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HALL'S ALGEBRA.

By the Author of this Work,

AN

ELEMENTARY TREATISE

ON THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

SECOND EDITION, 12*s*. 6*d*.

THE

ELEMENTS

OF

ALGEBRA;

CHIEFLY INTENDED FOR

SCHOOLS, AND THE JUNIOR CLASSES IN COLLEGES.

BY THE

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LONDON:
JOHN W. PARKER, WEST STRAND.

M.DCCC.XL.

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS.

PREFACE.

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49 line 22		568	575.
— — 24		153	133.
72 — 11		21000	2100.
84 — 14		$4\sqrt{x}$	$2\sqrt{x}$.
85	Ex. 4.	$11x^2$	$13x^2$.
—	Ex. 11.	$\frac{5}{4}$	$\frac{4}{5}$.
—	Ex. 17.	$\frac{4}{3}$	$\frac{160}{3}$.
86	Ex. 22.	$\frac{b}{a-1}$	$\frac{a-1}{b}$.
—	Ex. 26.	$x = 8$	$x = \frac{31}{4}$.
89	Ex. 65.	$\frac{x^2}{(1-\sqrt{1+x})^2}$	$\frac{x^2}{(1+\sqrt{1+x})^2}$.
93 — 20		49	46.
112	Ex. 12.	127500	−127500.
127	Ex. 31.	$y = 9$	$y = 27$.
129	Ex. 56.	100	80.

PREFACE.

The present work is chiefly intended for beginners in Algebra; and it will be found to contain very numerous examples adapted to each subject introduced into it.

Yet although this intention of making the young algebraist skilful in what may be termed the mechanical part of his science has never been lost sight of, still the algebraic rules have been fully explained, and in general rigorously proved; so that the learner may not only become expert in the use of symbols, but also may be enabled to give a reason for each step of the investigations he pursues. In a book of so moderate a size as this is, a part only of Algebra could be included; but the Table of Contents will shew that a uniform system has been adopted, and that the subjects treated on, are not only necessary, but of the highest importance.

King's College, London,
March 7, 1840.

TABLE OF CONTENTS.

PAGE

INTRODUCTION 1

CHAPTER I.

Addition and Subtraction 7
Multiplication and Division 14

CHAPTER II.

Algebraic Fractions 33
Greatest Common Measure 45
Least Common Multiple 54
Involution and Evolution 57

CHAPTER III.

Simple Equations 79
Quadratic Equations 98
Problems 131

CHAPTER IV.

Surds 154
Binomial Quadratic Surds 160

CHAPTER V.

Ratio 167
Proportion 172
Variation 178

CHAPTER VI.

PAGE
Arithmetic Series 182
Geometric Series 190
Harmonic Progression 199

CHAPTER VII.

Permutations 205
Combinations 208

CHAPTER VIII.

The Binomial Theorem 211
APPENDIX 231

INTRODUCTION.

1. In Algebra, numbers and magnitudes are represented by the letters of the alphabet, and their relation and connexion expressed by certain signs, which used instead of words, abbreviate the expressions. The numbers or magnitudes which are known or given, are denoted by the first letters of the alphabet, as A, B, C, &c., a, b, c, &c.; but those which are to be discovered by some of the letters x, y, z, v, u, w. Known quantities are also denoted by the Greek letters, α, β, γ, δ, &c. and the unknown by θ, ϕ, ψ, &c.

The signs chiefly made use of are the following:

$$+ \quad - \quad \times \quad \div \quad =.$$

(1) $+$ which is read *plus*, serves to mark the addition of two or more numbers. Thus $25+6$, read 25 plus 6; signifies that 25 is to be increased by 6; and $a+b$ means that the number expressed by a, is to be increased by the number expressed by b.

(2) $-$, read *minus*, which is placed between two numbers shews that the latter is to be taken away from the former: thus $24-19$, or 24 minus 19, means that 19 is to be taken from 24, and $a-b$ expresses the excess of a over b.

(3) $\times$, the sign of multiplication, which is placed between two numbers; thus 12×18 is read, 12 multiplied by 18, or more briefly 12 into 18: and $a \times b$ represents the product of a and b. Sometimes a point (.) is placed between two letters as $a.b$, but never between two numbers, lest it should be taken for the decimal point. But in general when the factors of a product are letters, neither $\times$ nor (.) is used, and the product a by b is written ab; when, however, both

the factors are numbers, the sign × must be placed between each: for should we write 56 instead of 5×6, the product five times six would be confounded with fifty-six.

(4) ÷ the sign of division, thus $36 \div 12$ read 36 divided by 12, means the quotient of 36 by 12, and $a \div b$, is the quotient of a by b; but this is more commonly expressed by writing a above b and drawing a line between them; thus $\frac{a}{b}$.

(5) = the sign of equality, thus $5 + 7 = 12$, read 5 plus 7 equal 12; and if the sum of a and b was equal to the excess of c above d, it would be expressed by $a + b = c - d$.

These are the principal algebraical signs, and to them may be added, the sign of inequality >, by which we shew that one quantity is greater or smaller than another;

thus $a > b$ is read a greater than b,

and $a < b$ is read a less than b.

The opening being always turned to the larger quantity.

2. We now proceed to explain some algebraical abbreviations. Instead of writing $a + a + a + a + a$, when the sum of 5 numbers each equal to a is required, we put $5a$. In the same manner $13a$, expresses the sum of thirteen numbers each equal to a, the numbers 5 and 13 are called coefficients. And $7ab$ which indicates that the product of a by b is to be taken 7 times, has 7 for its coefficient.

The coefficient therefore is a number written to the left hand of the quantity expressed by a letter or letters: and shews how many times the quantity expressed by the letter or letters ought to be taken.

Sometimes indeed a letter is called the coefficient, thus a is the coefficient of ax, $3b$ of $3bx$.

We must also observe that when unity is the coefficient, it is never expressed in writing, thus we put a and not $1a$;

but where we have to add or to subtract a we must not forget that its coefficient is unity.

3. Again, if a number as a be multiplied by itself, instead of writing $a \times a$, or aa, we put a^2; the 2 which is written above a and a little to the right hand, is called the *index* or exponent of a, and is intended to shew that the product is composed of two factors each equal to a. In the same manner $a \times a \times a$ is written a^3, and $a \times a \times a \times a$, is written a^4. Also, a^2 is called a squared, or a to the second power, a^3 is called a cubed or a to the third power, and a^5, a to the fifth power. The utility of the exponent and coefficient in Algebra may be thus shewn. Let it be required to express a product composed of 4 factors equal to a, of 3 factors equal to b, and of 2 equal to c: we then write $a^4b^3c^2$ instead of $aaaabbbcc$: and if also we wished to express that this last result should be taken 7 times we should prefix 7; and write $7a^4b^3c^2$, instead of repeating $aaaabbbcc$ seven times.

4. The sign $\sqrt{}$ is placed over a quantity when its root is to be extracted, the particular root being indicated by a small figure placed on the left of this sign; thus $\sqrt[3]{a}$, is read the cube root, or third root of a, and $\sqrt[4]{b}$, is read the fourth root of b; but the square root of a is written $\sqrt{a}$ and not $\sqrt[2]{a}$, the number in this root being always omitted.

5. The vinculum — or brackets ( ), the former of which being placed above or the latter enclosing two or more algebraical terms, shews that the quantities are to be taken as one sum. Thus $3\overline{a+b}$ or $3(a+b)$, means that the sum of a and b is to be multiplied by 3, and thus if $5b$ is taken from $5a$, which is expressed by $5a - 5b$, it may also be written $5(a-b)$ or $5.\overline{a-b}$.

6. These are the chief signs and abbreviations made use of in Algebra; and to them may be added $\therefore$ therefore,

and $\because$ since. Also : is to; thus $a : b$ expresses the ratio of a to b. And :, ::, : used between the terms of a proportion, thus $a : b :: c : d$ which is read a is to b as c is to d.

7. The use of some of these signs may now be shewn in the solution of the following question: "If the sum of two numbers be 13, and their difference 3, what are the numbers?"

First, if the smaller of the two numbers were known, the larger would be found by adding 3 to it. Let therefore the smaller be called x, then the larger is $x + 3$, and their sum will be $x + x + 3$, or $2x + 3$. But this sum is by the question equal to 13, and thus we have an equality or as it is technically termed an equation:

$$\text{viz.} \quad 2x + 3 = 13.$$

Now as $2x$ increased by 3, is equal to 13, $2x$ must be equal to 13 diminished by 3, or $2x$ must equal 10; and if twice a number equal 10, the number itself must be the half of 10 or 5, i.e. $x = 5$; and therefore the smaller number being 5, the larger must equal 8.

In fact $8 + 5 = 13$; and $8 - 5 = 3$.

The process written algebraically would stand thus:

Let x be the smaller number;

$$\therefore x + 3 \text{ is the larger;}$$

$$\therefore 2x + 3 \text{ is their sum;}$$

$$\therefore 2x + 3 = 13;$$

$$\therefore 2x = 10;$$

$$\therefore x = \frac{10}{2} = 5.$$

$$\text{And } x + 3 = 8.$$

8. The method used to obtain the answer to the preceding question is obviously applicable to every other of

the same kind; instead however of multiplying examples, with different numbers, it might be shewn that, if the sum of the numbers $= S$, and their difference $= D$,

$$\text{the greater number} = \frac{S}{2} + \frac{D}{2},$$

$$\text{the less number} = \frac{S}{2} - \frac{D}{2};$$

i.e. the greater $= \frac{1}{2}$ the sum $+ \frac{1}{2}$ the difference of the numbers,

the less $= \frac{1}{2}$ the sum $- \frac{1}{2}$ the difference;

thus in the preceding example, $S = 13$ and $D = 3$,

$$\text{then } \frac{13}{2} + \frac{3}{2} = \frac{16}{2} = 8 = \text{the greater},$$

$$\text{and } \frac{13}{2} - \frac{3}{2} = \frac{10}{2} = 5 = \text{the less}.$$

Here then we perceive one great advantage of Algebra, and it consists in this. Algebra not only affords methods for the solution of particular questions, but investigates rules by which all questions of the same kind may be solved; i.e. it arrives at a general answer, such as $\frac{S}{2} + \frac{D}{2}$, which will suit every question, whatever numbers consistent with the question we put for S and D.

Another, and it may be, a principal advantage of Algebra, is the clearness with which it represents to the eye, and so transmits to the mind, the quantity whose value you wish to determine. Thus in the previous question, among the suppositions made and amidst the calculations consequent upon them; x the quantity whose value is sought, constantly appears, and claims the attention of the computer, and at the same time relieves the burden under which the mind labours while a question of this kind is attempted to be solved by ordinary language.

The word *solution* or untying just used, means in Algebra, the mode of arriving at the answer to a question.

There are two kinds of questions in Algebra—Problems and Theorems.

In the problem, we have to find a number or quantity, which combined with certain given numbers or quantities, has some given value.

In the theorem, we have to shew what relation constantly exist between given and known quantities.

Thus the question, "To find two numbers whose sum is 13, and difference 3", is a problem.

And if we could shew that a and b being two numbers of which a is the greater

$$a = \frac{a+b}{2} + \frac{a-b}{2},$$

we should have a theorem.

CHAPTER I.

ALGEBRAICAL DEFINITIONS.

9. EVERY quantity written in algebraical language is called an algebraic quantity or an algebraic expression: these are divided into simple terms, and compound expressions. Simple terms are those when one letter or one combination of letters only is used, such are

$$5a, \quad 7b, \quad 8ab, \quad 9abc, \quad 12ab^2.$$

Compound expressions are those when two or a greater number of simple terms are connected by the signs + or −, as

$$a+b, \quad 7a^2+5ab+3bc, \quad 4a^2-6bc, \text{ \&c.}$$

Also an algebraical expression consisting of one term, is called a Mononomial; of two terms, a Binomial; of three terms, a Trinomial; and of many terms a Polynomial.

Also *positive* terms are those to which the sign + is prefixed, and *negative* terms are those which have the sign − before them. The first term of an algebraic expression, if positive, need not have the sign + before it, but the negative sign must never be omitted. Algebraical quantities are also called *like* or *unlike;* like when they involve the same letters; and unlike when the simple terms involve different letters. Thus $7a$, $5a$, are like quantities, and so also are $12ab^2$, $8ab^2$; but $7a$, $5b$, $12ab$, $13ac^2$, &c. are unlike quantities. In general the letters are written in the order of the alphabet, thus we write abc, and not bca.

SECTION I.

ADDITION AND SUBTRACTION.

ADDITION OF LIKE AND UNLIKE QUANTITIES.

10. The addition of unlike terms is performed by connecting them together with their proper signs. Thus the sum of

$$5a,\ 3b, \text{ and } 4c \text{ is } 5a + 3b + 4c.$$

The addition of like terms is performed by taking the sum of the coefficients when the coefficients have the same sign, and the difference of the coefficients with its proper sign, when they have different signs, and writing the sum or difference before the letter. Thus,

$$7a + 12a = 19a, \text{ and } 4a^2 - 9a^2 + 3a^2, \text{ or } 7a^2 - 9a^2 = -2a^2;$$

for there is a positive quantity $7a^2$, and a negative quantity $9a^2$, and since $9a^2$ is the same as $7a^2$ and $2a^2$, therefore $-9a^2$ is the same as $-7a^2$ and $-2a^2$, hence the whole sum is $7a^2 - 7a^2 - 2a^2$, and since $7a^2 - 7a^2$ equals zero, the result is $-2a^2$.

The usual method of proceeding in addition is, to write the expressions whether simple or compound under each other, and add or subtract the coefficients of like terms as in the addition of numbers, thus,

$5a$			
$7a$	$a + b + 3c$	$8a^2 - 9x^2 + 5$	$-a + b$
$3a$	$2a + 2b + c$	$7a^2 + 4x^2 - 3$	$+2a - 3b$
a	$3a + b + 2c$	$-11a^2 + 3x^2 + 13$	$-a + 2b$
$16a$	$6a + 4b + 6c$	$4a^2 - 2x^2 + 15$	* *

We may remark that it is not necessary to write the same letters directly under the corresponding letter above it, but in finding the sum, we must take care to connect together all the coefficients which involve the same letter or the same literal product. Example:

$$\begin{array}{r}
4x^2 - 8xy + 5y^2 - 7 \\
4xy - 7y^2 + 8x^2 - 13 \\
2y^2 + 5x^2 - 7xy + 23 \\
16 - 3xy + 4y^2 - 7x^2 \\
\hline
10x^2 - 14xy + 4y^2 + 19 \\
\hline
\end{array}$$

For the coefficient of x^2, is $4 + 8 + 5 - 7 = 10$, of xy it is $-8 + 4 - 7 - 3$ or -14, of y^2, $5 - 7 + 2 + 4$ or 4, and $-7 - 13 + 23 + 16$ equals 19.

The vinculum or bracket is very useful in expressing the sum of quantities when the coefficients are literal.

Ex. Find the sum of $ax^2 + by^2 + cz$, $2bx^2 + cy^2 + az$, and $x^2 + y^2 - z$; writing the expressions down under each other.

$$\begin{array}{c}
ax^2 + by^2 + cz \\
2bx^2 + cy^2 + az \\
x^2 + y^2 - z \\
\hline
(a + 2b + 1)x^2 + (b + c + 1)y^2 + (c + a - 1)z \\
\hline
\end{array}$$

For we observe that x^2 is multiplied by a, $2b$, and 1, y^2 by b, c, and 1, z by c and a, and also that z is to be subtracted, hence its whole coefficient is $c + a - 1$, whence we obtain the result given above.

SUBTRACTION.

11. In subtraction when the quantities are like we take the difference between the coefficients, and write after this difference, the literal quantity: but when the quantities are unlike, the sign $-$, is placed between them.

Then if $5a$ is to be taken from $7a$ the result is $2a$; but if $4b$ is to be subtracted from $7a$, the result is written $7a-4b$. The operations may be thus represented.

From	$7a$	$7a$	$8a+4b+7c$	a
Take	$5a$	$4b$	$4a+\ b+3c$	a
	$2a$	$7a-4b$	$4a+3b+4c$	0

The preceding examples contain no difficulty: had we however to subtract $b-c$ from a, the result $a-b+c$, requires explanation.

First, having to subtract $b-c$, if we only subtract b, the result $a-b$, is obviously too little, for the quantity b, which has been taken from a, ought to be diminished by c, before the subtraction be effected: we have in fact subtracted a quantity too great by c, and therefore to obtain a true result the difference $a-b$ must be increased by c, or the difference is, as we have before written it, $a-b+c$: and we see that we should have obtained the same result, had we changed the signs of the terms of $b-c$ and then added the quantity so changed to a. Next, had we to subtract a negative quantity from another as $-b$, from a: what is the result?

It is clear that the difference between the two quantities will be unaltered, if we add the same quantity to both. Add therefore b, when a becomes $a+b$, and $-b$ becomes $-b+b$ or 0; and 0 taken from $a+b$, leaves $a+b$: or $a-(-b)=a+b$.

From these considerations we obtain the following general rule: "Change the signs of all the terms of the quantity to be subtracted, or of the lower line, and then proceed as in addition*." Thus,

* The following proof may also be given:

Since $a=a-b+c+b-c$, for $-b+b=0$ and $+c-c=0$, if now $(b-c)$ be taken from each side we shall have

$$a-(b-c)=a+b-c.$$

$4a-3b$	$17a^2-4ab+5$	$-16a^2+25ab$	a	$-a$
$2a+2b$	$-2a^2+2ab-3$	$-12a^2-5ab$	$-a$	$+a$
$2a-5b$	$19a^2-6ab+8$	$-4a^2+30ab$	$2a$	$-2a$

12. We shall now give a few more examples of addition and subtraction.

Ex. Find the sum and difference of $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

$$\begin{array}{ll} \text{To} & a^2+2ab+b^2 \\ \text{Add} & a^2-2ab+b^2 \\ \hline & 2a^2 \quad +2b^2 \end{array} \qquad \begin{array}{ll} \text{From} & a^2+2ab+b^2 \\ \text{Take} & a^2-2ab+b^2 \\ \hline & 4ab \end{array}$$

$$\begin{array}{ll} \text{To} & a^3+3a^2b+3ab^2+b^3 \\ \text{Add} & a^3-3a^2b+3ab^2-b^3 \\ \hline & 2a^3 \quad +6ab^2 \end{array} \qquad \begin{array}{ll} \text{From} & a^3+3a^2b+3ab^2+b^3 \\ \text{Take} & a^3-3a^2b+3ab^2-b^3 \\ \hline & 6a^2b \quad +2b^3 \end{array}$$

$$\begin{array}{ll} \text{To} & \frac{a^2}{2}-ax+\frac{x^2}{4} \\ \text{Add} & \frac{a^2}{3}+2ax+\frac{x^2}{2} \\ \hline & \frac{5}{6}a^2+ax-\frac{x^2}{4} \end{array} \qquad \begin{array}{ll} \text{From} & \frac{a^2}{2}-ax+\frac{x^2}{4} \\ \text{Take} & \frac{a^2}{3}+2ax-\frac{x^2}{2} \\ \hline & \frac{a^2}{6}-3ax+\frac{3x^2}{4} \end{array}$$

In the last two examples the coefficient of a^2 in the upper line is $\frac{1}{2}$ and in the lower it is $\frac{1}{3}$ and $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ and $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$. Also $\frac{1}{4}$ and $-\frac{1}{2}$ are the coefficients of x^2, and

$$\frac{1}{4}+\left(-\frac{1}{2}\right)=\frac{1}{4}-\frac{1}{2}=\frac{1}{4}-\frac{2}{4}=-\frac{1}{4} \text{ and } \frac{1}{4}-\left(-\frac{1}{2}\right)=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}.$$

These two examples require to be well understood by the learner.

13. It may perhaps be useful to him to give numerical values to the letters: thus, suppose $a = 3$, $b = 2$, and $c = 1$.

$$\text{Then} \quad 5a + 3b - 2c = 15 + 6 - 2 = 19,$$
$$2a^2 + 6ab - 13bc = 18 + 36 - 26 = 28.$$

And as examples he may prove that

$$a^2 + 2ab + b^2 = 25, \quad a^2 - 2ab + b^2 = 1,$$
$$ab + ac + bc + b^2 = 15,$$
$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 36,$$
$$a^3 + 3a^2b + 3ab^2 + b^3 = 125,$$
$$a^3 + 3ac^2 - 3a^2c - c^3 = 8.$$

Having proved these results, he may proceed to the following examples in addition and subtraction.

EXAMPLES—ADDITION.

(1) The sum of $3a + 2b$, $5a + 6b$, $a + 3b$ and $b + a$ is $10a + 12b$.

(2) Of $a + 2b + 3c$, $2a + 3b + c$, and $3a + b + 2c$ is $6(a + b + c)$.

(3) Of $7a - 5c + 2b$ and $2a - 3c + 5b$ is $9a - 8c + 7b$.

(4) Of $5a + 3b + 2c - 5$, and $3a - 2b + 2c - 2$ is $8a + b + 4c - 7$.

(5) Of $-6a + 3b$, $-2a - 8b$, and $8a + 5b$ is 0.

(6) Of $2b + 3c - a$, $4a - 8b + 2c$, and $-6c + 4b - 3a$ is $-2b - c$ or $-(2b + c)$.

(7) Of $7a - 6b + 8c - 5$, $6a - 7b - 5c + 8$, $5a - 8b + 6c - 7$ and $8a + 5b - 7c + 6$ is $26a - 16b + 2c + 2$.

(8) Of $25a^4 - 16a^4 + 8a^4 - 7a^4 + 12a^4$ is $22a^4$.

(9) Of $8x^2-17xy+12y^2$, $4x^2+3xy-9y^2$, $7y^2-2xy-8x^2$ and x^2+xy+y^2 is $5x^2-15xy+11y^2$.

(10) Of $5a^4b+3a^2b^2c-7ab$, $-6a^4b+2a^2b^2c+17ab$, and $9a^4b-8a^2b^2c-10ab$ is $8a^4b-3a^2b^2c$.

(11) Of $3ax^2+2by^2-8$, $-ax^2+2cy^2-10$, and bx^2+ay^2+20 is $(2a+b)x^2+(2b+2c+a)y^2+2$.

(12) Of ax^3+bx^2+cx+d, $-3ax^3+cx^2+2bx+a$, $4x^3+3x^2+2x+1$, and $2bx^3-2ax^2-ax-b$ is $(4-2a+2b)x^3-(2a-3b-3)x^2+(2+2b+c-a)x+d+a+1-b$.

(13) Of $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{5}c$, $\frac{1}{4}a-\frac{1}{5}b-\frac{1}{3}c$ and $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c$ is $\frac{13a}{12}-\frac{17b}{60}+\frac{11c}{30}$.

(14) Of $\frac{5x^2}{2}-\frac{7y^2}{3}$, $\frac{-4x^2}{3}+\frac{5y^2}{2}$, $\frac{7x^2}{4}-\frac{3y^2}{2}$ is $2\frac{11}{12}x^2-1\frac{1}{3}y^2$.

SUBTRACTION.

(1) The difference between $3a-7b+4c$, and $2a-3b+2c$ is $a-4b+2c$.

(2) Between $4a-2b+3c$ and $3a+4b-c$ is $a-6b+4c$.

(3) Between $13a-2b+9c-3d$ and $-4a-6b+9c-10d$ is $17a+4b+7d$.

(4) Between $-2a-3b+2c$ and $2a+b-2c$ is $-4a-4b+4c$.

(5) Between $5b-3a+150c-80d$ and $17b-18a+210c-120d$ is $15a-12b-60c+40d$.

(6) Between $6a+2b-(3a+b)$ and $2a+4b-(4a-b)$ is $5a-4b$.

(7) Between $a+b$ and $\frac{a}{2}-\frac{b}{2}$ is $\frac{a}{2}+\frac{3b}{2}$.

(8) Between $\frac{a}{3}+\frac{b}{2}-\frac{c}{4}$ and $\frac{a+b}{4}-\frac{c}{8}$ is $\frac{a+3b}{12}-\frac{c}{8}$.

(9) Between $\frac{x^2}{2}-\frac{5x}{6}-\left(\frac{3x}{4}-x^2\right)$ and $3x-\frac{x^2}{8}-(2x+3x^2)$ is $4\frac{5}{8}x^2-2\frac{7}{12}x$.

(10) Between $4ax^2+bx+c$ and $3x^2-2x+5$ is $(4a-3)x^2+(2+b)x+c-5$.

(11) Between ax^3+bx^2+cx+d and ex^3+fx^2-gx-h is $(a-e)x^3+(b-f)x^2+(c+g)x+d+h$.

(12) Between $ax^m-bx^n-cx^p$ and $-2ax^m+cx^n-bx^p$ is $3ax^m-(b+c)x^n+(b-c)x^p$.

SECTION II.

MULTIPLICATION AND DIVISION OF ALGEBRAIC QUANTITIES.

14. PREVIOUS to the investigation of rules for the multiplication and division of algebraic quantities, it will be necessary that the beginner should have clear views of the nature of the positive and negative sign; his earnest attention to the next article, which treats of these signs, is consequently demanded.

ON THE SIGNS + AND −.

15. We have already seen that when b is to be added to a, the result is algebraically expressed by $a+b$: and that when b is to be subtracted from a the result is written $a-b$: and thus the signs + and − are the marks of the opposite

operations, addition and subtraction. But the signs + and − are not limited to the interpretation of these rules only; their general relative signification being *opposition*. Whatever quality or affection is expressed by $+a$, the contrary is expressed by $-a$. Thus if $+a$ represent property, $-a$ would represent debt, if $+a$ be the length of a line or curve drawn from a given point to the right hand $-a$ would represent an equal distance taken from the same point towards the left hand. If $+a$ represent a distance measured from a point upwards $-a$ will represent an equal distance measured from the same point downwards. Thus, as in some thermometers, the freezing point is zero. Then $+20^0$, would express a state of temperature 20 degrees above the freezing point; and -20^0 a state 20 degrees below it.

16. Bearing in mind this definition of the opposite qualities of + and −, let us examine the result of the multiplication of $+a$ or $-a$ by $+b$ or $-b$. And here we may observe that a and b, though standing for things and not merely for numbers; i.e. for distances, areas, money and other things similar to these, yet may be considered as numbers, which are multipliers of the units of these quantities: thus if a be the length of a rod which is a feet long, one foot is the unit, and the length is a times one foot: if a represent pounds sterling, then £1. is the unit and a is the number of times that £1. is to be taken: and if a be lbs. avoirdupois, a is the number of lbs. taken.

This being said we may remark, that the product of a and b independent of the algebraic signs will always be ab.

(1) $+a \times +b$ will $= +ab$ or ab: for considering a to be a number, a must be added b times or the number added must $= +ab$.

(2) Hence $+a \times -b$ must $= -ab$: for $-b$ multiplied into a must give a result directly opposite to that arising from multiplying $+a$ by $+b$.

(3) $-a \times +b$ should give a result directly opposite to $+a \times +b$, i. e. the result must be $-ab$.

(4) $-a \times -b$ must give a result directly opposite to that of $-a \times +b$, or to $-ab$, and this opposite result is $+ab$; $\therefore -a \times -b = +ab$.

Hence we see that + multiplied into + and − into −, give results affected by the positive sign, and that + into −, and − into +, give results affected by the negative sign; and hence we obtain the following rule of signs.

When two quantities, having like signs, are multiplied together, the sign of the product is +; and when the quantities have unlike signs, the sign of the product is −.

MULTIPLICATION.

17. The multiplication of simple unlike terms is performed by multiplying the numerical coefficients of the two terms together, and writing the letters in order after the product of the numbers, taking care to prefix the proper sign.

And the rule of signs is this,

+ into + and − into − give +,
+ into − and − into + give −.

Thus $7a$ multiplied by $5b$, is $35ab$; and $8ab$ multiplied by $-2cd$, is $-16abcd$; in working examples the multiplier is usually written under the multiplicand; thus

$12a$	$-16a$	$18ab$	$-9ac$
$3b$	$-\ 7c$	$-\ 3c$	$5bd$
$36ab$,	$112ac$,	$-54abc$,	$-45abcd$.

18. Again, if many unlike terms connected by the signs + and −, are to be multiplied by a simple term, the product is found by multiplying each term of the multiplicand by

the multiplier, and collecting together the simple products affected with their proper signs.

Thus, if $2a+3b-5c$, is to be multiplied by $6d$, each term of the trinomial must be multiplied by $6d$, and the sum of the partial products, gives the whole product required; in practice the multiplier is written under the left hand term as below, and the result obtained in a single line,

Multiply	$2a+3b-5c$
By	$6d$
Product	$12ad+18bd-30cd$

Ex.	$5a-6b-8c$	$-11a+5b-12$
	$9d$	$-3c$
	$45ad-54bd-72cd$	$33ac-15bc+36c$

19. If the multiplier consist of more terms than one or be a compound quantity, we might multiply the multiplicand by each term of the multiplier separately, and then add together the products so obtained, for the complete result; but this would entail upon us the trouble of writing the multiplicand as often as there were terms in the multiplier. In general therefore the multiplier is written under the multiplicand, and every term of the latter is first multiplied by the first term of the former, reckoning from the left hand, and this result is written in one line; a similar product is formed by multiplying the multiplicand by the second term of the multiplier, and this result is written under the former, but its first term is placed under the second term of the first product; similarly we proceed with the third and fourth and remaining terms of the multiplier, and all the separate products being found, the sum of all the terms thus arising will give the product required. The following examples will shew how the multiplication is effected.

Ex. Multiply $3a+5b+2c$ by $5d+3$.

$$\begin{array}{lr} & 3a+5b+2c \\ & 5d+3 \\ \hline \text{Multiplying by } 5d & 15ad+25bd+10cd \\ \text{Multiplying by } 3 & +9a+15b+6c \\ \hline \text{Product} = & 15ad+25bd+10cd+9a+15b+6c. \end{array}$$

Again in the following examples,

$$\begin{array}{l} 2a-3d \\ 3b+2c \\ \hline 6ab-9bd \\ \quad +4ac-6cd \\ \hline 6ab+4ac-9bd-6cd \end{array} \qquad \begin{array}{l} 5x-7 \\ 7y+9 \\ \hline 35xy-49y \\ \quad +45x-63 \\ \hline 35xy+45x-49y-63. \end{array}$$

20. In the preceding examples, the same letter has never occurred in both factors; this case which constantly happens remains to be mentioned. Thus had we to find the product of $7a^2b$ by $8a^2b^2$ the result would be $56a^2ba^2b^2$; but if we consider that a^2 is the product of two a's or $= a \times a$, then $a^2 \times a^2$ is the product of four a's or $= a \times a \times a \times a$ which is agreed to be written a^4, also since $b \times b^2$ is the same as $b \times b \times b$ or b^3, the whole product becomes $56a^4b^3$.

21. So again, if $2a^3$ is multiplied by $4a^7$: then since a^3 is the product of three a's and a^7 the product of seven a's; $\therefore a^3 \times a^7$ will be the product of ten a's or $= a^{10}$.

And, $\therefore 2a^3 \times 4a^7 = 8a^{10}$.

Hence, also since the product of m a's or

$a \times a \times a \times a \ldots\ldots$ to m factors $= a^m$.

And the product of n a's is for the same reason a^n. Therefore the product of a^m by a^n will be that of $(m+n)$ a's, or may be written a^{m+n};

$$\text{i.e. } a^m \times a^n = a^{m+n}.$$

Whence we obtain this rule: To find the product of the powers of the same letter. 'Write the letter, with an index, equal to the sum of the indices of the factors.'

Observe that the index of the simple power is unity, or that $a = a^1$.

22. We shall begin with simple terms to illustrate the preceding rule.

Multiply	$5a^2$	$6ab$	$-12a^3b^2c$	$-9x^2yz^3$
By	$7a^3$	$-7ab$	$8ab^2c^3$	$-11x^2y^2z^2$
	$35a^5$	$-42a^2b^2$	$-96a^4b^4c^4$	$99x^4y^3z^5$

Next let the multiplicand be compound,

$$\begin{array}{l} 2a^2 - 3ab + 2b^2 \\ 3a \\ \hline 6a^3 - 9a^2b + 6ab^2 \end{array} \qquad \begin{array}{l} a^5 - 5a^4b + 15a^2b^2 - 30ab^4 \\ -2a \\ \hline -2a^6 + 10a^5b - 30a^3b^2 + 60a^2b^4 \end{array}$$

Lastly let both multiplier and multiplicand be compound.

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \quad + ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array} \qquad \begin{array}{l} a - 2b \\ a - 2b \\ \hline a^2 - 2ab \\ \quad - 2ab + 4b^2 \\ \hline a^2 - 4ab + 4b^2 \end{array} \qquad \begin{array}{l} a^2 - ab + b^2 \\ a + b \\ \hline a^3 - a^2b + ab^2 \\ \quad + a^2b - ab^2 + b^3 \\ \hline a^3 \quad * \quad * \quad + b^3 \end{array}$$*

* "When in the addition the positive and negative quantities are equal, and there is therefore no result to be put down, an asterisk * is generally put to mark the circumstance."

$$\begin{array}{lllll} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \quad * \quad -b^2 \\ \hline \end{array} \qquad \begin{array}{l} x^4+x^3y+x^2y^2+xy^3+y^4 \\ x-y \\ \hline x^5+x^4y+x^3y^2+x^2y^3+xy^4 \\ \quad -x^4y-x^3y^2-x^2y^3-xy^4-y^5 \\ \hline x^5 \quad * \quad * \quad * \quad * \quad -y^5 \\ \hline \end{array}$$

$$\begin{array}{l} 2a^3-3a^2b+4ab^2-5b^3 \\ 2a^2+3ab+4b^2 \\ \hline 4a^5-6a^4b+8a^3b^2-10a^2b^3 \\ \qquad +6a^4b-9a^3b^2+12a^2b^3-15ab^4 \\ \qquad\qquad +8a^3b^2-12a^2b^3+16ab^4-20b^5 \\ \hline 4a^5 \quad * \quad +7a^3b^2-10a^2b^3+ab^4 \quad -20b^5 \\ \hline \end{array}$$

$$\begin{array}{l} a^2-\frac{a}{2}+\frac{1}{4} \\ a^2+\frac{a}{3}+\frac{1}{9} \\ \hline a^4-\frac{a^3}{2}-\frac{a^2}{4} \\ \quad +\frac{a^3}{3}-\frac{a^2}{6}+\frac{a}{12} \\ \qquad +\frac{a^2}{9}-\frac{a}{18}+\frac{1}{36} \\ \hline a^4-\frac{a^3}{6}+\frac{7a^2}{36}+\frac{a}{36}+\frac{1}{36} \\ \hline \end{array}$$

For the coefficient of a^3 is $-\frac{1}{2}+\frac{1}{3}$ or $-\frac{3}{6}+\frac{2}{6}=-\frac{1}{6}$: that of a^2 is $\frac{1}{4}-\frac{1}{6}+\frac{1}{9}$ or $\frac{9}{36}-\frac{6}{36}+\frac{4}{36}=\frac{7}{36}$: of x it is $\frac{1}{12}-\frac{1}{18}=\frac{3}{36}-\frac{2}{36}=\frac{1}{36}$ whence the results obtained above.

DIVISION.

23. Division is the inverse of multiplication; for if a quantity be first multiplied and then divided by the same quantity, its value is unaltered: and to divide one simple term by another is to find how often the latter is contained in the former: or since the dividend is equal to the product of the divisor and the quotient, we may define division to be, 'The having given one factor of a quantity to find the remaining factor.'

Thus $4abc \div$ by $2c$ or $\frac{4abc}{2c} = 2ab.$

Since $2c \times 2ab = 4abc$: and $\frac{18ab}{6b} = 3a$, since the product of $6b$ and $3a$ is $18ab$.

24. When powers of the same letter occur both in the divisor and dividend, the index of the less must be taken from that of the greater.

Thus $\frac{a^5}{a^2} = a^3$.

For a^5 is the product of five a's and a^2 of two a's, and therefore striking two from each we leave $a \times a \times a$ or a^3 in the numerator and unity in the denominator.

Also for the same reason $\frac{a^2}{a^5} = \frac{1}{a^3}$.

Hence in the division of mononomials, "We must first strike out the letters common to both divisor and dividend, then take the difference between the indices of the same letter, and multiply the result by the numerical quotient if any."

25. The rules for the algebraical signs may be obtained from those in multiplication. For the dividend being equal to the product of the divisor and quotient, the sign of the quotient may be easily determined. For since

+ arises from + into + or − into −

− .... + into − or − into +

If the dividend and divisor have the same sign, the quotient is +
.... different signs, it is −

or we may give the rule this practical form,

$$\frac{+}{+} \text{ and } \frac{-}{-} \text{ give } +$$

$$\text{And } \frac{+}{-}, \quad \frac{-}{+} \text{ give } -$$

The following examples will illustrate the preceding rules and remarks,

$7a)35a^2bc$ $\quad -2ab)16ab$ $\quad 8a^2b)-128a^6b^3$ $\quad -3bc)-27abc$

$5abc$ $\quad -8$ $\quad -16a^4b^2$ $\quad 9a$

When the numerical coefficient of the divisor is not contained in that of the dividend it is better to write the quantities in the form of fractions.

$$\text{Thus } 85a^3b^2 \div 25a^2b = \frac{85a^3b^2}{25a^2b} = \frac{17ab}{5}.$$

$$\text{And } 32abc \div 48a^3c = \frac{32abc}{48a^3c} = \frac{2b}{3a^2}.$$

26. When the dividend is a compound quantity and the divisor a simple term, the quotient is found by dividing every term of the dividend by the divisor, as in the following examples.

$5a)25a^2-15ab$ $\quad 4b)12a^2b-16b^3$ $\quad -2ab)8a^2b^2-14abc$

$5a-3b$ $\quad 3a^2-4b^2$ $\quad -4ab+7c$

27. When the divisor is also a compound quantity "arrange the terms both of the divisor and dividend according to the powers of the *same* letter, beginning with the *highest* power in each. Divide the first term of the dividend by the first term of the divisor, and set down the quotient by itself: multiply every term of the divisor by

this quotient, and subtract the product so arising from the dividend: then considering the remainder, should there be any, to be a new dividend, again divide its first term by the first term of the divisor, and add their quotient to the former, multiply the divisor a second time by this quotient and subtract this product, from the former remainder, and so proceed till the division be completed, or till no term of the remainder be divisible by the first term of the divisor."

The following example will illustrate the rule.

Divide $2a^2+4ab+2b^2$ by $a+b$

$$
\begin{array}{l}
a+b)\,2a^2+4ab+2b^2\,(2a+2b \\
\quad\quad 2a^2+2ab \\
\hline
\quad\quad\quad\quad 2ab+2b^2 \\
\quad\quad\quad\quad 2ab+2b^2 \\
\hline
\quad\quad\quad\quad * \quad * \\
\hline
\end{array}
$$

Here $2a^2$ divided by a gives $2a$ the first quotient, then $a+b$ the divisor, multiplied by $2a = 2a^2+2ab$; subtract this from the dividend and the remainder is $2ab+2b^2$.

Again, $2ab$ divided by a gives $2b$, the second quotient, and $2b$ multiplied by $a+b$ produces $2ab+b^2$, which subtracted from the first remainder, leaves no second remainder and the division is completed.

(Ex. 2.)

$$
\begin{array}{l}
x-3)\,x^3-6x^2+11x-6\,(x^2-3x+2 \\
\quad\quad x^3-3x^2 \\
\hline
\quad\quad\quad -3x^2+11x \\
\quad\quad\quad -3x^2+\ \ 9x \\
\hline
\quad\quad\quad\quad\quad 2x-6 \\
\quad\quad\quad\quad\quad 2x-6 \\
\hline
\quad\quad\quad\quad\quad * \quad * \\
\hline
\end{array}
$$

In this example it may be observed that the whole remainder is not brought down after the first subtraction;

this is done to save writing, as -6 was not wanted in the second subtraction; a little practice will point out the reason of this omission better than the laying down of an explicit rule.

(Ex. 3.)

$$x^2+2ax+a^2)\,x^4+4x^3a+6x^2a^2+4xa^3+a^4\,(x^2+2xa+a^2$$
$$x^4+2x^3a+x^2a^2$$
$$\overline{\quad 2x^3a+5x^2a^2+4xa^3}$$
$$2x^3a+4x^2a^2+2xa^3$$
$$\overline{\quad x^2a^2+2xa^3+a^4}$$
$$x^2a^2+2xa^3+a^4$$
$$\overline{\quad * \quad * \quad *}$$

(Ex. 4.) $a-3b)\,a^4-81b^4\,(a^3+3a^2b+9ab^2+27b^3$

$$a^4-3a^3b$$
$$\overline{\quad 3a^3b}$$
$$3a^3b-9a^2b^2$$
$$\overline{\quad 9a^2b^2}$$
$$9a^2b^2-27ab^3$$
$$\overline{\quad 27ab^3-81b^4}$$
$$27ab^3-81b^4$$
$$\overline{\quad * \quad *}$$

(Ex. 5.)

$$2a^3-5ab^2+2b^3)\,4a^6-25a^2b^4+20ab^5-4b^6\,(2a^3+5ab^2-2b^3$$
$$4a^6-10a^4b^2+4a^3b^3$$
$$\overline{\quad 10a^4b^2-4a^3b^3-25a^2b^4+20ab^5}$$
$$10a^4b^2-25a^2b^4 \qquad +10ab^5$$
$$\overline{\quad -4a^3b^3+10ab^5-4b^6}$$
$$-4a^3b^3+10ab^5-4b^6$$
$$\overline{\quad * \quad * \quad *}$$

$$\frac{x^2}{2} - x + 3\,\Big)\,\frac{3x^5}{4} - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27\,\Big(\frac{3x^3}{2} - 5x^2 + \frac{x}{4} + 9$$

$$\begin{array}{l}
\underline{\dfrac{3x^5}{4} - \dfrac{3x^4}{2} + \dfrac{9x^3}{2}} \\
\quad - \dfrac{5x^4}{2} + \dfrac{41}{8}x^3 - \dfrac{43}{4}x^2 \\
\quad \underline{- \dfrac{5x^4}{2} + 5x^3 - 15x^2} \\
\qquad \dfrac{1}{8}x^3 + \dfrac{17}{4}x^2 - \dfrac{33}{4}x \\
\qquad \underline{\dfrac{1}{8}x^3 - \dfrac{x^2}{4} + \dfrac{3x}{4}} \\
\qquad\quad \dfrac{9x^2}{2} - 9x + 27 \\
\qquad\quad \underline{\dfrac{9x^2}{2} - 9x + 27} \\
\qquad\quad * \quad * \quad *
\end{array}$$

28. In the preceding examples, the divisor is contained exactly in the dividend, and the remainder is consequently nothing: we shall now give examples in which there is a remainder.

$$\begin{array}{l}
x - a\,\big)\,x^3 + a^3\,\big(x^2 + ax + a^2 \\
\qquad \underline{x^3 - ax^2} \\
\qquad + ax^2 \\
\qquad\quad \underline{ax^2 - a^2x} \\
\qquad\qquad a^2x + a^3 \\
\qquad\qquad \underline{a^2x - a^3} \\
\qquad\qquad\quad 2a^3 \text{ the remainder.}
\end{array}$$

The division cannot now be carried on by the former process, but still it may be continued, the next term of the quotient being $\frac{2a^3}{x}$, in fact, there will be a series of fractions in the quotient, which series may be continued, to any length.

Again, divide 1 by $1-x$,

$$1-x)\ 1\ \ (1+x+x^2+x^3+\&c.$$
$$\underline{1-x}$$
$$x$$
$$\underline{x-x^2}$$
$$x^2$$
$$\underline{x^2-x^3}$$
$$x^3$$
$$\underline{x^3-x^4}$$
$$\underline{x^4}\ \text{remainder}$$

Here the division may be continued for ever.

Divide $x+2$ by $x+1$.

$$x+1)\ x+2\ (1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\&c.$$
$$\underline{x+1}$$
$$1$$
$$\underline{1+\frac{1}{x}}$$
$$-\frac{1}{x}$$
$$\underline{-\frac{1}{x}-\frac{1}{x^2}}$$
$$\frac{1}{x^2}$$
$$\underline{\frac{1}{x^2}+\frac{1}{x^3}}$$
$$\underline{\frac{1}{x^3}}\ \text{remainder.}$$

Such expressions as $1 + x + x^2 + x^3 + \&c.$ and $1 + \frac{1}{x} - \frac{1}{x^2} + \&c.$ are called infinite quotients, not because their value is of necessity very great, for indeed they are sometimes very small: but because they do not terminate.

29. It is useful to know that the difference both of the odd and even powers of a and b is always divisible by $a - b$: thus

$$(a^2 - b^2) \div (a - b) = a + b$$
$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2$$
$$(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3$$
$$(a^5 - b^5) \div (a - b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4,$$

and so on; but the difference of the even powers is only divisible by $a + b$. Thus

$$(a^2 - b^2) \div (a + b) = a - b$$
$$(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3$$
$$(a^6 - b^6) \div (a + b) = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5.$$

Also since $(a + b) \times (a - b) = a^2 - b^2$.

Therefore the sum of two quantities multiplied by their difference, equals the difference of their squares.

And the difference between two numbers is equal to the product of the sum and difference of their square roots; for a and b are the square roots of a^2 and b^2.

Thus $(5 + 3) \times (5 - 3) = 25 - 9 = 16.$

And $81 - 49 = (9 + 7) \times (9 - 7) = 32.$

The reader may supply other examples; and let him verify the assertions in the preceding part of this article.

EXAMPLES IN MULTIPLICATION.

(1) The product of $15ab$ and $30cd = 450abcd$.

(2) $3a^2bc \times 12abc^2 \times 5a^3b^2c^2 = 180a^6b^4c^5$.

(3) $3a^3 \times 4a^4 \times 5a^5 \times 6a^6 = 360a^{18}$.

(4) $\frac{5}{2}a^3b^4 \times \frac{6}{7}a^2b \times \frac{42}{9}ab^3 = 10a^6b^8$.

(5) $(3a^2 - 5ab + 2b^2) \times 4a = 12a^3 - 20a^2b + 8ab^2$.

(6) $(8a^3 - 6ab + 12b^3) \times 3ab^2 = 24a^4b - 18a^2b^3 + 36ab^4$.

(7) $(2a^4 - 12a^3 + 4a^2 + 2) \times 2a^2 = 4a^6 - 24a^5 + 8a^4 + 4a^2$.

(8) $\left(\frac{8}{3}a^2b - \frac{16}{9}ab^2 + \frac{3}{5}b^3\right) \times \frac{27}{4}ab = 18a^3b^2 - 12a^2b^3 + \frac{81}{20}ab^4$.

(9) $(a^{n-1}b - a^{n-2}b^2 + ab^{n-1}) \times ab = a^nb^2 - a^{n-1}b^3 + a^2b^n$.

(10) $(2a - 3)(a + 2) = 2a^2 + a - 6$.

(11) $(3a^2 - 2ab)(2a - 4b) = 6a^3 - 16a^2b + 8ab^2$.

(12) $(4a^2 - 6a + 9)(2a + 3) = 8a^3 + 27$.

(13) $(a^2 + ab^3)(a^4 - a^3b^3) = a^6 - a^4b^6$.

(14) $(1 + 2x + 3x^2)(1 + 2x + 3x^2) = 1 + 4x + 10x^2 + 12x^3 + 9x^4$.

(15) $(x^2 + 16x + 60)(x + 2) = x^3 + 18x^2 + 92x + 120$.

(16) $(1 + x + x^2 + x^3 + x^4)(1 - x) = 1 - x^5$.

(17) $(a^2 + 2b^2)(a + 2b) = a^3 + 2a^2b + 2ab^2 + 4b^3$.

(18) $(x^2 + 3x + 2)(x + 3) = x^3 + 6x^2 + 11x + 6$.

(19) $(x^2 - 7x + 5)(x + 8) = x^3 + x^2 - 51x + 40$.

(20) $(3a^2 - 5ab + 2b^2)(a^2 - 7ab) = 3a^4 - 26a^3b + 37a^2b^2 - 14ab^3$.

(21) $(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \times (a - b) = a^5 - b^5$.

(22) $(4a^4 - 3a^3b + 5ab^2 - 2b^4) \times (2a^2 - 4b^2) = 8a^6 - 6a^5b - 16a^4b^2 + 10a^3b^2 + 12a^3b^3 - 4a^2b^4 - 20ab^4 + 8b^6$.

(23) $(x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4)(x - 2y) = x^5 - 32b^5$.

(24) $(a^6 - 2a^4b^2 + 3a^2b^4)(2a^2 - 3ab + 4b^2) = 2a^8 - 3a^7b + 6a^5b^3 - 2a^4b^4 - 9a^3b^5 + 12a^2b^6$.

(25) $\left(\frac{x^2}{2} - \frac{x}{3} + \frac{1}{4}\right)\left(\frac{2x}{3} - \frac{1}{2}\right) = \frac{x^3}{3} - \frac{17x^2}{36} + \frac{x}{3} - \frac{1}{8}$.

(26) $(a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2) = a^4 + 4b^4$.

(27) $(2a^2 - 3ab - 4b^2)(3a^2 - 2ab + b^2) = 6a^4 - 13a^3b - 4a^2b^2 + 5ab^3 - 4b^4$.

(28) $(a^2 + ab + b^2)(a + b) = a^3 + 2a^2b + 2ab^2 + b^3$.

(29) $(a^2 - ab + b^2)(a - b) = a^3 - 2a^2b + 2ab^2 - b^3$.

(30) $(a + b + c)(a + b - c) = a^2 + 2ab + b^2 - c^2$.

(31) $(a^2 + b^2 + c^2 - ab - ac - bc)(a + b + c) = a^3 + b^3 + c^3 - 3abc$.

(32) $(a^3 + 2a^2b + 2ab^2 + b^3)(a^3 - 2a^2b + 2ab^2 - b^3) = a^6 - b^6$.

(33) $(1 - 3x + 3x^2 - x^3)(1 - 2x + x^2) = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$.

(34) $(2a^3bc - 3a^2b^2c + 5abc^3) \times (3ab^3c - 5ab^2c^2 - 7a^3bc) = -14a^6b^2c^2 + 21a^5b^3c^2 + a^4b^2c^2(6b^2 - 10bc - 35c^2) - 3a^3b^4c^2(3b - 5c) + 5a^2b^3c^4(3b - 5c)$.

(35) $(x^2 - px + q)(x - a) = x^3 - (p + a)x^2 + (q + ap)x - aq$.

(36) $(x^2 - px + q)(x^2 + px - r) = x^4 + (q - r - p^2)x^2 + (pr + pq)x - rq$.

(37) $(a^m + b^m + c^m)(a^n + b^n + c^n) = a^{m+n} + b^{m+n} + c^{m+n} + a^m b^n + a^m c^n + b^m a^n + b^m c^n + c^m a^n + b^n c^m$.

(38) $(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + acd + bcd)x + abcd$.

(39) $(a^2 + b^2 + c^2 + d^2)(a^2 + b^2 - c^2 - d^2) = a^4 + b^4 + 2a^2b^2 - 2c^2d^2 - c^4 - d^4$.

(40) $(a^2 - b^2 + c^2 - d^2)(a^2 + b^2 - c^2 - d^2) = a^4 + d^4 - b^4 - c^4 + 2b^2c^2 - 2a^2d^2$.

(41) $\left(\frac{5x^2}{2} + 3ax - \frac{7a^2}{3}\right) \times \left(2x^2 - ax - \frac{a^2}{2}\right) = 5x^4 + \frac{7ax^3}{2} - \frac{107}{12}a^2x^2 + \frac{5a^3x}{6} + \frac{7a^4}{6}$.

(42) $\left(\frac{15b^2}{a^6} - \frac{7b^4}{a^5} + \frac{6b^6}{a^4}\right)\left(\frac{8b^2}{a^2} - \frac{3b^4}{a}\right) = \frac{120b^4}{a^8} - \frac{101b^6}{a^7} + \frac{69b^8}{a^6} - \frac{18b^{10}}{a^5}$.

EXAMPLES OF DIVISION.

(1) The quotient of $6a - 8b + 4c$ by $2 = 3a - 4b + 2c$.

(2) $a^3 - 2a^2b + 4ab^2$ by $a = a^2 - 2ab + 4b^2$.

(3) $12a^3 - 20a^2b + 8ab^2$ by $4a = 3a^2 - 5ab + 2b^2$.

(4) $24a^4b - 18a^2b^3 + 36ab^4$ by $6ab = 4a^3 - 3ab^2 + 6b^3$.

(5) $9a^2bc - 12ab^2c + 15abc^2$ by $3abc = 3a - 4b + 5c$.

(6) $4a^2 - 6a^2c + 8abc$ by $6ab = \frac{2a}{3b} - \frac{ac}{b} + \frac{4}{3}c$.

(7) $a^2 + 4ab + 4b^2$ by $a + 2b = a + 2b$.

(8) $18a^2 - 8b^2$ by $3a - 2b = 6a + 4b$.

(9) $x^3 + 6x^2 + 11x + 6$ by $x + 2 = x^2 + 4x + 3$.

(10) $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b = a^2 + 2ab + b^2$.

(11) $a^3 + a^2b - ab^2 - b^3$ by $a - b = a^2 + 2ab + b^2$.

(12) $3a^4 - 26a^3b + 37a^2b^2 - 14ab^3$ by $a^2 - 7ab = 3a^2 - 5ab + 2b^2$.

(13) $a^3 + b^3$ by $a + b = a^2 - ab + b^2$.

(14) a^5+b^5 by $a+b=a^4-a^3b+a^2b^2-ab^3+b^4$.

(15) $2a^4-13a^3b+31a^2b^2-38ab^3+24b^4$ by $2a^2-3ab+4b^2=a^2-5ab+6b^2$.

(16) a^4+4b^4 by $a^2-2ab+2b^2=a^2+2ab+2b^2$.

(17) $a^4+4a^2b^2+16b^4$ by $a^2+2ab+4b^2=a^2-2ab+4b^2$.

(18) $4a^4-9a^2b^2+6ab^3-b^4$ by $2a^2-3ab+b^2=2a^2+3ab-b^2$.

(19) $27a^4-6a^2+\frac{1}{3}$ by $3a^2+2a+\frac{1}{3}=9a^2-6a+1$.

(20) $-40y^5+68xy^4+25x^2y^3+21x^3y^2-18x^4y-56x^5$ by $5y^2-6xy-8x^2=-8y^3+4xy^2-3x^2y+7x^3$.

(21) $56a^4-59a^3-73a^2+95a-25$ by $7a^3-3a^2-11a+5=8a-5$.

(22) $10a^3+11a^2b-15a^2c-19abc+3ab^2+15bc^2-5b^2c$ by $5a^2+3ab-5bc=2a+b-c$.

(23) $a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+28a^2b^6-8ab^7+b^8$ by $a^3-3a^2b+3ab^2-b^3=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

(24) $x^3-(a+p)x^2+(q+ap)x-aq$ by $x-a=x^2-px+q$.

(25) $\frac{x^3}{3}-\frac{17x^2}{36}+\frac{x}{3}-\frac{1}{8}$ by $\frac{2x}{3}-\frac{1}{2}=\frac{x^2}{2}-\frac{x}{3}+\frac{1}{4}$.

(26) $5a^4+\frac{7ba^3}{2}-\frac{107}{12}a^2b^2+\frac{5ab^3}{6}+\frac{7b^4}{6}$ by $2a^2-ab-\frac{b^2}{2}=\frac{5a^2}{2}+3ab-\frac{7b^2}{3}$.

(27) $x^4-\frac{19}{6}a^2x^2+\frac{a^3x}{3}+\frac{a^4}{6}$ by $x^2-2ax+\frac{a^2}{2}=x^2+2ax+\frac{a^2}{3}$.

(28) $a^2-b^2-c^2+d^2-2ad+2bc$ by $a-b+c-d=a+b-c-d$.

(29) $a^4 + b^4 + 2a^2b^2 - 2c^2d^2 - c^4 - d^4$ by $a^2 + b^2 - c^2 - d^2$ $= a^2 + b^2 + c^2 + d^2$.

Cases in which the division does not terminate.

(30) $\dfrac{a}{1+x} = a - ax + ax^2 - ax^3 + ax^4 - \&c.$

(31) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \&c.$

(32) $\dfrac{1}{1-2x+x^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$

(33) $\dfrac{1}{1+2x+x^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \&c.$

(34) $\dfrac{a}{x+1} = \dfrac{a}{x} - \dfrac{a}{x^2} + \dfrac{a}{x^3} - \dfrac{a}{x^4} + \&c.$

(35) $\dfrac{x+a}{x-b} = 1 + \dfrac{a+b}{x} + \dfrac{ab+b^2}{x^2} + \dfrac{ab^2+b^3}{x^3} + \&c.$

(36) $\dfrac{1-3x-2x^2}{1-4x} = 1 + x + 2x^2 + 2 . 4x^3 + 2 . 4^2 . x^4 + \&c.$

CHAPTER II.

ALGEBRAIC FRACTIONS, GREATEST COMMON MEASURE, INVOLUTION, EVOLUTION.

SECTION I.

ALGEBRAIC FRACTIONS.

30. Our notions of Algebraic fractions are best derived from those of the ordinary numerical ones. Thus as $\frac{5}{7}$ means that the unit, whatever it may be, has been divided into 7 parts, of which 5 have been taken: so the fraction $\frac{a}{b}$ conveys to the mind the same idea; viz. that the unit has been divided into b parts, and a of these parts have been taken.

This being the case, we have only to extend to Algebra the rules to which we have been accustomed in Arithmetic, in order to reduce the algebraic fractions, to others having the same denominator: to add them together, to find their difference, product and quotient; before however we do so, we shall premise a few important propositions.

31. Since $\frac{a}{b}$ represents the quotient of a divided by b, and if we multiply the quotient by the divisor, the result should be the dividend;

$$\therefore \frac{a}{b} \times b = a:$$

or if a fraction be multiplied by its denominator, the product is the numerator.

32. Again, since $b \times \frac{a}{b} = a$, as we have just seen; therefore multiplying both by another quantity m,

$$mb \times \frac{a}{b} = ma;$$

$$\therefore \frac{a}{b} = \frac{ma}{mb};$$

i.e. if the numerator and denominator of a fraction be multiplied by the same quantity, its value remains unaltered.

Cor. Hence also since $\frac{ma}{mb} = \frac{a}{b}$; if the numerator and denominator of a fraction be divided by the same quantity, its value remains unaltered.

Examples. Thus $\frac{5x}{3} \times 3 = 5x$,

$$\frac{x}{2y} = \frac{2x}{4y} = \frac{3x}{6y},$$

and $$\frac{16x}{28y} = \frac{8x}{14y} = \frac{4x}{7y}.$$

33. To reduce fractions having different denominators to others having a common denominator, "Multiply each numerator into all the denominators except its own, for new numerators, and all the denominators together for a new denominator."

The principle of the rule is this: "if we multiply the numerator and denominator of a fraction by the same quantity, the value of the fraction remains unaltered:" a proposition we have just proved, and instead of following the strict letter of the rule, we will use this principle in the following examples.

(Ex. 1.) Reduce $\frac{9x}{5}$ and $\frac{7x}{11}$ to fractions having a common denominator.

$$\frac{9x}{5} = \frac{9x}{5} \times \frac{11}{11} = \frac{99x}{55},$$

$$\frac{7x}{11} = \frac{7x}{11} \times \frac{5}{5} = \frac{35x}{55}.$$

And $\frac{99x}{55}$, $\frac{35x}{55}$ are the fractions required.

(Ex. 2.) Reduce $\frac{a}{a+b}$ and $\frac{b}{a-b}$ to fractions having a common denominator.

$$\frac{a}{a+b} = \frac{a}{a-b} \times \frac{a-b}{a-b} = \frac{a^2-ab}{a^2-b^2},$$

$$\frac{b}{a-b} = \frac{b}{a-b} \times \frac{a+b}{a+b} = \frac{a^2+ab}{a^2-b^2};$$

$\therefore \frac{a^2-ab}{a^2-b^2}$ and $\frac{a^2+ab}{a^2-b^2}$ are the fractions required.

(Ex. 3.) Let $\frac{2a}{3b}$, $\frac{4a}{7}$ and $\frac{5a}{8}$ be the fractions.

$$\frac{2a}{3b} = \frac{2a}{3b} \times \frac{7\times 8}{7\times 8} = \frac{112a}{168b},$$

$$\frac{4a}{7} = \frac{4a}{7} \times \frac{3b\times 8}{4b\times 8} = \frac{96ab}{168b},$$

$$\frac{5a}{8} = \frac{5a}{8} \times \frac{3b\times 7}{3b\times 7} = \frac{105ab}{168b};$$

$\therefore \frac{112a}{168b}$, $\frac{96ab}{168b}$ and $\frac{105ab}{168b}$ are the fractions required.

(Ex. 4.) Reduce $\frac{5x}{3}$, $\frac{7x}{5}$ and $\frac{3x}{10}$, to equivalent fractions having the same denominator.

As it is always better to have the fractions in the lowest terms, we will multiply each fraction by such numbers as will make the denominator the least common denominator.

We see that 10 is a multiple of 5, and 3×10 or 30 contains all the denominators. If then we multiply the numerator and denominator of the first fraction by 10, of the second by 6, and of the third by 3, each will have the same denominator.

For $\frac{5x}{3} \times \frac{10}{10} = \frac{50x}{30}$; $\frac{7x}{5} \times \frac{6}{6} = \frac{42x}{30}$; $\frac{3x}{10} = \frac{9x}{30}$; and $\frac{50x}{30}$, $\frac{42x}{30}$ and $\frac{9x}{30}$ are the fractions required.

(Ex. 5.) Let $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ be the fractions;

$$\frac{a+b}{a-b} = \frac{a+b}{a-b} \times \frac{a+b}{a+b} = \frac{a^2+2ab+b^2}{a^2-b^2},$$

$$\frac{a-b}{a+b} = \frac{a-b}{a+b} \times \frac{a-b}{a-b} = \frac{a^2-2ab+b^2}{a^2-b^2};$$

$\therefore$ $\frac{a^2+2ab+b^2}{a^2-b^2}$ and $\frac{a^2-2ab+b^2}{a^2-b^2}$ are the equivalent fractions.

34. Every integral quantity as a may be reduced to the form of a fraction by placing unity under it.

Thus $\frac{a}{1} = a$.

And it may be made to have any denominator as b by multiplying both a and 1 by b.

Thus $\frac{a}{1} = \frac{ab}{b}$. Since $\frac{ab}{b} = a$.

35. Hence to reduce a mixed quantity to a fraction, multiply the integral quantity by the denominator of the

fraction, and add to this product the numerator of the fraction, placing the denominator under both for a common denominator.

(Ex. 1.) $a+\frac{b^2}{a}=\frac{a^2+b^2}{a}.$

(Ex. 2.) $3x+\frac{5x-2}{7}=\frac{21x+5x-2}{7}=\frac{26x-2}{7}.$

(Ex. 3.) $a+b+\frac{a^2+b^2}{a-b}=\frac{a^2-b^2+a^2+b^2}{a-b}=\frac{2a^2}{a-b}.$

36. So conversely to reduce a fraction to a quantity which is partly integral and partly fractional, we must observe what terms of the numerator are divisible by the denominator, and these divided will give the integral part; the remaining terms of the numerator with the denominator written under them will be the fractional part.

(Ex. 4.) $\frac{4x^2-8x+3}{4x}=\frac{4x^2-8x}{4x}+\frac{3}{4x}=x-2+\frac{3}{4x}.$

(Ex. 5.) $\frac{2a^3+6a^2b+3b^2}{2a^2}=\frac{2a^3+6a^2b}{2a^2}+\frac{3b^2}{2a^2}=a+3b+\frac{3b^2}{2a^2}.$

(Ex. 6.) $a+b+\frac{b^2}{c}=\frac{ac+bc+b^2}{c}.$

(Ex. 7.) $a+\frac{ab}{a-b}=\frac{a^2}{a-b}.$

(Ex. 8.) $\frac{10x^2-15xy+7y^2}{5x}=2x-3y+\frac{7y^2}{5x}.$

(Ex. 9.) $\frac{2a^2-2ab+4b^2}{a-b}=2a+\frac{4b^2}{a-b}.$

(Ex. 10.) $a^2+ab+b^2+\frac{b^3}{a-b}=\frac{a^3}{a-b}.$

37. To find the sum and difference of two algebraic fractions.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions.

$$\text{Then let } \frac{a}{b} = x \text{ and } \frac{c}{d} = y;$$

$$\therefore\ a = bx \text{ and } c = dy;$$

$$\therefore\ ad = bdx \text{ and } cb = bdy;$$

$$\therefore\ ad \pm cb = bdx \pm bdy = bd\,.(x \pm y);$$

$$\therefore\ x \pm y = \frac{ad \pm cb}{bd}.$$

Whence this rule "Reduce the fractions to others having the same denominator, and take the sum or difference of the new numerators.

(Ex. 1.) Find the sum of, and difference between, $\frac{9x}{5}$ and $\frac{7x}{11}$;

$$\because \text{ their equivalent fractions are } \frac{99x}{55} \text{ and } \frac{35x}{55};$$

$$\therefore\ \text{sum} = \frac{134x}{55}: \qquad \text{difference} = \frac{64x}{55}.$$

(Ex. 2.) Find the sum of and difference between $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

$$\text{Since } \frac{a}{a+b} = \frac{a^2 - ab}{a^2 - b^2} \text{ and } \frac{b}{a-b} = \frac{ab + b^2}{a^2 - b^2};$$

$$\therefore\ \text{sum} = \frac{a^2 + b^2}{a^2 - b^2}: \quad \text{difference} = \frac{a^2 - 2ab - b^2}{a^2 - b^2}.$$

(Ex. 3.) Find the difference between $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$:

$$\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a^2 + 2ab + b^2) - (a^2 - 2ab + b^2)}{a^2 - b^2} = \frac{4ab}{a^2 - b^2}.$$

38. To find the product of two or more fractions, "Multiply the numerators together for a new numerator, and the denominators together for a new denominator:" to prove this.

If $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions,

$$\text{let } \frac{a}{b}=x \text{ and } \frac{c}{d}=y.$$

$$\text{Then } a=bx \text{ and } c=dy;$$

$$\therefore\ ac=bd\,.\,x\,.\,y \text{ or dividing by } bd,$$

$$\frac{ac}{bd}=x\times y=\frac{a}{b}\times\frac{c}{d},$$

which proves the truth of the rule just enunciated.

Ex. Find the product of $\frac{a+b}{a+2b}$ and $\frac{a-b}{a+3b}$.

$$\text{Product} = \frac{(a+b)(a-b)}{(a+2b)(a+3b)} = \frac{a^2-b^2}{a^2+5ab+6b^2}.$$

39. To find the quotient of one fraction by another, "Invert the divisor and proceed as in multiplication."

$$\text{For if } \frac{a}{b}=x \text{ and } \frac{c}{d}=y;$$

$$\text{and } \therefore\ a=bx \text{ and } c=dy;$$

$$\therefore\ ad=bdx \text{ and } cb=bdy;$$

$$\therefore\ \frac{bdx}{bdy}=\frac{ad}{cb}.$$

$$\text{But } \frac{bdx}{bdy}=\frac{x}{y}=\frac{a}{b}\div\frac{c}{d};$$

$$\therefore\ \frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c},$$

which is the rule.

(Ex. 1.) Find the quotient of $\frac{7x}{5}$ by $\frac{3x}{10}$,

$$\frac{7x}{5}\div\frac{3x}{10}=\frac{7x}{5}\times\frac{10}{3x}=\frac{70x}{15x}=\frac{14}{3}.$$

(Ex. 2.) Divide $\frac{a}{b}$ by $\frac{a^2}{a-b}$;

$$\frac{a}{b} \div \frac{a^2}{a-b} = \frac{a}{b} \times \frac{a-b}{a^2} = \frac{1}{b} \times \frac{a-b}{a} = \frac{a-b}{ab}.$$

(Ex. 3.) Divide $\frac{a}{b}$ by $\frac{1}{b}$,

$$\frac{a}{b} \div \frac{1}{b} = \frac{a}{b} \times \frac{b}{1} = \frac{ab}{b} = a.$$

(Ex. 4.) Divide $\frac{1}{a}$ by $\frac{1}{b}$,

$$\frac{1}{a} \div \frac{1}{b} = \frac{1}{a} \times \frac{b}{1} = \frac{b}{a}.$$

The Examples 3 and 4 are important, and are constantly illustrated in Arithmetic, and will be useful hereafter.

Thus, divide $\frac{2}{3}$ by $\frac{1}{3}$: quotient $= \frac{2}{3} \times \frac{3}{1} = 2$.

Divide $\frac{1}{2}$ by $\frac{1}{3}$: result is $\frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$.

40. The following are simple examples in subtraction and addition.

(Ex. 1.) Add $\frac{3a}{2bc}$ to $\frac{2c}{5d}$: Ans. $\frac{15ad + 4bc^2}{10bcd}$.

(Ex. 2.) Add $\frac{2ac}{b}$ to $\frac{3bf}{2g}$: Ans. $\frac{4acg + 3b^2f}{2bg}$.

(Ex. 3.) Add $\frac{17x^2}{5y}$ to $\frac{8y^2}{3x}$: Ans. $\frac{51x^3 + 40y^3}{15xy}$.

(Ex. 4.) Add $\frac{3x}{5}$, $\frac{4x}{7}$ and $\frac{2x}{3}$: Ans. $\frac{193x}{105}$.

(Ex. 5.) Add $\frac{x}{x-2}$ to $\frac{x}{x-3}$: Ans. $\frac{2x^2-5x}{x^2-5x+6}$.

(Ex. 6.) From $\frac{x}{x-2}$ take $\frac{x}{x-3}$: Ans. $\frac{-x}{x^2-5x+6}$.

(Ex. 7.) From $\frac{13ab-2c}{12ab}$ take $\frac{12ab}{13ab+2c}$:

Ans. $\frac{15a^2b^2-4c^2}{156a^2b^2+24abc}$.

(Ex. 8.) From $\frac{2x+1}{3x+2}$ take $\frac{4x+5}{5x+4}$:

Ans. $-\frac{2x^2+10x+6}{15x^2+22x+8}$.

(Ex. 9.) Add $\frac{x}{x+1}$, $\frac{2}{x+2}$ and $\frac{3}{x+3}$:

Ans. $\frac{x^3+10x^2+23x+12}{x^3+6x^2+11x+6}$.

(Ex. 10.) Add $\frac{2}{2x+1}$, $\frac{3}{3x+2}$ and $\frac{-8}{4x+3}$:

Ans. $\frac{8x+5}{24x^3+46x^2+29x+6}$.

41. The following are examples in multiplication and division.

(Ex. 1.) $\frac{5x}{7}\times\frac{6x}{35}=\frac{30x^2}{245}=\frac{6x^2}{49}$.

(Ex. 2.) $\frac{2x+1}{3}\times\frac{2x-1}{7}=\frac{4x^2-1}{21}$.

(Ex. 3.) $\frac{a+b}{a}\times\frac{a^2}{a-b}=\frac{a^2+ab}{a-b}$.

(Ex. 4.) $\frac{4x^2-1}{8}\times\frac{2x+1}{2x-1}=\frac{4x^2+4x+1}{8}$.

(Ex. 5.) $\frac{2a^2}{a^2-b^2} \times \frac{(a+b)^2}{4b^2a^2} = \frac{a+b}{2b^2(a-b)}.$

(Ex. 6.) $\frac{5x}{6} \div \frac{6x}{7} = \frac{35}{36}.$

(Ex. 7.) $\frac{x^2-9}{2x+1} \div \frac{x+3}{2} = \frac{2x-6}{2x+1}.$

(Ex. 8.) $\frac{a+b}{a-b} \div \frac{a-b}{a+b} = \frac{a^2+2ab+b^2}{a^2-2ab+b^2}.$

(Ex. 9.) $\frac{a+1}{b} \div \frac{1+\frac{1}{a}}{b} = a.$

(Ex. 10.) $\frac{1}{a} + \frac{1}{b} \div \frac{1}{a} - \frac{1}{b} = \frac{b+a}{b-a}.$

(Ex. 11.) $\frac{17-4bc}{16a^4-12a^3b+8a^2} \times 4a^2 = \frac{17-4bc}{4a^2-3ab+2}.$

(Ex. 12.) $\left(a+b+\frac{b^2}{a}\right) \div \left(a+b+\frac{a^2}{b}\right) = \frac{b}{a}.$

42. We shall now give a few examples of the Reduction of Fractions, in which the foregoing rules will be further illustrated: and afterwards add some examples for practice.

(Ex. 1.) Find the value of $\frac{2a}{a^2-b^2} + \frac{1}{a+b} - \frac{1}{a-b}.$

First $\frac{1}{a+b} - \frac{1}{a-b} = \frac{a-b}{a^2-b^2} - \frac{a+b}{a^2-b^2} = \frac{-2b}{a^2-b^2};$

$$\therefore \frac{2a}{a^2-b^2} + \frac{1}{a+b} - \frac{1}{a-b} = \frac{2a}{a^2-b^2} - \frac{2b}{a^2-b^2} = \frac{2(a-b)}{a^2-b^2} = \frac{2}{a+b}.$$

For $a^2-b^2=(a-b)(a+b)$ and the numerator and denominator are both divisible by $a-b$.

(Ex. 2.) Find the value of $\frac{13a-5b}{4}-\frac{7a-2b}{6}-\frac{3a}{5}$.

Since 60 is divisible by 4, 5, and 6, if we multiply the numerator and denominator of the first fraction by 15, of the second by 10, and of the third by 12, each will have 60 for its denominator, and we have

$$\frac{195a-75b}{60}-\frac{70a-20b}{60}-\frac{36a}{60}=\frac{195a-75b-(70a-20b)-36a}{60}$$

$$=\frac{195a-106a-75b+20b}{60}=\frac{89a-55b}{60}.$$

(Ex. 3.) Find the value of $\frac{x}{1-x}+\frac{x^2}{(1-x)^2}-\frac{x^3}{(1-x)^3}$.

Multiply both terms of the first fraction by $(1-x)^2$, and of the second by $1-x$, we shall then have

$$\frac{x(1-x)^2+x^2(1-x)-x^3}{(1-x)^3}=\frac{x-2x^2+x^3+x^2-x^3-x^3}{(1-x)^3}=\frac{x-x^2-x^3}{(1-x)^3}.$$

(Ex. 4.) Multiply $a-\frac{x^2}{a}$ by $\frac{a}{x}+\frac{x}{a}$.

$$\begin{array}{l} a-\frac{x^2}{a} \\ \frac{a}{x}+\frac{x}{a} \\ \hline \frac{a^2}{x}-x \\ \qquad +x-\frac{x^3}{a^2} \\ \hline \frac{a^2}{x} \quad * \quad -\frac{x^3}{a^2}=\frac{a^4-x^4}{a^2x}. \end{array}$$

For $-\frac{x^2}{a}\times\frac{a}{x}=-\frac{ax^2}{ax}=-x$ and $a\times\frac{x}{a}=x$.

(Ex. 5.) Multiply $x^2+2+\frac{1}{x^2}$ by $x+\frac{1}{x}$.

$$\begin{array}{l} x^2+2+\frac{1}{x^2} \\ x+\frac{1}{x} \\ \hline x^3+2x+\frac{1}{x} \\ \quad +x+\frac{2}{x}+\frac{1}{x^3} \\ \hline x^3+3x+\frac{3}{x}+\frac{1}{x^3}. \\ \hline \end{array}$$

(Ex. 6.) Divide $\frac{a}{a-b}+\frac{b}{a+b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$.

$$\frac{a}{a-b}+\frac{b}{a+b}=\frac{a^2+ab}{a^2-b^2}+\frac{ab-b^2}{a^2-b^2}=\frac{a^2+2ab-b^2}{a^2-b^2},$$

$$\frac{a}{a-b}-\frac{b}{a+b}=\frac{a^2+ab}{a^2-b^2}-\frac{ab-b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2};$$

$$\therefore \text{quotient}=\frac{a^2+2ab-b^2}{a^2-b^2}\times\frac{a^2-b^2}{a^2+b^2}=\frac{a^2+2ab-b^2}{a^2+b^2}.$$

(Ex. 7.) $\frac{a^2-b^2}{5}\times\frac{15a^2}{a+b}=3a(a-b)$.

(Ex. 8.) $\frac{a+b}{2}+\frac{a-b}{2}=a$.

(Ex. 9.) $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}=\frac{85a-20b}{24}$.

(Ex. 10.) $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}=\frac{a^2c+b^2c+a^2b+c^2b}{abc}$.

(Ex. 11.) $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}=\frac{bc-ac+ab}{abc}$.

(Ex. 12.) $\frac{a}{a+x} + \frac{x}{a-x} = \frac{a^2+x^2}{a^2-x^2}.$

(Ex. 13.) $\frac{a^2+b^2}{a^2-b^2} + \frac{a}{a+b} + \frac{b}{a-b} = \frac{2a}{a+b}.$

(Ex. 14.)

$$\frac{a+b}{a-b} + \frac{a^2-b^2}{a^2+b^2} - \frac{a^2+b^2}{a^2-b^2} = \frac{a^4+2a^3b-2a^2b^2+2ab^3+b^4}{a^4-b^4}.$$

(Ex. 15.) $\frac{a-b}{ab} + \frac{c-a}{ac} + \frac{b-c}{bc} = 0.$

(Ex. 16.) $\frac{1}{2(x+1)} - \frac{4}{x+2} + \frac{9}{2(x+3)} = \frac{x^2}{x^3+6x^2+11x+6}.$

(Ex. 17.) $\frac{x^3-a^3}{x^3+a^3} \div \frac{x-a}{x+a} = \frac{x^2+ax+a^2}{x^2-ax+a^2}.$

(Ex. 18.) $\frac{x^2-(a+b)x+ab}{x^2-(a-b)x-ab} \times \frac{x+b}{x-b} = 1.$

(Ex. 19.) $\left(\frac{2x}{a} + \frac{a}{2x}\right)\left(\frac{2x}{a} + \frac{3a}{2x}\right) = \frac{4x^2}{a^2} + 4 + \frac{3a^2}{4x^2}.$

(Ex. 20.) $\left(\frac{a^3}{b^3} + 3\frac{a}{b} + 3\frac{b}{a} + \frac{b^3}{a^3}\right) \div \left(\frac{a}{b} + \frac{b}{a}\right) = \frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}.$

SECTION II.

GREATEST COMMON MEASURE.

43. When a number divides each of two other numbers without leaving a remainder, it is said to be a common measure of the two, and to be the greatest when no number greater than itself is contained in both of the numbers.

Thus of the numbers 36 and 48, 4 is said to be a common measure, but 12 is the *greatest* common measure, since no number greater than 12 will divide both 36 and 48.

44. And as numbers, so algebraical terms and expressions may have greatest common measures; and the method by which these are found differs but little from that which is used in finding the greatest common measure of numbers.

When both of the algebraical quantities are simple terms, the greatest common measure is easily discovered.

Thus if $6a^2bc$ and $2ac^2d$ be two algebraic terms, we see that $2ac$ will divide both, and as no other combination of letters will, therefore $2ac$ is the greatest divisor. And again, to find the greatest common measure of $32\,a^3bc$ and $56\,a^2b^2d$: we first observe that 8 and 8 only will divide the numerical coefficients, and then that a^2 and b are found in both terms: hence the greatest common measure is $8a^2b$.

45. The chief object we have in finding the greatest common measure is to reduce fractions to their lowest terms. In the following simple examples we strike out the numerical and literal factors which are obviously common to both numerator and denominator, and the fraction is reduced without noticing the value of the greatest common divisor.

$$\text{Thus, } \frac{6a^2bc}{9abc^2} = \frac{2a^2bc}{3abc^2} = \frac{2a}{3c};$$

$$\text{Again, } \frac{12ax^2}{16a^2x^3} = \frac{3}{4ax}, \text{ and } \frac{9ab^2}{15abc} = \frac{3b}{5c},$$

$$\frac{5ax^2}{30bx} = \frac{ax}{6b}, \text{ and } \frac{32ax^2y^2}{16a^2xz} = \frac{2xy^2}{az},$$

$$\frac{52ab^5x^2}{13a^2b^3x^3} = \frac{4b^2}{ax}, \text{ and } \frac{85ax^2y^2z^2}{35a^2x^3yz} = \frac{17yz}{7ax},$$

$$\frac{15a^3c^2 - 25a^7}{5a^2bc + 55a^5b^2} = \frac{3ac^2 - 5a^5}{bc + 11a^3b^2}, \text{ and } \frac{ax + x^2}{3bx - cx} = \frac{a + x}{3b - c},$$

$$\frac{14a^2 - 7ab}{10ac - 5bc} = \frac{7a}{5c}, \text{ and } \frac{12a^2x^2 + 2ax^3}{18ab^2x + 3b^2x^2} = \frac{2ax^2}{3b^2},$$

$$\frac{5a^2 + 5ax}{a^2 - x^2} = \frac{5a}{a + x}, \text{ and } \frac{a^3 - x^3}{(a - x)^2} = \frac{a^2 + ax + x^2}{a - x}.$$

46. We shall now investigate the truth of the rule which is given in all books of arithmetic, for finding the greatest common measure of two numbers a and b.

And to do so we must prove the two following propositions:

(1) If any number as x measures a it will measure any multiple of a: for if $a = nx$, therefore $ma = mnx$, or if x be contained in a, n times, it is contained in ma, mn times.

(2) If x measure both a and b it will measure $a \pm b$.

For if $a = mx$ and $b = nx$, $a \pm b = mx \pm nx = (m \pm n) . x$, i.e. x is contained in $a \pm b$, $m \pm n$ times.

These being premised, let b be $< a$; divide a by b and let the quotient be p and the remainder c; again divide b by c, and let the quotient be q and the remainder d, then divide c by d, let the quotient be r, and let there be no remainder. This process may be thus exhibited.

$$\begin{array}{l} b)\ a\ (p \\ \quad \underline{pb} \\ \quad c)\ b\ (q \\ \qquad \underline{qc} \\ \qquad d)\ c\ (r \\ \qquad\quad \underline{dr} \\ \qquad\quad \underline{0} \end{array}$$

Then by the principle of division,

$$a = pb + c,$$
$$b = qc + d,$$
$$c = rd.$$

Whence we see that d, the last divisor, measures c, therefore qc, and therefore $qc + d$, i.e. b, therefore pb and $pb + c$, or a, hence d measures both a and b. Also d is the greatest common measure: for let D be the greatest.

Then D measures a and b, and therefore a and pb, and therefore $a-pb$ or c, therefore qc, and therefore $b-qc$, or d, therefore the greatest common measure of a and b measure d: but d measures a and b, therefore d is that greatest common measure or $D \doteq d$: for no number greater than itself can divide d.

47. Again, to find the greatest common measure of three numbers, a, b, c.

Let D be the greatest common measure of a and b,

and E .... of D and c.

Then E is .... of a, b, c.

For every common measure of a and b measures D, therefore E measures a and b, and it is the greatest that measures D and c, therefore it is the greatest which measures a, b, c.

48. And finally, to reduce fractions to their lowest terms, we must divide the numerator and denominator by their greatest common measure, and the fraction will be in its lowest terms.

For let a and b be the numerator and denominator of a fraction and let D be their greatest common measure:

and let $a=pD$ and $b=qD$;

$$\therefore \frac{a}{b}=\frac{p.D}{q.D}=\frac{p}{q}.$$

and p and q have no common measure, therefore $\frac{p}{q}$ is the fraction, $\frac{a}{b}$ in its lowest terms: this article applies both to numbers and to algebraical quantities.

EXAMPLES.

49. Find the greatest common measure of 365 and 320.

```
320) 365 (1
     320
     ---
      45) 320 (7
          315
          ---
            5) 45 (9
               45
               --
                *
               --
```

∴ 5 is the greatest common measure.

Find the greatest common measure of 114, 102 and 822.

```
102) 114 (1
     102
     ---
      12) 102 (8
           96
           --
            6) 12 (2
               12
               --
                *
               --
```

and 6 is the greatest common measure of 102 and 114.

```
6) 822
   ---
   137
   ---
```

∴ 6 is the greatest common measure of 102, 114 and 822.

Exs. Reduce $\frac{483}{568}$, $\frac{1020}{1596}$, $\frac{7631}{26415}$ and $\frac{8398}{29393}$ to their lowest terms.

Ans. $\frac{21}{25}$; $\frac{85}{153}$; $\frac{13}{45}$; $\frac{2}{7}$.

50. Next to find the greatest common measure of two algebraical expressions.

Let A and B be two algebraical quantities arranged according to the powers of some one letter beginning with the highest, and suppose the highest power in B not to exceed the highest power in A.

Make B the divisor, P the quotient, and Cc the remainder, where C is a compound term and c a simple factor, obviously not contained in B.

Again, make B the dividend and C the divisor, Q the quotient and Dd the remainder, where D is a compound quantity, and d a simple factor, not contained in C.

Lastly, let C be the dividend, D the divisor, R the quotient, and suppose that there is no remainder, then D will be the greatest common measure required. The work may be written down thus:

$$
\begin{array}{l}
B)\ A\ (P \\
\quad\ \underline{PB} \\
\quad\ \ Cc \\
C)\ \overline{B}\ (Q \\
\quad\ \underline{QC} \\
\quad\ \ Dd \\
D)\ \overline{C}\ (R \\
\quad\ \underline{RD} \\
\quad\ \ \ 0
\end{array}
$$

We see that

$$C = RD,$$

$$B = QC + Dd,$$

$$A = PB + Cc;$$

Hence D will measure C; $\therefore QC$; $\therefore QC + Dd$, and $\therefore B$; $\therefore PB$, and $\therefore PB + Cc$ or A, i. e. D is a factor of A and

B; also the greatest common measure of A and B will measure A and B, and $\therefore$ $A - PB$ or Cc; $\therefore$ $B - QC$ or Dd, and $\therefore$ D; $\therefore$ D must be the greatest common measure.

51. The following example will illustrate the process. Find the greatest common divisor of $a^2 - 5ab + 4b^2$, and $a^3 - a^2b + 3ab^2 - 3b^3$.

(Ex. 1.) Here $A = a^3 - a^2b + 3ab^2 - 3b^3$; $B = a^2 - 5ab + 4b^2$;

$$a^2 - 5ab + 4b^2)\ a^3 - a^2b + 3ab^2 - 3b^3\ (a + 4b = P$$
$$\underline{a^3 - 5a^2b + 4ab^2}$$
$$4a^2b - ab^2 - 3b^3$$
$$\underline{4a^2b - 20ab^2 + 16b^3}$$
$$19ab^2 - 19b^3 = Cc.$$

But $19ab^2 - 19b^3 = 19b^2(a - b)$ and as neither 19 nor b^2 will divide either A or B, $a - b$ will be the common measure if there be one; now divide B by $a - b$:

$$a - b)\ a^2 - 5ab + 4b^2\ (a - 4b$$
$$\underline{a^2 - ab}$$
$$-4ab + 4b^2$$
$$\underline{-4ab + 4b^2}$$
$$*\quad *$$

and $\therefore$ $a - b$ is the greatest common measure sought.

52. In the application of the rule just given it is frequently found convenient to multiply one of the terms A or B by a number which is not a factor of the other: this however will not produce any effect upon the greatest common divisor: For if D be the greatest common divisor

of A and B, so that $A = pD$ and $B = qD$, and if B be multiplied by n whence $nB = nqD$; D is still the greatest common measure of A and nB, so long as n is not a factor of A.

(Ex. 2.) Find the greatest common measure of $7a^2 - 23ab + 6b^2$, and $5a^3 - 18a^2b + 11ab^2 - 6b^3$.

The former term is obviously the divisor, and the first term of the quotient would be $\frac{5a}{7}$; to avoid therefore this fraction we shall multiply the dividend by 7, thus

$$\begin{array}{l} 5a^3 - 18a^2b + 11ab^2 - 6b^3 \\ 7 \\ \hline \end{array}$$

$$7a^2 - 23ab + 6b^2)\ 35a^3 - 126a^2b + 77ab^2 - 42b^3\ (5a$$

$$\begin{array}{l} 35a^3 - 115a^2b + 30ab^2 \\ \hline -11a^2b + 47ab^2 - 42b^3 \end{array}$$

Multiply by 7, $\quad 7$

$$\begin{array}{l} -77a^2b + 329ab^2 - 294b^3\ (-11b \\ -77a^2b + 253ab^2 - \ 66b^3 \\ \hline \qquad 76ab^2 - 228b^3 \\ \text{or } 76b^2(a - 3b). \end{array}$$

and as neither 76 nor b^2 are divisors of the quantities, we try $a - 3b$,

$$a - 3b)\ 7a^2 - 23ab + 6b^2\ (7a - 2b$$

$$\begin{array}{l} 7a^2 - 21ab \\ \hline \quad -2ab + 6b^2 \\ \quad -2ab + 6b^2 \\ \hline \quad\ \ * \quad * \\ \hline \end{array}$$

53. When a simple term divides either the numerator or the denominator of a fraction, it may be omitted in the process: but when it divides both, it must be reckoned part of the common measure although it be left out when the division is performed.

(Ex. 3.) Find the greatest common measure of

$$\frac{15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4}{12a^3b^2+38a^2b^3+16ab^4-10b^5}.$$

Here a is a common factor of the upper, and $2b^2$ of the lower line; omitting them, the quantities become, being arranged as divisor and dividend,

$$6a^3+19a^2b+8ab^2-5b^3)\,15a^4+10a^3b+4a^2b^2+6ab^3-3b^4$$

$$2$$

$$30a^4+20a^3b+8a^2b^2+12ab^3-6b^4\,(5a$$

$$30a^4+95a^3b+40a^2b^2-25ab^3$$

$$-75a^3b-32a^2b^2+37ab^3-6b^4$$

$$2$$

$$-150a^3b-64a^2b^2+74ab^3-12b^4\,(-25b$$

$$-150a^3b-475a^2b^2-200ab^3+125b^4$$

$$411a^2b^2+274ab^3-137b^4$$

or, $137b^2(3a^2+2ab-b^2)$.

$$3a^2+2ab-b^2)\,6a^3+19a^2b+8ab^2-5b^3\,(2a+5b$$

$$6a^3+\ \ 4a^2b-2ab^2$$

$$15a^2b+10ab^2-5b^3$$

$$15a^2b+10ab^2-5b^3$$

* * *

$\therefore\ 3a^2+2ab-b^2$ is the greatest common measure required.

(Ex. 4.) Find the same for the quantities $2x^3+3x^2+x$ and x^3-x^2-2x.

Here x divides both quantities and we therefore find the common measure of $2x^2+3x+1$, and x^2-x-2 which by the preceding method is, $x+1$, and therefore the complete common measure is x^2+x.

THE LEAST COMMON MULTIPLE OF TWO QUANTITIES.

54. A common multiple of two numbers is that number which is divided by both; and the least common multiple that least number which is so divisible.

Thus 48 is a common multiple of 8 and 12, for both 8 and 12 will measure 48, and 24 is the least common multiple, since there is no number less than it, which is divisible both by 8 and 12: and the quotients arising from the division of 24 by 8 and 12 are 3 and 2, which have no common divisor except unity.

55. But to find the least common multiple of any two quantities a and b: let D be their greatest common measure, and let $a=pD$ and $b=qD$ where p and q have no common divisor.

Then since $pqD=pD\times q$ or $qD\times p$,

i. e. $=a\times q$ or $b\times p$;

therefore pqD is divisible by a and b respectively, and the quotients q and p which arise from these divisions, have no common divisor; therefore pqD is the least common multiple of a and b.

$$\text{But } pqD=\frac{pD\times qD}{D}=\frac{a\times b}{D},$$

or the least common multiple of two quantities is equal to the product of the quantities, divided by their greatest common measure.

56. Next to find the least common multiple of three quantities a, b, c.

Let m be the least common multiple of a and b, and M that of m and c; then M will be the least common multiple required. For since m and therefore a and b measure M, and M is the least multiple of m and c, it is the least quantity which can be measured by a, b, and c. In the same manner may the least common multiple of a greater number of quantities be found.

(Ex. 1.) Find the least common multiple of 24 and 16.

Here 8 is the greatest common measure $= D$;

$$\therefore \quad m = \frac{24 \times 16}{8} = 3 \times 16 = 48.$$

(Ex. 2.) Find M, for 12, 18 and 20,

D for 12 and 18 is 6;

$$\therefore \quad m = \frac{12 \times 18}{6} = 2 \times 18 = 36,$$

and D for 36 and 20 is 4;

$$\therefore \quad M = \frac{36 \times 20}{4} = 9 \times 20 = 180;$$

$\therefore$ 180 is the least multiple of 12, 18, and 20.

EXAMPLES.

(1) The greatest common measure of $x^2 + 2x - 3$ and $x^2 + 5x + 6$ is $x + 3$.

(2) Of $x^2 - 8x + 15$ and $x^2 + 2x - 15$ is $x - 3$.

(3) Of $a^2 - 2ax + x^2$ and $a^2 - x^2$ is $a - x$.

(4) Of $a^3 + x^3$ and $a^2 + 2ax + x^2$ is $a + x$.

(5) Of $x^3 - 2x^2$ and $x^2 - 4x + 4$ is $x - 2$.

(6) Of $x^3 - 6x^2 + 11x + 6$ and $x^3 + 4x^2 + x - 6$ is $x - 1$.

(7) Of $3x^2-22x+32$ and $x^3-11x^2+32x-28$ is $x-2$.

(8) Of $x^2-(a-b)x-ab$ and $x^2-(a+b)x+ab$ is $x-a$.

(9) Of $x^4-9x^3+29x^2-39x+18$ and $4x^3-27x^2+58x-39$ is $x-3$.

(10) Of x^4-a^4 and $x^3+bx^2-a^2x-a^2b$ is x^2-a^2.

(11) Of $3x^2-38x+119$ and $x^3-19x^2+119x-245$ is $x-7$.

(12) Of $x^3+x^2y+xy+y^2$ and x^4-y^2 is x^2+y.

(13) Of $x^8+x^6y^2+x^2y+y^8$ and x^4-y^4 is x^2+y^2.

(14) Of $9x^3+53x^2-9x-18$ and $x^2+11x+30$ is $x+6$.

(15) Of $2x^3+x^2-8x+5$ and $7x^2-12x+5$ is $x-1$.

(16) Of $2x^4-4x^3+8x^2-12x+6$ and $3x^4-3x^3-6x^2+9x-3$ is $(x-1)^2$.

(17) Of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$ is $5x^2-1$.

(18) Of $ac+bd+ad+bc$ and $af+2bx+2ax+bf$ is $a+b$.

(19) Of $a^2+b^2+c^2+2ab+2ac+2bc$ and $a^2-b^2-c^2$ $2bc$ is $a+b+c$.

(20) Of $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$ and $3x^2-(4a+2b)x+2ab+a^2$ is $x-a$.

(21) Of $3a^4-a^2b^2-2b^4$ and $10a^4+15a^3b-10a^2b^2-15ab^3$ is a^2-b^2.

(22) Of $x^4-ax^3+(b-1)x^2+ax-b$ and $x^4-bx^3+(a-1)x^2+bx-a$ is x^2-1.

(23) The least common multiple of $2a$, $4a^2$ and $3ab$ is $12a^2b$.

(24) Of $8a^2$, $12a^3$ and $20a^4$ is $120a^4$.

(25) Of $a+x$, a^2-x^2, and $(a-x)^2$ is $a^3-a^2x-ax^2+x^3$.

(26) $\frac{x^2+2x-3}{x^2+5x+6}$ reduced to its lowest terms $=\frac{x-1}{x+2}$.

(27) $\frac{x^2-9x+20}{x^2+6x-55}=\frac{x-4}{x+11}$.

(28) $\frac{9x^3+53x^2-9x-18}{x^2+11x+30}=\frac{9x^2-x-3}{x+5}$.

(29) $\frac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}=\frac{3a+5b}{3c-1}$.

(30) $\frac{2x^2+3x+1}{x^2-x-2}=\frac{2x+1}{x-2}$.

(31) $\frac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}=\frac{a-2b}{a+b-c}$.

(32) $\frac{x^8+x^6y^2+x^2y+y^3}{x^4-y^4}=\frac{x^6+y}{x^2-y^2}$.

(33) $\frac{3x^3-3x^2y+xy^2-y^3}{4x^2-xy+3y^2}=\frac{3x^2+y^2}{4x+3y}$.

(34) $\frac{ab+2a^2-3b^2-4bc-ac-c^2}{9ac+2a^2-5ab+4c^2+8bc-12b^2}=\frac{a-b-c}{a-4b+4c}$.

SECTION III.

INVOLUTION AND EVOLUTION.

INVOLUTION.

57. Involution is the finding of powers of a number or quantity, or is the method by which the square, cube, fourth power, &c. of any quantity is to be found.

We have already seen that the power of any simple term is expressed by writing the number of the power above it, which number is called the index: thus

$a^2, a^3, a^4, a^5 \ldots\ldots a^n$ respectively express the second, third, fourth, fifth, and n^{th} powers of a.

And to find the cube of a^2 we write $(a^2)^3$,

$$\text{but } (a^2)^3 = a^2 \times a^2 \times a^2 = a^6 = a^{2\times 3};$$

i.e. we multiply the index 2 by 3.

So to find the m^{th} power of a^n we write $(a^m)^n$.

$$\text{But } (a^m)^n = a^m \times a^m \times a^m \times a^m \ldots\ldots \text{to } n \text{ terms},$$

$$= a^{m+m+m} + \text{\&c. to } n \text{ terms},$$

$$= a^{mn}.$$

Hence we have this rule, to find the n^{th} power of any quantity we must multiply the index of that quantity by n.

Hence,

$$\text{the square of } a^{10} = (a^{10})^2 = a^{20},$$

$$\text{the fourth power of } a^2b^3 = (a^2b^3)^4 = a^8b^{12},$$

$$\text{the cube of } 3a^6b^4c^2 = 27a^{18}b^{12}c^6.$$

58. Hence may we find the simplest form of such an expression as $\{(a^m)^n\}^p$.

$$\text{For } (a^m)^n = a^{mn}; \quad \therefore \{(a^m)^n\}^p = (a^{mn})^p = a^{mnp};$$

or in examples of this description the new index is equal to the product of the separate indices.

(Ex. 1.) $\{(a^2)^3\}^4 = a^{2\times 3\times 4} = a^{24}$.

(Ex. 2.) $\left\{\left(\dfrac{a^3b^2}{c^2}\right)^2\right\}^3 = \dfrac{a^{18}\,b^{12}}{c^{12}}$.

59. When the quantity to be involved is negative, the result will be negative when the index is an odd number,

and positive when it is even. For the product of an odd number of negative quantities is negative; but of an even number it is positive.

Thus $(-a)^5 = -a^5$ but $(-a)^4 = (-a)^2 \times (-a)^2 = a^4$.

And $\{(-a^2)^3\}^2 = (-a^6)^2 = a^{12}$,

but $\{(-a^2)^3\}^5 = (-a^6)^5 = -a^{30}$.

60. Next to find the powers of a binomial.

$$(1) \quad (a+b)^2 = (a+b) \times (a+b) = a^2 + 2ab + b^2,$$

by which we see that the square of any two terms is equal to the sum of the squares of the two terms together with twice the product of the two terms.

Thus if $a = 7$ and $b = 6$ or $a + b = 13$.

Then

$(a+b)^2$ or $(13)^2 = 7^2 + 2 \times 7 \times 6 + 6^2 = 49 + 84 + 36 = 169$.

And by this formula may the squares of numbers often be found in a very convenient manner: thus if it be required to find the square of 79, which may be written $70+9$: then

$$(70+9)^2 = 70^2 + 2 \times 70 \times 9 + 9^2 = 4900 + 1260 + 81 = 6241.$$

Also since $(a+1)^2 = a^2 + 2a + 1$; we may, having given the square of one number as a, find that of the next succeeding or $a+1$, by adding $2a+1$ to a^2, and so we may form a table of squares.

Then as $10^2 = 100$;

$$\therefore (10+1)^2 = 100 + 2 \,.\, 10 + 1 = 121,$$

$$(11+1)^2 = 121 + 2 \times 11 + 1 = 144,$$

$$(12+1)^2 = 144 + 2 \times 12 + 1 = 169,$$

$$(13+1)^2 = 169 + 2 \times 13 + 1 = 196;$$

and we may readily either by multiplication or by the rule just enunciated, prove that;

$$\therefore\ (a+2b)^2 = a^2 + 4ab + 4b^2,$$

$$(2x+3y)^2 = 4x^2 + 12xy + 9y^2,$$

$$\left(x+\frac{p}{2}\right)^2 = x^2 + px + \frac{p^2}{4},$$

$$\left(\frac{x}{2}+\frac{y}{4}\right)^2 = \frac{x^2}{4} + \frac{xy}{3} + \frac{y^2}{9}.$$

Also $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$ or the square of the difference of two terms equals the sum of the squares of the two terms diminished by twice their product;

$$\therefore\ (2a-5b)^2 = 4a^2 - 20ab + 25b^2;$$

$$\therefore\ (6x-7y)^2 = 36x^2 - 24xy + 49y^2,$$

$$\left(x-\frac{p}{2}\right)^2 = x^2 - px + \frac{p^2}{4}.$$

61. Hence also may we find the square of a binomial $a+b+c$, for considering $b+c$ as one quantity we have by the rule, placing a vinculum over $b+c$,

$$(a+\overline{b+c})^2 = a^2 + 2a(b+c) + (b+c)^2$$

$$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$$

$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

To take a numerical example of this formula. Let the square of 432 be required:

the number must be written $400 + 30 + 2$,

and here $a = 400$, $b = 30$, and $c = 2$;

$$\therefore\ (432)^2 = (400)^2 + (30)^2 + 2^2 + 2\times 400\times 30 + 2\times 400\times 2 + 2$$
$$\times\, 30 \times 2 = 160000 + 900 + 4 + 24000 + 1600 + 120 = 186624.$$

62. So also as $a+b+c+d$ may be written $\overline{a+b}+\overline{b+c}$;

$$\therefore\ (\overline{a+b}+\overline{c+d})^2=(a+b)^2+2\,\overline{a+b}\,.\,\overline{c+d}+\overline{c+d}|^2$$

$$\text{but}\quad (a+b)^2=a^2+2ab+2b^2,$$

$$2(a+b)(c+d)=2ac+2ad+2bc+2bd,$$

$$(c+d)^2=c^2+2cd+d^2;$$

$$\therefore\ (a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc$$
$$+2bd+2cd;$$

or the square of the sum of four quantities a, b, c, d, is equal to the sum of the squares of the four quantities + the product of $2a$, into all the letters that follow a, + the product of $2b$ into the letters that follow b, + $2c$ into d. It is however better for the beginner to verify the last two results by the actual multiplication of $a+b+c$ into $a+b+c$; and of $a+b+c+d$ into $a+b+c+d$.

63. The cube of $(a+b)$ or $(a+b)^3$ may be found by multiplying $(a+b)$ into itself *twice*.

$$(a+b)^3=(a+b)\,.\,(a+b)\,.\,(a+b)=(a^2+2ab+b^2)(a+b)$$
$$=a^3+3a^2b+3ab^2+b^3.$$

And $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

The student should render himself familiar with these equivalent expressions for $(a+b)^3$, and $(a-b)^3$.

64. Ordinary multiplication will lead to the following results; namely,

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4,$$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4,$$

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5,$$

$$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$$

65. **As additional examples of involution, prove that**

$$(2x+1)^3 = 8x^3 + 12x^2 + 6x + 1.$$

$$(2x^2 - 3y^2)^3 = 8x^6 - 36x^4y^2 + 54x^2y^4 - 27y^6.$$

$$(2a+3b)^4 = 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$$

$$(5-4x)^4 = 625 - 2000x + 2400x^2 - 1280x^3 + 256x^4.$$

$$(3ac-2bd)^5 = 243a^5c^5 - 810a^4c^4bd + 1080a^3c^3b^2d^2 - 720a^2c^2b^3d^3$$
$$+ 240acb^4d^4 - 32b^5d^5.$$

EVOLUTION.

66. Evolution means the extraction of roots; and is consequently the inverse of Involution.

The rules which it gives for finding the square, cube or n^{th} roots of any quantity as a, are derived from the methods by which the second, third, &c. powers of a are obtained.

Thus $\because\ a \times a = a^2$; $\therefore\ a$ is the square root of a^2.

And $\because\ a \times a \times a = a^3$; $\therefore\ a$ is the cube root of a^3.

And $\because\ a^2 \times a^2 = a^4$; $\therefore\ a^2$ is the square root of a^4.

$\because\ a^4 \times a^4 \times a^4 = a^{12}$; $\therefore\ a^4$ is the cube root of a^{12}.

And thus we see, that to find the square root of any power of a we must divide the index of a by 2, to find the cube root of any power, the index must be divided by 3, and so on: and to find the n^{th} root we must divide the index by n: for to shew the truth of this last assertion:

$$\because\ a^m \times a^m \times a^m \times \ldots \text{to } n \text{ terms} = a^{mn};$$

$$\therefore\ \sqrt[n]{a^{mn}} = a^m = a^{\frac{mn}{n}}.$$

Ex. Thus
$$\sqrt{4a^4b^2c^2} = 2a^2bc,$$
$$\sqrt[3]{27a^6b^3} = 3a^2b,$$
$$\sqrt[4]{\frac{16a^4b^8}{c^{12}}} = \frac{2ab^2}{c^3}.$$

67. Had we then to find the square root of the simple power of a, since $a = a^1$, we should have by the application of the foregoing rule, the square root of a represented by $a^{\frac{1}{2}}$, in the same manner $\sqrt[3]{a}$, might be written $a^{\frac{1}{3}}$, and the fourth root of a, $a^{\frac{1}{4}}$.

So also the cube root of a^2, may be put $a^{\frac{2}{3}}$,

and the fourth root of a^3 may be written $a^{\frac{3}{4}}$.

These remarks are very important, and will if fully understood be exceedingly useful hereafter.

68. But it may be said that since the definition of a square root, as of a, implies that it is a quantity which multiplied by itself should produce a; how can we say that

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a?$$

We have already seen that when m and n are integers $a^m \times a^n$ may be written a^{m+n}, or that the index of the product of the powers of the same quantity is found by adding together the indices of the factors.

Let us assume this rule to be true, when m and n are not integers but fractions, and see whether the result we arrive at through this assumption, be true or false.

To do so let us take the instance before us.

Now if $a^{\frac{1}{2}}$ truly represent $\sqrt{a}$; $\therefore$ $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$ should equal a.

But by the rule of indices $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$;

$\therefore$ the rule of indices is in this case correct;

and at the same time we see that $\sqrt{a}$ may be truly written $a^{\frac{1}{2}}$.

And similarly since by the same rule of indices,

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{3}{3}} = a;$$

and that by the definition of a cube root,

$$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a;$$

$\therefore$ $\sqrt[3]{a}$ and $a^{\frac{1}{3}}$ may each represent the cube root of a.

And hence the third root of a^4 and the fifth root of a^6 may be written $a^{\frac{4}{3}}$ and $a^{\frac{6}{5}}$ respectively.

And hence also, whenever we meet with a quantity as a having a fractional index, we must remember that the numerator of the fraction expresses the *power* to which the quantity is to be raised, and the denominator the root which is to be extracted.

Thus $a^{\frac{m}{n}}$ is to be understood as meaning the n^{th} root of the m^{th} power of a.

69. It will greatly aid our full comprehension of these things if we compute some numerical expressions.

$$\text{Thus,}\quad (4)^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = (\sqrt{4})^3 = 2^3 = 8,$$

$$\text{and}\quad (8)^{\frac{5}{3}} = (8^{\frac{1}{3}})^5 = (\sqrt[3]{8})^5 = 2^5 = 32,$$

$$(32)^{\frac{4}{5}} = (32^{\frac{1}{5}})^4 = (\sqrt[5]{32})^4 = 2^4 = 16.$$

$$\text{Also,}\quad (27)^{\frac{7}{3}} = 19683: \quad\text{and}\quad (625)^{\frac{5}{4}} = 3125.$$

70. These remarks upon indices may make this a fit place to explain the meaning of such a term as a^{-m}, i. e. a quantity with a negative index.

In fact a^{-m} is another method of writing $\frac{1}{a^m}$: and this being so we ought to have $a^m \times a^{-m} = 1$.

But if the law of indices be universally true

$$a^m \times a^{-m} = a^{m-m} = a^0,$$

and therefore a^0 ought to be equal to 1.

Now by the rule of division for powers of the same letter

$$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}.$$

$$\text{Let } m = n; \therefore a^m \div a^m = \frac{a^m}{a^m} = 1;$$

$$\text{but } a^{m-m} \text{ becomes } a^0; \therefore a^0 = 1.$$

$$\text{Hence, } a^m \times a^{-m} = 1, \text{ and } \therefore a^{-m} = \frac{1}{a^m}.$$

71. We may now compute expressions of the following kind.

(Ex. 1.) $4a^{-4} \times 5a^{-3} \times 2a^4 \times a^6 = 40a^{10-7} = 40a^3.$

For this is the same as writing

$$\frac{4}{a^4} \times \frac{5}{a^3} \times 2a^4 \times a^6 = \frac{40a^{10}}{a^7} = 40\,a^3.$$

(Ex. 2.) $2a^{-2} \times 3a^{-3} \times 4a^{-5} \times 5a^6 = 120a^{-4} = \frac{120}{a^4}.$

(Ex. 3.) $-3a^{-2}b^2c^{-1} \times 5a^{-4}b^{-6}c^3 = -15a^{-6}b^{-4}c^2 = \frac{15c^2}{a^6b^4}.$

(Ex. 4.) $\frac{18a^{-5}b^3}{7c^{-2}d^{-6}} \times \frac{4a^6b^{-5}}{9c^3d^9} = \frac{8ab^{-2}}{7cd^3} = \frac{8a}{7b^2cd^3}.$

72. And here we may remark that as

$$+a \times +a = a^2 \text{ and } -a \times -a \text{ also} = a^2;$$

$$\therefore \sqrt{a^2} = a \text{ or } -a.$$

But as $+a \times +a \times +a = a^3$ and $-a \times -a \times -a = -a^3$;

$$\therefore \sqrt[3]{a^3} = a \text{ and } \sqrt[3]{-a^3} = -a,$$

$$\text{so also } \sqrt[4]{a^8} = +a^4 \text{ or } -a^4.$$

$$\text{But } \sqrt[3]{a^9} = a^3 \text{ and not also } -a^3;$$

i. e. the square root of a number is either positive or negative, but the cube root of a positive number is + and of a negative number is −.

A negative number has no square root; for no number multiplied by itself can give a negative result; and such a term as $\sqrt{-9}$ or $\sqrt{-a}$ is called an impossible root.

EXTRACTION OF THE SQUARE ROOT OF COMPOUND ALGEBRAICAL QUANTITIES.

73. The extraction of the square root of a compound quantity is the finding of that factor which multiplied by itself will produce the original quantity.

Thus as we have already seen that

$$a^2 + 2ab + b^2 = (a+b)^2 = (a+b)\,(a+b);$$

$$\therefore\ \sqrt{a^2 + 2ab + b^2} \text{ is } a+b.$$

We must therefore have some method by which the factor $a+b$ may be derived from $a^2 + 2ab + b^2$.

Now the trinomial may be written $a^2 + (2a+b)b$: if therefore we can derive a from a^2, and subtracting a^2 divide the remainder by $2a+b$, the thing is done.

But as $\sqrt{a^2}$ is a, a may be found by extracting the square root of the first term: put therefore a by itself as the first term of the root, then take a^2 from $a^2 + 2ab + b^2$, and the remainder is $2ab + b^2$: now to derive b, double the first term of the root, i. e. take $2a$: divide $2ab$ by $2a$, the quotient is b, the quantity required; write this after a with its proper sign, and $a+b$ is the root required; but to be certain that b is the proper quantity, add b to $2a$, and writing $2a+b$ as a divisor, multiply it by b and subtract the product $2ab + b^2$ from the first remainder, and the result is nothing; the process is in general performed thus,

$$
\begin{array}{rl}
 & a^2+2ab+b^2\,(a+b \\
 & a^2 \\
\hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
 & * \quad * \\
\hline
\end{array}
$$

The principle of the rule is obviously this; we subtract, and here at two steps, from $a^2+2ab+b^2$ the square of $a+b$.

(Ex. 1.) Extract the square root of $4x^4+4a^2x^2+a^4$.

$$
\begin{array}{rl}
 & 4x^4+4a^2x^2+a^4\,(2x^2+a^2 \\
 & 4x^4 \\
\hline
4x^2+a^2) & 4a^2x^2+a^4 \\
 & 4a^2x^2+a^4 \\
\hline
 & * \quad * \\
\hline
\end{array}
$$

(Ex. 2.)

$$
\begin{array}{rl}
 & 81+36a+4a^2\,(9+2a \\
 & 81 \\
\hline
18+2a) & 36a+4a^2 \\
 & 36a+4a^2 \\
\hline
 & * \quad * \\
\hline
\end{array}
$$

(Ex. 3.)

$$
\begin{array}{rl}
 & 25x^2-5x+\frac{1}{4}\left(5x-\frac{1}{2}\right. \\
 & 25x^2 \\
\hline
10x-\frac{1}{2}) & -5x+\frac{1}{4} \\
 & -5x+\frac{1}{4} \\
\hline
 & * \quad * \\
\hline
\end{array}
$$

Here $-5x \div 10x = \frac{-5x}{10x} = -\frac{1}{2}$.

(Ex. 4.)

$$
\begin{array}{rl}
& \frac{9a^2}{4} - ab + \frac{b^2}{9} \left(\frac{3a}{2} - \frac{b}{3} \right. \\
& \frac{9a^2}{4} \\
\hline
3a - \frac{b}{3} \Big) & -ab + \frac{b^2}{9} \\
& -ab + \frac{b^2}{9} \\
\hline
& * \quad *
\end{array}
$$

74. When there is a remainder after the second subtraction, it is a proof either that the root consists of more than two terms, or that there is no factor which will answer the requisite conditions; in either case we continue the process, as the following examples will shew.

(Ex. 5.)

$$
\begin{array}{rl}
& a^2 + 2ab - 2ac - 2bc + b^2 + c^2 \left(a + b - c \right. \\
& a^2 \\
\hline
2a + b \big) & 2ab - 2ac - 2bc + b^2 + c^2 \\
& 2ab \qquad\qquad\qquad + b^2 \\
\hline
2a + 2b - c \big) & -2ac - 2bc + c^2 \\
& -2ac - 2bc + c^2 \\
\hline
& * \quad * \quad *
\end{array}
$$

Observe, that to obtain the third simple root $-c$, we double $a + b$, for a divisor: but we only divide $-2ac$, by $2a$.

(Ex. 6.)

$$
\begin{array}{rl}
& a^4 + 2a^3 + 3a^2 + 2a + 1 \left(a^2 + a + 1 \right. \\
& a^4 \\
\hline
2a^2 + a \big) & 2a^3 + 3a^2 + 2a + 1 \\
& 2a^3 + a^2 \\
\hline
2a^2 + 2a + 1 \big) & 2a^2 + 2a + 1 \\
& 2a^2 + 2a + 1 \\
\hline
& * \quad * \quad *
\end{array}
$$

(Ex. 7.)

$$
\begin{array}{rl}
 & 49x^4 - 28x^3 - 17x^2 + 6x + \frac{9}{4}\left(7x^2 - 2x - \frac{3}{2}\right. \\
 & 49x^4 \\
\hline
14x^2 - 2x) & -28x^3 - 17x^2 \\
 & -28x^3 + 4x^2 \\
\hline
14x^2 - 4x - \frac{3}{2}) & -21x^2 + 6x + \frac{9}{4} \\
 & -21x^2 + 6x + \frac{9}{4} \\
\hline
 & * \quad * \quad * \\
\hline
\end{array}
$$

In the two following examples, the root cannot be exactly found.

(Ex. 8.) Find $\sqrt{1+x}$ and $\sqrt{a^2-x}$.

$$
\begin{array}{rl}
 & 1 + x \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \text{\&c.}\right. \\
 & 1 \\
\hline
2 + \frac{x}{2}) & x \\
 & x + \frac{x^2}{4} \\
\hline
2 + x - \frac{x^2}{8}) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
\hline
2 + x - \frac{x^2}{4} + \frac{x^3}{16}) & \frac{x^3}{8} - \frac{x^4}{64} \\
 & \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{64} + \frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64} + \frac{x^5}{64} - \frac{x^6}{256} \\
\hline
\end{array}
$$

(Ex. 9.) $a^2 - x \left(a - \frac{x}{2a} - \frac{x^2}{8a^3} - \frac{x^3}{16a^5} - \&c. \right.$

$$
\begin{array}{rl}
 & a^2 - x \\
 & a^2 \\
\hline
2a - \frac{x}{2a} \Big) & -x \\
 & -x + \frac{x^2}{4a^2} \\
\hline
2a - \frac{x}{a} - \frac{x^2}{8a^3} \Big) & -\frac{x^2}{4a^2} \\
 & -\frac{x^2}{4a^2} + \frac{x^3}{8a^4} + \frac{x^4}{64a^6} \\
\hline
2x - \frac{x}{a} - \frac{x^2}{4a^3} - \frac{x^3}{16a^5} \Big) & -\frac{x^3}{8a^4} - \frac{x^4}{64a^6} \\
 & -\frac{x^3}{8a^4} + \frac{x^4}{16a^6} + \frac{x^5}{64a^8} + \frac{x^6}{256a^{10}} \\
\hline
 & -\frac{5x^4}{64a^6} - \frac{x^5}{64a^8} - \frac{x^6}{256a^{10}}
\end{array}
$$

75. From the square root we proceed to the cube root, and as in the rule given for finding the former, we derive the rule from involution, so do we in the cube root.

$$\text{Now } (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$

$$\therefore\ a + b = \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3};$$

and $\therefore\ a+b$ must be obtained from $a^3 + 3a^2b + 3ab^2 + b^3$.

Now a is found from a^3; $\because\ \sqrt[3]{a^3} = a$; and to find b we must divide $3a^2b$, by $3a^2$, or by 3 times the square of the first term of the root, also since

$$3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b;$$

if we write $3a^2 + 3ab + b^2$ for a divisor and multiply it by b, the product will be equal to the remainder, which arises from subtracting a^3 from the original cube; the process may be thus exhibited:

(Ex. 1.) $a^3+3a^2b+3ab^2+b^3\,(a+b$

$$\begin{array}{r l}
 & a^3 \\ \hline
3a^2+3ab+b^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\ \hline
 & * \quad * \quad * \\ \hline
\end{array}$$

(Ex. 2.) $a^3+12a^2+48a+64\,(a+4$

$$\begin{array}{r l}
 & a^3 \\ \hline
3a^2+12a+16) & 12a^2+48a+64 \\
 & 12a^2+48a+64 \\ \hline
 & * \quad * \quad * \\
\end{array}$$

For here $b=\frac{12a^2}{3a^2}=4;\quad \therefore\ 3ab=12a$, and $b^2=16$.

(Ex. 3.) $a^6-6a^5+15a^4-20a^3+15a^2-6a+1\,(a^2-2a+1$

$$\begin{array}{r l}
 & a^6 \\ \hline
3a^4-6a^3+4a^2) & -6a^5+15a^4-20a^3 \\
 & -6a^5+12a^4-8a^3 \\ \hline
3a^4-12a^3+15a^2-6a+1) & 3a^4-12a^3+15a^2-6a+1 \\
 & 3a^4-12a^3+15a^2-6a+1 \\ \hline
 & * \quad * \quad * \quad * \quad * \\ \hline
\end{array}$$

In the first division $b=-\frac{6a^5}{3a^4}=-2a$ and the divisor is

$$3a^4+3a^2\times-2a+(-2a)^2=3a^4-6a^3+4a^2.$$

In the second division $b=\frac{3a^4}{3a^4}=1$ and the divisor is

$$3(a^2-2a)^2+3(a^2-2a)\times1+1=3a^4-12a^3+12a^2+3a^2-6a+1.$$

76. In a similar manner may the roots of higher powers be found, but in general other processes are used, and we shall not proceed further, except to remark, that as the fourth power is the square of the square, i.e. $(a+b)^4 = \{(a+b)^2\}^2$ we may find the fourth root of a quantity, by first extracting the square root and then the square root of that root.

77. The rule for extracting the square and cube root of numbers is founded upon the preceding algebraical investigations, an example will shew this.

Extract the square root of 582169, or what is the same thing 580000 + 21000 + 69.

The nearest root is 700.

$$
\begin{array}{rr}
 & 582169\ (\overset{a}{700} + \overset{b}{60} + \overset{c}{3} \\
 & a^2 = 490000 \\ \hline
\overset{2a}{1400} + \overset{b}{60}) & 92169 \\
 & 87600 \\ \hline
\overset{2a}{1400} + \overset{2b}{120} + \overset{c}{3}) & 4569 \\
 & 4569 \\ \hline
 & * \\ \hline
\end{array}
$$

$\therefore$ 763 is the root.

We ordinarily in books of arithmetic proceed thus:

$$
\begin{array}{rl}
 & 5\dot{8}2\dot{1}6\dot{9}\ (763 \\
 & 49 \\ \hline
146) & 921 \\
 & 876 \\ \hline
1523) & 4569 \\
 & 4569 \\ \hline
 & * \\ \hline
\end{array}
$$

Extract the cube root of 884736.

$$884736\,(\overset{a}{90}+\overset{b}{6}$$
$$729000$$

$$3a^2 = 24300)\,155736$$

$$145800 = 3a^2b$$
$$9720 = 3ab^2$$
$$216 = b^3$$
$$155736$$

EXAMPLES.

(1) The square root of $64a^4b^2c^2 = 8a^2bc$, of $61009a^8b^6c^2 = 247a^4b^3c^2$, of $\frac{81}{16}a^{-4}b^{-2}c^6 = \frac{9}{4}a^{-2}b^{-1}c^3$.

(2) The cube root of $125a^3b^3 = 5ab$:

of $389017a^9b^{15}c^{12} = 73a^3b^5c^4$.

(3) $\sqrt{16x^2+40x+25} = 4x+5$, and $\sqrt{256a^4+96a^2+9} = 16a^2+3$.

(4) $\sqrt{(a^2+4ab+4b^2)} = a+2b$.

(5) $\sqrt{(4a^2+12ab+9b^2)} = 2a+3b$.

(6) $\sqrt{(x^4-32a^2x^2+256a^4)} = x^2-16a^2$.

(7) $\sqrt{(x^2-x+\frac{1}{4}} = x-\frac{1}{2}$.

(8) $\sqrt{x^2-px+\frac{p^2}{4}} = x-\frac{p}{2}$.

(9) $\sqrt{\left(\frac{9a^4}{4}+2a^2b^2+\frac{4b^4}{9}\right)} = \frac{3a^2}{2}+\frac{2b^2}{3}$.

4

(10) $\sqrt{\left(\frac{a^2}{b^2}+2+\frac{b^2}{a^2}\right)}=\frac{a}{b}+\frac{b}{a}$.

(11) $\sqrt{\frac{25a^2b^2}{4}-\frac{5abc^2}{3}+\frac{c^4}{9}}=\frac{5}{2}ab-\frac{c^2}{3}$.

(12) $\sqrt{\frac{a^2}{b^2}-\frac{4a}{3c}+\frac{4b^2}{9c^2}}=\frac{a}{b}-\frac{2b}{3c}$.

(13) $\sqrt{(a^2+b^2+c^2+2ab+2ac+2bc)}=a+b+c$.

(14) $\sqrt{(9a^2-12ab+24ac-16bc+4b^2+16c^2}=3a-2b+4c$.

(15) $\sqrt{\left(4a^2-12ab+ab^2+9b^2-\frac{3b^3}{2}+\frac{b^4}{16}\right)}=2a-3b+\frac{b^2}{4}$.

(16) $\sqrt{\left(36x^4-36x^3+17x^2-4x+\frac{4}{9}\right)}=6x^2-3x+\frac{2}{3}$.

(17) $\sqrt{\left(x^4+8x^2+24+\frac{16}{x^4}+\frac{32}{x}\right)}=x^2+4+\frac{4}{x^2}$.

(18) $\sqrt{(9a^2-6ab+30ac+6ad+b^2-10bc-2bd+25c^2+10cd+d^2)}=3a-b+5c+d$.

(19) The cube root of $1+6x+12x^2+8x^3=1+2x$.

(20) $\sqrt[3]{(8a^3-84a^2x+294ax^2-343x^3)}=2a-7x$.

(21) $\sqrt[3]{(a^3+24a^2b+192ab^2+512b^3)}=a+8b$.

(22) $\sqrt[3]{(a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3)}=a+b+c$.

(23) $\sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-$ &c.

(24) $\sqrt{a^2-x^2}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-$ &c.

(25) $\sqrt{a^2+x^2}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+$ &c.

CHAPTER III.

EQUATIONS.

78. A PRINCIPAL object of Algebra, is the determination of quantities which were before unknown, by means of quantities which are known. To effect this, the conditions of the questions must be expressed in algebraic terms; and the equality between any two such expressions, is termed an equation. The two algebraical expressions are called members of the equation:

Thus if $2x + 4 = x + 10$; such an equality is an equation, and $2x + 4$ and $x + 10$ are the members of the equation.

To solve an equation is, to find the value or values of the unknown quantity or quantities, which cause the equality between its members; thus, since 6 put for x, will make $2x + 4$, and $x + 10$ equal to each other, the finding of $x = 6$ is called the solution of the equation $2x + 4 = x + 10$.

79. Equations may be divided into three classes—Identical, Formulaic and Algebraic.

Identical equations are those in which the two members are evidently the same.

Thus $18 = 12 + 6$ and $2(x + 1) = 2x + 2$ are identical equations.

Formulaic are those in which the second member is only the expanded expression of the first, or which may be derived from it by direct algebraic processes.

$$\text{Thus } (a + x)^2 = a^2 + 2ax + x^2,$$

$$\text{and } \frac{a^3 + x^3}{a + x} = a^2 - ax + x^2,$$

are formulaic equations, and may be verified by giving any numbers whatever to a and x.

Algebraic equations, are those in which the equality is only true for fixed values of the required or unknown quantity; thus $2x+4=x+10$, which is only true for $x=6$, is an equation of this kind: and it is in fact to equalities of this description that the name equation is usually applied. Our attention will at present be directed solely to equations of this class.

80. Algebraic equations are also divided into literal and numerical equations: in the former the coefficients are letters, in the latter, they are numbers: thus, of the two equations

$$ax+b=cx+d,$$

$$\text{and } 2x+7=3x+2,$$

the former is a literal and the latter a numerical equation.

It sometimes happens that the second member of an equation is wanting, such is the case in the two equations

$$2x-4=0, \quad \text{and} \quad ax-b=0.$$

We may here remark that the symbol = which is placed between the two members of an equation, sometimes only expresses the result of an algebraical operation, and not an absolute identity: thus when we say that

$$\frac{a}{1-x}=a+ax+a^2x^2+\&c.$$

the second member which is the quotient, when a is divided by $1-x$, is not equal to the first, unless the remainder found after any division be also taken into account.

81. Equations are also divided into degrees, dependent upon the powers of the unknown quantity.

Equations of the first degree, or simple equations, contain only the first power of the unknown quantity or quantities. Equations of the second degree, commonly called quadratic equations, involve the square as well as the simple power. Equations of the third degree, or cubic equations, contain the cube as well as the lower powers of the unknown quantity. And similarly are there equations of higher degrees. But as we intend to confine our attention to those of the first and second degree, or at most to those which may be reduced to their form, we need not carry our classification further.

As instances we may take the following equations:

$ax + b = c$ is of the first degree

$ax^2 + bx = c$ .... second ...

$ax^3 + bx^2 + cx = d$ .... third ...

Equations are also called, of one, two, three or more, unknown quantities, according to the number of unknown quantities which are to be determined.

Equations are also determinate or indeterminate.

Determinate equations are those in which the number of distinct equations is the same as the number of the unknown quantities. Indeterminate equations have a less number of equations than unknown quantities. In this chapter we shall treat only of the former.

AXIOMS.

82. (1) If the same number or quantity be added to or subtracted from, each member of an equation, the equality still subsists.

(2) If each member of an equation be multiplied, or divided by the same number, the equality still subsists.

RULES FOR THE SOLUTION OF EQUATIONS.

83. Rule 1. Quantities may be transposed from one side of an equation to the other by changing their algebraic sign.

Thus, if $2x-5=13$; then by Axiom I. add 5 to both sides; $\therefore\ 2x+5-5=13+5$: or since $5-5=0$, $2x=13+5$.

In which we see that the -5 of the left-hand side of the first equation, becomes $+5$ on the right-hand side of the second equation.

Also if $x+a=b$, then subtracting a from both sides,

$$x+a-a=b-a; \text{ or since } a-a=0,\ x=b-a.$$

But if the whole of the right-hand side of the equation be transposed to the left, and the whole of the left-hand side of the equation to the right; the signs need not be changed.

For if $ax+b=cx+d$; $\therefore\ cx+d=ax+b$.

Also if all the signs of the terms be changed, the equation is still true.

For if $x-a=y-b$; therefore by transposition, $b-y=a-x$ and therefore by what has been just said $a-x=b-y$, an equation in which all the signs are different from those of the original equation.

RULE 2. When the terms of an equation are fractional, the equation may be multiplied by the denominators of the fractions, and reduced to an integral form. Thus if

$$\frac{2x}{3}+2=12,$$

$$2x+6=36,$$

by multiplying by 3, and the equation is true by Axiom II.

When there are more fractions than one, we may multiply at once by the product of all the denominators. Thus if

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{5}=31;$$

$\therefore$ multiplying by $2\times3\times5=30$ we have,

$$\frac{30x}{2}+\frac{30x}{3}+\frac{30x}{5}=31\times30;$$

$$\therefore\quad 15x+10x+6x=930.$$

When the denominators have a common multiple or a number divisible by each: it is more convenient to multiply the equation by it: thus if

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=15.$$

Then since 12 is the common multiple of 2, 3, 4 and 6, we have by multiplying the equation by 12,

$$6x+4x+3x+2x=180;$$

the application of this rule is called, clearing the equation of fractions.

Rule 3. When both sides of the equation are divisible by the same number of quantity: divide by the common factor, and the equation is still true: thus if

$$12x=96,$$

dividing by 12, we have

$$x=8.$$

Rule 4. Both members of an equation may be involved to the same power, and the same root of each may be extracted.

Thus if $x^2=81$; $\therefore x=9$,

and if $\sqrt{x}=3$; $\therefore x=9$.

If $\sqrt{x^2+1}=2x+4$;

$\therefore x^2+1=4x^2+16x+16.$

SOLUTION OF SIMPLE EQUATIONS.

84. Having separately illustrated each of the foregoing rules, we may now combine them, and with the aid of the axioms, form a general rule for the solution of simple equations.

(1) Divide every term by the greatest common measure, if any.

(2) Clear the equation of fractions.

(3) Transpose all the unknown quantities to the left-hand side of the equation, and all the known quantities to the right-hand side.

(4) Collect into one sum the various coefficients of the unknown quantity.

(5) Divide both sides by the coefficient so arising and the result is the answer required.

(Ex. 1.) Let $7x - 3 = 5x + 15$, required x;

$$\therefore\ 7x - 5x = 3 + 15, \text{ Rule 1.}$$
$$\therefore\ 2x = 18;$$
$$\therefore\ x = 9.$$

If we wish to verify this result, we must write 9 for x in the original equation, when the first member becomes

$7 \times 9 - 3$, or $63 - 3$, or 60, and the second becomes

$5 \times 9 + 15$, or $45 + 15$ or 60, as it ought.

(Ex. 2.) Let $8x - 15 = 6x + 1$, find x;

$$\therefore\ 8x - 6x = 1 + 15;$$
$$\therefore\ 2x = 16;$$
$$\therefore\ x = \frac{16}{2} = 8.$$

(Ex. 3.) $5(x + 1) + 6(x + 2) = 9(x + 3)$, find x;

$$\therefore\ 5x + 5 + 6x + 12 = 9x + 27,$$
$$11x - 9x = 27 - 17;$$
$$\therefore\ 2x = 10; \quad \therefore\ x = 5.$$

(Ex. 4.) Let $ax + b = c$, find x;

$$\therefore\ ax = c - b;$$
$$\therefore\ x = \frac{c - b}{a}.$$

(Ex. 5.) Let $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$; find x.

Multiply both sides by 12, the least common multiple of 2, 3, 4, then

$$6x+4x=3x+84;$$

$$\therefore\ 7x=84;\ \therefore\ x=12.$$

(Ex. 6.) Let $\frac{3x+1}{2}+\frac{5x+1}{3}-\frac{9x+3}{4}=2$, find x.

Multiply every term by 12;

$$18x+6+20x+4-27x-9=24;$$

$$\therefore\ 38x-27x=24+9-6-4;$$

$$11x=23;\ \therefore\ x=2\tfrac{1}{11}.$$

It may here be observed that when a negative sign is placed before a fraction, we must take care to change the sign of every term of the numerator when the denominator is got rid of; for the fraction

$$-\frac{9x+3}{4}\text{ is the same as }-\left(\frac{9x}{4}+\frac{3}{4}\right)\text{ or }-\frac{9x}{4}-\frac{3}{4},$$

which two fractions when multiplied by 12, become $-27x-9$; in the next example other instances of this kind will be given.

(Ex. 7.) $$\frac{5x-3}{6}-\frac{24-8x}{3}-\frac{12x-4}{8}=4.$$

Multiply by 24;

$$\therefore\ 20x-12-192+64x-36x+12=96;$$

$$\therefore\ 84x-36x=96+12+192-12;$$

$$\therefore\ 48x=300-12=288,$$

$$x=\frac{288}{48}=6.$$

(Ex. 8.) Let $\frac{ax}{b} + \frac{bx}{a} = x + c$, find x;

$$\therefore\ a^2x + b^2x = abx + abc;$$

$$\therefore\ a^2x + b^2x - abx = abc;$$

$$\therefore\ (a^2 + b^2 - ab)x = abc;$$

$$\therefore\ x = \frac{abc}{a^2 + b^2 - ab}.$$

(Ex. 9.) Let $\frac{16}{x} = \frac{9}{13 + x}$, find x;

$$\therefore\ 208 + 16x = 9x;$$

$$\therefore\ 16x - 9x = -208;$$

$$\therefore\ 7x = -208;$$

$$\therefore\ x = \quad 29\tfrac{3}{7}.$$

(Ex. 10.) Let $\frac{3x + 7}{14} - \frac{2x - 7}{21} + 2\frac{3}{4} = \frac{x - 4}{4}$,

writing $\frac{11}{4}$ instead of $2\frac{3}{4}$, and then multiplying by 84,

$$18x + 42 - 8x + 28 + 231 = 21x - 84;$$

$$\therefore\ 18x - 8x - 21x = -231 - 42 - 28 - 84;$$

$$\therefore\ -11x = -385;\quad \therefore\ 11x = 385;\quad \therefore\ x = 35.$$

(Ex. 11.) Let $\frac{10x + 17}{18} - \frac{12x + 2}{11x - 8} = \frac{5x - 4}{9}$.

Multiply by 18;

$$\therefore\ 10x + 17 - \frac{216x + 36}{11x - 8} = 10x - 8;$$

$$\therefore\ 25 = \frac{216x + 36}{11x - 8}, \text{ by transposition};$$

$$\therefore\ 275x - 200 = 216x + 36;$$

$$\therefore\ 59x = 236;\quad \therefore\ x = 4.$$

(Ex. 12.) Let $\frac{\sqrt{5x}}{3} + 12 = 17$;

$$\therefore \frac{\sqrt{5x}}{3} = 5;$$

$$\therefore \sqrt{5x} = 15;$$

$$\therefore 5x = 225; \quad \therefore x = 45.$$

(Ex. 13.) Let $\sqrt{12+x} = 2 + \sqrt{x}$;

$$\therefore 12 + x = 4 + 4\sqrt{x} + x;$$

$$\therefore 8 = 4\sqrt{x};$$

$$\therefore \sqrt{x} = 2; \quad \therefore x = 4.$$

(Ex. 14.) $\frac{5\sqrt[3]{11x-17}}{4} + \frac{3}{8} = 4\frac{1}{8}$;

$$\therefore \frac{5\sqrt[3]{11x-17}}{4} + \frac{3}{8} = \frac{33}{8};$$

$$\therefore 10\sqrt[3]{11x-17} + 3 = 33;$$

$$\sqrt[3]{11x-17} = 3; \quad \therefore 11x - 17 = 27;$$

$$\therefore 11x = 44; \quad \therefore x = 4.$$

(Ex. 15.) $\frac{(\sqrt{x}+a)}{\sqrt{x}+b} = \frac{\sqrt{x}+c}{\sqrt{x}+d}$, find x.

$$\therefore (\sqrt{x}+a)(\sqrt{x}+d) = (\sqrt{x}+c)(\sqrt{x}+b);$$

$$\therefore x + (a+d)\sqrt{x} + ad = x + (b+c)\sqrt{x} + bc;$$

$$\therefore \sqrt{x}(a+d-b-c) = bc - ad;$$

$$\therefore x = \frac{(bc-ad)^2}{(a-b+d-c)^2}.$$

(Ex. 16.) Let $x - 7 = \sqrt{49 + \sqrt{121x^2 + x^4}}$;

$$\therefore x^2 - 14x + 49 = 49 + \sqrt{121x^2 + x^4};$$

$$\therefore x^2 - 14x = \sqrt{121x^2 + x^4} = x\sqrt{121 + x^2};$$

$$\therefore\ x - 14 = \sqrt{121 + x^2};$$

$$\therefore\ x^2 - 28x + 196 = 121 + x^2;$$

$$\therefore\ 28x = 75; \qquad \therefore\ x = 2\tfrac{19}{28}.$$

(Ex. 17.) Let $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$ find x;

$$\therefore\ x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2;$$

$$\therefore\ x\sqrt{a^2 + x^2} = a^2 - x^2;$$

$$\therefore\ a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4;$$

$$\therefore\ 3a^2x^2 = a^4; \qquad \therefore\ 3x^2 = a^2;$$

$$\therefore\ x^2 = \frac{a^2}{3}, \text{ and } x = \frac{a}{\sqrt{3}}.$$

(Ex. 18.) Let $\dfrac{x - a}{\sqrt{x} + \sqrt{a}} = b + \dfrac{\sqrt{x} - \sqrt{a}}{3}$.

Since $x - a = (\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a})$;

$$\therefore\ \frac{x - a}{\sqrt{x} + \sqrt{a}} = \sqrt{x} - \sqrt{a} = b + \frac{\sqrt{x} - \sqrt{a}}{3};$$

$$\therefore\ 3\sqrt{x} - 3\sqrt{a} = 3b + \sqrt{x} - \sqrt{a},$$

$$4\sqrt{x} = 3b + 2\sqrt{a};$$

$$\therefore\ \sqrt{x} = \frac{3b + 2\sqrt{a}}{4};$$

$$\therefore\ x = \left(\frac{3b + 2\sqrt{a}}{4}\right)^2.$$

EXAMPLES.

(1) $2x + 7 = 3x + 2$; $\therefore x = 5$.

(2) $11x - 18 = 5x + 6$; $\therefore x = 4$.

(3) $7x + 20 - 3x = -50 + 4x + 60 + 8x$; $\therefore x = \frac{5}{4}$.

(4) $2x^2 + 13x = 11x^2 - 9x$; $\therefore x = 2$.

(5) $3x + 4 = \dfrac{5x + 4}{2} + 8$; $\therefore x = 12$.

(6) $\dfrac{x}{5} + \dfrac{x}{3} = x - 7$; $\therefore x = 15$.

(7) $\dfrac{x}{2} + \dfrac{x}{4} + \dfrac{3x}{8} = 9$; $\therefore x = 8$.

(8) $7x - 8\frac{3}{4} = 5\frac{1}{4} + 3x$; $\therefore x = 4$.

(9) $1\frac{1}{2} - \dfrac{2x}{3} = \dfrac{1}{4} + \dfrac{x}{2}$; $\therefore x = \frac{15}{14}$.

(10) $\dfrac{64}{x} - 7 = 9$; $\therefore x = 4$.

(11) $2 + x = 5 - 2\frac{3}{4}x$; $\therefore x = \frac{5}{4}$

(12) $\dfrac{30 + x}{x} - 5 = \dfrac{6}{x}$; $\therefore x = 6$.

(13) $\dfrac{5x + 3}{x - 1} + \dfrac{2x - 3}{2x - 2} = 9$; $\therefore x = 3\frac{1}{2}$.

(14) $\dfrac{x + 5}{4} + \dfrac{x + 4}{5} - \dfrac{x + 13}{8} = 3\frac{1}{40}$; $\therefore x = 8$.

(15) $\dfrac{x}{2} - \dfrac{5x + 4}{3} = 7 - \dfrac{8x - 2}{3}$; $\therefore x = 6$.

(16) $\dfrac{45(2 - x)}{6 + x} + 2 = 15$; $\therefore x = \frac{6}{29}$.

(17) $\dfrac{3x + 1}{5} - \dfrac{7x + 2}{10} + \dfrac{3x}{4} - \dfrac{7x}{8} = -12$; $\therefore x = \frac{4}{3}$.

(18) $3.25x - 5.1 + x - .75x = 3.9 + .500x$;

$\therefore x = 3.$

(19) $(2+x)(8+x) - 22 = \frac{3}{2} + x^2$; $\therefore x = \frac{3}{4}.$

(20) $ax - bx = d - cx$; $\therefore x = \dfrac{d}{a+c-b}.$

(21) $\dfrac{b^2+x^2}{bx} = c + \dfrac{x}{b}$; $\therefore x = \dfrac{b}{c}.$

(22) $\dfrac{a^2}{x} = ab + b + \dfrac{1}{x}$; $\therefore x = \dfrac{b}{a-1}.$

(23) $(a+x)(b+x) - a(b+c) = \dfrac{a^2c}{b} + x^2$;

$\therefore x = \dfrac{ac}{b}.$

(24) $a^2x + b^3 = b^2x + c^3$; $\therefore x = \dfrac{a^2+ab+b^2}{a+b}.$

(25) $\dfrac{3x}{5} - \dfrac{7x}{10} + \dfrac{3x}{4} - \dfrac{7x}{8} = -15$; $\therefore x = 66\frac{2}{3}.$

(26) $\dfrac{x-5}{11} + \dfrac{2x-3}{6} = x + 2 - \dfrac{46x+133}{66}$;

$\therefore x = 8.$

(27) $\dfrac{2x+3}{17} - \dfrac{5x+7}{21} + \dfrac{6x+1}{29} = \dfrac{5x+3}{87}$;

$\therefore = 6\frac{43}{75}.$

(28) $\dfrac{2}{x} + \dfrac{3}{x} + \dfrac{4}{x} + \dfrac{5}{x} = 56$; $\therefore x = \frac{1}{4}.$

(29) $\dfrac{2}{x+1} + \dfrac{5}{2x+2} + \dfrac{6x-6}{x^2-1} = 2\frac{5}{8}$; $\therefore x = \frac{1}{3}.$

(30) $\dfrac{6x+13}{5} - \dfrac{3x+5}{5x-25} = \dfrac{2x}{5}$; $\therefore x = 20.$

(31) $\dfrac{8x+5}{14}+\dfrac{7x-3}{6x+2}=\dfrac{16x+15}{28}+\dfrac{2\frac{1}{4}}{7}$;

$\therefore x = 1\frac{1}{8}$.

(32) $\dfrac{ax+b}{c}+\dfrac{ax+b}{cx+b}=\dfrac{2ax+d}{2c}+\dfrac{b}{c}$; $\therefore x=\dfrac{bd-2cd}{2ac-cd}$.

(33) $\dfrac{7x+5}{3}-\left(2x-\dfrac{3x-7}{14}\right)=5$; $\therefore x = 7$.

(34) $4x-\dfrac{x-2}{2}-\left\{2x-\left(\dfrac{\dfrac{3x}{2}-\dfrac{16-\dfrac{x+4}{2}}{3}}{6}\right)\right\}=\dfrac{3x+6}{2}$;

$\therefore x = 10$.

(35) $\dfrac{7x+\dfrac{13}{2}}{10}+\dfrac{11x-\dfrac{x-\dfrac{3}{2}}{2}}{12}=\dfrac{3x+1}{5}+\dfrac{43x-\dfrac{3-8x}{2}}{11}$;

$\therefore x = \frac{1}{2}$.

(36) $\dfrac{2x+1}{3}-\dfrac{3x-2}{5}+\dfrac{5x-\dfrac{x+\dfrac{8}{5}}{2}}{6}=8$;

$\therefore x = 11$.

(37) $\dfrac{7x}{x-1}=\dfrac{6x^2+x}{x+1}+\dfrac{3x+6x^3}{x^2-1}$; $\therefore x=\dfrac{-11}{12}$.

(38) $\dfrac{2x^2+1}{9x^2-16}=\dfrac{x}{4+3x}-\dfrac{1}{9}$; $\therefore x=\dfrac{7}{36}$.

(39) $ax+b=cx+d$; $\therefore x=\dfrac{d-b}{a-c}$.

(40) $\dfrac{3c(a-x)}{3a+x}+a=\dfrac{5a}{4}$; $\therefore x=\dfrac{3a}{4}$.

(41) $\dfrac{x}{a}+\dfrac{x}{b}+\dfrac{x}{c}=1$. $\therefore x=\dfrac{abc}{ab+ac+bc}$.

(42) $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=1;$ $\quad\therefore x=\frac{1}{a+b+c}.$

(43) $x+a+b+c=\frac{x^2+a^2+b^2+c^2}{a+b-c+x};\ \therefore x=\frac{c^2-ab}{a+b}.$

(44) $\frac{3ab}{6}+\frac{4ac}{5}-\frac{2cx}{3}=\frac{3ac}{4}+2ab-6cx;$

$$\therefore x=\frac{a}{c}\cdot\left(\frac{70b-3c}{320}\right).$$

(45) $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)\,b^2x}{a\,(a+b)^2}=3cx+\frac{bx}{a};$

$$\therefore x=\frac{ab}{a+b}.$$

(46) $\sqrt{x^2+16}=5;$ $\quad\therefore x=3.$

(47) $\sqrt{x+16}=2+\sqrt{x};$ $\quad\therefore x=9.$

(48) $\sqrt{x+13}=2+\sqrt{x-11};$ $\quad\therefore x=36.$

(49) $\sqrt{x^2+5x-2}=x+2;$ $\quad\therefore x=6.$

(50) $\sqrt[6]{x^2+11x+5}=\sqrt[3]{x+5};$ $\quad\therefore x=20.$

(51) $3\sqrt[3]{3x-5}=2\sqrt[3]{11x-17};$ $\quad\therefore x=\frac{1}{7}.$

(52) $\frac{x-4}{2+\sqrt{x}}=5\sqrt{x}-8+\frac{3\sqrt{x}}{2};$ $\quad\therefore x=\frac{144}{121}.$

(53) $\frac{x-a}{\sqrt{a}+\sqrt{x}}=\frac{\sqrt{x}-\sqrt{a}}{3}+2\sqrt{a};$ $\quad x=16a.$

(54) $x+\sqrt{x^2-2ax+b^2}=a+b;$ $\quad x=\frac{a^2+2ab}{2b}.$

(55) $\sqrt{x+225}-\sqrt{x-424}=11;$ $\quad x=1000.$

(56) $\frac{\sqrt{4x+5}+\sqrt{x}}{\sqrt{4x+5}-\sqrt{x}}=2;$ $\quad\therefore x=1.$

(57) $\dfrac{x+\sqrt{x^2-16}}{x-\sqrt{x^2-16}}=4.$ $\qquad x=5.$

(58) $\sqrt{\dfrac{x+4}{x-4}}+\sqrt{\dfrac{x-4}{x+4}}=\dfrac{10}{3};$ $\qquad x=5$

(59) $\dfrac{x}{\sqrt{a^2+x^2}}=\dfrac{c-x}{\sqrt{b^2+(c-x)^2}};$ $\qquad x=\dfrac{ac}{a+b}.$

(60) $\dfrac{\sqrt{x}+16}{\sqrt{x}+4}=\dfrac{\sqrt{x}+32}{\sqrt{x}+12};$ $\qquad x=64.$

(61) $\sqrt{3+x}+\sqrt{x}=\dfrac{5}{\sqrt{3+x}};$ $\qquad x=\frac{4}{7}.$

(62) $\sqrt{7+x}+\sqrt{x}=\dfrac{28}{\sqrt{7+x}};$ $\qquad x=9.$

(63) $\sqrt{a^2+x^2}=\sqrt[4]{b^4+x^4};$ $\qquad \therefore x=\sqrt{\dfrac{b^4-a^4}{2a^2}}.$

(64) $\sqrt{a^2+bx}+\sqrt{a^2-bx}=2c;$ $\qquad \therefore x=\dfrac{2c}{b}\sqrt{a^2-c^2}.$

(65) $x-4=\dfrac{x^2}{(1-\sqrt{1+x})^2};$ $\qquad \therefore x=8.$

(66) $8\sqrt{3x}+\dfrac{243+324\sqrt{3x}}{16x-3}=16x+3;$

$\therefore x=3.$

(67) $4+x=\sqrt{16+x\sqrt{144+x^2}};$ $\qquad \therefore x=5.$

(68) $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}};$ $\qquad \therefore x=\dfrac{b^2-4a^2}{4a}.$

(69) $\dfrac{1}{x}+\dfrac{1}{a}=\sqrt{\dfrac{1}{a^2}+\sqrt{\dfrac{1}{b^2x^2}+\dfrac{1}{x^4}}};$ $\qquad \therefore x=\dfrac{ab^2}{a^2-b^2}.$

(70) $\sqrt[3]{a+x}+\sqrt[3]{a-x}=\sqrt[3]{b};$ $\qquad \therefore x=\sqrt{a^2-\dfrac{(b-2a)^3}{27b}}.$

SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

85. When there are two equations between two unknown quantities, such as $ax+by=m$, and $cx+dy=n$; in which a, b, c, d, m and n may be any numbers whatever, it is sufficiently evident, that if we obtain from these two equations, two values of x, and equate them, there will arise an equation involving y only, from which y may be found; and y being known, x may also be determined, by substituting the known value of y in either of the equations.

In fact, $\because\ ax+by=m$, and $cx+dy=n$;

$$\therefore\ x=\frac{m-by}{a} \text{ and } x=\frac{n-dy}{c};$$

$$\therefore\ \frac{m-by}{a}=\frac{n-dy}{c};$$

$$\therefore\ mc-bcy=na-ady;$$

$$\therefore\ y=\frac{mc-na}{bc-ad};$$

$$\therefore\ x=\frac{m-by}{a}=\frac{1}{a}\left(m-\frac{mbc-nba}{bc-ad}\right)=\frac{nb-md}{bc-ad}.$$

As another example let us take

$$2x+3y=13; \text{ and } 3x+5y=21;$$

$$\therefore\ x=\frac{13-3y}{2} \text{ and } x=\frac{21-5y}{3};$$

$$\therefore\ \frac{13-3y}{2}=\frac{21-5y}{3};$$

$$\therefore\ 39-9y=42-10y; \quad \therefore\ y=3.$$

$$x=\frac{13-3y}{2}=\frac{13-9}{2}=\frac{4}{2}=2.$$

86. The method just given is sufficient to solve any equation of this kind; but in general another process, that of elimination, is made use of.

Elimination means, the getting rid of a thing, and applied to these equations, is the getting rid of one of the unknown quantities, so that the equation may be reduced to another, containing only one unknown quantity; to effect this purpose we have the following rule, to eliminate x.

"Multiply the upper equation by the coefficient of x in the lower equation, and then multiply the lower equation by the coefficient of x in the upper equation: take the difference between these products, and the result will contain y only.

(Ex. 1.) Let $6x + 7y = 46$............(1)

and $5x + 3y = 27$............(2)

(1) × 5; $\therefore 30x + 35y = 230$

(2) × 6; $\therefore 30x + 18y = 162$

$17y = 68; \quad \therefore y = 4;$

and $5x + 3y = 27; \quad \therefore 5x = 27 - 12; \quad \therefore x = 3.$

(Ex. 2.) Let $ax + by = m$. ....(1)

and $cx + dy = n$ ....(2)

$\therefore acx + bcy = mc;$

$\therefore acx + ady = na;$

$\therefore (bc - ad)y = mc - na;$

$$\therefore y = \frac{mc - na}{bc - ad},$$

and x may be found as before.

(Ex. 3.) Let $\frac{x}{7} + 7y = 99$,

and $\frac{y}{7} + 7x = 51$;

$$\therefore \quad x + 49y = 99 \times 7$$
$$y + 49x = 51 \times 7$$

$$\therefore \quad 50x + 50y = 150 \times 7 \quad \text{by addition;}$$
$$\therefore \quad x + y = 21.$$

And $48y - 48x = 48 \times 7$ by subtraction;

$$\therefore \quad y - x = 7;$$
$$\text{and } \because \quad y + x = 21;$$

$$\therefore \quad 2y = 28; \quad \therefore \quad y = 14,$$
$$2x = 14; \quad \therefore \quad x = 7.$$

(Ex. 4.) $$\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5 \ldots\ldots\ldots(1)$$

$$\frac{5y-7}{2} + \frac{4x-3}{6} = 18 - 5x \ldots\ldots\ldots(2).$$

From (1) $28 + 4x - 10x + 5y = 60y - 100$;

$$\therefore \quad 55y + 6x = 128 \quad \ldots(3).$$

From (2) $15y - 21 + 4x - 3 = 108 - 30x$;

$$\therefore \quad 15y + 34x = 132 \ldots(4).$$

Multiply (3) by 3 and (4) by 11: for $55 = 5 \times 11$ and $15 = 3 \times 5$.

$$165y + 18x = 384,$$
$$165y + 374x = 1452;$$

$$\therefore \quad 356x = 1068; \quad \therefore \quad x = 3.$$

But $15y = 132 - 34x = 132 - 102 = 30$; $\therefore \quad y = 2$.

87. To the two methods which have been given, may be added a third, which may be thus enunciated: "Substitute the value of x obtained from one equation, for x, in the other equation": this method, which is very obvious, is really contained in the first.

The principle of the three rules is briefly this: that as the two equations hold contemporaneously, i. e. are true for the same values of x and y at the same time, if in either of the equations we put for x or y, their values in other terms, the equations are still true.

We must be careful to see that the two equations are independent of each other, for had we the equations

$$2x + 3y = 14, \text{ and } 4x + 6y = 28,$$

the latter which is merely the double of the former, will be of no use in the determination of the particular values of x and y which render the equation $2x + 3y = 14$, a just equation.

EXAMPLES.

(1) $\left.\begin{array}{r} 5x + 3y = 74 \\ 3x + 2y = 49 \end{array}\right\};$ $\therefore$ $x = 10$, $y = 8$.

(2) $\left.\begin{array}{r} 13x - 17y + 54 = 0 \\ 7x + 28 - 9y = 0 \end{array}\right\};$ $\therefore$ $x = 5$, $y = 7$.

(3) $\left.\begin{array}{r} 5x + 7y = 43 \\ 11x + 9y = 69 \end{array}\right\};$ $\therefore$ $x = 3$, $y = 4$.

(4) $\left.\begin{array}{r} 8x - 21y = 33 \\ 6x + 35y = 177 \end{array}\right\};$ $\therefore$ $x = 12$, $y = 3$.

(5) $\left.\begin{array}{r} x + 10y = 123 \\ y + 10x = 141 \end{array}\right\};$ $\therefore$ $x = 13$, $y = 11$.

(6) $\left.\begin{array}{r}\frac{x}{6}+\frac{y}{5}=\frac{x}{2}+2 \\ \frac{x}{4}+\frac{y}{3}=\frac{3y}{10}+4\end{array}\right\};$ $\quad x=12,\quad y=30.$

(7) $\left.\begin{array}{r}\frac{4}{5+y}=\frac{5}{12+x} \\ 2x+5y=35\end{array}\right\};$ $\quad x=2,\quad y=\frac{31}{5}.$

(8) $\left.\begin{array}{r}2x+6 : 3y+2 :: 9 : 7 \\ 8x-4=9y\end{array}\right\};$ $\quad x=5,\quad y=4.$

(9) $\left.\begin{array}{r}x+y-8-0 \\ \frac{x-y}{2}+\frac{2x-3}{3}+\frac{4}{3}=0\end{array}\right\};$ $\quad x=2\frac{1}{5},\quad y=5\frac{4}{5}.$

(10) $\left.\begin{array}{r}\frac{2x}{3}+\frac{y+2x}{2}=8-\frac{9y-10}{12}+\frac{3x+7}{4} \\ \frac{y-3x}{6}=\frac{25}{6}-2x\end{array}\right\}$ $\quad x=2,\quad y=7.$

(11) $\left.\begin{array}{r}(x+7).(y+5)=(x-2)(y+15)+78 \\ (x+5).(y+7)=(y+3)(x+17)-80\end{array}\right\}$ $\quad x=5,\quad y=7.$

(12) $\left.\begin{array}{r}(x+a).(y+b)=(x-b).(y+a)+ab \\ ax=by+a^2\end{array}\right\}$ $\quad x=a.\left(\frac{a^2+ab-b^2}{a^2+b^2}\right),\quad y=a^2\left(\frac{b-2b}{a^2+b^2}\right).$

(13) $\left.\begin{array}{r}\frac{10+6y-4x}{6x-9y+3}=\frac{4}{3} \\ \frac{126+8x-17y}{100-12x+7y}=\frac{35}{13}\end{array}\right\};$ $\quad x=8,\quad y=5.$

(14) $\left.\begin{array}{l}\dfrac{5x}{9}+9y=91 \\ \dfrac{5y}{9}+9x=167\end{array}\right\};$ $\quad x=18,\ y=9.$

(15) $\left.\begin{array}{l}\dfrac{45x}{23}+y=91 \\ \dfrac{45y}{23}+x=113\end{array}\right\};$ $\quad x=23,\ y=46.$

(16) $\left.\begin{array}{l}\dfrac{2}{x}+\dfrac{3}{y}+\dfrac{1}{4}=\dfrac{18}{y}-\dfrac{5}{x} \\ \dfrac{2}{x}-\dfrac{1}{y}=\dfrac{1}{5}\left(\dfrac{1}{x}+\dfrac{1}{y}\right)+\dfrac{1}{12}\end{array}\right\};$ $\quad x=12,\ y=18.$

88. When there are three unknown quantities as, x, y, z, and three distinct equations between them, we may either by substitution, or elimination, reduce them to two equations involving two only of the three unknown quantities; and these two may then be found by the methods already given. Thus, between the first and second equations, we may eliminate z, and have an equation involving x and y. Again, between the first and third, or second and third equations, we may also eliminate z, and have a second equation involving x and y only; from these two, x and y may be found; and z may then be determined by substituting the values of x and y in any one of the three original equations.

(Ex. 1.)
$$7x+10y+5z=42 \ldots\ldots\ldots\ldots(1)$$
$$13x+6y+2z=31 \ldots\ldots\ldots\ldots(2)$$
$$11x+14y+8z=63 \ldots\ldots\ldots\ldots(3).$$

Multiply (2) by 5 and (1) by 2;

$$\therefore\ 65x+30y+10z=155$$
$$14x+20y+10z=84$$
$$\therefore\ 51x+10y=71$$

Multiply (2) by 4 and write (3) under the product;

$$\therefore\ 52x + 24y + 8z = 124,$$
$$11x + 14y + 8z = 63;$$

$$\therefore \text{ subtracting } 41x + 10y = 61.$$
$$\text{But } 51x + 10y = 71;$$

$$\therefore\ 10x = 10; \quad \therefore x = 1.$$

But $51x + 10y = 71$; $\therefore 10y = 71 - 51 = 20$; $\therefore y = 2$.

And $2z = 31 - 6y - 13x = 31 - 12 - 13 = 6$; $\therefore z = 3$.

(Ex. 2.) Let $xy = 2(x + y)$; $xz = 3(x + z)$; $yz = 4(y + z)$ find x, y, and z.

Since $xy = 2\ (x + y)$; $\therefore\ 1 = \frac{2 . (x+y)}{xy} = 2\left(\frac{1}{y} + \frac{1}{x}\right)$;

$$\therefore\ \frac{1}{x} + \frac{1}{y} = \frac{1}{2} \ldots\ldots (1).$$

$$\text{Similarly } \frac{1}{x} + \frac{1}{z} = \frac{1}{3} \ldots\ldots (2).$$

$$\text{and } \frac{1}{y} + \frac{1}{z} = \frac{1}{4} \ldots\ldots (3).$$

Subtract (2) from (1);

$$\therefore\ \frac{1}{y} - \frac{1}{z} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

$$\text{But } \frac{1}{y} + \frac{1}{z} = \qquad = \frac{1}{4};$$

$$\therefore\ \frac{2}{y} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}; \qquad \therefore\ y = \frac{24}{5}.$$

$$\frac{2}{z} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}; \qquad \therefore\ z = 24.$$

$$\text{and } \frac{1}{x} = \frac{1}{2} - \frac{1}{y} = \frac{12}{24} - \frac{5}{24} = \frac{7}{24}; \qquad \therefore\ x = \frac{24}{7}.$$

EXAMPLES.

(1) $\left.\begin{aligned} x+y&=50\\ x+z&=28\\ y+z&=42\end{aligned}\right\},$ $x=18,\ y=32,\ z=10.$

(2) $\left.\begin{aligned} 3x+5y&=76\\ 4x+6z&=108\\ 5z+7y&=106\end{aligned}\right\},$ $x=12,\ y=8,\ z=10.$

(3) $\left.\begin{aligned} x+y+z&=26\\ x+y-z&=-6\\ x-y+z&=12\end{aligned}\right\},$ $x=3,\ y=7,\ z=16.$

(4) $\left.\begin{aligned} 3x+5y-4z&=25\\ 5x-2y+3z&=46\\ 3y+5z-x&=62\end{aligned}\right\};$ $x=7,\ y=8,\ z=9.$

(5) $\left.x+\frac{y}{2}=100;\ y+\frac{z}{3}=100;\ z+\frac{x}{4}=100\right\};$ $x=64,\ y=72,\ z=84.$

(6) $\left.\begin{aligned} 3x+4y-5z&=32\\ 4x-5y+3z&=18\\ 5x-3y-4z&=2\end{aligned}\right\},$ $x=10,\ y=8,\ z=6.$

(7) $\left.\begin{aligned} 5x-6y+4z&=15\\ 7x+4y-3z&=19\\ 2x+y+6z&=46\end{aligned}\right\},$ $x=3,\ y=4,\ z=6.$

(8) $\left.\begin{aligned} \frac{x}{2}+\frac{y}{3}+\frac{z}{7}&=22\\ \frac{x}{3}+\frac{y}{5}+\frac{z}{2}&=31\\ \frac{x}{4}+\frac{y}{2}+\frac{z}{3}&=32\end{aligned}\right\},$ $x=12,\ y=30,\ z=42.$

(9) $\dfrac{xy}{x+y} = 70, \quad \dfrac{xz}{x+z} = 84, \quad \dfrac{yz}{y+z} = 140.$

Ans. $x = 105$; $y = 210$; $z = 420$.

(10) $\left.\begin{aligned} \frac{2}{x} - \frac{5}{3y} + \frac{1}{z} &= \frac{85}{27} \\ \frac{1}{4x} + \frac{1}{y} + \frac{2}{z} &= \frac{443}{72} \\ \frac{5}{6x} - \frac{1}{y} + \frac{4}{z} &= \frac{433}{36} \end{aligned}\right\},$ $\quad x = 6,\ y = 9,\ z = \frac{1}{3}.$

(11) $\left.\begin{aligned} x - 9y + 3z - 10u &= 21 \\ 2x + 7y - z - u &= 683 \\ 3x + y + 5z + 2u &= 195 \\ 4x - 6y - 2z - 9u &= 516 \end{aligned}\right\},$ $\quad x = 100,\ y = 60,\ z = -13,\ u = -50.$

(12) $\left.\begin{aligned} \frac{x}{3} + \frac{y}{5} + \frac{2z}{7} &= 58 \\ \frac{5x}{4} + \frac{y}{6} + \frac{z}{3} &= 76 \\ \frac{x}{2} + \frac{3z}{8} + \frac{u}{5} &= 79 \\ y + z + u &= 248 \end{aligned}\right\},$ $\quad x = 12,\ y = 30,\ z = 168,\ u = 50.$

QUADRATIC EQUATIONS.

89. **Quadratic equations are of two kinds.**

(1) Pure quadratics, which are of the form $ax^2 = b$, or which do not involve the simple power of the unknown quantity.

(2) Adfected quadratics, which are of the form $ax^2 + bx = c$, a, b, and c being any numbers whatever, positive or negative.

90. **The pure quadratic equation is thus solved:**

$$ax^2 = b; \quad \therefore \; x^2 = \frac{b}{a}; \quad \therefore \; x = \pm\sqrt{\frac{b}{a}};$$

or x has two values $+\sqrt{\frac{b}{a}}$ and $-\sqrt{\frac{b}{a}}$; for either of these values when put for x in the original equation $x^2 - \frac{b}{a} = 0$, satisfy the given condition, that one term should be equal to the other.

(Ex. 1.) $5x^2 = 125; \quad \therefore \; x^2 = 25; \quad \therefore \; x = \pm 5.$

(Ex. 2.) $7x^2 + 18 = 4x^2 + 450; \quad \therefore \; x = \pm 12.$

(Ex. 3.) $\frac{5x^2}{3} + 12 = \frac{8x^2}{7} + 37\frac{2}{3}; \quad \therefore \; x = \pm 7.$

SOLUTION OF THE ADFECTED QUADRATIC EQUATION.

91. Let $ax^2 + bx = c$ be the equation, or dividing by a.

$$x^2 + \frac{b}{a}x = \frac{c}{a}.$$

Instead of $\frac{b}{a}$ put p, and for $\frac{c}{a}$ put q;

$$\therefore \; x^2 + px = q.$$

Now let a quantity r be added to both sides, so that $x^2 + px + r$ shall be a complete square; for then if the roots of both sides be taken, the equation will be reduced to a simple equation.

But to find r. We see that in every complete square of a binomial, as $x^2 + 2ax + a^2$, which is the square of $x + a$; four times the product of the first and third terms equals the square of the middle term: for $4a^2 \times x^2 = (2ax)^2$;

$$\therefore \; 4rx^2 = (px)^2 = p^2x^2; \quad \therefore \; r = \frac{p^2}{4} = \left(\frac{p}{2}\right)^2;$$

or the quantity to be added to make $x^2 + px$ a complete square, is the square of half the coefficient of x; making this addition to both sides, we have

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}.$$

And since by the ordinary rules, the square root of

$$x^2 + px + \frac{p^2}{4} \text{ is } x + \frac{p}{2};$$

$$\therefore\ x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} + q}.$$

Hence we have this rule to complete the square. Add to each member of the equation, the square of half the coefficient of x; and to extract the square root of the square so completed, add to x, half the coefficient of x in the original equation.

Ex. Complete the squares of the following quantities:

(1) $x^2 + 8x$. Ans. $x^2 + 8x + 16$.

(2) $x^2 - 8x$. Ans. $x^2 - 8x + 16$.

(3) $x^2 - 3x$. Ans. $x^2 - 3x + \frac{9}{4}$.

(4) $x^2 + 5x$. Ans. $x^2 + 5x + \frac{25}{4}$.

Also the square roots of the squares are respectively

$$x + 4, \quad x - 4, \quad x - \frac{3}{2} \text{ and } x + \frac{5}{2}.$$

It may be observed that both the positive and negative signs are prefixed to the root of the second member of the equation, the reason for which may be thus shewn;

$$\text{make } \frac{p^2}{4} + q = m^2.$$

Then since $+m \times +m$ and $-m \times -m$ are both equal to m^2; therefore the square root of m^2 may be either $+m$ or $-m$; and therefore that of $\frac{p^2}{4}+q$ may be either

$$+\sqrt{\frac{p^2}{4}+q} \text{ or } -\sqrt{\frac{p^2}{4}+q},$$

and both satisfy the conditions of the equation.

92. Resuming the equation, since

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}}; \qquad \therefore\ x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q};$$

or x has two values, viz.

$$-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}} \text{ and } -\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$$

Let α and β respectively represent these two values;

$$\therefore\ \alpha=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}},$$

$$\beta=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$$

Therefore, $\alpha+\beta=-p$, or the sum of the values of x is equal to the coefficient of the second term with its sign changed: this proposition is true of equations of any dimensions whatever.

$$\text{Again, } \alpha\beta=\frac{p^2}{4}-\left(q+\frac{p^2}{4}\right)=-q.$$

Hence, if we write the equation under the form

$$x^2+px-q=0,$$

we see that in a quadratic equation, the last term is equal to the product of the values of x. This proposition may be thus generalized for an equation of any degree whatever; the last term is equal to the product of the values of x with their signs changed.

93. Again, since

$$x-\alpha=x+\frac{p}{2}-\sqrt{\frac{p^2}{4}+q}, \text{ and } x-\beta=x+\frac{p}{2}+\sqrt{\frac{p^2}{4}+q};$$

$$\therefore\ (x-\alpha).(x-\beta)=\left(x+\frac{p}{2}\right)^2-\left(\frac{p^2}{4}+q\right)=x^2+px-q.$$

Hence if α and β be separately put for x, they will separately make $x^2+px-q=0$, or satisfy the conditions of the equation. But no other terms but α and β will do this; for these only can make either $x-\alpha=0$, or $x-\beta=0$.

These terms, α and β, are called roots of the equation, and hence we say that a quadratic can only have two roots. We may remark that these values are not true at the same time, unless

$$q=-\frac{p^2}{4}, \text{ and then } \sqrt{q+\frac{p^2}{4}}=0, \text{ and then } \alpha=\beta.$$

And also that, if α be a root of a quadratic, $x-\alpha$ will divide the equation, without leaving any remainder, for the quotient will be $x-\beta$.

94. When q is a positive quantity, both values of x are real; but if q be negative, and

$$\therefore\ x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q},$$

they are real only so long as $\frac{p^2}{4}$ is not less than q. When q is $>\frac{p^2}{4}$, $\frac{p^2}{4}-q$ is a negative quantity and its square root cannot be found, and both α and β are impossible quantities.

Such a circumstance will in general point out to us, that the conditions of the question are incapable of being fulfilled. As, for instance, if it were proposed to find two numbers whose sum $= 6$ and product $= 12$, we should have $p = -6$, and $q = -12$; for the equation arising would be $x^2 - 6x = -12$, and $\frac{p^2}{4} - q = 9 - 12 = -3$. In fact there are no such numbers.

95. And now we shall give a few useful rules for the solution of quadratic equations; first observing that we shall seldom find the equation in so simple a form as

$$x^2 + px = q;$$

fractions must be reduced, and terms transposed, before we can begin its solution. The following steps are in general necessary.

(1) Clear the equation of fractions.

(2) Transpose the terms involving x^2 and x to the left hand and the numbers to the right hand side of the equation.

(3) Divide every term by the coefficient of x^2.

(4) If x^2 be negative, change all the signs of the equation.

(5) Complete the square.

(6) Extract the square root of both sides, and a simple equation remains; whence x may be found.

EXAMPLES.

(Ex. 1.) Let $x^2 + 6x = 91$; find x.

Add 9 or $\left(\frac{6}{2}\right)^2$ to both sides;

$$\therefore \; x^2 + 6x + 9 = 91 + 9 = 100;$$

$$\therefore \quad x + 3 = \pm 10;$$

$$\therefore \quad x = -3 \pm 10 = 7 \text{ or } -13.$$

(Ex. 2.) Let $x^2 - 5x = 24$; find x.

Add $\left(\frac{-5}{2}\right)^2$ or $\frac{25}{4}$ to both sides;

$$\therefore \quad x^2 - 5x + \frac{25}{4} = 24 + \frac{25}{4} = \frac{121}{4};$$

$$\therefore \quad x - \frac{5}{2} = \pm \frac{11}{2}; \qquad \therefore \quad x = 8 \text{ or } -3.$$

(Ex. 3.) Let $x^2 - x = 72$.

Here $\left(\frac{-1}{2}\right)^2 = \left(\frac{p}{2}\right)^2$ or $\frac{1}{4} = \frac{p^2}{4}$;

$$\therefore \quad x^2 - x + \frac{1}{4} = \frac{1}{4} + 72 = \frac{289}{4};$$

$$\therefore \quad x - \frac{1}{2} = \pm \frac{17}{2}; \qquad \therefore \quad x = \frac{1}{2} \pm \frac{17}{2} = 9 \text{ or } -8.$$

(Ex. 4.) Let $3x^2 - 2x = 65$; find x;

$$\therefore \quad x^2 - \frac{2x}{3} = \frac{65}{3};$$

$$\therefore \quad x^2 - \frac{2x}{3} + \frac{1}{9} = \frac{65}{3} + \frac{1}{9} = \frac{196}{9};$$

$$\therefore \quad x - \frac{1}{3} = \pm \frac{14}{3}; \qquad \therefore \quad x = 5 \text{ or } -\frac{13}{3}.$$

(Ex. 5.) Let $\frac{40}{x-5} + \frac{27}{x} = 13$;

$$\therefore \quad 40x + 27x - 135 = 13x^2 - 65x;$$

$$\therefore \quad 13x^2 - 132x = -135;$$

$$\therefore\ x^2 - \frac{132x}{13} = -\frac{135}{13};$$

$$\therefore\ x^2 - \frac{132}{13}x + \frac{4356}{169} = \frac{4356}{169} - \frac{135}{13} = \frac{2601}{169};$$

$$\therefore\ x - \frac{66}{13} = \pm\frac{51}{13};\quad \therefore\ x = \frac{66 \pm 51}{13} = 9 \text{ or } \frac{15}{13}.$$

(Ex. 6.) Let $adx - acx^2 = bcx - bd;$

$$\therefore\ acx^2 - adx + bcx = bd;$$

$$\therefore\ x^2 - \frac{ad - bc}{ac}x = \frac{bd}{ac};$$

$$\therefore\ x^2 - \frac{ad-bc}{ac}x + \left(\frac{ad-bc}{2ac}\right)^2 = \frac{(ad-bc)^2}{4a^2c^2} + \frac{bd}{ac} = \frac{(ad+bc)^2}{4a^2c^2};$$

$$\therefore\ x - \frac{ad-bc}{2ac} = \pm\frac{ad+bc}{2ac};$$

$$\therefore\ x = \frac{ad-bc}{2ac} \pm \frac{ad+bc}{2ac} = \frac{2ad}{2ac} \text{ or } \frac{-2bc}{2ac};$$

$$\text{i.e. } x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

96. The equations hitherto solved have, by very obvious reductions, assumed the form of $x^2 + px = q$; but there are many others which may be solved as quadratics; thus every equation of the form $x^{2n} + px^n = q$, may be so solved.

Since by putting $x^n = y$; $\therefore\ x^{2n} = y^2$, we have

$$y^2 + py = q,$$

which is of the required form, and this remark is true whether the indices be integral or fractional, so long as the index of x in one term, is half the index of x in the other.

$$\text{Thus,}\quad ax^8 + bx^4 = c,$$

$$ax^{\frac{4}{3}} + b^{\frac{2}{3}} = c,$$

$$ax^{\frac{1}{3}} + bx^{\frac{1}{6}} = c,$$

$$ax^{\frac{1}{2}} + bx^{\frac{1}{4}} = c,$$

and many others may be solved as quadratics.

97. Also algebraic expressions of the form

$$(ax^2 + bx + c)^{2n} + p\,(ax^2 + bx + c)^n = q,$$

$$\text{and}\quad (ax^{2m} + bx^m + c)^{2n} + p\,(ax^{2m} + bx^m + c)^n = q,$$

may be solved as quadratics; since if in the former of these equations we put $(ax^2 + bx + c)^n = y$, and in the latter

$$(ax^{2m} + bx^m + c)^n = y,$$

we have for both

$$y^2 + py = q.$$

Other reductions must be left to the ingenuity which is gained only by practice.

(Ex. 1.) Let $x^4 - 74x^2 = -1225$;

$$\therefore\ x^4 - 74x^2 + 1369 = 1369 - 1225 = 144;$$

$$\therefore\ x^2 = 37 \pm 12 = 49 \text{ or } 25; \quad \therefore\ x = \pm 7 \text{ or } \pm 5.$$

(Ex. 2.) Let $x^{\frac{4}{3}} + 7x^{\frac{2}{3}} = 44$;

$$\therefore\ x^{\frac{4}{3}} + 7x^{\frac{2}{3}} + \frac{49}{4} = 44 + \frac{49}{4} = \frac{225}{4};$$

$$\therefore\ x^{\frac{2}{3}} + \frac{7}{2} = \pm\frac{15}{2}; \quad \therefore\ x^{\frac{2}{3}} = \pm\frac{15}{2} - \frac{7}{2};$$

$$\therefore\ x^{\frac{2}{3}} = 4 \text{ or } -11; \quad \therefore\ x^{\frac{1}{3}} = \pm 2 \text{ or } \sqrt{-11}; \quad \therefore\ x = \pm 8 \text{ or } (-11)^{\frac{3}{2}}.$$

(Ex. 3.) Let $\sqrt[4]{x+13}+5\sqrt{x+13}=22$; find x;

$$\therefore \sqrt{x+13}+\frac{1}{5}\sqrt[4]{x+13}=\frac{22}{5};$$

$$\therefore \sqrt{x+13}+\frac{1}{5}\sqrt[4]{x+13}+\frac{1}{100}=\frac{22}{5}+\frac{1}{100}=\frac{441}{100};$$

$$\therefore \sqrt[4]{x+13}=\pm\frac{21}{10}-\frac{1}{10}=2 \text{ or } -\frac{11}{5};$$

$$\therefore x+13=16 \text{ or } \left(\frac{11}{5}\right)^4; \quad \therefore x=3, \text{ or } \left(\frac{11}{5}\right)^4-13.$$

(Ex. 4.) Let $5x-7x^2+8\sqrt{7x^2-5x+1}=8$; find x.

Make $\sqrt{7x^2-5x+1}=y$; $\therefore 7x^2-5x+1=y^2$;

$$\therefore -y^2+1+8y=8;$$

$$\therefore y^2-8y+16=16-7=9;$$

$$\therefore y=\sqrt{7x^2-5x+1}=4\pm 3=7 \text{ or } 1;$$

$$\therefore 7x^2-5x+1=49 \text{ or } 1.$$

Let $7x^2-5x+1=49$;

$$\therefore x^2-\frac{5x}{7}+\frac{25}{196}=\frac{25}{196}+\frac{48}{7}=\frac{1369}{196};$$

$$\therefore x=\frac{5}{14}\pm\frac{37}{14}=3 \text{ or } -\frac{16}{7}.$$

And if $7x^2-5x+1=1$; $\therefore 7x^2-5x=0$; $\therefore x=0$ or $\frac{5}{7}$.

(Ex. 5.) Let $\frac{x}{a+x}+\frac{a+x}{x}=2b$; find x.

Instead of clearing the equation of fractions, multiply each term by the first; then

$$\left(\frac{x}{a+x}\right)^2+1=2b\,\frac{x}{a+x}; \quad \text{make } y=\frac{x}{a+x};$$

$$\therefore y^2-2by=-1; \quad \therefore y^2-2by+b^2=b^2-1;$$

$$\therefore\ y = b \pm \sqrt{b^2 - 1} = b + \sqrt{b^2 - 1} \text{ or } \frac{1}{b + \sqrt{b^2 - 1}}.$$

Make $b + \sqrt{b^2 - 1} = \beta$;

$$\therefore\ \frac{x}{a + x} = \beta \text{ or } \frac{1}{\beta};\quad \therefore\ x = \frac{a\beta}{1 - \beta} \text{ or } \frac{-a}{1 - \beta}.$$

There are many artifices used in the solution of equations, which can only be learnt by practice; the use of some of these we now add, but we must refer the reader to Bland's Algebraical Problems, for a more complete exhibition of them.

(Ex. 6.) Let $4x^4 + \frac{x}{2} = 4x^3 + 33$; find x;

$$\therefore\ 4x^4 - 4x^3 + x^2 - x^2 + \frac{x}{2} = 33, \text{ adding and subtracting } x^2;$$

$$\therefore\ (2x^2 - x)^2 - \tfrac{1}{2}(2x^2 - x) = 33;$$

$$\therefore\ (2x^2 - x)^2 - \tfrac{1}{2}(2x^2 - x) + \frac{1}{16} = 33 + \frac{1}{16} = \frac{529}{16};$$

$$\therefore\ 2x^2 - x = \pm\frac{23}{4} + \frac{1}{4} = 6 \text{ or } -\frac{11}{2},$$

$$\text{whence } x = 2 \text{ or } -\frac{3}{2}, \text{ or } \frac{1 \pm \sqrt{-43}}{4}.$$

(Ex. 7.) Let $\frac{x}{4} = \frac{\sqrt{x} - 12}{x - 18}$; find x;

$$\therefore\ x^2 - 18x = 4\sqrt{x} - 48;$$

$$\therefore\ x^2 - 14x + 49 = 4x + 4\sqrt{x} + 1;$$

$$\therefore\ x - 7 = 2\sqrt{x} + 1;\quad \therefore\ x - 2\sqrt{x} + 1 = 9;$$

$$\therefore\ \sqrt{x} - 1 = \pm 3;\quad \therefore\ x = 16 \text{ or } 4;$$

observe that if the equation be verified by putting the values for x in it, when $x = 4$, $\sqrt{x} = -2$.

(Ex. 8.) Let $\sqrt{x} - \frac{8}{x} = \frac{7}{\sqrt{x} - 2}$; find x;

$$\therefore\ x - 2\sqrt{x} - \frac{8\sqrt{x}}{x} + \frac{16}{x} = 7;$$

$$\therefore\ x + 8 + \frac{16}{x} - 2\left(\sqrt{x} + \frac{4}{\sqrt{x}}\right) = 15;$$

$$\therefore\ \left(\sqrt{x} + \frac{4}{\sqrt{x}}\right)^2 - 2\left(\sqrt{x} + \frac{4}{\sqrt{x}}\right) + 1 = 16;$$

$$\therefore\ \sqrt{x} + \frac{4}{\sqrt{x}} = 1 \pm 4 = 5 \text{ or } -3,$$

whence $x = 16$ or 1, or $\frac{1 \pm 3\sqrt{-7}}{2}$.

(Ex. 9.) Let $x^3 = 6x + 9$; find x;

$$\therefore\ x^4 = 6x^2 + 9x \text{ multiplying by } x;$$

$$\therefore\ x^4 + 3x^2 = 9x^2 + 9x;$$

$$\therefore\ x^4 + 3x^2 + \frac{9}{4} = 9x^2 + 9x + \frac{9}{4} = 9\left(x^2 + x + \frac{1}{4}\right);$$

$$\therefore\ x^2 + \frac{3}{2} = \pm 3\left(x + \frac{1}{2}\right);$$

$$x^2 = 3x, \text{ and } x = 0 \text{ or } x = 3,$$

or $x^2 + 3x = -3$; $\therefore\ x = \frac{-3 \pm \sqrt{-3}}{2}$.

(Ex. 10.) Let $x^3 - x^2 = 4$; find x;

$$\therefore\ x^4 - x^3 = 4x;$$

$$\therefore\ x^4 + 2x^2 + 1 - (x^3 + x) = 2x^2 + 3x + 1;$$

$$\therefore\ (x^2 + 1)^2 - x(x^2 + 1) + \frac{x^2}{4} = \frac{9x^2}{4} + 3x + 1;$$

$$\therefore\ x^2+1-\frac{x}{2}=\pm\left(\frac{3x}{2}+1\right);$$

$$\therefore\ x^2=2x, \text{ and } x=0, \text{ or } x=2,$$

$$\text{and } x^2+x=-2, \text{ whence } x=\frac{-1\pm\sqrt{-7}}{2}.$$

(Ex. 11.) Let $x^4-8x^3+10x^2+24x+5=0$,

the first and second terms are the same as those of $(x-2)^4$, and $(x-2)^4=x^4-8x^3+24x^2-32x+16$

$$=x^4-8x^3+10x^2+24x+5+14x^2-56x+11$$

$$=14x^2-56x+11=14(x^2+4x)+11.$$

Let $x-2=y$; $\therefore\ x=y+2$ and $x^2-4x=y^2-4$;

$$\therefore\ y^4=14(y^2-4)+11;$$

$$\therefore\ y^4-14y^2+49=11+49-56=4;$$

$$\therefore\ y^2=7\pm2=9 \text{ or } 5;$$

$$\therefore\ y=\pm3 \text{ or } \pm\sqrt{5}, \text{ and } x=5 \text{ or } -1, \text{ or } 2\pm\sqrt{5}.$$

(Ex. 12.) Let $3\sqrt{112-8x}=19+\sqrt{3x+7}$; find x.

Let $3x+7=u$; $\therefore\ x=\dfrac{u-7}{3}$; $\therefore\ 112-8x=\dfrac{392-8u}{3}$;

$$\therefore\ 3\sqrt{\frac{392-8u}{3}}=19+\sqrt{u};$$

$\therefore\ 1176-24u=361+38\sqrt{u}+u$; $\therefore\ 25u+38\sqrt{u}=815$;

whence $\sqrt{u}=5$; $\therefore\ u=25$ and $x=6$.

(Ex. 13.) Given $\{(x-2)^2-x\}^2-(x-2)^2=90-x$; find x.

Make $(x-2)^2-x=u$; $\therefore\ \{(x-2)^2-x\}^2=u^2$;

$$\therefore\ u^2-u=90;\ \therefore\ u=10 \text{ or } -9.$$

Let $u=10$; $\therefore (x-2)^2-x=10$; $\therefore x=6$ or -1.

If $u=-9$; $\therefore (x-2)^2-x=-9$ and $x=\dfrac{5\pm 3\sqrt{-3}}{2}$.

(Ex. 14.) Let $x^4+4x^3+4x+1=\dfrac{57x^2}{4}$; find x.

Divide every term by x^2;

$$\therefore\ x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)=\frac{57}{4}.$$

Let $x+\dfrac{1}{x}=u$; $\therefore x^2+\dfrac{1}{x^2}=u^2-2$;

$$\therefore\ u^2-2+4u=\frac{57}{4};\quad \therefore\ u^2+4u+4=\frac{81}{4};$$

$$\therefore\ u=\pm\frac{9}{2}-2=\frac{5}{2}\text{ or }-\frac{13}{2}.$$

If $x+\dfrac{1}{x}=\dfrac{5}{2}$, we have $x=2$ or $\dfrac{1}{2}$.

If $x+\dfrac{1}{x}=-\dfrac{13}{2}$, we have $x=\dfrac{-13\pm\sqrt{143}}{4}$.

EXAMPLES.

(1) $x^2+8x=20$; $\therefore x=2$ or -10.

(2) $x^2+16x=80$; $\therefore x=4$ or -20.

(3) $x^2+7x=78$; $\therefore x=6$ or -15.

(4) $x^2+3x=28$; $\therefore x=4$ or -7.

(5) $x^2-10x=24$; $\therefore x=12$ or -2.

(6) $x^2-8x=20$; $\therefore x=10$ or -2.

(7) $x^2-5x=6$; $\therefore x=6$ or -1.

(8) $x^2 + x = 30$; $\therefore x = 5$ or -6.

(9) $9x^2 + 9x = 4$; $\therefore x = \frac{1}{3}$ or $-\frac{4}{3}$.

(10) $-x^2 + 33x = 272$; $\therefore x = 17$ or 16.

(11) $x^2 + 25x = 1250$; $\therefore x = 25$ or -50.

(12) $x^2 - 1000x = 127500$; $\therefore x = 250$ or -150.

(13) $-x^2 + x = \dfrac{6}{25}$; $\therefore x = \frac{2}{5}$ or $\frac{3}{5}$.

(14) $4x^2 - 26x = 2x - 48$; $\therefore x = 4$ or 3.

(15) $3x^2 - 30x = 9(x - 12)$; $\therefore x = 9$ or 4.

(16) $17x^2 - 19x = 30$; $\therefore x = 2$ or $-\dfrac{15}{17}$.

(17) $\dfrac{x}{2} + \dfrac{2}{x} = \dfrac{x}{4} + \dfrac{3}{2}$; $\therefore x = 2$ or 4.

(18) $x^2 - 4x = -1$; $\therefore x = 2 \pm \sqrt{3}$.

(19) $3x^2 + 5x = 2$; $\therefore x = \frac{1}{3}$ or -2.

(20) $4x - 3x^2 = 6x - 8$; $\therefore x = \frac{4}{3}$ or -2.

(21) $\dfrac{4x}{7} - \dfrac{2x^2}{3} = \dfrac{10x}{3} - \dfrac{20}{7}$; $\therefore x = -5$ or $\frac{6}{7}$.

(22) $\dfrac{65x}{2} - \dfrac{10x^2}{11} = \dfrac{13}{2} - \dfrac{2x}{11}$; $\therefore x = \frac{1}{5}$ or $35\frac{3}{4}$.

(23) $\dfrac{x}{x+8} = \dfrac{x+3}{2x+1}$; $\therefore x = 12$ or -2.

(24) $\dfrac{x+3}{x-4} + \dfrac{x-4}{4} = 2 + \dfrac{4x+7}{5x}$; $\therefore x = 9$ or -3.

(25) $\dfrac{x}{x+60} = \dfrac{7}{3x-5}$; $\therefore x = 14$ or -10.

(26) $x + \dfrac{24}{x-1} = 3x - 4$; $\therefore x = 5$ or -2.

(27) $\frac{x+3}{x}+7\cdot\frac{x}{x+3}=\frac{23}{4}$; $\therefore x=4$ or $\frac{7}{4}$.

(28) $\frac{x+5}{x+9}+\frac{x+3}{x+7}=1\frac{19}{99}$; $\therefore x=2$ or $-\frac{81}{10}$.

(29) $\frac{x-4}{x-3}+\frac{x-3}{x-4}=2\frac{1}{20}$; $\therefore x=8$ or -1.

(30) $x+\frac{1}{x}=\frac{4}{\sqrt{3}}$; $\therefore x=\sqrt{3}$ or $\frac{1}{\sqrt{3}}$.

(31) $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-\frac{13}{2}$; $\therefore x=8$ or $13\frac{22}{31}$.

(32) $\frac{\frac{1}{2}}{\frac{x}{3}+\frac{x}{4}-1}+\frac{3x}{2x-6}=3$; $\therefore x=12$ or 2.

(33) $\frac{x^2+2}{x^2-2}+9\frac{x^2-2}{x^2+2}=7\frac{13}{77}$; $\therefore x=\pm 3$.

(34) $\frac{x+\frac{1}{x}}{x-\frac{1}{x}}+\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{13}{4}$; $\therefore x=3$ or $-\frac{7}{5}$.

(35) $\frac{4x}{x+1}+\frac{6x}{x+2}=\frac{9x}{x+3}+1$; $\therefore x=.3549$ or -1.5367.

(36) $\frac{3x-2}{x-4}+\frac{2x-1}{x-2}=6+\frac{50}{x^2-6x+8}$;

$\therefore x=10$ or -9.

(37) $\frac{x}{x+a}=\frac{b}{x-b}$; $\therefore x=b\pm\sqrt{ab+b^2}$.

(38) $\sqrt{x+a}-\sqrt{x+b}=\sqrt{2x}$; $\therefore x=\frac{a+b}{2}\pm\sqrt{\frac{a^2+b^2}{2}}$.

(39) $x^2+\sqrt{x}=\frac{72}{x}$; $\therefore x=4$ or $\sqrt[3]{81}$.

(40) $x^2 + \frac{2}{3}\sqrt{3x^2+3x} = 46 - x$;

$\therefore x = 3$, or -4, or $-\frac{1}{2} \pm \frac{\sqrt{777}}{6}$.

(41) $x^2 - 3x + 7\sqrt{11x - 2x^2 + 2} = \frac{5x}{2} + 21$;

$\therefore x = 2$ or $\frac{7}{2}$ or $\frac{11 \pm \sqrt{-663}}{4}$.

(42) $x^2 + x + 2\sqrt{x^2 + x + 4} = 20$;

$\therefore x = 3$ or -4 or $\frac{-1 \pm \sqrt{129}}{2}$.

(43) $\sqrt{x} + \frac{4}{\sqrt{x}} = 5$; $\therefore x = 16$ or 1.

(44) $6x + \frac{12}{x} = 5\sqrt{3}\sqrt{1 + x^2}$; $\therefore x = \pm\sqrt{3}$.

(45) $\frac{\sqrt{x}+1}{\sqrt{x}-1} + 5\frac{\sqrt{x}-1}{\sqrt{x}+1} = \frac{9}{2}$; $\therefore x = 9$ or $\frac{49}{9}$.

(46) $x^4 - 25x^2 = -144$; $\therefore x = \pm 3$ or ± 4.

(47) $x^4 - 7x^2 = 8$; $\therefore x = \pm 2\sqrt{2}$ or $\pm\sqrt{-1}$.

(48) $x^4 - (2bc + 4a^2)x^2 = -b^2c^2$;

$\therefore \ = \pm\sqrt{bc + 2a^2 \pm 2a\sqrt{bc + a^2}}$.

(49) $5x^4 + 7x^2 = 6732$; $\therefore x = \pm 6$.

(50) $9x^6 - 11x^3 = 488$; $\therefore x = 2$ or $\sqrt[3]{-\frac{61}{9}}$.

(51) $x^3 - x^{\frac{3}{2}} = 15500$; $\therefore x = 25$ or $(124)^{\frac{2}{3}}$.

(52) $8\sqrt{x} + 21\sqrt[4]{x} = 74$; $\therefore x = 16$ or $\left(\frac{37}{8}\right)^4$.

(53) $x^{\frac{5}{6}} + x^{\frac{5}{3}} = 1056$; $\therefore x = 64$ or $(-33)^{\frac{6}{5}}$.

(54) $x^{\frac{4}{3}} - \frac{1}{2}x^{\frac{2}{3}} = -112$; $\therefore x = 8$ or $(-14)^{\frac{3}{4}}$.

(55) $x^2+4x+\frac{4}{x^2}=\frac{17x-8}{x}$; $\therefore x=2$, or 1, or $\frac{-7\pm\sqrt{41}}{2}$.

(56) $\frac{x}{2}=(\sqrt{1+x}-1)(\sqrt{1-x}+1)$; $\therefore x=\frac{24}{25}$ or 0.

(57) $7\sqrt{3x-6}=3\sqrt{3x+1}+3$; $\therefore x=5$.

(58) $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$; $\therefore x=9$.

(59) $\frac{x}{x^2+x+5}+\frac{5}{\sqrt{x^2+x+5}}=\frac{116}{25x}$; $\therefore x=4$ or $-\frac{20}{9}$.

(60) $\{(x+3)^2+x+3\}^2-7(x+3)^2=711+7x$;

$\therefore x=2$ or -9.

(61) $x^2+\frac{4}{x^2}+6x+\frac{12}{x}=23$; $\therefore x=2$ or 1.

(62) $\sqrt[3]{x+784}-\sqrt[3]{x-208}=8$; $\therefore x=216$ or -792.

(63) $cx^2-2cx\sqrt{d}=dx^2-cd$; $\therefore x=\frac{\sqrt{cd}}{\sqrt{c}\pm\sqrt{d}}$.

(64) $x^3-2x=4$; $\therefore x=2$ or $-1\pm\sqrt{-1}$.

(65) $2x^3-x^2=1$; $\therefore x=1$ or $\frac{1}{2}(-1\pm\sqrt{-1})$.

(66) $x^3-3x=2$; $\therefore x=2$ or -1.

(67) $\frac{x^2+1}{x}+\frac{7}{2}\cdot\frac{x+1}{\sqrt{x}}=13$; $\therefore x=4$ or $\frac{1}{4}$.

(68) $x^4-2x^3-2x^2+3x=108$;

$\therefore x=4$, or -3, or $\frac{1\pm\sqrt{-23}}{2}$.

(69) $(x+2)^2+(x+2)=20$; $\therefore x=2$ or -7.

(70) $\frac{x}{x+4}+\frac{4}{\sqrt{x+4}}=\frac{21}{x}$; $\therefore x=12$ or -3.

(71) $x^4-8x^2-24x=32$; $\therefore x=4$, or -2, or $-1\pm\sqrt{-3}$.

(72) $\dfrac{2x+\sqrt{x}}{2x-\sqrt{x}}+3\dfrac{2x-\sqrt{x}}{2x+\sqrt{x}}=4;\quad \therefore\ x=1.$

(73) $(x-4)^2+2(x-4)=\dfrac{2}{x}-1;\ \therefore\ x=2$ or $2\pm\sqrt{3}.$

(74) $x-3=\dfrac{3+4\sqrt{x}}{x};\qquad \therefore\ x=\frac{1}{2}(7\pm\sqrt{13}).$

QUADRATIC EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

98. These may be divided, as quadratic equations with one unknown quantity have been, into two classes:

(1) Pure Quadratic,

(2) Adfected Quadratic Equations.

99. Pure Quadratic Equations may in general be reduced to the solution of one of the pairs of equations.

(1) $\left.\begin{matrix} x+y=a \\ xy=b \end{matrix}\right\}.$ (2) $\left.\begin{matrix} x-y=c \\ xy=b \end{matrix}\right\}.$ (3) $\begin{matrix} x+y=a, \\ x-y=c. \end{matrix}$

Our attention therefore will be directed to the solution of the two former: since that of the third is sufficiently obvious.

(Ex. 1.) To solve $x+y=a$, and $xy=b$; to do so we shall endeavour to obtain $x-y$.

Squaring the first equation;

$\therefore\ x^2+2xy+y^2=a^2,$

and $4xy\qquad =4b$, multiplying 2nd by 4;

$\therefore\ x^2-2xy+y^2=a^2-4b$, by subtraction,

or $(x\ \ y)^2=a^2-4b;$

$$\therefore \quad x - y = \sqrt{a^2 - 4b}.$$

$$\text{But } x + y = \quad a;$$

$$\therefore \quad 2x = a + \sqrt{a^2 - 4b},$$

$$2y = a - \sqrt{a^2 - 4b};$$

whence x and y may be found.

In a similar manner the equation (2) may be solved, the only difference being, that since $x + y$ is to be found, $4xy$ is added to the square of $x - y$; for

$$(x + y)^2 = (x - y)^2 + 4xy.$$

(Ex. 2.) To solve $x^2 + y^2 = d^2$, and $x + y = a$;

$$\therefore \quad x^2 + 2xy + y^2 = a^2 \; \ldots\ldots \; (1).$$

$$\text{But } x^2 + \quad y^2 = d^2 \; \ldots\ldots \; (2).$$

Take (2) from (1) $\therefore \; 2xy = a^2 - d^2 \; \ldots\ldots \; (3).$

Take (3) from (2) $\therefore \; x^2 - 2xy + y^2 = 2d^2 - a^2;$

$$\therefore \quad x - y = \sqrt{2d^2 - a^2},$$

$$\text{and } x + y = a;$$

$$\therefore \quad 2x = a + \sqrt{2d^2 - a^2}, \text{ and } 2y = a - \sqrt{2d^2 - a^2}.$$

(Ex. 3.) Solve the equations $x^2 + y^2 = 73$, and $x + y = 11$;

$$\therefore \quad x^2 + 2xy + y^2 = 121 \; \ldots\ldots \; (1).$$

$$\text{But } x^2 \quad + y^2 = \; 73 \; \ldots\ldots \; (2);$$

$$\therefore \; 2xy = \; 48 \; \ldots\ldots \; (3);$$

$$\therefore \quad x^2 - 2xy + y^2 = \; 25;$$

$\therefore \; x - y = \pm 5$, and $x + y = 11$; $\therefore \; x = 8$ or 3, $y = 3$ or 8.

(Ex. 4.) Given $x^2+y^2=85$, and $xy=42$; find x and y.

(1) Add twice the second to the first;

$$\therefore\ x^2+2xy+y^2=169;\quad \therefore\ x+y=\pm 13.$$

(2) Subtract twice the second from the first;

$$\therefore\ x^2-2xy+y^2=1;\quad \therefore\ x-y=\pm 1;$$

$$\therefore\ 2x=\pm 14;\quad \therefore\ x=\pm 7,\text{ and } 2y=\pm 12;\quad \therefore\ y=\pm 6.$$

(Ex. 5.) Given $x+y=30$, and $xy=209$; find x and y;

$$\therefore\ x^2+2xy+y^2=900,$$

$$\text{and}\quad 4xy\quad =836;$$

$$\therefore\ x^2-2xy+y^2=\ 64;$$

$$\therefore\ x-y=\pm 8,\text{ and } x+y=30;\quad \therefore\ 2x=38\text{ or }22,$$

and $2y=22$ or 38; $\therefore\ x=19$ or 11; $y=11$ or 19.

(Ex. 6.) Given $x^3+y^3=91$, and $x+y=7$; find x and y.

Divide the first equation by the second;

$$\therefore\ x^2-xy+y^2=13\ \ldots\ldots\ (1).$$

But $x^2+2xy+y^2=49\ \ldots\ldots\ (2)$, by squaring 2nd.

Subtract (1) from (2) $\therefore\ 3xy=36$; $\therefore\ xy=12\ \ldots\ (3)$.

Subtract (3) from (1) $\therefore\ x^2-2xy+y^2=1$;

$$\therefore\ x-y=\pm 1,\text{ and } x+y=7;$$

$\therefore\ 2x=8$ or 6, $2y=6$ or 8, $x=4$ or 3, $y=3$ or 4.

(Ex. 7.) Given $x^2+xy+y^2=91$, and $x+\sqrt{xy}+y=13$.

Divide the former by the latter equation;

$$\therefore\ x-\sqrt{xy}+y=\ 7\ \ldots\ldots\ (1).$$

$$\text{But } x+\sqrt{xy}+y=13\ \ldots\ldots\ (2).$$

Take (1) from (2) $\therefore\ 2\sqrt{xy}\quad =\ 6\ \ldots\ldots\ (3)$;

$$\therefore\ x+y=10,$$

$$x^2+2xy+y^2=100,$$

Square (3) $\therefore\quad 4xy \quad = 36;$

$$\therefore\ x^2-2xy+y^2=64;\quad \therefore\ x-y=\pm 8.$$

$\therefore\ 2x=18$ or 2, $2y=2$ or 18; $\therefore\ x=9$ or 1; $y=1$ or 9.

(Ex. 8.) Let $x^{\frac{1}{4}}+y^{\frac{1}{5}}=6$, and $x^{\frac{3}{4}}+y^{\frac{3}{5}}=126$, find x and y.

Divide the second equation by the first,

$$x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{5}}+y^{\frac{2}{5}}=21\ldots\ldots(1).$$

But $x^{\frac{1}{2}}+2x^{\frac{1}{4}}y^{\frac{1}{5}}+y^{\frac{2}{5}}=36\ldots\ldots(2)$;

$$\therefore\ 3x^{\frac{1}{4}}y^{\frac{1}{5}}=15;\quad \therefore\ x^{\frac{1}{4}}y^{\frac{1}{5}}=5\ldots\ldots(3).$$

Take (3) from (1) $\therefore\ x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{\frac{1}{5}}+y^{\frac{2}{5}}=16;$

$$\therefore\ x^{\frac{1}{4}}-y^{\frac{1}{5}}=\pm 4,\ \text{ and }\ x^{\frac{1}{4}}+y^{\frac{1}{5}}=6;$$

$$\therefore\ 2x^{\frac{1}{4}}=10 \text{ or } 2;\quad \therefore\ x^{\frac{1}{4}}=5 \text{ or } 1;\quad \therefore\ x=625 \text{ or } 1,$$

$$2y^{\frac{1}{5}}=2 \text{ or } 10;\quad \therefore\ y^{\frac{1}{5}}=1 \text{ or } 5;\quad \therefore\ y=1 \text{ or } 3125.$$

(Ex. 9.) Given $(x-y)(x^2-y^2)=160$, and $(x+y)(x^2+y^2)=580$, find x and y; multiply both the equations out;

$$\therefore\ x^3-x^2y-xy^2+y^3=160\ldots\ldots(1)$$

$$x^3+x^2y+xy^2+y^3=580\ldots\ldots(2).$$

Take (1) from (2) $\therefore\ 2x^2y+2xy^2=420\ldots.(3).$

Add (3) to (2) $\qquad \therefore\ x^3+3x^2y+3xy^2+y^3=1000.$

Extract the cube root; $\therefore\ x+y=10$.

From (3) $xy(x+y)=210$; $\therefore\ xy=21$.

Whence from $x+y=10$, and $xy=21$, we have $x=7$ or 3,

$$y=3 \text{ or } 7.$$

ADFECTED QUADRATIC EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

100. Equations of this class are frequently very complicated; we shall confine ourselves to the more simple cases; such equations are commonly solved, by substituting in one of the equations the values of x or y obtained from the other equation; the problem is then reduced to one, in which a single unknown quantity is to be found.

(Ex. 1.) Let $3x^2+2xy+2x=20$, and $5x-3y=7$.

From the 2nd, $3y=5x-7$; $\therefore\ xy=\frac{5x^2-7x}{3}$.

Substituting for xy in the first, we have,

$$3x^2+\frac{10x^2-14x}{3}+2x=20; \quad \therefore\ 19x^2-8x=60;$$

$$\therefore\ x=2 \text{ or } -\frac{30}{19}, \text{ and } y=1 \text{ or } -\frac{283}{57}.$$

(Ex. 2.) Let $5x-2y=4$, and $4x^2-3y^2=-11$; find x and y.

From 1st, $y=\frac{5x}{2}-2$; $\therefore\ y^2=\frac{25x^2}{4}-10x+4$;

$$\therefore\ 4x^2-\frac{75x^2}{4}+30x-12=-11; \quad \therefore\ 59x^2-120x=-4;$$

$$\therefore\ x=2 \text{ or } \frac{2}{59}, \text{ and } y=3 \text{ or } -\frac{113}{59}.$$

Sometimes however particular methods are more useful, as we shall see in the following examples.

(Ex. 3.) Given $x^4+y^4=17$, and $x+y=3$; find x and y.
Raise the second equation to the fourth power;

$$\therefore\ x^4+4x^3y+6x^2y^2+4xy^3+y^4=81.$$

$$\text{But}\ x^4 \qquad\qquad\qquad +y^4=17;$$

$$\therefore\ 4x^3y+6x^2y^2+4xy^3 = 64,$$

$$\text{or}\quad xy(2x^2+3xy+2y^2) = 32\ldots.(1).$$

Take twice the square of the second, and multiply by xy;

$$\therefore\ xy(2x^2+4xy+2y^2)=18xy\ldots\ldots(2).$$

Take (1) from (2) $\therefore\ xy\times xy=18xy-32$;

$\therefore\ (xy)^2-18xy+81=81-32=49$; $\therefore\ xy=2$ or 16;

and $\because\ x+y=3$, if $xy=2$; $x=2$ or 1, $y=1$ or 2;
if we take $xy=16$, the values of x and y are impossible.

(Ex. 4.) Given $x^5+y^5=33$, and $x+y=3$; find x and y.
Divide the former equation by the latter;

$$\therefore\ x^4-x^3y+x^2y^2-xy^3+y^4=11\ldots\ldots(1).$$

$$\text{But}\ x^4+4x^3y+6x^2y^2+4xy^3+y^4=81\ldots\ldots(2).$$

Take (1) from (2) $\therefore\ 5x^3y+5x^2y^2+5xy^3=70$;

$$\therefore\ xy(x^2+xy+y^2)=14.$$

But $x^2+2xy+y^2=9$; $\therefore\ x^2+xy+y^2=9-xy$;

$\therefore\ xy(9-xy)=14$; $\therefore\ (xy)^2-9xy=-14$;

$\therefore\ xy=\frac{9}{2}\pm\frac{5}{2}=2$ or 7, and $\because\ x+y=3$;

$\therefore$ taking $xy=2$, we have $x=2$ or 1, $y=1$ or 2.

(Ex. 5.) Given $x+y+\sqrt{x+y}=12$, and $xy=20$; find x and y.

Complete the square of the former equation by adding $\frac{1}{4}$;

$$\therefore\ (x+y)+\sqrt{x+y}+\frac{1}{4}=12+\frac{1}{4}=\frac{49}{4};$$

$$\therefore\ \sqrt{x+y}=\pm\frac{7}{2}-\frac{1}{2}=3 \text{ or } -4;$$

$$\therefore\ x+y=9 \text{ or } 16, \text{ and } xy=20,$$

and from $x+y=9$ and $xy=20$; $x=4$ or 5, $y=5$ or 4;

from $x+y=16$ and $xy=20$; $x=8\pm\sqrt{34}$, $y=8\mp\sqrt{34}$.

(Ex. 6.) Given $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}=\frac{27}{4}$, and $x^2+y^2=20$.

Since $\left(\frac{x}{y}+\frac{y}{x}\right)^2=\frac{x^2}{y^2}+2+\frac{y^2}{x^2}$; add two to each of the members of the former equation, and then $\frac{1}{4}$ to complete the square;

$$\therefore\ \left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{4}=\frac{27}{4}+2+\frac{1}{4}=9;$$

$$\therefore\ \frac{x}{y}+\frac{y}{x}=\pm 3-\frac{1}{2}=\frac{5}{2} \text{ or } -\frac{7}{2}.$$

Let $\frac{x}{y}+\frac{y}{x}=\frac{5}{2}$; $\therefore\ \frac{x^2+y^2}{xy}$ or $\frac{20}{xy}=\frac{5}{2}$;

$$\therefore\ xy=8, \text{ and } 2xy=16;$$

$$\therefore\ x^2+2xy+y^2=36, \text{ and } x^2-2xy+y^2=4;$$

$$\therefore\ x+y=\pm 6, \text{ and } x-y=\pm 2;\ \therefore\ x=\pm 4,\ y=\pm 2.$$

(Ex. 7.) Given $3x^2+4y^2=7xy$ and $x^{\frac{3}{2}}-\frac{2y^2}{9}=yx^{\frac{1}{2}}$;

$$\therefore\ 3x^2-7xy=-4y^2;\quad \therefore\ x^2-\frac{7}{3}xy=-\frac{4y^2}{3};$$

$$\therefore\ x^2 - \frac{7}{3}xy + \frac{49}{36}y^2 = \frac{49}{36}y^2 - \frac{4y^2}{3} = \frac{y^2}{36};$$

$$\therefore\ x = \frac{7y}{6} \pm \frac{y}{6} = \frac{4y}{3} \text{ or } y.$$

But from the second equation, $x\sqrt{x} - y\sqrt{x} = \frac{2y^2}{9}$;

$$\therefore\ \sqrt{x}\,(x-y) = \frac{2y^2}{9}; \quad \therefore\ \sqrt{x} \times \frac{y}{3} = \frac{2y^2}{9}; \quad \therefore\ \sqrt{x} = \frac{2y}{3};$$

$$\therefore\ x = \frac{4y^2}{9} = \frac{4y}{3}; \quad \therefore\ y = 3, \text{ and } x = 4.$$

From $x = y$ we have $x - y = 0$; $\therefore\ y^2 = 0$; $\therefore\ y = 0$, and $x = 0$.

101. When the equations are homogeneous, i. e. when the sum of the indices of the unknown quantities is the same in every term, it is frequently convenient to substitute vy for one of the quantities as x, and then either by dividing one of the equations by the other or by other obvious methods, an equation will result involving v only.

(Ex. 1.) Given $ax^2 + bxy = c$, and $a_1x^2 + b_1y^2 = c_1$*; find x and y.

Let $x = vy$;

$$\therefore\ av^2y^2 + bvy^2 = c \quad (1), \quad \text{and } a_1v^2y^2 + b_1y^2 = c_1 \quad (2).$$

Multiply (1) by c_1, and (2) by c, then

$$c_1av^2y^2 + c_1bvy^2 = cc_1 = ca_1v^2y^2 + cb_1y^2;$$

$$\therefore\ (c_1a - ca_1)v^2 + c_1bv = cb_1,$$

* Letters marked as a_1, b_1, c_1, and which are read a one, b one, c one; are frequently made use of in long and complicated expressions, for the symmetry which they give to the results; but they have no value different from that of the letters which are without a mark.

a quadratic equation from which v may formed, and therefore y and x.

(Ex. 2.) $2x^2 - 3xy + 4y^2 = 24, \quad 3x^2 - 5y^2 = 28,$

$$x = vy; \therefore (2v^2 - 3v + 4)y^2 = 24; \ (3v^2 - 5)y^2 = 28;$$

$$\therefore 7y^2 \times (2v^2 - 3v + 4) = 168 = 6y^2(3v^2 - 5);$$

$$\therefore 14v^2 - 21v + 28 = 18v^2 - 30; \ \therefore 4v^2 + 21v = 58;$$

$$\text{whence } v = 2; \ \therefore y^2 = \frac{28}{3v^2 - 5} = \frac{28}{7} = 4;$$

$$\therefore y = \pm 2 \text{ and } x = vy = \pm 4.$$

(Ex. 3.) Given $x^3y^2 + xy^4 = 156$, and $2x^3y^2 - x^2y^3 = 144$.

Here substitute for y instead of x, as the equations for v will be more simple; $\therefore$ make $y = vx$;

$$\therefore x^5.(v^2 + v^4) = 156, \text{ and } x^5.(2v^2 - v^3) = 144;$$

$$\therefore \frac{v^2 + 1}{2 - v} = \frac{156}{144} = \frac{13}{12}; \ \therefore 12v^2 + 13v = 14;$$

whence $v = \frac{2}{3}$, and $x = 3$, and $y = 2$.

EXAMPLES.

(1) $\left.\begin{aligned} x^2 + y^2 &= 549 \\ x^2 - y^2 &= 99 \end{aligned}\right\};$ $\quad x = \pm 18,\ y = \pm 15.$

(2) $x + y = 41;\ xy = 420;$ $\quad x = 21$ or 20, $y = 20$ or 21.

(3) $x - y = 5;\ xy = 36;$ $\quad x = 9$ or -4, $y = 4$ or -9.

(4) $\left.\begin{aligned} x+y&=20 \\ x^2+y^2&=202 \end{aligned}\right\};$ $\quad x=11$ or 9, $y=9$ or 11.

(5) $\left.\begin{aligned} x-y&=2 \\ x^2+y^2&=394 \end{aligned}\right\};$ $\quad x=15$ or -13, $y=13$ or -15.

(6) $\left.\begin{aligned} x^2+y^2&=113 \\ xy&=56 \end{aligned}\right\};$ $\quad x=\pm 8$, $y=\pm 7$.

(7) $\left.\begin{aligned} x^2+y^2&=\frac{97}{36} \\ xy&=1 \end{aligned}\right\};$ $\quad x=\frac{3}{2}$ or $\frac{2}{3}$, $y=\frac{2}{3}$ or $\frac{3}{2}$.

(8) $\left.\begin{aligned} x^2-y^2&=99 \\ x+y&=33 \end{aligned}\right\};$ $\quad x=18$, $y=15$.

(9) $\left.\begin{aligned} x^2-y^2&=105 \\ x-y&=7 \end{aligned}\right\};$ $\quad x=11$, $y=4$.

(10) $\left.\begin{aligned} x+y&=11 \\ x^3+y^3&=407 \end{aligned}\right\};$ $\quad x=7$ or 4, $y=4$ or 7.

(11) $\left.\begin{aligned} x^3+y^3 : x^3-y^3 &:: 9 : 7 \\ x^2y-y^2x&=16 \end{aligned}\right\};$ $\quad x=4$, $y=2$.

(12) $\left.\begin{aligned} xy+y&=10 \\ x^2y^2+y^2&=68 \end{aligned}\right\};$ $\quad x=4$, $y=2$.

(13) $\left.\begin{aligned} y+x\sqrt{y}&=21 \\ x^2y+y^2&=225 \end{aligned}\right\};$ $\quad x=4$, $y=9$.

(14) $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{5}{6} \\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{13}{36} \end{aligned}\right\};$ $\quad x=2$ or 3, $y=3$ or 2.

(15) $$\left.\begin{array}{r} x^2+y^2 : xy :: 13 : 6 \\ x^2-y^2=20 \end{array}\right\};\qquad \begin{array}{l} x=\pm 6, \\ y=\pm 4. \end{array}$$

(16) $$\left.\begin{array}{r} x^3+y^3=1001 \\ x^2y+xy^2=110 \end{array}\right\};\qquad \begin{array}{l} x=10 \text{ or } 1, \\ y=1 \text{ or } 10. \end{array}$$

(17) $$\left.\begin{array}{r} x^3-y^3=2375 \\ x-y=5 \end{array}\right\};\qquad \begin{array}{l} x=15 \text{ or } -10, \\ y=10 \text{ or } -15. \end{array}$$

(18) $$\left.\begin{array}{r} x^2-y^2=105 \\ xy=44 \end{array}\right\};\qquad \begin{array}{l} x=11 \text{ or } -4, \\ y=4 \text{ or } -11. \end{array}$$

(19) $$\left.\begin{array}{r} (x+y)(x^2+y^2)=520 \\ (x-y)\,.\,(x^2-y^2)=40 \end{array}\right\};\qquad \begin{array}{l} x=6 \text{ or } 4, \\ y=4 \text{ or } 6. \end{array}$$

(20) $$\left.\begin{array}{r} x^2+xy+y^2=91 \\ x+\sqrt{xy}+y=13 \end{array}\right\};\qquad \begin{array}{l} x=9 \text{ or } 1, \\ y=1 \text{ or } 9. \end{array}$$

(21) $$\left.\begin{array}{r} x^2+xy=4500 \\ xy+y^2=3600 \end{array}\right\};\qquad \begin{array}{l} x=50 \text{ or } 40, \\ y=40 \text{ or } 50. \end{array}$$

(22) $$\left.\begin{array}{r} x^2+xy=84 \\ x^2-y^2=24 \end{array}\right\};\qquad \begin{array}{l} x=\pm 7, \\ y=\pm 5. \end{array}$$

(23) $$\left.\begin{array}{r} x^3+y^3=152 \\ x^2-xy+y^2=19 \end{array}\right\};\qquad \begin{array}{l} x=5 \text{ or } 3, \\ y=3 \text{ or } 5. \end{array}$$

(24) $$\left.\begin{array}{r} x^2+y^2+xy=208 \\ x+y=16 \end{array}\right\};\qquad \begin{array}{l} x=12 \text{ or } 4, \\ y=4 \text{ or } 12. \end{array}$$

(25) $$\left.\begin{array}{r} x^3-y^3=7\,xy \\ x-y=2 \end{array}\right\};\qquad \begin{array}{l} x=4 \text{ or } -2, \\ y=2 \text{ or } -4. \end{array}$$

(26) $$\left.\begin{array}{r} \dfrac{x^2}{y}-\dfrac{y^2}{x}=8\,\dfrac{2}{3} \\ x-y=2 \end{array}\right\};\qquad \begin{array}{l} x=3 \text{ or } -1, \\ y=1 \text{ or } -3. \end{array}$$

(27) $\left.\begin{array}{r} 2x - 3y = 2 \\ 8x^3 - 27y^3 = 37xy \end{array}\right\};$ $\quad x = 4$ or -3, $y = 2$ or $-\frac{8}{3}$.

(28) $\left.\begin{array}{l} x - y = \sqrt{x} + \sqrt{y} \\ x^{\frac{3}{2}} - y^{\frac{3}{2}} = 37 \end{array}\right\};$ $\quad x = 16$ or 9, $y = 9$ or 16.

(29) $\left.\begin{array}{l} x^2 + y^2 + \sqrt{x^2 + y^2} = 30 \\ x + y : xy :: 7 : 12 \end{array}\right\};$ $\quad x = 4$ or 3, $y = 3$ or 4.

(30) $\left.\begin{array}{l} x^2y + xy^2 = 30 \\ \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \end{array}\right\};$ $\quad x = 3$ or 2, $y = 2$ or 3.

(31) $\left.\begin{array}{l} x^{\frac{1}{4}} + y^{\frac{1}{2}} = 5 \\ x^{\frac{2}{4}} + y^{\frac{2}{2}} = 13 \end{array}\right\};$ $\quad x = 16$, $y = 9$.

(32) $\left.\begin{array}{r} x - y = 208 \\ \sqrt[3]{x} - \sqrt[3]{y} = 4 \end{array}\right\};$ $\quad x = 216$, $y = 8$.

(33) $\left.\begin{array}{r} x^2 + y^2 + x + y = 922 \\ \sqrt{xy} = 20 \end{array}\right\};$ $\quad x = 25$, $y = 16$.

(34) $\left.\begin{array}{l} x^2y^2 = 180 - 8xy \\ x + 3y = 11 \end{array}\right\};$ $\quad x = 5$ or 6, $y = 2$ or $\frac{5}{3}$.

(35) $\left.\begin{array}{r} \frac{2}{x} + \frac{3}{y} = 8 \\ 7xy = 6 \end{array}\right\};$ $\quad x = 2$ or $\frac{1}{7}$, $y = \frac{3}{7}$ or 6.

(36) $\left.\begin{array}{r} x + y + \sqrt{x + y} = 12 \\ x^2 + y^2 = 41 \end{array}\right\};$ $\quad x = 5$ or 4, $y = 4$ or 5.

(37) $\left.\begin{array}{r} 5xy + 3x + 7y = 639 \\ 9\dfrac{x}{y} = 16\dfrac{y}{x} \end{array}\right\};$ $\quad x = 12$ or $-\dfrac{71}{5}$, $\quad y = 9$ or $-\dfrac{213}{10}$.

(38) $\left.\begin{array}{r} 5x - 7y = 19 \\ 7x^2 - 5y^2 = 403 \end{array}\right\};$ $\quad x = 8$, $\quad y = 3$.

(39) $\left.\begin{array}{r} x^2 + y^2 + x + y = 330 \\ x^2 - y^2 + x - y = 150 \end{array}\right\};$ $\quad x = 15$ or -16, $\quad y = 9$ or -10.

(40) $\left.\begin{array}{l} x^4 + y^4 = 337 \\ x + y = 7 \end{array}\right\};$ $\quad x = 4$ or 3, $\quad y = 3$ or 4.

(41) $\left.\begin{array}{l} x^5 + y^5 = 3157 \\ x + y = 7 \end{array}\right\};$ $\quad x = 5$ or 2, $\quad y = 2$ or 5.

(42) $\left.\begin{array}{r} x + y + xy = 34 \\ x^2 + y^2 - x - y = 42 \end{array}\right\};$ $\quad x = 6$ or 4, $\quad y = 4$ or 6.

(43) $\left.\begin{array}{r} xy(x^2 + y^2) = 300 \\ x^2 - y^2 = 7 \end{array}\right\};$ $\quad x = \pm 4$ or $\pm\sqrt{-3}$, $\quad y = \pm 3$ or $\pm\sqrt{-4}$.

(44) $\left.\begin{array}{r} x + \sqrt{1 - y^2} = 1 \\ y + \sqrt{1 - x^2} = \sqrt{3} \end{array}\right\};$ $\quad x = \dfrac{1}{2}$, $\quad y = \dfrac{\sqrt{3}}{2}$.

(45) $\left.\begin{array}{r} xy(x^2 + y^2) = 1820 \\ x + y = 11 \end{array}\right\};$ $\quad x = 7$ or 4, $\quad y = 4$ or 7.

(46) $\left.\begin{array}{l} x^3 - y^3 = 19 \\ \dfrac{x}{y} - \dfrac{y}{x} = \dfrac{5}{6} \end{array}\right\};$ $\quad x = 2$, $\quad y = 3$.

(47) $\left.\begin{aligned} x+y&=xy \\ x+y+x^2+y^2&=15\tfrac{3}{4}\end{aligned}\right\};$ $\quad x=3$ or $\frac{3}{2}$, $y=\frac{3}{2}$ or 3.

(48) $\left.\begin{aligned} x^{\frac{1}{2}}+y^{\frac{1}{5}}&=7 \\ x^2+y^{\frac{4}{5}}&=641\end{aligned}\right\};$ $\quad x=25$, $y=32$.

(49) $\left.\begin{aligned} x^2+y^2&=34 \\ x^2-xy&=10\end{aligned}\right\};$ $\quad x=5$, $y=3$.

(50) $\left.\begin{aligned} x^2+xy&=66 \\ xy-y^2&=5\end{aligned}\right\};$ $\quad x=6$, $y=5$.

(51) $\left.\begin{aligned} x^2+3xy+4y^2&=14 \\ 3x^2+4xy+5y^2&=25\end{aligned}\right\};$ $\quad x=2$, $y=1$.

(52) $\left.\begin{aligned} x^4-x^2+y^4-y^2&=84 \\ x^2+x^2y^2+y^2&=49\end{aligned}\right\};$ $\quad x=\pm 3$, $y=\pm 2$.

(53) $\left.\begin{aligned} \frac{y}{x}+\frac{3x}{x+y}&=\frac{x^2-y^2}{y} \\ \frac{x}{y}-\frac{x+y}{x}&=\frac{y}{x}\end{aligned}\right\};$ $\quad x=\frac{5}{3}$, $y=\frac{5}{6}$.

(54) $\left.\begin{aligned} x^m y^n&=a \\ x^p y^q&=b\end{aligned}\right\};$ $\quad x=\left(\frac{a^q}{b^n}\right)^{\frac{1}{mq-np}}$, $y=\left(\frac{b^m}{a^p}\right)^{\frac{1}{mq-np}}$.

(55) $\left.\begin{aligned} x^2+y^2&=\frac{820}{xy} \\ x^2-y^2&=9\end{aligned}\right\};$ $\quad x=\pm 5$, $y=\pm 4$.

(56) $\left.\begin{aligned} x^2+y^2&=100 \\ xy-x-y&=20\end{aligned}\right\};$ $\quad x=8$, $y=4$.

(57) $\left.\begin{aligned} 2y^2 &= xy + 2 \\ 4x^2 &= xy + 30 \end{aligned}\right\};$ $\quad x = \pm 3,\ y = \pm 2.$

(58) $\left.\begin{aligned} \frac{120}{y} - \frac{120}{x} &= 6 \\ \frac{120}{y+1} - \frac{120}{x+1} &= 4 \end{aligned}\right\};$ $\quad x = 5,\ y = 4.$

(59) $\left.\begin{aligned} 5 + \sqrt{xy} &= 4 + \sqrt{y+2} \\ 8 + x &= \sqrt{128 + 64y} \end{aligned}\right\}$ $\quad x = 4,\ y = \frac{1}{4}.$

(60) $\left.\begin{aligned} x^2 + 2y^2 &= x\sqrt{y} + 2xy\sqrt{y} \\ x^2 - 2y^2 &= 256 - x\sqrt{y} \end{aligned}\right\}$ $\quad y = 4,\ x = \pm 2.$

(61) $\left.\begin{aligned} x^2 + y^2 + z^2 &= 50 \\ y^2 + 14 &= 2xz \\ 5z &= 3x \end{aligned}\right\};$ $\quad x = 5,\ y = 4,\ z = 3.$

(62) $\left.\begin{aligned} x + y + z &= 21 \\ x^2 + y^2 + z^2 &= 273 \\ x : y &:: y : z \end{aligned}\right\};$ $\quad x = 1,\ y = 4,\ z = 16.$

(63) $\left.\begin{aligned} xy + xz &= 80 \\ xy + yz &= 72 \\ xz + yz &= 56 \end{aligned}\right\};$ $\quad x = \pm 8,\ y = \pm 6,\ z = \pm 4.$

(64) $\left.\begin{aligned} x^2 + xy + xz &= 20 \\ y^2 + xy + yz &= 30 \\ z^2 + zy + zx &= 50 \end{aligned}\right\};$ $\quad x = \pm 2,\ y = \pm 3,\ z = \pm 5.$

(65) $\frac{xyz}{x+y} = 4\frac{4}{5}, \quad \frac{xyz}{y+z} = 3\frac{3}{7}, \quad \frac{xyz}{z+x} = 4;$

$\therefore\ x = \pm 2, \quad y = \pm 3, \quad z = \pm 4.$

PROBLEMS PRODUCING EQUATIONS.

102. When a problem is proposed, the first thing to be done is to express its conditions in algebraic language, and then from the given relation between the unknown and known quantities of the question, to form an equation or equations; this being effected, the unknown quantity or quantities must be determined, and then the solution of the problem is completed.

So that in every problem, there are two distinct operations to be performed. (1) To make the equation or equations. (2) To solve it or them when made.

The latter, which is generally the less difficult task, has been already abundantly illustrated; of the former we proceed to give examples.

PROBLEMS PRODUCING SIMPLE EQUATIONS.

(Ex. 1.) Find that number to which if 80 be added, the sum shall be 5 times the original number.

Let x be the required number;

$$\therefore\ x + 80 = \text{number} + 80;$$

$$\therefore\ x + 80 = 5x; \qquad \therefore\ x = 20,$$

and we see that $20 + 80 = 100 = 5 \times 20$.

(Ex. 2.) Two men, A and B, are partners in trade and gain £267.; A's gain is twice B's; what is the gain of each?

$$\text{Let } x = B\text{'s gain};$$

$$\therefore\ 2x = A\text{'s gain};$$

$$\therefore\ x + 2x = \text{the whole gain} = 267;$$

$$\therefore\ x = 89 \text{ and } 2x = 178;$$

$\therefore$ A's gain is £178., and B's gain is £89.

(Ex. 3.) £1100. were divided among 4 persons A, B, C, D; B had twice as much as A; C as much as A and B; D as much as C and B; what had each?

Let x be what A had;

$$\therefore 2x = B\text{'s share},$$

$$3x = C\text{'s share},$$

$$5x = D\text{'s share};$$

$$\therefore x + 2x + 3x + 5x = 11x = 1100; \quad \therefore x = 100;$$

$\therefore$ £100., £200., £300., and £500. are the respective shares.

(Ex. 4.) Find two numbers whose sum is 30, and difference $= \frac{1}{2}$ the less.

Let $2x$ be the less; $\therefore 30 - 2x =$ the greater;

$$\therefore 30 - 4x = \text{the difference} = \tfrac{1}{2}(2x) = x;$$

$$\therefore 5x = 30; \quad \therefore x = 6; \quad 2x = 12, \text{ and } 30 - 2x = 18,$$

and the numbers are 12 and 18.

(Ex. 5.) A and B start from two towns which are 216 miles apart, with the intention of meeting; A travels 20 miles and B 16 miles each day. In how many days will they meet?

If $x =$ the number of days required,

$20x =$ distance travelled by A,

$16x =$ that by B;

$$\therefore 20x + 16x = 216; \quad \therefore x = 6.$$

(Ex. 6.) £3800. is divided among A, B, C; B's share amounts to $\frac{2}{3}$rds, and C's to $\frac{3}{4}$ths of what A had. What is each share?

Let $x = A$'s share;

$$\therefore \frac{2x}{3} \text{ and } \frac{3x}{4} \text{ are } B\text{'s and } C\text{'s respectively;}$$

$$\therefore x + \frac{2x}{3} + \frac{3x}{4} = 3800; \quad \therefore 29x = 12 \times 3800;$$

$\therefore x = 2400 = A$'s share, and £1600. and £1800. are B's and C's shares.

Another solution which avoids fractions.

Let $12x = A$'s share;

$$\therefore 8x \text{ and } 9x \text{ are } B\text{'s and } C\text{'s;}$$

$$\therefore 12x + 8x + 9x = 29x = 3800; \quad \therefore x = 200;$$

$\therefore$ £2400., £1600. and £1800. are the three shares.

(Ex. 7.) A and B together can reap a field of corn in 12 days; A alone could do it in 20 days. In how many days could B alone reap it?

Let $W =$ the whole work done;

$\therefore \frac{W}{12} =$ the part which A and B together can do in one day,

and $\frac{W}{20} =$ the part done by A in one day;

$\therefore \frac{W}{12} - \frac{W}{20} =$ part done by B in one day.

Let $x =$ number of days in which B can alone reap the field;

$\therefore \frac{W}{x} = B$'s day's work;

$$\therefore \frac{W}{12} - \frac{W}{20} = \frac{W}{x}; \quad \therefore \frac{1}{12} - \frac{1}{20} = \frac{1}{x}; \quad \therefore x = 30.$$

(Ex. 8.) One hand of a watch makes a revolution in a hours, the other in b hours; if they both start from the same point, when will they be again coincident?

Let C = circumference described by the hands of the watch;

$\therefore \frac{C}{a}$ and $\frac{C}{b}$ respectively equal the spaces moved over by each in an hour;

$$\therefore \frac{C}{a} - \frac{C}{b} = \text{the separation in an hour,}$$

$$\text{and } \frac{2C}{a} - \frac{2C}{b} = \ldots\ldots\ldots\ldots\ldots\ 2 \text{ hours,}$$

$$\text{and } \frac{xC}{a} - \frac{xC}{b} = \ldots\ldots\ldots\ldots\ldots\ x \text{ hours,}$$

but when this arc is equal to the circumference they again coincide; $\therefore x$ may be found from the equation;

$$\therefore \frac{xC}{a} - \frac{xC}{b} = C;$$

$$\therefore \frac{x}{a} - \frac{x}{b} = 1; \quad \therefore x = \frac{ab}{b-a}.$$

Ex. If the hands of a watch are together at 12 when will they be together again?

Here $a = 11$ and $b = 12$; $\therefore x = \frac{12}{11} = 1\frac{1}{11}$ of an hour.

The same method is applicable when the hands do not start from the same point.

(Ex. 9.) A cistern can be filled by 3 pipes; the first will fill it in 80 minutes, the second in 3 hours and 20 minutes, and the third in 5 hours, in what time will it be filled, if the 3 pipes are opened at once?

Let W = the capacity of the vessel;

x = the time in minutes of filling;

$\therefore \frac{W}{x}$ = the quantity poured in by the 3 pipes in a minute;

and $\frac{W}{80}$ = quantity by first pipe, $\frac{W}{200}$ by the second pipe, and $\frac{W}{300}$ by the third pipe;

$$\therefore \frac{W}{80} + \frac{W}{200} + \frac{W}{300} = \frac{W}{x};$$

$$\therefore \frac{1}{x} = \frac{1}{20}\left(\frac{1}{4} + \frac{1}{80} + \frac{1}{15}\right) = \frac{1}{20} \times \frac{25}{60} = \frac{1}{48};$$

$$\therefore x = 48 \text{ minutes.}$$

(Ex. 10.) There are two numbers in the proportion of 2 : 3; but if 24 be added to each, their sums will be as 8 : 9.

Let $2x$, and $3x$ be the numbers.

Then $2x + 24 : 3x + 24 :: 8 : 9$;

$\therefore 18x + 216 = 24x + 192$; $\quad \therefore 6x = 24$; $\quad \therefore x = 4$,

and the numbers are 8 and 12.

(Ex. 11.) A mass of copper and tin weighs 80 lbs.; and for every 7 lbs. of copper there are 3 lbs. of tin; how much copper must be added to the mass, that for every 11 lbs. of copper there may be 4 lbs. of tin?

Let $7x$ and $3x$ be the original quantities of copper and tin;

$$\therefore 10x = 80; \quad \therefore x = 8,$$

and 56 and 24 are the lbs. of copper and tin of which the mass 80 lbs. consisted.

Let y = the copper to be added;

$$\therefore 56 + y : 24 :: 11 : 4;$$

$$\therefore 224 + 4y = 264; \quad \therefore 4y = 40; \quad \therefore y = 10,$$

i. e. 10 lbs. of copper must be added.

(Ex. 12.) *A* bought eggs at 18*d.* a dozen, but had he bought 5 more for the same money, they would have cost him $2\frac{1}{2}d.$ a dozen less; how many eggs did he buy?

Let $x =$ number of eggs he bought;

$$\therefore\ 12 : x :: 18 : \frac{18x}{12} = \text{money he paid},$$

and $12 : x+5 :: 15\frac{1}{2} : 31\frac{(x+5)}{24} =$ money paid on second supposition.

But the money is the same in both cases;

$$\therefore\ \frac{18x}{12} = \frac{31(x+5)}{24};\quad \therefore\ 36x = 31x + 155;\quad \therefore\ x = 31.$$

(Ex. 13.) *A* and *B* have together £9800.; *A* invests the sixth part of his property in business, and *B* the fifth part, and then each has the same sum remaining. How much has each?

Let $6x$ and $5y$ represent *A*'s and *B*'s property;

$\therefore\ 6x - x = 5x = A$'s property after investing the sixth part,

$5y - y = 4y = B$'s.............................fifth part;

$$\therefore\ 6x + 5y = 9800,$$

$$\text{and }\ 5x = 4y;$$

$\therefore\ x = 800$, and $y = 1000$, and *A* has £4800. and *B* £5000.

(Ex. 14.) *A* and *B* can do a certain work in 16 days; they work together for 4 days, when *A* turns idle and *B* finishes it in 36 days more; in what time would each do it separately?

Let x and y be the days in which *A* and *B* would perform the work alone, and let W be the work.

Then $\frac{W}{x}+\frac{W}{y}=$ the part they would together perform in one day

$$=\therefore \frac{W}{16};$$

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{16}\ldots\ldots\ldots\ldots(1).$$

$$\text{But} \quad \frac{4W}{x}+\frac{4W}{y}+\frac{36W}{y}=W;$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{9}{y}=\frac{1}{4}\ldots\ldots\ldots(2).$$

Take (1) from (2) $\therefore \frac{9}{y}=\frac{1}{4}-\frac{1}{16}=\frac{3}{16}; \quad \therefore y=48,$

$$\text{and} \quad \frac{1}{x}=\frac{1}{16}-\frac{1}{48}=\frac{2}{48}=\frac{1}{24}; \quad \therefore x=24.$$

(Ex. 15.) There is a number consisting of two digits, which if divided by the sum of its digits the quotient $=4$; but if 27 be added to it, the number will be inverted.

Let x be the digit in the ten's place,

y .... unit's place;

$$\therefore 10x+y = \text{the number},$$

$$\text{and} \quad \frac{10x+y}{x+y}=4; \quad \therefore 2x=y.$$

$$\text{And} \quad 10x+y+27=10y+x;$$

$$\therefore 9y-9x=27; \quad \therefore y-x=3;$$

$\therefore x=3,\ y=6,$ and the number is 36.

(Ex. 16.) There are in three boxes A, B, C, 162 sovereigns, and in order that there may be the same sum in each box, I take out from A, and put into B and C, half as much as they already contained. I then take out of *B*, and put into *A* and *C*, half of what each contained, and I do the same to *C*; and then my object is effected. How much did each box contain at first?

Let $2x$ and $2y$ be what *B* and *C* held at first,

and $3z+x+y$ ....*A* had;

$$\therefore\ 3x+3y+3z=162;\quad \therefore\ x+y+z=54.$$

Now taking, according to the question, $x+y$ out of *A*, and putting x into *B*, and y into *C*, the boxes will respectively contain

$$3z,\quad 3x,\quad 3y;$$

then putting $\frac{3z}{2}$ into *A*, and $\frac{3y}{2}$ into *C*, the state will be

$$\frac{9z}{2},\quad 3x-\frac{3z+3y}{2},\quad \frac{9y}{2}.$$

Now putting $\frac{9z}{4}$ into *A*, and $\frac{3x}{2}-\frac{3z+3y}{4}$ into *B*, each box will contain one-third of the whole sum;

$$\therefore\ \frac{9z}{2}+\frac{9z}{4}=54;\quad \therefore\ z=8;\quad \therefore\ x+y=46.$$

$$\text{And}\quad 3x-\frac{3z+3y}{2}+\frac{3x}{2}-\frac{3z+3y}{4}=54;$$

$$\therefore\ \frac{3x}{2}-\frac{z+y}{2}-\frac{z+y}{4}=\frac{3x}{2}-\frac{3y}{4}-6=18;$$

$$\therefore\ x=26,\ \text{and}\ y=20;$$

$\therefore\ 3z+x+y=70$, what *A* held; $2x=52$, what *B* held, $2y=40$, what *C* held.

EXAMPLES.

(1) What is that number, the double of which is as much above 50; as its half is below it? Ans. 40.

(2) A man bequeathed £330. to three persons; *A* had twice as much as *C*, and *B* as much as *A* and *C* together; what had each? *A* had £110; *B* £165; *C* £55.

(3) Divide 100 into 2 such parts, that if the one be divided by 15 and the other by 5, the sum of the quotients = 10.

Ans. 75 and 25.

(4) A person has 55 coins, consisting of crowns and shillings, and their amount is £7. 3*s*. How many has he of each kind? Ans. 22 crowns and 33 shillings.

(5) A farm of 864 acres is divided among the three sons of a farmer; *A* has 5 acres for *B*'s 11 acres, and *C* has as much as *A* and *B* together; what number had each?

Ans. *A*, 135; *B*, 297; *C*, 432.

(6) A garrison consists of 2100 men; there are 10 times as many foot soldiers, and three times as many artillery as there are cavalry. How many were there of each?

Ans. 150 cavalry; 1500 foot; 450 artillery.

(7) *A* says to *B*, if you give me £50. I shall have twice as much as you had; but if I give you £50. each will have the same sum. How much had each?

Ans. *A* £250.; *B* £150.

(8) Find that number, which when multiplied by 5; and 24 taken from the product, and this difference divided by 6, and 13 added to the quotient, there will still be the same number. Ans. 54.

(9) A had £100. and B £48.; B gave a certain sum away and A twice as much, and then A had three times as much as B had. What did A give away?

Ans. £88.

(10) A sum of money is to be divided among 5 persons, A, B, C, D, and E; B received £10. less than A; C £16. more than B; D £5. less than C; E £15. more than D; and the shares of the last two equal the sum of the shares of the other three. Find the whole sum, and how much each received?

Ans. £118. = whole sum. The shares are £21., £11., £27., £22. and £37.

(11) A person bought an equal number of sheep, cows and oxen, for £660., each sheep cost £3.; each cow £12.; each ox £18. Find the number bought of each?

Ans. 20.

(12) How much tea worth 4*s.* 6*d.* per lb. must be mixed with 50 lbs. of tea worth 6*s.* per lb., that the mixture may be worth 5*s.* per lb.? Ans. $x = 100$.

(13) Find a number, such that if it be added to its half, the sum shall be as much above 80 as its third shall be below 30. Ans. 60.

(14) A man by his will left his property among his three sons, in the following manner; the eldest to have £1000. less than the half, the second £800. less than the third, and the youngest £600. less than the fourth of the property. Required the whole property and the portion of each son?

Ans. Whole property £28,800; the three shares were, £13,400., £8,800., £6,600.

(15) Find that number, to which if 15, 27, and 45 be added, the first sum shall be to the second as the second to the third? Ans. 9.

(16) An officer wishes to form his troops into a solid square, but he finds in doing so, that he has 60 over; he then forms a column with 6 men more in front than before, but the number of ranks less by 4, he then has only 4 men remaining. How many had he? Ans. 1660.

(17) A person in a foreign town wishes to exchange a sovereign for 25 pieces of the two kinds of coin used there; and he finds that 30 of the one, or 15 of the other, is equivalent to a sovereign. How many must he have of each?

Ans. 20 and 5.

(18) A sum of money is divided among 4 persons, *A, B, C, D*; *A* has £3000. less than the half; *B*, £1000. less than the third; *C*, £600. more than the fifth; and *D* $\frac{1}{3}$rd of what the other three had. What was the property and the share of each?

Ans. Property was £12,000., and each had £3000.

(19) Tea at *5s. 3d.* per lb., is mixed with tea at *4s. 3d.* and 100 lbs. of the mixture are sold for £25. *5s.* How much was there of each? Ans. 80 and 20.

(20) A person wishes to sell a watch by means of a lottery; if the tickets be sold at *5s.* each, he would lose £5.; but if the price be *6s.*, he would gain £4. Find the price of the watch and the number of tickets?

Ans. Number = 180; price of watch is £50.

(21) *A* puts £5500. out to interest at 4 per cent., and $4\frac{1}{2}$ years after £8000. out at 5 per cent. In how many years will the interest received from the one equal that from the other sum? Ans. 10.

(22) The circumference of each of the two fore wheels of a carriage is $5\frac{1}{4}$ feet, and of each of the hind wheels is $7\frac{1}{8}$ feet; when the fore wheel has made 2000 more revolutions than the hind wheel, what is the distance travelled.

Ans. 7 miles 980 yards.

(23) A cistern can be filled by 3 pipes; by the first in 2 hours, by the second in 3 hours, and by the third in 4 hours. In what time will the cistern be filled, if the 3 pipes are opened at once? Ans. In $55\frac{5}{13}$ minutes.

(24) *A* could reap a field in 20 days, but if *B* assisted him for 6 days, he could reap it in 16 days. In how many days could *B* finish it alone? Ans. 30 days.

(25) *A, B, C* are to build a wall; *A* can build 8 cubic feet in 6 days, *B* 10 cubic feet in 5 days, *C* 12 cubic feet in 4 days. In how many days will they build a wall containing 228 cubic feet? Ans. 36 days.

(26) In a bag containing sovereigns and shillings, there are three times as many sovereigns as shillings; but if 8 sovereigns and as many shillings be taken away, there will be five times as many sovereigns as shillings. How many were there of each? Ans. 48 sovereigns, and 16 shillings.

(27) *A* has 3 pieces of metal of the same size; 5 cubic inches of the first weigh $69\frac{3}{4}$ oz., of the second $3\frac{1}{8}$ cubic inches weigh 41 oz., of the third $4\frac{2}{7}$ cubic inches weigh 91 oz. The weight of the 3 is $949\frac{2}{3}$ oz., what is the size of each?

Ans. 20 cubic inches.

(28) *A, B, C* engage in business and gain £5020., of which *C* receives £2570. for his share, but *B* contributed at first half as much again as *A*, and *C* £300. more than *A* and *B* together. How much did each contribute?

Ans. *A* £2450.; *B* £3675.; *C* £6425.

(29) I sold a horse for £60., and by so doing lost 20 per cent. on the purchase money, but I expected to have made 10 per cent. For how much was the horse sold below its estimated value? Ans. £22. 10*s*.

(30) When will the hands of a watch be together between 3 and 4? Ans. $16\frac{4}{11}$ minutes past 3.

(31) When will the hands be together again, when one makes a revolution in 10 hours and the other in 12?

Ans. 60 hours.

(32) *A* starts from a certain place and travels 27 miles a day; 2 days afterwards *B* sets out and travels so as to overtake *A* in 6 days. How many miles a day did *B* travel?

Ans. 36.

(33) A courier who travels $31\frac{1}{2}$ miles every 5 hours has set out 8 hours; when another, who travels $22\frac{1}{2}$ miles every 3 hours is sent after him to overtake him. When will the second overtake the first. Ans. In 42 hours.

(34) The quantity of water which flows from an orifice is proportional to the product of the area of the orifice and the velocity of the water. Now there are 2 orifices in a reservoir, the areas being as 5 : 13, and the velocities are as 8 : 7, and from one there issued in a certain time 561 cubic feet more than from the other. How much water did each orifice discharge in this time? Ans. 440 and 1001 cubic feet.

(35) *A* sets out from London to York, *B* from York to London, *A* arrives in York 9 hours, and *B* in London 16 hours after they met. In what time did each perform the journey? Ans. *A* in 21 hours, *B* in 28 hours.

(36) *A* performs a journey at a certain rate; had he travelled $\frac{1}{2}$ a mile an hour quicker, he would have performed the journey in $\frac{4}{5}$ of the time; but had he travelled $\frac{1}{2}$ a mile

slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance and his rate.

Ans. Distance is 15 miles; rate 2 miles per hour.

(37) *A* and *B* are in trade together, with different sums; but if £50. be added to *A*'s property, and £20. be taken from *B*'s, they will have the same sum; also if *A*'s property were 3 times and *B*'s 5 times as great as each really is, they would have together £2350. How much has each? Ans. *A* has £250.; *B* £320.

(38) *A* has 2 vessels with wine in them, and finds that $\frac{2}{5}$ths of the first contains 96 gallons less than $\frac{3}{4}$ths of the second, and that $\frac{5}{8}$ths of the second contains as much as $\frac{4}{9}$ths of the first. How much did each vessel hold?

Ans. 720 and 512 gallons.

(39) There is a fraction such that if 1 be added to the numerator, and the numerator to the denominator, its value $=\frac{1}{4}$; but if the denominator be increased by unity and the numerator by the denominator, its value $=\frac{8}{7}$; find it. Ans. $\frac{3}{13}$.

(40) The sum of two numbers is 13, and the difference of their squares $=65$; find them. Ans. 9 and 4.

(41) A grocer has two kinds of tea, 8 lbs. of the first and 19 lbs. of the second are sold for £18. 4*s.* 2*d.*; and 20 lbs. of the first and 16 lbs. of the second, for £25. 16*s.* 8*d.* How much does a pound of each cost?

Ans. 15*s.* 10*d.*, and 12*s.* 6*d.*

(42) There is a number consisting of two digits, and which divided by the sum of its digits, gives a quotient 7; but if the digits be written in an inverse order, and the number so arising be divided by the sum of the digits increased by 4, the quotient $=3$; find the number. Ans. 84.

(43) A and B can perform a certain work in 20 days; after working together 5 days, A falls ill, and B completes the work in 36 days more, in what time would each have done it alone? Ans. A in $34\frac{2}{7}$th days, B in 48 days.

(44) A mass of tin and lead weighing 120 lbs. in vacuo, loses 14 lbs. when weighed in water; and it is known that 37 lbs. of tin lose 5 lbs. and 23 lbs. of lead lose 2 lbs. in water. What are the respective weights of tin and lead?

Ans. 74 lbs. of tin, and 46 lbs. of lead.

(45) Three labourers A, B, C, are to do a certain work; A and B working together can do it in 12 days; A and C in 15 days; B and C in 20 days. In what time would each do it alone? And what time would it take to finish it, if they all three work together?

Ans. A in 20 days, B 30 days, C 60 days; all three in 10 days.

(46) A finds that $7\frac{1}{2}$ lbs. of coffee, 23 lbs. of sugar, and $12\frac{1}{4}$ lbs. of tea cost £7. 3*s.* 6*d.*; that 9 lbs. of coffee, 7 lbs. of sugar, and 3 lbs. of tea, cost £2. 18*s.*, and that 2 lbs. of coffee, $5\frac{1}{2}$ lbs. of sugar, and 4 lbs. of tea, cost £2. 3*s.* 6*d.* What does each cost per lb.?

Ans. Coffee costs 3*s.*, sugar 1*s.*, and tea 8*s.*

(47) A, B, and C have £96. between them; A who has most, gives to B and C as much as they already had; in the same manner B gives to A and C, and C to A and B; it was then found that each had the same. How much had each at first? Ans. A £52., B £28., and C £16.

(48) A number consisting of 3 digits when divided by the sum of the digits $+9$, gives a quotient 19; also the middle digit $=\frac{1}{2}$ the sum of the first and third; and if 198 be added to the number, we obtain a number with the same digits but in an inverted order. What is the number?

Ans. 456.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

(1) Find that number, whose square exceeds its simple power by 306.

Let x be the number;

$$\therefore\ x^2 - x = 306, \text{ by the question};$$

$$\therefore\ \left(x - \frac{1}{2}\right)^2 = 306 + \frac{1}{4} = \frac{1225}{4} = \left(\frac{35}{2}\right)^2;$$

$$\therefore\ x = \frac{1}{2} \pm \frac{35}{2} = 18 \text{ or } -17.$$

Both answers satisfy the algebraical conditions of the question; but the result (-17) tells us that the algebraic is more general than the ordinary language, and it is the answer to this question, "Find that number which added to its square, the sum will be 306."

(2) A person buys some pieces of cloth for £60. Had he bought 3 more for the same sum, each piece would have cost him £1. less. How many did he buy?

Let x = number he bought;

$$\therefore\ \frac{60}{x} = \text{price of each piece};$$

$$\therefore\ \frac{60}{x+3} = \text{price had he bought 3 more for £60};$$

$$\therefore\ \frac{60}{x} - \frac{60}{x+3} = 1; \quad \therefore\ 180 = x^2 + 3x; \quad \therefore\ x = 12,$$

or he bought 12 pieces at £5. each.

But x also $= -15$; what is the meaning of the negative value?

Here $\frac{60}{x} = -4$, and $\frac{60}{x+3} = -5$, or the prices he would have given are $-$£4. and $-$£5.; in other words, since buying and selling are opposite operations, the result -15, is the answer to this question, "A person *sells* cloth for £60.; had he sold 3 pieces *less* for the same sum, he would have *gained* £1. *more*."

(3) A person sells a horse for £144. and gains as much per cent. as the horse cost him. What did the horse cost him?

Let x = original cost of horse;

$$\therefore 100 : x :: x : \frac{x^2}{100} = \text{the gain};$$

$$\therefore x + \frac{x^2}{100} = 144; \quad \therefore x = 80 \text{ or } -180,$$

the negative root -180 may be easily explained.

(4) There is a number consisting of 3 digits, the last of which is double of the first; if the number be divided by the sum of the digits, the quotient is 22, but if divided by a third of the product of the last two increased by 4 the quotient is also 22. Find the number.

Let x, y and $2x$ be the digits;

$$\therefore 100x + 10y + 2x \text{ is the number};$$

$$\therefore \frac{102x + 10y}{3x + y} = 22\ldots(1), \quad \text{and} \quad \frac{102x + 10y}{\frac{2xy}{3} + 4} = 22\ldots(2).$$

From (1) $y = 3x$, from (2) $51x + 5y = \left(\frac{xy}{3} + 2\right)22$;

$$\therefore 22(x^2 + 2) = 66x; \quad \therefore x^2 - 3x = -2;$$

$\therefore x = 2$ or 1, $y = 6$ or 3, and 264 and 132 are the numbers.

EXAMPLES.

(1) What is that number whose half multiplied by its $\frac{1}{4}$th part $= \frac{225}{2}$? Ans. 30.

(2) Find two numbers in the proportion of 2 : 3, the sum of whose squares = 208. Ans. 8 and 12.

(3) There are two numbers whose product = 450, and quotient = 2. Find them. Ans. 30 and 15.

(4) Find two numbers which shall be as 3 to 4, and the difference of whose cubes : sum of their squares :: 37 : 5.

Ans. 15 and 20.

(5) Find that number the square root of which exceeds its fourth root by 12. Ans. 256.

(6) There are two numbers whose difference = 8, and product 240. Find them. Ans. 12 and 200.

(7) The difference between the sides of a rectangular field is 100 yards, and the distance between two opposite corners is 500 yards. Find the length of the sides and the area of the field.

Ans. Area of field 120,000 square yards; 400 and 300 the length of the sides.

(8) Divide 185 into two such parts that the difference of their square roots shall be = 3. Ans. 64 and 121.

(9) The difference between two numbers multiplied by the greater = 16, but by the less = 12. Find them.

Ans. 8 and 6.

(10) The sum of two numbers multiplied by the greater = 40, and difference multiplied by the less = 6. Find them.

Ans. 5 and 3.

(11) A pedestrian having to walk 45 miles, finds that if he increases his speed $\frac{1}{2}$ a mile an hour, he will perform his task $1\frac{1}{4}$ hour sooner than if he walked at his usual rate. What is that rate? Ans. 4 miles per hour.

(12) A charitable person is about to distribute £6. among some poor persons, when two others come in, and thus each person's share is diminished by 2*s*. Find the number of persons relieved. Ans. 12.

(13) A person bought a number of sheep for £80., if he had bought 4 more for the same money, he would have paid £1. less for each. How many did he buy? Ans. 16.

(14) A man left by his will £46800. to be divided equally among his children; two of them die before the division of the property is made, and consequently each child receives £1950. more than it was originally entitled to. Required the number of children. Ans. 8.

(15) A tradesman bought cloth at Leeds, and has to pay for insurance and carriage 4 per cent.; he sells it for £390. and gains as much per cent. as the 12th part of the purchase money amounts to. What did he give for the cloth?

Ans. £300.

(16) There are two numbers whose difference is 10, and if 600 be divided by each, the difference of the quotients also = 10. Find them. Ans. 20 and 30.

(17) The sum of two numbers = 80; and if they be divided alternately by each other, the sum of the fractions = $3\frac{1}{3}$. Find the numbers. Ans. 20 and 60.

(18) Divide 10 into two such parts that their product may exceed their difference by 22. Ans. 6 and 4.

(19) The sum of the squares of two numbers = 41, and their product exceeds their sum by 11. Find them.

Ans. 5 and 4.

(20) A man sold a horse for £96. and gained as much per cent. as the horse cost him. What was the price of the horse? Ans. £60.

(21) A bought certain pieces of cloth for £45., but had he received 3 pieces more for the same money, each piece would have cost him 15*s*. less. How many pieces did he buy? Ans. 12.

(22) The joint stock of two partners is £1000., one leaves his money in the partnership for 12 months, the other for 18 months; but each takes £990. for capital and profit. What stock did each furnish? Ans. £450. and £550.

(23) A and B set out at the same time; A from C to go to D, and B from D to go to C; they meet on the road, when it appears that A has travelled 30 miles more than B, and that at the rate he is travelling he will reach D in 4 days, and that B will arrive at C in 9 days. Find the distance of C from D. Ans. 150 miles.

(24) Find two numbers such that their sum, product and difference of their squares may be equal.

Ans. 1.680 + and 2.6180 + nearly.

(25) Divide 134 into 3 such parts that the sum of the first, twice the second and three times the third = 278, and the sum of the squares = 6036. Ans. 40, 44, 50.

(26) The sum of two numbers = 11, and the sum of their 5th powers = 17831. Required the numbers.

Ans. 4 and 7.

(27) The product of 4 consecutive numbers is 840. Find them. Ans. 4, 5, 6, 7.

(28) There are three numbers, and the product of the squares of the first and second divided by the third number = 8; the product of the squares of the first and third

divided by the second = 64; and the product of the squares of the second and third divided by the first number = 512. Find the numbers. Ans. 2, 4, 8.

(29) If to a certain number you add 1578, and from the same number you take 142, the difference between the cube roots of the numbers so obtained = 10. Find the number. Ans. 150.

(30) There is a number consisting of 3 digits, in which the sum of the squares of the digits = 104; the square of the middle digit exceeds twice the product of the other two by 4, and if 594 be taken from the number the digits will be inverted. Ans. 862.

(31) There are two numbers whose difference is 2, and their product multiplied by their sum is 12; what are they?

Ans. 3 and 1.

(32) A person has a certain number of crowns, half-sovereigns and sovereigns, and if the crowns became sovereigns, and the sovereigns crowns, the difference between what he would have and what he had is £4. 10*s*.; also the number of crowns : the number of half-sovereigns :: that number to the number of sovereigns; and the sum of the squares of the crowns and sovereigns increased by 12, equals 5 times the square of the number of half-sovereigns. How many crowns, half-sovereigns and sovereigns had he?

Ans. 2 crowns, 4 half-sovereigns and 8 sovereigns.

CHAPTER IV.

SURDS AND IMAGINARY QUANTITIES.

103. NUMERICAL roots of the form $\sqrt{2}$, $\sqrt[3]{7}$, $\sqrt{17}$, $\sqrt[4]{63}$, and algebraical ones, such as $\sqrt{a^2+x}$, $\sqrt[3]{a^3+a^2x}$, which cannot be expressed in a finite number of terms, and therefore have no known ratio to unity, are called irrational quantities or surds.

And terms such as $\sqrt{-a^2}$ and $a+\sqrt{-b^2}$, already mentioned (Art. 73.) are called imaginary or impossible quantities.

But although these surds cannot be exactly determined, yet while they retain their form, they can be added to or subtracted from other surds, and may be multiplied or divided as other algebraical quantities.

DEF. Similar surds are there which have the same quantity under the root.

Thus $2\sqrt{2}$ and $5\sqrt{2}$ are similar surds; as also are $4a\sqrt[3]{b}$, and $3c\sqrt[3]{b}$.

RULE 1. When different surds are or can be reduced to similar surds, their sum or difference may be found, by taking the sum or difference of the coefficients of the common surd.

(Ex. 1.) Find the sum and difference of $\sqrt{72}$ and $\sqrt{128}$,

$$\sqrt{72}=\sqrt{36\times 2}=6\sqrt{2} \text{ and } \sqrt{128}=\sqrt{64\times 2}=8\sqrt{2};$$

$$\therefore \sqrt{128}+\sqrt{72}=14\sqrt{2} \text{ and } \sqrt{128}-\sqrt{72}=2\sqrt{2}.$$

(Ex. 2.) $\sqrt{243}+\sqrt{27}+\sqrt{48}=16\sqrt{3}$.

(Ex. 3.) $\sqrt{24}+\sqrt{54}-\sqrt{96}=\sqrt{6}$.

(Ex. 4.) $\sqrt[3]{128}+\sqrt[3]{686}+\sqrt[3]{16}-4\sqrt[3]{250}=-7\sqrt[3]{2}$.

(Ex. 5.)

$$2\sqrt{8a^3}-7a\sqrt{18a}+5\sqrt{72a}-\sqrt{50ab^2}=(13a-5b)\sqrt{2a}.$$

(Ex. 6.) $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2}-5\sqrt[3]{128}=8\sqrt[3]{2}$.

(Ex. 7.) Find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$;

$$7\sqrt{\frac{27}{50}}=7\sqrt{\frac{9\times 3}{25\times 2}}=\frac{21}{5}\sqrt{\frac{3}{2}}=\frac{21}{5}\sqrt{\frac{6}{4}}=\frac{21}{5}\frac{\sqrt{6}}{2}=\frac{21}{10}\sqrt{6};$$

$$3\sqrt{\frac{2}{3}}=3\sqrt{\frac{6}{9}}=3\frac{\sqrt{6}}{3}=\sqrt{6}; \quad \therefore \text{ Sum}=\frac{31}{10}\sqrt{6}.$$

Prove that

$$12\sqrt[3]{\frac{1}{4}}+3\sqrt[3]{\frac{1}{32}}=\frac{27}{4}\sqrt[3]{2}, \text{ and } 8\sqrt{\frac{3}{4}}-\frac{1}{2}\sqrt{12}-2\sqrt{\frac{3}{16}}=\frac{5}{2}\sqrt{3}.$$

104. When the quantities do not involve the same surd, their sum or difference is expressed by placing the proper sign between them; thus to add $3\sqrt{b}$ to $5\sqrt{a}$, we put $5\sqrt{a}+3\sqrt{b}$.

MULTIPLICATION AND DIVISION OF SURDS.

105. The product of two or more simple surds, is found by multiplying together the quantities under the root as in ordinary multiplication, and placing the sign of the root over the product.

Thus $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ and $\sqrt{3}\times\sqrt{7}=\sqrt{21}$.

This rule, however, only applies to surds of the same dimension, i. e. having the same root to be extracted; when the product of different roots is to be found, fractional indices must be put for the roots and then the fractions reduced to a common denominator, which denominator will express the root of the product.

Thus to find the product of $\sqrt{a}$ and $\sqrt[3]{a}$,

$\sqrt{a} = a^{\frac{1}{2}}$, and $\sqrt[3]{a} = a^{\frac{1}{3}}$; also $\because \frac{1}{2}$ and $\frac{1}{3} = \frac{3}{6}$ and $\frac{2}{6}$ respectively;

$$\therefore \sqrt{a} \times \sqrt[3]{a} = a^{\frac{3}{6}} \times a^{\frac{2}{6}} = a^{\frac{5}{6}} = \sqrt[6]{a^5}.$$

$$\text{And } \sqrt[n]{a} \times \sqrt[m]{a} = a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{m}{mn}} \times a^{\frac{n}{mn}} = a^{\frac{m+n}{mn}} = \sqrt[mn]{a^{m+n}}.$$

$$\text{And } \sqrt[3]{a} \times \sqrt[4]{b} = a^{\frac{1}{3}} \times b^{\frac{1}{4}} = a^{\frac{4}{12}} \times b^{\frac{3}{12}} = \sqrt[12]{a^4} \times \sqrt[12]{b^3} = \sqrt[12]{a^4 b^3}.$$

106. To divide one surd by another of the same denomination, divide as in common division and place the sign of the root over the quotient.

$$\text{Thus } \sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}};\ \sqrt{32} \div \sqrt{12} = \sqrt{\frac{32}{12}} = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}},$$

$$\sqrt[4]{a^3 b} \div \sqrt[4]{a b^3} = \sqrt[4]{\frac{a^3 b}{a b^3}} = \sqrt[4]{\frac{a^2}{b^2}} = \sqrt{\frac{a}{b}}.$$

$$\text{But } \sqrt{a} \div \sqrt[3]{a} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}} = \frac{a^{\frac{3}{6}}}{a^{\frac{2}{6}}} = a^{\frac{1}{6}} = \sqrt[6]{a}.$$

We may here remark that $\sqrt{a}$ multiplied by $\sqrt{a}$ is a, and that $\sqrt{-a} \times \sqrt{-a}$ is $-a$; for in the latter case $\sqrt{-a} \times \sqrt{-a} = \sqrt{(-a)^2} = (-a)^{\frac{2}{2}} = -a$.

And here we may observe that it is frequently convenient to put rational quantities under the form of surds.

Then $a = \sqrt{a^2} = \sqrt[3]{a^3}$ and $2a\sqrt{3b} = \sqrt{4a^2} \times \sqrt{3b} = \sqrt{12a^2 b}$.

Whence we see that to put a rational quantity under the form of a surd, we must raise the quantity to the power expressed by the root, and place the sign of the root over it; also conversely every impossible expression as $\sqrt{-a^2}$ may be put under the form $\sqrt{a^2 \times -1} = \sqrt{a^2}\sqrt{-1} = a\sqrt{-1}$, and thus also $a \pm \sqrt{-b^2}$ may be written

$$a \pm \sqrt{b^2} \times \sqrt{-1} = a \pm b\sqrt{-1};$$

this latter expression may be taken as the general type of an imaginary quantity.

107. Next to find a multiplier which will make a given binomial surd as $a + \sqrt{b}$, or $\sqrt{a} + \sqrt{b}$, rational.

Since $x^2 - y^2 = (x + y)(x - y)$.

If $x = a$ and $y = \sqrt{b}$, then $a^2 - b = (a + \sqrt{b})(a - \sqrt{b})$,

or if $x = \sqrt{a}$ and $y = \sqrt{b}$, then $a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

If therefore $a + \sqrt{b}$, or $\sqrt{a} + \sqrt{b}$ be the given surd, the multiplier is $a - \sqrt{b}$ or $\sqrt{a} - \sqrt{b}$.

Ex. Let $7 + \sqrt{3}$ and $\sqrt{7} + \sqrt{3}$ be two surds, then $7 - \sqrt{3}$ and $\sqrt{7} - \sqrt{3}$ are their multipliers, and the respective products will be $49 - 3$, and $7 - 3$, or 46 and 4.

108. Next to reduce a fraction of the form $\frac{c}{a + \sqrt{b}}$, to one with a rational denominator.

Multiply both numerator and denominator by $a-\sqrt{b}$ and he fraction becomes

$$\frac{c}{a+\sqrt{b}}\times\frac{a-\sqrt{b}}{a-\sqrt{b}}=\frac{ca-c\sqrt{b}}{a^2-b}.$$

Ex. Reduce $\frac{2}{\sqrt{2}+\sqrt{5}}$ to a fraction with a rational denominator.

$$\frac{2}{\sqrt{2}+\sqrt{5}}=\frac{2(\sqrt{5}-\sqrt{2})}{5-2}=\frac{2}{3}\cdot(\sqrt{5}-\sqrt{2}).$$

109. If the surd be of the form $\sqrt[3]{a}+\sqrt[3]{b}$, then since

$$a+b=(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}),$$

the multiplier will be $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

Ex. Find the multiplier which will make $\sqrt[3]{2}-\frac{1}{2}\sqrt[3]{3}$ a rational quantity.

Since $\frac{1}{2}\sqrt[3]{3}=\sqrt[3]{\frac{3}{8}}$; $\therefore$ $a=2$, and $b=\frac{3}{8}$, and the multiplier is $2^{\frac{2}{3}}-\left(\frac{3}{4}\right)^{\frac{1}{3}}+\left(\frac{3}{8}\right)^{\frac{2}{3}}$, and the product is $=2-\frac{3}{8}=\frac{13}{8}$.

110. To find the multiplier which will make $\sqrt{a}+\sqrt{b}+\sqrt{c}$ a rational quantity.

Since $(\sqrt{a}+\sqrt{b}+\sqrt{c})\times(\sqrt{a}+\sqrt{b}-\sqrt{c})=(\sqrt{a}+\sqrt{b})^2-c$

$$=a+b-c-2\sqrt{ab},$$

and if $2d$ be put $=a+b-c$, then $2d-2\sqrt{ab}$ will become rational if it be multiplied by $d+\sqrt{ab}$;

$\therefore$ the multiplier required is $(a+b-c+\sqrt{ab})(\sqrt{a}+\sqrt{b}-\sqrt{c})$.

Ex. Let $\sqrt{2}+\sqrt{3}+\sqrt{5}$ be the surd; find the multiplier.

Here $a=2,\ b=3,\ c=5;\ \therefore\ a+b-c-\sqrt{ab}=\sqrt{6};$

$\therefore$ the multiplier is $\sqrt{6}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})=\sqrt{12}+\sqrt{18}-\sqrt{30},$

and if the surd be multiplied by it the result $=12$.

EXAMPLES.

(1) $(7+2\sqrt{6})\times(9-5\sqrt{6})=3-17\sqrt{6}.$

(2) $\left(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}\right)\times\left(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}\right)=\frac{43}{3}+13\sqrt{\frac{7}{6}}.$

(3) $(\sqrt{2}+\sqrt{3})^4=49+20\sqrt{6}.$

(4) $(2\sqrt{8}+3\sqrt{5}-7\sqrt{2})\,(\sqrt{72}-5\sqrt{20}-2\sqrt{2})$
$$=42\sqrt{10}-174.$$

(5) $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{5}=\sqrt[12]{648000}.$

(6) $\sqrt[7]{\frac{4}{3}}\times\sqrt[3]{\frac{1}{2}}\times\sqrt[1]{6}=\sqrt[42]{\frac{2}{27}}.$

(7) $(c\sqrt{a}+d\sqrt{b})\times(c\sqrt{a}-d\sqrt{b})=c^2a-d^2b.$

(8) $(\sqrt{2}+3\sqrt{\tfrac{1}{2}})\div\tfrac{1}{2}\sqrt{\tfrac{1}{2}}=10.$

(9) $(\sqrt{3}+\sqrt{2})\div(\sqrt{3}-\sqrt{2})=5+2\sqrt{6}.$

(10) $(3\sqrt{5}-2\sqrt{2})+(2\sqrt{5}-\sqrt{18})=\dfrac{18+5\sqrt{10}}{2}.$

(11) Shew that $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\dfrac{\sqrt{30}+3\sqrt{2}+2\sqrt{3}}{12}.$

(12) $\dfrac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}=\sqrt{6}+\sqrt{2}+\sqrt{5}.$

(13) Multiply $a^{\frac{5}{2}}-2a^{2}b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}-8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}-32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}$. Ans. $a^{3}-64b^{2}$.

(14) And $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$. Ans. $a-b^{2}$.

(15) Divide $x^{\frac{5}{2}}-x^{2}-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}}$ by $x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2$.

Ans. $x-x^{\frac{1}{2}}$.

(16) And $256x-y^{4}$ by $4x^{\frac{1}{4}}-y$.

Ans. $64x^{\frac{3}{4}}+16x^{\frac{1}{2}}y+2x^{4}y^{2}+y^{3}$.

(17) Shew that $(x^{2}-x\sqrt{2}+1)(x^{2}+x\sqrt{2}+1)=x^{4}+1$.

(18) $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}+\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\dfrac{2(a^{2}-b^{2})}{a^{2}+b^{2}}.$

(19) $\dfrac{\sqrt{x^{2}+1}+\sqrt{x^{2}-1}}{\sqrt{x^{2}+1}-\sqrt{x^{2}-1}}+\dfrac{\sqrt{x^{2}+1}-\sqrt{x^{2}-1}}{\sqrt{x^{2}+1}+\sqrt{x^{2}-1}}=2x^{2}.$

(20) $\dfrac{1}{x-1}+\dfrac{2}{2x+1-\sqrt{-3}}+\dfrac{2}{2x+1+\sqrt{-3}}=\dfrac{3x^{2}}{x^{3}-1}.$

BINOMIAL QUADRATIC SURDS.

111. Expressions of the form $A+\sqrt{B}$ or $\sqrt{A}+\sqrt{B}$, are called binomial surds; such are found in the solution of some quadratic equations; as for instance, in $x^{4}-2px^{2}=q$. For then $x^{2}=p+\sqrt{p^{2}+q}$, which will agree with the former of the expressions if p be rational and $p^{2}+q$ not a perfect

square, and with the latter if p be a surd quantity. This being the case $x=\sqrt{p+\sqrt{p^2+q}}$, a complicated expression which it is sometimes possible to put under the more convenient form of $a \pm \sqrt{b}$, where b is a rational quantity and a rational or irrational.

That this is true may be thus shewn:

$$(2+\sqrt{3})^2 = 7+4\sqrt{3}; \quad \therefore \sqrt{7+4\sqrt{3}} = 2+\sqrt{3}.$$

$$(\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}; \quad \therefore \sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}.$$

We shall now proceed to shew that $\sqrt{A+\sqrt{B}}$, where B is rational and A either rational, or a quadratic surd, can always be exhibited under the form above mentioned, when A^2-B is a perfect square; but before we do this we must premise the following propositions.

(1) No quantity can be partly rational and partly a quadratic surd.

For if possible let $\sqrt{x}=a+\sqrt{b}$; $\therefore$ squaring both sides,

$$x=a^2+2a\sqrt{b}+b; \quad \therefore \sqrt{b}=\frac{x-(a^2+b)}{2a},$$

or a surd quantity equals a rational quantity, which is contrary to the supposition of surds.

(2) If $x+\sqrt{y}=a+\sqrt{b}$, be an equation between rational quantities and surds; then $x=a$; and $\sqrt{y}=\sqrt{b}$.

For if x does not $=a$, let $x=a+m$;

$$\therefore a+m\sqrt{y}=a+\sqrt{b}; \quad \therefore m+\sqrt{y}=\sqrt{b},$$

i.e. a quantity which is partly rational and partly a quadratic surd is equal to a quadratic surd, which has been shewn to be impossible in the last article.

(3) The product of two surds which have not the same irrational part, is irrational.

For if not let the product of $\sqrt{x}$ and $\sqrt{y} = mx$;

$$\therefore \quad xy = m^2x^2; \quad \therefore \quad y = m^2x; \quad \therefore \quad \sqrt{y} = m\sqrt{x},$$

or $\sqrt{y}$ involves the same surd as $\sqrt{x}$, which is contrary to the supposition; hence $\sqrt{xy}$ is still a surd.

112. These propositions being premised, let us assume that $\sqrt{A} + \sqrt{B}$ is equal to $\sqrt{x} + \sqrt{y}$, where one or both of the quantities $\sqrt{x}$, and $\sqrt{y}$, are quadratic and different surds, and therefore $\sqrt{xy}$ is of necessity a surd;

$$\therefore \quad \sqrt{A + \sqrt{B}} = \sqrt{x} + \sqrt{y};$$

$$\therefore \quad A + \sqrt{B} = x + 2\sqrt{xy} + y;$$

$$\therefore \quad x + y = A; \text{ and } 2\sqrt{xy} = \sqrt{B};$$

whence by squaring both of the equations, we have

$$x^2 + 2xy + y^2 = A^2$$

$$4xy = B;$$

$$\therefore \quad x^2 - 2xy + y^2 = A^2 - B \text{ or } (x - y)^2 = A^2 - B.$$

Now let $A^2 - B$ be a perfect square $= C^2$;

$$\therefore \quad x - y = C, \text{ but } x + y = A; \quad \therefore \quad x = \frac{A+C}{2}; \quad \therefore \quad y = \frac{A-C}{2};$$

$$\therefore \quad \sqrt{x} + \sqrt{y} = \sqrt{A + \sqrt{B}} = \sqrt{\frac{A+C}{2}} + \sqrt{\frac{A-C}{2}}.$$

Ex. Find $\sqrt{31 + 10\sqrt{6}}$.

Here $A = 31$, $\sqrt{B} = 10\sqrt{6}$; $\therefore$ $A^2 - B = C^2 = 961 - 600 = 361$;

$$\therefore\ A+C=50,\quad A-C=12;\quad \therefore\ x=25,\ y=6;$$

$$\therefore\ \sqrt{x}+\sqrt{y}=\sqrt{31+10\sqrt{6}}=5+\sqrt{6}.$$

The result has been obtained by substitution, but it would be better for the learner to proceed thus:

Let $\sqrt{31+10\sqrt{6}}=\sqrt{x}+\sqrt{y}$; $\therefore\ 31+10\sqrt{6}=x+2\sqrt{xy}+y$;

$$\therefore\ x+y=31,\ \text{and}\ 2\sqrt{xy}=10\sqrt{6};$$

$$\therefore\ x^2+2xy+y^2=961$$

$$4xy \quad =600;$$

$$\therefore\ x^2-2xy+y^2=(x-y)^2=361;$$

$$\therefore\ x-y=19,\ \text{but}\ x+y=31;\quad \therefore\ 2x=50,\quad 2y=12;$$

$$\therefore\ \sqrt{x}+\sqrt{y}=\sqrt{31+10\sqrt{6}}=5+\sqrt{6}.$$

COR. Since $x+y=A$, and $2\sqrt{xy}=\sqrt{B}$; $\therefore$ by subtraction,

$$x-2\sqrt{xy}+y \ \text{ or }\ (\sqrt{x}-\sqrt{y})^2=A-\sqrt{B};$$

$$\therefore\ \sqrt{x}-\sqrt{y}=\sqrt{A-\sqrt{B}}.$$

Hence if a binomial surd of the form $\sqrt{A-\sqrt{B}}$ be proposed, we must equate it to $\sqrt{x}-\sqrt{y}$, and then proceed as above.

Also, that if $\sqrt{A+\sqrt{B}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{A-\sqrt{B}}=\sqrt{x}-\sqrt{y}$.

113. Binomial surds of the form $\sqrt{a\sqrt{c}+\sqrt{bc}}$ may also be reduced by the same method; for

$$a\sqrt{c}+\sqrt{bc}=\sqrt{c}(a+\sqrt{b});\ \therefore\ \sqrt{a\sqrt{c}+\sqrt{bc}}=\sqrt[4]{c}\ \sqrt{a+\sqrt{b}};$$

and $\sqrt{a+\sqrt{b}}$ may be found as before, when $\sqrt{a^2-b}$ is a complete square.

Ex. Find $\sqrt{\sqrt{18}+\sqrt{4}}$,

$$\sqrt{18}+\sqrt{4} = 3\sqrt{2}+2 = \sqrt{2}\,(3+\sqrt{2}).$$

Also $\sqrt{3+\sqrt{2}} = (1+\sqrt{2})$, by the preceding method;

$$\therefore\ \sqrt{\sqrt{18}+\sqrt{4}} = \sqrt[4]{2}\,.\,(1+\sqrt{2}) = \sqrt[4]{2}\,(1+\sqrt[4]{4}) = \sqrt[4]{2}+\sqrt[4]{8}.$$

114. The square root of the imaginary quantity $A+\sqrt{-B}$, may be similarly exhibited when A^2+B is a perfect square.

For let $\sqrt{A+\sqrt{-B}} = \sqrt{x}+\sqrt{y}$;

$$\therefore\ x+y = A, \text{ and } 2\sqrt{xy} = \sqrt{-B};$$

$$\therefore\ x^2+2xy+y^2 = A^2, \text{ and } 4xy = -B;$$

$$\therefore\ x^2-2xy+y^2 = A^2+B;\ \therefore\ x-y = \sqrt{A^2+B},$$

and if A^2+B be a perfect square C^2;

$$\therefore\ x-y = C, \text{ and } x+y = A;$$

$$\therefore\ x = \frac{C+A}{2}, \text{ and } y = -\frac{C-A}{2}, \text{ for } C \text{ is} > A;$$

$$\therefore\ \sqrt{x}+\sqrt{y} = \sqrt{\frac{C+A}{2}}+\sqrt{-\left(\frac{C-A}{2}\right)}$$

$$= \sqrt{\frac{C+A}{2}}+\sqrt{\frac{C-A}{2}}\,.\,\sqrt{-1}.$$

Cor. 1. Similarly we shall find that

$$\sqrt{A-\sqrt{-B}} = \sqrt{\frac{C+A}{2}}-\sqrt{\frac{C-A}{2}}\,\sqrt{-1}.$$

Hence

$$\sqrt{A+\sqrt{-B}}+\sqrt{A-\sqrt{-B}} = 2\sqrt{\frac{C+A}{2}} = \sqrt{2\,(C+A)};$$

and therefore the sum of two imaginary quantities related to each other as these are, will give a real and sometimes a rational result.

COR. 2. Also since $\sqrt{\frac{C+A}{2}}$ and $\sqrt{\frac{C-A}{2}}$ are both real quantities, we may put a_1 and b_1 for them, and thus we see that

$$\sqrt{A \pm \sqrt{-B}} = a_1 \pm b_1 \sqrt{-1}.$$

(Ex. 1.) Find $\sqrt{1 + 4\sqrt{-3}}$.

Here $A = 1$, $\sqrt{-B} = 4\sqrt{-3}$; $\therefore B = 48$; $\therefore A^2 + B = 49$;

$$\therefore x - y = \sqrt{A^2 + B} = 7, \; x + y = 1; \; \therefore 2x = 8, \; 2y = -6;$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{1 + 4\sqrt{-3}} = 2 + \sqrt{-3}.$$

(Ex. 2.) Extract the square root of $2a\sqrt{-1}$.

Here $\sqrt{x} + \sqrt{y} = \sqrt{0 + 2a\sqrt{-1}} = \sqrt{A + \sqrt{-B}}$;

$$\therefore A = 0, \; \sqrt{-B} = 2a\sqrt{-1}; \; \therefore B = 4a^2 \text{ and } C = 2a;$$

$$\therefore x - y = 2a, \; x + y = 0; \; \therefore 2x = 2a \text{ and } 2y = -2a;$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{a} + \sqrt{-a} = \sqrt{a} + \sqrt{a}(\sqrt{-1}) = \sqrt{a}(1 + \sqrt{-1}).$$

115. We may also sometimes extract the square root of an expression such as $m + \sqrt{p} + \sqrt{q} + \sqrt{r}$, by assuming the root equal to $\sqrt{x} + \sqrt{y} + \sqrt{z}$; for

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 = x + y + z + 2\sqrt{xy} + 2\sqrt{xz} + 2\sqrt{yz},$$

and if $x + y + z = m$, then making $2\sqrt{xy} = \sqrt{p}$, $2\sqrt{xz} = \sqrt{q}$, and $2\sqrt{yz} = \sqrt{r}$, x, y and z may be found.

Ex. Find the square root of $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$.

Here $\sqrt{xy}=\sqrt{6}$, $\sqrt{xz}=\sqrt{10}$, $\sqrt{zy}=\sqrt{15}$;

$\therefore\ xy=6,\ xz=10,\ yz=15;\ \therefore\ x^2y^2z^2=900;\ \therefore\ xyz=30;$

$$\therefore\ x=\frac{xyz}{xz}=2,\quad y=\frac{xyz}{xz}=3,\quad z=\frac{xyz}{xy}=5,$$

and $x+y+z=10$; $\therefore\ \sqrt{2}+\sqrt{3}+\sqrt{5}$ is the required root.

The equation $x+y+z=m$ which is called the equation of condition, must be verified, before we can say that $\sqrt{x}+\sqrt{y}+\sqrt{z}$ is the true root.

EXAMPLES.

(1) $\sqrt{11\pm 6\sqrt{2}}=3\pm\sqrt{2}$.

(2) $\sqrt{7\pm 4\sqrt{3}}=2\pm\sqrt{3}$.

(3) $\sqrt{3\pm 2\sqrt{2}}=\sqrt{2}\pm 1$.

(4) $\sqrt{36\pm 10\sqrt{11}}=5\pm\sqrt{11}$.

(5) $\sqrt{23\pm 8\sqrt{7}}=4\pm\sqrt{7}$.

(6) $\sqrt{\frac{9}{4}\pm\sqrt{2}}=\sqrt{2}\pm\frac{1}{2}$.

(7) $\sqrt{5\pm\sqrt{24}}=\sqrt{3}\pm\sqrt{2}$.

(8) $\sqrt{\sqrt{27}+2\sqrt{6}}=\sqrt[4]{12}+\sqrt[4]{3}$.

(9) $\sqrt{3\sqrt{5}+\sqrt{40}}=\sqrt[4]{20}+\sqrt[4]{5}$.

(10) $\sqrt{\sqrt{32}-\sqrt{24}}=\sqrt[4]{18}-\sqrt[4]{2}$.

(11) $\sqrt{31 \pm 42\sqrt{-2}} = 7 \pm 3\sqrt{-2}$.

(12) $\sqrt{-83 - 60\sqrt{-3}} = 5 - 6\sqrt{-3}$.

(13) $\sqrt{+\sqrt{-1}} = \dfrac{1}{\sqrt{2}}(1 + \sqrt{-1})$.

(14) $\sqrt{-\sqrt{-1}} = \dfrac{1}{\sqrt{2}}(1 - \sqrt{-1})$.

(15) $\sqrt{\dfrac{a^2}{b^2} - d + \dfrac{2a\sqrt{d}}{b}\sqrt{-1}} = \dfrac{a}{b} + \sqrt{d}\sqrt{-1}$.

(16) $\sqrt{8\sqrt{-1}} = 2 + 2\sqrt{-1}$.

(17) $\sqrt{13 + 2\sqrt{10} + 4\sqrt{3} + 2\sqrt{30}} = \sqrt{2} + \sqrt{5} + \sqrt{6}$.

(18) $\sqrt{16 + 6\sqrt{2} + 4\sqrt{3} + 2\sqrt{6} + 2\sqrt{10} + 2\sqrt{15} + 2\sqrt{30}}$

$$= \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6}.$$

CHAPTER V.

RATIO, PROPORTION, AND VARIATION.

RATIO.

116. RATIO is the relation which quantities bear to each other with regard to magnitude.

Thus if one magnitude be two-thirds of another magnitude, they are said to be in the ratio of 2 : 3. For if both be divided into respectively equal parts, the former will contain two and the latter three of these equal parts. And thus the ratio of 2 : 3 and the fraction $\frac{2}{3}$ express the same idea; for $\frac{2}{3}$ indicates that unity has been divided into 3 equal parts, and two having been taken, that portion will be to the whole as $\frac{2}{3} : \frac{3}{3}$, or as 2 : 3.

We may therefore either take $a : b$ or the fraction $\frac{a}{b}$ to express a ratio, the fraction equally exhibiting with $a : b$ the multiple, part, or parts, that a is of b.

The former term of a ratio is called the antecedent, the latter the consequent.

A ratio is said to be of greater or less inequality, according as the antecedent is greater or less than the consequent, and of equality when these are equal.

Both terms of a ratio may be multiplied or divided by the same number without altering the value of the ratio.

Thus $a : b$ and $ma : mb$ are equal ratios.

$$\text{For } \frac{a}{b} = \frac{ma}{mb}.$$

Also $a : b$ and $\frac{a}{m} : \frac{b}{m}$ are equal ratios.

$$\text{Since } \frac{a}{b} = \frac{a}{m} \div \frac{b}{m}.$$

117. If to both terms of the ratio $a : b$, the quantity x be added, the resulting ratio will be *greater* or *less* than the former, according as a is *less* or *greater* than b.

$$\text{For } \frac{a}{b} \text{ is} > \text{ or } < \frac{a+x}{b+x}.$$

$$\text{If } \frac{ab+ax}{b(b+x)} > \text{ or } < \frac{ab+bx}{b(b+x)}.$$

$$\text{If } ab+ax > \text{ or } < ab+bx.$$

$$\text{If } ax > \text{ or } < bx.$$

$$\text{If } a > \text{ or } < b.$$

Which shews the truth of the proposition.

Thus if unity be added to both terms of the ratio $5 : 7$, by which it becomes $6 : 8$ or $3 : 4$, then

$$\frac{5}{7} \text{ or } \frac{20}{28} \text{ is} < \frac{3}{4} \text{ or } \frac{21}{28}.$$

But if unity be added to both terms of $7 : 5$, then

$$\frac{7}{5} \text{ or } \frac{21}{15} \text{ is} > \frac{8}{6} \text{ or } \frac{4}{3} \text{ or } \frac{20}{15}.$$

118. A ratio is reduced to its lowest terms by dividing both the antecedent and consequent by their greatest common measure.

(Ex. 1.) Reduce $275 : 325$ to its lowest terms.

The greatest common measure is 25, and $275 = 25 \times 11$;

$$325 = 25 \times 13;$$

$$\therefore\ 11 : 13 \text{ is the ratio required.}$$

(Ex. 2.) Reduce to its lowest terms the ratio

$$a^2 - 3ax + 2x^2 : a^2 + ax - 2x^2.$$

Here $a - x$ is the common measure, and after having divided both terms by it, the ratio becomes $a - 2x : a + 2x$.

119. The duplicate ratio of two quantities is the ratio of their squares; the triplicate ratio, that of their cubes; the quadruplicate ratio, that of their fourth powers, and so on.

Thus $a^2 : b^2$ is the duplicate ratio of $a : b$,

$a^3 : b^3$ ... triplicate ... of $a : b$,

$a^4 : b^4$ ...quadruplicate... of $a : b$.

And the subduplicate ratio of two quantities is the ratio of their square roots; the subtriplicate, that of their cube roots.

Thus $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is the subduplicate, and $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is the subtriplicate ratio of $a : b$.

Also $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the sesquiplicate ratio of $a : b$.

(Ex. 1.) The bases of two similar triangles are $4\frac{1}{2}$ feet and $5\frac{1}{2}$ feet respectively. Compare their areas.

The areas are by Euclid (6.19) in the duplicate ratio of their bases, therefore as

$$(4\tfrac{1}{2})^2 : (5\tfrac{1}{2})^2 :: \left(\frac{9}{2}\right)^2 : \left(\frac{11}{2}\right)^2 :: 81 : 121.$$

(Ex. 2.) Spheres are in the triplicate ratio of their radii; let therefore the magnitude of two spheres be compared whose radii are 2 and 3 feet respectively.

They are therefore as $2^3 : 3^3 :: 8 : 27$.

(Ex. 3.) The diameter of the Earth being 8000 miles, and that of the Moon 2000 miles; compare the magnitudes of the Earth and Moon.

The Earth : the Moon $:: (8000)^3 : (2000)^3 :: 64 : 1$.

(Ex. 4.) Newton proved that the times of the planet's revolutions were in the sesquiplicate ratio of their mean distances. Suppose that Herschel is 84 years in completing the circuit of his orbit. Find his mean distance from the Sun.

Call the Earth's distance d, D that of Herschel;

$$\therefore 1 \text{ year} : 84 \text{ years} :: d^{\frac{3}{2}} : D^{\frac{3}{2}};$$

$$\therefore D = (84)^{\frac{2}{3}} d,$$

or more than 19 times d, or since d is about 95,000,000 miles, the planet's mean distance exceeds 1800 millions of miles.

By *mean*, we here understand half the sum of the greatest and least distances.

120. Composition of ratios. If there be any number of ratios, and all the antecedents be multiplied together, and also all the consequents, the ratio which arises is said to be compounded of the simple ratios.

Thus if $a : b$ and $c : d$ be two ratios. Then $ac : bd$ is the ratio which is compounded of the ratios of $a : b$, and $c : d$.

Thus in Euclid, Book VI. Prop. 23, the ratio of $K : M$ is said to be compounded of the ratios of $K : L$ and $L : M$; for the compound ratio is $KL : LM$ or dividing both terms by L, is $K : M$.

Also the ratio of $ac : bd$ is called the sum of the ratios of $a : b$ and $c : d$. Hence it is that the ratio of $a^2 : b^2$ which is the sum of the two equal ratios, each $a : b$, is said to be double of the ratio of $a : b$, and $a^3 : b^3$ triple of the ratio of $a : b$.

(Ex. 1.) Compound the ratios of $2 : 3$, $3 : 4$, and $6 : 5$.

It is better to write the ratios as fractions, then

$$\frac{2}{3} \times \frac{3}{4} \times \frac{6}{5} = \frac{2 \times 6}{4 \times 5} = \frac{3}{5}.$$

(Ex. 2.) Compound the ratios of

$$a^2 : a^2 - x^2; \quad a + x : a - x; \quad \text{and } a^3 - x^3 : a^3,$$

$$\frac{a^2}{a^2 - x^2} \times \frac{(a+x)}{(a-x)} \times \frac{a^3 - x^3}{a^3}$$

$$= \frac{1}{a - x} \times \frac{a^2 + ax - x^2}{a} = \frac{a^2 + ax + x^2}{a^2 - ax}.$$

For $a^2 - x^2 = (a - x) \,.\, (a + x)$,

and $a^3 - x^3 = (a - x) \,.\, (a^2 + ax + x^2)$.

(Ex. 3.) Compound the duplicate ratio of $2 : 3$; the triplicate of $3 : 4$; and the subduplicate of $64 : 36$.

Ans. $1 : 4$.

(Ex. 4.) If $x : y$ in the duplicate ratio of $a : b$; and $a : b$ in the subduplicate ratio of $a + x : a - y$ then $2x : a$ will equal $x - y : y$.

121. The ratios of the squares of high numbers which do not differ much from each other, may conveniently be found by a rule, sufficiently accurate for every practical purpose, the truth of which we shall now investigate.

Let a, and $a+x$ represent two large numbers, differing but little from each other, and where consequently x is small in comparison with a, then

$$\because \frac{a+x}{x} = 1 + \frac{x}{a}; \quad \therefore \frac{(a+x)^2}{a^2} = 1 + \frac{2x}{a} + \frac{x^2}{a^2}.$$

Now x being very small, and a great, $\frac{x}{a}$ is a very small quantity, and $\frac{x^2}{a^2}$ still smaller, and may be neglected, since it will be found in practice to affect the result by an inappreciable quantity.

$$\text{Hence } \frac{(a+x)^2}{a^2} = 1 + \frac{2x}{a} = \frac{a+2x}{a};$$

i.e. $(a+x)^2 : a^2$ is equal to $a+2x : a$, or the ratio of the squares of two high numbers, nearly equal to each other, may be found, by adding to the smaller twice their difference, and comparing this sum with the smaller of the two.

Thus to find the ratio of $(1000.1)^2$ to $(1000)^2$.

$$\text{Here } x = .1 \text{ or } \frac{1}{10} \text{ and } 2x = .2;$$

$\therefore (1000.1)^2 : (1000)^2 = 1000.2 : 1000$ or as $10002 : 10000$, very nearly.

In this example

$$\frac{x^2}{a^2} = \frac{(.1)^2}{(1000)^2} = \frac{.01}{1,000,000},$$

or is only one hundred-millioneth part of unity.

122. In the same manner since

$$\frac{(a+x)^3}{a^3} = \left(1+\frac{x}{a}\right)^3 = 1 + \frac{3x}{a} + \frac{3x^2}{a^2} + \frac{x^3}{a^3}$$

$$= 1 + \frac{3x}{a} \text{ nearly}$$

$$= \frac{a+3x}{a}.$$

Therefore $(a+x)^3 : a^3$ is represented by $a+3x : a$, and $(a+x)^4 : a^4$ by $a+4x : a$.

And also since $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + \&c.$;

$$\therefore \frac{\sqrt{a+x}}{\sqrt{a}} = \sqrt{1+\frac{x}{a}} = 1 + \tfrac{1}{2}\frac{x}{a} - \tfrac{1}{8}\frac{x^2}{a^2} + \&c.$$

$$= 1 + \tfrac{1}{2}\frac{x}{a} \text{ nearly}$$

$$= \frac{a+\frac{1}{2}x}{a} \text{ nearly;}$$

$$\therefore \sqrt{a+x} : \sqrt{a} \text{ equals } a + \tfrac{1}{2}x : a \text{ nearly.}$$

And similarly may the comparison of other roots of high numbers not much differing from each other be effected.

PROPORTION.

123. When two ratios as $a : b$ and $c : d$ are equal, the terms are said to be in proportion to each other, and a is said to have to b, the ratio that c has to d; or a is that multiple, part, or parts, of b that c is of d, and it is then written $a : b :: c : d$. Thus proportion arises from the equality of ratios.

Here a and d are called the extremes and b and c the means.

124. Since $a : b$ is expressed by $\frac{a}{b}$ and $c : d$ by $\frac{c}{d}$ if there be a proportion between a, b, c, and d; i. e. if

$$a : b :: c : d,$$

$$\text{then } \frac{a}{b} = \frac{c}{d}.$$

Multiply both sides by bd;

$$\therefore \frac{abd}{b} = \frac{cbd}{d}; \quad \therefore ad = cb;$$

i. e. if four quantities be proportionals the product of the extremes equals the product of the means; and conversely, if the product of two quantities be equal to the product of other two quantities, the four are proportionals.

For let $ad = bc$, then divide both sides by bd;

$$\therefore \frac{a}{b} = \frac{c}{d} \text{ or } a : b :: c : d.$$

Hence given a, b, c any three of the quantities, the fourth d may be found.

$$\text{For } ad = bc; \quad \therefore d = \frac{bc}{a},$$

which symbollically expresses the Rule of Three in arithmetic, viz. multiply the second and third terms together, and divide by the first.

When the two mean terms of a proportion are equal, the quantities are said to be in continued proportion, i. e. if $a : b :: b : c$, a, b, c are in continued proportion; but then $ac = b^2$ or the product of the extremes is equal to the square of the mean, and b is called a mean proportional between a and c.

Again, if $a : b :: c : d$, and $c : d :: e : f$,

then $a : b :: e : f$.

$$\text{For } \frac{a}{b}=\frac{c}{d} \text{ and } \frac{c}{d}=\frac{e}{f}; \therefore \frac{a}{b}=\frac{e}{f};$$

$$\text{i.e. } a : b :: e : f.$$

Also if $a : b :: c : d$, then $a^m : b^m :: c^m : d^m$.

$$\text{For } \frac{a}{b}=\frac{c}{d}; \therefore \frac{a^m}{b^m}=\frac{c^m}{d^m};$$

$$\text{and } \therefore a^m : b^m :: c^m : d^m.$$

125. (1°.) If $a : b :: c : d$, then $a : c :: b : d$; i.e. if 4 quantities are proportionals they are also so when taken alternately.

$$\text{For } \frac{a}{b}=\frac{c}{d}; \therefore \frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{b}{c}, \text{ or } \frac{a}{c}=\frac{b}{d};$$

$$\text{i.e. } a : c :: b : d.$$

(2°.) If $a : b :: c : d$, then $b : a :: d : c$, or quantities are also proportionals when taken inversely.

For $\frac{a}{b}=\frac{c}{d}$; and since if numbers are equal their reciprocals are also equal;

$$\therefore \frac{b}{a}=\frac{c}{d}; \quad \text{i.e. } b : a :: c : d.$$

(3°.) Next $a+b : b :: c+d : d$.

For since $\frac{a}{b}=\frac{c}{d}$; $\therefore \frac{a}{b}+1=\frac{c}{d}+1$, by adding unity to each side;

$$\therefore \frac{a+b}{b}=\frac{c+d}{d}; \quad \therefore a+b : b :: c+d : d,$$

this is termed *componendo*.

(4°.) And $a-b : b :: c-d : d$.

$$\text{For } \frac{a}{b}-1=\frac{c}{d}-1; \quad \therefore \frac{a-b}{b}=\frac{c-d}{d};$$

$$\therefore a-b : b :: c-d : d,$$

this is called *dividendo*.

(5°.) And $a+b : a-b :: c+d : c-d$.

$$\text{For } \frac{a+b}{b}=\frac{c+d}{d}, \text{ and } \frac{a-b}{b}=\frac{c-d}{d};$$

$$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d};$$

$$\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d};$$

$$\therefore a+b : a-b :: c+d : c-d.$$

(6°.) When there are 3 quantities, a, b, c, in continued proportion, the first shall be to the third in the duplicate ratio of the first to the second, or if $a : b :: b : c$;

$$\therefore a : c :: a^2 : b^2.$$

$$\text{For } \because a : b :: b : c; \quad \therefore \frac{a}{b}=\frac{b}{c};$$

$$\therefore \frac{a}{b}\times\frac{a}{b}=\frac{b}{c}\times\frac{a}{b}; \quad \therefore \frac{a^2}{b^2}=\frac{ab}{cb}=\frac{a}{c};$$

$$\therefore a : c :: a^2 : b^2;$$

(7°.) Again, if there be 4 magnitudes in continued proportion, the first shall have to the fourth the triplicate ratio of the first to the second, or if $a : b :: b : c :: c : d$, then $a^3 : b^3 :: a : d$.

$$\text{For } \because a : b :: b : c; \quad \therefore \frac{a^2}{b^2}=\frac{a}{c};$$

$$\text{and} \because b : c :: c : a; \quad \therefore \frac{b}{c} = \frac{c}{d}.$$

$$\text{But } \frac{a}{b} = \frac{b}{c}; \quad \therefore \frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a^2}{b^2} \times \frac{a}{b} = \frac{a}{c} \times \frac{c}{d}; \quad \therefore \frac{a^3}{b^3} = \frac{a}{d};$$

$$\text{i. e. } a : d :: a^3 : b^3.$$

(8°.) If $a : b :: c : d$, and a be the greatest, d will be the least, for $\frac{a}{b} = \frac{c}{d}$; $\therefore \frac{a}{c} = \frac{b}{d}$ and a is $> c$; $\therefore b$ is $> d$.

$$\text{Also } a + d \text{ is } > b + c.$$

$$\text{For } a : a - b :: c : c - d;$$

$$\therefore a : c :: a - b : c - d.$$

$$\text{And } a \text{ is } > c; \quad \therefore a - b > c - d; \quad \therefore a + d > b + c,$$

or the sum of the greatest and least terms in a proportion is greater than the sum of the other two.

126. It frequently happens when the ratio of two numbers or two magnitudes is required, that there is no fraction $\frac{c}{d}$ which is exactly equal to $\frac{a}{b}$; one case is when a and b are two numbers having no common divisor. It also happens when the ratio of the square roots of two numbers which are not both complete squares is required. Thus $\sqrt{2} : \sqrt{3}$ cannot be exactly expressed, since neither $\sqrt{2}$ nor $\sqrt{3}$ can be found in a terminating decimal, therefore no two numbers can be found which will be in the precise ratio required; whenever this occurs the quantities are said to be incommensurable.

But although the exact ratio between two such quantities cannot be found, yet an approximate one may be obtained, which may be made sufficiently correct for every practicable purpose.

As an instance of incommensurable quantities, we may take that of the circumference and the diameter of the circle.

If the diameter of the circle be divided into equal parts, and we try to ascertain by measurement, how many of such equal parts there are in the circumference, we shall find that there is not an exact number of such parts in it; that if the circle be small, the circumference will be a little larger than three diameters; and if we take a larger circle and divide the diameter into 10 parts, the circumference will contain more than 31, but less than 32 such parts; and if the diameter be divided into a hundred parts, that the circumference will have more than 314 but less than 315 such parts. In fact, if the diameter of the circle be called unity, the circumference is nearly = 3.14159, &c.

$$\therefore \frac{\text{circumference}}{\text{diameter}} = \frac{3.14159}{1} \text{ nearly.}$$

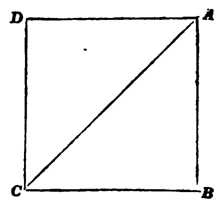
Hence, if we stop successively at the end of the first, second, third, &c. decimals, the required ratio will be represented by one of the fractions

$$\frac{3.1}{1}, \frac{3.14}{1}, \frac{3.141}{1}, \text{\&c. or by } \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \text{\&c.,}$$

and the greater the number of terms in the fraction, the more nearly will the fraction represent the true ratio; the particular circumstances of the question will guide us to the selection of the fraction which may be used without sensible error.

Thus, if the diameter be one foot, the second fraction is sufficiently exact, for the circumference is $\frac{314}{100}$ feet = 3 feet, 1 inch .68; and the error does not exceed the 80th part of an inch. But if the diameter be very great, as that of the Earth, which is not much less than 8000 miles, the fraction to be used must have many terms of decimals, that the error may not be appreciable.

Also the diagonal and the side of a square are incommensurable quantities.



For $AC^2 = AB^2 + BC^2 = 2AB^2$;

$$\therefore AC = AB\sqrt{2};$$

$$\therefore \frac{AC}{AB} = \frac{\sqrt{2}}{1} = \frac{1.41421 \text{ \&c.}}{1};$$

and remarks similar to what have been just used apply here.

VARIATION.

127. When one quantity depends in any manner whatever upon the change in magnitude of another, the former is said to vary as the latter.

Then, if $Y = aX$ and a be constant, Y is said to vary *directly* as X; or as it is written $Y \propto X$, the symbol $\propto$ being read, varies as; for as X increases or decreases, Y increases or decreases; in this case, if we change Y into y, and if x be the corresponding new value of X,

$$y = ax, \text{ and } \because Y = aX; \therefore Y : y :: X : x,$$

and thus we can change a variation into a proportion, when we know other corresponding values of Y and X.

Again, if $Y = \frac{a}{X}$, then since Y decreases or increases according as X increases or decreases, Y is said to vary inversely as X, or $Y \propto \frac{1}{X}$; and

$$Y : y :: \frac{1}{X} : \frac{1}{x} :: x : X.$$

As instances of the two preceding cases of variation we may take the following: We know that the quantity of work done (W) increases as the number of workmen (A) is increased, or technically $W \propto A$, but that the time (T) of doing a certain work will decrease, as the number of workmen increase or $T \propto \frac{1}{A}$.

128. If $y = ax + bx^2$, then y is said to vary as the sum of two quantities, one of which $\propto x$, and the other as x^2; the constant coefficients a and b, may be found from contemporaneous values of y and x.

(Ex. 1.) Let y vary as x, and let 2 and 3 be respective values of y and x, at the same time. Find y in terms of x;

$$\because\ y \propto x;\quad \therefore \text{ let } y = ax;\ \text{ but if } y = 2,\ x = 3;$$

$$\therefore\ 2 = 3a;\quad \therefore\ a = \frac{2}{3};\quad \therefore\ y = \frac{2x}{3}.$$

(Ex. 2.) Let $y = p + q$; where $p \propto x$ and $q \propto \frac{1}{x}$; and 6 and 1, 9 and 2 are contemporaneous values of y and x. Find y in terms of x.

Let $p = ax$; $q = \frac{b}{x}$; $\therefore\ y = ax + \frac{b}{x}$.

But if $x = 1$, $y = 6$; and if $x = 2$, $y = 9$;

$$\therefore\ 6 = a + b;\ \text{ and }\ 9 = 2a + \frac{b}{2};$$

$$\therefore\ a = 4;\ b = 2,\ \text{ and } y = 4x + \frac{2}{x}.$$

129. If $Y \propto XZ$, Y is said to vary jointly as X and Z; for the increase or decrease of Y will depend upon the increase or decrease of both X and Z.

Prop. If $Y \propto X$ when Z is constant,

and $Y \propto Z$ when X is constant;

then $Y \propto XZ$ when both X and Z vary.

Let y_1 and x be corresponding values of Y and X,

y and z .... y_1 and Z.

$$\text{Then}\quad Y : y_1 :: X : x,$$
$$y_1 : y :: Z : z;$$
$$\therefore\ Y : y :: XZ : xz.$$

The preceding proposition is the foundation of the rule given in the books of arithmetic, for the solution of questions both in the single and double rule of three. For suppose that, a certain effect is to be produced, or work to be done (W) by a number of persons (A), and in a time (T), it is clear that the work done, or the effect produced, will be changed if we alter either the time or the number of workmen; that W depends not only on A, but also on T; and A and T are independent quantities; hence $W \propto AT$, and therefore if w be another work done by (a) agents in time (t) that

$$W : w :: A.T : a.t.$$

We will however give an independent proof of this proposition.

Suppose that V was a work done by a persons in time T.

Then since W and V are done by different agents in the same time T; and V and w are done by the same number of persons, in different times,

$$\text{then}\quad W : V :: A : a,$$
$$\text{and}\quad V : w :: T : t;$$
$$\therefore\ W : w :: A.T : a.t, \quad\text{or}\quad W \propto AT.$$

Ex. If 252 men can dig a trench 210 yards long, 3 wide and 2 deep, in 5 days of 11 hours long; in how many days of 9 hours each, will 22 men dig a trench 420 yards long, 5 wide, and 3 deep?

Here $W = 210 \times 3 \times 2$; $w = 420 \times 5 \times 3$,

$$A = 252,\ a = 22;\quad T = 5 \times 11,\ t = 9 \times x;$$

$$\therefore\ 210 \times 3 \times 2 : 420 \times 5 \times 3 :: 252 \times 5 \times 11 : 22 \times 9 \times x;$$

$$1 : 5 :: 28 \times 5 : 2 \times x;$$

$$\therefore\ x = 14 \times 5 \times 5 = 350.$$

Cor. If $T = t$, or if $A = a$, then either $W : w :: A : a$, or $W : w :: T : t$, which is the common rule of Three; in which if W represent work done, A may be the number of persons who perform it; or T may be the time in which it is done: again, W may be goods, and A their cost, and so on. We must always take care that the first and second terms *be of the same denomination;* the third and fourth must consequently be of the same denomination also; it is through inattention to this obvious rule that many errors are made in the school arithmetics.

Ex. What quantity of cloth, at 6*s*. 8*d*. per yard, may be bought for 20 guineas.

Here the thing required is the number of yards.

Let $W = 1$ yard, $w = x$; then as 6*s*. 8*d*. $= \frac{1}{3}$£. and 20 guineas = £21.

$$1 : x :: \tfrac{1}{3} : 21 :: 1 : 63;$$

$$\therefore\ x = 63 \text{ yards}.$$

CHAPTER VI.

ARITHMETICAL, GEOMETRICAL AND HARMONICAL PROGRESSION.

130. A SERIES is a collection of numbers, connected together by the signs + or −, and in which any one term may be derived from those which precede it, by a rule, which is called the law of the series; thus

$$1 + 4 + 7 + 10 + 13 + \&c.$$

$$2 + 4 + 8 + 16 + 32 + \&c.$$

are series, in the former of which any term may be derived from that which precedes it, by adding 3, thus $7 = 4 + 3$; in the latter the third term 8, is found by taking the double of the second term 4; and the same rule applies to the fourth and succeeding terms.

The subject of series is extensive and difficult. Our attention will be confined to arithmetic and geometric series only; of which those above are respectively instances.

ARITHMETIC SERIES.

131. If the difference between any two consecutive terms of a series be the same through the whole extent of the series, the series is called arithmetic, and the terms are said to be in arithmetic progression.

Thus, $1 + 3 + 5 + 7 + 9 + \&c.$, and $12 + 9 + 6 + 3 + \&c.$, are arithmetic series, the former being an increasing, and the latter a decreasing series. The number by which each term differs from the succeeding one is called the common difference, which in the former series is 2; and in the latter is -3.

132. Hence the first term a and the common difference b being known, the other terms of the series may successively be derived; for since the difference between any two successive terms is b;

$$\therefore \text{2nd term} = \text{1st term} + b = a + b,$$

$$\text{3rd term} = \text{2nd term} + b = a + 2b,$$

$$\text{4th term} = \text{3rd term} + b = a + 3b, \text{ and so on,}$$

and as we see that the coefficient of b in any one term is less by unity than the place of the term in the series;

$$\therefore \text{the } n\text{th term} = a + (n-1)\,b;$$

and therefore the general form of an arithmetic series is

$$a + (a+b) + (a+2b) + (a+3b) + \&c. \quad + \{a + (n-1)\,b\},$$

the series being supposed to consist of n terms. Hence if l be the last term, we have

$$l = a + (n-1)\,b.$$

From the formula just obtained any term of the series may be found; thus

$$\text{the 20th term} = a + (20-1)\,b = a + 19b;$$

$$\text{100th term} = a + (100-1)\,b = a + 99b.$$

(Ex. 1.) Find the 50th term of $5 + 8 + 11 + \&c.$

Here $a = 5$; $b = 3$; $n = 50$;

$$\therefore \text{50th term} = a + 49b = 5 + 147 = 152.$$

(Ex. 2.) Find the 10th term of $12 + 9 + 6 + 3 + \&c.$

Here $a = 12$; $b = -3$; $n = 10$;

$$\therefore \text{10th term} = a + 9b = 12 - 27 = -15.$$

Cor. Since the last term is l, the last but one will be $l-b$, the last but two $l-2b$, and so on;

or since the last or nth term $= a + (n-1)\,b$;

$$\therefore\ (n-1)\text{th}\ \ldots = a + (n-2)\,b,$$

$$(n-2)\text{th}\ \ldots = a + (n-3)\,b;$$

$$\therefore\ (n-r)\text{th}\ \ldots = a + \{n-(r+1)\}\,b.$$

133. Prop. Given the first term a, the common difference b, and the number of terms n, find S, the sum of the series. Here

$$S = a + (a+b) + (a+2b) + \&\text{c.}\quad + (l-2b) + (l-b) + l,$$

and then writing the series in an inverse order,

$$S = l + (l-b) + (l-2b) + \&\text{c.}\quad + (a+2b) + (a+b) + a.$$

Whence by adding together the terms which are vertically opposite,

$$2S = (l+a) + (l+a) + (l+a) + \&\text{c.}\quad + (l+a) + (l+a) + (l+a);$$

and since $l+a$ is repeated n times;

$$\therefore\ 2S = (l+a)\,.\,n; \qquad \therefore\ S = (l+a)\,\frac{n}{2},$$

or the sum of an arithmetic series is found, by adding together the first and last terms, and multiplying their sum by half the number of terms.

Cor. 1. This rule may be put under a more convenient form.

For $l + a = a + (n-1)\,.\,b + a = 2a + (n-1)\,.\,b$;

$$\therefore\ S = \{2a + (n-1)\,b\}\,\frac{n}{2} = na + \frac{n\,.\,(n-1)\,b}{2}.$$

COR. 2. The two equations,

$$l = a + (n-1)b, \text{ and } S = na + n\frac{(n-1)}{2}b,$$

are sufficient for the solution of all questions respecting arithmetic series,

(Ex. 1.) Find the sum of $1 + 3 + 5 + 7 +$ &c. to 100 terms, and the last term.

$a = 1$, $\quad \therefore l = a + (n-1)b = 1 + 99 \times 2 = 199$,

$b = 2$,

$n = 100$; $\quad S = na + \frac{n \cdot (n-1) \cdot b}{2} = 100 + \frac{100 \times 99 \times 2}{2}$

$$= 100 + 9900 = 10{,}000 = (100)^2.$$

If the sum be required to n terms,

$$S = n + \frac{n(n-1) \cdot 2}{2} = n + n^2 - n = n^2;$$

or the sum of the first n odd numbers = the square of the number of terms.

(Ex. 2.) Find the sum of $12 + 10 + 8 +$ &c. to 20 terms.

$a = 12$, $\quad \therefore S = 20 \times 12 + \frac{20 \times 19}{2} \times -2$

$b = -2$,

$n = 20$; $\quad = 240 - 380 = -140$.

The series being a decreasing one, some of the terms become negative, and their sum exceeds that of the positive terms.

(Ex. 3.) The last term of a series is 29, the number of terms = 10 and common difference = 3; find the first term.

$l = 29$,

$b = 3$, $\quad$ But $l = a + (n-1)b$; $\therefore 29 = a + 9 \times 3$; $\therefore a = 2$.

$n = 10$.

(Ex. 4.) The sum = 155, first term = 2, common difference = 3; find n.

$$S = 155 = na + \frac{n.(n-1)b}{2} = 2n + \frac{n^2 - n}{2} \times 3;$$

$$\therefore\ 3n^2 + n = 310;\quad \therefore\ n^2 + \frac{n}{3} + \frac{1}{36} = \frac{310}{3} + \frac{1}{36} = \frac{3721}{36};$$

$$\therefore\ n + \frac{1}{6} = \frac{61}{6};\quad \therefore\ n = 10.$$

134. Find an arithmetic mean between two numbers, a and c.

Let m be the mean; $\therefore$ a, m and c are in arithmetic progression; $\therefore\ m - a = c - m$; $\therefore\ m = \frac{a+c}{2}$.

Next find p arithmetic means between a and l.

This is the same thing as having given, a the first and l the last term, and $p+2$ the number of terms, to find the common difference.

Let b be the common difference;

$$\therefore\ l = a + (n-1)b;\ \text{but}\ n = p + 2;\quad \therefore\ n - 1 = p + 1;$$

$$\therefore\ l - a = (p+1).b;\quad \therefore\ b = \frac{l-a}{p+1}.$$

(Ex. 1.) Find an arithmetic mean between 4 and 8.

Here $m = \frac{4+8}{2} = 6$, and 4, 6, 8 are in arithmetical progression.

(Ex. 2.) Place 5 means between 1 and 3.

$$\text{Here}\ b = \frac{3-1}{5+1} = \frac{2}{6} = \frac{1}{3};$$

$$\therefore\ 1 + \frac{1}{3},\quad 1 + \frac{2}{3},\quad 1 + \frac{3}{3},\quad 1 + \frac{4}{3},\quad 1 + \frac{5}{3};$$

$$\text{or}\ \frac{4}{3},\ \frac{5}{3},\ 2,\ \frac{7}{3},\ \frac{8}{3}\ \text{are the means.}$$

135. In any arithmetic series the sum of any two terms equidistant from each end of the series is a constant quantity.

$$\text{Thus the third term } = a + 2b,$$

$$\text{the last but two } = l - 2b;$$

$$\therefore \text{ third + last but two } = l + a;$$

$$\text{or generally, the } (1+r)^{\text{th}} \text{ term } = a + rb,$$

$$\text{and the } (l-r)^{\text{th}} \text{ term } = l - rb;$$

$$\therefore (1+r)^{\text{th}} + (l-r)^{\text{th}} = l + a.$$

136. We may here add the sum of the series,

$$1^2 + 2^2 + 3^2 + 4^2 + \text{ \&c. } + n^2.$$

$$\text{Let } S = 1^2 + 2^2 + 3^2 + 4^2 + \text{ \&c. } + n^2,$$

$$S_1 = 1^2 + 2^2 + 3^2 + 4^2 + \text{ \&c. } + n^2 + (n+1)^2;$$

$$\therefore S_1 - S = (n+1)^2 = n^2 + 2n + 1,$$

or the difference between the sum of n terms and that of $n+1$ terms is $(n+1)^2$; but if we assume that

$$S = An^3 + Bn^2 + Cn,$$

$$\text{and } \therefore S_1 = A(n+1)^3 + B(n+1)^2 + C(n+1);$$

$$\therefore S_1 - S = A.(3n^2 + 3n + 1) + B(2n+1) + C$$

$$= 3An^2 + (3A + 2B)n + A + B + C,$$

and $S_1 - S$ will have the same value in both instances, if

$$3A = 1, \text{ or } A = \frac{1}{3}, \quad 3A + 2B = 2, \text{ or } 2B = 1, \text{ or } B = \frac{1}{2},$$

$$\text{and if } A + B + C = 1, \text{ or } C = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6};$$

$$\therefore S = 1^2 + 2^2 + 3^2 + \text{ \&c. } + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \frac{n.(2n^2 + 3n + 1)}{6} = \frac{n.(n+1)(2n+1)}{2.3}.$$

Ex. Find the sum of the squares of the natural number to 100 terms.

$$S = \frac{100 \times 101 \times 201}{6} = 150 \times 101 \times 67 = 338350.$$

Cor. In the same manner may the sum of the cubes of the natural numbers be found, and it will be seen that

$$1^3 + 2^3 + 3^3 + \text{ \&c. } + n^3 = (1 + 2 + 3 + \text{ \&c. } + n)^2.$$

EXAMPLES.

(1) The sum of $1 + 2 + 3 + 4 +$ &c. to 50 terms $= 1275$.

(2) .... $2 + 5 + 8 +$ &c. to 17 ... $= 442$.

(3) .... $7 + \frac{29}{4} + \frac{15}{2} +$ &c. to 16 ... $= 142$.

(4) .... $12 + 8 + 4 +$ &c. to 20 ... $= -520$.

(5) .... $\frac{1}{3} + \frac{5}{6} + \frac{4}{3} +$ &c. to 12 ... $= 37$.

(6) .... $\frac{23}{3} + \frac{20}{3} + \frac{17}{3} +$ &c. to 15 ... $= 10$.

(7) .... $1 + 2 + 3 + 4 +$ &c. to n ... $= \frac{n(n+1)}{2}$.

(8) .... $1 + 3 + 5 +$ &c. to 100 ... $= 10,000$.

(9) .... $\frac{2}{3} + \frac{1}{2} + \frac{1}{3} +$ &c. to 9 ... $= 0$.

(10) .... $(8^2+9) + (8^2+11) + (8^2+13) +$ &c. to 9 terms $= 9^3$.

(11) .... $2\frac{1}{2} + 2\frac{3}{8} + 2\frac{5}{8} +$ &c. to 100 ... $= 1075$.

(12) .... $-7 - 4 - 1 +$ &c. to 8 ... $= 28$.

(13) The sum of $-10-12-14-$ &c. to 6 terms $=-90$.

(14) .... $\frac{1}{2}+\frac{3}{8}+\frac{1}{4}+$ &c. to 20 ... $=-\frac{55}{4}$.

(15) The sum of a series is 105, the common difference 1, the number of terms 14; find the first term.

Ans. 1.

(16) The first term is $2\frac{1}{2}$, the last $35\frac{1}{2}$, the sum 190; find the number of terms. Ans. 100.

(17) Insert 3 arithmetic means between 2 and 14.

Ans. 5, 8, 11.

(18) Place 5 between 1 and -1;

Ans. $\frac{2}{3}$, $\frac{1}{3}$, 0, $-\frac{1}{3}-\frac{2}{3}$.

(19) There are n arithmetic means between 3 and 17, and the last is 3 times as great as the first; find the number of means. Ans. 6.

(20) The sum of $116+108+100+$ &c. is 800; find the number of terms. Ans. 10.

(21) The sum of an arithmetic series is 507, the last term is 75, the common difference 6; find the number of terms. Ans. 13.

(22) The sum is $146\frac{1}{4}$, last term $15\frac{3}{4}$, number of terms 30; find the common difference and first term.

Ans. $\frac{3}{4}$, -6.

(23) How many terms of the series $15+\frac{44}{3}+\frac{43}{3}+$ &c. must be taken to make 200? Ans. 16 or 75.

(24) The 4th term of a series is 29, the 7th is 50; find the first term and common difference.

Ans. 8 and 7.

(25) A body of soldiers is drawn up in the form of a solid equilateral wedge, and the outer rank contains 180 men; find the number of soldiers. Ans. 16290.

(26) Find the number in a hollow wedge, the ranks of which are 3 deep; the outer rank containing n persons.

Ans. $9n - 36$.

GEOMETRIC PROGRESSION.

137. If there be a series represented by

$$a + b + c + d + e + f + \&c.,$$

in which the ratio of any one term to that immediately preceding it, is the same throughout the whole extent of the series, i. e. if

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}, \text{ \&c.; then } a, b, c, \text{ \&c.}$$

are said to be in geometrical progression.

As instances, we may take the following series,

$$1 + 2 + 4 + 8 + \&c., \quad 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \&c.;$$

in the former of which each term is twice, and in the latter half, of that immediately preceding it; or the ratio of the consecutive terms is in the one case 2, and in the other $\frac{1}{2}$.

Such a ratio is called a common ratio.

138. The term b, c, d, &c. may be more conveniently expressed in values of a and the common ratio;

Let $r =$ the common ratio;

$$\therefore \frac{b}{a} = r; \quad \therefore b = ar; \quad \frac{c}{b} = r; \quad \therefore c = br = ar^2,$$

$$\frac{d}{c} = r; \quad \therefore d = cr = ar^2 \times r = ar^3;$$

or if a be the first term and r the common ratio, ar, ar^2, ar^3, ar^4, &c. will respectively be the second, third, fourth, &c. terms, and the series (S) may be written

$$S = a + ar + ar^2 + ar^3 + \&c.$$

It may be observed, that the index of r in any term is less by unity than its place in the series; thus,

the third term $= ar^2$; the fourth $= ar^3$, and so on;

and thus the n^{th} term $= ar^{n-1}$;

or if n be the number of terms, and l the last term,

$$l = ar^{n-1}.$$

139. To find the sum of a geometric series,

$$S = a + ar + ar^2 + ar^3 + \&c. + ar^{n-2} + ar^{n-1};$$

$$\therefore \; Sr = ar + ar^2 + ar^3 + \&c. + ar^{n-2} + ar^{n-1} + ar^n,$$

by multiplying the upper line by r; now subtract;

$$\therefore \; S - Sr = a - ar^n; \quad \therefore \; Sr - S = ar^n - a = a(r^n - 1);$$

$$\therefore \quad S = a \cdot \frac{r^n - 1}{r - 1}.$$

This expression, with $l = ar^{n-1}$, will be sufficient to give the mathematical solution of all questions respecting geometric series; but when n is a high number, the computation of r^n by ordinary multiplication is almost impracticable; and recourse must be had to other methods which we shall hereafter explain.

Ex. Find the sum of $1 + 2 + 2^2 + 2^3 + \&c.$ to 10 terms.

Here $a = 1$; $r = 2$; $n = 10$;

$$S = \frac{2^{10} - 1}{2 - 1} = 2^{10} - 1 = 1024 - 1 = 1023;$$

$$l = 2^9 = 512.$$

140. When r is a fraction less than unity, and $\therefore r^n < 1$, each term of the series will be less than that which immediately precedes it; and as n increases, the value of the corresponding term will decrease; and when the number of terms is very great we may obtain a more convenient expression for the sum of such a series than the one in the preceding article. For $\because r$ and r^n are < 1;

$$\therefore S = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r},$$

and n being supposed very great, r^n becomes a fraction with a very large denominator, and when reduced to a decimal, the decimal point is succeeded by many cyphers; the value therefore of $\frac{ar^n}{1-r}$ becomes almost inappreciable; but when n is indefinitely increased, $\frac{ar^n}{1-r}$ is indefinitely decreased and has no sensible value; and thus the value of the series, $a + ar + ar^2 + ar^3 +$ &c. continued *ad infinitum*, when r is a proper fraction, is expressed by $\frac{a}{1-r}$; or representing such a series by Σ; $\Sigma = \frac{a}{1-r}$.

Ex. Find the value of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c., to infinity.

$$\text{Here } a = \tfrac{1}{2},\ r = \tfrac{1}{2},\ 1 - r = \tfrac{1}{2};\ \therefore \Sigma = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

The value Σ is the limit to which the series constantly tends, but to which it can never be said to be exactly equal, but from which it differs only by a quantity insensible to calculation. We may shew what the error is in the above series when we take Σ to be equal to the sum of 1000 terms; for

$$S = \Sigma - \frac{\frac{1}{2} \cdot \frac{1}{2^{1000}}}{\frac{1}{2}} = \Sigma - \frac{1}{(2)^{1000}};$$

i. e. S differs from Σ by a fraction whose denominator consists of more than 300 figures, or by a decimal the first significant figure of which is preceded by at least 300 cyphers; and as when n is greater, the error is less, enough has been said to shew that the formula $\Sigma = \frac{a}{1-r}$ may represent the sum of an infinite geometric series with sufficient exactness.

141. To find a geometric mean between two quantities a and c; let m be the mean; therefore a, m, and c are in geometrical progression;

$$\therefore \frac{m}{a} = \frac{c}{m}; \qquad \therefore m^2 = ac; \qquad \therefore m = \sqrt{ac}.$$

Also $\because \frac{m}{a} = \frac{c}{m}$; $\qquad \therefore \frac{a}{m} = \frac{m}{c}$; $\qquad \therefore a : m :: m : c$;

or quantities in geometrical progression are in continued proportion.

142. An arithmetic mean is greater than a geometric.

For if so, then $\frac{a+c}{2} > \sqrt{ac}$; $\quad \therefore a^2 + 2ac + c^2 > 4ac$;

$$\therefore a^2 - 2ac + c^2 > 0; \text{ or } (a-c)^2 > 0;$$

which it is, whatever be the magnitude of a or c.

143. To insert p geometric means between a and l; this is to find $p+2$ quantities in geometrical progression; a and l being respectively the first and last terms; we must therefore find r.

Now $l = ar^{n-1}$; and $n = p + 2$; $\therefore n - 1 = p + 1$

$$\therefore l = ar^{p+1}; \quad \therefore r^{p+1} = \frac{l}{a}; \quad \therefore r = \left(\frac{l}{a}\right)^{\frac{1}{p+1}}.$$

Ex. Find 4 means between 2 and 64.

$$a = 2; \quad l = 64; \quad p = 4;$$

$$\therefore r = \sqrt[5]{\frac{64}{2}} = \sqrt[5]{32} = 2, \text{ hence } ar, ar^2, ar^3, ar^4,$$

or 4, 8, 16, 32 are the means.

144. Recurring decimals are instances of geometrical progression *ad infinitum*, and may be solved by the rules previously given.

(Ex. 1.) Find the fraction which is equivalent to .333, &c. to infinity.

$$S = .333 \ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \&c.,$$

which compared with $a + ar + ar^2 + \&c.$, gives $a = \frac{3}{10}$, $r = \frac{1}{10}$,

$$\text{and } S = \frac{a}{1 - r} = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}.$$

(Ex. 2.) Let $S = .25313131$, &c., multiplying by 100, to separate the non-recurring from the recurring decimal,

$$100\, S = 25.3131 \ldots \&c. = 25 + s;$$

$$\therefore s = \frac{31}{100} + \frac{31}{10000} + \&c. = \frac{31}{100} \frac{1}{1 - \frac{1}{100}} = \frac{31}{99},$$

$$100\, S = 25 + \frac{31}{99} = \frac{2506}{99}; \quad \therefore S = \frac{2506}{9900}.$$

(Ex. 3.) Let $S = .abbb$ &c. where a contains m places of decimals, and b, n places;

$$\therefore\ 10^m S = a.bbb = a + \frac{b}{10^n} + \frac{b}{10^{2n}} + \&c.$$

$$= a + \frac{b}{10^n} \cdot \frac{1}{1 - \frac{1}{10^n}} = a + \frac{b}{10^n - 1};$$

$$\therefore\ S = \frac{a}{10^m} + \frac{b}{10^m(10^n - 1)} = \frac{a\,.\,10^n + b - a}{10^m(10^n - 1)}.$$

Thus if $a = 25$, $b = 31$, $m = 2$, $n = 2$,

$$S = \frac{25 \times 10^2 + 31 - 25}{10^2(10^2 - 1)} = \frac{2500 + 6}{100 \times 99} = \frac{2506}{9900}.$$

145. To find the sum of $\frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} + \frac{a+3b}{r^4} + \&c.$ the numerators of all the fractions being in arithmetic, and the denominators in geometric progression.

$$\text{Let } S = \frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} + \&c. + \frac{a+(n-2)b}{r^{n-1}} + \frac{a+(n-1)b}{r^n};$$

$$\therefore\ \frac{S}{r} = \qquad \frac{a}{r^2} + \frac{a+b}{r^3} + \&c. + \frac{a+(n-2)b}{r^n} + \frac{a+(n-1)b}{r^{n+1}}.$$

$$\therefore\ S - \frac{S}{r} = \frac{a}{r} + \left(\frac{b}{r^2} + \frac{b}{r^3} + \&c. + \frac{b}{r^n}\right) - \frac{a+(n-1)b}{r^{n+1}}.$$

Now to find $\frac{b}{r^2} + \frac{b}{r^3} + \frac{b}{r^4} + \&c. + \frac{b}{r^n}$; this is a geometric series of which the first term $= \frac{b}{r^2}$, $\frac{1}{r}$ is the common ratio and $n - 1$ the number of terms; therefore its sum is

$$\frac{b}{r^2} \cdot \left(\frac{\frac{1}{r^{n-1}} - 1}{\frac{1}{r} - 1}\right) = \frac{b}{r^n} \cdot \left(\frac{r^{n-1} - 1}{r - 1}\right);$$

$$\therefore\ S\cdot\left(\frac{r-1}{r}\right)=\frac{a}{r}+\frac{b}{r^n}\cdot\left(\frac{r^{n-1}-1}{r-1}\right)-\frac{a+(n-1)b}{r^{n+1}};$$

$$\therefore\ S=\frac{a}{r-1}+\frac{b}{r^{n-1}}\cdot\left\{\frac{r^{n-1}-1}{(r-1)^2}\right\}-\frac{a+(n-1)b}{r^n(r-1)}.$$

COR. 1. To find the same when r is a proper fraction and n infinitely great.

$$S=\frac{a}{r-1}+\frac{b}{(r-1)^2}-\frac{b}{r^{n-1}}\cdot\frac{1}{(r-1)^2}-\frac{a+(n-1)b}{r^n(r-1)};$$

and the two latter fractions being omitted since they are inappreciable;

$$\therefore\ \Sigma=\frac{a}{r-1}+\frac{b}{(r-1)^2}.$$

COR. 2. If a be the first and $\dfrac{a+b}{r}$ the second term, we must multiply Σ by r for the sum of the series, which therefore is

$$\frac{ar}{r-1}+\frac{br}{(r-1)^2}.$$

(Ex. 1.) Find the sum of $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+$ &c. to infinity.

Here $a=1,\ \ b=1,\ \ r=2;$

$$\therefore\ \Sigma=\frac{a}{r-1}+\frac{b}{(r-1)^2}=\frac{1}{1}+\frac{1}{1}=2.$$

(Ex. 2.) Sum $1+2r+3r^2+4r^3+$ &c. to infinity,

Here taking the second expression and writing $\frac{1}{r}$ for r;

$$\because\ a=1,\ \ b=2;$$

$$\therefore\ \Sigma=\frac{\frac{1}{r}}{\frac{1}{r}-1}+\frac{\frac{1}{r}}{\left(\frac{1}{r}-1\right)^2}=\frac{1}{1-r}+\frac{r}{(1-r)^2}=\frac{1}{(1-r)^2};$$

or better thus:

$$\Sigma = 1 + 2r + 3r^2 + 4r^3 + \&c.$$

$$\therefore\ \Sigma r = \quad + r + 2r^2 + 3r^3 + \&c.$$

$$\therefore\ \Sigma - \Sigma r = 1 + r + r^2 + r^3 + \&c. = \frac{1}{1-r};\ \therefore\ \Sigma = \frac{1}{(1-r)^2}.$$

(Ex. 3.) Sum $1 + \frac{3}{r} + \frac{5}{r^2} + \frac{7}{r^3} + \&c.$

$$\Sigma = 1 + \frac{3}{r} + \frac{5}{r^2} + \frac{7}{r^3} + \&c. \text{ to infinity.}$$

$$\frac{\Sigma}{r} = \quad \frac{1}{r} + \frac{3}{r^2} + \frac{5}{r^3} + \&c.$$

$$\therefore\ \Sigma\left(1 - \frac{1}{r}\right) = 1 + \frac{2}{r} + \frac{2}{r^2} + \frac{2}{r^3} + \&c. = 1 + 2.\left(\frac{\frac{1}{r}}{1 - \frac{1}{r}}\right)$$

$$= 1 + \frac{2}{r-1} = \frac{1+r}{r-1};$$

$$\therefore\ \Sigma = \frac{(1+r)r}{(r-1)^2}.$$

EXAMPLES.

(1) The sum of 1 + 2 + 4 + &c. to 12 terms = 4095.

(2) .... of 1 + 4 + 16 + &c. to 8 ... = 21,845.

(3) .... of 1 + 3 + 9 + &c. to 7 ... = 1093.

(4) .... of 4 + 12 + 36 + &c. to 10 ... = 118096.

(5) .... of 9 + 6 + 4 + &c. to 8 ... = $25\frac{230}{243}$.

(6) .... of 5 + 20 + 80 + &c. to 8 ... = 109,225.

(7) The sum of $\frac{2}{3} + \frac{1}{3} + \frac{1}{6}$ + &c. to infinity $= \frac{4}{3}$.

(8) .... of $\frac{2}{3} - \frac{1}{3} + \frac{1}{6}$ − &c. .... $= \frac{4}{9}$.

(9) .... of $9 + 6 + 4$ + &c. .... $= 27$.

(10) .... of $6 + 2 + \frac{2}{3}$ + &c. .... $= 9$.

(11) .... of $\frac{1}{4} - \frac{1}{8} + \frac{1}{16}$ − &c. .... $= \frac{1}{6}$.

(12) .... of $\frac{3}{2} + 1 + \frac{2}{3}$ + &c. .... $= \frac{9}{2}$.

(13) .... of $100 + 40 + 16$ + &c. ...... $= 166\frac{2}{3}$.

(14) .... of $a + b + \frac{b^2}{a}$ + &c. .... $= \frac{a^2}{a - b}$.

(15) Insert 3 geometric means between 4 and 64.
Ans. 8, 16, 32.

(16) Insert 4 between $\frac{1}{3}$ and 81. Ans. 1, 3, 9, 27.

(17) The difference between two numbers = 12; and the arithmetic : to the geometric mean :: 5 : 4; find the numbers. Ans. 4 and 16.

(18) The sum of a series to infinity is 2, and the sum of the squares of the terms of the same series is $\frac{4}{3}$; find a and r. Ans. $a = 1$, $r = \frac{1}{2}$.

(19) If $S = 1 + R + R^2$ + &c. to infinity, and

$$s = 1 + r + r^2 + r^3 + \text{\&c. to infinity};$$

find the sum of $1 + Rr + R^2r^2$ + &c. to infinity.

Ans. $\frac{Ss}{S + s - 1}$.

(20) If P be the product, S the sum, and S_1 the sum of the reciprocals of n quantities in geometrical progression; prove that $P^2 = \left(\frac{S}{S_1}\right)^n$.

HARMONIC PROGRESSION.

146. Quantities are said to be in harmonic progression when any 3 consecutive terms being taken; the first is to the third as the difference between the first and second is to the difference between the second and the third.

Thus, if a, b, c be the consecutive terms in a series, then if $a : c :: a - b : b - c$, a, b, c, are in harmonical progression.

147. To find an harmonic mean between two quantities a and c.

Let H be the mean, then a, H and c are in harmonic progression;

$$\therefore a : c :: a - H : H - c;$$

$$\therefore aH - ac = ac - aH; \quad \therefore H = \frac{2ac}{a + c}.$$

Cor. Hence if G equal a geometric mean, and A an arithmetic mean between the same quantities; then

$$\because G = \sqrt{ac} \text{ and } \frac{a + c}{2} = A; \quad \therefore a + c = 2A;$$

$$\therefore H = \frac{2G^2}{2A} = \frac{G^2}{A} \text{ or } A . H = G^2;$$

$$\therefore A : G :: G : H.$$

Cor. Hence since A is $>$ G, G is $>$ H.

148. If quantities be in harmonical progression, their reciprocals are in arithmetical progression.

Let a, b, c, d be in harmonical progression;

$$\therefore \; a : c :: a-b : b-c;$$

$$\therefore \; a.(b-c) = c.(a-b);$$

or dividing both sides by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}; \quad \therefore \; \frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{c};$$

and in the same manner since b, c, d are in harmonical progression,

$$\frac{1}{b} - \frac{1}{c} = \frac{1}{c} - \frac{1}{d};$$

and therefore the differences between the consecutive numbers

$$\frac{1}{a}, \; \frac{1}{b}, \; \frac{1}{c}, \; \frac{1}{d},$$

being equal, these numbers are in arithmetical progression.

Hence, to insert p harmonic means between a and l, we must insert p arithmetic means between $\frac{1}{a}$ and $\frac{1}{l}$. There is no method by which the sum of an harmonic series can be found.

Prob. There are four numbers, the first three in arithmetical, and the last three in harmonical progression; prove that 1st : 2nd :: 3rd : 4th.

Let a, b, c, d be the numbers.

Then $\because a, b, c$ are in arith. prog.; $2b = a + c$;

and $\because b, c, d$ are in harm. prog. $c = \frac{2bd}{b+d}$;

$$\therefore \; c = \frac{(a+c)d}{b+d}; \quad \therefore \; cb + cd = ad + cd;$$

$$\therefore \; ad = bc; \text{ and } \therefore \; a : b :: c : d.$$

EXAMPLES.

(1) Find an harmonic mean between 6 and 12.
Ans. 8.

(2) Insert two between 2 and 5. Ans. $\frac{1}{2}$, $\frac{2}{7}$.

(3) An arithmetic mean between two numbers : geometric :: 5 : 4, and the difference between the geometric and harmonic means $= \frac{6}{5}$; find the number. Ans. 3 and 12.

(4) What is the 4th term of the harmonic series 2, 3, 6.
Ans. infinity.

(5) The 5th term of an harmonic series is $\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the intermediate terms.

Ans. $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$.

(6) The sum of three terms of an harmonic series is 11, and the sum of their squares is 49; find the numbers.

Ans. 2, 3, 6.

PROBLEMS IN ARITHMETIC AND GEOMETRIC PROGRESSION.

149. (Ex. 1.) The sum of five numbers in arithmetic progression is 30 and the sum of their squares 220; find the numbers.

Let $x - 2y$, $x - y$, x, $x + y$, $x + 2y$ be the numbers;

$$\therefore 5x = 30; \quad \therefore x = 6 \text{ the middle number,}$$

$$\text{and } (x - 2y)^2 + (x - y)^2 + x^2 + (x + y)^2 + (x + 2y)^2 = 220;$$

$$\therefore 5x^2 + 10y^2 = 220; \quad \therefore x^2 + 2y^2 = 44;$$

$$\therefore 2y^2 = 8, \; y = \pm 2, \text{ and the numbers are}$$

$$6 \mp 4, \; 6 \mp 2, \; 6, \; 6 \pm 2, \; 6 \pm 4 \text{ or } 2, 4, 6, 8, 10.$$

(Ex. 2.) There are four numbers in arithmetic progression, and the sum of the squares of the extremes is 101, and of the means 65, find them.

Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers;

$$\therefore (x-3y)^2+(x+3y)^2=2x^2+18y^2=101,$$

$$(x-y)^2+(x+y)^2=2x^2+2y^2=65;$$

$$\therefore 16y^2=36; \quad \therefore y=\pm\frac{3}{2}; \quad \therefore x^2=\frac{121}{4}; \quad \therefore x=\frac{11}{2};$$

$\therefore$ the numbers are 1, 4, 7, 10.

Observe when there is an odd number of terms in arithmetic progression, the common difference must be y, and the middle term $=x$; but when there is an even number, the common difference must be $2y$, and the two middle terms, $x-y$, and $x+y$.

(Ex. 3.) There are 4 numbers in geometrical progression, the sum of the extremes is 18, the sum of the means is 12; find the numbers.

Let x, xy, xy^2, xy^3, be the members;

$$\therefore x+xy^3=18, \text{ and } xy+xy^2=12.$$

Dividing one equation by the other;

$$\therefore \frac{1+y^3}{y+y^2}=\frac{18}{12}; \quad \therefore \frac{1-y+y^2}{y}=\frac{3}{2};$$

$$\therefore 2y^2-5y=-2; \quad \therefore y=2; \quad \therefore x+8x=18; \quad \therefore x=2,$$

and the numbers are 2, 4, 8, 16.

(Ex. 4.) There are 4 numbers in arithmetical progression, which being increased by 2, 4, 8, and 15 respectively, the sums shall be in geometrical progression.

Let x, xy, xy^2, xy^3, be the numbers when increased;

$\therefore$ $x-2$, $xy-4$, xy^2-8, xy^3-15 are in arithmetical progression;

$$\therefore \text{1st} + \text{3rd} = 2 \times \text{2nd}; \text{ and } \text{2nd} + \text{4th} = 2 \times \text{3rd};$$

$$\therefore x + xy^2 - 10 = 2xy - 8; \quad \therefore x - 2xy + xy^2 = 2;$$

$$\therefore xy + xy^3 - 19 = 2xy^2 - 16; \quad \therefore xy - 2xy^2 + xy^3 = 3;$$

$$\therefore xy(1 - 2y + y^2) = 3, \ (1); \quad x(1 - 2y + y^2) = 2; \ (2)$$

$$(1) \div (2) \quad y = \frac{3}{2}, \text{ and } x\left(1 - 3 + \frac{9}{4}\right) = 2;$$

$$\therefore x = 8; \ xy = 12; \ xy^2 = 18; \ xy^3 = 27;$$

and subtracting 2, 4, 8, and 15 from these numbers, the remainders, 6, 8, 10, 12, are the numbers required.

EXAMPLES.

(1) The sum of 3 numbers in arithmetical progression is 30, and the sum of their squares is 308; find the numbers.

Ans. 8, 10, 12.

(2) There are 4 numbers in arithmetical progression, their sum is 24, and their product 945; find the numbers.

Ans. 3, 5, 7, 9.

(3) There are 3 numbers in geometrical progression whose sum is 31; and the sum of the 1st and 2nd : sum of 1st and 3rd :: 3 : 13; find them. Ans. 1, 5, 25.

(4) A traveller starts from a town, and travels 1 mile the first day, 2 the second, 3 the third, and so on; five days after another traveller leaves the same place, by the same road, and travels 12 miles a day. On what day will he overtake the first traveller?

Ans. 8th day, and on the 15th they will again be together.

(5) There are 3 numbers in arithmetical progression whose sum is 18; but if you multiply the first term by 2, the second by 3, and the third by 6, the products will be in geometrical progression; find them. Ans. 3, 6, 9.

(6) The sum of the 4th powers of 3 successive numbers is 353; find the numbers. Ans. 2, 3, 4.

(7) There are 4 numbers in arithmetical progression, their common difference is unity, and their product 360; find the numbers. Ans. 3, 4, 5, 6.

(8) The sum of 9 numbers in arithmetical progression is 45, and the sum of their squares is 285; find the numbers.

Ans. the first 9 numbers.

(9) There are 4 numbers in geometrical progression, the sum of the first and third is 10, the sum of the second and fourth is 30; find them. Ans. 1, 3, 9, 27.

(10) Find 3 numbers in geometrical progression whose sum is 7 and the sum of their cubes is 73. Ans. 1, 2, 4.

CHAPTER VII.

PERMUTATIONS AND COMBINATIONS.

150. The different arrangements that can be made of any number of quantities are called their Permutations.

Thus *ab*, and *ba* are the Permutations of *a* and *b*, and *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, are those of *abc*.

Instead of taking all the letters at once, let a certain number only be taken, then such a permutation is called a variation; thus of the letters *a*, *b*, *c*, the variations taken two and two together are, *ab*, *ba*, *ac*, *ca*, *bc*, *cb*.

151. To find the number of variations of n things taken 2 and 2 together.

If *a*, *b*, *c*, *d*, &c., be the n things; first write the $n-1$ things *b*, *c*, *d*, &c., by themselves, and then place *a* before each, in the following manner:

ab, *ac*, *ad*, *ae*, *af*, &c.

and we shall obviously have $(n-1)$ variations where *a* stands first; then write *a*, *c*, *d*, *e*, &c., by themselves, and afterwards place *b* before each letter, and there will be $(n-1)$ variations, where *b* stands first; and so on for each letter, and as there are n letters, each of which may stand first, so will there be n sets of variations of $(n-1)$ pairs of letters, or there will be $n.(n-1)$ variations; i. e. if V_2 represent the variations of n things taken 2 and 2 together,

$$V_2 = n.(n-1).$$

Next, to find the variations of n things taken 3 and 3 together; first leave out *a*; then the number of variations of the remaining $(n-1)$ things, taken 2 and 2 together, will be $(n-1).(n-2)$; if now a be placed before each of these, there will be $(n-1).(n-2)$ variations of things taken 3 and 3 together, where a stands first; so also will there be $(n-1).(n-2)$ variations of things taken 3 and 3 together, where b stands first; and so on for all the n letters, and therefore the whole number of variations or $V_3 = n(n-1)(n-2)$.

Hence it will readily appear that if V_4, V_5, &c. be the variations taken 4 and 4, 5 and 5 together, that

$$V_4 = n(n-1)(n-2)(n-3),$$

$$V_5 = n(n-1)(n-2)(n-3)(n-4);$$

and ∴ if the things be taken r and r together,

$$V_r = n(n-1)(n-2).(n-3)\ldots\{n-(r-1)\};$$

for we see that the negative number in the last factor is less by unity than the number of things taken together.

Cor. Hence we may find the permutations of n things; for these are but the variations of n things taken, all, i.e. n and n together;

making $r = n$, the last factor is $n-(n-1) = 1$,

and $V_n = n(n-1)(n-2)\ldots 3.2.1$;

or the number of permutations (P) of n things is equal to the product of the first n digits.

Ex. Find the number of permutations of the 6 letters a, b, c, d, e, f;

$$\text{number} = 6.5.4.3.2.1 = 720.$$

152. In the preceding remarks we have supposed that each letter or quantity is different; but when some are the same, the result in the last article requires modification.

Thus, as the permutations of a and b are ab, ba when a and b are different; but they become aa, i.e. one only when $b = a$; so the total number of variations found on the supposition of each quantity being different, must be divided by 2, when two of them, as a, b, become equal; for as in every arrangement of the letters there was originally found, both a and b, when these become equal every term must be twice repeated.

And again since, abc may be permuted six or 2×3 different ways, so long as a, b, c are different, but only in one way when $a = b = c$; we must divide V_n by $2 . 3$, in order to find the permutations of n things three of which are equal; or

$$P = \frac{n . (n-1)(n-2) \ldots 3 . 2 . 1}{2 . 3} = n . (n-1)(n-2) \ldots 4.$$

Next, if r of the letters be equal, and if P be the number on this supposition; then since, if the letters had been all different, the number of permutations would have been

$$n(n-1)(n-2) \ldots (3 . 2 . 1);$$

and because the r letters, if different, would have formed $1 . 2 . 3 \ldots r$ permutations, and therefore combined with P, would have formed

$$1 . 2 . 3 \ldots r \times P \text{ permutations; hence}$$

$$1 . 2 . 3 \ldots r \times P = n(n-1)(n-2) \ldots 3 . 2 . 1;$$

$$\therefore P = \frac{n(n-1)(n-2) \ldots 3 . 2 . 1}{1 . 2 . 3 \ldots r}.$$

COR. Had there been q letters equal, and also r letters equal; then we should similarly find that

$$P = \frac{n(n-1)(n-2) \ldots 3 . 2 . 1}{1 . 2 \ldots p . 1 . 2 \ldots q . 1 . 2 \ldots r \ldots}.$$

153. When a letter can be repeated, i.e. when a can stand before a, the permutations are styled variations with

repetitions; thus of the letters *a, b, c,* if we have another *a* to stand before each of the letters *a, b, c,* we have *aa, ab, ac,* or 3 variations, where *a* stands first; and, if we have another *b* and *c,* we shall have 3 variations where *b* stands first, and 3 where *c* stands first; and there will be 3×3 or 3^2 variations of such things taken 2 and 2 together.

So if there be *n* things *a, b, c,* &c. taken 2 and 2 together where repetitions are allowed; there will be *n* variations where *a* is first, *n*, where *b* is first, and so on for each of the *n* letters; and therefore the whole number of variations $= n \times n = n^2$.

And, if the letters be taken 3 and 3 together, and repetition be allowed, as there are n^2 variations when taken 2 and 2, and each letter may be placed before each variation of 2 and 2; the whole number taken 3 and 3 will be $n \times n^2 = n^3$.

And, if they be taken 4 and 4 there will be *n* times as many variations as when the same letters are taken 3 and 3, i.e. there will be $n \times n^3 = n^4$ variations; and if they be taken *n* and *n* together, there will be n^n variations.

Hence, if it be required to find the total number of such variations, when taken 1 and 1, 2 and 2, 3 and 3, &c. *n* and *n* together, the sum

$$= n + n^2 + n^3 + n^4 + \&c. + n^n = n\left(\frac{n^n - 1}{n - 1}\right).$$

Ex. How many throws can be made with two dice; the number equal the number of variations with repetitions that can be made with the 6 numbers 1, 2, 3, 4, 5, 6; $= 6 \times 6 = 36$.

It is supposed that 1, 6, and 6, 1 are different throws.

Def. The different collections that can be made of any number of things without regard to their order, are called their combinations.

Then ab, ba which form two variations, make but one combination; and abc, which may be permuted 6 different ways, is but one combination. And if there be n things, their variations are $n(n-1)$; but as each combination, as ab, contains two variations, therefore the combinations taken

$$2 \text{ and } 2 \text{ or } C_2 = \frac{n(n-1)}{2}.$$

Again, if they be taken 3 and 3 together, as each combination a, b, c contains 2.3 variations; and the variations are $n(n-1)(n-2)$;

$$\therefore\ C_3 = \text{the number of combinations} = \frac{n(n-1)(n-2)}{2.3}.$$

And C_r the number of combinations taken r and r together, will be, following the same reasoning,

$$= \frac{n(n-1)(n-2)\ldots\{n-(r-1)\}}{1\ .\ 2\ .\ 3\ \ldots\ r}.$$

The following examples will illustrate the preceding theory.

(Ex. 1.) Find the number of permutations that can be made out of the letters of the word Algebra.

Had the letters been different the number $= 7.6.5.4.3.2$,

but there are 2 a's, and $\therefore$ divide by 1×2,

and the number $= 7.6.5.4.3 = 2520$.

(Ex. 2.) In how many ways can we write the term $a^3b^4c^2$?

There are 9 letters; and 3 a's, 4 b's and 2 c's;

$$\therefore \text{number} = \frac{9.8.7.6.5.4.3.2}{2.3 \times 2.3 \times 4 \times 2} = 1260.$$

(Ex. 3.) In how many terms will a^3 stand first? the number will be equal to the number of permutations of

$$b^4c^2 = \frac{6.5.4.3.2}{2.3.4 \times 2} = 15.$$

EXAMPLES.

(1) How many changes can be rung with 5 bells out of 8? Ans. 6720.

(2) How often can 8 persons change their places at dinner so as not to preserve the same order? Ans. 40320.

(3) In how many different ways can the letters in the algebraic expression $a^3 b^7 c^4 d$ be written? Ans. 1801800.

(4) In the permutations of a, b, c, d, e, f, g, find how many begin with cd. Ans. 120.

(5) How many different throws can be made with 6 dice? Ans. 46656.

(6) In how many ways may the letters of the words Calculus and Institution be written? Ans. 5040; and 554400.

(7) There are 4 companies of soldiers, in each of which there are 12 men; in how many ways may 12 men be chosen, one being selected out of each company? Ans. 20736.

(8) Into how many different triangles may a decagon be divided by drawing lines from the angular points?

Ans. 120.

(9) Find the permutations of the letters in the word Proposition. Ans. 1663200.

(10) There are 4 sets of different things, one containing 4, another 6, the third 8, and the fourth 10; how many different combinations can be formed of them, taking 4 together? Ans. 1920.

(11) The whole number of combinations of n things taken 1 and 1, 2 and 2, 3 and 3, &c., together $= 2^n - 1$.

(12) The total number of combinations of $2n$ things : total number of n things :: 129 : 1; find n. Ans. 7.

CHAPTER VIII.

THE BINOMIAL THEOREM.

154. In the article on Involution, the successive powers of the binomial $a \pm b$ have been found, by means of ordinary multiplication. There is, however, a theorem called the Binomial Theorem, by the aid of which, the mechanical labour of multiplication may in great measure be got rid of, and the expansion of $(a+b)^n$ be written down at once; in fact we shall see that

$$(a+b)^n = a^n + n a^{n-1} b + n\frac{(n-1)}{2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} b^3 + \&c.$$

But before we proceed to prove the truth of this theorem, it is necessary to prove the following proposition.

If $a + bx + cx^2 + dx^3 + \&c. = A + Bx + Cx^2 + Dx^3 + \&c.$ for every possible value of x; then shall $a = A$, $b = B$, $c = C$, &c., or the coefficients of like powers of x shall be equal.

For as any value may be put for x, let $x = 0$;

$\therefore\ a = A$; and taking a and A from the original equation,

$$bx + cx^2 + dx^3 + \&c. = Bx + Cx^2 + Dx^3 + \&c.;$$

$$\text{or } b + cx + dx^2 + \&c. = B + Cx + Dx^2 + \&c.;$$

$$\therefore \text{ if } x = 0;\ b = B;\ \text{and then } c = C;\ d = D, \&c.$$

155. This theorem is exceedingly useful in many algebraical operations; particularly in finding the terms of an infinite quotient: this is done by assuming a quotient with unknown coefficients, and hence the method is called that of Indeterminate Coefficients.

Ex. Find the first four terms of the quotient of

$$\frac{1-3x+2x^2}{1+x+x^2}.$$

Let $\dfrac{1-3x+2x^2}{1+x+x^2} = a+bx+cx^2+dx^3+\&c.;$

then multiplying both sides by $1+x+x^2$,

$$\begin{aligned}1-3x+2x^2 = a + bx + cx^2 + dx^3 + \&c.\\ + ax + bx^2 + cx^3 + \&c.\\ + ax^2 + bx^3 + \&c.\end{aligned}$$

Hence equating the coefficients of the like powers of x,

$a=1$; $b+a=-3$; $\therefore b=-4$; $c+b+a=2$; $\therefore c=5$;

$$d+b+c=0; \quad \therefore d=-b-c=-1;$$

$$\therefore \frac{1-3x+2x^2}{1+x+x^2} = 1-4x+5x^2-x^3+\&c.$$

(Ex. 2.) If $x=y-y^2$ find y in terms of x.

$$\text{Let } y = ax+bx^2+cx^3+dx^4+\&c.;$$

$$\therefore y^2 = a^2x^2+2abx^3+(b^2+2ac)x^4+\&c.;$$

$$\begin{aligned}\therefore x = y-y^2 = ax+(b-a^2)x^2+(c-2ab)x^3\\ +(d-b^2-2ac)x^4+\&c.;\end{aligned}$$

$\therefore a=1$, $b-a^2=0$; $\therefore b=a^2=1$, $c-2ab=0$; $\therefore c=2$,

$$d-b^2-2ac=0; \quad \therefore d=b^2+2ac=5;$$

$$\therefore y = x+x^2+x^3+5x^4+\&c.$$

156. If the simple equation $a+bx=A+Bx$ be true for every value of x; we may shew that $a=A$ and $b=B$, by a method less liable to objection than the one just used for the general equation.

For since any value may be put for x,

$$\text{let } x = m; \quad \therefore\ a + bm = A + Bm;$$

subtract this from the original equation;

$$\therefore\ b(x - m) = B.(x - m); \quad \therefore\ b = B, \text{ and } \therefore\ a = A.$$

In the same manner, if $a + bx + cx^2 = A + Bx + Cx^2$, we may, by taking two values for x, shew that $a = A$, $b = B$, $c = C$.

157. The equation $a + bx = A + Bx$ just considered, differs greatly from those treated in the chapter on equations; there x is supposed to be a determinate, here an indeterminate quantity; in the former case it is dependent upon the constants of the equation and the coefficients; in the latter it is altogether independent of them; thus if x is to be found, we have $x = \dfrac{A - a}{b - B}$.

But if x be any thing whatever, i.e. is indeterminate, $A = a$, $b = B$, and $x = \dfrac{0}{0}$; a singular result, and which we see is in this case the symbol of an indeterminate quantity.

Sometimes x equal to a fraction $\dfrac{P}{Q}$ becomes $\dfrac{0}{0}$, when a particular value is put for the unknown quantity in the numerator and denominator; such a fraction is called a vanishing fraction; but the true value of x may be found from the fraction, as we shall see hereafter.

And here we may mention two other values $\dfrac{0}{a}$, and $\dfrac{a}{0}$, which are of frequent occurrence.

Since $\dfrac{b}{a} = b \times \dfrac{1}{a}$; if $b = 0$; $\dfrac{0}{a} = 0 \times \dfrac{1}{a} = \dfrac{1}{a} \times 0$, which by the principles of multiplication $= 0$.

Next $\dfrac{a}{0}$ is to be explained; we know that the value of a fraction as $\dfrac{a}{b}$ depends upon the relative and not upon the absolute values of a and b. That if a contain b many times, the value of the fraction will be great, although a be not a large quantity. Thus if a be a foot and b the $\frac{1}{1000}$th part, $\frac{a}{b}=\frac{1000}{1}$; and if b be the $\frac{1}{100,000}$th part of a foot, $\frac{a}{b}=\frac{100,000}{1}$. Hence also we may see that the value of a fraction increases, as its denominator decreases, and becomes infinitely great when the denominator is infinitely small; hence as 0 may represent the infinitely small state of a quantity, $\frac{a}{0}$ will be equal to an infinitely large quantity, or $\frac{a}{0}=\infty$.

158. We shall now proceed to establish the truth of the Binomial Theorem, or to prove that

$$(a+b)^n = a^n + na^{n-1}b + n\frac{(n-1)}{2}a^{n-2}b^2$$
$$+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3 + \&c.,$$

whatever be the value of n.

But first $\because\ a+b=a\left(1+\frac{b}{a}\right)$;

$$\therefore\ (a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n(1+x)^n,$$

where $x=\frac{b}{a}$;

hence if we can prove the truth of the expansion of $(1+x)^n$ we shall obtain that of $(a+b)^n$, by multiplying $(1+x)^n$ by a^n, and rewriting $\frac{b}{a}$ for x; this being allowed we shall attempt to shew that

$$(1+x)^n = 1 + nx + n\frac{(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}.x^3 + \&c.$$

The proof may be divided into two parts.

1°. To shew that $(1+x)^n = 1 + nx + \&c.$

2°. To find the general law of the coefficients.

(1°) To shew that the coefficient of the second term of the expansion of $(1+x)^n$ is n; whether n be integral or fractional, positive or negative.

Let the index be positive and integral; then since by multiplication we know that

$$(1+x)^2 = 1 + 2x + \&c.$$

$$(1+x)^3 = 1 + 3x + \&c.,$$

let us assume $(1+x)^{n-1} = 1 + (n-1)x + \&c.$;

$$\therefore (1+x)^n = \{1 + (n-1)x + \&c.\}(1+x) = 1 + nx + \&c.,$$

by multiplication.

Hence if the rule be true for any one index $n-1$, it is true for the next superior index n. Now by multiplication we find that it is true for the index 3, it is therefore true for $n=4$; therefore for $n=5$, and hence by continued inductions it is always true for n, integral and positive.

(2°) Let n be a fraction $= \frac{p}{q}$.

And let $(1+x)^{\frac{p}{q}} = 1 + ax + \&c. = 1 + Ax$, where Ax represents all the terms by which x is multiplied;

$$\therefore (1+x)^p = (1+Ax)^q;$$

$$\therefore 1 + px + \&c. = 1 + qAx + \&c. = 1 + x(qa + \&c.).$$

Hence equating coefficients of x, $p = qa$; $\therefore a = \frac{p}{q}$.

$$\text{And } (1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + \&c. \ldots$$

Lastly let n be negative; but then

$$(1+x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+\&c.}=1-nx+\&c. \text{ by division;}$$

$$\text{and } \therefore (1+x)^n=1+nx+\&c., \text{ whatever } (n) \text{ be.}$$

$$\text{Hence } \therefore (a+b)^n=a^n(1+nx+\&c.)=a^n\left(1+n\frac{b}{a}+\&c.\right)$$
$$=a^n+na^{n-1}b+\&c.,$$

and the first two terms of the series are determined.

Let $(1+x)^n=1+nx+A_2x^2+A_3x^3+A_4x^4+\&c.$, where A_2, A_3, A_4, &c. depend upon n.

For x put $x+z$;

$$\therefore (1+x+z)^n=1+n(x+z)+A_2(x+z)^2+A_3(x+z)^3$$
$$+A_4(x+z)^4+\&c.$$

$$\text{But } \because (a+b)^n=a^n+na^{n-1}b+\&c.;$$

$$\therefore (x+z)^2=x^2+2xz+\&c.$$
$$(x+z)^3=x^3+3x^2z+\&c.;$$

$$\therefore (1+x+z)^n=1+nx+A_2x^2+A_3x^3+A_4x^4+\&c.$$
$$+(n+2A_2x+3A_3x^2+4A_4x^3+\&c.)z+\&c.$$
$$=(1+x)^n+(n+2A_2x+3A_3x^2+\&c.)z+\&c. \quad (\alpha).$$

But $\because (1+x+z)^n=\{(1+x)+z\}^n$, considering $1+x$ as one term;

$$\therefore \{(1+x)+z\}^n=(1+x)^n+n(1+x)^{n-1}z+\&c. \quad (\beta);$$

$\therefore$ equating the coefficients of z in (α) and (β)

$$n+2A_2x+3A_3x^2+4A_4x^3+\&c.=n(1+x)^{n-1},$$

multiply both sides by $1+x$, and we have

$$\left.\begin{array}{l} n+2A_2x+3A_3x^2+4A_4x^3+\&c. \\ \quad +nx+2A_2x^2+3A_3x^3+\&c. \end{array}\right\} = n\,(1+x)^n$$

$$= n\,(1+nx+A_2x^2+A_3x^3+\&c.)$$

hence equating coefficients of the same powers of x,

$$2A_2+n=n^2;\quad \therefore\ 2A_2=n^2-n=n\,.\,(n-1);\quad \therefore\ A_2=n\frac{(n-1)}{2},$$

$$3A_3+2A_2=nA_2;\quad \therefore\ 3A_3=A_2\,(n-2);$$

$$\therefore\ A_3=A_2\frac{n-2}{3}=\frac{n(n-1)(n-2)}{2\,.\,3}.$$

Also $4A_4=nA_3-3A_3=A_3(n-3)$;

$$\therefore\ A_4=\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4},$$

and $5A_5=nA_4-4A_4=A_4(n-4)$;

$$\therefore\ A_5=A_4\frac{(n-4)}{5}.$$

And thus $A_r=A_{r-1}\left\{\dfrac{n-(r-1)}{r}\right\}=A_{r-1}\left(\dfrac{n+1-r}{r}\right)$;

$$\therefore\ (1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{2\,.\,3}x^3+\&c.$$

and $\therefore$ putting $\dfrac{b}{a}$ for x, $(a+b)^n$

$$=a^n\left\{1+n\,.\frac{b}{a}+\frac{n(n-1)}{2}\frac{b^2}{a^2}+\frac{n(n-1)(n-2)}{2\,.\,3}\,.\frac{b^3}{a^3}+\&c.\right\}$$

$$=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2\,.\,3}a^{n-3}b^3+\&c.,$$

which is the theorem required.

COR. 1. If $-b$ be put for b; then since the odd powers of $(-b)$ are negative, and the even powers positive,

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2$$

$$-\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}a^{n-3}b^3 + \&c.;$$

$$\therefore (a+b)^n + (a-b)^n$$

$$= 2\left\{a^n + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}a^{n-4}b^4 + \&c.\right\};$$

$$\text{and } (a+b)^n - (a-b)^n$$

$$= 2\left\{na^{n-1}b + \frac{n(n-1)(n-2)}{2\ .\ 3}a^{n-3}b^3 + \&c.\right\}$$

COR. 2. It may be observed that the sum of the indices of a and b in each term $= n$.

COR. 3. If n be positive and integral, the series will terminate and consist of $n+1$ terms.

For since $A_3 = A_2 \cdot \frac{n-2}{3}$ and $A_4 = A_3 \cdot \frac{n-3}{4}$, we see that each coefficient is derived from the preceding one, by multiplying by a factor which is less by unity than the least of the preceding factors, and by dividing by the number of terms; at length therefore there will be a factor $n-n$ or 0, and the coefficient $=0$ and the series will terminate; we may find the term thus;

$$\because A_r = A_{r-1}\left\{\frac{n-(r-1)}{r}\right\}; \text{ if } A_r = 0,$$

$n-(r-1)=0$, or $r=n+1$; $\therefore A_{n+1}=0$, and A_n is the last coefficient, and $A_n x^n$ the last term; hence, since A_r is assumed to be the coefficient of the $(1+r)^{\text{th}}$ term; therefore A_n is that of the $(1+n)^{\text{th}}$ term; or the number of terms $=1+n$.

159. To find the general term of the series.

Let this be the $(1+r)^{\text{th}}$ term, then its coefficient will be A_r and (r) will be the index of x; for this index is less by unity than the place of the term in the series; thus A_2x^2 is the third term; now we have seen that

$$A_2 = \frac{n(n-1)}{2};$$

$$A_3 = A_2\left(\frac{n-2}{3}\right);$$

$$A_4 = A_3\left(\frac{n-4}{4}\right);$$

$$A_5 = A_4\left(\frac{n-5}{5}\right)\ldots\ldots$$

$$\text{and } A_r = A_{r-1}\left(\frac{n-r+1}{r}\right);$$

$\therefore$ by multiplying these terms together,

$$A_r = \frac{n.(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{1\ .\ 2\ .\ 3\ .\ 4\ \ldots\ldots\ r};$$

$$\therefore (1+r)^{\text{th}} \text{ term of } (1+x)^n = A_r x^n = A_r\left(\frac{b}{a}\right)^r$$

$$= \frac{n(n-1)\ldots\ldots(n-r+1)}{1\ .\ 2\ \ldots\ldots\ r}\left(\frac{b}{a}\right)^r,$$

and $\therefore$ the general or $(1+r)^{\text{th}}$ term of $(a+b)^n = a^n \times A_r.x^r$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ r}.\,a^{n-r}b^r.$$

160. Hence to find, when n is positive and integral, the $(1+n)^{\text{th}}$ or last term, let $r=n$;

$\therefore\ n-r=0;\quad \therefore\ a^0=1,\ b^r=b^n,$ and $n-r+1=1,$

$$\text{and } A_n=\frac{n(n-1)(n-2)\ldots\ldots 1}{1\ .\ 2\ .\ 3\ \ldots\ldots n}=1;$$

$\therefore\ (1+n)^{\text{th}}$ term $=b^n$, the same as the first term, only b is written for a.

Again, the last term but one or the n^{th} term is found by putting $r=n-1$; $\therefore\ n-r+1=2$;

$$\therefore\ A_{n-1}=\frac{n(n-1)(n-2)\ldots\ldots 3.2}{1\ .\ 2\ .\ 3\ \ldots\ldots(n-1)}=n$$

$$= \text{coefficient of the second term,}$$

$$\text{and } A_{n-1}a^{n-r}b^r=nab^{n-1}.$$

In the same manner the coefficient of the last term but two, or $A_{n-2}=\frac{n(n-1)}{1.2}=A_2$, the coefficient of the third term and of the last term but three or $A_{n-3}=\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}=A_3$, that of the fourth term, and so on, or the coefficients of terms equidistant from each extremity of the series are equal.

Also we see that the last term but two $=\frac{n(n-1)}{2}a^2b^{n-2}$, and that this as well as the last term but one may be derived from the third and second terms by interchanging the letters a and b.

161. When n is a fraction the series does not terminate; for r being a whole number $n-(r-1)$ can never $=0$. Also if n be negative the series is infinite, for putting $-n$ for n, we have $A_r=-A_{r-1}\frac{n+r-1}{r}$ which cannot vanish; for $n+r$ being the sum of two whole numbers always exceeds unity.

162. Finally in making use of the general or $(1+r)^{\text{th}}$ term of a series, it may be observed that the number of terms in

the numerator, as also the number in the denominator, is equal to r, the index of b, which is also the number by which the first index of a has been diminished.

163. When n is even the number of terms is odd, and the middle term is the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term, which is

$$\frac{n(n-1)(n-2)(n-3)\ldots\left\{n-\left(\frac{n}{2}-1\right)\right\}}{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\frac{n}{2}}\,x^{\frac{n}{2}}$$

$$=\frac{n(n-1)(n-2)(n-3)\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\,.\,4\ldots\ldots\frac{n}{2}}\,x^{\frac{n}{2}};$$

$$=2^{\frac{n}{2}}.\left\{\frac{n(n-1)(n-2)\ldots\left(\frac{n}{2}+1\right)\cdot\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\ldots 2\,.\,1}{2\,.\,4\,.\,6\ldots n\cdot\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\ldots 2\,.\,1}\right\}x^{\frac{n}{2}}$$

$$=2^{\frac{n}{2}}.\left\{\frac{(n-1)(n-3)\ldots 3\,.\,1}{\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\ldots 2\,.\,1}\times\frac{n(n-2)\ldots 4\,.\,2}{2\,.\,4\ldots(n-2)\,n}\right\}x^{\frac{n}{2}}$$

$$=2^{\frac{n}{2}}.\frac{1\,.\,3\,.\,5\ldots(n-1)}{1\,.\,2\,.\,3\ldots\frac{n}{2}}\,x^{\frac{n}{2}}.$$

If n be odd there will be two terms, the $\left(\frac{n-1}{2}+1\right)^{\text{th}}$ and the $\left(\frac{n+1}{2}+1\right)^{\text{th}}$, which will be found to have equal coefficients; for from the general form

$$A_r=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots r},$$

and then putting $\frac{n-1}{2}$ and $\frac{n+1}{2}$ successively for r,

$$A_{\frac{n-1}{2}} = \frac{n(n-1)(n-2)\ldots\left(n-\frac{n-1}{2}+1\right)}{1\ .\ 2\ .\ 3\ \ldots\ \frac{n-1}{2}}$$

$$= \frac{n(n-1)(n-2)\ldots\frac{n+3}{2}}{1\ .\ 2\ .\ 3\ \ldots\frac{n-1}{2}};$$

$$A_{\frac{n+1}{2}} = \frac{n(n-1)(n-2)\ldots\left(n-\frac{n+1}{2}+1\right)}{1\ .\ 2\ .\ 3\ \ldots\ \frac{n+1}{2}}$$

$$= \frac{n(n-1)(n-2)\ldots\frac{n+1}{2}}{1\ .\ 2\ .\ 3\ \ldots\frac{n+1}{2}}$$

$$= \frac{n(n-1)(n-2)\ldots\frac{n+3}{2}}{1\ .\ 2\ .\ 3\ \ldots\frac{n-1}{2}} = A_{\frac{n-1}{2}}.$$

164. Next to find the sum of the coefficients; since

$$(1+x)^n = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \&c. + A_1 x^{n-1} + A_n x^n.$$

Let $x = 1$;

$\therefore\ (1+1)^n = 2^n = 1 + n + A_2 + A_3 + A_4 + \&c. =$ sum required.

Again writing $-x$ instead of x,

$$(1-x)^n = 1 - A_1 x + A_2 x^2 - A_3 x^3 + A_4 x^4 - \&c.$$

If x be made again $= 1$, then since $(1-1)^n = 0$, we have

$$1 - A_1 + A_2 - A_3 + A_4 - \&c. = 0;$$

$$\therefore\ 1 + A_2 + A_4 + A_6 + \&c. = A_1 + A_3 + A_5 + \&c.,$$

or the sum of the coefficients in the odd places is equal to those in the even.

Thus, if we take the coefficients of $(1+x)^7$

the sum $= 1+7+21+35+35+21+7+1=128=2^7$.

Again, the coefficients of $(1+x)^6$ are

1, 6, 15, 20, 15, 6, 1.

And $1+15+15+1=6+20+6$.

(Ex. 1.) Find the expansion of $(2x+3y)^5$.

Here $2x=a$, $3y=b$, $n=5$.

$$\therefore\ (2x+3y)^5=(2x)^5+5(2x)^4 3y+\frac{5.4}{1.2}(2x)^3(3y)^2$$

$$+\frac{5.4.3}{1.2.3}(2x)^2(3y)^3+\frac{5.4.3.2}{1.2.3.4}(2x)(3y^4)$$

$$+\frac{5.4.3.2.1}{1.2.3.4.5}(3y)^5$$

$$=32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5.$$

(Ex. 2.) Find the expansion of $(1+x)^{\frac{1}{2}}$.

Here $a=1$, $b=x$, $n=\frac{1}{2}$;

$$\therefore\ (1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}x^3+\&c.$$

$$=1+\tfrac{1}{2}x-\frac{1.1}{2.4}x^2+\frac{1.1.3}{2.4.6}x^3-\frac{1.1.3.5}{2.4.6.8}x^4+\&c.$$

(Ex. 3.) Find $(1-x)^{-\frac{1}{2}}$; here $a=1$, $b=-x$, $x=-\frac{1}{2}$;

$$\therefore\ (1-x)^{-\frac{1}{2}}=1-\tfrac{1}{2}(-x)+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1.2}(-x)^2$$

$$+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}(-x)^3+\&c.$$

$$=1+\tfrac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\&c.$$

(Ex. 4.) $(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + a^8.$

(Ex. 5.) $(5+4x)^4 = 625 + 2000x + 2400x^2 + 1280x^3 + 256x^4.$

(Ex. 6.) $(3-2x^2)^6 = 729 - 2916x^2 + 4860x^4 - 4320x^6 + 2160x^8 - 576x^{10} + 64x^{12}.$

(Ex. 7.) $\left(\frac{x}{2}+2y\right)^6 = \frac{x^6}{64} + \frac{3}{8}x^5y + \frac{15}{4}x^4y^2 + 20x^3y^3 + 60x^2y^4 + 96xy^5 + 64y^6.$

(Ex. 8.) $(3ac-2bd)^5 = 243a^5c^5 - 810a^4c^4bd + 1080a^3c^3b^2d^2 - 720a^2c^2b^3d^3 + 240acb^4d^4 - 32b^5d^5.$

(Ex. 9.) Find the 6th term of $(x^3+3xy)^9$;

$$\text{the } (1+r)^{\text{th}} \text{ term} = \frac{n(n-1)(n-2)\ldots(n+1-r)}{1\,.\,2\,.\,3\,\ldots\,r}\,a^{n-r}b^r.$$

Here $n=9$, $r=5$, $a=x^3$, $b=3xy$;

$$\therefore (1+5^{\text{th}}) = \frac{9.8.7.6.5}{1.2.3.4.5}(x^3)^4(3xy)^5 = 9\times7\times2\,.\,x^{12}\times243x^5y^5$$

$$= 30618x^{17}y^5.$$

(Ex. 10.) 8th term of $(1+x)^{11} = 330x^7$.

(Ex. 11.) 5th ... of $\left(\frac{x}{2}+2y\right)^7 = 70x^3y^4$.

(Ex. 12.) 6th ... of $(x-y)^{30} = -142506a^{25}b^5$.

(Ex. 13.) 5th ... of $(a^2-b^2)^{12} = 495a^{16}b^8$.

(Ex. 14.) Find the expansion of $\left(1+\frac{1}{n}\right)^n$ when n is very great.

Here $a=1$; $b=\frac{1}{n}$;

$$\therefore \left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{2.3}\frac{1}{n^3} + \&c.$$

$$= 1 + 1 + \frac{1}{1.2}\frac{n-1}{n} + \frac{1}{2.3}\frac{n^2-3n+2}{n^2} + \&c.$$

$$= 1 + 1 + \frac{1}{1.2} - \frac{1}{1.2}\frac{1}{n} + \frac{1}{2.3} - \frac{3}{2.3}.\frac{1}{n} + \frac{2}{2.3}\frac{1}{n^2} + \&c.$$

$$= 2 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c. \quad -\frac{1}{2n} - \frac{1}{2n} + \frac{1}{3n^2} - \&c.;$$

but as n is very large, the fractions $\frac{1}{n}$, $\frac{1}{n^2}$ are inappreciable;

$\therefore \left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{1.2} + \frac{1}{2.3} + \&c. = 2.71828 \ldots$ a particular number designated by ϵ.

165. To find $\sqrt{N}$ when N is very nearly a square or a high number.

Let $N = a^2 + x$; $\quad \therefore \sqrt{N} = \sqrt{a^2+x} = a\sqrt{1+\frac{x}{a^2}}$.

Now $\left(1+\frac{x}{a^2}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\frac{x}{a^2}$ very nearly; $\because x$ is small and a great;

$$\therefore \sqrt{N} = a\left(1+\frac{1}{2}\frac{x}{a^2}\right) = a + \frac{x}{2a}.$$

From which we obtain this practical rule, 'having found the square root of the integer part of a number, consisting partly of integers and partly of decimals, the remainder of the root may be found by dividing $N - a^2$; i. e. x by $2a$.'

Ex. Find the square root of 124.25.

Here $N = 121 + 3.25 = a^2 + 3.25$; $\therefore a = 11$,

$$\frac{x}{2a} = \frac{3.25}{22} = .1114; \quad \therefore \sqrt{N} = 11.1114.$$

Again if $N = a^n + b$ and $\sqrt[n]{N}$ be required,

then $\sqrt[n]{N} = \sqrt[n]{a^n\left(1 + \frac{b}{a^n}\right)} = a\left(1 + \frac{b}{a^n}\right)^{\frac{1}{n}}$

$$= a\left\{1 + \frac{1}{n} \cdot \frac{b}{a^n} + \frac{1}{n} \cdot \frac{\left(\frac{1}{n} - 1\right)}{2} \cdot \frac{b^2}{a^{2n}} + \&c.\right\}$$

$$= a\left\{1 + \frac{b}{n \cdot a^n} - \frac{n-1}{2 \cdot n^2} \cdot \frac{b^2}{a^{2n}} + \frac{(n-1)(2n-1)}{2 \cdot 3 \cdot n^3} \cdot \frac{b^3}{a^{3n}} - \&c.\right\}$$

Ex. Find the 5th root of 35.

$$\text{Here } N = 35 = 32 + 3 = 2^5\left(1 + \frac{3}{2^5}\right);$$

$$\therefore a = 2, \quad b = 3, \quad n = 5;$$

$$\therefore \sqrt[5]{35} = 2\left\{1 + \frac{3}{5 \times 32} - \frac{4 \cdot 3^2}{2 \times 5^2 \times (32)^2} + \frac{4 \cdot 9 \cdot 3^3}{2 \times 3 \times 5^3 \times (32)^3} - \&c.\right\}.$$

166. In the preceding example, since the series continues to infinity, we obtain only an approximate value for the required root; and as the denominators increase at a much more rapid rate than the numerators, a few terms only need be taken for practical purposes; still it may be required to know what is the error, or what is the limit in amount of the error occasioned by neglecting the remaining terms of the series. To do this, let R be the root required, and as the terms are alternately positive and negative,

let $R = a - b + c - d + e - f + g - h + \&c.$, and let

$R' = a - b + c - d + e - f$,

$R_i = a - b + c - d + e - f + g$.

Then since the terms continually decrease, $a - b$, $c - d$, $e - f$, $g - h$, &c. are all positive, and therefore R' which contains three only of these differences will be $< R$, and as for the same reason, all the pairs of terms after g, as $-h + k$, $-l + m$, &c. will be all negative, R_i will be $> R$;

and therefore the true value of the series lies between R' and $R_{,}$ or

$$a-b+c-d+e-f,$$

$$\text{and } a-b+c-d+e-f+g,$$

or the error committed by the omission of any number of the terms of a converging series, is less than the first term of the omitted part of the series.

Thus in the preceding example if we compute the root from 5 terms, the error will be less than the 6th term, which is

$$\frac{(n-1)(2n-1)(3n-1)(4n-1)}{2\ .\ 3\ .\ 4\ .\ 5n^5}\cdot\frac{q^5}{p^{5n}},$$

or substituting, is less than

$$\frac{4\times 9\times 14\times 19}{2\times 3\times 4\times 5}\cdot\frac{1}{5^5}\cdot\frac{3^5}{32^5}<\frac{9^3\times 7\times 19}{5^6\times 32^5}<\frac{96957}{524,288,000,000}.$$

167. To find the greatest coefficient and the greatest term of an infinite series.

Since $A_r = A_{r-1}\dfrac{n-r+1}{r}$, A_r will be $> A_{r-1}$, so long as $\dfrac{n-r+1}{r}$ is > 1; the first value therefore of r which will make $\dfrac{n-r+1}{r} < 1$, will indicate that the preceding coefficient is the greatest; to find r,

$n-r+1$ is that number next $> r$,

or $n+1$ .... $> 2r$,

or r next less than $\dfrac{n+1}{2}$;

$\therefore$ $(1+r)^{\text{th}}$ coefficient is next $<\dfrac{n+3}{2}$.

Thus if $n=8$, $\dfrac{n+3}{2}=\dfrac{11}{2}$ and the next less integer is 5, and we find by trial that the 5th is the greatest coefficient.

If n be an even integer the greatest coefficient is the $\left(\frac{n}{2}+1\right)^{\text{th}}$; if n be an odd integer there will be two coefficients, the $\frac{n+1}{2}^{\text{th}}$ and the $\frac{n+3}{2}^{\text{th}}$ each greater than any other.

Also in a converging series, if the greatest term be $A_{r-1}x^{r-1}$, then since the next term is $A_{r-1}\frac{n-r+1}{r}x^r = A_r x^r$;

$$\therefore \frac{A_r x^r}{A_{r-1}x^{r-1}} = \frac{n-r+1}{r}x \text{ must be} < 1;$$

$$\therefore (n+1)x < r + rx; \quad \therefore r > (n+1)\frac{x}{1+x}.$$

(Ex.) Find the greatest term of $\left(1+\frac{4}{3}\right)^{\frac{10}{3}}$.

$$\text{Here } r > \frac{13}{3}\,\frac{\frac{4}{3}}{1+\frac{4}{3}} > \frac{52}{21}; \quad \therefore r > 2; \quad \therefore r = 3,$$

and the 4th is the greatest term.

168. To find the sum of the squares of the coefficients of $(1+x)^n$;

$$\because (1+x)^n = 1 + nx + A_2x^2 + \text{\&c.} + A_2x^{n-2} + nx^{n-1} + x^n;$$

$$\therefore (1+x)^n = x^n + nx^{n-1} + A_2x^{n-2} + \text{\&c.} + A_2x^2 + nx + 1;$$

$$\therefore (1+x)^{2n} = x^n + n^2x^n + (A_2)^2x^n + \text{\&c.} + (A_2)^2x^n + n^2x^n + x^n + \text{\&c.}$$

or the sum of the squares of the coefficients = the coefficient of x^n in the expansion of $(1+x)^{2n}$.

But the coefficient of x^n in that expansion is the $(1+n)^{\text{th}}$ coefficient; hence putting $r = n$ and $n = 2n$ in

$$A_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\,\ldots\,r};$$

$$\therefore\ (1+n)^{\text{th}} = \frac{2n\,(2n-1)\,(2n-2)\ldots(n+1)}{1\ .\ 2\ .\ 3\ \ldots\ n}$$

$$= \frac{2n\,(2n-1)(2n-2)\ldots(n+1)\,n\,(n-1)\ldots 3\,.\,2\,.\,1}{(1\,.\,2\,.\,3\ldots n)^2}$$

$$= 2^n\,\frac{n\,(2n-1)\,(n-1)\,(2n-3)(n-2)\,(2n-5)\ldots 3\,.\,1}{(1\,.\,2\,.\,3\ldots n)^2}$$

$$= 2^n\,\frac{(2n-1)\,(2n-3)\ldots 5\,.\,3\,.\,1\times n\,(n-1)\ldots 2\,.\,1}{(1\,.\,2\,.\,3\ldots n)^2}$$

$$= 2^n\,\frac{1\,.\,3\,.\,5\ldots(2n-1)}{1\,.\,2\,.\,3\ldots\ n},$$

which is the sum of the coefficients.

169. We shall conclude this chapter by the expansion of a^x, or shall prove the truth of the exponential theorem, that

$$a^x = 1 + Ax + \frac{A^2x^2}{1\,.\,2} + \frac{A^3x^3}{2\,.\,3} + \&c. + \frac{A^nx^n}{1\,.\,2\ldots n} + \&c.$$

$$\text{where } A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

For $a^x = (1 + a - 1)^x = (1 + c)^x$ where $c = a - 1$.

$$\text{But } (1+c)^x = 1 + xc + x\,\frac{x-1}{2}\,c^2 + \frac{x\,(x-1)\,(x-2)}{1\,.\,2\,.\,3}\,c^3 + \&c.$$

$$= 1 + xc + \frac{x^2 - x}{2}\,c^2 + \frac{x^3 - 3x^2 + 2x}{2\,.\,3}\,c^3 + \&c.$$

$$= 1 + x\left(c - \frac{c^2}{2} + \frac{c^3}{3} - \&c.\right) + x^2\left(\frac{c^2}{2} - \frac{c^3}{2} + \&c.\right)$$

$$= 1 + Ax + A_2x^2 + A_3x^3 + \&c.$$

$$\text{where } A = c - \frac{c^2}{2} + \frac{c^3}{3} - \&c. = (a-1) - \tfrac{1}{2}(a-1)^2 + \&c.$$

and A_2, A_3, &c. depend also upon powers of c.

Hence we may assume that

$$a^x = 1 + Ax + A_2x^2 + A_3x^3 + A_4x^4 + \&c.;$$

$$\therefore\ a^{2x} = 1 + 2Ax + 4A_2x^2 + 8A_3x^3 + 16A_4x^4 + \&c.$$

But $a^{2x} = a^x \times a^x = (a^x)^2$;

$$\therefore\ 1 + 2Ax + 4A_2x^2 + 8A_3x^3 + 16A_4x^4 + \&c.$$

$$= 1 + 2Ax + (A^2 + 2A_2)x^2 + (2A_3 + 2AA_2)x^3$$

$$+ (2A_4 + A_2^2 + 2AA_3)x^4 + \&c.$$

Hence equating coefficients of like powers of x,

$$4A_2 = A^2 + 2A_2;\quad \therefore\ A_2 = \frac{A^2}{2},$$

$$8A_3 = 2A_3 + 2AA_2;\quad \therefore\ 6A_3 = A^3;\quad \therefore\ A_3 = \frac{A^3}{2.3},$$

$$16A_4 = 2A_4 + A_2^2 + 2AA_3;\quad \therefore\ 14A_4 = \frac{A^4}{4} + \frac{A^4}{3};\quad \therefore\ A_4 = \frac{A^4}{2.3.4};$$

$$\text{and}\ \therefore\ a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4} + \&c.$$

COR. Let e be that value of a which makes $A = 1$;

$$\therefore\ e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{2.3} + \&c.$$

To find e let $x = 1$;

$$\therefore\ e = 1 + 1 + \frac{1}{1.2} + \frac{1}{2.3} + \&c. = 2.71828, \&c.$$

a value which we have had occasion to mention.

APPENDIX.

LOGARITHMS. SIMPLE AND COMPOUND INTEREST.

1. If N be a number such that $N = a^x$, then x is said to be the logarithm of N and a is called the base of the system of logarithms.

Hence a logarithm may be defined to be the index or power to which the base is to be raised that the result may be equal to a given number.

Instead of writing the word logarithm; log, or sometimes l. only is used.

Cor. Since $a = a^1$, and $1 = a^0$, it follows that the logarithm of the base is unity and the logarithm of unity is $= 0$, or $\log a = 1$ and $\log 1 = 0$.

In the tables in common use, the base is 10; and since $10 = 10^1$, $100 = 10^2$, $1000 = 10^3$, $10{,}000 = 10^4$, &c.; $\therefore 1 = \log 10$, $2 = \log 100$, $3 = \log 1000$, $4 = \log 10{,}000$, &c.

Hence the logarithm of a number between 1 and 10 will be a fraction < 1, the logarithm of a number between 10 and 100 will be $> 1, < 2$; between 100 and 1000 will lie between 2 and 3 and so on; and thus if $.x$ be the logarithm of 6, or $6 = 10^{.x}$; therefore $\because 60 = 10 \times 10^{.x} = 10^{1.x}$; $600 = 10^{2.x}$, &c.

$$\therefore \log 6 = .x;\ \log 60 = 1.x;\ \log 600 = 2.x, \text{ \&c.},$$

the integer number prefixed to the decimal is called the characteristic, the decimal part the mantissa; and we see that the characteristic (when positive) is always one less than the number of digits in the number whose logarithm is sought.

We have mentioned that in the ordinary tables the base is 10; also every number is supposed to be represented by 10^x; hence to find the logarithm of a number, is to obtain x from the equation $10^x = N$; this is done by means of series, involving the powers of N; the investigation of which we have not space to attempt.

From these series the logarithms are computed, at the expence of prodigious labour, and are then registered in tables to facilitate the calculations of others.

The invention of logarithms is due to Baron Napier; the base he used was the number 2.71828, &c., which we have before represented by e; the base 10 was adopted by Briggs, by whose name the logarithms to that base are sometimes called.

We have just seen that $0 = \log 1$; what is then the logarithm of 0? Since 0 represents the value of a fraction with an infinitely great divisor;

$$\therefore\ 0 = \frac{1}{\infty} = \textit{a fortiori}\ \frac{1}{a^{\infty}} = a^{-\infty};$$

$\therefore$ by the definition of a logarithm, $\log 0 = -\infty$.

The following propositions will exhibit the utility of logarithms; the use of the tables may in general be learnt from rules given in the prefaces to such works.

PROP. 1. The logarithm of the product of any numbers equals the sum of the logarithms of the factors.

Let N_1, N_2, N_3, &c. be any numbers,

x_1, x_2, x_3, &c. their logarithms;

$$\therefore\ N_1 = a^{x_1},\quad N_2 = a^{x_2},\quad N_3 = a^{x_3},\ \&c;$$

$$\therefore\ N_1 \times N_2 \times N_3 \times \&c. = a^{x_1} \times a^{x_2} \times a^{x_3}\ \&c. = a^{x_1+x_2+x_3+\&c.};$$

$$\therefore\ \log(N_1 \times N_2 \times N_3\ \&c.) = x_1 + x_2 + x_3 + \&c. = \log N_1 + \log N_2 + \log N_3 + \&c.$$

PROP. 2. The logarithm of the quotient of two numbers, is the difference between the logarithms of the dividend and divisor.

$$\text{For } \frac{N_1}{N_2} = \frac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2};$$

$$\therefore \log\left(\frac{N_1}{N_2}\right) = x_1 - x_2 = \log N_1 - \log N_2.$$

PROP. 3. The $\log(N_1^m) = m \log N_1$.

$$\text{For } \because N_1 = a^{x_1}; \quad \therefore N_1^m = a^{mx_1};$$

$$\therefore \log(N_1)^m = mx_1 = m \log N_1.$$

PROP. 4. The $\log(N_1)^{\frac{m}{n}} = \frac{m}{n} \log N_1$.

$$\text{For } N_1^{\frac{m}{n}} = a^{\frac{mx_1}{n}}; \quad \therefore \log(N_1)^{\frac{m}{n}} = \frac{m}{n} \log N_1.$$

COR. Hence if the logarithms of numbers be collected into tables, the multiplication or division of numbers may be effected by means of the addition or subtraction of their logarithms, and the involution or evolution of numbers by multiplying or dividing the logarithms of the power or root.

2. In the tables most used the decimal part of the logarithm is alone put down, and the characteristic is left to be added when wanted; this, as we shall see, prevents the tables from being of inconvenient size.

For if we have $\log N = x$; then

$$\log(N\,10^n) = \log N + \log 10^n = \log N + n,$$

$$\log\left(\frac{N}{10^n}\right) = \log N - \log 10^n = \log N - n;$$

or knowing the logarithm of N, we can find the logarithm of every number, whether an integer or a decimal, which has the same significant digits; thus as 5.621 is between 1 and 10, its log is between 0 and 1, and we find from the tables that

$$\therefore \log 5.621 = .7498136;$$
$$\therefore \log 56.21 = 1.7498136;$$
$$\log 562.1 = 2.7498136;$$
$$\log 56210 = 4.7498136;$$

for $5.621 \times 10 = 56.21$, and $5.621 \times 10^4 = 56210$.

$$\text{Also } \log .5621 = -\ 1.7498136;$$
$$\log .05621 = -\ 2.7498136.$$

$$\text{For } .5621 = \frac{5.621}{10}; \quad \therefore \log .5621 = \log 5.621 - \log 10.$$

$$\text{And } .05621 = \frac{5.621}{100}; \quad \therefore \log .05621 = \log 5.621 - \log 100.$$

Hence this rule, if the number have n digits before the decimal point, the characteristic is $n - 1$; and if the number be a decimal with $n - 1$ cyphers before the first significant figure, the characteristic is $-n$.

(Ex. 1.) Find $x = \log (37.48 \times 1.752 \times 406.5)$;

$$\therefore x = \log 37.48 + \log 1.752 + \log 406.5,$$

$$\log 37.48 = 1.5737996,$$
$$\log 1.752 = \ \ .2435341,$$
$$\log 406.5 = 2.6090605;$$
$$\therefore x = 4.4263942.$$

(Ex. 2.) Find $x = \log \frac{7}{15}$; $\quad \therefore x = \log 7 - \log 15,$

$$\log 7 = \ \ .8450980,$$
$$\log 15 = 1.1760913;$$
$$\therefore x = -\ 1.6690067.$$

(Ex. 3.) Find $x = \log \left(\frac{7}{3}\right)^{14}$; $\therefore x = 14 (\log 7 - \log 3)$,

$$\begin{aligned} \log 7 &= .8450980, \\ \log 3 &= .4771213, \\ &\overline{.3679767} \\ &\quad 14 \\ x &= \overline{5.1516738.} \end{aligned}$$

(Ex. 4.) Find $x = \log (954)^{\frac{12}{17}}$;

$$\therefore x = \frac{12}{17} \log (954) = 2.1052106.$$

(Ex. 5.) $2^x = 769$; find x;

$$\therefore x \log 2 = \log 769; \quad \therefore x = \frac{2.8859263}{.3010300} = 9.586839.$$

(Ex. 6.) Given $a^{mx} b^{nx} = c$; find x;

$$\therefore mx \log a + nx \log b = \log c;$$

$$\therefore x = \frac{\log c}{m \log a + n \log b} = \frac{\log c}{\log (a^m b^n)}.$$

SIMPLE INTEREST AND DISCOUNT.

3. Let P = the principal or sum lent,

r = interest of £(1) for one year,

n = time;

$\therefore nr$ = interest of £1. for the time (n),

and Pnr = interest of £P for the time (n),

or if I be the interest, I = principal × time × interest of £1. for one year.

Let c = rate per cent. i.e. interest of £100. for one year;

$$\therefore r = \frac{c}{100};$$

$\therefore I = \frac{P \times n \times c}{100}$, which is the rule given in the books of arithmetic; viz. multiply the principal by the time and rate, and divide the product by 100.

Ex. Find the interest on £120. for 3 years at 4 per cent.

$$\text{Here } r = \frac{4}{100} = .04;$$

$$\therefore I = 120 \times 3 \times .04 = 14.4 = £14.\ 8s.,$$

or by the common rule, $I = \frac{120 \times 3 \times 4}{100} = £14.\ \frac{40}{100}$ £14. 8*s*.

Cor. The amount (M) is the principal + the interest, or $M = P + Pnr$.

4. Discount is the allowance made for the payment of a sum before it becomes due; and the present worth of a sum due some time hence is the sum to be paid at once instead of at the remote period; hence discount is the difference between the amount due at the end of the time and the present worth.

Let M be a sum due at end of time (n),

P the present worth;

then it is clear that if P be put to interest, its amount ought in fairness to be equal to M;

$$\therefore P + Pnr = M; \ \therefore P = \frac{M}{1 + nr}.$$

$$\text{Also } D = M - P = M - \frac{M}{1 + nr} = \frac{Mnr}{1 + nr}.$$

COR. If $\frac{c}{100}$ be put for r, $P = \frac{M}{1 + \frac{nc}{100}} = \frac{100\,M}{100 + nc}$;

$$\therefore (100 + nc)\,P = 100 \times M;$$

$$\therefore 100 + nc : 100 :: M : P,$$

which is the rule given in the ordinary books of arithmetic.

Ex. Find the present worth of £216. due 2 years hence reckoning 4 per cent.

Here $M = 216$; $r = .04$; $n = 2$; $\therefore nr = .08$;

$$\therefore P = \frac{216}{1.08} = 200.$$

Also by the common rule $\because nc = 8$,

$$108 : 100 :: 216 : P = \frac{216 \times 100}{108} = 200.$$

$$\text{Discount} = 216 - 200 = £16.$$

COMPOUND INTEREST.

5. When the interest due at the end of a fixed period, as for instance a year, is added to the principal, and interest is charged upon their amount, the money is said to increase at compound interest.

To find the amount of £P. increasing at compound interest.

If r be the interest of £1. for one year;

$\therefore P + Pr$ will be the amount due at the end of first year.

Let $P + Pr = P_1$, and let P_2, P_3, P_4, &c. P_n be the amounts due at end of 2nd, 3rd, and nth years;

$$\therefore\ P_1 = P + Pr = P(1+r);$$

$$\therefore\ P_2 = P_1 + P_1 r = P_1(1+r),$$

$$P_3 = P_2 + P_2 r = P_2(1+r),$$

$$P_4 = P_3 + P_3 r = P_3(1+r),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_n = P_{n-1} + r\,P_{n-1} = P_{n-1}(1+r)$$

Hence by multiplying and leaving out the factors common to each side,

$$P_n = P(1+r)^n,$$

$$\text{or if } M = P_n;\ \ M = P(1+r)^n,$$

the values of M, or n are best found by logarithmic tables.

Ex. Find the amount of £100., in 40 years reckoning, 4 per cent.

$$\text{Here } P = 100,\ \ r = .04,\ \ n = 40;$$

$$\therefore\ M = 100\,(1.04)^{40};\ \ \therefore\ \log M = \log 100 + 40 \log(1.04),$$

$$\log 100 - 2;\ \ 40 \log(1.04) = .6813320;$$

$$\therefore\ \log M = 2.6813320;\ \ \therefore\ M = £480.\ 2s. \text{ very nearly.}$$

6. To find in how many years a sum of money will increase m-fold at compound interest.

$$\text{Here } M = mP = P(1+r)^n;\ \ \therefore\ m = (1+r)^n;$$

$$\therefore\ \log(m) = n \cdot \log(1+r);\ \ \therefore\ n = \frac{\log(m)}{\log(1+r)}.$$

Ex. In how many years will a sum of money double itself at 5 per cent.? Here $m = 2$, $r = .05$;

$$\therefore\ n = \frac{\log(2)}{\log(1.05)} = \frac{.3010300}{.0211893} = 14.2 \text{ years.}$$

7. **To find the amount when interest is reckoned half-yearly.**

If r = interest of £1. for 1 year, and n the number of years, then $\because\ 1 + \frac{r}{2}$ = amount of £1. at the end of the first payment, and as $2n$ is the whole number of payments,

$$M = P\left(1 + \frac{r}{2}\right)^{2n}.$$

Let $P = 100$, $r = .04$, and $n = 40$; $\therefore M = 100\,(1.02)^{80}$;

$\therefore \log M = \log(100) + 80 \log(1.02)$; $\therefore M =$ £487. 10*s.* 11*d.*
whence by reference to the former example, we find that there is a gain of £7. 8*s.* 11*d.* by receiving dividends half-yearly.

If the interest be received every second it becomes due; then if m = seconds in a year; $\therefore mn$ = number of payments, and $M = P\left(1 + \frac{r}{m}\right)^{mn} = P\left(1 + \frac{r}{m}\right)^{\frac{m}{r} \cdot nr}$.

But we have seen that $\left(1 + \frac{r}{m}\right)^{\frac{m}{r}} = e$ when m is very great;

$$\therefore M = Pe^{nr}.$$

8. **To find the present value of a sum M, due n years hence, reckoning compound interest.**

Let P = present worth; then P put out to interest ought to amount to M, i. e.

$$P(1 + r)^n = M; \quad \therefore P = \frac{M}{(1 + r)^n}.$$

If n be the fractional part of a year, which is the case in most transactions;

$$P = \frac{M}{1 + nr} \text{ nearly.}$$

It is usual in business to charge interest instead of discount; and when n is small the error is small also.

$$\text{For } P = \frac{M}{1+nr} = M(1-nr) \text{ nearly} = M - Mnr, \text{ nearly}$$

$$= M - \text{interest of } M, \text{ nearly}.$$

9. To find the amount of an annuity for any number of years.

Let A be the annual payment.

Then at end of 1st year A is due,

.... 2nd year A + the amount of A

$= A + A(1+r)$ or A_1 is due,

.... 3rd year $A + A_1(1+r)$

$= A + A(1+r) + A(1+r)^2 = A_2$ is due,

.... 4th year $A + A_2(1+r)$

$= A + A(1+r) + A(1+r)^2 + A(1+r)^3$ is due;

$\therefore$ whole amount due at the end of the n^{th} year

$= A + A(1+r) + A(1+r)^2 + \text{\&c.} + A(1+r)^{n-1}$;

$$\therefore M = A.\{1 + (1+r) + (1+r)^2 + (1+r)^3 + \text{\&c.} + (1+r)^{n-1}\}$$

$$= A.\left\{\frac{(1+r)^n - 1}{1+r-1}\right\}, \text{ \&c.}$$

10. To find the present value of an annuity.

Let P be the present value; then in n years P ought to amount to M, i.e. should $= P(1+r)^n$;

$$\therefore P(1+r)^n = A.\frac{(1+r)^n - 1}{r};$$

$$\therefore P = \frac{A}{r}.\left\{1 - \frac{1}{(1+r)^n}\right\}$$

COR. If the annuity be perpetual or arise from a freehold estate $n = \infty$, and $\frac{1}{(1+r)^n}$ disappears;

$$\therefore\ P = \frac{A}{r} = \frac{100A}{c}.$$

Ex. What is the value of an estate producing a rent A, money making 4 or 5 per cent.?

Here $c = 4$ or 5; $\therefore\ P = 25A$ or $20A$;

i. e. is either 25 times or 20 times the annual rent, or as it is said, is worth 25 or 20 years purchase, according as money is worth 4 or 5 per cent.

EQUATION OF PAYMENTS.

11. A sum £P is due at the end of a months, £Q is due at the end of b months, at what time should both be paid at once, that neither the borrower nor the lender should incur loss?

Let x = time at which both payments can be made, and which is $> a < b$.

Therefore Interest of £P for the time $x - a$ ought to be equal to the interest of £Q for the time $b - x$;

$$\text{or } P(x-a)\,r = Q(b-x)\,r;$$

$$\therefore\ x = \frac{Pa + Qb}{P+Q}.$$

This rule which is ordinarily used is not strictly true; for the discount and not the interest of Q should be computed; since Q is paid $b - x$ months before Q is due; to obtain an accurate value for x, we must say that the interest of P for time $x - a$, should equal the discount of Q for time $(b - x)$;

$$\therefore\; P(x-a)r = \frac{Q(b-x)r}{1+(b-x)r};$$

$$\therefore\; P(x-a) + Pr(x-a)(b-x) = Q(b-x),$$

a quadratic equation from which x may be found.

There are many things in Algebra, not as yet mentioned, but these must be left for a succeeding volume.

ERRATA.

page	*line*	*for*	*read*
6	6	relation	relations
11	15	$+\frac{x^2}{2}$	$-\frac{x^2}{2}$
16	8 the sign – is omitted before the word *into.*		