 89,

$\frac{1}{2}(a+c) = 99°$	sin. (ar. co.)	10.00538
$\frac{1}{2}(a-c) = 50°$	sin.	9.88425
$\frac{1}{2}B = 44°$	cotan.	0.01516
$\frac{1}{2}(A-C) = 38° 46' 10''$	tang.	9.90479

Neper's Analogies.

By § 90,

$\frac{1}{2}(a+c) =$	99°	cos. ar. co.)	10.80567 *n.*
$\frac{1}{2}(a-c) =$	50°	cos.	9.80807
$\frac{1}{2}B$	$=$ 44°	cotan.	0.01516
$\frac{1}{2}(A+C) =$	103° 12′ 31″	tang.	0.62890 *n.*

Ans. $A = 141° 59' 41''$,

$C = 64° 27' 21''$.

2. Given in a spherical triangle two sides = 13°, and = 9°, and the included angle = 176°; to find the other angles.

Ans. 2° 24′ 7″, and 1° 40′ 13″.

SPHERICAL ASTRONOMY.

SPHERICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE AND ITS CIRCLES.

1. *Astronomy* is the science which treats of the heavenly bodies.

2. *Mathematical Astronomy* is the science which treats of the positions and motions of the heavenly bodies.

The elements of position of a heavenly body are (Geo. § 8) distance and direction.

3. *Spherical Astronomy* regards only one of the elements of position, namely, direction, and usually refers all directions to the centre of the earth.

4. In spherical astronomy all the stars may, then, be regarded as at the same distance from the earth's centre upon the surface of a sphere, which is called the *celestial sphere.*

Upon this imaginary sphere are supposed to be drawn various circles, which are divided into the well known classes of *great* and *small* circles. [B. p. 47.]

"All angular distances on the surface of the sphere, to an eye at the centre, are measured by arcs of *great* circles." [B. p. 48.]

5. "*Secondaries* to a great circle are great circles which pass through its poles, and are consequently perpendicular to it." [B. p. 48.]

6. "If the plane of the *terrestrial equator* be produced to the celestial sphere, it marks out a circle called the *celestial equator;* and if the axis of the earth be produced in like manner, it becomes the *axis* of the celestial sphere; and the points of the heavens, to which it is produced, are called *the poles*, being the poles of the celestial equator."

"The star nearest to each pole is called the *pole star.*" [B. p. 48.]

7. "Secondaries to the celestial equator are called *circles of declination;* of these 24, which divide the equator into equal parts of 15° each, are called *hour circles.*"

"Small circles, parallel to the celestial equator, are called *parallels* of declination." [B. p. 48.]

The parallels of declination correspond, therefore, to the terrestrial parallels of latitude, and the circles of declination to the terrestrial meridians. A certain point of the celestial equator has been fixed by astronomers, and is called the *vernal equinox.* The circle of declination, which passes through the vernal equinox, bears the same relation to other circles of declination, which the first meridian does to other terrestrial meridians.

8. "The *declination* of a star is its angular distance from the celestial equator," measured upon its circle of declination. [B. p. 49.]

9. The *right ascension* of a star is the arc of the equator intercepted between its circle of declination and the vernal equinox. [B. p. 49.]

Right ascension is either estimated in degrees, minutes, &c. from 0° to 360°; or in hours, minutes, &c. of time, 15 degrees being allowed for each hour, as in Sph. Trig. § 3.

The positions of the stars are completely determined upon the celestial sphere, when their right ascensions and declinations are known. Catalogues of the stars have accordingly been given, containing their right ascensions and declinations. [B. Table viii. p. 80.]

10. "The *sensible horizon* is that circle in the heavens, whose plane touches the earth at the spectator."

"The *rational horizon* is a great circle of the celestial sphere parallel to the sensible horizon." [B. p. 48.]

11. The radius, which is drawn to the observer, is called the *vertical* line.

The point, where the vertical line meets the celestial sphere *above* the observer, is called the *zenith;* the opposite point, where this line meets the sphere *below* the observer, is called the *nadir.*

Hence the vertical line is a radius of the celestial sphere perpendicular to the horizon; and the zenith and nadir are the poles of the horizon. [B. p. 48.]

12. Circles whose planes pass through the vertical line are called *vertical* circles. [B. p. 48.]

The vertical circles are secondaries to the horizon.

13. The vertical circle at any place, which is also a circle of declination, is called the *celestial meridian* of that place. [B. p. 48.]

The plane of the celestial meridian of a place is the same with that of the terrestrial meridian.

14. The points, where the celestial meridian cuts the horizon, are called the *north* and *south* points. [B. p. 48.]

The north point corresponds to the north pole, and the south point to the south pole.

15. The vertical circle, which is perpendicular to the meridian, is called the *prime vertical.* [B. p. 48.]

16. The points, where the prime vertical cuts the horizon, are called the *east* and *west* points. [B. p. 48.]

"To an observer, whose face is directed towards the south, the east point is to his left hand, and the west to his right hand. Hence the east and west points are 90° distant from the north and south. These four are called the *cardinal* points."

"The meridian of any place divides the heavens into two hemispheres, lying to the east and west; that lying to the east is called the *eastern* hemisphere, and the other the *western* hemisphere."

17. The *altitude* of a star is its angular distance from the horizon, measured upon the vertical circle passing through the star. [B. p. 48.]

18. The *azimuth* of a star is the arc of the horizon intercepted between its vertical circle and the north or south point. [B. p. 48.]

A star may be found without difficulty, when its altitude and azimuth are known. But these elements of position are constantly varying.

CHAPTER II.

THE DIURNAL MOTION.

19. "Stars are distinguished into two kinds, *fixed* and *wandering*." [B. p. 45.]

Most of the stars are fixed, that is, retain constantly almost the same relative position; so that the same celestial globes and maps continue to be accurate representations of the firmament for many years. This is a fact of fundamental importance, and furnishes the fixed points for arriving at a complete knowledge of the celestial motions. Small changes of position have, indeed, been detected even in the fixed stars, as will be shown in the course of this treatise; but these changes are too small to disturb the general fact; they are, indeed, too small ever to have been detected, if the positions of the stars had been subject to great variations.

20. Of the wandering stars there are eleven, which are called *planets*. They are *Mercury* (☿), *Venus* (♀), *the Earth* (⊕), *Mars* (♂), *Vesta* (⚶), *Juno* (⚵), *Pallas* (⚴), *Ceres* (⚳), *Jupiter* (♃), *Saturn* (♄), and *Uranus* (♅). [B. p. 45.]

21. For the sake of remembering the stars with greater ease, they have been divided into groups called *constellations;* and to give distinctness to the constellations, they have been supposed to be circumscribed by

the outlines of some figure which they were imagined to resemble. [B. p. 45.]

The stars have also been distinguished according to their brilliancy, as of the *first*, *second*, &c. magnitude.

Proper names have been given to the constellations and to the most remarkable stars.

The catalogues and the maps of the stars are now so accurate, that no new star could appear without being detected; and any change in the place of any of the larger stars would be immediately discovered.

22. All the stars appear to have a common motion, by which they are carried round the earth from east to west in 24 hours. This rotation of the heavens, or of the celestial sphere, is called the *diurnal motion.*

By its diurnal motion, the celestial sphere rotates, with the most perfect uniformity, about its axis. The pole star would, therefore, if it were exactly at the pole, remain stationary; but since it is not exactly at the pole, it revolves in a very small parallel of declination about the stationary pole.

Any star in the equator revolves in the plane of the equator, and all other stars revolve in the planes of the parallels of declination in which they are situated.

If O (fig. 34.) is the place of the observer, $NESW$ his horizon, Z his zenith, P and P' the poles, the star which is at the distance from P,

$$PM = PM'$$

will appear to describe the circumference $MH'M'H$. It will rise in the east at H and set at H', if the distance PM'

from the pole is greater than the altitude PN of the pole. But if its distance from the pole

$$PL = PL'$$

is less than PN, the star will not set, but will describe a circle above the horizon; and if its distance from the pole

$$PG = PG'$$

is greater than the greatest distance PS from the pole to the horizon, the star will never rise so as to be seen by the observer at O, but will describe a circle below the horizon.

23. The time which it takes a star to pass from any position round again to the same position is called a *sideral* day, that is, literally, a star-day. This day is divided into 24 hours, and clocks regulated to this time are said to denote *sideral time.* [B. p. 147.]

24. Each point of the celestial equator passes the meridian once in a sideral day; and the arc contained between two hour circles passes it in a sideral hour. The sideral time, therefore, which has elapsed since the vernal equinox was upon the equator, is equal to the right ascension of the meridian expressed in time. [B. p. 208.]

The meridian changes its right ascension at each instant, precisely *as if* the celestial sphere were stationary, while the observer, with his meridian and zenith, is carried uniformly round the earth's centre from west to east once in a sideral day.

25. The angle MPB (fig. 35.), which the circle of declination of the star makes with the meridian, is called its *hour angle*.

While the star moves from the point M in the meridian to the point B with an uniform motion, the arc MP is carried to the position PB, and the angle MPB is described with an uniform motion. This angle converted into time is, then, the sideral time since the passage of the star over the meridian.

26. *Corollary.* The difference of the right ascensions of the star and of the meridian is the hour angle of the star.

27. The distance of a star from the east or west points of the meridian, at the time of its rising or setting, is the *amplitude* of the star. [B. p. 48.]

28. *Problem. To find the altitude and azimuth of a star, when its declination and hour angle are known, and also the latitude of the place.*

Solution. If P (fig. 35.) is the pole, Z the zenith, and B the star; we have

$PZ =$ polar dist. of zenith $=$ co. latitude $= 90° - L$,

$PN = 90° - PZ = L$,

$PB =$ polar dist. of star $= p$,

$=$ co. declination of star, when it is on the same side of the equator with the pole.

$= 90° +$ declination of star, when it is on the different side of the equator from the pole.

$= 90° \mp D$,

$ZB =$ zenith dist. of star $= z$,

$=$ co. altitude of star, when it is above the horizon.

$= 90° +$ depression of star, when it is below the horizon.

$ZPB = *$'s hour angle $= h$,

$PZB =$ azimuth of star counted from the direction of the elevated pole.

$= a =$ azimuth, when less than 90°,

$= 180° -$ azimuth, when greater than 90°.

There are, then, given in the spherical triangle PZB, the two sides PZ and PB, and the included angle ZPB; so that the side BZ and the angle PZB can be calculated by Sph. Trig. § 44.

If we let fall the perpendicular BC upon PZ,

$$\text{tang.} PC = \cos. h \text{ tang.} (90° \mp D) = \mp \cos. h \text{ cotan.} D \quad (312)$$

$$CZ = PZ - PC = 90° - (L + PC)$$

$$\text{or} \quad = PC - PZ = (L + PC) - 90°. \quad (313)$$

Hence, by (241),

$$\cos. PC : \sin. (L + PC) = \pm \sin. D : \cos. z; \quad (314)$$

in which formulas the upper sign is used when the star is upon the same side of the equator with the elevated pole, that is, when D and L are of the same name; and, by (242),

$$\sin. PC : \cos. (L + PC) = \text{cotan.} h : \text{cotan.} a. \quad (315)$$

29. *Corollary.* When the altitude and azimuth are both to be found, the calculation by the above method is as short as

To find a star's altitude.

by any other; but when, as is usually the case, the altitude only is required, the following method is preferable.

We have

$$PZ + PB = 180° - L \mp D = 180° - (L \pm D)$$

$$PB - PZ = \mp D + L = L \mp D);$$

whence, by (249) and (250),

$$\cos. z = -\cos. (L \pm D) + 2 \cos. D \cos. L (\cos. \tfrac{1}{2} h)^2 \quad (316)$$

$$\cos. z = \cos. (L \mp D) - 2 \cos. D \cos. L (\sin. \tfrac{1}{2} h)^2, \quad (317)$$

which may be used at once, and (317) may be calculated by the aid of the column of Rising in Table XXIII. The rule obtained from (317) is the same with that on p. 250 of the Navigator, remembering that when the star is above the horizon

$$\cos. z = \sin. *\text{'s alt.} \quad (318)$$

But when the star is below the horizon

$$\cos. z = -\sin. *\text{'s depression.} \quad (319)$$

30. *Corollary.* If the given hour angle is $6^h = 90°$, the problem is at once reduced to the solution of a right triangle. We in this case have, by Napier's Rules,

$$\cos. z = \sin. L \cos. p,$$

or

$$\sin. *\text{'s alt.} = \pm \sin. L \sin. D \quad (320)$$

$$\text{cotan. } a = \cos. L \text{ cotan. } p$$

$$\text{cotan. } *\text{'s azimuth} = \pm \cos. L \text{ tang. } D. \quad (321)$$

The upper sign is to be used in formulas (320) and (321), when the declination is of the same name with the latitude; otherwise the lower sign. In the former case, therefore, the star is above the horizon when its hour angle is six hours, and on the same side of the prime vertical with the elevated pole; but, in the latter case, it is below the horizon, and on the same side of the prime vertical with the depressed pole.

31. *Corollary.* If the star is in the celestial equator, as in (fig. 36.), we have in the right triangle BZQ,

$$BQ = BPQ = h$$

$$ZQ = L$$

$$QZB = 180° - a,$$

whence $$\cos. z = \cos. L \cos. h,$$

or $$\sin. \ast\text{'s alt.} = \cos. L \cos. h \qquad (322)$$

$$\text{cotan.} (180° - a) = \sin. L \text{ cotan. } h,$$

or $$\text{cotan. } a = - \sin. L \text{ cotan. } h. \qquad (323)$$

Hence, if the hour angle is less than six hours, the star which moves in the celestial equator is above the horizon, and on the same side of the prime vertical with the depressed pole; but if the hour angle is greater than six hours, this star is below the horizon, and on the same side of the prime vertical with the elevated pole.

32. *Corollary.* If the place is at the equator, as in (fig. 37.), the celestial equator ZE is the prime vertical, so that if the hour circle PB is produced to C, we have in the right triangle ZBC

$$ZC = ZPB = h$$

$$BZC = 90° - a$$

$$BC = D,$$

whence $$\cos. z = \cos. D \cos. h,$$

or $$\sin. \ast\text{'s alt.} = \cos. D \cos. h \qquad (324)$$

$$\text{cotan.} (90° - a) = \sin. h \text{ cotan. } D,$$

or $$\text{tang. } a = \sin. h \text{ cotan. } D; \qquad (325)$$

so that the star is above the horizon when the hour angle is less than six hours, and below the horizon when the hour angle is greater than six hours.

33. Examples.

1. Find the altitude and azimuth of Aldebaran to an observer at Boston, in the year 1830, when the hour angle of this star is $3^h\ 25^m\ 12^s$.

Solution. We find by tables VIII and LIV

$$D = 16°\ 11'\ \text{N.} \qquad L = 42°\ 21'\ \text{N.}$$

Hence

$h = 3^h\ 25^m\ 12^s$	log. col. Ris.		4.57375
$L = 42°\ 21'$	cos.		9.86867
$D = 16°\ 11'$	cos.		9.98244
		26599	4.42486
$L - D = 26°\ 10'$	nat. cos.	89752	
alt. $= 39°\ 10'$	nat. sin.	63153	sec. 10.11052
	$h = 51°\ 18'$		sin. 9.89233
	$D.$		cos. 9.98244
azimuth from South $= 75°\ 10'$			sin. 9.98529

To find a star's altitude and azimuth.

2. Find the altitude and azimuth of Aldebaran at Boston, in the year 1830, six hours after it has passed the meridian.

Solution. By formulas (320) and (321),

$L = 42° 21'$	sin.	9.82844	cos.	9.86867
$D = 16° 11'$	sin.	9.44516	tang.	9.46271
alt. $= 10° 49'$	sin.	9.27360		
azimuth from North $= 77° 54'$			cotan.	9.33138

3. Find the altitude and azimuth of a star in the celestial equator, to an observer at Boston, when the hour angle of the star is $3^h\ 25^m\ 12^s$.

Solution. By formulas (322) and (323),

$L = 42° 21'$	cos.	9.86867	sin.	9.82844
$h = 51° 18'$	cos.	9.79605	cotan.	9.90371
alt. $= 27° 31'$	sin.	9.66472		
azimuth from South $= 61° 39'$			cotan.	9.73215

4. Find the altitude and azimuth of Aldebaran to an observer at the equator, in the year 1830, when the hour angle of the star is $3^h\ 25^m\ 12^s$.

Solution. By formulas (324) and (325),

$D = 16° 11'$	cos.	9.98244	cotan.	10.53729
$h = 51° 18'$	cos.	9.79605	cotan.	9.90371
alt. $= 36° 54'$	sin.	9.77849		
azimuth from North $= 70° 5'$			tang.	10.44100

Altitude and azimuth.

5. Find the altitude and azimuth of Fomalhaut to an observer at Boston, in the year 1840, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . $= 11° 51'$.
Its azimuth from the South $= 15° 24'$.

6. Find the altitude and azimuth of Dubhe to an observer at Boston, in the year 1840, when its hour angle is $9^h\ 30^m$.

Ans. Its altitude . . $= 19° 14'$.
Its azimuth from the North $= 17° 15'$.

7. Find the altitude and azimuth of Fomalhaut to an observer at Boston, in the year 1840, when its hour angle is 6^h.

Ans. Its depression below the horizon $= 19° 58'$.
Its azimuth from the South $= 38° 31'$.

8. Find the altitude and azimuth of Dubhe to an observer at Boston, in the year 1840, when its hour angle is 6^h.

Ans. Its altitude . . $= 36° 44'$.
Its azimuth from the North $= 69° 3'$.

9. Find the altitude and azimuth of a star in the celestial equator to an observer at Stockholm, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . $= 25° 58'$.
Its azimuth from the South $= 34° 45'$.

10. Find the altitude and azimuth of a star in the celestial

equator, to an observer at Stockholm, when the hour angle is $9^h\ 30^m$.

Ans. Its depression below the horizon $= 23° 51'$.

Its azimuth from the North $= 41° 44'$.

11. Find the altitude and azimuth of Fomalhaut, to an observer at the equator, in the year 1840, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . $= 47° 45'$.

Its azimuth from the South $= 41° 4'$.

12. Find the altitude and azimuth of Dubhe, to an observer at the equator, in the year 1840, when its hour angle is $9^h\ 30^m$.

Ans. Its depression below the horizon $= 21° 24'$.

Its azimuth from the North $= 17° 30'$.

34. In the triangle ZPB (fig. 2.) other parts might be given instead of the two sides ZP, PB, and the included angle P, and the triangle might be resolved. Of the problems thus derived, we shall only, for the present, consider two cases.

35. *Problem. To find a given star's hour angle and altitude, when it is upon the prime vertical.*

Solution. The angle PZB is, in this case, a right angle, and if we use the preceding notation, we have

$$\cos. h = \text{cotan.}\ L\ \text{cotan.}\ p = \pm\ \text{cotan.}\ L\ \text{tang.}\ D \qquad (326)$$

$$\cos. z = \cos. p\ \text{cosec.}\ L,$$

or $$\sin. \star\text{'s alt.} = \pm \sin. D\ \text{cosec.}\ L; \qquad (327)$$

so that when the declination and latitude are of the same name, the hour angle is less than 6 hours, and the star is above the horizon; but when the declination and latitude are of different names, the hour angle is greater than 6 hours, and the star is below the horizon.

36. *Scholium.* The problem is, by Sph. Trig. § 27, impossible, when the declination is greater than the latitude; so that, in this case, the star is never exactly east or west of the observer.

37. *Scholium.* The problem is, by Sph. Trig. § 28, indeterminate, when the latitude and declination are both equal to zero; so that, in this case, the star is always upon the prime vertical.

38. Examples.

1. Find the hour angle and altitude of Aldebaran, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. The hour angle $= 4^h\ 45^m\ 44^s$.
The altitude $= 24°\ 26'$.

2. Find the hour angle and altitude of Fomalhaut, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. The hour angle . . $= 8^h\ 40^m\ 50^s$.
The depression below the horizon $= 48°\ 49'$.

3. Find the hour angle and altitude of Dubhe, when it is

exactly east or west of an observer at Boston, in the year 1840.

Ans. Dubhe is never upon the prime vertical of Boston.

4. Find the hour angle and altitude of Canopus, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. Canopus is never upon the prime vertical of Boston.

39. *Problem. To find the hour angle and amplitude of a star, when it is in the horizon.*

Solution. In this case the side ZB (fig. 35.) of the triangle ZPB is 90°. The corresponding angle of the polar triangle is, therefore, a right angle, and the polar triangle is a right triangle, of which the other two angles are

$$180° - PZ = 180° - (90° - L) = 90° + L,$$

and $$180° - PB = 180° - (90° \mp D) = 90° \pm D.$$

The hypothenuse of the polar triangle is $180° - h$, and the leg, opposite the angle, $90° \pm D$, is $180° - a$.

Hence, by Sph. Trig. § 40, and Pl. Trig. § 60 and 62,

$$-\cos. h = \pm \text{tang. } L \text{ tang. } D,$$

or $$\cos. h = \mp \text{tang. } L \text{ tang. } D \qquad (328)$$

$$-\cos. a = \mp \sin. D \sec. L,$$

or $$\cos. a = \pm \sin. D \sec. L; \qquad (329)$$

in which the upper sign is used when the latitude and declination have the same name, and the lower sign when they have different names; so that in the former case the hour angle is greater than 6 hours, and the azimuth is counted from the direction of the elevated pole; but in the latter case, the hour angle is less than 6 hours, and the azimuth is counted from the direction of the depressed pole. The amplitude is the difference between the azimuth a and 90°. Hence

$$\cos. *\text{'s azim.} = \sin. *\text{'s amp.} = \sin. D \sec. L. \quad (330)$$

40. *Scholium.* The problem is, by Sph. Trig. § 41, impossible, when the sum of the declination and latitude is greater than 90°; so that, in this case, the star does not rise or set.

41. Examples.

1. Find the hour angle and amplitude of Aldebaran, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. The hour angle $= 7^h\ 1^m\ 21^s$.
The amplitude $= 22°\ 9'$ N.

2. Find the hour angle and amplitude of Fomalhaut, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. The hour angle $= 3^h\ 51^m\ 18^s$.
The amplitude $= 43°\ 19'$ S.

3. Find the hour angle and amplitude of Dubhe, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. Dubhe neither rises nor sets at Boston.

4. Find the hour angle and amplitude of Canopus, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. Canopus neither rises nor sets at Boston.

CHAPTER III.

THE MERIDIAN.

42. The intersection of the plane of the meridian with that of the horizon is called the *meridian line.*

43. *Problem. To determine the meridian line.*

Solution. First Method. Stars obviously rise to their greatest altitude in the plane of the meridian; so that if their progress could be traced with perfect accuracy, and the instant of their rising to their greatest height be observed, the direction of the meridian line could be exactly determined. But stars, when they are at their greatest height, change their altitude so slowly, that this method is of but little practical value.

Second Method. A star is evidently at equal altitudes when it is at equal distances from the meridian on opposite sides of it. If, therefore, the direction and altitude of a star are observed before it comes to the meridian; and if its direction is also observed, when it has descended again to the same altitude, after passing the meridian; the horizontal line, which bisects the angle of the two horizontal lines drawn in the directions thus determined, is the meridian line.

Third Method. [B. p. 147.] The time which elapses between the superior and inferior passage of a star over the meridian is just half of a sideral day. If, then, a telescope were placed so as to revolve on a horizontal axis in the plane of the meridian, the two intervals of time between three successive passages of a star over the central wire, must be exactly equal. But if the vertical plane of the telescope is not that of the meridian, these two intervals will not be equal, and the position of the telescope must be changed until they become equal.

Thus, if $ZMmN$ (fig. 37.) is the plane of the meridian, $ZSsT$ that of the vertical circle described by the telescope, $MSWsmE$ the circle of declination described by the star about the pole P; this star will be observed at the points S and s instead of at the points M and m. Now the star describes the circle of declination with an uniform motion, and therefore the arc SP moves uniformly with the star around the pole, so that the angle SPM is proportional to the time of its description; that is, the angle SPM, reduced to time, denotes the sideral time of its description.

Let then

$T =$ the sideral time of describing the arc SM,

$t =$ the sideral time of describing the arc sm,

$I =$ interval from the observation at S to that at s,

$i =$ interval from the observation at s to that at S,

$\delta i =$ the difference of these two intervals ;

we have then, in sideral time,

$$I = 12^h - T - t = 12^h - (T + t)$$

$$i = 12^h + T + t = 12^h + (T + t)$$

$$\delta i = i - I = 2(T + t) ; \qquad (331)$$

so that if T and t were equal to each other, and they are nearly so in the case of the pole-star, we should have

Meridian determined by circumpolar stars.

$$\delta i = 4\,T = 4\,t$$

$$T = t = \tfrac{1}{4}\,\delta i;$$

that is, *the time of describing the arc* MS *or* $m\,s$ *is nearly one quarter part of the difference between the intervals.*

But the error of this result can be calculated without much difficulty. For this purpose, let

$L =$ the latitude of the place, $= 90° - PZ$,

$p =$ the polar distance of the star $= PS = Ps$,

$a =$ the azimuth of $ZST = TN = TZN$.

The arcs MS and $m\,s$ are so small, that they do not differ sensibly from the arcs of great circles drawn from S and s perpendicular to ZPN.

If, then, in the two right triangles PSM and ZSM, PM and ZM are the middle parts, SM, co. SZM, and co. SPM are the adjacent parts, so that

$$\sin. PM : \sin. ZM = \text{cotan.}\, SPM : \text{cotan.}\, SZM$$

$$= \frac{1}{\text{tang.}\, SPM} : \frac{1}{\text{tang.}\, SZM}$$

$$= \text{tang.}\, SZM : \text{tang.}\, SPM.$$

But $$ZM = ZP - PM = 90° - L - p$$

and the angles SZM and SPM are so small, that they are sensibly proportional to their tangents, whence

$$\sin. p : \cos. (p + L) = a : SPM, \qquad (332)$$

or $$a : SPM = \sin. p : \cos. p \cos. L - \sin. p \sin. L$$

$$= 1 : \text{cotan.}\, p \cos. L - \sin. L,$$

and if T is expressed in sideral hours

$$T.\,15^\circ = SPM = a\ \text{cotan.}\, p\ \text{cos.}\, L - a\ \text{sin.}\, L.$$

In like manner, we find

$$t.\,15^\circ = s\,P\,m = a\ \text{cotan.}\, p\ \text{cos.}\, L + a\ \text{sin.}\, L.$$

Hence, by (331)

$$(T + t)\,15^\circ = \tfrac{1}{2}\,\delta\,i\,.\,15^\circ = 2\,a\ \text{cotan.}\, p\ \text{cos.}\, L$$

$$a\ \text{cotan.}\, p\ \text{cos.}\, L = \tfrac{1}{4}\,\delta\,i\,.\,15^\circ$$

$$T.\,15^\circ = \tfrac{1}{4}\,\delta\,i\,.\,15^\circ - a\ \text{sin.}\, L$$

$$t.\,15^\circ = \tfrac{1}{4}\,\delta\,i\,.\,15^\circ + a\ \text{sin.}\, L$$

$$a = \tfrac{1}{4}\,\delta\,i\,.\,15^\circ\ \text{tang.}\, p\ \text{sec.}\, L \tag{333}$$

$$T = \tfrac{1}{4}\,\delta\,i - \tfrac{1}{4}\,\delta\,i\ \text{tang.}\, p\ \text{tang.}\, L$$

$$t = \tfrac{1}{4}\,\delta\,i + \tfrac{1}{4}\,\delta\,i\ \text{tang.}\, p\ \text{tang.}\, L,$$

so that the correction is

$$\tfrac{1}{4}\,\delta\,i\ \text{tang.}\, p\ \text{tang.}\, L, \tag{334}$$

which is to be added to the quarter interval at the lower transit; and to be subtracted from the quarter interval at the upper transit.

This correction is proportional to the quarter interval, so that if it is computed for any supposed value of this interval, it may be computed for any other interval by a simple proportion. Now table A, page 151, of the Navigator, is the value of this correction, when the interval is 1000^s. It may be observed, that it is not necessary that this time should be sideral time, because all the terms of the values of T and t are expressed in the same time, which may be that of the clock.

Table B [B. p. 151.]

The azimuth a is given in table B, [B. p. 151], and may be computed from the formula (333). But the interval in the formula is supposed to be sideral time, whereas the time of the table is that called *solar time*, to which clocks are usually regulated, and which is soon to be described; all that need be known for the present is, that an interval of sideral time is reduced to solar time by table LII of the Navigator, or by the formula

$$\frac{\text{an interval of solar time}}{\text{an interval of sideral time}} = 0.9972695. \qquad (335)$$

Fourth Method. [B. p. 149.] This method of determining the meridian is by means of two known circumpolar stars, which differ nearly 12 hours in right ascension. The upper passage of one of these stars is to be observed, and the lower passage of the other. Then any deviation in the plane of the instrument from the meridian, will evidently produce contrary effects upon the observed times of transit, exactly as in the upper and lower transits of the same star. The time, which elapses between the two observations, will differ from the time which should elapse by the sum of the effects of the deviation upon the two stars. In the use of this method, therefore, the time of the clock must be known, so that it can readily be reduced to sideral time.

The deviations in the time of passage of a star, corresponding to any azimuth, can be calculated by means of equation (332). For this formula give for the time of describing the arc SM

$$T . 15^\circ = a \cos. (p + L) \text{ cosec. } p,$$

or

$$T = \tfrac{1}{15}\, a \cos. (p + L) \text{ cosec. } p; \qquad (336)$$

which may be used if T is expressed in sideral seconds, and the arc a in seconds of space. But if T is expressed in solar time we have, by (335),

$$T = 0.0664846\, a \cos. (p + L) \operatorname{cosec}. p. \tag{337}$$

In the same way the value of t for an inferior passage is found to be

$$t = 0.0664846\, a \cos. (p - L) \operatorname{cosec}. p. \tag{338}$$

Now, since these values of T and t are proportional to the azimuth a, their values may be computed for a given value of the azimuth, as 1000″, and arranged in a table like Table C, p. 152, of the Navigator, and their values for any other azimuth can be obtained by a simple proportion.

Fifth Method. [B. p. 149.] This method consists in observing the transits of two stars, which differ but little in right ascension. The error in the position of the telescope is, in this case, equal to the difference in the errors of the observed transits, instead of the sum, as in the preceding method.

44. In making calculations where angles are introduced as factors, some labor, in reducing them to the same denomination, is often saved by means of a table of Proportional Logarithms, such as Table XXII of the Navigator.

This table was particularly designed for reducing lunar distances, given in the Nautical Almanac, for every 3 hours to any intermediate time. It contains, on this account, the logarithm of the ratio of 3 hours to each angle expressed in time; that is, if A is the angle

$$\text{Prop. log. } A = \text{log. } \frac{3^h}{A} = \text{log.} 3^h - \text{log.} A = \text{log.} 180^m - \text{log.} A$$

$$= \text{log. } 10800^s - \text{log. } A, \qquad (339)$$

so that if A in the second member is reduced to seconds,

$$\text{Prop. log. } A = 4.03342 - \text{log. } A \text{ in seconds}; \qquad (340)$$

neglecting the right hand figure, so as to retain only four decimal places. This agrees with the explanation of the table in the Introduction to the Navigator; and it is evident that it is immaterial whether the angles, whose ratios are sought, are given in time or in degrees, &c.

Suppose, now, that the logarithm of the ratio of two angles is sought, A and a; we have, evidently,

$$\text{log. } \frac{A}{a} = \text{log. } A - \text{log. } a = \text{Prop. log. } a - \text{Pr. log. } A; \; (341)$$

so that if this ratio, which we will denote by M, were known, and if a were known, A might be calculated by the formula

$$\text{Prop. log. } A = \text{Prop. log. } a - \text{log. } M$$

$$= \text{Prop. log. } a + (\text{ar. co.}) \text{ log. } M; \quad (342)$$

which is, therefore, the formula for calculating the value of A, given by the equation

$$A = a\,M. \qquad (343)$$

Finally, the use of formula (342) is facilitated by remembering that *the arithmetical complements of the logarithms of the sine, cosine, tangent, cotangent, secant, and cosecant of an angle are respectively the logarithms of its cosecant, secant, cotangent, tangent, cosine, and sine.*

45. Examples.

1. Calculate the proportional logarithm of 0° 5′ 45″.

Solution: By (340),		4.03342
0° 5′ 45″ = 345″.		2.53782
Prop. log. 5′ 45″	=	1.4956

2. Calculate the corrections of tables A and B, [B. p. 151.], as in table XXII, when the latitude is 42°, and the polar distance of the star 30°.

Solution. By means of proportional logarithms, and equations (333) and (334),

$\frac{1}{4}.1000^s = 4^m\ 10^s$	Prop. log.	1.6355		1.6355
$L = 42°$	cotan.	10.0456	cos.	9.8711
30°	cotan.	10.2386		10.2386
corr. A = $130^s = 2^m\ 10^s$	Pr. log.	1.9197		
0.0664846				8.8227
corr. B = 48′ 41″			Prop. log.	0.5679

3. Calculate the corrections of table C [B. p. 152.] for the pole star and the latitude of 30°, when the polar distance of this star is 1° 32′ 37″.

Determination of the meridian line.

Solution. By (337) and (338),

	0.0664846	8.82273	8.82273
$a = 1000''$		3.00000	3.00000
$p = 1° 32' 37''$	cosec.	11.56964	11.56964
$p + L = 31° 32' 37''$	cos.	9.93058	
$p - L = -28° 26' 22''$			9.94414
corr. C upper trans. $= 2103^s$		3.32295	
corr. C lower trans. $= 2170^s$			3.33651

4. An observer at Boston in the year 1840, wishing to determine his meridian line, observed three successive transits of β Cephei over the central vertical line of his transit instrument, by means of a clock regulated to solar time, and found them to occur as follows; the first upper transit at $7^h\ 45^m\ 28^s$ P. M., the next inferior transit the next day at $7^h\ 41^m$ A. M. the third transit at $7^h\ 41^m\ 32^s$ P. M. What were the times of the star's passing the meridian the second day? and what was the azimuth error in the position of the instrument?

Solution.

The first interval $= 19^h\ 41^m - 7^h\ 45^m\ 28^s = 11^h\ 55^m\ 32^s$.

The second interval $= 19^h\ 41^m\ 32^s - 7^h\ 41^m = 12^h\ 0^m\ 32^s$.

Hence $\delta i = 5^m = 300^s$.

Now $L = 42°\ 21'$, $D = 69°\ 52'$, $p = 20°\ 8'$.

Hence, by tables A and B,

$$\text{corr. } A = 83^s \times 0.3 = 25^s,$$

$$\text{corr. } B = 31'\ 6'' \times 0.3 = 9'\ 19'';$$

so that the error in the time of the upper transit is

$$\tfrac{1}{4}.300^s - 25^s = 75^s - 25^s = 50^s,$$

and the error in the time of the lower transit is

$$\tfrac{1}{4}.300^s + 25^s = 75^s + 25^s = 100^s = 1^m\ 40^s.$$

The times of the star's passing the meridian the second day were, then,

$$7^h\ 41^m + 1^m\ 40^s = 7^h\ 42^m\ 40^s \text{ A. M.}$$

and $$7^h\ 41^m\ 32^s - 50^s = 7^h\ 40^m\ 42^s \text{ P. M.}$$

The error in the azimuth of the instrument was 9′ 19″ to the west of north.

5. An observer at Boston, wishing to determine his meridian line, on the morning of January 1, 1840, observed, by means of a clock regulated to solar time, the superior transit of γ Ursæ Majoris at $5^h\ 6^m\ 54^s$ A. M., and the inferior transit of Polaris at $6^h\ 12^m\ 23^s$ A. M. What was the azimuth error in the position of the transit instrument?

Solution. The interval between these two transits is

$$6^h\ 12^m\ 23^s - 5^h\ 6^m\ 54^s = 1^h\ 5^m\ 29^s.$$

But, by the Nautical Almanac,

12^h + R. A. of Polaris	=	$13^h\ 1^m\ 59^s$
R. A. of γ Ursæ Majoris	=	$11^h\ 45^m\ 25^s$
Sideral Interval	=	$1^h\ 16^m\ 34^s$
Solar Interval	=	$1^h\ 16^m\ 22^s$
Observed Interval	=	$1^h\ 5^m\ 29^s$
Error of Interval	=	$10^m\ 53^s = 653^s$.

Now for 1000″ of azimuth error, and the latitude of Boston, Table C gives, since

Dec. of γ Ursæ Majoris . .	$= 54° 35'$
Error of lower trans. of Polaris .	$= 1866^s$
Error of upper trans. of γ Ursæ Majoris	$= 25^s$
Sum of errors	$= 1891^s$

Then the proportion

$$1891^s : 653^s = 1000'' : \text{azimuth error},$$

gives

$$\text{azimuth error} = 345'' = 5' \, 45'' \text{ W.}$$

6. An observer, at Boston, wishing to determine his meridian line, in the evening of December 17, 1839, observed by means of a clock regulated to solar time, the superior transit of α Cassiopeæ at $6^h \, 48^m \, 35^s$ P. M., and that of Polaris at $6^h \, 53^m \, 15^s$ P. M. What was the azimuth error in the position of the transit instrument?

Solution. By the Nautical Almanac,

R. A. of Polaris	$= 1^h \; 2^m \, 26^s$
R. A. of α Cassiopeæ	$= 0^h \, 31^m \, 28^s$
Sideral Interval	$= 0^h \, 30^m \, 58^s$
Solar Interval	$= 0^h \, 30^m \, 53^s$
Observed Interval	$= 0^h \; 4^m \, 40^s$
Error of Interval	$= 0^h \, 26^m \, 13^s = 1573^s$.

Now Table C gives, for 1000″ of azimuth error and the latitude of Boston, since

Determination of the meridian line.

Dec. of α Cassiopeæ	$= 55° 40'$
Error of trans. of Polaris	$= 1777^s$
Error of trans. of α Cassiopeæ	$= 26^s$
Diff. of errors	$= 1751^s$

Then, the proportion

$$1751^s : 1573^s = 1000'' : \text{azimuth error}$$

gives

$$\text{azimuth error} = 900'' := 1' 30'' \text{ E.}$$

7. Calculate the proportional logarithm of 0° 2′ 33″.

Ans. 1.8487.

8. Calculate the proportional logarithm of 2° 59′ 12″.

Ans. 0.0019.

9. Calculate the corrections of tables A and B, when the latitude is 54°, and the star's polar distance 20°.

Ans. Corr. A $= 125^s$.
Corr. B $= 38' 48''$.

10. Calculate the corrections of table C, when the latitude is 20°, and the polar distance 5°.

Ans. For the upper transit, corr. C $= 691^s$.
For the lower transit, corr. C $= 737^s$.

11. An observer at Boston, in the year 1840, wishing to determine his meridian line, observed three successive transits of Polaris, by means of a clock regulated to solar time. The first lower transit was observed at 6^h A. M., the next transit at

Determination of the meridian line.

$7^h\ 2^m\ 11^s$ P. M., and the second lower transit at $5^h\ 56^m\ 4^s$ A. M. What was the time of the star's passing the meridian the second morning? and what was the azimuth error in the position of the instrument?

Ans. The time of third merid. trans. was $5^h\ 58^m\ 11^s$ A. M.
The azimuth error $= 1'\ 8''$ W.

12. An observer at Boston, wishing to determine his meridian line by means of a clock regulated to solar time, observed the inferior transit of Polaris on April 4, 1839, at 0^h A. M., and the superior transit of η Ursæ Majoris at $0^h\ 53^m\ 59^s$ A. M. What was the azimuth error in the position of his transit instrument?

The R. A. of Polaris is $1^h\ 0^m\ 50^s$, that of η Ursæ Majoris is $13^h\ 41^m\ 14^s$, and the declination of η Ursæ Majoris is $50°\ 7'$ N.

Ans. The azimuth error $= 7'\ 18''$ W.

13. An observer at Boston, wishing to determine his meridian line, in the evening of May 1, 1839, observed by means of a clock regulated to solar time, the lower transit of Polaris at $9^h\ 49^m\ 22^s$ P. M., and that of α Cassiopeæ at $9^h\ 52^m$ P. M. What was the azimuth error of the instrument?

The R. A. of Polaris $= 1^h\ 0^m\ 56^s$.

The R. A. of α Cassiopeæ $= 0^h\ 31^m\ 22^s$.

The Dec. of α Cassiopeæ $= 55°\ 39'$ N.

Ans. The azimuth error $= 18'\ 23''$ W.

CHAPTER IV.

LATITUDE.

46. *Problem. To find the latitude of a place.*

Solution. The latitude of the place is evidently, from (fig. 34.), equal to the altitude of the pole; so that this problem is the same as to find the altitude of the pole, which would be done without difficulty if the pole were a visible point of the celestial sphere.

First Method. By Meridian Altitudes. [B. p. 166 – 175.]

Observe the altitude of a star at its transit over the meridian, and let

$A =$ the altitude of the star,

$A' =$ ✳'s dist. from point of horizon below the pole;

then, if the notation of § 28 is used, it is evident, from (fig. 34.), that

$$L = A' \mp p; \qquad (344)$$

the upper sign being used when the transit is a superior one, and the lower sign when it is an inferior one.

I. Suppose the observed transit to be a superior one; then, if it passes upon the side of the zenith opposite to the pole, we have

$$A' = 180° - A, \quad p = 90° \mp D,$$

and (344) becomes

$$L = 90^\circ - (A \pm D) = (90 - A) \pm D = z \pm D; \quad (345)$$

the upper sign being used when the declination and latitude are of the same name, and the lower sign when they are of different names.

But if the star passes upon the same side of the zenith with the pole, we have

$$A' = A, \quad p = 90^\circ - D,$$

and (344) becomes

$$L = (A + D) - 90^\circ = D - (90^\circ - A) = D - z. \quad (346)$$

II. If the transit is an inferior one, we have

$$A' = A, \quad p = 90^\circ - D,$$

and (345) becomes

$$L = (A - D) + 90^\circ = A + (90^\circ - D). \quad (347)$$

Equations (345) and (346) agree with the rule of Case I, [B. p. 166.], and (347) with Case II, [B. p. 167.]

III. If both transits are observed, and if A' and A are referred to the upper transits, and

$$A_1 = \text{the altitude at the lower transit},$$

we have, by (344),

$$L = A' - p$$
$$L = A_1 + p,$$

the sum of which is

$$L = \tfrac{1}{2}(A' + A_1); \quad (348)$$

so that the latitude is determined in this case without knowing the star's declination.

Second Method. By a Single Altitude.

Observe the altitude and the time of the observation.

I. If the star is considerably distant from the meridian, we have given in the triangle PBZ (fig. 35.), PB, BZ, and BPZ to find PZ, which may be solved by Sph. Trig. § 59, and gives, by the notation of § 28,

$$\text{tang. } PC = \cos. h \text{ tang. } p = \pm \cos. h \text{ cotan. } D \quad (349)$$

$$\begin{aligned} \cos. ZC &= \cos. PC . \cos. z \sec. p \\ &= \pm \cos. PC . \cos. z \text{ cosec. } D, \end{aligned} \quad (350)$$

in which the upper sign is used if the declination and latitude are of the same name, otherwise the lower sign.

$$\begin{aligned} 90^\circ - L = PZ &= PC \pm ZC \\ L &= 90^\circ - (PC \pm ZC); \end{aligned} \quad (351)$$

in which both signs may be used if they give values of L contained between 0° and 90°, and in this case other data must be resorted to, in order to determine which is the true value of L.

Scholium. The problem is, by Sph. Trig. § 61, impossible, if the altitude is greater than the declination, when the hour angle is more than six hours.

II. If the latitude is known within a few miles, it may be exactly calculated by means of (317), or

$$\cos. z = \cos. [90^\circ - (L + p)] - 2 \cos. L \cos. D (\sin. \tfrac{1}{2} h)^2. \quad (352)$$

Latitude found by a single altitude.

But if A is the star's observed altitude, and A_1 its meridian altitude at its upper transit, (344) gives

$$A_1 = L + p, \text{ or } = 180° - (L + p),$$

and (352) becomes, by transposition,

$$\sin. A_1 = \sin. A + 2 \cos. L \cos. D (\sin. \tfrac{1}{2} h)^2 ; \qquad (353)$$

from which the meridian altitude may be calculated by means of table XXIII, as in the Rule. [B. p. 200.]

III. A formula can also be obtained from (281), which is particularly valuable when the star is, as it always should be in these observations, near the meridian.

In this case we have in (281) applied to PBZ

$$2s = 90° - L + p + z = 180° - L + p - A$$

$$2s - 2PZ = L + p - A$$

$$= A_1 - A \text{ or } = 180° - (A_1 + A) \qquad (354)$$

$$2s - 2PB = 180° - L - p - A$$

$$= 180° - (A_1 + A) \text{ or } = A_1 - A ; \qquad (355)$$

and if these values are substituted in (281), after it is squared and freed from fractions, they give

$$(\sin. \tfrac{1}{2} h)^2 \cos. L \cos. D = \sin. \tfrac{1}{2} (A_1 - A) \cos. \tfrac{1}{2} (A_1 + A), \qquad (356)$$

or

$$\sin. \tfrac{1}{2} (A_1 - A) = (\sin. \tfrac{1}{2} h)^2 \cos. L \cos. D \sec. \tfrac{1}{2} (A_1 + A) ; \qquad (357)$$

and if, in the second member of this equation, the value of A_1 is used, which is obtained from the approximate value of the latitude, the difference between the observed and the meridian altitudes may be found at once; and this difference is to be added to the observed altitude to obtain the meridian altitude.

Single altitude near the meridian.

IV. If the star is very near the meridian, $\frac{1}{2}(A_1 - A)$ and $\frac{1}{2}h$ will be so small, that we may put

$$\frac{\sin.\frac{1}{2}(A_1-A)}{\sin.1''}=\frac{\frac{1}{2}(A_1-A)}{1''}=\frac{\frac{1}{2}(A'-A)}{1},\quad \frac{\sin.\frac{1}{2}h}{\sin.1^s}=\frac{1}{2}h,$$

or

$$\sin.\tfrac{1}{2}(A_1-A)=\tfrac{1}{2}(A'-A)\sin.1''$$

$$\sin.\tfrac{1}{2}h=\tfrac{1}{2}h\sin.1^s=\tfrac{15}{2}h\sin.1'';$$

which, substituted in (357), give, by supposing A_1 equal to A in the second member, which is very nearly the case,

$$A_1-A=\tfrac{15}{2}h^2\sin.1^s\cos.L\cos.D\sec.A_1. \qquad (358)$$

This value of $A_1 - A$ is proportional to h^2, so that if it were calculated for

$$h=1^s,$$

any other value might be calculated by multiplying by h^2. Now Table XXXII, of the Navigator, contains the values of $A_1 - A$ for all latitudes and for all declinations less than 24°, excepting a few latitudes in which the meridian transit of the observed body is too near the zenith for this observation to be accurate; and Table XXXIII contains all the values of h^2, where h is less than 13^m.

V. If the observed star is very near the pole, we have in (349)

$$\text{tang.}\,PC=\cos.h\,\text{tang.}\,p; \qquad (359)$$

so that as p is very small, PC must be likewise small, and we have

$$\cos.h=\frac{\text{tang.}\,PC}{\text{tang.}\,p}=\frac{PC}{p}$$

$$PC=p\cos.h; \qquad (360)$$

and, by Pl. Trig. § 22,

$$\cos. PC = 1, \quad \sin. D = \cos. p = 1,$$

whence, by (350), and (351),

$$\cos. ZC = \cos. z, \quad ZC = z,$$

$$L = 90° - PC - ZC = 90° - z - PC$$

$$= A - p \cos. h; \qquad (361)$$

so that p cos. h may be regarded as a correction to be subtracted from A when it is positive, that is, when the hour angle is less than 6 hours, or greater than 18 hours; and it is to be added when the hour angle is greater than 6 hours and less than 18 hours.

The table [B. p. 206.] for the pole star was calculated for the year 1840, when

its R. A. $= 1^h\ 2^m$; its dec. $= 88°\ 27'$ nearly.

Third Method. By Circummeridian Altitudes.

I. If several altitudes are observed near the meridian, each observation may be reduced separately by (357) and (358), and the mean of the resulting latitudes is the correct latitude.

II. But if (358) is used, the mean of the values of $A_1 - A$ is evidently obtained by multiplying the mean of the values of h^2 by the constant factor; and if to the mean of the values of $A_1 - A$, the mean of the values of A is added, the sum is the mean of the values of A_1, whence precisely the same mean of resulting latitude is obtained as by the former method, but with much less calculation.

III. If the star is changing its declination in the course of the observations, this change may, in all cases which can occur if the hour angle is small, be neglected in the value of cos. D. But the value of A_1 will not, in this case, be at each observation equal to the meridian altitude, but will differ from it by the difference of the star's declination. Let the change of the star's declination in one minute be denoted by δD, which is positive when the star is approaching the elevated pole; and if h is the star's hour angle at the time of observation, which is negative before the star arrives at the meridian and afterwards positive, the whole change of declination is $h \delta D$, so that the correct meridian altitude is

$$A_1 - h \delta D.$$

The mean of the values of the corrected meridian altitude is, therefore, equal to the mean of the values of A_1 diminished by the mean of the values of $h \delta D$; and, if H denotes the mean of the hour angles h (regard being had to their signs), the correct meridian altitude is the mean of the values of A_1 diminished by $H \delta D$.

Fourth Method. By Double Altitudes.

I. Let two altitudes of a star, which does not change its declination, be observed, and the intervening time. Then (fig. 39.) let Z be the zenith, P the pole, S and S' the positions of the star; join ZS, ZS', PS, PS', and $SS'M$; draw PT to the middle T of SS', join ZT, and draw ZV perpendicular to PT. Let

$$p = PS = PS' = 90° - D, \; SPS' = \text{elapsed time} = h$$

$$ST = A = S'T, \; PT = 90° - B$$

By double altitudes.

$$A_1 = 90° - ZS,\ A'_1 = 90° - ZS'$$
$$ZTP = T,\ ZT = F,\ ZV = C$$
$$TV = Z,\ PV = 90° - E;$$

in which D and B are positive, when the latitude and declination are of the same name, but negative, if they are of contrary names; Z is positive, if the zenith is nearer the elevated pole than the point M.

Now the triangle TPS gives

$$\sin. A = \sin. PS \sin. SPT = \cos. D \sin. \tfrac{1}{2} h$$

$$\cos. PS = \cos. PT \cos. A, \text{ or } \sin. D = \sin. B \cos. A, \quad (362)$$

$$\text{or} \quad \operatorname{cosec}. A = \sec. D \operatorname{cosec}. \tfrac{1}{2} h \quad (363)$$

$$\operatorname{cosec}. B = \cos. A \operatorname{cosec}. D. \quad (364)$$

The triangles ZTS and ZTS' give

$$\sin. A_1 = \cos. F \cos. A - \sin. F . \sin. A \sin. T, \quad (365)$$

$$\sin. A'_1 = \cos. F \cos. A + \sin. F . \sin. A \sin. T, \quad (366)$$

The sum and difference of which is, by (36) and (37),

$$\sin. \tfrac{1}{2} (A_1 + A'_1) \cos. \tfrac{1}{2} (A'_1 - A_1) = \cos. F \cos. A, \quad (367)$$

$$\sin. \tfrac{1}{2} (A'_1 - A_1) \cos. \tfrac{1}{2} (A_1 + A'_1) = \sin. F \sin. A \sin. T. \quad (368)$$

But triangle ZTV gives

$$\sin. C = \sin. F \sin. T, \quad (369)$$

$$\cos. F = \cos. C \cos. Z; \quad (370)$$

which, substituted in (367) and (368), give

$$\sin. C = \sin. \tfrac{1}{2} (A'_1 - A_1) \cos. \tfrac{1}{2} (A_1 + A'_1) \operatorname{cosec}. A, \quad (371)$$

$$\sec. Z = \cos. A \cos. C \sec. \tfrac{1}{2} (A_1 + A'_1) \operatorname{cosec}. \tfrac{1}{2} (A'_1 - A_1). \quad (372)$$

By double altitudes.

But $$PV = PT - TV,$$

or $$90° - E = 90° - B - Z$$

$$E = B + Z. \qquad (373)$$

Lastly, triangle ZPV gives

$$\cos. PZ = \cos. ZV \cos PV$$

$$\sin. L = \cos. C \sin. E. \qquad (374)$$

Equations (363, 364, 371 – 374) correspond to the rule and formula given in the Navigator. [B. p. 180.]

II. Another method of calculating the values of B, C, and Z has been given, which dispenses with A and one opening of the tables, and may therefore be preferred by some calculators, although it requires one more logarithm. Triangle TPS gives

$$\text{tang. } PT = \cos. \tfrac{1}{2} h \text{ tang. } PS,$$

or $$\text{cotan. } B = \cos. \tfrac{1}{2} h \text{ cotan. } D. \qquad (375)$$

The substitution of (364) in (371) gives

$$\sin. C = \cos. \tfrac{1}{2}(A_1 + A'_1) \sin. \tfrac{1}{2}(A'_1 - A_1) \sec. D \text{ cosec. } \tfrac{1}{2} h. \qquad (376)$$

Triangle PTS gives

$$\cos. A = \sin. D \text{ cosec. } B; \qquad (377)$$

which, substituted in (372), gives (378)

$$\sec. Z = \cos. C \sin. D \text{ cosec. } B \text{ cosec. } \tfrac{1}{2}(A_1 + A'_1) \sec. \tfrac{1}{2}(A'_1 - A_1).$$

Corollary. The hour angle ZPT is the mean between the hour angles ZPS and ZPS', and if we put

$$ZPT = H,$$

the triangle ZPV gives

$$\text{tang. } H = \text{tang. } C \text{ sec. } E, \tag{379}$$

as in B. p. 181.

III. *Douwes's Method.* When the latitude is known within a few miles. In this case let

$$L' = \text{the assumed latitude,}$$

and the triangles ZSP and $ZS'P$ give

$$\sin. A_1 = \sin. L' \sin. D + \cos. L' \cos. D \cos. ZPS, \tag{380}$$

$$\sin. A'_1 = \sin. L' \sin. D + \cos. L' \cos. D \cos. ZPS'; \tag{381}$$

whence, and by (39),

$$\sin. A'_1 - \sin. A_1 = \cos. L' \cos. D (\cos. ZPS' - \cos. ZPS)$$

$$= 2 \cos. L' \cos. D \sin. \tfrac{1}{2}(ZPS' + ZPS) \sin. \tfrac{1}{2}(ZPS' - ZPS)$$

$$= 2 \cos. L' \cos. D \sin. H \sin. \tfrac{1}{2} h$$

$$2 \sin. H = (\sin. A'_1 - \sin. A_1) \sec. L' \sec. D \text{ cosec. } \tfrac{1}{2} h, \tag{382}$$

$$ZPS = H - \tfrac{1}{2} h; \tag{383}$$

whence the hour angle ZPS corresponding to the observation at S' is known, and the latitude may be found by the method of a single altitude. The combination of the formulas (380, 381), and the method of computing the latitude by a single altitude, corresponds exactly to the rule given in the Navigator. [B. p. 185.]

The log. cosec. $\frac{1}{2} h$ is not only given in table XXVII, but also in table XXIII, where it is called the log. $\frac{1}{2}$ elapsed time of $\frac{1}{2} h$.

Table XXIII.

The value of

$$\log. 2 \sin. H - 5 = \log. \sin. H + \log. 2$$

$$= \log. \sin. H - \text{ar. co.} \log. 2 + 5$$

$$= \log. \sin. H - 4.69897$$

$$= 5.30103 = \log. \text{elapsed time of } H \quad (384)$$

is inserted in table XXIII, and is called the log. middle time of H. The 5 is subtracted from log. 2 sin. H, on account of the different values of the radius in tables XXIV and XXVII.

Scholium. When the calculated latitude differs much from the assumed latitude, the calculation must be gone over again, with the calculated latitude instead of the assumed latitude. This labor may be avoided by noticing, in the course of the original calculation, the difference which would arise from a change of 10′ in the value of the assumed latitude, and calculating the correction of the latitude by the rule of double position. The error of the hypothesis is in each case the excess of the calculated above the assumed latitude, and the proportion is

$$\text{diff. of errors : diff. of hyp.} = \text{least error : corr. of hyp.} \quad (385)$$

IV. If the star has changed its declination a little, during the interval between the observations, the second altitude will correspond to a declination D', a little different from D.

If D' is put instead of D in (381), and if A'_2 denotes the second observed altitude, A_1 being retained to denote what this second altitude would have been, if the declination had remained unchanged, (381) becomes

$$\sin. A'_2 = \sin. L' \sin. D' + \cos. L' \cos. D' \cos. ZPS'. \quad (385)$$

Table XLVI.

Now, if (381) multiplied by cos. D' is subtracted from (385) multiplied by cos. D, the remainder is

$$\cos. D \sin. A'_2 - \cos. D' \sin. A'_1 = \sin. L' \sin. (D' - D). \quad (386)$$

But if we put

$$D' - D = \delta D, \; A'_2 - A'_1 = \delta A_1 \quad (387)$$

we have, by (13) and (15),

$$\cos. D' = \cos. (D + \delta D) = \cos. D - \sin. \delta D . \sin. D \quad (388)$$

$$\sin. A'_2 = \sin. (A'_1 + \delta A_1) = \sin. A'_1 + \sin. \delta A_1 . \cos. A'_1, \quad (389)$$

which, substituted in (386), give

$$\sin. A'_1 \sin. D \sin. \delta D . + \cos. A'_1 \cos. D \sin. \delta A_1 = \sin. L' \sin. \delta D$$

$$\frac{\sin. \delta A_1}{\sin. \delta D} = \frac{\delta A_1}{\delta D} = \frac{\sin. L' - \sin. A'_1 \sin. D}{\cos. A'_1 \cos. D}, \quad (390)$$

and, by (34) and (35),

$$\delta A_1 = \frac{2 \sin. L' - \cos. (A'_1 - D) + \cos. (A'_1 + D)}{\cos. (A'_1 - D) + \cos. (A'_1 + D)} \delta D, \quad (391)$$

in which D is to be negative, when the latitude and declination are of contrary names. Hence the value of δA_1 can be computed by this formula, and thence

$$A'_1 = A'_2 - \delta A_1,$$

and in calculating δA_1, A'_2 may be substituted for A'_1. Since the value of δA_1 is proportional to δD, it may be computed for some assumed value of δD, a d arranged in a table like table XLVI of the Navigator, and the value of δA_1 can be computed from this table by a simple proportion. The value of A'_1 is thus found; the rest of the calculation can be conducted according to the preceding methods, as in B. p. 189.

By double altitudes of different stars.

V. If two stars are observed, whose declinations are quite different. Then, if P (fig. 40) is the pole, Z the zenith, S and S' the places of the star.

$$A_1 = 90° - ZS = \text{the less altitude,}$$

$$A'_1 = 90° - ZS' = \text{the greater altitude,}$$

$$D = 90° - PS = \text{the declination of star at } S,$$

$$D' = 90° - PS' = \text{the declination of star at } S',$$

$$H = SPS' = \text{hour angle} = \text{interv. of sideral time.}$$

Then, in the triangle PSS', PS, PS', and H are given to find

$$SS' = C, \text{ and } S'SP = 90° - F.$$

Next, in the triangle ZSS', the three sides are known, to find the angle

$$ZSS' = Z.$$

Hence $$ZSD = 90° - G = 90° - F - Z$$

$$G = F + Z.$$

Lastly, in the triangle ZSP, ZS, SP, and the included angle ZSP are given to find

$$ZP = 90° - L.$$

This solution is precisely similar to the Rule in B. p. 193; and it is easy to prove the rules for the s gns which are there given.

VI. If the distance SS' were observed, the angles ZSS' and $S'SP$ might be found from the triangle ZSS' and $S'SP$, in which the sides are all known, and the rest of the calculation would be as in the last method, and this method corresponds exactly to the Rule in B. p. 197.

47. Examples.

1. The correct meridian altitude of Aldebaran was found by observation, in the year 1838, to be 55° 45′, when its bearing was south; what was the latitude?

Solution	The zenith distance	=	34° 15′ N.
	The declination	=	16° 10′ N.
	The latitude	=	50° 25′ N.

2. The correct meridian altitude of Canopus was found by observation, in the year 1839, to be 16° 25′, when its bearing was south; what was the latitude?

Solution.	The zenith distance	=	73° 35′ N.
	The declination	=	52° 36′ S.
	The latitude	=	20° 59′ N.

3. The correct meridian altitude of Dubhe was found by observation, in the year 1830, to be 50° 45′, when its bearing was north; what was the latitude?

Solution.	The zenith distance	=	39° 15′ S.
	The declination	=	52° 36′ N.
	The latitude	=	13° 21′ N.

4. If the correct meridian altitude of Dubhe, at its greatest elevation, were found by observation, in the year 1830, to be 50° 45′, when its bearing was south; what would be the latitude?

Meridian altitudes.

Solution. The zenith distance = 39° 15′ N.
The declination = 52° 36′ N.

The latitude = 91° 51′ N.

The problem is impossible.

5. The correct meridian altitude of Dubhe, at its least elevation, was found by observation, in the year 1830, to be 50° 45′; what was the latitude?

Solution. The polar distance = 37° 24′.
The altitude = 50° 45′.

The latitude = 88° 09′ N.

6. The correct meridian altitudes of Dubhe, at its greatest and least elevation, which were on opposite sides of the zenith, were found by observation to be 41° 56′ and 53° 16′; what was the latitude?

Solution. The greatest altitude = 53° 16′.
The least altitude = 41° 56′.

Diff. of altitudes = 11° 20′.
180° — Diff. of altitudes = 168° 40′.
Latitude = 84° 20′ N.

7. The correct meridian altitudes of a northern star, at its greatest and least altitudes, which were on the same side of the zenith, were found by observation to be 12° 14′ and 72° 14′; what was the latitude?

Single altitude.

Solution.

Greatest alt.	$= 72° 14'$.
Least alt.	$= 12° 14'$.
Sum of alts.	$= 84° 28'$.
Latitude	$= 42° 14'$ N.

8. In a northern latitude, the altitude of Aldebaran was found by observation, in the year 1839, to be $25° 38'$, when its hour angle was $4^h\ 12^m\ 20^s$; what was the latitude?

Solution. By (349, 350, 351),

$h = 4^h\ 12^m\ 20^s$	cos.	9.65580		
$D = 16° 11'$	cotan.	10.53729	cosec.	10.55484
$90° - PC = 32° 40'$	cotan.	10.19309	sin.	9.73215
		$A = 25° 38'$	sin.	9.63610
$ZC = 33°\ 6'$			cos.	9.92309
$L = 65° 46'$ N.				

9. In lat. $65° 40'$ N. nearly, the altitude of Aldebaran was found by observation, in the year 1839, to be $25° 38'$, when its hour angle was $4^h\ 12^m\ 20^s$; what was the true latitude?

Solution. I.		$65° 40'$	cos.	9.61494
		$16° 11'$	cos.	9.98244
		$4^h\ 12^m\ 20^s$	log. Ris.	4.73823
	Nat. num.	21657		4.33567
$25° 38'$	Nat. sine	43261		
$49° 31'$ N.	Nat. cos.	64918		
$16° 11'$ N.				
$65° 42'$ N. = the latitude.				

Had the assumed latitude been taken 10′ more, the calculated latitude would have been 65° 48½′ N.; hence, by (385),

$$3\tfrac{1}{2} : 1\tfrac{1}{2} = 10' : 4' = \text{corr. of second hypothesis,}$$

or the latitude = 65° 46′ N., as in the preceding example.

II. By (357),

	$\frac{1}{2}h = 2^h\, 6^m\, 10^s$	2 log. sin.	9.43720
	$L = 65°\ 40'$	cos.	9.61494
	$D = 16°\ 11'$	cos.	9.98244
	$A_1 = 40°\ 31'$		
$A = 25°\ 38'$	$A' = 25°\ 38'$		
$A_1 - A = 14°\ 51'$	$\frac{1}{2}(A_1 + A) = 33°\ 4\tfrac{1}{2}'$	sec.	10.07678
$A_1 = 40°\ 29'$	$\frac{1}{2}(A_1 - A) = 7°\ 25\tfrac{1}{2}'$	sin.	9.11136

corr. $A_1 = 15°\ 2' =$ corr. lat. $= 65°\ 40' + 2' = 65°\ 42'$ as before.

10. Calculate the variation of a star's altitude in one minute from the meridian, when the declination is 12° N. and the latitude 5° N.

Solution. If $A_1 - A$ is required in seconds, (358) gives

$$A_1 - A = 450 \text{ sin. } 1^m \text{ cos. } L \text{ cos. } D \text{ sec. } A_1$$

by 450 sin. 1^m = log. 450 + log. sin 1^m

$= 2.65321 + 7.63982$	=		0.29303
$L = 12°$		cos.	9.99040
$D = 5°$		cos.	9.99834
$A_1 = 83°$		sec.	0.91411
$A_1 - A = 15''.7$, as in table XXXII.			1.19588

Single altitude near the meridian.

11. Calculate the tabular number for $11^m\ 48^s$ in table XXXIII.

Solution. $11^m\ 48^s = 708^s$ log. 2.85003

60^s log. 1.77815

1.07188

2

139.2, as in table XXXIII. 2.14376

12. In lat. 45° 28′ N. nearly, the correct altitude of Aldebaran was found by observation, in the year 1839, to be 60° 40′ 20″, when its hour angle was $7^m\ 17^s$. What was the true latitude, if the declination of Aldebaran was 16° 11′ 9″.2 N.?

Solution. From Table XXXII 2″.7

From Table XXXIII 53

2′ 23″.1 = 143″.1

60° 40′ 20″

Third alt. = 60° 42′ 43″.1

Dec. = 16° 11′ 9″.2

Lat. = 45° 28′ 26″.1 N.

13. In lat. 40° N. nearly, the sum of ten correct central altitudes of the sun, when its declination was 20° S. were 300° 20′. The hour angles of these observations were $4^m\ 15^s$, 3^m, $2^m\ 6^s$, $1^m\ 8^s$, 30^s, 50^s, $1^m\ 12^s$, $2^m\ 15^s$, $3^m\ 10^s$, $4^m\ 25^s$. What is the true latitude, if the change of declination is neglected?

Latitude by circummeridian altitudes.

Solution. The numbers of Table XXXIII are

4^m	15^s	gives	18.1
3	0		9.0
2	6		4.4
1	8		1.3
0	30		0.2
0	50		0.7
1	12		1.4
2	15		5.1
3	10		10.0
4	25		19.5

Sum = 69.7

Mean = 6.97

Table XXXII gives 1″.6

11″

Mean of observations = 30° 0′ 40″

Merid. alt. = 30° 0′ 51″

Dec. = 20° S.

Lat. = 39° 59′ 9″ N.

14. At Göttingen, in lat. 51° 32′ N. nearly, the correct central altitudes of the sun on the 11th of March, 1794, were by observation

Latitude by circummeridian altitudes.

34° 54′ 46″	when the hour angle was	— 9ᵐ 41ˢ
34 55 26		— 8 19
34 56 8		— 6 39
34 56 31		— 5 16
34 56 53		— 3 49
34 57 6		— 2 47
34 57 18		0 19
34 57 11		2 5
34 57 3		3 9
34 56 48		4 36
34 56 26		6 8

The sun's meridian declination was 3° 30′ 38″ S., and it was decreasing at the rate of 0″.98 in a minute. What is the true latitude?

Solution. The mean of the altitudes is 34° 56′ 30″.5 ; that of the numbers of Table XXXIII is 30″.0, which, multiplied by 1″.5 from Table XXXII, gives 45″.0

The mean of the hour angles is, regarding their signs, — 1ᵐ 50ˢ, which, multiplied by 0″.98, gives by (362), for the correction of the meridian altitude 1″.8

The meridian altitude = 34° 57′ 17″.3

The declination = 3° 30′ 38″ S.

The latitude = 51° 32′ 4″.7 N.

which agrees exactly with the calculations of Littrow in his Astronomy.

15. Calculate the correction for the altitude of the pole star [B. p. 206], when the right ascension of the zenith is $2^h\ 7^m$.

Solution. By (361),

$h = 2^h\ 7^m - 1^h\ 2^m = 1^h\ 5^m$	sec.	0.0177
$p = 1°\ 33'$	Prop. log.	0.2868
Corr. alt. $= 1°\ 29'$, as in the table,	Prop. log.	0.3045

16. When the right ascension of the zenith was $7^h\ 9\frac{1}{2}^m$, the altitude of the pole star was observed at Newburyport to be $42°\ 44'$. What is the latitude of Newburyport?

Solution. The correction of table $=\ \ 0°\ \ 3'$

Altitude . . $= 42°\ 44'$

Latitude . . $= 42°\ 47'$

17. Calculate the log. elapsed time and log. middle time of Table XXIII for $3^h\ 7^m\ 10^s$.

Solution. By Table XXVII and (384),

$3^h\ 7^m\ 10^s$ cosec. 0.13735 = log. elapsed time

5.30103

5.16368 = log. mid. time.

Variation of star's altitude.

18. Calculate the variation of the altitude of a star arising from the change of 100 seconds in the declination, when the latitude is 40°, the declination 10°, and the altitude 30°.

Solution. By (391),

$L' = 40°$,	$2 \times$ Nat. sin.	1.2856		1.2856
$A'_1 - D = 20°$	Nat. cos.	0.9397	-0.9397	0.9397
$A'_1 + D = 40°$	Nat. cos.	0.7660	0.7660	-0.7660
		1.7057	1.1119	1.4593

1.7057	(ar. co.)	9.7681	9.7681
$100'' \times 1.1119$		2.0661	
$100'' \times 1.4593$			2.1641
$65''$ = var. when D is +,		1.8142	
$86''$ = var. when D is —,			1.9322

19 The moon's correct central altitude was found, by observation, to be 53° 43′, when her declination was 14° 16′ N. After an interval, in which the hour angle was $1^h\ 44^m\ 15^s$, her correct central altitude was 42° 29′, and her declination 13° 52′ N. The latitude was 48° 50′ N. nearly; what was it exactly?

Solution. Table XLVI gives, for the second alt. 83″

Whole change of declination 24′

Correction of second altitude 20′

Corrected second alt. = 42° 49′, dec. = 14° 16′ N.

Latitude by double altitudes.

I. By Bowditch's first method.

$1^h\ 44^m\ 15^s$	cosec. 0.64675		
14° 16′	sec. 0.01360	cosec. 0.60830	
A	cosec. 0.66035	cos. 9.98936	cos. 9.98936
B = 14° 38′ N.		cosec. 0.59766	
	cos. 9.82326	½ sum alts. = 48° 16′	cosec. 0.12712
	sin. 8.97762	½ diff. alts. = 5° 27′	sec. 0.00197
C	sin. 9.46123	cos. 9.98103	cos. 9.98103
Z = 37° 19′ N.			sec. 0.09948
E = 51° 57′ N.		sin. 9.89624	
Latitude = 48° 55½′ N.		sin. 9.87727	

II. By the method (375 – 378).

$1^h\ 44^m\ 15^s$		cos. 9.98852	cosec. 0.64675	
14° 16′		cotan. 0.59469	sec. 0.01360	sin. 9.39170
B = 14° 38′ N.		cotan. 0.58321		cosec. 0.59753
	½ sum alts. = 48° 16′		cos. 9.82326	cosec. 0.12712
	½ diff. alts. = 5° 27′		sin. 8.97762	sec. 0.00197
C		cos. 9.98103	sin. 9.46123	cos. 9.98103
Z = 37° 18′ N.				sec. .09935
E = 51° 56′ N.		sin. 9.89614		
Lat = 48° 54½′ N.		sin. 9.87717		

Latitude by double altitudes.

III. By Douwes's method.

		48° 50′ sec.	0.18161
53° 43′ N. sin.	80610	14° 46′ sec.	0.01360
42° 49′ N. sin.	67965	log. ratio	0.19521
	12645	log.	4.10192
$\frac{1}{2}(1^h\ 44^m\ 15^s) =$	$52^m\ 7\frac{1}{2}^s$	log. el. time	0.64689
	$1^h\ 44^m\ 18^s$	log. mid. time	4.94402
	$52^m\ 11\frac{1}{2}^s$	log. ris.	3.41152
		log. ratio	0.19521
	1645	log.	3.21631
	80610		
34° 40′ N. N. cos.	82255		
14° 16′ N.			

Lat. = 48° 56′ N.

Had the latitude been supposed 10′ greater, the calculated latitude would have been 48° 55′ N.

Latitude by double altitudes.

IV. By Bowditch's fourth method.

	$1^h\ 44^m\ 15^s$ sec. 0.04657		tan. 9.68938
	14° 16′ N. tan. 9.40531	sin. 9.39170	
A =	15° 48′ S. tan. 9.45188	cosec. 0.56485	cos. 9.98326
	13° 52′ N.		
B =	1° 56′ S.	cos. 9.99975	cosec. 1.47190
C	25° 16′ cosec. 0.36961	cos. 9.95630	
		F = 4° 6′ N.	cotan. 1.14454
	53° 43′	*Z* = 51° 38′ N.	
		G = 55° 44′ N.	sin. 9.91720
	42° 29′ sec. 0.13225	sin. 9.82955	cotan. 0.03820
½ sum =	60° 44′ cos. 9.68920	*I* sec. 0.12936	tan. 9.95540
Rem. =	7° 1′ sin. 9.08692	*K* sin. 9.91823	*I* = 42° 4′ N.
	2) 19.27798	lat. sin. 9.87714	13° 52′ N.
½ *Z* =	25° 49′ N. sin. 9.63899	lat. = 48° 54′ N.	*K* = 55° 56′ N.

19. The correct meridian altitude of Aldebaran was, by observation, 56° 25′ 40″ bearing south, and its declination at the time of the observation was 16° 8′ 44″ N.; what was the latitude?

Ans. 49° 43′ 4″ N.

Latitude by meridian altitudes.

20. The correct meridian altitude of Sirius was 70° 59′ 33″ bearing north, and its declination 16° 28′ 9″ S.; what was the latitude?

Ans. 26° 28′ 36″ S.

21. The meridian altitude of the sun's centre was 25° 38′ 30″ bearing south, and its declination 22° 18′ 14″ S.; what was the latitude?

Ans. 42° 3′ 16″ N.

22. The meridian altitude of the planet Jupiter was 50° 20′ 8″ bearing south, and its declination 18° 47′ 37″ N.; what was the latitude?

Ans. 58° 27′ 29″ N.

23. The altitude of the pole star was 30° 1′ 30″ below the pole, and its polar distance 1° 38′ 2″; what was the latitude?

Ans. 31° 39′ 32″ N.

24. The altitude of Capella on the meridian below the pole was 9° 52′ 42″, and its polar distance 44° 11′ 33″; what was the latitude?

Ans. 54° 4′ 15″ N.

25. The meridian altitude of the sun's centre was 7° 9′ 11″ below the pole, and its declination 23° 8′ 17″ N.; what was the latitude?

Ans. 74° 0′ 54″ N.

26. The two meridian altitudes of a northern circumpolar star were 61° 49′ 13″ and 47° 34′ 27″; what was the latitude?

Ans. 54° 36′ 50″ N.

Latitude by single altitudes.

27. In a northern latitude, the altitude of the sun's centre was 54° 9′, when its hour angle was $32^m\ 40^s$, and its declination 11° 17′ N.; what was the latitude?

Ans. 46° 27′ N.

28. In latitude 49° 17′ N. nearly, the altitude of the sun's centre was 14° 15′, when its hour angle was $1^h\ 40^m$, and its declination 23° 28′ S.; what was the true latitude?

Ans. 48° 55′ N.

29. Calculate the variation of a star's altitude in one minute from the meridian, when the declination is 3° and the latitude 7°.

Ans. It is 27″.9 when the dec. and lat. are of the same name, and 11″.2 when they are of contrary names.

30. Calculate the tabular number for $12^m\ 59^s$ in Table XXXIII.

Ans. 168.6.

31. In lat. 50° 30′ N. nearly, the altitude of Sirius was 22° 59′ 36″, when its hour angle was $4^m\ 15^s$, and its declination 16° 29′ 11″ S.; what was the true latitude?

Ans. 50° 30′ 49″ N.

32. In lat. 20° 27′ N. nearly, the sum of seven altitudes of Sirius was 371° 21′; the hour angles of the observations were 7^m, $5^m\ 3^s$, $2^m\ 12^s$, 9^s, 3^m, $4^m\ 6^s$, $8^m\ 13^s$; what was the true latitude, if the declination of Sirius was 16° 29′ 30″?

Ans. 20° 26′ 18″ N.

Latitude by circummeridian altitudes.

33. In lat. 50° N. nearly, the sum of twelve central altitudes of the moon was 590°; the hour angles of the observations were $-9^m\,3^s$, $-7^m\,40^s$, $-6^m\,12^s$, $-5^m\,30^s$, $-3^m\,2^s$, -1^m, -12^s, 50^s, $1^m\,59^s$, 4^m, $7^m\,30^s$, 10^m; the moon's meridian declination was 19° 10′ 58″.4 N., and her change of declination for one minute 13″.875; what was the true latitude?

Ans. 59° 50′ 2″.3 N.

34. Calculate the correction for the altitude of the pole star [B. p. 206.], when the right ascension of the zenith is $9^h\,7^m$.

Ans. 48′.

35. The altitude of the pole star was 25° 9′, when the right ascension of the zenith was 21° 47′; what was the latitude?

Ans. 24° 8′ N.

36. Calculate the log. elapsed time and log. middle time of Table XXIII for $5^h\,49^m\,50^s$.

Ans. Log. elapsed time = 0.00001
Log. middle time = 5.30102

37. Calculate the variation of the altitude of a star arising from the change of 100 seconds in declination, when the latitude is 60°, the declination 20°, the altitude 30°, and the declination and latitude of the same name.

Ans. 85″.

38. Calculate the variation of the altitude of a star arising from the change of 100 seconds in declination, when the latitude is 50°, the declination 24°, and the altitude 20°.

Ans. It is 73″ when the lat. and dec. are of the same name, and 105″ when they are of contrary names.

Latitude by double altitudes.

39. The sun's correct central altitudes were found by observation to be 30° 13′ and 50° 4′; his declination was 20° 7′ N., and the interval of sideral time between the observations was $2^h\ 55^m\ 32^s$; the assumed latitude was 56° 29′ N.; what was the true latitude?

Ans. 56° 47′ N.

40. The sun's correct central altitude was 41° 33′ 12″, his declination 14° N.; after an interval of $1^h\ 30^m$, his correct central altitude was 50° 1′ 12″, and declination 13° 58′ 38″ N.; the assumed latitude was 52° 5′ N.; what was the true latitude?

Ans. 52° 5′ N.

41. The moon's correct central altitude was 55° 38′, her declination 0° 20′ S.; after an interval in which the hour angle was $5^h\ 30^m\ 49^s$, her correct central altitude was 29° 37′, and her declination 1° 10′ N.; the assumed latitude was 23° 25′ S.; what was the true latitude?

Ans. 23° 24′ S.

42. The sun's correct central altitude was 16° 6′, his declination 8° 18′ N.; after an interval in which the hour angle was 3^h, his correct central altitude was 42° 14′ 9″, and his declination 8° 15′ N.; the assumed latitude was 49° N.; what was the true latitude?

Ans. 48° 50′ N.

43. The moon's correct central altitude was 35° 21′, and her declination 5° 31′ 6″ S.; after an interval in which the hour angle was $2^h\ 20^m$, her correct central altitude was 70° 1′, and her declination 5° 28′ 54″ S.; the assumed latitude was 1° 30′ N.; what was the true latitude?

Ans. 1° 29′ N.

44. The altitude of Capella was 60° 45′ 36″, and her declination 45° 48′ 21″ N.; at the same instant, the altitude of Sirius was 17° 54′ 12″, and his declination 16° 28′ 40″ S.; the hour angle was $1^h\ 33^m\ 45^s$, and the latitude was about 53° 15′ N.; what was the true latitude?

Ans. 53° 19′ N.

45. The altitude of α Bootis was 50° 3′ 39″, and its declination 20° 10′ 56″ N.; the altitude of α Aquilæ was 33° 33′, and its declination 8° 22′ 35″ N.; the hour angle of the observations was $5^h\ 5^m\ 5\frac{1}{2}^s$, and the assumed latitude 38° 27′ N.; what was the true latitude?

Ans. 38° 28′ N.

46. The distance of the centres of the sun and moon was found, by observation, to be 75°; the sun's central altitude was 37° 40′; the moon's central altitude was 55° 20′; the sun's declination was 0° 17′ S.; the moon's declination was 0° 36′ N.; what was the latitude, supposing it to be north?

Ans. 23° 24′ N.

48. The method of determining the latitude by means of the pole star is so accurate in practice, that tables are given in the Nautical Almanac for correcting the observed altitude for differences of latitude, and changes in the right ascension and declination of the star.

The *first correction* of the Nautical Almanac corresponds to that of the Navigator, and is calculated by (361) for R. A. of Polaris $= 1^h\ 1^m\ 48^s.2$. (392)

$$\text{Dec. of Polaris} = 88^\circ\ 26'\ 54'' = D, \qquad (393)$$

which gives

$$p = 1° \; 33' \; 6'' = 5586'' \qquad (394)$$

$$\log. p = 3.74710 \qquad (395)$$

$$h = \text{R. A. of zenith} - 1^h \; 1^m \; 48^s.2. \qquad (396)$$

The *second correction* of the Nautical Almanac depends upon the latitude, and would vanish, if in (350) the values of p and PC were so small that we could put

$$\sin. D = \cos. p = \cos. PC = 1.$$

Now equation (350) is equivalent to the proportion

$$\cos. PC : \cos. p = \cos. ZC : \sin. A,$$

but, by (360),

$$ZC = 90° - L - PC = 90° - L - p \cos. h;$$

whence

$$\cos. PC : \cos. p = \sin. (L + p \cos. h) : \sin. A.$$

Hence, by the theory of proportions,

$$\frac{\cos. PC - \cos. p}{\cos. PC + \cos. p} = \frac{\sin. (L + p \cos. h) - \sin. A}{\sin. (L + p \cos. h) + \sin. A}, \qquad (397)$$

and, by (40), (41), and (360),

$$\text{tang.} [\tfrac{1}{2} p (1 - \cos. h)] \cdot \text{tang.} [\tfrac{1}{2} p (1 + \cos. h)] =$$

$$\frac{\text{tang.} \tfrac{1}{2} (L + p \cos. h - A)}{\text{tang.} \tfrac{1}{2} (L + p \cos. h + A)}; \qquad (398)$$

or, since p and $L + p \cos. h - A$ are very small, we may put

$$\text{tang.} [\tfrac{1}{2} p (1 - \cos. h)] = \tfrac{1}{2} p (1 - \cos. h) \,\text{tang.}\, 1''$$

$$\text{tang.} [\tfrac{1}{2} p (1 + \cos. h)] = \tfrac{1}{2} p (1 + \cos. h) \,\text{tang.}\, 1''$$

Corrections of the Nautical Almanac for Polaris.

$$\text{tang.}\ \tfrac{1}{2}(L+p\cos.h-A)=\tfrac{1}{2}(L+p\cos.h-A)\,\text{tang.}\,1''$$

$$\text{tan.}\tfrac{1}{2}(L+p\cos.h+A)=\text{tan.}[L+p\cos.h-\tfrac{1}{2}(L+p\cos.h-A)]$$
$$=\text{tang.}\,(L+p\cos.h),$$

which, substituted in (390), give

$$\tfrac{1}{2}p^2(1-\cos.^2h)\text{tang.}(L+p\cos.h)\,\text{tang.}1''=L+p\cos.h-A$$

or

$$L=p\cos.h+A+\tfrac{1}{2}p^2\sin.^2h\,\text{tang.}(L+p\cos.h)\,\text{tang.}\,1'' \quad (399)$$

so that

$$\tfrac{1}{2}p^2\sin.^2h\,\text{tang.}\,(L+p\cos.h)\,\text{tang.}\,1'' \quad (400)$$
$$\text{or}\ \tfrac{1}{2}p^2\sin.^2h\,\text{tang}\,L\,\text{tang.}\,1''$$

is the correction depending upon the latitude, and in calculating it we have

$$\log.\,(\tfrac{1}{2}p^2\,\text{tang.}\,1'')=7.49420+9.69897+4.68558$$
$$=1.87875 \quad (401)$$

and h is the same as in (396).

The *third correction* of the Nautical Almanac is the change in the value of the first correction arising from the changes in the declination and right ascension of the star. Thus if the declination is greater than that of (393) by δD, the value of p must be less by δD, and the correction $-p\cos.h$ is increased by

$$\delta D\cos.h. \quad (402)$$

Again, if the right ascension is greater than that of (392) by δR, the value of h must be less by δR, and the value of $-p\cos.h$ is increased by

$$-p\,[\cos.(h-\delta R)-\cos.h], \quad (403)$$

which, by (15), is equal to

Corrections of the Nautical Almanac for Polaris.

$$-p \sin. \delta R \sin. h = -p \delta R \sin. 1^s \sin. h$$

$$= 5580 \times 0.000075 \times \delta R \sin. h = 0''.4 \delta R \sin. h, \quad (404)$$

and the whole change is the sum of (402) and (404). The values of (402) and (404) are easily obtained from the tables of difference of latitude and departure. We may neglect the $1^m\ 48^s.2$ in the value of h (396), when we calculate these corrections, and take

$$h = \text{R. A. of zenith} - 1^h. \quad (405)$$

The third correction is sometimes positive and sometimes negative, but always less than $1'$, so that

$$1' + \text{the third correction}$$

is always positive; and this is the given sum in the Nautical Almanac; that is, the third correction is given $1'$ greater than its real value, so that it may always be positive. The latitude, obtained by means of the table of the Nautical Almanac, would then be $1'$ greater than its true value, if $1'$ were not subtracted agreeably to the rule given in the Almanac.

49. Examples.

1. Calculate the first correction of the Nautical Almanac when the R. A. of the zenith $= 4^h\ 20^m$.

Solution.	log. $p =$	3.74710
$L = 4^h\ 20^m - 1^h\ 1^m\ 48^s.2 = 3^h\ 18^m\ 11^s.8$	cos.	9.81219
1st corr. $= 3624'' = 1°\ 0'\ 24''$		3.55929

2. Calculate the second correction of the Nautical Almanac when R. A. of the zenith is $7^h\ 30^m$, and the latitude $50°$.

Solution.	log. $(\frac{1}{2} p^2 \text{ tang. } 1'')$	=	1.87875
$L = 6^h\ 28^m\ 11^s.8$		sin.²	9.99340
	50°	tang.	0.07619
2d corr. = 89″ = 1′ 29″			1.94834

3. Calculate the third correction of the Nautical Almanac for Dec. 31, 1839, when R. A. of zenith $= 14^h$.

Solution. $h = 14^h - 1^h = 13^h = 12^h + 1^h = 180° + 15°.$

Dec. of Polaris =	88° 27′ 46″.6,	R. A. =	$1^h\ 1^m\ 59^s.47$
$D =$	88° 26′ 54″		$1^h\ 1^m\ 48^s.2$
$\delta D =$	52″.6	$\delta R =$	$11^s.27$
		$0''.4\ \delta R =$	4″.5

By Table II., omitting the tenths of seconds in the result,

$$1' + \text{3d corr.} = 1' - 50''.8 + 1''.2 = 1' - 50'' = 10''.$$

4. The correct altitude of Polaris on June 25, 1839, was 47° 28′ 35″, when the Right ascension of the zenith was 6^h $18^m\ 30^s$; what was the latitude?

Solution.	Cor. Alt.	=	47° 28′ 35″
	First corr.	=	— 13′ 27″
	A + First corr.	=	47° 15′ 8″
	Second corr.	=	1′ 28″
	Third corr.	=	1′ 6″ — 1′
	Lat.	=	47° 16′ 42″ N.

Observer in motion.

5. Calculate the three corrections of the Nautical Almanac for Sept. 1, 1839, and latitude 70°, when the R. A. of zenith is 8^h. At this time, we have

Dec. of Polaris $= 88° 27' 8''.4$, its R. A. $= 1^h 2^m 21^s.32$.

Ans. The first correction $= 23' 23''$

The second correction $= 3' 15''$

$1' +$ The third correction $= 43''$.

6. The correct altitude of Polaris on March 6, 1839, when the R. A. of the zenith was $6^h 39^m 24^s$, was $46° 17' 28''$; find the latitude. The following is an extract from the tables of the Nautical Almanac sufficient for the present example.

	1st corr.	2d corr.	
		Lat. = 45°	Lat. = 50°
$6^h 30^m$	— 12′ 53″	1′ 14″	1′ 28″
$6^h 40^m$	— 8′ 51″	1′ 16″	1′ 30″

Third correction + 1′

	March 1,	April 1,
6^h	1′ 27″	1′ 27″
8^h	1′ 13″	1′ 19″

Ans. 46° 10′ 3″ N.

50. The observer has been supposed stationary in the preceding observations, but if he is in motion his second altitude will differ from the altitude for this time at the first station by the number of minutes by which the observer has approached the star or receded from it; so that the correction arising from this change of place is obviously computed by the method in [B. p. 183.]

Greatest altitude of a star in motion.

51. In observing the meridian altitude of a star, the position of the meridian has been supposed to be known; but if it were not known, the meridian altitude can be distinguished from any other altitude from the fact that it is the greatest or the least altitude; so that it is only necessary to observe the greatest or the least altitude of the star.

52. But if the star changes its declination, the greatest altitude ceases to be the meridian altitude. Let h denote the hour angle of the star at the time of observation. Then if the star did not change its declination, and if B were the number of seconds given by Table XXXII for the diminution of altitude in one minute from the meridian passage, $h^2 B$ would be the diminution of altitude in h minutes. But, since h is small, the altitude, at this time, is increased by the change of declination; so that if A is the number of minutes by which the star changes its declination in one hour, that is, the number of seconds by which it changes its declination in one minute, $h A$ will be the increase of altitude in the time h, so that the altitude at the time h exceeds the meridian altitude by

$$h A - h^2 B. \qquad (406)$$

If, then, h denotes the time of the greatest altitude, and $h + \delta h$ a time which differs very slightly from the greatest altitude; the greatest altitude exceeds the altitude at the time $h + \delta h$ by the quantity

$$\begin{aligned}(h A - h^2 B) &- [(h + \delta h) A - (h + \delta h)^2 B] \\ &= \delta h [(- A + 2 B h) + B \delta h], \qquad (407)\end{aligned}$$

and δh can be supposed so small that $B \delta h$ may be insensible, and (407) becomes

$$\delta h (- A + 2 B h). \qquad (408)$$

Greatest altitude of a star in motion.

Now $-A+2Bh$ cannot be negative, because h is supposed to correspond to the greatest altitude, and cannot be less than the altitude at the time $h+\delta h$. Neither can $-A+2Bh$ be positive, for the altitude at the time h exceeds that at the time $h-\delta h$ by the quantity

$$-\delta h(-A+2Bh),$$

which, in this case, would be negative, and the altitude at the time $h-\delta h$ would exceed the greatest altitude. Since, then, $-A+2Bh$ can neither be greater nor less than zero, we must have

$$-A+2Bh=0$$

or

$$h=\frac{A}{2B}, \qquad (409)$$

and this value of h, substituted in (406), gives

$$\frac{A^2}{2B}-\frac{A^2}{4B}=\frac{A^2}{4B} \qquad (410)$$

for the excess of the greatest altitude above the meridian altitude.

53. If the observer were not at rest, his change of latitude will affect his observed greatest altitude in the same way in which it would be affected by an equal change in the declination of the star; so that the calculation of the correction on this account may be made by means of (409) and (410) precisely as in [B. p. 169.].

54. Examples.

1. An observer sailing N.N.W. 9 miles per hour found, by observation, the greatest central altitude of the moon, bearing

south, to be 54° 18′; what was the latitude, if the moon's declination was 6° 30′ S., and her increase of declination per hour 16′.52?

Solution. ☽'s zenith dist. = 35° 42′ N.

☽'s dec. = 6° 30′ S.

Approx. lat. = 29° 12′ N.

☽'s increase of dec. per hour = 16′.52 S.

Ship's change of lat. = 8′.3

$A = 24.82, \ A^2 = 616.0$

By Table XXXII $B = 2.9, \ 4B = 11.6$

Corr. of gr. alt. = corr. of lat. = 52″ = 1′ nearly

Lat. = 29° 12′ + 1′ = 29° 13′ N.

2. An observer sailing south 12½ miles per hour found, by observation, the greatest central altitude of the moon bearing south, to be 25° 15′; what was the latitude, if the moon's declination was 1° 12′ N., and her increase of declination per hour 18′.5?

Ans. 66° 1′ N.

CHAPTER V.

THE ECLIPTIC.

55. The careful observation of the sun's motion shows this body to move nearly in the circumference of a great circle. This great circle is called *the ecliptic.* [B. p. 48.]

56. The angle which the ecliptic makes with the equator is called *the obliquity of the ecliptic.*

57. The points, where the ecliptic intersects the equator, are called the *equinoctial points;* or the *equinoxes.* The point through which the sun *ascends* from the southern to the northern side of the equator is called the *vernal equinox*, and the other equinox is called the *autumnal equinox.*

The points 90° distant from the ecliptic are called the *solstitial points*, or the *solstices.* [B. p. 49.]

58. The circumference of the ecliptic is divided into twelve equal parts, called *signs*, beginning with the vernal equinox, and proceeding with the sun from west to east.

The names of these signs are *Aries* (♈), *Taurus* (♉), *Gemini* (♊), *Cancer* (♋), *Leo* (♌,) *Virgo* (♍), *Libra* (♎), *Scorpio* (♏), *Sagittarius* (♐), *Capricornus* (♑), *Aquarius* (♒), *Pisces* (♓). The vernal equinox is therefore *the first point*, or beginning of Aries, and the autumnal equinox is the first point of Libra; the first six signs are north of the equator, and the last six south of the equator. The northern solstice is the first point of Cancer, and the southern solstice the first point of Capricorn. [B. p. 49.]

59. Secondary circles drawn perpendicular to the ecliptic are called *circles of latitude.*

The circle of latitude drawn through the equinoxes is called *the equinoctial colure.*

The circle of latitude drawn through the solstices is called *the solstitial colure.* [B. p. 49.]

Corollary. The solstitial colure is also a secondary to the equator, so that it passes through the poles of both the equator and the ecliptic.

60. Small circles, drawn parallel to the equator through the solstitial points, are called *tropics.*

The northern tropic is called the *tropic of Cancer;* the southern tropic the *tropic of Capricorn.*

Small circles, drawn at the same distance from the poles which the tropics are from the equator, are called *polar circles.*

The northern polar circle is called the *arctic circle,* the southern the *antartic.*

61. The *latitude of a star* is its distance from the ecliptic measured upon the circle of latitude, which

passes through the star. If the observer is supposed to be at the earth, the latitude is called *geocentric latitude;* but if he is at the sun, it is *heliocentric latitude.* [B. p. 49.]

62. The *longitude of a star* is the arc of the ecliptic contained between the circle of latitude drawn through the star and the vernal equinox. [B. p. 50.]

Corollary. The longitude and right ascension of the first point of Cancer are each equal to 6^h, and those of the first point of Capricorn are each equal to 18^h.

63. The *nonagesimal point* of the ecliptic is the highest point at any time.

Corollary. The distance of the nonagesimal from the zenith is therefore equal to the distance of the zenith from the ecliptic, that is, to the *celestial latitude of the zenith;* and the longitude of the nonagesimal is the *celestial longitude of the zenith.*

64. *Problem. To find the latitude and longitude of a star, when its right ascension and declination are known.*

Solution. Let P (fig. 35.) be the north pole of the equator, Z the north pole of the ecliptic, and B the star. Then EQW will be the equator, $NESW$ the ecliptic, and $NPZS$ the solstitial colure, so that the point S is the southern solstice, and N the northern solstice. Now if the arc PB be produced to cut the equator at M, and ZB to cut the ecliptic at L; the angle ZPB is measured by the arc QM, that is, by the difference of the right ascensions of Q and M, or by the difference of the ✱'s right ascension and 18^h; that is,

$$ZPB = 18^h - \text{R. A.} = 24^h - (6^h + \text{R. A.}) \qquad (411)$$

$$\text{or} \qquad = \text{R. A.} - 18^h = (\text{R. A.} + 6^h) - 24^h$$

$$\text{or} \qquad = 24^h + \text{R. A.} - 18^h = \text{R. A.} + 6^h.$$

In the same way

$$PZB = NL = \text{Long.} - 90° \qquad (412)$$

$$\text{or} \qquad = 360° - (\text{Long. } 90°)$$

$$\text{or} \qquad = -(\text{Long.} - 90°),$$

in which the first values of ZPB and PZB correspond to the star's being east of the solstitial colure; the second and third values to the star's being west of the colure. We also have

$$PB = 90° - \text{Dec.} \qquad (413)$$

$$BZ = 90° - \text{Lat.} \qquad (414)$$

$$PZ = 90° - ZQ = QS$$
$$= \text{obliquity of ecliptic} = \pm E, \qquad (415)$$

in which the declination and latitude are positive when north, and negative when south, and E has the same sign with R. A. — 12^h.

The present problem does not, then, differ from that of § 28, and if we put

$$\pm A = PC - 90°, \qquad (416)$$

in which the upper sign is used, when R. A. — 12^h is positive, and otherwise the lower sign, we have by (99, 105, and 98)

$$\text{tang. } PC = \mp \text{ cotan. } A = \cos. (\text{R. A.} + 6^h) \text{ cotan. Dec.}$$
$$= - \sin. \text{R. A. cotan. Dec.} \qquad (417)$$

To find the latitude and longitude of a star.

in which the signs are used as in (416); so that A and Dec. are always positive or negative at the same time. Instead of (417), its reciprocal may be used, which is

$$\mp \text{tang.} A = - \text{cosec. R. A. tang. Dec.} \qquad (418)$$

If, then,

$$B = E + A, \qquad (419)$$

we have

$$AP = \mp E - 90° \mp A = \mp B - 90° \qquad (420)$$

$$\text{or} \quad = 90° \pm A \pm E = 90° \pm B,$$

in which the upper or lower signs are used, as in (415). Hence

$$\cos. PC : \cos. AP = \mp \sin. A : \mp \sin. B = \sin. A : \sin. B$$
$$= \text{sin. Dec. : sin. Lat.} \qquad (421)$$

so that, since Dec. and A are both positive or both negative, B and Lat. must also be both positive or both negative. Again,

$$\sin. PC : \sin. PA = \cos. A : \pm \cos. B \qquad (422)$$
$$= \pm \text{cotang. (R. A} + 6^h) : \pm \text{cotan. (Long.} - 90°)$$
$$= \pm \text{tang. R. A.} : \pm \text{tang. Long.}$$

in which the signs may be neglected, and Long. is to be found in the same quadrant with R. A., unless the foot P of the perpendicular falls within the triangle; in which case the first value of AP (420) is used, so that B is obtuse. In this case, the longitude is in the adjacent quadrant on the same side of the solsticial colure with the right ascension. These results agree with the Rule in [B. p. 435.]

65. *Corollary.* The latitude and longitude of the zenith, that is, the zenith distance and longitude of the nonagesimal, might be found by the same method. But another rule can be

used, which is of peculiar advantage, where these quantities are often to be calculated for the same place. We have by (310) and (311), calling B the zenith, and putting

$$T = 24^h - ZPB \text{ or } = ZPB \qquad (423)$$

$$F = \tfrac{1}{2}(PZB - ZBP) \text{ or } = 180° - \tfrac{1}{2}(PZB - ZBP) \quad (424)$$

$$G = \tfrac{1}{2}(PZB + ZBP) \text{ or } = 180° - \tfrac{1}{2}(PZB + ZBP) \quad (425)$$

$$\text{tang.} F = -\text{cosec.} \tfrac{1}{2}(PB + PZ) \sin. \tfrac{1}{2}(PB - PZ) \text{cotan.} \tfrac{1}{2} T$$
$$= \text{tang.}(24^h - F) \quad (426)$$

$$\text{tang.} G = -\sec. \tfrac{1}{2}(PB + PZ) \cos. \tfrac{1}{2}(PB - PZ) \cot. \tfrac{1}{2} T \quad (427)$$

$$90° + F + G = PZB + 90° \text{ or } = 360° - PZB + 90° \quad (428)$$

$$= \text{Long. or } = 360° + \text{Long.} \qquad (429)$$

in which the first member of (426) is used when PB is greater than PZ, and the third when PB is less than PZ, that is, within the north polar circle; and the second members of (423, 424, 425, 428) correspond to the position of the zenith at the east of the solstitial colure, but the third members to the west of the colure.

Again, by (295),

$$\text{tang.} \tfrac{1}{2}(90° - \text{lat.}) = \text{tang.} \tfrac{1}{2} \text{ alt. nonagesimal}$$
$$= \cos. G \,.\, \sec. F \,\text{tang.} \tfrac{1}{2}(PB + PZ), \quad (430)$$

and the preceding formulas correspond to the rule in [B. p. 402.]

66. *Scholium.* The rule with regard to the value of G appears to be a little different, but the difference is only apparent;

for it follows from (427), that G and $12^h - \frac{1}{2}T$ are, at the same time, both acute or both obtuse, unless

$$\tfrac{1}{2}(PB + PZ) > 90°,$$

$$\text{or} \qquad PB > 180° - PZ, \qquad (431)$$

which corresponds to the south polar circle.

67. The abridged method of calculating the altitude and longitude of the nonagesimal [B. p. 403.], only consists in the previous computation of the values

$$A = \text{log.}\,[\text{cos.}\,(\tfrac{1}{2}PB - PZ)\ \text{sec.}\,\tfrac{1}{2}(PB + PZ)] \quad (432)$$

$$C = \text{log. tang.}\,\tfrac{1}{2}(PB + PZ) \quad (433)$$

$$B = \text{log. tang.}\,\tfrac{1}{2}(PB - PZ) - C \quad (434)$$

$$= \text{log.}\,[\text{tang.}\,\tfrac{1}{2}(PB - PZ)\ \text{cotan.}\,\tfrac{1}{2}(PB + PZ)]$$

$$= \text{log.}\,[\text{cosec.}\tfrac{1}{2}(PB + PZ)\,\text{sin.}\tfrac{1}{2}(PB - PZ)] - A,$$

whence

$$\text{log.}\,[\text{cosec.}\tfrac{1}{2}(PB + PZ)\text{sin.}\tfrac{1}{2}(PB - PZ)] = B + A \quad (435)$$

$$\text{and} \qquad \text{log. tang.}\,G = A + \text{log.}\,(-\,\text{cotan.}\,\tfrac{1}{2}T) \quad (436)$$

$$\text{log. tang.}\,F = A + B + \text{log.}\,(-\text{cotan.}\tfrac{1}{2}T) \quad (437)$$

$$= \text{log. tang.}\,G + B$$

$$\text{log. tang.}\,\tfrac{1}{2}\,\text{alt. non.} = \text{log. cos.}\,G + \text{log. sec.}\,F + C. \quad (438)$$

68. The rule in [B. p. 436.] for finding right ascension and declination, when the longitude and latitude are given, may be obtained by a process precisely similar to that for the rule before it.

69. Examples.

1. Calculate the latitude and longitude of the moon, when its right ascension is $4^h\ 42^m\ 56^s$, and its declination 27° 21′ 58″ N., and the obliquity of the ecliptic 23° 27′ 45″.

Solution.	27° 21′ 58″ N.		tang.	9.71400
$4^h\ 42^m\ 56^s$	tang.	0.45650	cosec.	0.02503
A = 28° 44′ 12″ N.	sec.	0.05708	tang.	9.73903
E = 23° 27′ 45″ S.				
B = 5° 16′ 27″ N.	cos.	9.99816	tang.	8.96524
long. = 72° 53′ 31″	tang.	0.51174	sin.	9.98034
lat. = 5° 2′ 33″ N.			tang.	8.94558

2. Calculate the values of A, B, and C for the obliquity 23° 27′ 40″, and the reduced latitude of 42° 12′ 2″ N.

Solution. Polar dist. = 47° 47′ 58″

47° 47′ 58″				
23° 27′ 40″				
½ sum = 35° 37′ 49″	sec.	0.09002	tang.	9.85536
diff. = 12° 10′ 9″	cos.	9.99013	tang.	9.33374
	A =	0.08015,	B =	9.47838

3. Calculate the altitude and longitude of the nonagesimal, when the right ascension of the meridian is $19^h\ 50^m$, the latitude 42° 12′ 2″ N., and the obliquity 23° 27′ 40″.

To find the latitude and longitude of a star.

Solution. $T = 19^h\ 50^m + 6^h - 24^h = 1^h\ 50^m$

$\frac{1}{2}(1^h\ 50^m)$	cotan. 0.61137		
	$A =$ 0.08015		
$G = 101°\ 30'\ 2''$	tang. 0.69152	cos. 9.29968	
90°	$B =$ 9.47838	$C =$ 9.85536	
$F = 124°\ 4'\ 5''$	tang. 0.16990	sec. 0.25167	
long. $= 315°\ 34'\ 7''$	$14°\ 18'\ 40''$	tang. 9.40671	

alt. $= 28°\ 37'\ 20''$.

4. Calculate the latitude and longitude of the moon, when its right ascension is $18^h\ 27^m\ 12^s$, and its declination $27°\ 49'\ 38''$ S., and the obliquity of the ecliptic $23°\ 27'\ 45''$.

Ans. The ☽'s long. $= 276°\ 1'\ 46''$
its lat. $= 4°\ 30'\ 26''$ S.

5. Calculate the values of A, B, and C for Albany, and the obliquity $23°\ 27'\ 40''$.

Ans. $A = 0.07967$
$B = 9.47573$
$C = 9.85333$

6. Calculate the longitude and altitude of the nonagesimal, when the obliquity of the ecliptic is $23°\ 27'\ 40''$, the latitude $42°\ 12'\ 2''$ N., and the R. A. of the meridian $10^h\ 10^m$.

Ans. The long. $= 138°\ 30'\ 23''$
alt. $= 61°\ 18'\ 49''$.

7. Calculate the moon's right ascension and declination, when its latitude is 5° 0′ 7″ N., its longitude 64° 54′ 1″, and the obliquity of the ecliptic 23° 27′ 45″.

Ans. Its R. A. $= 4^h\ 7^m\ 46^s$.
Its Dec. $= 26°\ 3'\ 1''$ N.

70. *Problem. To find the declination of a star.*

Solution. I. Observe its meridian altitude, and its declination is at once found by one of the equations [345 – 347.]

II. If the star does not set, and both its transits are observed, we have

$$p = 90° - \text{Dec.} = \tfrac{1}{2}\,(A_1 - A'). \qquad (438)$$

71. *Problem. To find the position of the equinoctial points.*

Solution. Since the right ascension of all stars is counted from the vernal equinox, and since the two equinoxes are 12^h apart, the present problem is the same as to find the right ascension of some one of the stars, which may afterwards serve as a fixed point for determining the right ascension of the other stars.

Observe the declination of the sun for several successive noons near the equinox, until two noons are found between which its declination has changed its sign; and observe also the instant of the sun's transit across the meridian on these days, by a clock whose rate of going is known. Then, by supposing the sun's motions in declination and right ascension to be uniform at this time, which they nearly are, the time of the equinox, that is, of the sun's being in the equator, is found by the proportion the whole change of declination : either declination = the sideral interval between the transits — 24^h : the sideral interval between the transit of the equinox and that of the sun at this declination; (439)

and this interval is the difference between the right ascensions of the sun at this declination and the equinox. If the passage of a star had been observed in the same day, the right ascension of the star would have been the interval of sideral time of its passage after that of the vernal equinox.

72. Examples.

1. If the sun's declination is found at one transit to be $7' \, 9''.5$ S., and at the next transit to be $16' \, 31''.1$ N.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 38^s.21$.

Solution.

$7' \, 9''.5 + 16' \, 31''.1 = 23' \, 40''.6 = 1420''.6$	ar. co.	6.84753
$16' \, 31''.1 = 991''.1$		2.99612
$3^m \, 38^s.21 = 218^s.21$		2.33887
☉'s R. A. $= 0^h \, 2^m \, 32^s.2$	$152^s.2$	2.18252

2. If the sun's declination is found at one transit to be $18' \, 38''.8$ S., and at the next transit to be $5' \, 3''.2$ N.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 38^s.4$?

Ans. $0^h \, 0^m \, 46^s.5$.

3. If the sun's declination is found at one transit to be $5' \, 57''.9$ N., and at the next transit to be $17' \, 26''.3$ S.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 35^s.71$?

Ans. $12^h \, 2^m \, 40^s.8$.

73. *Problem. To find the obliquity of the ecliptic.*

Solution. I. Observe the right ascension and declination of the sun, when he is nearly at his greatest declination; that is, when his right ascension is nearly 6^h or 18^h. If he were observed at exactly his greatest declination, the observed declination would obviously be the required obliquity. But for any other time, the sun's declination and right ascension are the legs of a right triangle, of which the obliquity of the ecliptic is the angle opposite the declination. Hence

$$\text{tang. }\odot\text{'s Dec.} = \text{sin. }\odot\text{'s R. A. tang. obliq.} \qquad (440)$$

Now if we put

$$h = \text{the diff. of }\odot\text{'s R. A. and R. A. of solstice,}$$

we have

$$\cos. h = \frac{\text{tang. }\odot\text{'s Dec.}}{\text{tang. Obliq.}} \qquad (441)$$

and by (277) and (278),

$$\frac{\text{sin. (obliq.} - \odot\text{'s dec.)}}{\text{sin. (obliq.} + \odot\text{'s dec.)}} = \frac{1 - \cos. h}{1 + \cos. h} = \frac{2 \sin.^2 \frac{1}{2} h}{2 \cos.^2 \frac{1}{2} h}$$

$$= \text{tang.}^2 \tfrac{1}{2} h \qquad (442)$$

$$\text{sin. (obliq.} - \odot\text{'s dec.)} = \text{(obliq.} - \odot\text{'s dec.) sin. } 1''$$

$$= \text{tang.}^2 \tfrac{1}{2} h \text{ sin. (obl.} + \odot\text{'s dec.)} \qquad (443)$$

$$\text{obl.} - \odot\text{'s dec.} = \text{cosec. } 1'' \text{ tan.}^2 \tfrac{1}{2} h \text{ sin. (obl.} + \odot\text{'s dec.)} \qquad (444)$$

$$= \tfrac{1}{4} h^2 \text{ cosec. } 1'' \text{ tan.}^2\, 1^s \text{ sin. (obl.} + \odot\text{'s dec.)}$$

and the second member of (444) may be regarded as a correction in seconds to be added to the ☉'s dec. to obtain the obliquity, and the obliquity in the second member need only be known approximately.

74. Examples.

1. The right ascensions and declinations of the sun on several successive days were as follows:

June 19,	R. A. $= 5^h\, 50^m\, 53^s$,	Dec. $= 23° 26' 45''.2$ N.
20	5 55 3	23 27 27 .3
21	5 59 12	23 27 44 .7
22	6 3 21	23 27 37 .3
23	6 7 31	23 27 4 .6

To find the obliquity of the ecliptic.

Solution. Assume for the obliquity the greatest observed declination, or 23° 27′ 45″, and the corrections of all the observations may be computed in the same way as that of the first, which is thus found,

$\frac{1}{4}$ cosec. $1''$ tang.^{2} $1^s = \frac{225}{4}$ tang. $1''$		6.43570
$h = 9^m\ 7^s = 547^s$	2 log.	5.47598
$23° 26' 45'' + 23° 27' 45'' = 46° 54' 30''$	sin.	9.86348
cor. dec. $= 59''.59$		1.77516
$23° 26' 45''.2$		
obliquity $= 23° 27' 44''.8$		$= 23° 27' 44''.8$

In the same way the 2d observation gives 23 27 44 .9
the 3d observation gives 23 27 45 .2
the 4th observation gives 23 27 45 .3
the 5th observation gives 23 27 45 .3

sum = 117 18 45 .5

The mean $= 23° 27' 45''.1$

To find the obliquity of the ecliptic.

2. The right ascensions and declinations of the sun on several successive days, were as follows:

Dec. 20	☉'s R. A. =	$17^h\,51^m\,14^s$	23° 26′ 48″.4 S.
21		17 55 40	23 27 30 .0
22		18 0 7	23 27 44 .0
23		18 4 33	23 27 29 .5
24		18 9 0	23 26 45 .5

what was the obliquity?

Ans. 23° 27′ 44″.7.

CHAPTER VI.

PRECESSION AND NUTATION.

74. The ecliptic is not a fixed but a moving plane, and its observed position in the year 1750 has been adopted by astronomers as a *fixed plane*, to which its situation at any other time is referred.

The motion of the ecliptic is shown by the changes in the latitudes of the stars.

75. Celestial motions are generally separated into two portions, *secular* and *periodical.*

Secular motions are those portions of the celestial motions which either remain nearly unchanged, or else are subject to a nearly uniform increase or diminution which lasts for so many ages, that their limits and times of duration have not yet been determined with any accuracy.

Periodical motions are those whose limits are small, and periods so short, that they have been determined with considerable accuracy.

76. The *true position* of a heavenly body, or of a celestial plane, is that which it actually has; its *mean*

position is that which it would have if it were freed from the effects of its periodical motions.

The mean position is, consequently, subject to all the secular changes.

77. The *mean ecliptic* has, from the time of the earliest observations, been approaching the plane of the equator at a little less than the half of a second each year, thus causing a diminution of the obliquity of the ecliptic.

Let NAA' (fig. 41.) be the fixed plane of 1750, and NA_1 the mean ecliptic for the number of years t after 1750. Let A be the vernal equinox of 1750, and AQ the equator. Let

$$\Pi = NA \text{ and } \pi = \text{the angle } ANA_1;$$

then, upon the authority of Bessel, the point of intersection N of the ecliptic, which is called the *node* of the ecliptic, with the fixed plane, has a retrograde motion, by which it approaches A at the annual rate of 5″.18, and if this point could have existed in 1750, its longitude would have been 171° 36′ 10″, so that

$$\Pi = 171° \; 36' \; 10'' - 5''.18 \, t. \tag{445}$$

Moreover, the angle which the mean ecliptic makes with the fixed plane increases at the annual rate of 0″.48892, but this rate of increase is itself decreasing at such a rate, that at the time t this angle is

$$\pi = 0''.48892 \, t - 0''.0000030719 \, t^2 \tag{446}$$

78. *Problem. To find the change of the mean latitude of a star, which arises from the motion of the ecliptic.*

Change of mean latitude.

Solution. Let

$L =$ the ✱'s lat. in 1750

$\delta L =$ its change of lat.

$A =$ its long. in $1750 - 171° \, 36' \, 10'' + 5''.18 \, t$ (447)

$=$ its long. referred to the node of the ecliptic

$\delta A =$ its change of long. from the node;

then, if Z (fig. 42) is the pole of the fixed plane, P that of the ecliptic, and B the star; we have

$$PZ = \pi, \; ZB = 90° - L, \; PB = 90° - L - \delta L$$

$$PZB = 90° + A, \; P = 90° - A - \delta A.$$

Draw ZC perpendicular to PB, and we have, since PZ, PC, and CZ are very small,

$$PC \doteq PZ \cos. P = \pi \sin. (A + \delta A)$$

or $$= \pi \sin. A$$

$$\cos. PZ : \cos. PC = \cos. PZ : \cos. BC$$

or $$PZ = BC$$

$$PC = PB - BZ = - \delta L = \pi \sin. A$$

$$\delta L = - \pi \sin. A \qquad (448)$$

$$= - (0''.48892 \, t - 0''.0000030719 \, t^2) \sin. A.$$

Again, the triangle ZPB gives, by (295),

$$\sin. \tfrac{1}{2}(PZB + P) : \cos. \tfrac{1}{2}(PZB - P) = \tan. \tfrac{1}{2}\pi : \tan. \tfrac{1}{2}(PB + PZ)$$

But

$$\tfrac{1}{2}(PZB + P) = 90° - \tfrac{1}{2}\delta A, \; \tfrac{1}{2}(PZB - P) = A + \tfrac{1}{2}\delta A,$$

whence $$\delta A = \pi \cos. A \text{ tang. } L \qquad (449)$$

$$= (0''.48892\, t - 0''.0000030719\, t^2) \cos. A \text{ tang. } L.$$

79. The *mean celestial equator* has a motion by which its node upon the fixed plane moves from the node of the ecliptic at the annual rate of about 50″, while its inclination to the fixed plane has a very small increase proportioned to the square of the time from 1750.

Thus, if AQ (fig. 41.) is the equator of 1750, and $A'Q'$ that for the time t, so that A is the vernal equinox of 1750, and APA_1 that for the time t.

Let $$\psi = AA', \quad \omega = NA'Q',$$

then A' moves from A at the annual rate of 50″.340499, and this rate is diminishing so that at the time

$$\psi = 50''.340499\, t - 0''.0001217945\, t^2, \qquad (450)$$

and the value of ω in the year 1750 was

$$\omega' = 23^\circ\ 28'\ 18'',$$

and is increasing at a rate proportioned to the square of the time, so that

$$\omega = \omega' + 0''.00000984233\, t^2. \qquad (451)$$

80. *Problem. To find the change of the mean obliquity of the ecliptic and that of longitude.*

Solution. Let (fig. 41.)

$$NA_1Q' = \omega_1, \quad NAA_1 = \psi_1 + \Pi;$$

Change of mean obliquity and longitude.

then, by (310) and (311),

$$\frac{\sin. [\varPi + \frac{1}{2}(\psi + \psi_1)]}{\sin. \frac{1}{2}(\psi - \psi_1)} = \frac{\text{tang.} \frac{1}{2}(\omega + \omega_1)}{\text{tang.} \frac{1}{2}\pi} \tag{452}$$

$$\frac{\cos. [\varPi + \frac{1}{2}(\psi + \psi_1)]}{\cos. \frac{1}{2}(\psi - \psi_1)} = \frac{\text{tang.} \frac{1}{2}(\omega_1 - \omega)}{\text{tang.} \frac{1}{2}\pi}. \tag{453}$$

Now in calculating the parts of $\psi_1 - \psi$ and $\omega_1 - \omega$, which are proportional to the time, we may, since ψ and ψ_1 differ but little as well as ω and ω_1, and since π is small, put

$$\varPi + \tfrac{1}{2}(\psi + \psi_1) = \varPi, \ \sin. \tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1) \sin. 1''$$

$$\text{tang.} \tfrac{1}{2}\pi = \tfrac{1}{2}\pi \text{ tang. } 1'' = \tfrac{1}{2}\pi \sin. 1'' = \tfrac{1}{2}(0''.48892)\, t \sin. 1''$$

$$\tfrac{1}{2}(\omega + \omega_1) = \omega', \text{ tang.} \tfrac{1}{2}(\omega_1 - \omega) = \tfrac{1}{2}(\omega_1 - \omega) \sin. 1'$$

$$\cos. \tfrac{1}{2}(\psi - \psi_1) = 1,$$

which, subtituted in (452) and (453), give

$$\psi - \psi_1 = 0''.48892\, t \sin. \varPi \text{ cotan. } \omega' \tag{454}$$

$$\omega_1 - \omega = 0''.48892\, t \cos. \varPi, \tag{455}$$

which are thus computed,

0″.48892		9.68924		9.68924
171° 36′ 10″	cos.	9.99532_n	sin.	9.16446
—0″.48368		9.68456_n		
	23° 28′ 18″		cotan.	0.36229
0″.164431				9.21599

that is, $\omega_1 - \omega = -0''.48368\, t$ (456)

$$\psi - \psi_1 = \quad 0''.164431\, t \tag{457}$$

Change of mean obliquity and longitude.

or $\omega_1 = 23° \, 28' \, 18'' - 0''.48368 \, t$ (458)

$\varphi_1 = 50''.340499 \, t - 0''.164431 \, t = 50''.176068 \, t.$ (459)

But, in computing the parts of $\omega_1 - \omega$ and $\psi - \psi_1$, which depend upon t^2, we need only retain the part depending upon t^2 in the value of tang. $\frac{1}{2}\pi$, and neglect these parts in the other terms of (452) and (453), we thus have

$$\sin. [\Pi + \tfrac{1}{2}(\psi + \psi_1)] = \sin. (\pi + 45''.08 \, t) \quad (460)$$

$$= \sin. \Pi + 45''.08 \, t \sin. 1''. \cos. \Pi$$

$$\cos. [\Pi + \tfrac{1}{2}(\psi + \psi_1)] = \cos. (\Pi) - 45''.08 \, t \sin. 1'' \sin. \Pi \quad (461)$$

$$\tan. \tfrac{1}{2}\pi = \tfrac{1}{2}\pi \sin. 1'' = \tfrac{1}{2} \sin. 1'' (0''.48892 \, t - 0''.0000030719 \, t^2 \quad (462)$$

$$\text{cotan.} \tfrac{1}{2}(\omega + \omega_1) = \text{cotan.} (\omega' - 0''.24184 \, t, \quad (463)$$

$$= \frac{1 + 0''.24184 \, t \sin. 1'' \text{ tang } \omega'}{\text{tang. } \omega' - 0''.24184 \, t \sin. 1''}$$

$$= \text{cotan.} \, \omega' + 0''.24184 \, t \sin. 1'' (1 + \text{cotan.}^2 \, \omega')$$

$$= \text{cotan.} \, \omega' + 0''.24184 \, t \sin. 1'' \text{ cosec.}^2 \, \omega'.$$

$$\cos. \tfrac{1}{2}(\psi - \psi_1) = 1, \ \sin. \tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1) \sin. 1''$$

$$\sin. \tfrac{1}{2}(\omega_1 - \omega) = \tfrac{1}{2}(\omega_1 - \omega) \sin. 1''$$

which, substituted in (452) and 453), give

$$\psi - \psi_1 = 0''.164431 \, t + 0''.48892 \, t^2 \sin. 1'' \, 45''.08 \cos. \Pi \text{ cotan.} \, \omega'$$

$$+ 0''.48892 \, t^2 \sin. 1'' \times 0''.24184 \sin. \Pi \text{ cosec.}^2 \, \omega' \quad (464)$$

$$- 0''.0000030719 \, t^2 \sin. \Pi \text{ cotan.} \, \omega'$$

$$\omega_1 - \omega = - 0''.48368 \, t - 0''.48892 \, t^2 \sin. 1'' \, 45''.08 \sin. \Pi$$

$$- 0''.0000030719 \, t^2 \cos. \Pi,$$

Change of mean obliquity and longitude.

which are thus computed,

	0″.48892	9.68924	
	1″	sin. 4.68557	
	45″.08	1.65398	
	171° 36′ 10″	sin. 9.16446	cos. 9.99532_n
−0″.000015605		5.19325	
+0″.000003039		0″.0000030719	4.48741
−0″.000012566			4.48273_n

0″.0000030719		4.48741			
171° 36′ 10″	sin.	9.16446	cos. 9.99532_n	sin.	9.16446
23° 28′ 18″	cotan.	0.36229	0.36229	cosec.²	0.79958
−0″.000001033		4.01416	sin. 1″ 4.68557		4.68557
		45″.08	1.65398		
		0″.48892	9.68924		9.68924
−0″.000243445			6.38640_n		
		0″.24184			9.38353
0″.000000528					3.72238
−0″.000243950					

so that $\psi - \psi_1 = 0''.164431\, t - 0''.000243950\, t^2$

$$\omega_1 - \omega = -0''.48368\, t - 0''.000012566\, t^2$$

$$\psi_1 = 50''.176068\, t - 0''.0001217945\, t^2 + 0''.000243950\, t^2$$

$$= 50''.176068\, t + 0''.000122156\, t^2 \qquad (465)$$

$$\omega_1 = 23^\circ\, 28'\, 18'' - 0''.48368\, t - 0''.000002724\, t^2 \qquad (466)$$

or more accurately, from Bessel's *Fundamenta Astronomiæ*,

$$\psi_1 = 50''.176068\, t + 0''.0001221483\, t^2 \tag{467}$$

$$\omega_1 = 23°\, 28'\, 18'' - 0''.48368\, t - 0''.00000272295\, t \tag{468}$$

These values were afterward changed by Bessel in his *Tabulæ Regiomontanæ* to

$$\psi = 50''.37572\, t - 0''.0001217945\, t^2 \tag{469}$$

$$\psi_1 = 50''.21129\, t + 0''.0001221483\, t^2 \tag{470}$$

$$\omega_1 = 23°\, 28'\, 18'' - 0''.48368\, t - 0''.00000272295\, t^2 \tag{471}$$

But these formulas were obtained from the physical theory, and are, as Bessel says, subject to errors, on account of the uncertainty with regard to some of the data; so that we shall adopt Poisson's formulas, because they agree in the variation of the obliquity almost exactly with Bessel's observations, and shall change the value of ω' to that determined by Bessel from observations; our formulas are, then,

$$\omega' = 23°\, 28'\, 17''.65 \tag{472}$$

$$\psi = 50''.37572\, t - 0.''00010905\, t^2 \tag{473}$$

$$\psi_1 = 50''.22300\, t + 0.''00011637\, t^2 \tag{474}$$

$$\omega = 23°\, 28'\, 17''.65 + 0''.00008001\, t^2 \tag{475}$$

$$\omega_1 = 23°\, 28'\, 17''.65 - 0''.45692\, t - 0''.000002242\, t^2 \tag{476}$$

If, now, the value of ψ_1 is added to that of δA (449), the resulting value is the total change of a star's mean longitude.

81. The backward motion ψ_1 of the equinoxes is called the *precession of the equinoxes.*

Change of mean equator.

82. *Problem. To find the intersection of the mean equator with the equator of* 1750 *and its inclination to it.*

Solution. Produce $A\,Q$ and $A'Q'$ (fig. 41.) till they meet at T, and let

$$AT = \Phi,\ A'T = \Phi',$$

and the triangle ATA' gives, by (291, 295, and 310),

$$\cos.\tfrac{1}{2}(\omega'-\omega) : \cos.\tfrac{1}{2}(\omega'+\omega) = \text{tang.}\tfrac{1}{2}\psi : \text{tang.}\tfrac{1}{2}(\Phi'-\Phi) \quad (477)$$

$$\sin.\tfrac{1}{2}(\omega'-\omega) : \sin.\tfrac{1}{2}(\omega'+\omega) = \text{tang.}\tfrac{1}{2}\psi : \text{tang.}\tfrac{1}{2}(\Phi'+\Phi) \quad (478)$$

$$\sin.\tfrac{1}{2}(\Phi'+\Phi) : \sin.\tfrac{1}{2}(\Phi'-\Phi) = \text{cotan.}\tfrac{1}{2}T : \cot.\tfrac{1}{2}(\omega'+\omega) \quad (479)$$

so that t^2 may be neglected in all the terms but ψ, and we have

$$1 : \cos.\omega' = \tfrac{1}{2}\psi \sin. 1'' : \tfrac{1}{2}(\Phi'-\Phi)\sin. 1'' \quad (480)$$

$$0 : \sin.\omega' = \tfrac{1}{2}\psi \sin. 1'' : \text{tang.}\tfrac{1}{2}(\Phi'+\Phi) \quad (481)$$

$$1 : \tfrac{1}{2}(\Phi'-\Phi)\sin. 1'' = \text{tang.}\omega' : \tfrac{1}{2}T\sin. 1''. \quad (482)$$

Hence

$$\tfrac{1}{2}(\Phi'+\Phi) = 90^\circ \quad (483)$$

$$\tfrac{1}{2}(\Phi'-\Phi) = \tfrac{1}{2}\psi\cos.\omega' \quad (484)$$

$$T = (\Phi'-\Phi)\,\text{tang.}\omega', \quad (485)$$

which are thus computed,

ω'	cos.	9.96249	cos. 9.96249
25″18786		1.40120	
23″.103		1.36369	
0″.000054525			5.73660
0″.000050013			5.69909
ω'	tang.	9.63771	9.63771
10″.032		1.00140	
0″.000021717			5.33680

so that

$$\Phi = 90° - 23''.103\,t + 0''.000050013\,t^2 \qquad (486)$$

$$T = 20''.0640\,t - 0''.000043434\,t^2. \qquad (487)$$

83. *Problem. To find the variation of a star's mean right ascension and declination.*

I. The variation, which arises from the change of the equator's inclination, may be found precisely in the same way in which the variations of latitude and longitude were found in § 78, for a similar change in the position of the ecliptic; so that formulas (448) and (449) give, by substituting for $\mathcal{A}$, L and π,

$$\mathcal{A} = \ast\text{'s R. A.} - 90° + 23''.103\,t = R - 90$$

$$L = \ast\text{'s Dec.} = D, \quad \pi = T$$

$$\delta D = - T \cos. R \qquad (488)$$

$$\delta R = T \sin. R \text{ tang. } D; \qquad (489)$$

or instead of counting the value of T and t from 1750, they may be reduced to the beginning of each year, and the square of t may then be neglected.

II. The variation in right ascension is to be increased by the change in the position of the equinox, arising from its precession, which is thus found. Had the ecliptic remained stationary, the equinox would have removed from A to A', so that if AP is perpendicular to the equator, we should have for the increase of right ascension by (475) and (484),

$$A'P = AA' \cos. AA'P = \psi \cos. \omega \qquad (490)$$

$$= (\Phi' - \Phi)$$

$$= 46''.206\,t - 0''.000100026\,t^2.$$

Change of mean right ascension.

But the equinox advances upon the equator from the motion of the ecliptic by the arc $A'A_1$, which is thus found. We have, by (291),

$$\cos.\tfrac{1}{2}(\omega_1 - \omega) : \cos.\tfrac{1}{2}(\omega_1 + \omega) = \text{tang.}\tfrac{1}{2}A'A_1 : \text{tang.}\tfrac{1}{2}(\psi - \psi_1)$$

But $\cos.\tfrac{1}{2}(\omega_1 - \omega) = 1$

$$\cos.\tfrac{1}{2}(\omega_1 + \omega) = \cos.(\omega' - 0''.22846\,t)$$
$$= \cos.\omega' + 0''.22846\,t \sin.1'' \sin.\omega'$$
$$\sec.\tfrac{1}{2}(\omega_1 + \omega) = \sec.\omega' - 0''.22846\,t \sin.1'' \sin.\omega' \sec.^2\omega'$$
$$\text{tang.}\tfrac{1}{2}A'A_1 = \tfrac{1}{2}A'A_1 \sin.1''$$
$$\text{tang.}\tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1)\sin.1''$$
$$= \tfrac{1}{2}\sin.1''\,(0''.15272\,t - 0''.00022542\,t^2)$$

whence
$$A'A_1 = 0''.15272\,t \sec.\omega'$$
$$- 0''.00022542\,t^2 \sec.\omega'$$
$$- 0''.22846\,t^2\; 0''15272\,t^2 \sin.1'' \text{tang.}\omega' \sec.\omega'$$

which is thus computed,

0″.15272		9.18390			9.18390
ω′	sec.	0.03751	0.03751		0.03751
0″.1665		9.22141			
		0″.00022542	6.35299		
0″.00024575			6.39050		
		0″.22846			9.35881
		ω′		tang.	9.63771
		1″		sin.	4.68557
0″					2.90350

so that

$$A'A_1 = 0''.1665\,t - 0''.00024575\,t^2, \qquad (491)$$

and, by (489) and (490),

$$\delta R = 46''.0395\,t + 0''.00016593\,t^2 + T \sin. R \tang. D. \qquad (492)$$

84. By the motions of precession and of diminution of the obliquity, the mean pole of the equator is carried round the pole of the ecliptic, gradually approaching it; but the true pole of the equator has a motion round the mean pole, which is called *nutation.* This motion is in an oval, at the centre of which is the mean pole, and is such that the position of the mean equinox differs from that of the true equinox by the longitude

$$\delta \text{ long.} = i \sin. ☊ + i_1 \sin. 2 ☊ + i_2 \sin. 2 ☽ + i_3 \sin. 2 ☉ \qquad (493)$$

where

☊ = the mean longitude of that point of intersection of the moon's orbit with the ecliptic, through which the moon ascends from the south to the north side of the ecliptic, and which is called the moon's ascending node,

☽ = the moon's true longitude,

☉ = the sun's true longitude.

The values of i, i_1, i_2, i_3 are given differently by different astronomers, and those which are, at present, adopted in the Nautical Almanac are

$$i = -17''.2985, \quad i_1 = 0''.2082 \qquad (494)$$
$$i_2 = -0''.2074, \quad i_3 = -1''.2550.$$

Nutation.

This nutation of the pole causes also the true obliquity of the ecliptic to change from the mean obliquity by the quantity

$$\delta_{\omega_1} = k \cos. ☊ + k_1 \cos. 2 ☊ + k_2 \cos. 2 ☽ + k_3 \cos. 2 \odot \quad (495)$$

in which the values of k &c., at present adopted in the Nautical Almanac, are

$$k = 9''.2500, \quad k_1 = -0''.0903 \quad (496)$$
$$k_2 = 0''.0900, \quad k_3 = 0''.5447.$$

85. *Corollary.* The effect of nutation upon the right ascensions and declinations of the stars may be computed by § 83, and the formulas which are obtained agree with those given in the Nautical Almanac, and which depend upon the terms, called C and D in the formulas for Reduction of the Almanac; these terms contain also the changes arising from the mean motion of the equinoxes, and the formulas are so reduced that t is counted from the beginning of each year.

86. Examples.

1. Find the mean obliquity of the ecliptic for the year 1840, and reduce the formulas for finding the variations of right ascension and declination to the beginning of that year.

Solution. In (476) let $t = 1840 - 1750 = 90$,

and it gives

$$\omega_1 = 23° \, 28' \, 17''.65 - 41''.12 - 0''.02 = 23° \, 27' \, 36''.51.$$

In (487, 488, and 492) let $t = 90 + t'$, and neglect the terms depending upon t'^2, so that

Change in right ascension and declination.

$$T = 30' \, 5''.76 - 0''.35 + 20''.0640 \, t' - 0''.0078 \, t'$$
$$= 30' \, 5''.41 + 20''.0562 \, t',$$

and the mean variations, counted from the beginning of the year, are

$$\delta' D = 20''.0562 \, t' \cos. R$$
$$\delta' R = 46''.0693 \, t' + 20''.0562 \, t' \sin. R \text{ tang. } D.$$

Finally, the variations arising from nutation are thus found. The change in the obliquity of the ecliptic gives at once, from (448) and (449), by referring the positions to the mean ecliptic instead of to that of 1750,

$$\delta' D = - \delta \omega_1 \sin. R$$
$$\delta' R = - \delta \omega_1 \cos. R \text{ tang. } D,$$

and the change in the position of the equinox gives by (485, 488, 489, and 490),

$$T = - \delta \varDelta \sin. \omega_1$$
$$\delta' D = \delta \varDelta \sin. \omega_1 \cos. R$$
$$\delta' R = \delta \varDelta \cos. \omega_1 + \delta \varDelta \sin. \omega_1 \sin. R \text{ tang. } D.$$

Hence, if we take

$$46''.0693 \, C = 46''.0693 \, t' + \delta \varDelta \cos. \omega_1$$
$$c = 46''.0693 + 20''.0562 \sin. R \text{ tang. } D$$
$$c' = 20''.0562 \cos. R$$
$$d = \cos. R \text{ tang. } D$$
$$d' = - \sin \, R$$

we have

$$C = t' + \frac{\cos. \omega_1}{46''.0693} \delta \varDelta = t' + \frac{\sin. \omega_1}{20''.0562} \delta \varDelta$$
$$= t' - 0.3448 \sin. ☊ + 0.00415 \sin. 2 ☊$$
$$- 0.00413 \sin. 2 ☽ - 0.02502 \sin. 2 ☉,$$

and the entire changes of declination and right ascension are

$$\delta' D = C c' - \delta \omega . d'$$

$$\delta' R = C c - \delta \omega . d,$$

which agree with the formulas in the Nautical Almanac, except in the coefficients of t', which are $46''.0206$ and $20''.0426$ instead of $46''.0693$ and $20''.0562$.

If, again, we take

$$f = 46''.0693\, C,$$

$$g \cos. G = 20''.0562\, C, \quad g \sin. G = - \delta \omega,$$

the above formulas become

$$\delta' D = g \cos. G \cos. R - g \sin. G \sin. R = g \cos. (G + R)$$

$$\delta' R = f + g \sin. R \cos. G \text{ tang. } D + g \sin. G \cos. R \text{ tang. } D$$

$$= f + g \sin. (R + G) \text{ tang. } D,$$

as in the Nautical Almanac.

2. Find the annual variations in the right ascension and declination of α Hydræ for the year 1840, and its true place for mean midnight at Greenwich, Jan. 1, 1840; its mean right ascension for Jan. 1, 1839, being $9^h\, 19^m\, 40^s.620$, and its declination $- 7° 57' 49''.50$, and using the numbers of the Nautical Almanac.

Solution.

$20''.0426$	1.30195	1.30195
$R = 9^h\,19^m\,40^s.620$	cos. 9.88374^n	sin. 9.80872
$\delta' D = -15''.335$	1.18569^n	
$D = -7°\,57'\,49''.50$		tang. 9.14584^n
$\delta R = 46''.0206 - 1''.8051$		0.25651^n
$= 44''.2155 = 2^s.948$		

Hence its mean place for Jan. 1, 1840, is

$$R = 9^h\ 19^m\ 43^s.568$$

$$D = -7°\ 58'\ 4''.83.$$

To calculate the effects of nutation, we have

$$☊ = 339°\ 40',\quad ☽ = 242°\ 30',\quad ☉ = 281°\ 15'$$

-0.3448 sin. $☊ = 0.1205,$	$9''.25$ cos. $☊ = 8''.673$
0.00415 sin. $2☊ = -0.0027,$	$-0''.0903$ cos. $2☊ = -0''.068$
-0.00413 sin. $2☽ = -0.0034,$	$0''.0900$ cos. $2☽ = -0''.032$
-0.02502 sin. $2☉ = 0.0096,$	$0''.5447$ cos. $2☉ = -0''.504$
$C = t' + 0.1240,$	$\delta\omega_1 = 8''.049$

$$C c' = c' t' + 20''.0426 \times 0.1240 \text{ cos. } R$$

$$= c' t' - 15''.335 \times 0.1240 = c' t' - 1''.901$$

$$-\delta\omega . d' = 8''.049 \text{ sin. } R = 5''.181$$

$$C c = c t' + 0.1240 \times 2^s.948 = c t' + 0^s.365$$

$$-\delta\omega d = -8''.049 \text{ cos. } R \text{ tang. } D = -0''.861 = -0^s.058,$$

whence the variations arising from nutation are

$$\delta' D = 3''.28, \quad \delta' R = 0^s.30,$$

and the true places are

$$D = -7° \; 58' \; 1''.55, \quad R = 9^h \; 19^m \; 43^s.87.$$

3. Find the mean obliquity of the ecliptic for the year 1950, and reduce the formulas for finding the variations of mean right ascension and declination to the beginning of that year.

Ans. $\omega_1 = 23° \; 26' \; 36''.18.$

$$\delta' D = 19''.8903 \, t' \cos. R$$

$$\delta' R = 46''.1059 \, t' + 19''.8903 \, t' \sin. R \text{ tang. } D.$$

4. Find the annual variations in the right ascension and declination of β Ursæ Minoris for the year 1839, and its true place for mean midnight at Greenwich, Aug. 9, 1839; its mean right ascension for Jan. 1, 1839, being $14^h \; 51^m \; 14^s.943$, its declination 74° 48′ 48″.89 N., the longitude of the moon's ascending node for Aug. 9, 1839, being 347° 17′, that of the moon 144° 2′, and that of the sun 136° 30′, and using the constants of the Nautical Almanac, which give for Aug. 9, 1839,

$$f = 32''.33, \quad g = 16''.70, \quad G = 327° \; 30'.$$

Ans. Var. in R. A. $= -0^s.277$; var. in Dec. $= 14''.71$;

and for Aug. 9, 1839,

$$R = 14^h \; 51^m \; 16^s.36$$

$$D = 74° \; 48' \; 32''.46.$$

Tables XL and XLIII.

5. Calculate the values of f, g, and G for April 1, 1839, mean midnight at Greenwich, when ☊ $= 354° \, 10'$, ☉ $= 11° \, 34'$, and ☽ is neglected.

Ans. $f = 12''.53, \quad g = 11''.05, \quad G = 299° \, 34'.$

In Table XL of the Navigator, the decimal is neglected, and 20 used instead of 20.0562. Table XLIII is calculated from the formulas of Bessel, which differ a little from those of Bailly used in the Nautical Almanac. The construction of these two tables is sufficiently simple from the calculations already given.

CHAPTER VII.

TIME.

87. The intervals between the successive returns of the mean place of a star to the meridian are precisely equal, and the mean daily motion of the star is perfectly uniform; so that sideral time is adapted to all the wants of astronomy. The instant, which has been adopted as the commencement of the sideral day, is *the upper transit of the vernal equinox.*

The length of the sideral day, which is thus adopted, differs therefore from the true sideral or *star* day by the daily change in the right ascension of the vernal equinox. But this change is annually about 50″ or $3^s.3$, so that the daily change is less than $0^s.01$, and is altogether insensible.

88. *Corollary.* The difference between the sideral time of different places is exactly equal to the difference of the longitude of the places.

89. The interval between two successive upper transits of the sun over the meridian is called a *solar day;* and the hour angle of the sun is called *solar time.* This is the measure of time best fitted to the common purposes of life.

Perigee.	Apogee.

The intervals between the successive returns of the sun to the meridian are not exactly equal, but depend upon the variable motion of the sun in right ascension, and can only be determined by an accurate knowledge of this motion.

90. The want of uniformity in the sun's motion in right ascension arises from two different causes.

I. The sun does not move in the equator but in the ecliptic.

II. The sun's motion in the ecliptic is not uniform.

The variable motion of the sun along the ecliptic, and its deviations from the plane of the mean ecliptic, cannot be distinctly represented, without reference to the variations of its distance from the earth, and to the nature of the curve which it describes. This portion of the subject, therefore, which involves the determination of the sun's exact daily position, that is, the calculation of its *ephemeris*, must be reserved for the *Physical Astronomy.* It is sufficient, for our present purpose, to know that the sun moves with the greatest velocity when it is nearest the earth, that is, in its *perigee;* and that it moves most slowly when it is farthest from the earth, that is, in its *apogee.*

91. The sun arrives at its perigee about 8 days after the winter solstice, and at its apogee about 8 days after the summer solstice. The mean longitude of the perigee at the beginning of the year 1800 was 279° 30′ 5″, and it is advancing towards the eastward at the annual rate of about 11″.8, so that, by adding the precession of the equinoxes, the annual increase of its longitude is about 62″.

92. To avoid the irregularity of time arising from the want of uniformity of the sun's motion, a fictitious sun, called a *mean sun*, is supposed to move uniformly in the ecliptic at such a rate, as to return to the perigee at the same time with the true sun. A *second mean sun* is also supposed to move in the equator at the same rate with the first mean sun, and to return to each equinox at the same time with the first mean sun.

We shall denote the first mean sun by $\odot_1$, and the second mean sun by $\odot_2$.

93. *Corollary.* The right ascension of the second mean sun is equal to the longitude of the first mean sun.

94. The time which is denoted by the second mean sun is perfectly uniform in its increase, and is called *mean time;* while that which is denoted by the true sun is called *true* or *apparent time;* the difference between mean and true time is called the *equation of time.*

95. The time which it takes the sun to complete a revolution about the earth is called a *year.*

The time which it takes the mean sun to return to the same longitude is the *common* or *tropical year.*

The time which it takes it to return to the same star is the *sideral year;* and the time which it takes it to return to the perigee is the *anomalistic year.*

Year.	Leap year.

The length of the mean tropical year is

$$Y = 365^d\ 5^h\ 48^m\ 47^s.808, \quad (497)$$

so that the daily mean motion of the sun is found by the proportion

$$Y : 1^d = 360^\circ : \text{daily motion} = 59'\ 8''.3302. \quad (498)$$

96. The fraction of a day is necessarily neglected in the length of the year in common life, and the common year is taken equal to 365^d. By this approximation, the error in four years amounts to

$$23^h\ 15^m\ 11^s.232 = 1^d - 44^m\ 48^s.768, \quad (499)$$

or nearly a day, and an additional day is consequently added to the fourth year, which is called the *leap year*. At the end of a century the remaining error amounts to nearly $-0^d.75$, which is noticed by the neglect of three leap years in four centuries. For practical convenience, those years are taken as leap years which are exactly divisible by 4, and the centurial years would thus be leap years, but only those are retained as leap years which are divisible by 400.

97. When the mean sun has returned to the same mean longitude, it has not returned to the same star, because the equinox from which the longitude is counted has retrograded by $50''.223$, so that the mean sun has $50''.223$ farther to go, and the time of describing this arc is the fourth term of the proportion

$$59'\ 8''.3302 : 1^d = 50''.223 : 20^m\ 22^s.786, \quad (500)$$

so that the length of the sideral year is

$$Y_1 = Y + 20^m\ 22^s.786 = 365^d\ 6^h\ 9^m\ 10^s.594. \quad (501)$$

98. The length of the mean solar day is also different from that of the sideral day, because when the $\odot_2$, in its diurnal motion, returns to the meridian, it is $59' \, 8''.3302$ advanced in right ascension; so that $360° \, 59' \, 8''.3302$ pass the meridian in a solar day, instead of 360°, which pass in a sideral day. Hence the excess of the solar day above the sideral day, expressed in solar time, is the fourth term of the proportion

$$360° \, 59' \, 8''.3302 : 59' \, 8''.3302 = 1^d : 0^d.0027305$$

$$\text{or} \quad 3^h \, 55^m.9094; \qquad (502)$$

that is, $\quad 1 \text{ sid. day} = 0.9972695 \text{ sol. day},$

or $\quad 24^h \text{ sid. time} = 23^h \, 56^m \, 4^s.0906 \text{ of solar time}; \quad (503)$

which agrees with (335) and the table for changing sideral to solar time in the Nautical Almanac, and with table LII of the Navigator.

In the same way this excess expressed in sideral time is the fourth term of the proportion

$$360° : 59' \, 8''.3302 = 1^d : 0^d.002738 \text{ or } 3^m \, 56^s.5554;$$

that is, $\quad 1 \text{ sol. day} = 1.002738 \text{ sid. day}, \qquad (504)$

or $\quad 24^h \text{ sol. time} = 24^h \, 3^m \, 56^s.5554 \text{ sid. time}; \qquad (505)$

which agrees with the table for changing solar to sideral time in the Nautical Almanac, and with table LI of the Navigator. The remainder of tables LI and LII, as well as the corresponding ones given in the Nautical Almanac, are calculated by simple proportions from the numbers which are given for 24^h.

The sideral day begins with the transit of the true vernal equinox. At the time of the transit of $\odot_2$, then, that is, at *mean noon*, we have

$$\begin{aligned}\text{the sid. time} &= \text{R. A. of } \odot_2 \text{ from the equinox} \\ &= \text{R. A. of } \odot_2 \text{ from mean equinox} \\ &\qquad + \text{Nutation of equinox in R. A.} \\ &= \text{sun's mean long} + \text{Nutation in R. A.} \quad (506)\end{aligned}$$

99. The sun's mean long. for Jan. 1, 1800, at Paris, was found by Bessel to be 279° 54′ 11″.36. Its longitude for Jan. 1, of any other year t, may thus be found. Let f be the remainder after the division of t by 4, the number of days, then, by which Jan. 1 of the year t is removed from Jan. 1, 1800, is

$$\begin{aligned}365\tfrac{1}{4}(t-f) + 365f &= t\,.\,365\tfrac{1}{4} - \tfrac{1}{4}f \\ &= Y.\,t + t\,.\,11^m\,12^s.192 - \tfrac{1}{4}f \qquad (507) \\ &= Y.\,t + t\,.\,0^d.00778 - \tfrac{1}{4}f.\end{aligned}$$

But in Yt days the sun's longitude increases exactly $t.360°$, which is to be neglected; and its increase in longitude is

$$59'\,8''.3302\,(t+0.00778-\tfrac{1}{4}f)=t.27''.61-f.14'47''.083, \quad (508)$$

or more accurately from Bes el, the mean longitude E, for the the first of January of the year 1800 + t at Paris, is

$$\begin{aligned}E = 279°\,54'\,1''.36 + t\,27''.605844 + t^2.\,0''.0001221805 \\ -f.\,14'\,47''.083. \qquad (509)\end{aligned}$$

The mean longitude is found for the first of January, for any other meridian by the following proportion, derived from the interval of time between the $\odot_2$'s passage over this meridian and that of Paris.

$$24^h : \text{long. from Par.} = 59'8''.3302 : \text{change in value of } E. \quad (510)$$

Sideral time converted to solar time.

The sun's mean longitude for any mean noon n of the year after that of the first of Jan. is

$$E + n \,.\, 59' \; 8''.3302. \tag{511}$$

Hence the sideral time of the mean noon n is

$$S = \frac{E}{15} + n \,.\, 3^m \, 56^s.555348 + \text{Nutation in R. A.} \tag{512}$$

so that the solar time of the transit of the equinox from the preceding noon is

$$24^h - S \text{ (converted into solar time).} \tag{513}$$

100. Examples.

1. Find the sideral interval which corresponds to 10^h of solar time.

Ans. $10 \; 1^m \, 38^s.5647.$

2. Find the solar interval which corresponds to 10^h of sideral time.

Ans. $9^h \, 58^m \, 21^s.7044.$

3. Find the sideral interval which corresponds to 10^m of solar time.

Ans. $10^m \, 1^s.6428.$

4. Find the solar interval which corresponds to 10^s of sideral time.

Ans. $9^m \, 58^s.3617.$

5. Find the sideral interval which corresponds to 10^s of solar time.

Ans. $10^s.0274.$

6. Find the solar interval which corresponds to 10^s of sideral time.

Ans. $9^s.9727$.

7. Find the sideral interval which corresponds to $0^s.85$ of solar time.

Ans. $0^s.85233$.

8. Find the solar interval which corresponds to $0^s.85$ of sideral time.

Ans. $0^s.84768$.

9. Find the sun's mean longitude at Greenwich for the mean noon of April 4, 1839, the sideral time at this noon, and the solar time of the transit of the vernal equinox from the preceding noon; the meridian of Greenwich is $9^m\ 21^s.5$ west of that of Paris.

Ans. The sun's mean longitude $= 12°\ 7'\ 3''.02$.

The sideral time of mean noon $= 48^m\ 31^s.27$.

Time of tran. of ver. equi. $=$ April 3d, $23^h\ 11^m\ 39^s.68$.

101. *Problem. To find the time by observation.*

Solution. First Method. By equal altitudes.

I. If the star does not change its declination. Observe the times when the star is at equal altitudes before and after passing the meridian; the arithmetical mean between these two times is the time of the star's passing the meridian, which compared with the known time of this passage, gives the error of the clock at this time, and the correction of this error gives the time of each observation.

Equal altitudes of sun.

II. When the declination of the star is changing, the time of the star's arriving at the observed altitude A is affected; thus if

$L =$ the latitude,

$D =$ the declination at the meridian,

$\delta D =$ the increase of declination from the meridian,

$h =$ the hour angle, supposing no change in the declination,

$\delta h =$ the increase of the hour angle in time,

we have, by (380),

$$\sin. A = \sin. L \sin. D + \cos. L \cos. D \cos. h \qquad (514)$$

$$= \sin. L \sin.(D + \delta D) + \cos. L \cos.(D + \delta D) \cos (h + \delta h)$$

$$= \sin. L \sin. D + \delta D \sin. 1'' \sin. L \cos. D + \cos. L \cos. D \cos. h$$

$$- \delta D \sin. 1'' \cos. L \sin. D \cos. h - 15 \delta h \sin. 1'' \cos. L \cos. D \sin. h,$$

whence

$$0 = \delta D \sin. L \cos. D - \delta D \cos L \sin. D \cos. h$$
$$- 15 \delta h \cos. L \cos D \sin. h$$

$$\varrho h = \tfrac{1}{15} \delta D \operatorname{tang.} L \operatorname{cosec.} h - \tfrac{1}{15} \delta D \operatorname{tang.} D \operatorname{cotan.} h$$

$$= \frac{\delta D}{15 \operatorname{cotan.} L \sin. h} - \frac{\delta D}{15 \operatorname{cotan.} D \operatorname{tang.} h}, \qquad (515)$$

and since the two observations are at nearly the same distance from the meridian, the value of δh is the same for both of them; so that their mean is augmented by δh, and δh is consequently to be subtracted from the mean of the observed times, in order to obtain the true time of the star's passing the meridian.

In calculating the value of δh, its two terms may be calculated separately. Now if $\delta' D$ is the daily variation of the star's declination, we have

$$\delta D = \frac{h\,\delta' D}{24^h} = \frac{2\,h\,\delta' D}{2 \times 24^h}, \qquad (516)$$

and in using proportional logarithms, the proportional logarithm of the hours and minutes of $2\,h$, which is the elapsed time, may be taken as if they were minutes and seconds, provided the same is done with the 24^h in the denominator. Finally, the value of δh is reduced from minutes and seconds to seconds and thirds by multiplying by 60, so that if M is taken for the denominator of either of the parts of (515), this part P is calculated by the formula

$$\text{Prop. log. } P = -\text{Prop. log. } \frac{2 \times 24^m \times 15}{60} + \text{log.} M + \text{Prop. log.} 2\,h$$
$$+ \text{Prop. log. } \delta' D, \qquad (517)$$

which agrees with [B. p. 219.], for

$$-\text{Prop. log. } \frac{2 \times 24^m \times 15}{60} = -\text{Prop. log. } 12^m = -1.1761$$
$$= 8.8239. \qquad (518)$$

III. If the altitude at the two observations had differed slightly, the mean time would require to be corrected; for this purpose, let

δA = the excess of the second altitude above the first,

δh = the increase of the hour angle, and we easily deduce from (514)

$$\cos. A\, \delta A = -15 \cos. L \cos. D \sin. h\, \delta h, \qquad (519)$$

so that

$$\delta h = -\frac{\cos. A\, \delta A}{15 \cos. L \cos. D \sin. h}. \qquad (520)$$

The time of the second observation being thus increased by δh, that of the mean is increased by $\frac{1}{2}\delta h$, which is, therefore, the correction to be subtracted from this mean.

The corrections (515) and (520) must be both of them applied when the star is changing its declination, and at the same time the observed altitudes are slightly different.

Second Method. By a single altitude. [B. p. 208–218.]

When a single altitude is observed, there are known in the triangle *PZB* (fig. 35.), the three sides, to find the hour angle *ZPB*, which is thus found by (277),

$$s = \tfrac{1}{2}(z + 90° - L + p) \qquad (521)$$

$$\cos. \tfrac{1}{2} h = \sqrt{\left(\frac{\sin. s \sin. (s - z)}{\sin. (90° - L) \sin. p}\right)}, \qquad (522)$$

which corresponds to [B. p. 210.]

The hour angle may also be found by (282), thus if we put

$$s' = \tfrac{1}{2}(A + L + p), \qquad (523)$$

we have

$$s = \tfrac{1}{2}(180° - A - L + p) = 90° - s' + p = 90° - A - L + s'$$

$$s - p = 90° - s', \quad s - (90° - L) = s' - A,$$

whence

$$\sin. \tfrac{1}{2} h = \sqrt{\left(\frac{\cos. s' \sin. (s' - A)}{\cos. L \sin. p}\right)}, \qquad (524)$$

which corresponds to [B. p. 209.]

Third Method. By the distance from a fixed terrestrial object.

If the position of the terrestrial object has been before determined, its hour angle and polar distance may be considered as known.

Hence, if T (fig. 40.) is the position of the terrestrial object projected upon the celestial sphere, P the pole, and S the star. Let the distance TS be observed, and let

$$PT = P, \quad PS = p, \quad TS = d,$$

$$TPZ = H, \quad TPS = h', \quad SPT = h,$$

$$s = \tfrac{1}{2}(P + p + d), \tag{525}$$

we have

$$\sin.\tfrac{1}{2}h' = \sqrt{\left(\frac{\sin.(s-p)\sin.(s-p)}{\sin.P\sin.p}\right)}, \tag{526}$$

or

$$\cos.\tfrac{1}{2}h' = \sqrt{\left(\frac{\sin.s\sin.(s-d)}{\sin.P\sin.p}\right)}, \tag{527}$$

$$h = H + h'.$$

If the polar distance and hour angle of the terrestrial object is not known, but only its altitude and azimuth, the polar distance and hour angle can be easily found by solving the triangle PZT.

Fourth Method. By a meridian transit. [B. p. 221.]

If the passage of a star is observed over the different wires of a transit instrument, the mean of the observed times is the time of the meridian transit, which should agree with the known time of this transit. This method surpasses all others in accuracy and brevity.

Fifth Method. By a disappearance behind a terrestrial object.

If the instant of a star's disappearance behind a vertical tower has been observed repeatedly with great care, the observed time of this disappearance may afterwards be used for correcting the chronometer. For this purpose, the position of the observer must always be precisely the same. Any change in the right ascension of the star does not affect the star's hour angle, that is, the elapsed time from the meridian transit; this change, consequently, affects the observed time exactly as if the observation were that of a meridian transit.

A small change in the declination of the star affects the hour angle, and therefore the time of observation. Thus, if P (fig. 44.) is the pole, Z the zenith, ZSS' the vertical plane of the terrestrial object; then if the polar distance PS is diminished by

$$RS = \delta D,$$

the hour angle ZPS is diminished by the angle

$$SS'P = \delta h.$$

But the $S'R$ is nearly perpendicular to SP, and the sides of $SS'R$ are so small, that their curvature may be neglected, whence

$$RS' = \delta D \text{ tang. } S = 15 \cos. D \,.\, \delta h,$$

so that

$$\delta h = \tfrac{1}{15} \, \delta D \text{ tang. } S \text{ sec. } D. \qquad (528)$$

To find the time.

102. Examples.

1. On July 25, 1823, in latitude 54° 20′ N., the sun was at equal altitudes, the observed interval was $6^h\ 1^m\ 36^s$; find the correction for the mean of the observed t mes. The sun's declination is 19° 48′, and his daily increase of declination 12′ 44″.

Solution.

		8.8239			8.8239
54° 20′	cotan.	9.8559,	19° 48′	cotan.	0.4437
$6^h\ 1^m\ 36^s$	sin.	9.8510		tang.	0.0030
$6^m\ 1^s$	P. L.	1.4759			1.4759
12′ 44″	P. L.	1.1503			1.1503
12^s 55		1.1570	$-2^s.28$		1.8968
$-2^s.28$					
$10^s.3$ = the required correction.					

2. On September 1, 1824, in latitude 46° 50′ N., the interval between the observations, when the sun was at equal altitudes, was $7^h\ 46^m\ 35^s$; the sun's declination was 8° 14′ N., and his daily increase of declination — 21′ 49″; what is the correction for the mean of the observations?

Ans. $16^s.5$.

3. On March 5, 1825, in latitude 38° 34′ N., the interval between the observations, when the sun was at equal altitudes, was $8^h\ 29^m\ 28^s$; the sun's declination was 6° 2′ S., and his daily increase of declination was 23′ 9″; what is the correction for the mean of the observations?

Ans. $15^s.4$.

To find the time.

4. On March 27, 1794, in latitude 51° 32′ N., the interval between the observations, when the sun was at equal altitudes, was $7^h\ 29^m\ 55^s$; the sun's declination was 2° 47′ N., and his daily increase of declination 23′ 26″; what is the correction for the mean of the observations?

Ans. $-21^s.7$.

5. In latitude 20° 26′ N., the altitude of Aldebaran, before arriving at the meridian, was found to be 45° 20′, and, after passing the meridian, to be 45° 10′; the interval between the observations was $7^h\ 16^m\ 35^s$, and the declination of Aldebaran was 16° 10′ N.; what is the correction for the mean of the observations?

Ans. 19^s.

6. In latitude 36° 39′ S., the sun's correct central altitude was found to be 10° 40′, when his declination was 9° 27′ N.; what was the hour angle?

Ans. $7^h\ 23^m\ 51^s$.

7. In latitude 13° 17′ N., the sun's correct central altitude was found to be 36° 37′, when his declination was 22° 10′ S.; what was the hour angle?

Ans. $9^h\ 17^m\ 8^s$.

8. In latitude 50° 56′ 17″ N., the zenith distance of a terrestrial object was found to be 90° 24′ 28″, and its azimuth 35° 47′ 4″ from the south; what were its polar distance and hour angle?

Ans. Its polar distance = 121° 6′ 43″

Its hour angle = $2^h\ 52^m\ 16^s$.

To find the time.

9. From the preceding terrestrial object three distances of the sun were found to be 78° 9′ 26″, 77° 39′ 26″, and 77° 29′ 26″, when his declination was 14° 7′ 13″ S.; what were the sun's hour angles, if he was on the opposite side of the meridian from the terrestrial object?

Ans. $2^h\ 45^m\ 49^s$, $2^h\ 43^m\ 27^s$, and $2^h\ 42^m\ 39^s$.

CHAPTER VIII.

LONGITUDE.

103. *Problem. To find the longitude of a place.*

First Method. By terrestrial measurement.

If the longitude of a place is known, that of another place, which is near it, can be found by measuring the bearing and distance; whence the difference of longitude may be calculated by the rules already given in Navigation.

Second Method. By signals.

The stars, by their diurnal motion, pass round the earth once in 24 sideral hours; hence they arrive at each meridian by a difference of sideral time equal to the difference of longitude. In the same way, the sun passes round the earth once in 24 solar hours; so that it arrives at each meridian by a difference of solar time equal to the difference of longitude. The difference of longitude of two places is, consequently, equal to their difference of time. Now if any signal, as the bursting of a rocket, is observed at two places; the instant of this event, as noticed by the clocks of the two places, gives their difference of time.

Third Method. By a chronometer.

The difference of time of two places can, obviously, be determined by carrying a chronometer, whose rate is well ascertained, from one place to the other; and if the chronometer did not change its rate during the passage, this method would be perfectly accurate.

Fourth Method. By an eclipse of one of Jupiter's satellites. [B. p. 252.]

The signal of the second method cannot be used, when the places are more than 20 or 30 miles apart; and, when the distance is very great, a celestial signal must be used, such as the immersion or emersion of one of Jupiter's satellites. For this purpose, the instant when any such event would happen to an observer at Greenwich is inserted in the Nautical Almanac; and the observer at any other place has only to compare the time of his observation with that of the Almanac to obtain his longitude from Greenwich.

Fifth Method. By an eclipse of the moon. [B. p. 253.]

The beginning or ending of an eclipse of the moon may also be substituted for the signal of the second method to determine the difference of time.

Sixth Method. By a meridian transit of the moon. [B. p. 431.]

The motion of the moon is so rapid, that the instant of its arrival at a given place in the heavens may be used for the signal. Of the elements of its position its right ascension is changing most rapidly, and this element is easily determined at the instant of its passage over the meridian by the difference of time between its passage and that of a known star. The instant of Greenwich time, when the moon's right ascension is equal to the observed right ascension, might be determined from the right ascension, which is given in the Nautical Almanac for every hour. But this computation involves the observation of the solar time, whereas the observed interval gives at once the sideral time of the observation.

The calculation is then more simple, by means of the table of Moon-Culminating stars given in the Nautical Almanac, in which the right ascensions of the suitable stars and of the moon's bright limb are given at the instant of their upper transits over the meridian of Greenwich, and also the right ascension of the moon's bright limb at the instant of its lower transit. Hence the difference between the right ascensions of the moon's limb, at two successive transits, is the change of its right ascension in passing from the meridian of Greenwich to that which is 12^h from Greenwich; so that if the motion in right ascension were perfectly uniform, the right ascension, which corresponded to a given meridian, or the meridian, which corresponded to a given right ascension, might be found by the following simple proportion,

$$12^h : \text{long. of place} = \text{diff. of right ascensions for } 12^h : \text{diff. of right ascensions for long. of place,} \quad (529)$$

in which the longitude of the place may be counted from the meridian 12^h from that of Greenwich, provided the change of right ascension for an upper transit is computed from the preceding right ascension, which is that of a lower transit at Greenwich, that is, if the place is in east longitude.

Let then $T =$ long., if west,

or $= 12^h$ — long. (if the long. is east);

and let $A =$ diff. of right ascension for the Greenwich transits, which immediately precede and follow the required or observed transit, By a moon's transit.

and let $\delta A =$ change of right ascension from the preceding Greenwich transit to the observed transit,

and we have, by (529),

$$12^h : T = A : \delta A, \qquad (530)$$

whence

$$\delta A \frac{A T}{12^h}, \text{ and } T = \frac{12^h \delta A}{A}, \qquad (531)$$

and if T is reduced to seconds, we have

$$\delta A = \frac{A T}{43200} \qquad (532)$$

$$\begin{aligned} \log. \delta A &= \log. A + \log. T + (\text{ar. co.}) \log. 43200 \\ &= \log. A + \log. T + 5.36452 \qquad (533) \end{aligned}$$

and

$$T = \frac{43200 \, \delta A}{A} \qquad (534)$$

$$\log. T = 4.63548 + \text{ar. co. } \log. A + \log. \delta A, \qquad (535)$$

and formulas (533) and (535) agree with the parts of the rules in the Naviga or, which depend upon A, and are independent of the want of uniformity in the moon's motion.

The corrections which arise from the change of the moon's motion may be calculated, on the supposition that this motion is uniformly increasing or decreasing, so that the mean motion for any interval is equal to the motion which it has at the middle instant of that interval. If we put, then,

$$B = \text{the increase of motion in } 12^h, \qquad (536)$$

A is not the mean daily motion for the interval of longitude T and the instant $\frac{1}{2} T$ after the meridian transit at Greenwich, but for the interval 12^h and the instant 6^h after this transit. The mean daily motion for the instant $\frac{1}{2} T$ is, therefore, By moon's transit. Table XLV.

$$A - \frac{(6^h - \frac{1}{2}\,T)\,B}{12^h}, \tag{537}$$

so that the correction for A is

$$-\frac{(6^h - \frac{1}{2}\,T)\,B}{12^h} = -\frac{(21600^s - \frac{1}{2}\,T)\,B}{43200}, \tag{538}$$

and the correction of δA in (532) is

$$\delta B = -\frac{T(21600^s - \frac{1}{2}T)}{(43200^s)^2}\,B = -\frac{T(43200 - T)}{2\,(43200)^2}\,B, \tag{539}$$

and the value of δB is easily calculated and put into tables, like Table XLV of the Navigator.

In correcting the value of T (534), the correction of δA is to be computed from Table XLV by means of the approximate value of T, and the correction of T is then found by the formula to be

$$\delta T = \frac{43200\,\delta B}{A}. \tag{540}$$

It only remains, to show how to find the value of B from the Nautical Almanac. Now if A' denotes the motion in right ascension for the 12^h interval of longitude, which precedes that to which A corresponds; and if A'' denotes the motion in right ascension for the 12^h interval of longitude which follows that of A; we have

$$2\,B = A'' - A'$$

$$B = \tfrac{1}{2}\,(A'' - A'), \tag{541}$$

and the calculation agrees entirely with that given in the Navigator.

When the longitude is small, or nearly 12^h, the correction for the variation of motion may be neglected, provided, instead of A, the motion is used which corresponds to the time of the nearest Greenwich transit. Now, in the Nautical Almanac, this motion is given for an hour's interval, of which the middle instant is that of the transit, so that if $H =$ this hourly motion, the motion for the time T may be found by the formula

$$1^h : T = H : \delta A,$$

whence

$$T = \frac{\delta A \times 1^h}{H} = \frac{2600^s \times \delta A}{H} \qquad (542)$$

$$\log. T = 3.55630 + \log. \delta A + (\text{ar. co.}) \log. H, \qquad (543)$$

which agrees with [B. p. 432.]

The formula (543) may be rendered more correct, if the value of H is taken for the instant $\frac{1}{2}$ T of longitude; and the value can be computed precisely in the same way in which the right ascension was computed for the time T, by noticing the want of uniformity in its increase; and the formula thus corrected is accurate for small differences of longitude.

Seventh Method. By a lunar distance.

The distance of the moon from the sun or a star may be used as the signal; but the true places of these bodies differ from their apparent places, as will be shown in succeeding chapters, so that the observed distance requires to be corrected; and the correction cannot be found without knowing the altitudes of the bodies. It is sufficient, for the present purpose, to know that the difference between the true and apparent places is only a difference of altitude, and not one of azimuth, and that the apparent place of the sun or a star is higher than its true place, while that of the moon is lower. The true dis-

By a lunar distance.

tance may, then, be calculated from the observed distance by one of the following methods.

I. Let Z (fig. 45.) be the zenith, S the apparent place of the sun or star, and S' the true place, M the apparent place of the moon, M' the true place; let

$a =$ the star's apparent alt. $= 90° - ZS$

$a' =$ its true alt. $= 90° - ZS'$

$b =$ the moon's app. alt. $= 90° - ZM$

$b' =$ its true alt. $= 90° - ZM'$

$E =$ the app. dist. $= SM$

$E' =$ the true dist. $= S'M'$

$Z =$ the angle Z

$\delta a = SS' = a - a'$

$\delta b = MM' = b' - b$

$\delta E = E' - E.$

Then the triangles ZSM and $ZS'M'$ give, by (273),

$$2(\cos.\tfrac{1}{2}Z)^2 = \frac{\cos. E + \cos.(a+b)}{\cos. a \cos. b} = \frac{\cos. E' + \cos.(a'+b')}{\cos. a' \cos. b'} \quad (544)$$

Let
$$\cos. m = \frac{\cos. a' \cos. b'}{2 \cos. a \cos. b}, \quad (545)$$

and we have, by (544),

$$\cos. E' + \cos.(a'+b') = 2\cos. m \cos. E + 2 \cos. m \cos.(a+b)$$
$$= \cos.(E+m) + \cos.(E-m) + \cos.(a+b+m) + \cos.(a+b-m)$$
$$\cos. E' = -\cos.(a'+b') + \cos.(E+m) + \cos.(E-m)$$
$$+ \cos.(a+b+m) + \cos.(a+b-m), \quad (546)$$

By a lunar distance.

whence E' can be found by a table of natural sines and cosines, when m has been found from (545).

II. In the same way by (279), we find

$$2(\sin.\tfrac{1}{2}Z)^2 = \frac{\cos.(a-b)-\cos.E}{\cos.a\cos.b} = \frac{\cos.(a'-b')-\cos.E'}{\cos.a'\cos.b'} \quad (547)$$

$$\cos.(a'-b')-\cos.E' = 2\cos.m\cos.(a-b)-2\cos.m\cos.E$$

$$= \cos.(a-b+m)+\cos.a-b-m)-\cos.(E+m)-\cos.(E-m)$$

$$\cos.E' = \cos.(a'-b')-\cos.(a-b+m)-\cos.(a-b-m)$$
$$+\cos.(E+m)+\cos.(E-m). \quad (548)$$

III. The correction may be separated into two parts, one of which depends only upon the sun or star, and the other upon the moon; and let

$\delta' E =$ the part of δE which depends upon the sun or star,

$\delta'' E =$ the part which depends upon the moon.

Now if the correction were only to be made for the moon, SM would be decreased to SM', whence

$$SM' = E + \delta'' E,$$

and if we put

$$S = ZSM, \quad M = ZMS,$$
$$s = \tfrac{1}{2}(a+b+E), \quad (549)$$

the triangles SMM' and SZM give

$$(\sin.\tfrac{1}{2}M)^2 = \frac{\sin.s\sin.(s-a)}{\sin.E\cos.b}$$

$$= \frac{\sin.[E+\tfrac{1}{2}(\delta''E-\delta b)]\sin.\tfrac{1}{2}(\delta''E+\delta b)}{\sin.\delta b\sin.E}$$

By a lunar distance.

$$= \frac{\delta'' E + \delta b}{2 \delta b} [1 + \tfrac{1}{2} \text{cotan. } E \sin. 1''. (\delta'' E - \delta b)] \quad (550)$$

$$60' + \delta'' E = (59' \, 42'' - \delta b) + \frac{\sin. s \sin. (s - a)}{\sin. E} \cdot \frac{2 \delta b}{\cos. b}$$

$$+ [18'' - \tfrac{1}{2} \text{cotan. } E \sin. 1'' [(\delta'' E)^2 - (\delta b)^2]. \quad (551)$$

The triangles $SS'M$ and SZM give, by (277) and (281),

$$(\cos. \tfrac{1}{2} S)^2 = \frac{\cos. (s - E) \sin. (s - a)}{\sin. E \cos. a}$$

$$= \frac{\sin. [E + \tfrac{1}{2} (\delta'' E - \delta a)] \sin. \tfrac{1}{2} (\delta'' E + \delta a)}{\sin. \delta a \sin. E}$$

$$= \frac{\delta a + \delta' E}{2 \delta a} \quad (552)$$

$$60' + \delta' E = (60' - \delta a) + \frac{\cos. (S - E) \sin. (s - a)}{\sin. E} \cdot \frac{2 \delta a}{\cos. a}. \quad (553)$$

If now $M'K$ and $S'L$ are drawn perpendicular to MS, and $S'L'$ to $M'S$, we have nearly

$$S'M' = E + \delta E = SM' + SL' = E + \delta'' E + SL'$$

$$\delta E = \delta'' E + SL' = \delta'' E + \delta' E + (SL' - \delta' E) \quad (554)$$

$$\delta' E = SL = \delta a \cos. S \quad (555)$$

$$SL' = \delta a \cos. (S'SL') = \delta a \cos. (S - MSM')$$

$$= \delta a \cos. S + \delta a \sin. S \sin. MSM'$$

$$= \delta' E + S'L \sin. MSM' \quad (556)$$

$$SL' - \delta' E = S'L \sin. MSM'. \quad (557)$$

But from $M'SK$,

$$\sin. MSM' = \frac{\sin. M'K}{\sin. E} = \frac{M'K \sin. 1''}{\sin. E}, \quad (558)$$

whence

$$SL' - \delta' E = \frac{S'L \times M'K . \sin. 1''}{\sin. E}, \qquad (559)$$

and

$$\delta E = \delta' E + \delta'' E + \frac{S'L \times M'K . \sin. 1''}{\sin. E} \qquad (560)$$

$$2^\circ + \delta E = (60' + \delta' E) + (60' + \delta'' E) + \frac{S'L \times M'K \sin 1''}{\sin. E} \qquad (561)$$

in which 1° is added to $\delta' E$ and $\delta'' E$ in order to render them positive. Now, of $60' + \delta' E$ (553), the part $60' - \delta a$ is given in table XVII or table XVIII; and the remaining term is computed by proportional logarithms, and is the first correction of the First Method of the Navigator. [B. p. 231.] The proportional logarithm of the factor $2 \delta a \sec. a$, is the logarithm of the table from which $60' - \delta a$ is taken.

In the same way, the two first terms of $60' + \delta'' E$ are taken from table XIX and (551). The remainder of (551) combined with the third term of (561), is computed and inserted in table XX of the Navigator.

In calculating table XX, the value of $\delta'' E$ is used, which is obtained from the two first terms of (551); and $S'L$ and $M'K$ are found from $S'SL$ and MKM' in which the sides are so small that their curvature may be neglected, and we have, nearly

$$S'L = \sqrt{(\delta a^2 - \delta' E^2)} \qquad (562)$$

$$M'K = \sqrt{(\delta b^2 - \delta'' E^2)}. \qquad (563)$$

IV. The calculation of the values of δa and δb will be fully explained in subsequent chapters; but we need only remark, in this place, that the value of δa for a star is given in table XII, for the sun it is the number of table XII diminished by that of table XIV, and for a planet, it is that of table XII, diminished by that of table X, A. The value of δb is obtained by the formula

$$\delta b = P \cos. b - \delta' b, \tag{564}$$

in which $\delta' b$ is the number of table XII, and P is the number taken from the Nautical Almanac, and which is called the horizontal parallax. In computing table XX, the value of P is taken at its mean of 57′ 30″.

In the formulas for the corrections, the zenith distances may be introduced instead of the altitudes, and if we put

$$90° - a = Z, \quad 90° - b = Z,$$
$$s_1 = z + Z + E, \tag{565}$$

we have, by neglecting the term depending upon the correction of table XX, as well as the other small quantities,

$$\cos.^2 \tfrac{1}{2} M = \frac{\sin.\, s_1 \sin.\,(s_1 - z)}{\sin.\, E \sin.\, Z}$$

$$= \frac{\sin.\,[E + \frac{1}{2}(\delta'' E + \delta b)] \sin.\, \frac{1}{2}(\delta b - \delta'' E)}{\sin.\, E \sin.\, \delta b}$$

$$= \frac{\delta b - \delta'' E}{2\,\delta b} \tag{566}$$

$$\delta'' E = \delta b - \frac{2 \sin.\, s_1 \sin.\,(s_1 - z)}{\sin.\, E \sin.\, Z}\, \delta b \tag{567}$$

$$\cos.^2 \tfrac{1}{2} S = \frac{\sin.\, s_1 \sin.\,(s_1 - Z)}{\sin.\, E \sin.\, z} = \frac{\delta' E + \delta a}{2\,\delta a}$$

$$\delta' E = -\delta a + \frac{2 \sin.\, s_1 \sin.\,(s_1 - Z)}{\sin.\, E \sin.\, z}\, \delta a. \tag{568}$$

Then the second term of the value of $\delta' E$ is the first correction of the Third Method of the Navigator [B. p. 242.]

and the second term of the value of $\delta'' E$ is the second correction of this method; and the computation from (564, 567, 568) agrees entirely with this method. The third correction is taken from table XX, as in the first method.

V. Draw ZN perpendicular to MS, so as to make SN acute. In the right triangles ZSN and ZSM let

$$B = 90° - SN, \quad B' = 90° + MN, \quad A = \tfrac{1}{2}(B' + B), \tag{569}$$

and we have

$$E = MN + SN = B' - B, \tag{570}$$

and, by Bowditch's Rules for oblique triangles,

$$\text{cos. } ZS : \text{cos. } ZM = \text{cos. } NS : \text{cos. } MN,$$

or

$$\text{sin } a : \text{sin. } b = \text{sin. } B : \text{sin. } B'; \tag{571}$$

and, by the theory of proportions,

$$\frac{\text{sin. } a + \text{sin. } b}{\text{sin. } b - \text{sin. } a} = \frac{\text{sin. } B + \text{sin. } B'}{\text{sin. } B' - \text{sin. } B},$$

that is,

$$\frac{\text{tang. } \frac{1}{2}(a+b)}{\text{tang. } \frac{1}{2}(b-a)} = \frac{\text{tang. } A}{\text{tang. } \frac{1}{2} E} \tag{572}$$

$$\text{tang. } A = \text{tang. } \tfrac{1}{2}(a+b) \text{ cotan. } \tfrac{1}{2}(b-a) \text{ tang. } \tfrac{1}{2} E \tag{573}$$

$$B' = A + \tfrac{1}{2} E, \quad B = A - \tfrac{1}{2} E, \tag{574}$$

and the right triangles ZSN, MZN, SLS', MKM', give

$$\text{cos. } S = \frac{\delta' E}{\delta a} = \text{cotan. } ZS \text{ tang. } SN = \text{tang. } a \text{ cotan. } B$$

$$-\text{cos. } M = \frac{\delta'' E}{\delta b} = -\text{cot. } ZM \text{ tang. } MN = \text{tang. } b \text{ cotan. } B'$$

$$\delta' E = \delta a \text{ tang. } a \text{ cotan. } B \tag{575}$$

$$\delta'' E = \delta b \text{ tang. } b \text{ cotan. } B', \tag{576}$$

and the formulas (573 – 576) correspond to the Fourth Method of the Navigator. [B. p. 243.]

It may be observed, that since cotan. $\frac{1}{2}(b - a)$ is the only term of (573) which can change its sign, A is acute when b is greater than a, and obtuse when b is less than a.

VI. The most important of corrections of the distance arise from that term of $\delta\, b$ (564), which depends upon the parallax. If we consider this, therefore, as the only correction of the moon's altitude, we may calculate the corrections of the distance arising from it by putting

$$\delta\, b = MM' = P \cos. b. \qquad (577)$$

The triangles ZSM and $M'MK$, give then

$$\cos. M = -\frac{\delta'' E}{P \cos. b} = \frac{\sin. a - \cos. E \sin. b}{\sin. E \cos. b} \qquad (578)$$

$$\delta'' E = - P \sin. a \operatorname{cosec}. E + P \operatorname{cotan}. E \sin. b, \qquad (579)$$

and if we put

$$\delta_1 E' = P \sin. a \operatorname{cosec}. E \qquad (580)$$

$$\delta_2 E = \pm P \operatorname{cotan}. E \sin. b, \qquad (581)$$

in which the signs are taken so that $\delta_2 E$ is always positive, we have

$$\delta'' E = - \delta_1 E \pm \delta_2 E \qquad (582)$$

$$10° + \delta'' E = (5° - \delta_1 E) + (5° \pm \delta_2 E). \qquad (583)$$

Now table XLVII is a common table of proportional logarithms, like table XXII; but the angle which is placed at the top of the table is

$$5° - \text{the angle of Table XXII}, \qquad (584)$$

and the angle at the bottom of the table is

$$5^\circ + \text{the angle of Table XXII}; \qquad (585)$$

so that the terms of (583) may be directly obtained from these tables; and this method of computing the corrections, which depend upon the moon's parallax, agrees with the second method of the Navigator. [B. p. 239.]

The remaining corrections may be computed from the formulas (567 and 568), and the corrections of table XX may be neglected, provided the value of E is corrected for the parallax. These combined corrections may be inserted in a table like table XLVIII, which serves for the star, and, by means of the part P, for the sun; or like tables XLIX and L, which serve for the planets. In calculating those tables, the moon's horizontal parallax is taken at its mean value of 57′ 30″; and the planet's or sun's parallax in altitude is obtained from the formula

$$\delta' a = - P \cos. a,$$

in which P is the horizontal parallax. The value of P, used in the construction of the part P of table XLVIII, is 8″.6; that used for table XLIX is 35″; and since these corrections are proportional to the parallax, they are easily reduced to any other parallax. This reduction is actually made in table L.

VII. The value of $\delta'' E$ (578), might be found by the formula

$$\delta'' E = \frac{2 \sin. a - \sin. (b + E) - \sin. (b - E)}{2 \sin. E} P, \qquad (586)$$

which is easily calculated by means of the table of natural sines and cosines.

VIII. The true distance may be obtained from observation by either of the preceding methods, and the time of the observation must be compared with the time when the distance is the same to an observer at Greenwich. Now this latter time can be obtained from the Nautical Almanac by precisely the same process of interpolation, which has been applied to the changes of right ascension. The distances are given in the Nautical Almanac for every three hours, and the proportional logarithm of the difference of these distances. If, then, the distance increases uniformly at the rate of increase, F for every three hours; the interval T, at which it has increased by the quantity F', is found by the proportion

$$F : F' = 3^h : T \tag{587}$$

$$\text{Prop. log. } T = \text{Prop. log. } F' - \text{Prop. log. } F + \text{Prop. log. } 3^h. \tag{588}$$

But

$$\text{Prop. log. } 3^h = 0; \tag{589}$$

and if we put

$$\text{Prop. log. } F = Q, \tag{590}$$

(588) becomes

$$\text{Prop. log. } T = \text{Prop. log. } F' - Q. \tag{591}$$

If the distance increased uniformly, the value of Q would be invariable; but Q is variable, and must be regarded as belonging to the middle instant of the interval to which it belongs; and it increases while the distance decreases, and the reverse. Let then

$\delta Q =$ the decrease of Q in three hours,

$\delta T =$ the correction of T, arising from the change of Q,

and the value of Q for the interval T is

$$Q + \frac{90^m - \frac{1}{2} T}{180^m} \delta Q = Q + \delta' Q, \tag{592}$$

Lunar distance from Nautical Almanac.

so that by (591) and (340)

$$\text{Prop. log. } (T + \delta T) = \text{Prop. log. } T - \delta' Q \qquad (593)$$

$$\log. (T + \delta T) = \log. T + \delta' Q \qquad (594)$$

$$\log. (T + \delta T) - \log. T = \log. \left((1 + \frac{\delta T}{T}\right) = \delta' Q. \qquad (595)$$

But if in (196) and (204) we substitute

$$\frac{\delta T}{T} = 2 m, \qquad (596)$$

we have, by logarithms and (595),

$$\log. e = \frac{T}{\delta T} \log. \left(1 + \frac{\delta T}{T}\right) = \frac{T \delta' Q}{\delta T}, \qquad (597)$$

so that by (592) and (208)

$$\delta T = \frac{T \delta' Q}{\log. e} = \frac{(180^m - T)\, T \delta Q}{2 \times 180^m. \times 0.434} \qquad (598)$$

$$= \frac{(180^m - T)\, T \delta Q}{156^m}; \qquad (599)$$

and the table [B. p. 245.] for correcting by second differences may be calculated by this formula; and, in order to obtain the value of δT expressed in seconds, the factor T should be expressed in seconds, while $(180^m - T)$ is expressed in minutes; and it must not be forgotten, that the proportional logarithms are decimals.

IX. When the distance is observed for a star, whose distance is not given in the Nautical Almanac, the Greenwich time of the observation can be found approximately by adding the assumed longitude, if west to the observed time, or subtracting it if east; or the time can be taken from the chronometer if it is regulated to Greenwich time.

Lunar distance not given in the Nautical Almanac.

Find, in the Nautical Almanac, the right ascension and declination of the star and the declination of the moon, for this time. Then, if T and S (fig. 43.) are supposed to be the moon and star, and P the pole of the equator, D and D' their declinations, disregarding their names, so that their polar distances are $90° \pm D$ and $90° \pm D'$, and if R' is their difference of right ascensions, we have, when their declinations are of the same name, by putting

$$S = \tfrac{1}{2}(D + D' + E) \tag{600}$$

$$\cos. \tfrac{1}{2} R' = \cos. \tfrac{1}{2} SPT = \sqrt{\left(\frac{\cos. S \cos. (S-E)}{\cos. D \cos. D'}\right)}. \tag{601}$$

But if the declinations are of the same name,

$$\sin. \tfrac{1}{2} R' = \sin. \tfrac{1}{2} SPT = \sqrt{\left(\frac{\sin. S \sin. (S-E)}{\cos. D \cos. D'}\right)}, \tag{602}$$

and the right ascension of the moon being thus found, the Greenwich time, when it has this right ascension, is easily found from the moon's hourly ephemeris in the Nautical Almanac, and this method is the same with that in [B. p. 428.]

X. The latitudes and longitudes may be used instead of the right ascensions and declinations, and the calculation will be as in [B. p. 427.] The variation of daily motion is, in this case, to be had regard to, precisely as explained in (536–541).

XI. The distances of the Nautical Almanac can be calculated from the right ascensions and declinations of the sun, moon, and stars, or their latitudes and longitudes, by resolving the triangles TPS (fig. 43.) by either of the methods which have been given, when two sides and the included angle are known, as in [B. p. 434.]

Lunar distances.

In calculating the distance of the sun and moon, the latitude of the sun may be usually neglected; so that if *SR* (fig. 46.) is an arc of the ecliptic, *S* the sun's place, *M* the moon's, and *MR* perpendicular to *SR*,

$$MR = L = \text{the moon's latitude,}$$

$$SR = L_1 = \text{the diff. of long. of ☉ and ☽,}$$

and $$\cos. E = \cos. SM = \cos. L \cos. L_1, \qquad (603)$$

as in [B. p. 433.]

It would, however, be rather more accurate to take

$$L = \text{the diff. of lat. of ☉ and ☽.}$$

XII. The determination of the longitude by solar eclipses and occultations will be reserved for another chapter.

104. Examples.

1. Calculate the correction of table XLV, when

$$T = 1^h\ 50^m, \text{ and } B = 9^m = 540^s.$$

Solution.	$1^h\ 50^m$ P. L. $1^m\ 50^s$	ar. co.	8.0080
$12^h - 1^h\ 50^m = 10^h\ 10^m$	P. L. $10^m\ 10^s$	ar. co.	8.7519
	2 P. L. 12^m		2.3522
$\frac{1}{2} B = 270^s$			2.4314
corr. $= 34^s.9$			1.5435

2. Calculate the correction of table XLV, when

$$T = 3^h\ 10^m, \text{ and } B = 11^m.$$

Ans. $64^s.1$.

To find the moon's right ascension.

3. Find the right ascension of the moon's bright limb, Sept. 25, 1839, at the time of the transit over the meridian of New York. The right ascensions of the moon for the two preceding and the two following transits at Greenwich are

Sept. 25.	Moon II. L. T.	2^h	0^m	$36^s.69$
	Moon II. U. T.	2	30	38 .08
Sept. 26.	Moon II. L. T.	3	1	33 .18
	Moon II. U. T.	3	33	19 .89

Ans. $2^h\ 43^m\ 14^s.4$.

4. At a place in west longitude, Oct. 25, 1839, the moon's bright limb passed the meridian $10^m\ 6^s.83$ sideral time, before the star C. Tauri; find the longitude of the place of observation.

The right ascension of the star C. Tauri was $5^h\ 43^m\ 16^s.84$, and those of the moon

Oct. 25.	Moon II. L. T.	4^h	43^m	$53^s.55$
	Moon II. U. T.	5	16	28 .40
Oct. 26.	Moon II. L. T.	5	52	51 .91
	Moon II. U. T.	6	26	40 .00

Ans. 70° 25′ 30″ W.

5. Find the moon's parallax in altitude, and the correction and logarithm of table XIX, when the altitude is 40° 40′, and the horizontal parallax is 58′.

Tables XVII, XVIII, XIX, XX.

Solution.	58′	P. L.	0.4918
	40° 40′	sec.	0.1200
Parallax in alt. =	44′	P. L.	0.6118
By Table XII. Refrac. =	1′ 6″		9.6990
Corr. = 16′ 48″* = 59′ 42″ —	42′ 54″	P. L.	0.6228
Log. of Table XIX. =			0.2018

6. Find the correction and logarithm of Table XVII, for a star, when the altitude is 13° 15′.

Ans. Corr. = 56′ 2″, Log. = 1.3433.

7. Find the correction and logarithm of Table XVII, for Venus or Mars, when the parallax is 20″, and the altitude 24° 30′.

Ans. Corr. = 58′ 14″, Log. = 1.6647.

8. Find the correction and logarithm of Table XVIII, when the altitude is 56°.

Ans. Corr. = 59′ 26″, Log. = 1.9544.

9. Find the correction and logarithm of Table XIX, when the altitude is 70°, and the horizontal parallax 54′.

Ans. Corr. = 41′ 34″, Log. = 0.2299.

* The numbers of Table XIX are so disposed in the Navigator, that the corrections of proportional parts of parallax are all additive. This is effected by placing each number opposite that parallax, which is 10″ less than the one to which it belongs. There is, therefore, a correction for 0″ of parallax.

Auxiliary angle in lunar distances.

10. Compute the value of the auxiliary angle m, in the first and second methods of correcting the lunar distance, when the moon's apparent altitude is 40° 40′, its horizontal parallax 58′, and the sun's apparent altitude 70°.

Solution. The values of m might be computed directly from (545), but it is more convenient to obtain it by some process of approximation. For this purpose let

$$m = 60° + \delta m,$$

and we have

$$2 \cos. (60° + \delta m) = \frac{\cos. (b + \delta b)}{\cos. b} \frac{\cos. (a - \delta a)}{\cos. a}$$

$$= 2 \cos. 60° \cos. \delta m - 2 \sin. 60° \sin. \delta m \qquad (604)$$

$$= (\cos. \delta b - \text{tang. } b \sin. \delta b) (\cos. \delta a + \text{tang. } a \sin. \delta a),$$

in which we may put

$$2 \cos. 60° = 1, \cos. \delta b = 1 - 2 \sin.^2 \tfrac{1}{2} \delta b = 1 - \tfrac{1}{2} \delta b^2 \sin.^2 1''$$

$$\cos. \delta m = 1 - \tfrac{1}{2} \delta m^2 \sin.^2 1'',$$

and (604) becomes

$$2 \delta m \sin. 60° = \delta b \text{ tang. } b - \delta a \text{ tang. } a \qquad (605)$$

$$+ \tfrac{1}{2}(\delta b^2 - \delta m^2) \sin. 1''.$$

But if we take

$$e = 2 \delta b \sec. b \text{ and } e' = 2 \delta a \sec. a,$$

Prop. log. e is the logarithm of Table XIX, and Prog. log. e' is the corresponding logarithm for the sun, star, or planet, and by (605),

$$\delta m \doteq \tfrac{1}{4} e \sin. b \text{ cosec. } 60° - \tfrac{1}{4} e' \sin. a \text{ cosec. } 60° \qquad (606)$$

$$+ \tfrac{1}{2}(\delta b^2 - \delta m^2) \sin. 1'' \text{ cotan. } 60°,$$

whence in the present case

e	P. L.	0.2018	e'	P. L.	2.0173
40° 40′	cosec.	0.1860	70°	cosec.	0.0270
60°	sin.	9.9375			9.9375
1° 25′ 7″		0.3253	1′ 53″		1.9818

approx. $\delta m = \frac{1}{4}(1°\,25'\,7'' - 1'\,53'') = \frac{1}{4}(1°\,23'\,4'') = 20'\,46'' = 1246''$

$$\delta b = 42'\,54'' = 2574''$$

$\delta b + \delta m = 3820$		3.5821
$\delta b - \delta m = 1328$		3.1232
$1''$	sin.	4.6856
60°	cotan.	9.7614
corr. $\delta m = 7'' = \frac{1}{2}(14'')$		1.1523

$$\delta m = 20'\,46'' + 7'' = 20'\,53''.$$

11. Compute the value of the auxiliary angle m, when the moon's apparent altitude is 25° 30′, the horizontal parallax 60′, and the star's apparent altitude 10.°

Ans. 60° 13′ 48″.

12. Find the correction of Table XX, when the distance is 25°, the sun's altitude 10°, and the moon's altitude 25°.

Table XX.

Solution. We should find, in this case,

$\delta b = 50' \, 6''$		$\delta a =$	$5' \; 6''$
$\delta'' E = -27' \, 22''$		$\delta' E =$	$-3' 15''$
$\delta b - \delta'' E = 1° 17' 28'' = 4648''$		$\delta a - \delta' E =$	$8' 21'' = 501''$
$\delta b + \delta'' E = 22' 44''$		$\delta a + \delta' E =$	$1' 51'' = 111''$
$22' 44'' =$	P. L. 0.8986		0.899
$1° 17' 28'' = 4648''$	(ar. co.) 6.3327		P. L. 0.366
25°	tang. 9.6687		2 sin. 9.252
1″	cosec. 5.3144		1″ 2 cosec. 0.629
$1' 6'' = 66''$	2.2144		501″ (ar. co.) 7.300
$\frac{1}{2}(66'') = 33''$			111″ (ar. co.) 7.955
			2)6.401
$24'' =$		$18'' + 6''$	3.200
$57'' =$ corr. Table XX.			

13. Calculate the correction of Table XX, when the distance is 120°, the sun's altitude 20°, and the moon's altitude 10°.

Ans. 10″.

14. Calculate the corrections of Tables XLVIII, XLIX, and L, when the apparent distance is 28°, the moon's apparent altitude 38°, the planet's apparent altitude 18°, and its horizontal parallax 16″

Tables XLVIII, XLIX, L.

Solution.

57′ 30″	P. L.	0.4956			0.4956
18°	cosec.	0.5100	38°	cosec.	0.2107
28°	sin.	9.6716		tang.	9.7257
5° — 1st. cor. = 4° 22′ 9″		0.6772	5° + 2d cor. = 6° 6′ 34″		0.4320
6° 6′ 34″			moon's par. in alt. = 43′		
28°			moon's approx. alt. = 38° 43′		
28° 29′ = approx. dist.					
18°			43′ + 29′ = 72′ = 4320″	ar.co.	6.3645
38° 43′			43′ — 29′ = 14′	P. L.	1.1091
28° 22′ = ½ sum	tang.	9.73235	28°	tang.	9.7257
10° 22′ = ½ diff.	cotan.	0.73771	1″	cosec.	5.3144
½ (28°) = 14°	tang.	9.39677	2)34″		2.5137
A = 36° 21′		9.86683	17″		
1st ang. = 22° 21′	tang.	9.6140			9 614
18°	cotan.	0.4882			0.488
By Table XII 2′ 54″	P. L.	1.7929	T. X, A. 33″.	P. L.	2.515
2′ 17″		1.8951	25″ = cor. T. XIX		2.617
2d ang. = 50° 21′	tang.	0.0816	$\frac{16}{35}$ × 25″ = 11″ = cor. T. L		
38°	cotan.	0.1072			
By Table XII 1′ 13″	P. L.	2.1701			
47″		2.3589			

Cor. Table XLVIII = 2″ 17″ — 47′ + 17″ = 1′ 47″.

True from apparent distance.

15. Calculate the corrections of Tables XLVIII, XLIX, and L, when the apparent distance is 60°, the moon's apparent altitude 50°, the planet's apparent altitude 30°, and its horizontal parallax 30.″

Ans. Cor. Table XLVIII $= 1' \, 24''$

XLIX $= -21''$

L $= -18''$.

16. Find the correction of the table [B. p. 245.] for the interval of $2^h \, 30^m$, and the difference of the Proportional Logarithms equal to 88.

Ans. 15^s.

17. If the observed distance were 45° 34′ 10″, the moon's apparent altitude 22° 19′, its horizontal parallax 60′ 19″, the planet's apparent altitude 42° 12′, its horizontal parallax 15″.3; what is the true distance.

Solution. I. In this case $m = 60° \, 12' \, 28''$

$a = 42° \, 12'$ $\quad \delta a = 51''$ $\quad a' = 42 \; 11 \; 9$

$b = 22 \; 19$ $\quad \delta b = 53' \, 31''$ $\quad b' = 23 \; 12 \; 31$

$a' + b' = 65 \; 23 \; 40''$ — N. cos. $= -0.41637$ $\quad E = 45 \; 34 \; 10$

$E + m = 105 \; 46 \; 38$ N. cos. $= -0.27189$

$a + b + m = 124 \; 43 \; 28$ N. cos. $= -0.56964$

-1.25790

$E - m = -14° \, 38' \, 18''$ N. cos. $= 0.96754$

$a + b - m = 4 \; 18 \; 32$ N. cos. $= 0.99718$

$E' = 45 \; 1 \; 21$ N. cos. $= 0.70682$

Lunar distance corrected.

II.	$a - b + m =$	80° 5′ 28″	— N. cos. =	— 0.17208
	$a - b - m =$	— 40 19 28	— N. cos. =	— 0.76239
	$E + m =$	105 46 38	N. cos. =	— 0.27189
				— 1.20636
	$a' - b' =$	18° 58′ 38″	N. cos. =	0.94567
	$E - m =$	— 14 38 18	N. cos. =	0.96754
	$E' =$	45 1 14	N. cos. =	0.70685

III. $s = \frac{1}{2}(a + b + E) = 55° 2' 35''$ sec. 0.2420

$E =$	45° 34′ 10″	sin.	9.8538		9.8538
$s - a =$	12 50 35	cosec.	0.6532		0.6532
$s - E =$	9 28 25	sec.	0.0060	6′ 11″. T. XIX.	0.1920

59′ 8″. Table XVII	1.8907	20 37	P. L. 0.9410
43″	P. L. 2.4037	31	Table XX.
59°.51		27′ 19″	

$$E' = 45° 34' 10'' + 59' 51'' + 27' 19'' - 2° = 45° 1' 20''.$$

Clearing lunar distances.

IV. $Z = 47° 48'$ $z = 67° 41'$

$s_1 = 80° 31' 35''$	cosec. 0.0060			
$E = 45\ 34\ 10$	sin. 9.8538			
	9.6990			
	9.5588			9.5588
$Z = 47° 48'$	sin. 9.8697	$z = 67° 41'$		sin. 9.9662
$s_1 - z = 12° 50' 35''$	cosec. 0.6532			
$s_1 - Z = 32° 43' 35''$				cosec. 0.2671
$\delta a = 51''$	P. L. 2.3259	$\delta b = 53' 21''$		P. L. 0.5281
1st cor. $= 42''$	P. L. 2.4076	2d cor. $1° 26' 7''$		0.3202
$\delta b = 53' 21''$		$\delta a =$	$51''$	

$E' = 54'\ 3'' + 45° 34' 10'' + 31'' - 18'' - 1° 26' 58'' = 45° 1' 28''$

V. $\frac{1}{2}(a + b)$	$= 32° 15' 30''$	tang.	9.80014
$\frac{1}{2}(a - b)$	$= 9\ 56\ 30$	cotan.	0.75626
$\frac{1}{2} E$	$= 22\ 47\ 5$	tang.	9.62330
A	$= 123\ 28\ 15$	tang.	0.17970

1st ang. $= 100° 51' 10''$	tang. 0.7175		
2d ang. $= 146° 15' 20''$			tang. 9.8250
$a = 42° 12'$	cotan. 0.0425	$b = 22° 19'$	cotan. 0.3867
$\delta a = 51''$	P. L. 2.3259	$\delta b = 53' 21''$	P. L. 0.5281
1st cor. $= -7''$	P. L. 3.1859		
2d cor. $= -32' 46''$			P. L. 0.7398

$E' = 45° 34' 10'' - 7'' - 32' 46'' + 31'' - 18'' = 45° 1' 20''$.

VI.	60′ 19″	P. L. 0.4748			0.4748
	$a = 42° 12′$	cosec. 0.1728	$b = 22° 19′$	cosec.	0.4205
	$E = 45° 34′ 10″$	sin. 9.8538		tang.	0.0086
1st cor. =	4° 3′ 16″	0.5014	2d cor. = 5° 22′ 28″		0.9039

Cor. Table XLVIII = 1′ 33″

$E' = 45° 34' 10'' + 4° 3' 16'' + 5° 22' 28'' + 1' 33'' - 10° = 45° 1' 27''.$

VII.

$a =$	42° 12′	N. sin.	0.67172		
$b + E =$	67° 53′ 10″,	½N. sin.	— 0.46322	60′ 19″ P. L.	0.4748
$b - E =$	— 23° 15′ 10″,	½N. sin.	0.19739		
			0.40589	ar. co.	0.3916
$E =$	45° 34′ 10″			sin.	9.8538
Cor. Table XLVIII =	1′ 33″			cor. = — 34′ 17″	0.7302

$E' = 45° 34' 10'' + 1' 33'' - 34' 17'' = 45° 1' 26''.$

18. The apparent distance of the sun and moon is 95° 50′ 33″; the moon's apparent altitude is 35° 45′ 4″, its horizontal parallax is 54′ 24″; the sun's apparent altitude is 70° 48′ 1″; what is the true distance?

Ans. 95° 44′ 29″.

19. The apparent distance of a star from the moon is 31° 13′ 26″; the moon's apparent altitude is 8° 26′ 13″, its horizontal parallax is 60′, the star's apparent altitude is 35° 40′; what is the true distance?

Ans. 30° 23′ 56″.

Lunar distances.

20. Find the Greenwich time, Oct. 3, 1839, when the moon's distance from the sun was 38° 12′ 9″.

Solution.

Distance 1839, Oct. 3, 15^h		38° 59′ 21″	P. L. 0.3180
		38 12 9	
18^h P. L. 3189		47′ 12″	P. L. 0.5813
3180	$T =$ $1^h\,38^m\,10^s$		P. L. 0 2633
9	cor. T. $=$ -2^s		

Greenwich time $=$ $16^h\,38^m\;8^s$.

21. Find the Greenwich time, Jan. 2, 1839, when the moon's distance from Aldebaran was 70° 45′ 13″.

1839. Jan. 2, 9^h Greenw. Time, Dist. $=$ 69° 26′ 29″

P. L. $=$ 0.2852

12^h P. L. $=$ 0.2863

Ans. $12^h\,31^m\,47^s$.

22. The correct distance of the moon from β Corvi, 1839, April 3d, $11^h\,20^m$, in longitude 70° W by account, was 54° 8′ 15″; what was the longitude?

Lunar distances.

Solution.	$54° \; 8' 15''$	Gr. T. $= 11^h 20^m + 4^h 40^m = 16^h$		
☽'s Dec. $=$	$26 \; 48 \; 52$	by N. A.	sec.	0.04941
✱'s Dec. $=$	$22 \; 30 \; 11$		sec.	0.03439
$\frac{1}{2}$ sum $=$	$51° 27' 18''$		cos.	9.79198
Dist. — $\frac{1}{2}$ sum $=$	$2 \; 24 \; 36$		cos.	9.99962
				2)19.87540
	$3^h 59^m 42^s$		cos.	9.93770
✱'s Dec. $=$	$12 \; 25 \; 56$			
☽'s Dec. $=$	$16^h 25^m 38^s$	$=$ Greenw. Time $= 16^h$		

Long. $= 16^h - 14^h 20^m = 4^h 40^m = 70°$, as supposed.

23. The correct distance of the moon from Castor, 1839, Nov. $29^d \; 19^h$, in longitude 45° W. by account, was 78° 3′; what was the longitude?

Greenwich, 1839,

Nov. $29^d \; 21^h$, ☽'s R. A. $= 12^h 15^m 16^s.5$, Dec. $= 3° 48' 31''$ S.
22^h, ☽'s R. A. $= 12 \; 17 \;\; 2 .9$, Dec. $= 4 \;\; 2 \; 39$ S.
Castor's R. A. $= 7 \; 24 \; 24 .4$, Dec. $= 32 \; 14 \;\; 2$ N.

Ans. 44° 18′ W.

24. Find the distance of the moon from the sun, 1839, August 12^d, Greenwich time at mean noon.

☉'s R. A. $= 9^h 25^m 51^s.72$, Dec. $= 15° \; 7' 51''.5$ N.
☽'s R. A. $= 11 \; 42 \; 23 .48$, Dec. $= 0 \; 57 \; 27 .9$ N.

Ans. 36° 33′ 14″.

Lunar distances.

25. Find the distance of the moon from the sun, 1839, August 14^d, Greenwich time at mean noon.

$$\odot\text{'s R. A.} = 9^h\,33^m\,24^s.57, \quad \text{Dec.} = 14^\circ\,31'\,28''.2\text{ N.}$$

$$☽\text{'s R. A.} = 13\;\;8\;\;27\,.62, \quad \text{Dec.} = 10\;\;25\;\;54\,.5\text{ S.}$$

Ans. 58° 50′ 38″

CHAPTER IX.

ABERRATION.

105. The apparent position of the stars is affected by two sources of optical deception, so that they are not in the direction in which they appear to be.

The first of these sources is the motion of the earth, and the corresponding correction is called *aberration.*

Aberration, like the earth's motion, is either *annual* or *diurnal.*

106. *Problem. To find the aberration of a star.*

Solution. The apparent direction of a star is obviously that of the telescope, through which the star is seen. Let S (fig. 47.) be the star, and O the place of the observer at the instant of observation; SO is the true direction of the star, or the path of the particle of light which proceeded from the star to the observer, and it would be the direction of the telescope if he were stationary. But if he is moving in the direction OP, the direction of the telescope OT must be such, that the end T was at the point R, in the line OS, at the same instant in which the particle of light was at this point. The length RT is, therefore, the distance gone by the observer while the light is describing the line OR.

If, then, we put

$V =$ the velocity of light,

$v =$ the earth's velocity,

$I = TOP = RTO$,

$\delta I = - ROT =$ the aberration from the true place,

$$m = \frac{v}{V \sin. 1''}, \tag{607}$$

we have,

$$V : v = OR : TR = \sin. I : - \delta I \sin. 1''$$

$$\delta I = - m \sin. I. \tag{608}$$

107. *Problem. To find the annual aberration in latitude and longitude.*

Solution. The earth is moving in the plane of the ecliptic at nearly right angles to the direction of the sun. Hence if TP (fig. 48.) is the ecliptic, T the point towards which the earth is moving, S the true star, S' the apparent star,

$\odot =$ the sun's longitude,

$\mathit{A} =$ the star's longitude, $\delta \mathit{A} =$ the aberration in long.

$L =$ the star's latitude, $\delta L =$ the aberration in lat.

we have

$$ST = I,\ SP = L,$$

$$\text{long. of } T = \odot - 90°,\ PT = \odot - 90° - \mathit{A} = \mathit{A}_1$$

$$PP' = \delta \mathit{A} = TP - TP',\ \delta L \doteq S'P' - SP$$

$$\cos. T = \text{cotan.}\, I \,\text{tang.}\, \mathit{A}_1 = \text{cotan.}\, (I + \delta I)\, \text{tang.}\, (\mathit{A}_1 - \delta \mathit{A}),$$

whence
$$\frac{\tan. (\mathit{A}_1 - \delta \mathit{A})}{\text{tang.}\, \mathit{A}_1} = \frac{\text{tang.}\, (I + \delta I)}{\text{tang.}\, I}, \tag{609}$$

Aberration in latitude and longitude.

and, by (287 and 288),

$$\frac{\sin. \delta A}{\sin. (2 A_1 - \delta A)} = - \frac{\sin. \delta I}{\sin. (2 I + \delta I)}, \qquad (610)$$

or omitting δA and δI in the denominators, and reducing by means of (608),

$$\delta A = - \frac{\sin. 2 A_1}{\sin. 2 I} \delta I = - \frac{\sin. A_1 \cos. A_1}{\sin. I \cos. I} \delta I$$

$$= m \frac{\sin. A_1 \cos. A_1}{\cos. I}. \qquad (611)$$

But $\cos I = \cos. A_1 \cos. L,$ (612)

whence $\delta A = m \sin. A_1 \sec. L$ (613)

$$= - m \cos. (\odot - A) \sec. L.$$

We also have

$$\sin. T = \frac{\sin L}{\sin. I} = \frac{\sin. (L + \delta L)}{\sin (I + \delta I)}, \qquad (614)$$

whence

$$\sin. L \sin. (I + \delta I) = \sin. I \sin. (L + \delta L), \qquad (615)$$

and $\delta L = \frac{\sin. L \cos. I}{\cos. L \sin. I} \delta I$

$$= - m \operatorname{tang}. L \cos. I$$

$$= - m \cos. A_1 \sin. L$$

$$= - m \sin. (\odot - A) \sin. L. \qquad (616)$$

108. *Problem. To find the annual aberration in distance and direction from the vernal equinox.*

Aberration from vernal equinox.

Solution. Let A (fig. 48.) be the vernal equinox, and let

$$M = SA, \quad \delta M = \text{aberration of } M,$$
$$N = SAT, \quad \delta N = \text{aberration of } N.$$

Now we have

$$\delta M = \delta I \cos. AST = \frac{\sin. \odot - \cos. M \cos. I}{\sin. M \sin. I} \delta I$$

$$= - m \frac{\sin. \odot - \cos. M \cos. I}{\sin. M}. \qquad (617)$$

But

$$\cos. I = \sin. \odot \cos. M - \cos. \odot \sin. M \cos. N, \qquad (618)$$

whence if we put

$$B = - m \sin. \odot \qquad (619)$$

$$C = - m \cos. \odot, \qquad (620)$$

we have

$$\delta M = B \sin. M + C \cos. M \cos. N.$$

Again; the triangles ASS' and ATS' give by (243),

$$\sin. AST = \frac{\sin. M \odot \delta N}{\delta I} = - \frac{\cos. \odot \sin. N}{\sin. I} \qquad (621)$$

$$\delta N = m \cos. \odot \frac{\sin. N}{\sin. M} = - \frac{C \sin. N}{\sin. M}. \qquad (622)$$

109. *Problem. To find the annual aberration in right ascension and declination.*

Solution. If AT (fig. 48.) were the equator, we should have

$$D = SP, \quad R = AP,$$

Aberration in right ascension and declination.

and if we put

$$N_1 = SAP, \quad \omega = \text{obliquity of ecliptic,}$$

we have

$$N_1 = N + \omega,$$

and the triangles ASP, $AS'P'$ give

$$\sin. D = \sin M \sin. N_1 \tag{623}$$

$$\sin. (D - \delta D) = \sin. (M - \delta M) \sin. (N_1 - \delta N) \tag{624}$$

$$\cos. D\, \delta D = \sin. M \cos. N_1\, \delta N + \cos. M \sin. N_1\, \delta M \tag{625}$$

$$= B \sin. M \cos. M \sin. N_1$$

$$- C(\sin. N \cos. N_1 - \cos.^2 M \sin. N_1 \cos. N),$$

and if we put

$$A = C \cos. \omega \tag{626}$$

$$b' = \sin. M \cos. M \sin N_1 \sec. D \tag{627}$$

$$a' = -(\sin. N \cos. N_1 - \cos.^2 M \sin. N_1 \cos. N) \sec. D \sec. \omega, \tag{628}$$

we have

$$\cos. M = \cos. D \cos. R \tag{629}$$

$$\text{cotan.} N_1 = \sin. R \text{ cotan.} D \tag{630}$$

$$\sin. M \cos. N_1 = \frac{\sin. D \cos. N_1}{\sin. N_1} = \sin. D \text{ cotan.} N_1 \tag{631}$$

$$= \sin. D \sin. R \text{ cotan.} D = \sin. R \cos. D$$

$$b' = \sin. D \cos. R \cos. D \sec. D = \sin. D \cos. R \tag{632}$$

$$a' = -[\sin.(N - N_1) + \sin.^2 M \sin. N_1 \cos. N] \sec. D \sec. \omega$$

$$= [\sin. \omega - \sin.^2 M \sin.^2 N_1 \sin. \omega] \sec. D \sec. \omega$$

$$- \sin.^2 M \sin. N_1 \cos. N_1 \cos. \omega \sec\ D \sec. \omega$$

Aberration in right ascension and declination.

$$= (1 - \sin.^2 D) \sin. \omega \sec. D \sec. \omega - \sin. M \sin. D \cos. N_1 \sec. D$$

$$= \cos. D \tan. \omega - \sin. R \sin. D \qquad (633)$$

$$\delta D = A a' + B b'. \qquad (634)$$

Again, we have

$$\cos. M = \cos. R \cos. D \qquad (635)$$

$$\cos. (M + \delta M) = \cos. (R + \delta R) \cos. (D + \delta D)$$

$$\cos. D \sin. R \, \delta R = \sin. M \, \delta M - \cos. R \sin. D \, \delta D$$

$$= B (\sin.^2 M - b' \cos. R \sin. D)$$

$$+ A (\sin. M \cos. M \cos. N \sec. \omega - a' \cos. R \sin. D), \qquad (636)$$

and if we put

$$a = (\sin. M \cos. M \cos. N \sec. \omega - a' \cos. R \sin. D) \sec. D \operatorname{cosec.} R$$

$$b = (\sin.^2 M - b' \cos. R \sin. D) \sec. D \operatorname{cosec.} R,$$

we have

$$a \cos. D \sin. R = \sin. M \cos. M \cos. N_1 + \sin. R \cos. R \sin.^2 D$$

$$+ (\sin. M \cos. M \sin. N_1 - \cos. R \sin. D \cos. D) \tan. \omega$$

$$= \sin. R \cos. R (\cos.^2 D + \sin.^2 D)$$

$$+ (\sin. M \sin. N_1 \cos. R \cos. D - \cos. R \sin. M \sin. N_1 \cos. D) \tan. \omega$$

$$= \sin. R \cos. R$$

$$a = \cos. R \sec. D \qquad (637)$$

$$b \cos. D \sin. R = 1 - \cos.^2 M - \sin.^2 D \cos.^2 R$$

$$= 1 - \cos.^2 D \cos.^2 R - \sin.^2 D \cos.^2 R$$

$$= 1 - \cos.^2 R = \sin.^2 R$$

$$b = \sin. R \sec. D \qquad (638)$$

$$\delta R = A a + B b, \qquad (639)$$

and formulas (619, 620, 632, 633, 634, 637, 638, 639) agree with those given in the Nautical Almanac for finding the annual aberration.

110. *Corollary* The value of m, which is used in the Nautical Almanac, is

$$m = 20''.3600,$$

which gives

$$m \cos. \omega = 20''.3600 \cos. 23° \, 27' \, 36''.98 = 18''.6768.$$

111. *Scholium.* In the values of the aberration in right ascension and declination, each term consists of two factors, one of which is the same each instant for all the stars, and the other is the same for each star, during several years.

112. *Corollary.* If in (634) and (639) we put

$$i = A \tan. \omega \qquad (640)$$

$$B = h \cos. H \qquad (641)$$

$$A = h \sin. H; \qquad (642)$$

they become

$$\delta D = i \cos. D - h \sin. H \sin. R \sin. D + h \cos. H \cos. R \sin. D$$

$$= i \cos. D + h \cos. (H + R) \sin. D \qquad (643)$$

$$\delta R = h \sin. H \cos. R \sec. D + h \cos. H \sin. R \sec. D$$

$$= h \sin. (H + R) \sec. D, \qquad (644)$$

which agree with the formulas in the Nautical Almanac.

113. We have from (619 – 639)

$$\delta R = \sec. D \, [- m \cos. \omega \cos. \odot \cos. R - m \sin. \odot \sin. R] \quad (645)$$

Table XLII.

$$= \sec. D \left[-\tfrac{1}{2} m (\cos. \omega + 1)(\cos. \odot \cos. R + \sin. \odot \sin. R) \right.$$
$$\left. + \tfrac{1}{2} m (1 - \cos. \omega)(\cos. \odot \cos. R - \sin. \odot \sin. R) \right]$$
$$= \sec. D [-m \cos.^2 \tfrac{1}{2} \omega \cos. (R - \odot) + m \sin.^2 \tfrac{1}{2} \omega \cos. (R + \odot)],$$

and if we put

$$Q = R - \odot, \quad Q' = R + \odot \tag{646}$$

$$n = -m \cos.^2 \tfrac{1}{2} \omega, \quad n' = m \sin.^2 \tfrac{1}{2} \omega, \tag{647}$$

(645) becomes

$$\delta R = \sec. D \, (n \cos. Q + n' \cos. Q'), \tag{648}$$

and the values of n cos. Q and n' cos. Q' may be put in tables like Parts I and II of Table XLII of the Navigator.

Again, we have

$$\delta D = \sin. D \, (m \cos. \omega \sin. R \cos. \odot - m \cos. R \sin. \odot)$$
$$- m \sin. \omega \cos. \odot \cos. D$$
$$= \sin. D \, [\tfrac{1}{2} m (\cos. \omega + 1) \sin. Q - \tfrac{1}{2} m (1 - \cos. \omega) \sin. Q']$$
$$- \tfrac{1}{2} m \sin. \omega \, [\cos. (\odot + D) + (\cos. \odot - D)]$$
$$= \sin. D [-m \cos.^2 \tfrac{1}{2} \omega \cos.) Q + 90°) + \tfrac{1}{2} m \sin.^2 \tfrac{1}{2} \omega \cos. (Q' + 90°)]$$
$$- \tfrac{1}{2} m \sin. \omega \, [\cos. (\odot + D) + \cos. (\odot - D)]$$
$$= \sin. D \, [-n \cos. (Q + 90°) + n' \cos. (Q' - 90°)]$$
$$- \tfrac{1}{2} m \sin. \omega \, [\cos. (\odot + D) + \cos. (\odot - D)], \tag{649}$$

and the values of

$$-\tfrac{1}{2} m \sin. \omega \cos. (\odot + D) \text{ and } -\tfrac{1}{2} m \sin. \omega \cos. (\odot - D)$$

may be put in a table like Part III of Table XLII. The rules for finding the variations in right ascension and declination are then the same as in the explanation of this table.

Table XLI.

114. In constructing Table XLII, the values of m and ω were taken

$$m = 20'', \quad \omega = 23° \, 27' \, 28'', \tag{650}$$

whence

$$n = -19''.173, \quad n' = 0''.827, \tag{651}$$

$$-\tfrac{1}{2} m \sin. \omega = -3''.9814. \tag{652}$$

115. By putting

$$☉ - \varLambda = P, \tag{653}$$

we have, by (613 and 615),

$$\delta L = - m \cos. (P - 90°) \sin. L \tag{654}$$

$$\delta \varLambda = - m \cos. P \sec. L, \tag{655}$$

so that if the values of

$$- m \cos. P$$

are inserted in tables like Table XLI of the Navigator, the variations of latitude and longitude are found by the rule given in the explanation of this table.

116. If the star is nearly in the ecliptic, the aberration in latitude may be neglected, and the aberration in longitude will be by (655)

$$\delta \varLambda = - m \cos. P. \tag{656}$$

117. *Problem. To find the diurnal aberration in right ascension and declination.*

Solution. Let

$v' =$ the velocity of a point of the equator, arising from the earth's rotation,

$$m' = \frac{v'}{V \sin. 1''}. \tag{657}$$

Aberration from the motion of the star.

The velocity of the observer is evidently in proportion to the circumference which he describes in a day, that is, to the radius of this circumference, or to the cosine of the latitude.

$$\text{The velocity of the observer} = v' \cos. \text{lat.}$$

Now, the diurnal motion is parallel to the equator, whence the formulas (613) and (616) may be referred at once to the present case by putting

$$Z = \text{the right ascension of the zenith,}$$

and changing m into $m' \cos. \text{lat.}$, $\odot - A$ into $Z - R$, and L into D; whence the diurnal aberrations in right ascension and declination are

$$\delta' R = - m' \cos. (Z - R) \sec. D \cos. \text{lat.} \qquad (658)$$

$$\delta' D = - m' \sin. (Z - R) \sin. D \cos. \text{lat.} \qquad (659)$$

118. The value of m' is nearly

$$m' = 0''.31. \qquad (660)$$

119. *Problem. To find the aberration which arises from the motion of a planet.*

Solution. The most important planets revolve about the sun almost uniformly in circles, and in the plane of the ecliptic. At the instant, then, of the light's reaching the earth, the planet has advanced in its orbit by a distance proportioned to its velocity, and to the time which the light takes in reaching the earth. Let then S (fig. 49.) be the sun, and $O_1 O_1'$ perpendicular to $O_1 S$ the path of the planet; and put

$$v_1 = \text{the velocity of the planet,}$$

$$m_1 = \frac{v_1}{V \sin. 1''}, \quad P_1 = OO_1S,$$

$$r = OS, \quad r_1 = O_1S,$$

we have

$$\delta_1 A = -O_1 O O_1' = -\frac{O_1 O_1' \cos. P_1}{O O_1 \sin. 1''} = -m_1 \cos. P_1. \quad (661)$$

But it will be shown in Theoretical Astronomy that

$$v^2 : v_1^2 = r_1 : r;$$

hence

$$m^2 : m_1^2 = v^2 : v_1^2 = r_1 : r$$

$$m : m_1 = \sqrt{r_1} : \sqrt{r}$$

$$m_1 = m \sqrt{\frac{r}{r_1}} \quad (662)$$

$$\delta_1 A = -m \sqrt{\frac{r}{r_1}} \cos. P_1; \quad (663)$$

and this aberration being combined with (656) gives the whole aberration in longitude, from which a table, like Table XXXIX of the Navigator, may be constructed.

120. Examples.

1. Find the values of log. A, log. B, h, H, and i for May 1, 1839, when ☉ $= 40° 23' 52''$.

Ans. log. $A = 1.1498_n$
log. $B = 1.1248_n$
$h = 19''.42$
$H = 226° 40'$
$i = -6''.13$

2. Find the values of log. a, log. b, log. a', log. b' for Altair in the year 1839.

Annual aberration.

Solution.

$R = 19^h\ 42^m\ 55^s$		cos.	9.63760		sin.	9.95466_n
$D = 8°\ 26'\ 52''$		sec.	0.00474		sec.	0.00474
		log. a =	9.64234		log. b =	9.95940_n

R	cos. 9.63760		sin. 9.95466_n	
D	sin. 9.16704		sin. 9.16704	cos. 9.99526
	log. b' = 8.80464,	0.13234	9.12170_n	ω tan. 9.63747
		0.42927		9.63273
		$a' = 0.56161$		log. $a' = 9.74947$

3. Find the values of log. a, log. b, log. a', log. b' for Regulus in the year 1839; for this star

$$R = 9^h\ 59^m\ 48^s, \quad D = 12°\ 45'\ 7''.$$

Ans. log. $a = 9.94816_n$
log. $b = 9.71048$
log. $a' = 9.49516$
log. $b' = 9.28122_n$

4. Find the numbers of the different parts of Table XLII for the argument $7^s\ 20° = 230°$.

Ans. 12″.32 for Part I,
— 0″.53 for Part II,
2″.56 for Part III.

Annual aberration.

5. Find the number of Table XLI for $7^s\ 20°$.

Ans. $12''.9$.

6. Find the aberration in right ascension and declination of Altair for May 1, 1839.

Solution. I.

	A	1.1498_n			1.1498_n
	a	9.6423		a'	9.7494
$-6''.20$		0.7921_n	$-7''.93$		0.8992_n
	B	1.1248_n			1.1248_n
	b	9.9594_n		b'	8.8046
$12''.14$		1.0842	$-0''.85$		9.9294_n
$33 = 5''.9 = 0^s.39$			$\delta D = -8''.78$		

II.

$H + \alpha = 162°\ 44' + 360°$	sin. 9.4725		cos. 9.9800_n	
$h = 19''.42$	1.2882		1.2882	
$D = 8°\ 27'$	sec. 0.0047		sin. 9.1670	
$\delta R = 5''.83 = 0^s.39$	$0.7654,$	$-2''.72$	0.4352_n	
	i cos. $D =$	$-6''.06$		
	$\delta D =$	$-8''.78$		

Annual aberration.

III.

$R - \odot = 255° \; 40' = 8^s \, 15° \, 40'$ P. I $= 4''.75$

$R + \odot = 76° + 360 = 2^s \, 16° + 12^s$ P. II $= 0''.20$

	$4''.95$		0.6946
D		sec.	0.0047
$\delta R = 5'' = 0^s.33$			0.6993

$8^s \, 15° \, 40' + 3^s = 11^s \, 15° \, 40'$ P. I. $- 18''.57$

$2^s \, 16° + 3^s = 5^s \, 16°$ P. II. $- 0''.80$

	$- 19''.37$		1.2871_n
D		sin.	9.1670
	$- 2''.85$		0.4541_n

$\odot + D = 48° = 1^s \, 18°$ $- 2''.66$

$\odot - D = 32° = 1^s \, 2°$ $- 3''.38$

$\delta D = - 8''.89$

7. Find the aberration in right ascension and declination of Regulus for May 1, 1839.

Ans. By Naut. Alm. $\delta R = 0^s.38$

$\delta D = -1''.87$

By the Navigator $\delta R = 0^s.38$

$\delta D = -1''.91$

8. Find the aberration of Regulus in latitude and longitude for May 1, 1839.

Ans. $\delta \Lambda = 6''.5$

$\delta L = 0''.15$

Aberration of the planets.

9. Find the aberration of Venus in longitude, when the difference of longitude of Venus and the sun is 45°.

Solution.	r		0.0000		0.0000
	r_1	ar. co.	0.1407	$\frac{1}{2}$ (ar. co.)	0.0703
$P = 45°$		sin.	9.8495	20″ log.	1.3010
$P_1 =$		sin.	9.9902	cos.	9.3214
					0.6927

— 5″ when P_1 is acute, + 5″ when P_1 is obtuse,
—14″ from Table XLI —14″

$\delta A =$ —19″ when P_1 is ac., = — 9″ when P_1 is obtuse.

10. Find the aberration of each of the planets in longitude, when the difference of longitude of the sun and planet is 15°. The value of log. r_1 for each of the planets is

For Mercury	9.5878	is the mean value,
Venus	9.8593	
The Earth	0.0000	
Mars	0.1829	
Jupiter	0.7161	
Saturn	0.9795	
Uranus	1.2829	

Ans.	For Mercury	— 43″	when P_1 is acute,
		4″	when P_1 is obtuse,
	Venus	— 41″	when P_1 is acute,
		3″	when P_1 is obtuse,
	Mars	35″	
	Jupiter	28″	
	Saturn	26″	
	Uranus	24″	

Diurnal aberration.

11. Find the diurnal aberration of right ascension and declination of Polaris for Jan. 1, 1839, and latitude 45°, when the hour angle is $0^h\ 30^m$.

Solution.	$0''.31$		9.4914		9.4914
	45°	cos.	9.8495		9.8495
$D = 88°\ 27'$		sec.	1.5678	sin.	9.9998
$0^h\ 30^m$		cos.	9.9963	sin.	9.1157
$\delta' R = -8''.04 = -0^s.53$			0.9050	$\delta' D = 0''.03$	8.4564

12. Find the diurnal aberration of δ Ursæ Minoris in right ascension and declination for Jan. 1, 1839, and latitude 0°, when the star is upon the meridian.

Dec. of δ Ursæ Minoris $= 86°\ 35'$.

Ans. $\delta' R = -0^s.35$

$\delta' D = 0.$

CHAPTER X.

REFRACTION.

121. Light proceeds in exactly straight lines, only in the void spaces of the heavens; but when it enters the atmosphere of a planet, it is sensibly bent from its original direction according to known optical laws, and its path becomes curved. This change of direction is called *refraction;* and the corresponding change in the position of each star is the *refraction of that star.*

122. *Problem. To find the refraction of a star.*

Solution. Let O (fig. 50.) be the earth's centre, A the position of the observer, AEF the section of the surface formed by a vertical plane passing through the star. It is then a law of optics, that

Astronomical Refraction takes place in vertical planes, so as to increase the altitude of each star without affecting its azimuth.

Let, now, ZBH be the section of the upper surface of the upper atmosphere formed by the vertical plane, SB the direction of the ray of light which comes to the eye of the observer. This ray begins to be bent at B, and describes the curve BA, which is such, that the direction AC is that at which it enters the eye. Let, now, Ratio of sines in the law of refraction.

$\varphi = ZAC =$ the ∗'s apparent zenith distance,

$r =$ the refraction,

$=$ the diff. of directions of AC and BS,

$= SBL - S'CL$

$u = COZ,$

and we have

$$LCS' = \varphi - u,$$
$$SBL = \varphi - u + r.$$

Again, it is a law of optics that *the ratio of the sines of the two angles LBS and ZAS′ is constant for all heights, and dependent upon the refractive power of the air at the observer.*

Denote this ratio by n, and we have

$$\frac{\sin.(\varphi - u + r)}{\sin.\varphi} = n, \tag{664}$$

and if

U and $R =$ the values of u and r at the horizon,

we have

$$\frac{\sin.(\varphi - u + r)}{\sin.\varphi} = n = \cos.(U - R). \tag{665}$$

whence

$$\frac{\sin.\varphi - \sin.(\varphi - u + r)}{\sin.\varphi + \sin.(\varphi - u + r)} = \frac{1 - \cos.(U - R)}{1 + \cos.(U - R)} \tag{666}$$

$$\frac{\text{tang.}\,\frac{1}{2}(u - r)}{\text{tang.}\,[\varphi - \frac{1}{2}(u - r)]} = \text{tang.}^2\,\tfrac{1}{2}(U - R) = N, \tag{667}$$

and since $\frac{1}{2}(u-r)$ is small,

$$\tfrac{1}{2}(u-r) = N \text{ tang. } [\varphi - \tfrac{1}{2}(u-r)]. \qquad (668)$$

Again, to find u, the triangle COA gives

$$\frac{\sin.(\varphi - u)}{\sin.\varphi} = \frac{OA}{OC}. \qquad (669)$$

Now the point C is at different heights for different zenith distances of the star; but this difference in the values of OC is small, and may be neglected in this approximation; so that

$$\frac{\sin.(\varphi - u)}{\sin.\varphi} = \cos. U = \frac{OA}{OK}, \qquad (670)$$

$$\frac{\sin.\varphi - \sin.(\varphi - u)}{\sin.\varphi + \sin.(\varphi - u)} = \frac{1 - \cos. U}{1 + \cos. U} \qquad (671)$$

$$\tan.\tfrac{1}{2}u = \text{tang.}^2 \tfrac{1}{2} U \tan.(\varphi - \tfrac{1}{2}u). \qquad (672)$$

and since u is small,

$$\tfrac{1}{2}u = \text{tang.}^2 \tfrac{1}{2} U \text{ tang.} (\varphi - \tfrac{1}{2}u) \qquad (673)$$

which, compared with this rough value of $\frac{1}{2}(u-r)$ from (668),

$$\tfrac{1}{2}(u-r) = N \tan.(\varphi - \tfrac{1}{2}u) \qquad (674)$$

gives

$$u = \frac{r}{1 - N \cot.^2 \frac{1}{2} U} = N' r, \qquad (675)$$

and if we put

$$m = \frac{2N}{N' - 1} \qquad (676)$$

$$p = \tfrac{1}{2}(N' - 1), \qquad (677)$$

we have, by (668),

$$\tfrac{1}{2}(u - r) = p\,r \qquad (678)$$

$$r = m \tan.(\varphi - p\,r), \qquad (679)$$

and the values of m and p must be determined by observation; their mean values, as found by Bradley, and adopted in the Navigator, are

$$m = 57''.035, \quad p = 3, \qquad (680)$$

by which Table XII is calculated.

123. The variation in the values of m and p for different altitudes of the star can only be determined from a knowledge of the curve which the ray of light describes. But this curve depends upon the law of the refractive power of the air at different heights; and this law is not known, so that the variations of m and p must be determined by observation. At altitudes greater than 12 degrees, the mean values of m and p are found to be nearly constant, and observations at lower altitudes are rarely to be used.

124. The mean values of m and p, which are given in (680), correspond to

the height of the barometer = 29.6 inches, (681)

the thermometer = 50° Farenheit. (682)

Now the refraction is proportional to the density of the air; but, at the same temperature, the density of the air is proportional to its elastic power, that is, to the height of the barom-emeter. If then

Table XXXVI.

$h =$ the height of the barometer in inches,

$r =$ the refraction of Table XLI,

$\delta r =$ the correction for the barometer;

we have

$$r : r + \delta r = 29.6 : h \tag{683}$$

$$29.6\, \delta r = (h - 29.6)\, r \tag{684}$$

$$\delta r = \frac{(h - 29.6)}{29.6}\, r, \tag{685}$$

whence the corresponding correction of Table XXXVI is calculated.

Again, the density of the air, for the same elastic force, increases by one four hundredth part for every depression of 1° of Fahrenheit; hence the refraction increases at the same rate, so that if

$\delta' r =$ the correction for the thermometer,

$f =$ the temperature in degrees of Fahrenheit,

we have

$$\delta' r = \frac{50 - f}{400}, \tag{686}$$

whence the corresponding correction of Table XXXVI is calculated.

125. Examples.

1. Find the refraction, when the altitude of the star is 14°, and the corrections for this altitude, when the barometer is 31.32 inches, and the thermometer 72° Fahrenheit.

Solution.	15″.035 log.	1 75614			
	76° tan.	0.60323			
1st app. $r = 228''.7 = 3' 48''.7$		2.35937			
	15″.035	1.75614			
$76° - 3r = 75° 48' 34''$	tan.	0.59711			
2d app. $r = 226'' = 3' 46''$		2.35325			2.353
$31.32 - 29.6 = 1.72$		0.235	$50 - 72 = -22$		1.342_n
29.6	ar. co.	8.529	400	ar. co.	7.398
$\delta r = 13''$		1.117	$\delta' r = -12''$		1.093_n

2. Find the refraction, when the altitude of the star is 50°, and the corrections for this altitude, when the barometer is 31.66 inches, and the thermometer 36°.

Ans. The refraction $= 48''$
Correction for barometer $= 3''$
Correction for thermom. $= 2''$

3. Find the refraction, when the altitude of the star is 10°, and the corrections for this altitude, when the barometer is 27.80 inches, and the thermometer 32°.

Ans. The refraction $= 5' 15''$
Correction for barometer $= -19''$
Correction for thermom. $= 15''$

126. *Problem. To find the radius of curvature of the path of the ray of light in the earth's atmosphere.*

Path of the ray of light.

Solution. By the *radius of curvature* is meant the radius of the circular arc, which most nearly coincides with the curve. Now this radius may be found with sufficient accuracy, by regarding the whole curve AB as the arc of a circle; and if we put

$$r_1 = \text{the radius of curvature,}$$

$$R_1 = OA = \text{the earth's radius,}$$

we have

$$AC : R_1 = \sin. u : \sin. (\varphi + u), \tag{687}$$

or, nearly,

$$AB : R_1 = u \sin. 1'' : \sin. \varphi$$

$$AB = \frac{R_1 u \sin. 1''}{\sin. \varphi}. \tag{688}$$

Again, the radii of the arc AB, which are drawn to the points A and B, are perpendicular to the tangents AS' and BS, so that the angle which they make with each other is

$$S'AS = r;$$

that is, r is the angle at the centre, which is measured by the arc AB, consequently

$$AB = r_1 \sin. r = r_1 r \sin. 1'', \tag{689}$$

whence

$$r_1 = \frac{u R_1}{r \sin. \varphi}. \tag{690}$$

But, by (678),

$$u = 7 r, \tag{691}$$

whence
$$r_1 = \frac{7 R_1}{\sin. \varphi}, \tag{692}$$

so that at the horizon

$$r_1 = 7 R_1. \tag{693}$$

as in (225, 226).

Dip of the horizon.

127. *Problem. To find the dip of the horizon.*

Solution. The *dip of the horizon* is the error of supposing the apparent horizon to be only 90° from the zenith, whereas it is more than 90°. If O (fig. 51.) is the centre of the earth, B the position of the observer at the height AB above the surface, O' the centre of curvature of the visual ray BT, which just touches the earth's surface at T, BH' perpendicular to $O'B$, is the direction of the apparent horizon and

$$\delta H = HBH' = OBO' = \text{the dip.}$$

The triangle BOO' gives

$$BO' : OO' = \sin. BOO' : \sin. \delta H = \sin. BOH' : \sin \delta H,$$

or, since $BO' = 7\,BO$ nearly, and $OO' = 6\,BO$,

and δH and BOH' are small,

$$7 : 6 = BOH' : \delta H$$

$$\delta H = \tfrac{6}{7}\,BOH' = \tfrac{6}{7}\,\frac{AH'}{AO \sin. 1''}. \qquad (694)$$

But, by (227), we have, if we put

$$R = AO, \; h = AB$$

$$\tfrac{6}{7}\,AH' = \tfrac{6}{7}\,\sqrt{(\tfrac{7}{3}\,R\,h)}$$

$$= 2\,\sqrt{(\tfrac{3}{7}\,R\,h)} \qquad (695)$$

whence

$$\delta H = \frac{2}{\sqrt{(\tfrac{7}{3}\,R)} \sin. 1''}\,\sqrt{h}, \qquad (696)$$

and

$$\log. \delta H = \log. 2 - \log. (\sqrt{\tfrac{7}{3}\,R}) - \log. \sin, 1'' + \tfrac{1}{2} \log. h$$

$$= 1.77128 + \tfrac{1}{2} \log. h, \qquad (697)$$

which is the same with the formula, given in the preface to the Navigator, for calculating Table XIII.

128. *Problem. To find the dip of the sea at different distances from the observer.*

Solution. Let O (fig. 52.) be the centre of the earth, B the observer at the height

$$h = AB \text{ (in feet)}$$

above the sea, and A' the point of the sea which is observed at the distance

$$d = AA' \text{ (in sea miles)} = AOA'$$

from B; and let

$$M = \text{the length of a sea mile in feet.}$$

If the radius OA is produced to B', so that

$$A'B' = AB,$$

the point B' will be elevated by refraction nearly as much as the point A'. But the visual ray BB' will, from the equal heights of B and B', be perpendicular to the radius OC, which is half way between B and B', so that the dip of B' is, by (694),

$$\delta B = \tfrac{6}{7} BOC = \tfrac{3}{7} AOA' = \tfrac{3}{7} d. \qquad (698)$$

The dip of the point A' will be greater than B' by the angle

$$i = B'BA,$$

which it subtends at B, and which is found with sufficient accuracy by the formula

$$\sin. i = \frac{A'B'}{A'B} = \frac{h}{Md} = i \sin. 1' \qquad (699)$$

$$i = \frac{h}{M \sin. 1' d}. \qquad (700)$$

But, by (228),

$$M = \frac{\pi R}{10800'} \qquad (701)$$

Twilight.

$$\frac{1}{M \sin. 1'} = \frac{10800'}{\pi R \sin. 1'} = 0.56514, \tag{702}$$

so that the dip of A' is

$$\delta A = \tfrac{3}{7} d + 0.56514 \frac{h}{d}, \tag{703}$$

which is the same with the formula, given in the preface to the Navigator, for calculating Table XVI.

129. Refraction, by elevating the stars in the horizon, will affect the times of their rising and setting; and the star will not set until its zenith distance is

$$90^\circ + \text{horizontal refraction},$$

and the corresponding hour angle is easily found by solving the triangle PZB (fig. 35.)

130. Another astronomical phenomenon, connected with the atmosphere, and dependent upon the combination of reflection and refraction is the *twilight*, or the light, before and after sunset, which arises from the illuminated atmosphere in the horizon. This light begins and ends when the sun is about 18° below the horizon; so that the time of its beginning or ending is easily calculated, from the triangle PZB (fig. 35.)

131. Examples.

1. Find the dip of the horizon, when the height of the eye is 20 feet.

Ans. $264'' = 4' \, 24''$.

Dip.

2. Find the dip of the sea at the distance of 3 miles, when the height of the eye is 30 feet.

Solution. $\frac{3}{7} \times 3 = \frac{9}{7} = 1'.3$

$$0.56514 \times \frac{30}{3} = 5'.6$$

$$\text{dip} = 7'.$$

3. Find the dip of the sea at the distance of $2\frac{1}{2}$ miles, when the height of the eye is 40 feet.

Ans. 10′.

4. Find the dip of the sea at the distance of $\frac{1}{4}$ of a mile, when the height of the eye is 30 feet.

Ans. 68′.

CHAPTER XI.

PARALLAX.

132. The fixed stars are at such immense distances from the earth, that their apparent positions are the same for all observers. But this is not the case with the sun, moon, and planets; so that, in order to compare together observations taken in different places, they must be reduced to some one point of observation. The point of observation which has been adopted for this purpose is the earth's centre; and the difference between the apparent positions of a heavenly body, as seen from the surface or the centre of the earth, is called its *parallax*.

133. *Problem. To find the parallax of a star.*

Solution. Let O (fig. 53.) be the earth's centre, A the observer, S the star, and OSA, being the difference of directions of the visual rays drawn to the observer and the earth's centre, is the parallax. Now since SAZ is the apparent zenith distance of the star, and SOZ is its distance from the same zenith to an observer at O, the parallax

$$OSA = p$$

is the excess of the apparent zenith distance above the true zenith distance. If, then,

$z = SAZ$, $R = OA =$ the earth's radius,

$r = OS =$ the distance of the star from the earth's centre,

we have $$P : R = \sin. Z : \sin. p,$$

or $$\sin. p = \frac{R \sin. z}{r}, \qquad (704)$$

or $$p = \frac{R \sin. z}{r \sin. 1''}. \qquad (705)$$

134. *Corollary.* If P is the horizontal parallax, we have

$$\sin. P = \frac{R}{r}, \qquad (706)$$

or $$P = \frac{R}{r \sin. 1''}; \qquad (707)$$

whence $$\sin. p = \sin. P . \sin. z, \qquad (708)$$

or $$p = P . \sin. z, \qquad (709)$$

which agrees with (564) and Tables X. A., XIV, and XXIX, are computed by this formula, combined, in the last table, with the refraction of Table XII.

135. *Corollary.* In common cases, the value of the horizontal parallax can be taken from the Nautical Almanac; but, in eclipses and occultations, regard must be had to the length of the earth's radius, which is different for different places. *The curvature of the earth is such that the radius diminishes as we recede from the equator proportionally to the square of the sine of the latitude; the whole diminution at the pole being about $\frac{1}{300}$th part of this radius.*

Now the horizontal parallax is, by (707), proportional to the earth's radius, so that it diminishes at the same rate, from the equatorial value which is given in the Nautical Almanac. Hence, if

$\delta R =$ the diminution of R for the latitude L,

$d P =$ that of P,

$R =$ the radius at the equator,

$P =$ the parallax at the equator,

we have

$$\delta R = \tfrac{1}{300} R \sin.^2 L$$

$$= \tfrac{1}{600} R (1 - \cos. 2 L) \qquad (710)$$

$$\delta P = \tfrac{1}{300} P \sin.^2 L$$

$$= \tfrac{1}{600} P (1 - \cos. 2 L), \qquad (712)$$

and if P is expressed in minutes, while δP is expressed in seconds, (711) becomes

$$\delta P \text{ in seconds} = \tfrac{1}{10} (P \text{ in minutes}) (1 - \cos. 2 L), \quad (712)$$

which agrees with the formulas for calculating the reduction of parallax given in the explanation to Table XXXVIII of the Navigator.

136. In reducing delicate observations to the centre of the earth, it must be observed that the centre is not exactly in the direction of the vertical. Thus, if O (fig. 54.) is the earth's centre, P the pole, A the observer, Z the zenith, ZAL the vertical, Z' the point where the radius OA produced meets the celestial sphere, Z' is called the *true zenith*, and Z the *apparent zenith*. The angle ZAZ', which is the difference between the polar distance of the true and apparent zenith is called the *reduction of the latitude*, and must be subtracted from the angle ALE, or the latitude to attain the angle AOE,

or the direction of the observer from the earth's centre. The angle AOE is called *the reduced latitude*, and is to be substituted for the latitude in reducing delicate observations to the centre of the earth.

137. *Problem. To find the reduction of the latitude.*

Solution. Draw OB (fig. 54.) parallel to AE; with OA as a radius describe the arc AR. The angle

$$\delta L = AOB = OAL$$

is the reduction of the latitude, and is so small, that the arcs AB and AR may be regarded as straight lines, and the triangle ABR as a right triangle; and since the sides AB and AR are perpendicular, respectively, to AL and AO, we have

$$\delta L = ABR.$$

If now, we put

$m =$ the difference of the polar and equatorial radii divided by the equatorial radius $= \frac{1}{300}$, (712)

we have, by neglecting the very small terms $m \delta L^2$, δL^3, m^3,

$$R' = OA = R(1 - m \sin^2 L) \quad (713)$$

$$\begin{aligned} OB &= R[1 - m \sin.^2 (L + \delta L)] \\ &= R[1 - m(\sin.^2 L + \sin. \delta L \cos. L)^2] \\ &= R(1 - m \sin.^2 L - 2m \sin. L \cos. L \sin. \delta L) \quad (714) \end{aligned}$$

$$RAB = R' - OB = 2mR \sin. L \cos. L \, \delta L \quad (715)$$

$$AB = R' \sin \delta L = R \sin. \delta L (1 - m \sin.^2 L) \quad (716)$$

$$\text{tang.} \, \delta L = \sin. \delta L = \frac{BR}{AB} = \frac{2m \sin. L \cos. L}{1 - m \sin.^2 L}, \quad (717)$$

whence

$$\sin.\,\delta L - m \sin.\,\delta L \sin.^2 L = 2\,m \sin.\,L \cos.\,L \qquad (718)$$

$$\sin.\,\delta L = 2\,m \sin.\,L\,(\cos.\,L + \tfrac{1}{2} \sin.\,\delta L \sin.\,L)$$

$$= 2\,m \sin.\,L\,(\cos.\,L + \sin.\,\tfrac{1}{2}\,\delta L \sin.\,L)$$

$$= 2\,m \sin.\,L \cos.\,(L - \tfrac{1}{2}\,\delta L) \qquad (719)$$

$$\delta L = \frac{2\,m}{\sin.\,1''} \sin.\,L \cos.\,(L - \tfrac{1}{2}\,\delta L)$$

$$= \frac{2}{300 \sin.\,1''} \sin.\,L \cos.\,(L - \tfrac{1}{2}\,\delta L)$$

$$= \frac{2}{5 \sin.\,1'} \sin.\,L \cos.\,(L - \tfrac{1}{2}\,\delta L), \qquad (720)$$

from which δL is easily calculated.

138. *Corollary.* By putting in the last term of (719), for $\sin.\,\delta L$, the approximate value

$$\sin.\,\delta L = 2\,m \sin.\,L \cos.\,L, \qquad (721)$$

it becomes

$$\sin.\,\delta L = 2\,m \sin.\,L\,(\cos.\,L + m \sin.^2 L \cos.\,L)$$

$$= 2\,m \sin.\,L \cos.\,L\,(1 + m \sin.^2 L) \qquad (722)$$

$$\delta L = \tfrac{2}{5} \operatorname{cosec.} 1' \sin.\,L \cos.\,L\,(1 + \tfrac{1}{300} \sin.^2 L) \qquad (723)$$

139. *Corollary.* We have, by (56),

$$\cot.(L - \delta L) = \frac{1 + \tan.\,L \tan.\,\delta L}{\tan.\,L - \tan.\,\delta L} = \frac{1 + \tan.\,L \tan.\,\delta L}{\tan.\,L(1 - \cot.\,L \tan.\,\delta L)}$$

$$= \text{cotan.}\,L\,\frac{1 + \text{tang.}\,L\,\text{tang.}\,\delta L}{1 - \text{cotan.}\,L\,\text{tang.}\,\delta L}, \qquad (724)$$

and if we put

$$n = 2\,m\,(1 + m\,\sin.^2 L) = 2\,m + 2\,m^2\,\sin.^2 L$$
$$= 2\,m + m^2 - m^2\,\cos.\,2\,L, \qquad (725)$$

we have

$$\text{tang.}\,L = \sin.\,\delta L = n\,\sin.\,L\,\cos.\,L \qquad (726)$$
$$1 + \text{tang.}\,L\,\text{tang.}\,\delta L = 1 + n\,\sin.^2 L \qquad (727)$$
$$1 - \text{cotan.}\,L\,\text{tang.}\,\delta L = 1 - n\,\cos.^2 L \qquad (728)$$

$$\text{cotan.}\,(L - \delta L) = \text{cotan.}\,L\,(1 + n\,\sin.^2 L)\,(1 - n\,\cos.^2 L)^{-1}$$
$$= \text{cotan.}\,L\,(1 + n\,\sin.^2 L)\,(1 + n\,\cos.^2 L + \tfrac{1}{2}\,n^2\,\cos.^4 L)$$
$$= \text{cot.}\,L\,[1 + n\,(\sin.^2 L + \cos.^2 L) + n^2 \cos.^2 L\,(\sin.^2 L + \cos.^2 L)]$$
$$= \text{cotan.}\,L\,(1 + n + \tfrac{1}{2}\,n^2\,\cos.^2 L)$$
$$= \text{cotan.}\,L\,(1 + n + \tfrac{1}{2}\,n^2 + \tfrac{1}{4}\,n^2\,\cos.^2 L)$$
$$= \text{cotan.}\,L\,(1 + 2\,m + 3\,m^2 + m^2\,\cos.^2 L), \qquad (729)$$

and the term $m^2\,\cos.^2 L$ is so small, that it may be neglected, whence we have

$$\text{cotan.}\,(L - \delta L) = (1 + 2\,m + 3\,m^2)\,\text{cotan.}\,L$$
$$= 1.0067\,\text{cotan.}\,L, \qquad (730)$$

which agrees with the formula given in the explanation of Table XXXVIII in the Navigator, and which must be calculated by means of Tables of 7 places of decimals.

140. *Problem. To find the parallax in latitude and longitude.*

Solution. Let Z (fig. 55.) be the zenith, P the pole of the ecliptic, and M' the apparent place of the place of the body, whose parallax is sought, and M its true place. Let also

Parallax in latitude and longitude.

$N = PZ =$ the zenith distance of the pole,

$=$ the altitude of the nonagesimal,

$A = 90° - ZM' =$ the apparent altitude,

$L = 90° - PM =$ the true latitude of the body,

$D = ZPM$ $=$ the true diff. of long. of the body and the zenith,

$P =$ the horizontal parallax,

$p = P \cos. A = MM' =$ the parallax in altitude,

$\delta D = ZPM' - ZPM =$ the parallax in longitude,

$\delta L = PM' - PM =$ the parallax in latitude,

$L' = L + \delta L.$

The triangles PMM' and ZPM' give

$$\delta D = \frac{p \sin. M}{\cos. L} = \frac{p \sin. B \sin. (D + \delta D)}{\cos. A \cos. L}$$

$$= P \sin. B \sec. L \sin. (D + \delta D) \qquad (731)$$

Again, the triangles ZPM and ZPM' give

$$\cos. Z = \frac{\sin. L - \cos. B \sin. (A + p)}{\sin. B \cos. (A + p)} = \frac{\sin. L' - \cos. B \sin. A}{\sin. B \cos. A} \qquad (732)$$

whence

$$\sin. L \cos. A - \sin. L' \cos. (A + p) = \cos. B \sin. (A + p) \cos. A$$
$$- \cos. B \sin. A \cos. (A + p)$$
$$= \cos. B \sin. p. \qquad (733)$$

But $\sin. L = \sin. (L' - \delta L) = \sin. L' \cos. \delta L - \cos. L' \sin. \delta L$

$$= \sin. L' - \cos. L' \sin. \delta L - 2 \sin. L' \sin.^2 \tfrac{1}{2} \delta L$$
$$= \sin. L' - \cos. L' \sin. \delta L - \tfrac{1}{2} \sin. L' \sin.^2 \delta L \qquad (734)$$

$$\cos. (A + p) = \cos. A - \sin. A \sin. p - \tfrac{1}{2} \cos. A \sin.^2 p, \qquad (735)$$

whence

$$\cos. L' \cos. A\, \delta L = -(\cos. B - \sin. A \sin. L')p$$
$$+ \tfrac{1}{2} \sin. L' \cos. A\, (p \sin. p - \delta L \sin. \delta L). \quad (736)$$

But

$$\sin. A = \cos. B \sin. L' + \sin. B \cos. L' \cos. (D + \delta D) \quad (737)$$

whence

$$\cos. B - \sin. A \sin. L' = \cos. B - \cos. B \sin.^2 L'$$
$$- \sin. B \sin. L' \cos. L' \sin. (D + \delta D)$$
$$= \cos. B \cos.^2 L' - \sin. B \sin. L' \cos. L' \cos. (D + \delta D), \quad (738)$$

and also $\cos. B - \sin. A \sin. L' = \cos. M' \cos. L' \cos. A, \quad (739)$
so that, for a first approximation,

$$\delta L = - \cos. M' p \quad (740)$$

$$\delta L \sin. \delta L = \cos.^2 M' p \sin. p \quad (741)$$

$$p \sin. p = \delta L \sin. \delta L = \sin.^2 M' p \sin. p$$
$$= \sin. M' \cos. L' \sin. \delta D . p$$
$$= \frac{\cos. L' \sin. B \sin. D\, \delta D}{\cos. A} p, \quad (742)$$

and (736) becomes

$$\delta L = - \cos. B \cos. L' . P + \sin. B \sin. L' . P [\cos. (D + \delta D)$$
$$+ \tfrac{1}{2} \sin. D\, \delta D]$$
$$= - \cos. B \cos. L' . P + \sin. B \sin. L' \cos. (D + \tfrac{1}{2} \delta D) P, \quad (743)$$

and formulas (731) and (743) agree with the rule in the Navigator [B. p. 404].

141. *Problem. To find the parallax in right ascension and declination.*

Parallax in right ascension and declination.

Solution. Formulas (731) and (743) may be applied immediately to this case, by putting

$B =$ the altitude of the equator $=$ the co-latitude,

$L =$ the true declination,

$L' =$ the apparent declination,

$D =$ the right ascension of the body diminished by that of the zenith $=$ the hour angle of the body.

$\delta L =$ the parallax in declination,

$\delta D =$ the parallax in right ascension.

142. *Corollary.* The value of δL may, in this case, be found by a somewhat different process, which is quite convenient when the altitude of the body is required to be calculated. Draw PN to bisect the angle MPM', and draw MH and $M'H'$ perpendicular to PN, and we have nearly.

$$\begin{aligned}\delta L &= HH' = HN + H'N' \\ &= MN \cos. N + M'N \cos. N \\ &= (MN + M'N) \cos. N = MM' \cos. N \\ &= P \cos. A \cos. N. \qquad (744)\end{aligned}$$

Now, in the triangle PZN, we have

$$PZ = \text{co-lat. } ZPN = D + \tfrac{1}{2}\,\delta D,$$

and may take

$$PN = 90° - L, \; ZN = 90° - A,$$

so that PZ, ZPN, and PN are given, to find ZN and N. This method of calculating the parallax in right ascension and declination is precisely that used in [B. p. 443] for calculating from the relative parallax the corrections for right ascension and declination.

143. The *apparent diameter* of a heavenly body is the angle which its disc subtends.

144. *Problem. To find the apparent semidiameter of a heavenly body.*

Solution. Let O' (fig. 56.) be the centre of the heavenly body, A the observer, and AT the tangent to the disc of the body. The angle TAO' is the apparent semidiameter. Let

$$R_1 = O'T$$
$$\sigma = O'AT$$
$$r = AO',$$

we have
$$\sin. \sigma = \frac{O'T}{AO'} = \frac{R_1}{r}. \quad (745)$$

Hence, by (fig. 53.), if A is the apparent altitude of the body,

$$\sin. \sigma = \frac{R_1 \sin. p}{R \cos. (A - p)} \quad (746)$$

$$\sigma = \frac{R_1}{R} p \sec. (A - p). \quad (747)$$

145. *Corollary.* If Σ is the horizontal semidiameter, we have

$$\Sigma = \frac{R_1}{R} P, \quad (748)$$

which is also the semidiameter, as seen from the earth's centre.

Now
$$R_1 = 0.2725\, R \quad (749)$$
$$R = 3.67\, R_1 \quad (750)$$

whence
$$\log. \frac{R_1}{R} = 9.43537, \text{ (ar. co.)} = 0.5646, \quad (751)$$

Augmentation of semidiameter.

so that formula (748) agrees with [B. p. 443. No. 10 of the Rule].

146. *Corollary.* If $\delta\,\sigma$ is the augmentation of the semidiameter for the altitude A, we have by (747), and putting $A' = A - b$,

$$\sigma = \frac{p\,R_1}{R\cos.(A+p)} = \frac{P R_1 \cos.(A'-p)}{R\cos.A'}$$

$$= \Sigma\,\frac{\cos.(A'-p)}{\cos.A'}$$

$$= \Sigma + \Sigma\,\frac{\sin.p\,\sin.A'}{\cos.A'}$$

$$= \Sigma + \Sigma\,\sin P\,\sin.A \qquad (752)$$

$$\delta\,\sigma = \Sigma\,\sin.P\,\sin.A = \Sigma\,P\,\sin.1''\,\sin.A$$

$$= \frac{R_1}{R}\,P^2\,\sin.1''\,\sin.A. \qquad (753)$$

Now for the mean horizontal parallax of 57′ 30″, we have

$$\log.\frac{R_1}{R}\,P^2\,\sin.1'' = 1.19658 \qquad (754)$$

$$\frac{R_1}{R}\,P^2\,\sin.1'' = 15.72, \qquad (755)$$

agreeing very nearly with the explanation to Table XV of the Navigator.

147. *Corollary.* The augmentation can also be calculated without determining the altitude. Thus, from (752)

$$\delta\,\sigma = \Sigma\left(\frac{\cos.A}{\cos.A'} - 1\right). \qquad (756)$$

But from (fig. 55.) and (731)

$$\cos. A = \sin. ZM' = \frac{\sin.(D + \delta D).\cos.(L - \delta L)}{\sin. Z} \quad (757)$$

$$\cos. A' = \sin. ZM = \frac{\sin. D \cos. L}{\sin. Z} \quad (758)$$

$$\frac{\cos. A}{\cos. A'} - 1 = \frac{\sin.(D + \delta D).\cos.(L - \delta L)}{\sin. D \cos. L} - 1$$

$$= \frac{\cos. D \cos.(L - \delta L) \delta D}{\cos. L. \sin. D} + \frac{\cos.(L - \delta L)}{\cos. L} - 1$$

$$= \frac{P.\cos. D.\sin. B \sin.(D + \delta D)}{\cos. L \sin. D} + \frac{\cos.(L - \delta L)}{\cos. L} - 1$$

Now the latitude of the moon is so small, that, in the first term, we may put

$$\cos. L = 1, \quad (760)$$

which gives by (756), and putting

$$H = \Sigma P. \cos. D \sin. B \quad (761)$$

$$H' = \Sigma \left(\frac{\cos.(L - \delta L)}{\cos. L} - 1 \right) \quad (762)$$

$$\delta\sigma = H + H \cos. D \,\delta D + H'$$

$$= H + H.P.\cos. D \sin. B + H'$$

$$= H + \frac{H^2}{\Sigma} + H'. \quad (763)$$

Now we have by (761) and (762)

$$H = \tfrac{1}{2} \Sigma. P. [\sin.(B + D) + \sin.(B - D)] \quad (764)$$

$$H' = \Sigma (\text{tang.} L. \delta L + \cos. \delta L - 1), \quad (765)$$

and formulas (763 to 765) agree with the method of calculating the augmentation of the semidiameter given in Table

XLIV of the Navigator. The three first parts of this table are calculated for the value of Σ,

$$\Sigma = 16' = 960'',$$

whence $\frac{1}{2}\Sigma . P = 8''.18.$

The fourth part of the table is the correction which arises from the difference between the actual value of Σ and that assumed in the three former parts. If we put

$$\delta' \sigma = \text{the value of } \delta \sigma \text{ for } \Sigma = 16',$$

we have, by (755) and (748),

$$\delta \sigma : \delta' \sigma = \Sigma^2 : (16')^2 \qquad (766)$$

$$\delta \sigma = \frac{\Sigma^2}{256} \delta' \sigma$$

$$= \delta' \sigma + \left(\frac{\Sigma^2}{256} - 1\right) \delta' \sigma$$

$$= \delta' \sigma + \frac{\Sigma^2 - 256}{256} \delta' \sigma$$

$$= \delta' \sigma + \frac{(\Sigma + 16)(\Sigma - 16)}{256} \delta' \sigma, \qquad (767)$$

as in the explanation of this table.

148. Examples.

1. Find a planet's parallax in altitude, when its horizontal parallax is 25″, and its altitude 30°.

Ans. 22″.

2. Find the moon's parallax in latitude and longitude, when her horizontal parallax is 59′ 10″.3; her latitude 3° 7′ 19″ S.,

Parallax in latitude and longitude.

her longitude 44° 36′ 16″; the altitude of the nonagesimal 37° 56′ 14″, its longitude 25° 27′ 16″, the latitude of the place 43° 17′ 18″ N.

Solution.

Reduced parallax = 59′ 10″.3 — 5″.3 = 59′ 5″ = 3545″

Reduced latitude = 43° 17′ 18″ — 11′ 27″ = 43° 5′ 51″

D = 44° 36′ 16″ — 25° 27′ 16″ = 19° 9′

3545		3.54962			3.54962		3.550
37° 56′ 14″	sin.	9.78873		cos.	9.89691	sin.	9.789
3° 7′ 19″	sec.	0.00064	3° 7′ 19″	cos.	9.99936		
		3.33899	46′ 32″		3.44589		
19° 9′	sin.	9.51593	3° 53′ 51″		9.99899		
12′		2.85492	46′ 30″		3.44552		
19° 21′	sin.	9.52027	3° 53′ 41″			sin.	8.831
δD = 12′ 3″		2.85926		19° 15′		log.	9.975
19° 21′ 3″			—2′ 20″				2.145

δL = 44′ 10″

3. Find the moon's parallax in latitude and longitude, when her horizontal parallax is 60′ 5″.9; her latitude 1° 30′ 12″ N., her longitude 130° 17′, the altitude of the nonagesimal 85° 14′, its longitude 125° 17′, the latitude of the place 46° 11′ 28″.4 N.

Ans. Parallax in longitude = 5′ 18″

Parallax in latitude = 3′ 30″.5.

Augmentation of semidiameter.

4. Calculate the parts of Table XLIV, when the argument of the first part is $3^s\ 19° = 109°$; that of the second $12''.4$, the moon's true latitude $1°\ 20'$ N., the moon's parallax in latitude $50'$, the sum of the three first parts $13''$, and the moon's horizontal semidiameter $14'\ 50''$.

Solution. $8''.1845 \sin. 109° = 7''.74 =$ Part I.

$$\text{Part II} = \frac{(12''.4)^2}{960''} = 0''.16.$$

$$\text{Part III} = 960'' [\sin. 50' \text{ tang. } 1° 20' - 1 + \cos. 50']$$

$$= 960'' [\sin. 50' \text{ tang. } 1° 20' - 2 \sin.^2 25']$$

$$= 960'' [0.00023] = 0''.22.$$

$$\text{Part IV} = -13'' \times \frac{30' 50'' \times 1' 10''}{256'} = -\frac{13'' \times 30.83 \times 1.17}{256}$$

$$= -1''.83.$$

5. Calculate the parts of Table XLIV, when the argument of the first part is $2^s\ 16°$, that of the second $15.''5$, the moon's true latitude $3°$ S., the moon's parallax in latitude $30'$, the sum of the three first parts $11''$, and the moon's horizontal semidiameter $15'\ 20''$.

Ans. Part I $=$ $7''.94$

Part II $=$ $0\ .25$

Part III $= -0\ .48$

Part IV $= -0\ .90$

6. Calculate the number of Table XV, when the altitude is $45°$.

Ans. $11''$.

Augmentation of semidiameter.

7. Calculate the augmentation of the moon's semidiameter in Example 2; when the horizontal semidiameter is 16′ 50″.

Solution. Part I $= 6''.87 + 2''.58 =$ 9″.45

Part II $=$ 0 .09

Part III $=$ —1 .02

sum $=$ 8″.52

Part IV $=$ 0 .91

augmentation $=$ 9″.43

8. Calculate the augmentation of the moon's semidiameter in Example 3, when the horizontal semidiameter is 15′ 30″.

Ans. 15″.52.

CHAPTER XII.

ECLIPSES.

149. A solar *eclipse* is an obscuration of the sun, arising from the moon's coming between the sun and the earth; and occurs therefore at the time of new moon.

It is *central* to an observer, when the centre of the moon passes over the sun's centre. It is *total*, when the moon's apparent disc is larger than the sun's, and totally hides the sun. It is annular, when the moon's apparent disc is smaller than the sun's, but is wholly projected upon the sun's disc.

The *phase* of an eclipse is its state as to magnitude.

150. An *occultation of a star or planet* is an eclipse of this star or planet by the moon.

A *transit* of Venus or Mercury is an eclipse of the sun by one of these planets.

151. *Problem. To find when a solar eclipse will take place.*

Solution. Let O (fig. 57.) be the sun's centre, and O_1 the moon's centre at the time of new moon, and let

$$\beta = \text{the latitude of the moon at new moon}$$
$$= OO_1.$$

When a solar eclipse will happen.

Let ON be the ecliptic, and N the moon's node, so that NO_1 is the moon's path. Let

$N =$ the inclination of the moon's orbit to the ecliptic;

Draw OP perpendicular to the moon's orbit, and if, when the moon arrives at P, the sun arrives at O', the least distance of the centres of sun and moon is nearly equal to $O'P$. Now the triangle OPO' gives

$$OP = \beta \cos. N = \beta - \beta (1 - \cos. N)$$
$$= \beta - 2\beta \sin.^2 \tfrac{1}{2} N = \beta - \tfrac{1}{2} \beta \sin.^2 N$$

$n =$ ratio of the sun's mean motion divided by the moon's

$$= \tfrac{1}{12} \text{ nearly}, \quad (768)$$

we have $OO' = n \times O_1 P = n \beta \sin. N.$

Draw $O'R$ perpendicular to OP, and we have nearly

$$OR = OP - O'P = OO' \sin. N$$
$$= n \beta \sin.^2 N.$$

Hence

$$O'P = \beta - (\tfrac{1}{2} + n) \beta \sin.^2 N = \beta - \tfrac{7}{12} \beta \sin.^2 N. \quad (769)$$

The apparent distance of the centres of the sun and moon is affected by parallax, and the true distance is diminished as much as possible for that observer, who sees the sun and moon in the horizon, and OP vertical, in which case the diminution is equal to the difference of the horizontal parallaxes of the sun and moon. Let, then,

$P =$ the moon's horizontal parallax,

$\pi =$ the sun's horizontal parallax,

$\Delta =$ the apparent distance of the centres,

we have

$$\text{the least apparent dist.} = OP - (P - \pi)$$
$$= \beta - \tfrac{7}{12} \beta \sin.^2 N - P + \pi. \quad (770)$$

Now, an eclipse will take place, when this least apparent distance of the centres is less than the sum of the semidiameters of the sun and moon. Thus, let

$$s = \text{the moon's semidiameter,}$$

$$\sigma = \text{the sun's semidiameter.}$$

In case of an eclipse, we must have

$$\beta - \tfrac{7}{12}\beta \sin.^2 N - P + \pi < s + \sigma, \qquad (771)$$

or

$$\beta < P - \pi + s + \sigma + \tfrac{7}{12}\beta \sin.^2 N. \qquad (772)$$

152. *Corollary*. We have, by observation,

the greatest value of P	=	61′ 32″,
the least value	=	52′ 50″,
the mean value	=	57′ 11″,
the greatest value of π	=	9″,
the least value	=	8″,
the greatest value of s	=	16′ 46″,
the least value	=	14′ 24″,
the mean value	=	15′ 35″,
the greatest value of σ	=	16′ 18″,
the least value	=	15′ 45″,
the mean value	=	16′ 1″,
the greatest value of N	=	5° 20′ 6″,
the least value	=	4° 57′ 22″,
the mean value	=	5° 8′ 44″.

Now, in the last term of (772) we may put for N its mean value, and for β its mean value obtained by supposing it equal to the preceding terms, which gives

$$\beta = P - \pi + s + \sigma = 88' \, 38'' = 5318'' \qquad (773)$$

$$\tfrac{7}{12} \beta = 3102''$$

$$\sin. N = \sin. 5^\circ \, 8' \, 44'' = 0.09, \ \sin.^2 N = 0.008$$

$$\tfrac{7}{12} \beta \sin.^2 N = 25'', \qquad (774)$$

whence (772) becomes

$$\beta < P - \pi + s + \sigma + 25''. \qquad (775)$$

153. *Corollary.* If, in (775), the greatest values of P, s, and σ, and the least value of π are substituted, the limit

$$\beta < 1^\circ \, 34' \, 52''$$

is the greatest limit of the moon's latitude at the time of new moon, for which an eclipse can occur.

154. *Corollary.* If, in (775), the least values of P, s, and σ, and the greatest value of π are substituted, the limit

$$\beta < 1^\circ \, 23' \, 15''$$

is the least limit of the moon's latitude at the time of new moon, for which an eclipse can fail to occur.

155. *Problem. To find the places where a given phase of a solar eclipse is first and last seen.*

Solution. The distance of the centres of the sun and moon will first be reduced to a given apparent distance Δ, at that place where the moon is vertically above the sun and the lower limb of the moon just beginning to rise. Let

P' = the relative horizontal parallax of the sun and moon,

$$= P - \pi, \qquad (776)$$

in which it is advisable to take for P its reduced value for the latitude of 45°, because the latitude of the required place is not known.

For the time of new moon, let

$D =$ the moon's declination,

$d =$ the sun's declination,

$R =$ the diff. of right ascension of sun and moon,

$=$ the moon's right ascension — the sun's,

$D_1 =$ the relative hourly motion in declination,

$=$ the moon's motion — the sun's,

$R_1 =$ the relative motion in right ascension.

Let S (fig. 58.) be the sun, M the moon, MM' the moon's relative path, that is, the path which it would describe if the sun were stationary, and the moon's motion were the relative motion; let SP be perpendicular to MM', and N be the north pole. The zenith of the place is in the line SMZ, which joins the centres of the sun and moon, and at a distance SZ of about 90° from them. Let

$$i = \text{the angle } CSP$$

$$k = MSP$$

$$p = SP.$$

Join NP and draw PR perpendicular to NS, we have

$$RPC = i$$

$$\frac{PNR}{CR} = \frac{R_1}{D_1} = \frac{PNR}{PR \tan. i}$$

$$= \frac{1}{\sin. NR \tan. i} = \frac{1}{\cos. D \tan. i}$$

whence

$$\tan. i = \frac{D_1}{R_1 \cos. D} \qquad (777)$$

$$p = CS . \cos. i = (D - d) \cos. i \qquad (778)$$

$$PR = p \sin. i, \; CR = p \sin. i \tan. i, \; PC = p \tan. i.$$

Let then

$t =$ the interval between the moon's passing from P to C,

$$= \frac{CR}{D_1} = \frac{p \sin. i}{D_1} \tan. i, \qquad (779)$$

let

$$c = \frac{p \sin. i}{D_1} \times 3600'' \qquad (780)$$

$$t \text{ (in seconds)} = c \tan. i. \qquad (781)$$

Again,

let $\Delta' = MS =$ the true dist. of centres of sun and moon,

we have $$\Delta' = \Delta + P' \qquad (782)$$

$$\cos. k = \frac{p}{\Delta'} \qquad (783)$$

$$MP = p \tan. k, \qquad (784)$$

let $\tau =$ time of describing MP,

we have $$\tau = \frac{t . MP}{PC} = c \tan. k \qquad (785)$$

$$a = MSN = -i \mp k, \qquad (786)$$

the positive values of a being reckoned towards the east, so that the upper sign corresponds to the beginning, the lower to the end of the eclipse.

Finally, if $L =$ the latitude $= 90° - ZN$

$h =$ the hour angle after noon $= ZNS$,

the triangle ZNS gives, by § 39,

$$\sin. L = \cos. d \cos. a \tag{787}$$

$$\tan. h = -\frac{\tan. a}{\sin. d}. \tag{788}$$

156. *Corollary.* The value of $\varDelta$ is for the beginning or ending of an eclipse,

$$\varDelta = s + \sigma; \tag{789}$$

for the beginning or ending of total darkness in a total eclipse,

$$\varDelta = s - \sigma; \tag{790}$$

for the forming or breaking up of the ring in an annular eclipse,

$$\varDelta = \sigma - s; \tag{791}$$

for the central phase,

$$\varDelta = 0. \tag{792}$$

157. *Problem. To find the places for which a given phase of the eclipse is seen at sunrise or sunset.*

Solution. Let M (fig. 59.) be the centre of the true moon, at any time after the first formation of the phase $\varDelta$ and before its end, S that of the sun, m that of the apparent moon affected by relative parallax. Since the sun and moon are in the horizon, we have

$$Mm = P',$$

also $$mS = \varDelta.$$

Places where eclipse begins at sunrise or sunset.

The zenith Z is in the line Mm, at the distance

$$ZS = 90°$$

from S, let N be the north pole. Join MN, and draw Ng perpendicular to NS, and the right triangle NMG gives, by putting

$$D_0 = \text{the declination of } g$$

$$\cos R = \text{tang.} Ng \text{ cotan.} MN$$

$$= \text{tang.} D \text{ cotan.} D_0 = \frac{\tan. D}{\tan. D_0}, \quad (793)$$

whence, by (287) and (52),

$$\frac{\sin.(D_0 - D)}{\sin.(D + D_0)} = \frac{1 - \cos. R}{1 + \cos. R} = \text{tang.}^2 \tfrac{1}{2} R, \quad (794)$$

or, since $D_0 - D$ and R are small,

$$D_0 - D = \tfrac{1}{4} R^2 \sin. 1'' \sin. 2D \quad (795)$$

$$D_0 = D + \tfrac{1}{4} R^2 \sin. 1'' \sin. 2D. \quad (796)$$

Let now, $x_0 = gS, \quad y_0 = Mg, \quad S = MSg,$

and we have $x_0 = D_0 - d$ (797)

$$y_0 = R . \cos. D_0 \quad (798)$$

$$\tan. S = \frac{y_0}{x_0} \quad (799)$$

$$\Delta' = x_0 \sec. S. \quad (800)$$

Now since ZM and ZS are nearly quadrants, they are nearly parallel at their extremities M and S, so that if

$$b = MSZ = m MS, \quad (801)$$

and $$q = \frac{P' + \Delta}{2}, \quad q' = \frac{P' - \Delta}{2}, \quad (802)$$

we have $$\sin.\tfrac{1}{2}b = \sqrt{\frac{(q - \frac{1}{2}\Delta')(\frac{1}{2}\Delta' - q')}{P'.\Delta}}, \tag{803}$$

whence the triangle NZS gives

$$ZNS = S \mp m \tag{804}$$

$$\sin. L = \cos.(S \mp m)\cos. d, \tan. h = -\tan.(S \mp m)\operatorname{cosec} d. \tag{805}$$

158. *Corollary.* Since Mm may be taken on either side of MS, there are two places for each place M, except when

$$\Delta' = \Delta + P', \tag{806}$$

which corresponds to the beginning or to the ending of the phase, or when

$$\Delta' < P' - \Delta. \tag{807}$$

159. *Corollary.* When the nearest approach Δ' of the true centres is less than $P' - \Delta$, the places at which the phase Δ are seen in the horizon are upon two different oval curves.

160. *Corollary.* When the nearest approach Δ' of the true centres is greater than $P' - \Delta$, the places at which the phase Δ are seen in the horizon are all upon one curve, which intersects itself, and is formed like a figure 8 much distorted; and in this case this curve is the northern or the southern boundary of the eclipse.

161. *Corollary.* The values of x_0 and y_0 might be found more easily, but less accurately, by the formulæ

$$\text{tang. } k = \frac{t}{c} \tag{808}$$

$$\Delta' = p \sec. k \tag{809}$$

Limits of a solar eclipse on the earth.

$$S = -i \mp k \tag{810}$$

$$x = \Delta' \cos. S, \; y_0 = \Delta' \sin. S, \tag{811}$$

and from the approximate value of L, obtained by this process, an accurate value of P' may be found, which may be used in the calculation by the process before given.

162. *Problem. To find the curves of extreme northern and southern places at which a phase is seen.*

Solution. When the nearest approach is greater than $P' - \Delta$, one of the limiting curves is, as in § 130, the northern or southern portion of the rising and setting curve, accordingly as the moon passes to the north or to the south of the sun. The other limiting curve consists of those places, at which the nearest approach of the apparent centres is equal to Δ; and these are the places which compose both the limiting curves, when the nearest approach is less than $P' - \Delta$. The eastern and western limiting curves are always those of rising and setting, and at the points where the rising and setting curves cease to be the limiting curves, the phase Δ is one of nearest approach, and at the same time is in the horizon. We have, then, only to consider at present the places where the phase Δ is one of nearest approach.

For this purpose, let M (fig. 60.) be the true moon's centre, m the apparent relative moon's centre, S the sun's centre, N the north pole, Z the zenith of the place; draw $m\,r$, Mg perpendicular to NS. Let

$$D' = \text{the declination of } m$$

$$R' = m\,N\,S$$

$$z = Zm, \; M = NmZ, \; h = ZNS$$

$$MNm = \text{relative parallax in right asc.} = R - R'$$
$$= P' \cos. L \sec. D \sin. (h - R') \qquad (812)$$

$$x = mh = D - D' = \text{relative paral. in dec.} = P' \sin. z \cos. M =$$
$$= P' [\sin. L \cos. D' - \cos. L \sin. D' \cos. (h - R')] \quad (813)$$
$$y = hM = (R - R') \cos. D = P' \cos. L \sin. (h - R')$$
$$= P' \sin. z \sin. M \quad (814)$$

$$x_0 = Sg, \quad y_0 = Mg.$$

Now, by the diurnal motion, the angle h will increase for the instant δt, of an hour, by the quantity

$$15° \, \delta t,$$

and the changes in the other terms of x and y will be too small to be sensible in these small quantities; so that the increments of x and y will be, by (13) and (15),

$$\delta x = 15° \, P' \delta t \cos. L \sin. D' \sin. (h - R') = 15° \, y \, \delta t \sin. D' \quad (815)$$
$$\delta y = 15° \, P' \delta t \cos. L \cos. (h - R') \quad (816)$$
$$= 15° \, P' \delta t (\cos. z \cos. D' - \sin. z \sin. D' \cos. M)$$
$$= 15° \, P' \delta t \cos. z \cos. D' - 15° \, x \, \delta t \sin. D'. \quad (817)$$

Again, if

$$u = Sr, \quad v = mr, \quad i' = rSM, \quad (818)$$

we have

$$u = \Delta \cos. i' = x - x_0, \quad v = \Delta \sin. i' = y_0 - y, \quad (819)$$

and if mm' is the apparent relative orbit of the moon, it must be perpendicular to Sm, because m is the point of nearest approach. Hence if m' is the place of the moon at the end of the instant δt, we have

$$\delta u = \delta x - \delta x_0 = - mm' \sin. i'$$
$$= 15° \, \delta t \sin. D' (y_0 - \Delta \sin. i') - \delta x_0 \quad (820)$$

Limits of a phase upon the earth.

$$\delta v = \delta y_0 - \delta y = m\,m' \cos. i' = -15^\circ P' \delta t \cos. z \cos. D'$$
$$+ 15^\circ (x_0 + D \cos. i')\, \delta t \sin. D' + \delta y_0, \quad (821)$$

whence $\quad m\,m' \sin. i' = -\delta u \cos. i' = d\,v \sin. i'$

$$= -15^\circ \delta t \sin. D' (y_0 \cos. i' - \varDelta \sin. i' \cos. i') + \delta x_0 \cos. i' \quad (822)$$
$$= \quad 15^\circ \delta t \sin. D' (x_0 \sin. i' + \varDelta \sin. i' \cos. i')$$
$$+ \delta y_0 \sin. i' - 15^\circ P' \delta t \sin. i' \cos. z \cos. D'.$$

Now D' differs so little from d, that d may be substituted for it in this equation, and we have also

$$\frac{\delta x_0}{\delta t} = \text{the hourly motion in relative declination} = D_1$$

$$\frac{\delta y_0}{\delta t} \sec. D = \text{the hourly motion in relative dec.} = R_1,$$

and if we put

$$A = \frac{R_1 \cos. D}{15^\circ \sin. 1''}, \quad B = \frac{D_1}{15^\circ \sin. 1''} \quad (823)$$

(822) becomes, by dividing by $15^\circ \delta t \sin. 1''$,

$$P' \sin. i' \cos. z \cos. d = (A + x_0 \sin. d) \sin. i'$$
$$- (B - y_0 \sin. d) \cos. i'. \quad (824)$$

Let now λ and ν be so taken, that

$$A + x_0 \sin. d = \lambda P' \cos. z \cos. d \cos. \nu$$
$$B - y_0 \sin. d = \lambda P' \cos. z \cos. d \sin. \nu, \quad (826)$$

and (824) becomes

$$\cos. z \sin. i' = \lambda \sin. (i' - \nu) \quad (827)$$

$$\cos. z = \frac{\lambda \sin. (i' - \nu)}{\sin. i'} = \lambda \cos. \nu - \lambda \sin. \nu \cot. i'. \quad (828)$$

To find i', its value may be, at first, assumed as equal to i, as it is nearly, because the true relative orbit PM is nearly parallel to the apparent relative orbit $m\,m'$. Hence u and v are found by (819), and thence

$$D' = d \mp u \tag{829}$$

$$R' = \pm\, v \sec. D' \tag{830}$$

$$D_0 = D + \tfrac{1}{4}(R - R')^2 \sin. 2D \tag{831}$$

$$y = (R - R') \cos. D_0 \tag{832}$$

$$x = D_0 - D' \tag{833}$$

$$\tan. M = \frac{y}{x} \tag{834}$$

$$\sin. z = \frac{x}{P' \cos. M} \tag{835}$$

and from this value of z, i' may be found by means of (828), which gives

$$\cot. i' = \cot. \nu - \frac{\cos. z}{\lambda \sin. \nu}, \tag{836}$$

or if φ is taken, so that

$$\sin. \varphi = \sqrt{\frac{\cos. z}{2\lambda \cos. \nu}} \tag{837}$$

$$\cot. i' = \cot. \nu - \frac{2\lambda \cos. \nu \sin.^2 \varphi}{\lambda \sin. \nu} \tag{838}$$

$$= \cot. \nu\,(1 - 2\sin.^2 \varphi) = \cot. \nu \cos. 2\varphi, \tag{839}$$

whence new values of z and M may be computed. Then the triangle NMZ can be solved by the usual process, and will give the values of

$$h - R' = ZNM \text{ and } L = 90° - NZ.$$

163. *Corollary.* There are two points, m and m', at which we should have

$$\Delta = S\,m = S\,m',$$

and therefore two zeniths, Z and Z', which correspond to the two values (829–835).

164. *Corollary.* If t is the time of the phase Δ counted from the middle of the eclipse,

$$p = SP$$

the perpendicular upon the orbit, and

$$k = MSP, \; k' = M\,m\,S,$$

we have, nearly, by (785),

$$\tan. k' = \frac{MP}{P\,m} = \frac{p \tan. k}{p \pm \Delta} = \frac{p\,t}{c\,(p \pm \Delta)} \qquad (840)$$

$$\sin. Z = \frac{M\,m}{P'} = \frac{p \pm \Delta}{P' \cos. k'} \qquad (841)$$

$$M = (-i) \mp k', \qquad (842)$$

which gives a rough method of computing Z and M.

165. *Problem. To find the duration of a phase upon the earth.*

Solution. At the first and last points we have

$$Z = 90°, \quad PM = P',$$

$$\cos. k' = \frac{p \pm \Delta}{P'} \qquad (843)$$

$$t = \text{semiduration} = c \tan. k \qquad (844)$$

$$= \frac{c\,.\,PM}{PS} = \frac{c\,.\,P'}{p} \sin. k'. \qquad (845)$$

166. *Problem. To find the places where the eclipse is central.*

Solution. For these places m and S coincide, so that

$$MS = \Delta = P' \sin. Z \tag{846}$$

$$\sin. Z = \frac{\Delta}{P'}, \tag{847}$$

so that in the triangle ZSN, the two sides ZS, NS, and the included angle S are given to find NZ and ZNS.

167. *Corollary.* For one place the eclipse will be central at noon, and for this place we have, obviously,

$$\Delta = \text{diff. dec.}$$

$$\sin. Z = \frac{\Delta}{P'} \tag{848}$$

$$L = d + Z \tag{849}$$

west long. of place = app. Greenw. time of cent. eclipse.

168. *Problem. To calculate the time of the beginning or ending of a given phase of a solar eclipse for a given place.*

Solution. Find for a supposed time near the required time, such as the time of new moon, the relative parallaxes in right ascension and declination of the sun and moon, and their relative right ascension and declination. Hence their apparent relative right ascension and declination is found by simple addition or subtraction.

Let $D =$ their apparent relative declination,

$R =$ their apparent relative right ascension,

$d =$ the sun's declination,

$W =$ the distance apart of their apparent centres,

$\varDelta =$ the phase,

$A =$ the angle which W makes with the parallel of declination, and we have

$$D = W \sin. A, \; R \cos. d = W \cos. A \tag{850}$$

so that

$$\tan. A = \frac{R \cos. d}{D} \tag{851}$$

$$W = D \operatorname{cosec.} A = R \cos. d \sec. A. \tag{852}$$

If $$W = \varDelta,$$

the supposed time is that of the beginning or ending of the phase. But if W differ from $\varDelta$, find another apparent distance W' of the centres for a time a little after the former one. Then we have $W - W' : W - \varDelta =$ diff. of supposed times : the correction which is to be added to the first supposed time to obtain the required time. If this correction is large, a new computation must be made, using the time just obtained as a new supposed time.

169. *Corollary.* The time of the phase of an eclipse or occultation might also be calculated by the following process.

Let $R_1 =$ the relative hourly motion in right ascension,

$D_1 =$ that in declination ;

then let S (fig. 62.) be the centre of the sun, and M that of the moon at the supposed time, CS the hour circle, A the moon's centre at the beginning of the phase, B at the end; we have, then,

$$\tan. CSM = \tan. S = \frac{CM}{CS} = \frac{R \cos. d}{D} \tag{853}$$

$$\tan. i = \tan. CMI = \tan. FSI = \frac{CI}{CM} = \frac{D_1}{R_1 \cos. d} \tag{854}$$

$$SM = W = y \operatorname{cosec.} S = x \sec. S \tag{855}$$

$$SP = p = W \cos. PSM = W \cos. (S + i) \tag{856}$$

$$\cos. k = \cos. PSA = \cos. PSB = \frac{p}{\Delta} \tag{857}$$

$$a = ASM = S + i + k \tag{858}$$

$$b = BSM = k - (S + i). \tag{859}$$

Then let $t_1 =$ the interval of moon's passing from A to M,

$t_2 =$ the time from M to B,

and we have

$$t_1 = \frac{AM \cos. i}{y_1} = \frac{W \sin. ASM \cos. i}{y_1 \sin. MAS} = \frac{W \sin. a \cos. i}{y_1 \cos. k} \tag{860}$$

$$t_2 = \frac{W \sin. b \cos. i}{y_1 \cos. k}. \tag{861}$$

170. *Corollary.* This method may be used by substituting latitude and longitude for declination and right ascension; and in this case the sun's latitude is zero, so that the formulas agree with the rule in the Navigator [B. p. 425].

171. *Corollary.* For the beginning or end of the eclipse the phase is

$\Delta =$ the sum of the horizontal semidiameters of the sun and moon increased by the augmentation of the moon's semidiameter. (862)

For the beginning or end of total darkness in a total eclipse,

$\Delta =$ diff. of semidiam. + aug. of ☽'s semidiam. (863)

For the formation or breaking up of the ring in an annular eclipse,

$\Delta =$ diff. of semidiam. — aug. of ☽'s semidiam. (864)

172. *Problem. To find the greatest magnitude of the eclipse at any place.*

Solution Let

$D =$ the relative apparent declination at the beginning of the phase Δ,

$A =$ the angle which the line joining the centre of the apparent sun and moon makes with the circle of declination at this time,

D' and $A' =$ the values of D and A at the end of the phase Δ,

$\Delta_0 =$ the nearest approach of the centres,

we have, by (fig. 61.), in which MNM' is the moon's apparent relative orbit, S the sun, SD the circle of declinations,

$$SM = SM' = \Delta, \quad SN = \Delta_0,$$

$$\sin. A = \frac{D}{\Delta}, \ \sin. A' = \frac{D'}{\Delta'}, \qquad (865)$$

$$\Delta_0 = \Delta \cos. \tfrac{1}{2} (A' - A). \qquad (866)$$

173. The calculation of occultations is the same as that of solar eclipses, except that the star has no parallax, and its disc is insensible. The calculation of transits of planets over the disc of the sun is the same as that of a solar eclipse, except that the planet is to be substituted for the moon.

174. *Problem. To find when a lunar eclipse will happen.*

Solution. The solution is the same as in § 168, except that the semidiameter of the earth's shadow at the distance of the moon is to be substituted for that of the sun; and the change in the position and apparent magnitude of the moon from parallax may be neglected, because when the earth's shadow falls upon the moon, the moon is eclipsed to all who can see it. Now if S (fig. 61.) is the sun, E the earth, GF the semidiameter of the sun's shadow at the moon, we have

$$\text{the app. semi.} = FEG = EFL - EIF = P - EIF$$
$$= P - (KES - EKI)$$
$$= P - \sigma + \pi,$$

or rather, this would be the apparent semidiameter, if it were not for the earth's atmosphere, which increases the breadth of the shadow about $\frac{1}{60}$th part; so that

$$\text{the app. semidiam.} = \tfrac{61}{60}\,(P - \sigma + \pi),$$

and therefore, in order that an eclipse must happen, we must have, by (762),

$$\beta = \text{the latitude at the time of full moon},$$
$$\beta < \tfrac{61}{60}\,(P + \pi - \sigma) + s + \tfrac{7}{12}\,\beta\,\sin.^2 N. \qquad (867)$$

Lunar eclipse.

175. *Corollary.* In the last term of (867), we may put for N its mean value, and for β its mean value obtained by supposing it equal to the preceding terms, which gives

$$\beta = 57' \, 35'' = 3455'', \quad \tfrac{7}{12} \beta = 2015''$$
$$\sin.^2 N = 0.008, \quad \tfrac{7}{12} \beta \sin.^2 N = 16'',$$

whence (867) becomes

$$\beta < \tfrac{61}{60} (P + \pi - \sigma) + s + 16''. \qquad (868)$$

176. *Corollary.* If, in (868), the greatest values of P, π, and s are substituted, and the least value of σ, the limit

$$\beta < 63' \, 45''$$

is the greatest limit of the moon's latitude at the time of full moon, for which an eclipse can occur.

177. *Corollary.* If, in (868), the least values of P, π, and s are substituted, and the greatest value of σ, the limit

$$\beta < 51' \, 57''$$

is the least limit at which an eclipse can fail to occur.

178. *Problem. To calculate when a given phase of a lunar eclipse will occur.*

Solution. Let

$D =$ the relative declination of the moon referred to the centre of the shadow at time of full moon,

$d =$ the declination of the centre of the shadow $= -$ sun's declination,

$R =$ the relative right ascension,

$=$ ☽'s R. A. $-$ ☉'s R. A. $\pm$ 180°,

$W =$ the dist. of centres at this time,

$\Delta =$ the given phase.

Find W, as in the case of the solar eclipse, by the equations

$$\text{tang.}\, A = \frac{D}{R \cos.\, d} \tag{869}$$

$$W = D \,\text{cosec.}\, A. \tag{870}$$

In the same way, find another value of W' for another time a little different from that of full moon, and finish the computation as in the case of the solar eclipse.

179. *Corollary.* The same method might be used if longitudes and latitudes were substituted for right ascensions and declinations.

178. *Corollary.* This eclipse might also be calculated by the process of § 169, and the result is the same as the calculation in [B. p. 417].

179. *Corollary.* At the beginning or end of the eclipse, we have

$$\Delta = \tfrac{61}{60}\,(P + \pi - \sigma) + s. \tag{871}$$

180. *Problem. Given the latitude of the place and the apparent time of the beginning or end of a phase of a solar eclipse, to find the longitude of the place.*

Solution. From the supposed longitude of the place, find the Greenwich time of the observation, and for this time find the places of the sun and moon, and their relative parallaxes; and hence the Greenwich time of the phase. The difference between the Greenwich time and the observed time is the longitude.

Longitude from solar eclipse.

181. *Corollary.* Instead of calculating the parallaxes for the rapidly varying positions of the moon, the moon may be supposed to be at the distance of the stars, while the sun is supposed to be at a distance equal to the real distance of the moon. But in this case the effect of parallax, by bringing the sun's limb into contact with the moon's, can only be obtained by supposing the observer to be in the situation of his antipodes, so that the parallax may be reversed. In this case the sun's diameter must be diminished, just as the moon's diameter is really increased.

182. *Corollary.* The sun changes its place so slowly, that its right ascension, corrected for relative parallax, as in the preceding corollary, cannot differ much from that of the moon, at the time of conjunction in right ascension. For a time, then, when the moon's true right ascension is nearly that of the sun's corrected for relative parallax, find the values of

$R =$ the relative right ascension $+$ parallax in R. A.

$D =$ the relative declination $+$ correction for declin.

$d_0 =$ the sun's declination corrected for relative paral.

$R_1 =$ the hourly variation of R,

$D_1 =$ the hourly variation of D,

$D_0 =$ the diff. between the values of D at the time t, and of the time of app. conjunction in right ascension,

$t_0 =$ the time of apparent conjunction.

Then we have, by (fig. 62.), if S is the sun's place corrected for relative parallax, and BMA the moon's true orbit.

$$t_0 = t + \frac{R}{R_1}, \quad D_0 = D + \frac{R}{R_1} D_1 = SI, \tag{872}$$

$$\text{tan. } CMI = \text{tang. } i = \frac{D_1}{R_1 \text{ cos. } d} \tag{873}$$

Longitude from solar eclipse.

$$p = PS = D_0 \cos. i \tag{874}$$

$$\cos. FSA = \cos. a = \frac{p}{\varDelta} = \frac{D_0 \cos. i}{\varDelta} \tag{875}$$

$$AI = \frac{\varDelta \sin. ASI}{\sin. AIS} = \frac{\varDelta \sin. (a + i)}{\cos. i}, \tag{876}$$

interval of moon's passing from A to I

$$= \frac{AI \cos. i}{R_1 \cos. d} = \frac{A \sin. (a+i)}{R_1 \cos. d}, \tag{877}$$

$$\text{time of obs. phase} = t_0 + \frac{\sin. (a+i)}{R_1 \cos. d}, \tag{878}$$

which agrees with [B. p. 463, from No. 12 to the end of the rule].

183. *Corollary.* The diminution of the semidiameter required by § 181 is, by (753),

$$\delta \sigma = \sigma . P . \sin. 1'' \sin. A, \tag{879}$$

or if σ and P are expressed in minutes,

$$\delta \sigma = 3600 . \sigma . P . \sin. 1'' \sin. A$$

$$= \tfrac{1}{100} . \sigma . P . 360000 \sin. 1'' \sin. A. \tag{880}$$

But $$360000 \sin. 1'' = 1.76, \tag{881}$$

whence 1.76 sin. A is the factor of the table [B. p. 443].

184. *Corollary.* The latitude of the moon changes so slowly, that its latitude at the time of the phase may be regarded as the same with its latitude at the supposed time. The sun is also in the ecliptic; so that if we put

$L =$ the moon's apparent latitude,

$\varDelta =$ the apparent relative longitude,

we have

$$\Delta = \sqrt{(\Delta^2 - L^2)} = \sqrt{[(\Delta + L)(\Delta - L)]}, \quad (882)$$

from which the apparent and true longitude of the moon at the time of conjunction may be computed, as in [B. p. 413], and thence the longitude of the place.

185. *Corollary.* If both the beginning and end of the eclipse are observed, and if

R = the app. relative right ascen. at the beginning,

R' = that at the end,

D = the apparent relative dec. at the beginning,

D' = that at the end,

we have

$$\text{tang. } i = \frac{D' - D}{(R' - R) \text{ cos. } d}.$$

The dist. gone by the moon $= E = (R' - R)$ cos. d sec. i.

Then, if A (fig. 62.) is the place of the moon at the beginning, B at the end of the eclipse, the three sides AS, AB, and BS of the triangle ASB are given, and hence AF, ASF can be found, whence

$$a = ASI = ASF + FSI,$$

and $$R = AI.\text{ cos. } i \text{ sec. } d = \frac{AS.\text{ sin. } a}{\text{sin. } AIS} \text{ cos. } i \text{ sec. } d$$

$$= \Delta \text{ sin. } a \text{ sec. } d,$$

whence the apparent right ascension of the moon at the instant of the commencement of the eclipse can be found, and thence its true right ascension and the Greenwich time of the beginning.

186. *Corollary.* The method of the preceding corollary is the same as that [B. p. 407]; except that latitudes and longitudes are used, and the sun is in the ecliptic.

187. *Corollary.* The calculation of the longitude by means of occultations is the same as that by means of solar eclipses, as in [B. pp. 410, 414, 446].

188. Examples.

1. To find when and where the different phases of the eclipse of Sept. 18, 1838, begin and end upon the earth.

Solution.

Greenw. mean t. of new moon		=	$7^h\ 56^m\ 38^s.2$	
☽'s true declination	$= D$	=	$2°\ 43'\ 52''.3$	
☉'s true declination	$= d$	=	$1°\ 49'\ 15''.5$	
	$D - d$	=	$54'\ 36''.8$	$= 3276''.8$
☽'s hourly motion in dec.		=	$-14'\ 10''.5$	
☉'s hourly motion in dec.		=	$-\ 0'\ 58''.3$	
relative motion in dec	$= D_1$	=	$-13'\ 12''.2$	$= -792''.2$
☽'s motion in right ascension		=	$26'\ 0''.5$	
☉'s motion in right ascension		=	$2'\ 14''.7$	
rel. motion in right asc.	$= R_1$	=	$23'\ 45''.8$	$= 1425''.8$

Solar eclipse of Sept. 18, 1838.

D_1		2.89883_n			ar. co	7.10117_n
R_1	ar. co.	6.84594				
D	sec.	0.00049	$D-d$	3.51545		3.55630
$i=-29^\circ\,5'\,4''$	tan.	9.74526_n	cos.	9.94147	sin.	9.68673_n
$p=2863''.7$				3.45692		3.45692
c		3.80112				3.80112
$t=-3518^s$		3.54638_n				

$=-58^m\,38^s$. time of middle $=7^h\,56^m\,38^s-t=8^h\,55^m\,16^s$.

Now for the phases, we have

☽'s equatorial horizon. paral.	$=$	$53'\,53''.7$
☉'s equatorial horizon. paral.	$=$	$8''.5$
Relative parallax for equator	$=$	$53'\,45''.2$
Reduction for lat. 45°	$=$	$5''.3$
Relat. par. for lat. 45°	$=P'=$	$53'\,39''.9=3219''.9$
☽'s true semidiameter	$=s=$	$14'\,41''.2$
☉'s true semidiameter	$=\sigma=$	$15'\,57''.1$

For first contact $\Delta'=P'+s+\sigma=84'\,18''.2=5058''.2$

p.		3.45692		
Δ'	ar. co.	6.29600	c	3.80112
$k=55^\circ\,31'$	cos.	9.75292	tan.	0.16314
$-i=29^\circ\;\;5'$		$\tau=9210^s=2^h\,33^m\,20^s$		3.96426

$a=-26^\circ\,26'$, m. t. of begin. $=$ time of mid. $-\tau=6^h\,21^m\,56^s$

app. t. $=6^h\,21^m\,56^s+5^m\,55^s=6^h\,27^m\,51^s=96^\circ\,58'$

$b=\;\;84^\circ\,36'$, time of end $=$ time of mid. $+\tau=11^h\,28^m\,36^s$

app. t. $=11^h\,28^m\,36^s+6^m=11^h\,34^m\,36^s=173^\circ\,39'$

Solar eclipse of Sept. 18, 1838.

a	cos.	9.95204		tan.	9.69647_n
d	cos.	9.99978		cosec.	1.49793
lat. = 63° 31′ N.	sin.	9.95182	266° 20′	tan.	1.19440

at beginning long. = 266° 20′ — 96° 58′ = 169° 22′ E.
lat. = 63° 31′ + 9′ = 63° 40′ N.

b	cos.	8.97363		tan.	1.02444
d	cos.	9.99978		cosec.	1.49793
lat = 5° 24′ N.	sin.	8.97341	90° 11′	tan.	2.52237_n

at end long. = 173° 39′ — 90° 11′ = 83° 28′ W.
lat. = 5° 24′ + 2′ = 5° 26′ N.

For central eclipse,

P		3.45692		
P'	ar. co.	6.49215	c	3.80112
k = 27° 13′	cos.	9.94907	tan.	9.71107
$-i$ = 29° 5′		$\tau = 3252^s = 54^m\ 12^s$		3.51219
a = 1° 52′				
b = 56° 18′				

time of begin. = t. of mid. $- \tau = 8^h\ 1^m\ 4^s$

app. time $= 8^h\ 1^m\ 4^s + 5^m\ 57^s = 8^h\ 7^m\ 1^s = 121° 45'$

time of end = time of middle $- \tau = 9^h\ 49^m\ 28^s$

app. time $= 9^h\ 49^m\ 28^s + 5^m\ 59^s = 9^h\ 55^m\ 27^s = 148° 52'$

a	cos.	9.99977		tan.	8.51310
d	cos.	9.99978		cosec.	1.49793
lat. = 87° 24′ N.	sin.	9.99955	134° 17′	tan.	0.01103

at beginning long. = 134° 17′ — 121° 45′ = 12° 32′ E.

Solar eclipse of Sept. 18, 1838.

$b.$	cos. 9.74417		tan.	0.17593
$d.$	cos. 9.99978		cosec.	1.49793
lat. = 33° 41′ N.	9.74395	91° 13′	tan.	1.67386

at end long. = 148° 52′ — 91° 13′ = 57° 39′ W.

lat. = 33° 41′ + 10′ = 33° 51′ N.

2. Find the places where the eclipse of Sept. 18, 1838, begins or ends in the horizon at 8^h Greenwich mean time.

Solution. $g = \frac{1}{2}(P' + \varDelta) = 4139''$

$q' = \frac{1}{2}(P' - \varDelta) = 919''.1$

time from middle eclipse $= 55^m\ 16^s = 3316^s = t$

t		3.52061			
c	ar. co.	6.19888	$p.$ 3.45692		
$k = 27°\ 40'$	tan.	9.71949	sec. 0.05273		
$-i = 29°\ \ 5'$		$\varDelta' = 3233''$	3.50965	ar. co.	6.49035
$S = \ 1°\ 15'$		$\frac{1}{2}\varDelta' - 9' = 697''.4$			2.84348
		$q - \frac{1}{2}\varDelta' = 2522''.5$			3.40183
		P'		ar. co.	6.49215
					2)19.22781
$m = 48°\ 32'$		$\frac{m}{2} = 24°\ 16'$		sin.	9.61390

$S - m = -47°\ 17'$	cos. 9.83147		tan.	0.03465_n
d	cos. 9.99978		cosec.	1.49793
lat. = 42° 41′ N.	sin. 9.83125	268° 19′	tan.	1.53158

Solar eclipse of Sept. 18, 1838.

$$\text{Lat.} = 42° \, 41' + 11' = 42° \, 52' \text{ N.}$$

$$\text{Greenwich app. time} = 8^h \, 5^m \, 57^s = 121° \, 29'$$

$$\text{long.} = 268° \, 19' - 121° \, 29' = 146° \, 50' \text{ E.}$$

and at this place the eclipse is rising.

$S + m = 49° \, 47'$	cos.	9.81002	tan.	0.07286
d	cos.	9.99978	cosec.	1.49793
Lat. $= 40° \, 12'$ N.		9.80980	$91° \, 32'$	1.57078_n

$$\text{long.} = 121° \, 29' - 91° \, 32' = 29° \, 57' \text{ W.}$$

$$\text{lat.} = 40° \, 12' + 11' = 40° \, 23' \text{ N.}$$

and at this place the eclipse is setting.

3. Find the place on the southern limit of the eclipse of Sept. 18, 1831, which corresponds to the Greenwich mean time of 8^h.

Solution. Since the altitude is not known, the increase of the moon's semidiameter is not known, but it may be supposed at first to be 6″, which is about its mean value. Hence

$$\Delta = \sigma + s + 6'' = 30' \, 43''.3 = 1844''.3$$

$$p - \Delta = 1019''.4$$

Δ		3.26583		3.26583
i	cos.	9.94147	sin.	9.68673_n
$u = 26' \, 52''$		3.20730		
$d + u = 2° \, 16' \, 7'' = D'$			sec.	0.00034
$R' = 897''.2 = 14' \, 57''$				2.95290_n

Solar eclipse of Sept. 18, 1838.

p		3.45692		
c	ar. co.	6.19888	P'. ar. co.	6.49215
$p - \Delta$	ar. co.	6.99166		3.00834
E		6.64746	W cos.	9.50049
t		3.52061		
$k' = 55° 49' 11''$		0.16807	sec.	0.25042
$-i = 29° 5' 4''$		$z = 34° 18'$	sin.	9.75091
$M = -26° 44' 7''$	cos.	9.95089	tan.	9.70219_n
$Z =$	tan.	9.83388		
$\theta = 31° 21' 2''$	tan.	9.78477	sin.	9.71623
$\theta + D = 33° 37' 9''$	tan.	9.82274	sec.	0.07947
$h - R' =$	cos.	9.97950	tan.	9.49789_n
$L = 32° 23'$	tan.	9.80224		

For a more accurate determination.

$$P' = 53' 45''.2 - 3''.2 = 53' 42'' = 3222''$$
$$s = 14' 41''.2 + 11''.6 = 14' 52''.8$$
$$\Delta = s + \sigma = 30' 49''.9 = 1849''.9$$

$15°$ sin. $1''$ ar. co.	0.58204		0.58204	
D_1	2.89883_n	R_1	3.15406	
$B = -3026$	3.48087_n	D cos.	9.99951	
$A = 5440.1$			3.73561	

For 8^h Gr. time we have			$D =$ ☽'s dec. $=$	$2° 43' 5''$
$\frac{1}{2} R . 48''$	2 log.	3.362	$d =$ ⊙'s dec. $=$	$1° 49' 12''$
$1''$	sin.	4.686	$R =$ rel. R.A. $=$	$50''.1$
$2D = 5° 26'$	sin.	8.976	$D_0 = D$	
$D_0 - D = 0$		6.924	$x = D_0 - d =$	$53' 83'' = 3233'$

Solar eclipse of Sept. 18, 1838.

R.		1.69984			
D	cos.	9.99951			
$y_0 = 50''$		1.69935	x_0		3.50961
d	sin.	8.50187			8.50187
$2''$		0.20122	$102''.7$		2.01148
$B - 3026''$			$A = 5446''.1$		0.00022
$-3028''$		3.48116_n	$5542''.8$		3.74372
$5542''.8$	ar. co.	6.25628	P'	ar. co.	6.49187
ν	tan.	9.73744_n		sec.	0.05671
$2\varphi = 58° 40'$	sec.	0.28396		λ	0.29252
$i' = -46° 25'$	tan.	0.02137_n	z	cos.	9.91703
			2	ar. co.	9.69897
$2\varphi =$	sec.				2)19.38019
$\varphi = 29° 20'$				sin.	9.69004

i'	cos.	9.83848	sin.	9.85996_n	
$\varDelta$		3.26926		3.26926	
$u = 21' 21''$		3.10774			
$d + u = 2° 10' 33''$			sec.	0.00032	
$D = 2° 43' 5''$	$-R' = -1347''.5$			3.12954	
$x = 32' 32'' = 1952''$,	$R - R' = 1297''.4$				3.11307_n
	D		cos.		9.99951
y					3.11258_n

Solar eclipse of Sept. 18, 1838.

y		3.11258_n		3.11258_n		
P'	ar. co.	6.49187	x ar. co.	6.71175		
M	cosec.	0.25566	tan.	9.82433	cos.	9.92002
z	sin.	9.86011			tan.	0.02160
$\theta =$	$41° 16' 24''$		sin.	9.81834	tan.	9.94162
$\theta + D' =$	$43° 20'$		sec.	0.13824	tan.	9.97472
$h - R' =$	$-30° 8'$		tan.	9.78091	cos.	9.93250
$h =$	$-31° 30' 8''$		$L =$	$38° 56'$ N.	tan.	9.90722
$H =$	$122°$		long. $=$	$153° 20'$ W.		

lat. $= 38° 56' + 11' = 39° 7'$ N.

4. Find where the solar eclipse of Sept. 18, 1838, is central for the Greenwich mean time of 9^h.

$t = 4^m 44^s = 284^s$				2.45332
p		3.45692	c ar. co.	6.19888
$k = 2° 34'$	sec.	0.00044	tan.	8.65220
Δ'		3.45736		
P'	ar. co.	6.49215		
Z	sin.	9.94951	tan.	0.29106
$s = 31° 39'$	tan.	9.78987	cos.	9.93007
$\theta = 59°$	sin.	9.93304	tan.	0.22113
$\theta + d = 60° 49'$	sec.	0.31193	tan.	0.25298
$h = 47° 18'$	tan.	0.03484	cos.	9.83140
	$L = 50° 32'$		tan.	0.08438

Solar eclipse of Sept. 18, 1838.

For a more accurate determination.

$P' = 53' 45''.2 - 6''.2 = 53' 39'' = 3219''$			
$D = 2° 28' 54''$, $d = 1° 48' 13''$,		D	cos. 9.99959
$R = 25' 5''.4 = 1505''.4$			3.17765
$x_0 = D - d = 40' 41'' - 2441''$		3.38757	6.61243
S	cos. 9.93013	sec. 0.06987	tan. 9.78967
Δ'		3.45744	
P'.		ar. co. 6.49228	
z	tan. 0.29208	sin. 9.94972	
$\theta = 59° 3' 24''$	tan. 0.22221		sin. 9.93332
$\theta + d = 60° 51' 37''$		tan. 0.25375	sec. 0.31252
$h = 47° 20'$		cos. 9.83100	tan. 0.03551
$L = 50° 33'$		tan. 0.08475	

lat. $= 50° 33' + 11' = 50° 44'$ N.

long. $= 47° 20' - 136° 29' = 89° 9'$ W.

Calculate the solar eclipse of September 18, 1838, for the city of New York.

Calculation for 9^h Gr. mean time, by the principles of § 182.

Gr. app. t. $= 9^h 5^m 57^s.4$ $L = 40° 42' 40'' - 11' 20'' = 40° 31' 20''$

N. Y. long. $= 4^h 56^m 4^s.5$

h $= 4^h 9^m 52^s.9$ P.M. $P' = 53' 45''.2 - 4''.6 = 53' 40''6$.

Solar eclipse of Sept. 18, 1838.

$P'. = 3220''.6$		3.50794				
$L.$	cos.	9.88090		9.88090		9.88090
15	ar. co.	8.82391				
$2h = 8^h\,19^m\,45^s.8$	sin.	9.94782				
S'		2.16057				
D	sec.	0.00041				
$2^m\,24^s.9$		2.16098				
$2h' = 8^h\,17^m\,21^s$, $h' = 4^h\,8^m\,40^s.5$, log. Ris.				4.72684	sin.	9.94661
$L - d = 38°\,43'\,7''$ N.	cos.	78022	d cos.	9.99979		
		40507		4.60753		
$22°\,2'$ N.	sin.	37515	cos.	9.96706	sec.	0.03294
M			cos.	9.83794	sin.	9.86045
			$P'.$	3.50794		
$\delta d = 34'\,15''.6$ N.				3.31294	S'	2.16057
$d' = 2°\,22'\,29''$			cos.	9.99963	sec.	0.00037
$\delta R = 2^m\,24^s.9 = 36'\,13'' = 2173''$						2.16094
$R - \delta R = -668''$				2.82478_n		
$D - d' = 6'\,26'' = 385''$			ar. co.	7.41454		2.58546
$S = -60°\,1'$			tan.	0.23895_n	sec.	0.30125
W						2.88671

Solar eclipse of Sept. 18, 1838.

$D_1 = 792''.7$		2.89911_n		
$R_1 = 1425''$	ar. co.	6.84619		
d	sec.	0.00037		
$i = -29° 6'$	tan.	9.74567_n	$W.$	2.88671
$S + i = -89° 7'$			cos.	8.18798
$p.$				1.07469

For beginning of eclipse.

$\Delta = 30' 38''.3 = 1838''.3$			ar. co.	6.73565
p				1.07469
$k = 89° 38'$	sec.	2.18966	cos.	7.81034
$W.$		2.88691		
$3600''$		3.55630		
$i =$	cos.	9.94137		
R_1	ar. co.	6.84619		
d	sec.	0.00037		
$a = 31'$	sin.	7.95508		
$t_1 = -40^m$		3.37568		

mean time at N. Y. $3^h 24^m$, Gr. m. t. $= 8^h 20^m$.

For a more accurate calculation.

Gr. mean time $= 8^h 20^m$			$h = 3^h 29^m 52^s.3$
$15 P'$ cos. L		2.21275	$d = 1° 48' 52''.9$
$2h = 6^h 59^m 44^s.6$	sin.	9.89928	$R = 555''.0$
S'		2.11203	
$D = 2° 38' 21''.1$	sec.	0.00046	
$2^m 9^s.6$		2.11249	

Solar eclipse of Sept. 18, 1838.

	L	cos.	9.88090		9.88090
$2h' = 6^h 57^m 35^s$	$h' = 3^h 28^m 47^s.5$	log. Ris.	4.58778	sin.	9.89770
$L - d = 38° 42' 27''$	N. cos. .78033	d. cos.	9.99978		
	.29407		4.46846		
$29° 5' 42''$	N. sin. .48626	cos.	9.94142	sec.	0.05858
M		cos.	9.86114	sin.	9.83718
		P'.	3.50794		
$\delta d = 34' 4''.1$ N.			3.31050	S'	2.11203
$d' = 2° 22' 57''$		cos.	9.99962	sec.	0.00038
$\delta R = 2^m 9^s 55 = 32' 23''.2 = 1943''.2$					2.11243
$R - \delta R = -1388''.2$			3.14245_n		
$D - d' = 15' 24''.1 = 924''.1$		ar. co.	7.03428		2.96572
$S = -56° 19' 32''$		tan.	0.17635_n	sec.	0.25612
$D_1 = -792.5$			2.89900_n	W.	3.22184
$R_1 = 1425.3$		ar. co.	6.84610		
d'.		sec.	0.00038		
$i = -29° 5' 48''$		tan.	9.74548_n		
$S + i = -85° 25' 20''$				cos.	8.90207
					2.12391
$\Delta = 30' 30''.8 = 1830''.8$				ar. co.	6.73736
$k = 85° 50'$				cos.	8.86127

Solar eclipse of Sept. 18, 1838.

k	sec.	1.13873
W		3.22184
3600″		3.55630
i	cos.	9.94141
R_1 cos. d	ar. co.	6.84648
$a =$ 24′ 40″	sin.	7.85583
$t_1 = -12^m\ 28^s$		2.56059

mean time at N. Y. $3^h\ 17^m\ 20^s$, Gr. time $= 8^h\ 13^m\ 24^s$.

For a still more accurate calculation.

Gr. mean time	$8^h\ 13^m\ 30^s$	$h = 3^h\ 23^m\ 22^s.3$		
$15\, P'$ cos. L	2.21279	$d = 1°\ 48'\ 59''.3$		
$2\,h = 6^h\ 46^m\ 44^s.6$	sin. 9.88954	$R = 401.1$		
S'	2.10233			
$D = 2°\ 39'\ 53''.2$	sec. 0.00046			
$2^m\ 6^s.7$	2.10279	L. cos.	9.88090	9.88090
$2\,h' = 6^h\ 44^m\ 37^s.9$	$h' = 3^h\ 22^m\ 18^s$	log. Ris.	4.56222	sin. 9.88790
$L - d = 38°\ 42'\ 21''$ N.	cos. 78035	d cos.	9.99978	
	27727		4.44290	
30° 12′ 7″ N.	sin. 50308	cos.	9.93664	sec. 0.06336
M.		cos.	9.86553	sin. 9.83216
		P'	3.50798	
$\delta D =$ 34′ 2″.4 N.			3.31015	

Solar eclipse of Sept. 18, 1838.

			S' 2.10233
$d' = 2° 23' 1''.7$		cos. 9.99962	sec. 0.00038
$\delta R = 2^m 6^s.68 = 31' 40''.2 = 1900''.2$			2.10271
$R - \delta R = -1499''.1$		3.17593_n	
$D - d' = 16' 51''.5 = 1011''.5$	ar. co.	6.99503	3.00497
$S = -55° 58' 2''$		tan. 0.17048	sec. 0.25207
		W. 3.25704	3.25704
$S + i = -85° 3' 50''$			cos. 8.93472
$(3600'' \cos. i) \div R_1 \cos. d = 0.34419\, \varDelta$			ar. co. 6.73753
$k = 85° 7' 32''$		sec. 1.07071	cos. 8.92929
$a = 3' 42''$		sin. 7.03193	
$t_1 = -50^s.6$		1.70387	

Mean Gr. time $= 8^h 12^m 39^s 4.$, N. Y. m. t. $= 3^h 16^m 34^s.9$.

For beginning of annular phase.

$\varDelta = 1' 15''.9 = 75''.9$			ar. co. 8.11976
W.		2.88671	p 1.07469
$k = 81°$	sec.	0.80555	cos. 9.19445
$W\, 3600'' \cos. i$		3.49767	
$R_1 \cos. d$	ar. co.	6.84656	
$a = -8° 7'$	sin.	9.14980_n	
$t_1 = 25^m 36^s$		3.18629_n	

Gr. mean time $= 9^h 25^m 36^s$.

Solar eclipse of Sept. 18, 1838.

For a more accurate calculation.

Gr. mean time $= 9^h\ 30^m$,		$h = 4^h\ 39^m\ 53^s.4$	
15 P' cos. L	2.21266	$d = 1°\ 47'\ 44''.7$	
$2h = 9^h\ 19^m\ 46^s.8$	sin. 9.97291	$R = 2217''.9$	
S'	2.18557		
$D = 2°\ 21'\ 48''.2$	sec. 0.00037		
$2^m\ 33^s.4$	2.18594	L. cos. 9.88090	9.88090
$2h' = 9^h\ 17^m\ 13^s.4$ $h' = 4^h\ 38^m\ 36^s.7$		log. Ris. 4.81444	sin. 9.97202
$L - d = 38°\ 43'\ 35''$	N. cos. 78015	d cos. 9.99979	
	49560	4.69513	
$16°\ 31'\ 56''$	N. sin. 28455	cos. 9.98166	sec. 0.01834
M		cos. 9.82528	sin. 9.87120
		P' 3.50785	
$\delta d = 34'\ 24''.4$		3.31479	S'. 2.18557
$d' = 2°\ 22'\ 19''.1$		cos. 9.99963	sec. 0.00037
$\delta R = 2301'' = 13^s.4$		$\delta R =$	2.18594
$R - \delta R = -83''.1$		1.91960_n	
$D - d' = -30''.9$		ar. co. 8.45967_n	1.48996_n
$S = -112°\ 40'\ 54''$		tan. 0.37890	sec. 0.41385_n
$D_1 = -793''$		2.89927_n	W 1.90381
$R_1 = 1423''.2$		ar. co. 6.84674	
d'		sec. 0.00037	
$i = -29°\ 8'\ 50''$		tan. 9.74638_n	

Solar eclipse of Sept. 18, 1838.

			W	1.90381
$S + i = -141° 49' 44''$			cos.	9.89551_n
	W	1.90381		1.79932_n
$3600'' \div R_1 \cos. d'$		0.40341	Δ ar. co.	8.14997
$k = 152° 51' 9''$	sec.	0.05071	cos.	9.94929_n
$a = 17° 1' 25''$	sin.	9.28151		
i	cos.	9.94120		
$t_1 = 38^s.1$		1.58064		

Gr. m. time $= 9^h 30^m 38^s.1$. N. Y. m. t. $= 4^h 34^m 33^s.6$.

In the same way we should find for the end of the annular phase.

$$\text{N. Y. mean time} = 4^h 38^m 12^s;$$

and for the end of the eclipse,

$$\text{N. Y. mean time} = 5^h 47^m 54^s.$$

5. If the beginning of the solar eclipse of Sept. 18, 1838, had been observed at New Orleans, in lat. 29° 57′ 45″ N., at 2^h 19^m $1^s.5$ mean time, what would be the longitude of New Orleans?

Solution. Let the supposed long. $= 6^h$

Greenwich mean time $= 8^h 19^m 1^s.5$, $h = 2^h 24^m 58^s.4$

$L =$ reduced lat. $= 29° 57' 45'' - 9' 55'' = 29° 47' 50''$

$P' = 53' 45''.3 - 2''.8 = 53' 42''.5 = 3222''.5$

$d = 1° 48' 53''.3$

$R = 530''.1$

Solar eclipse of Sept. 18, 1838.

P'		3.50820				
15.	ar. co.	8.82391				
L.	cos.	9.93841	cos.	9.93841	cos.	9.93841
$2h = 4^h\, 49^m\, 56^s.8$	sin.	9.77174				
S'		2.04226				
$D = 2°\, 38'\, 34''.8$	sec.	0.00046				
$2\delta h' = 1^m\, 50^s.3$		2.04272				
$2h' = 4^h\, 48^m\, 6^s.5,\ h' = 2^h\, 24^m\, 3^s.2$			l. R.	4.28132	sin.	9.76936
$L - d = 27°\, 58'\, 57''$	N. cos.	89115	d. cos.	9.99978		
		16577		4.21951		
$46°\, 10'\, 3''$	N. sin.	72538	cos.	9.84045	sec.	0.15955
M			cos.	9.83005	sin.	9.86732
			P'.	3.50820		
$\delta d = 25'\, 9''$				3.17870	S'.	2.04226
$d' = 2°\, 14'\, 2''.3$			cos.	9.99967	sec.	0.00033
$\delta R = 1654''.5 = 1110^s.3$						2.04259
$R = \delta R = -1124''.4$				3.05082_n		
$D - d' = 1472''.5$			ar. co.	6.83194		3.16806
$S = -37°\, 20'\, 16''$			tan.	9.88243_n	sec.	0.09960
					W	3.26766
$D_1 = -792''.4$				2.89894		
$R_1 = 1425''.4$			ar. co.	6.84607		
d'			sec.	0.00033		
$i = -29°\, 5'\, 20''$			tan.	9.74534		
$S + i = -66°\, 25'\, 36''$					cos.	9·60198
					p.	2.86964

Solar eclipse of Sept. 18, 1838.

i	cos.	9.94145	p.	2.86964
	W.	3.26766	Δ. ar. co.	6.73824
$k = 66° \; 5' \; 2''$	sec.	0.39212	cos.	9.60788
$a = -20' \; 34''$	sin.	7.77689		
R_1 cos. d'	ar. co.	6.84640		
		3.55630		
$t = 60^s.37 = 1^m \; 0^s.37$		1.78082		

long. $= 6^h \; 1^m \; 0^s.37 = 90° \; 15' \; 4''$.

For a more accurate calculation.

Gr. mean time $= 8^h \; 20^m$

$d = 1° \; 48' \; 52''.8$, $D = 2° \; 38' \; 21''.1$

$R = 555''.4$,

$d' = 2° \; 14' \; 1''.8$		cos.	9.99967		
$R - {}^{\delta}R = -1099''.1$			3.04104_n		
$D - d' =$	$1459''.3$	ar. co.	6.83585		3.16415
$S' = -36° \; 57' \; 52''$		tan.	9.87656_n	sec.	0.09745
			3.26160		3.26160
$S + i = -66° \; 3' \; 12''$				cos.	9.60840
$3600 \cos. i \div R_1 \cos. d'$			0.34415		2.87000
$a =$	$42''$	sin.	6.30882	Δ. ar. co.	6.73821
	$66° \; 3' \; 54''$	sec.	0.39179	cos.	9.60821
$t_1 = -2^s.02$			0.30636		

long. $=$ $90° \; 14' \; 7''$ W.

Solar eclipse of Sept. 18, 1838.

6. To find the times of the beginning and end of the annular eclipse of Sept. 18, 1838, at Washington, D. C., and the times of the formation and rupture of the ring.

Extracts from the Nautical Almanac.

Greenwich mean time, of Sept. 1838.

$17^d.0^h$	☉'s R. A. $= 11^h\ 38^m\ 25^s.08$,	Dec. $= 2°\ 20'\ 14''.3$ N.
18 .0	☉'s R. A. $= 11\ \ 42\ \ \ 0\ .61$,	Dec. $= 1\ \ 56\ 58\ .2$ N.
19 .0	☉'s R. A. $= 11\ \ 45\ \ 36\ .16$,	Dec. $= 1\ \ 33\ 39\ .5$ N.
20 .0	☉'s R. A. $= 11\ \ 49\ \ 11\ .75$,	Dec. $= 1\ \ 10\ 18\ .6$ N.
17 .0	equation of time $= 5^m\ 28^s.55$	
18 .0	equation of time $= 5\ \ 49\ .57$	
19 .0	equation of time $= 6\ \ 10\ .57$	
20 .0	equation of time $= 6\ \ 31\ .53$	
$18^d\ \ 6^h$	☽'s R. A. $= 11^h\ 39^m\ 49^s.64$,	Dec. $= 3°\ 11'\ 24''.6$ N.
18 7	☽'s R. A. $= 11\ \ 41\ \ 33\ .74$,	Dec. $= 2\ \ 57\ \ 14\ .9$ N.
18 8	☽'s R. A. $= 11\ \ 43\ \ 17\ .79$,	Dec. $= 2\ \ 43\ \ \ 4\ .6$ N.
18 9	☽'s R. A. $= 11\ \ 45\ \ \ 1\ .79$,	Dec. $= 2\ \ 28\ \ 53\ .9$ N.
18 10	☽'s R. A. $= 11\ \ 46\ \ 45\ .75$,	Dec. $= 2\ \ 14\ \ 42\ .6$ N.
18 11	☽'s R. A. $= 11\ \ 48\ \ 29\ .67$,	Dec. $= 2\ \ \ 0\ \ 31\ .0$ N.
18 0	☽'s semid. $= 14'\ 41''.5$	Hor. Par. $= 53'\ 54''.8$
18 12	☽'s semid. $= 14'\ 41''.1$	Hor. Par. $= 53'\ 53''.3$
	☉'s semid. $= 15'\ 57''.1$	Hor. Par. $= 8''.5$

Ans. Beginning of eclipse at $3^h\ \ 5^m\ 15^s.6$ W. mean time.
of ring at 4 23 46 .1
end of ring at 4 29 42 .3
of eclipse at 5 39 30 .3

Solar eclipse of May 15, 1836.

7. Calculate the time of the beginning and end upon the earth of the solar eclipse of May 15, 1836, and the places where it is first and last seen.

Extracts from the Nautical Almanac.

Greenwich mean time of May, 1836.

14^d	0^h	☉'s R. A. =	$3^h\ 25^m\ 5^s.13$	Dec. =	$18°\ 42'\ 21''.0$ N.
15	0		3 29 1 .93		18 56 35 .9 N.
16	0		3 32 59 .30		19 10 31 .6 N.
17	0		3 36 57 .25		19 24 7 .8 N.
14	0	equation of time =	$3^m\ 56^s.30$		
15	0	equation of time =	3 56 .05		
16	0	equation of time =	3 55 .24		
17	0	equation of time =	3 53 .86		
14	22	☽'s R. A. =	3 20 41 .89	Dec. =	18 40 52 .9 N.
14	23		3 22 41 .69		18 51 11 .4 N.
15	0		3 24 41 .69		19 1 24 .7 N.
15	1		3 26 41 .88		19 11 33 .0 N.
15	2		3 28 42 .26		19 21 36 .1 N.
15	3		3 30 42 .84		19 31 34 .0 N.
15	4		3 32 43 .62		19 41 26 .6 N.
15	5		3 34 44 .60		19 51 13 .9 N.
15	6		3 36 45 .77		20 0 55 .9 N.

14^d	12^h	☽'s semid. =	$14'\ 52''.3$	Hor. Par. =	$54'\ 34''.5$
15	0	☽'s semid. =	14 49 .9	Hor. Par. =	54 25 .6
15	12	☽'s semid. =	14 47 .7	Hor. Par. =	54 17 .7
		☉'s semid. =	$15'\ 49''.9$	Hor. Par. =	$8''.5$

Solar eclipse of May 15, 1836.

Ans. Time of begin. = $14^d\ 23^h\ 6^m\ 30^s$
in Long. = 76° 53′ W., and Lat. = 2° 10′ S.
Time of central begin. = $15^d\ 0^h\ 18^m\ 10^s$
in Long. = 98° 16′ W., and Lat. = 7° 58′ N.
Time of central end = $15^d\ 3^h\ 44^m\ 44^s$
in Long. = 52° 41′ E., and Lat. = 44° 50′ N.
Time of end = $15^d\ 4^h\ 56^m\ 24^s$
in Long. = 28° 51′ E., and Lat. = 35° 13′ N.

8. Find where the solar eclipse of May 15, 1836, is rising or setting at $15^d\ 0^h\ 30^m$ Greenwich mean time.

Ans. In Long. = 90° 44′ W. Lat. = 21° 32′ S.
and in Long. = 116° 42′ W. Lat. = 42° 29′ N.

9. Find the place which is upon the southern limit of the above eclipse's visibility at the Greenwich mean time of $15^d\ 3^h$.

Ans. Long. = 10° 11′ 36″ W. Lat. = 21° 41′ 24″ N.

10. Find the place which is upon the northern limit of the visibility of the annular phase of the solar eclipse of May 15, 1836, at the Greenwich mean time of $3^h\ 11^m\ 27^s$.

Ans. Long. = 3° 8′ 30″ W. Lat. = 56° 30′ 54″ N.

11. Find the place where the solar eclipse of May 15, 1836, is central at $3^h\ 11^m\ 27^s$ Greenwich mean time.

Ans. Long. = 3° 53′ 18″ W. Lat. = 58° 29′ 18″ N.

12. Find where the solar eclipse of May 15, 1836, is central at noon.

Ans. Long. = 36° 20′ W. Lat. = 49° 17′ N.

Solar eclipse of May 15, 1836.

12. Calculate the time of the beginning of the eclipse of May 15, 1836, for the Observatory of Edinburgh.

Ans. Beginning of eclipse $= 1^h\ 32^m\ 40^s$ Edinb. M. time.

13. Suppose the beginning of the solar eclipse of May 15, 1836, to be observed to take place at $1^h\ 36^m\ 35^s.6$ app. time, in latitude $55°\ 57'\ 20''$ N., and longitude about 12^m W.; find the longitude of the place of observation.

Ans. Long. $= 12^m\ 43^s.7$ W.

THE END.