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THE

ELEMENTS OF ALGEBRA,

DESIGNED FOR THE

USE OF COMMON SCHOOLS:

ALSO,

SERVING AS AN INTRODUCTION

TO THE

"TREATISE ON ALGEBRA."

BY GEORGE R. PERKINS, A. M.,

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PREFACE

TO THE SECOND EDITION.

Our Common Schools have reached that stage of advancement which enables them to pursue a more extended course of studies than formerly.

A few years ago Geographical Studies were hardly known in our Common Schools; now, every child is taught, at least, the elements of Geography. In the same way, Algebra is fast finding its proper place among other branches of an elementary education.

In a logical point of view, there is perhaps no science so well calculated to awaken a vigorous and rigid exercise of the reasoning powers as Mathematics.

Algebra and Geometry are the two great pillars of this science. Algebra, being more nearly allied to Arithmetic, may be made to precede the study of Geometry. Indeed, Algebra is a sort of universal Arithmetic, and affords great assistance to a clear and correct comprehension of the various Arithmetical rules in frequent use.

It cannot, however, be expected, at present, that our Common Schools should pursue Algebra to the same extent as pursued in our Colleges; still they may, to good advantage, acquire the more elementary portions of this branch of Mathematics.

With these views before me, I have endeavored to prepare a clear and concise exposition of the Elements of this important branch of Mathematics, which should be adapted to the present wants of our Common Schools.

The plan of this work is quite similar to that of my "Treatise on Algebra," and may in some respects be regarded as an Introduction to that work.

Under Chapter VII., I have introduced a method of elimination by *Indeterminate Multipliers*. This method was first given, I believe, by the celebrated LAGRANGE, in his *Mecanique Analytique*. I have given also, under this Chapter, an entirely *new method* of solving three simultaneous simple equations, which, from the peculiar process employed, may be called *The Chequer-Board Method*.

Under Chapter VIII., I have also thought it not out of place to give a few examples immediately after Permutations and Combinations, on the Theory of Probabilities.

Should these few Elements of Algebra be found to aid in elevating the standard of our Common School education, I shall feel that my object in preparing this book has been accomplished.

GEO. R. PERKINS.

UTICA, March, 1846.

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ELEMENTS OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND PRELIMINARY RULES.

DEFINITIONS AND SYMBOLS.

(*Article* 1.) ALGEBRA is that branch of Mathematics, in which the operations are performed by means of *figures*, *letters*, and *signs* or *symbols*.

(2.) In Algebra, quantities, whether given or required, are usually represented by letters. The first letters of the alphabet are, for the most part, used to represent known quantities; and the final letters are used for the unknown quantities.

(3.) The symbol $=$, is called the sign of *equality;* and denotes that the quantities between which it is placed, are equal or equivalent to each other. Thus, \$1=100 *cents*, which is read, one dollar equals one hundred cents. In the same way $a=b$, is read, a equals b. And the same for other similar expressions.

(4.) The symbol $+$, is called the sign of *addition*, or *plus;* and denotes that quantities between which it is placed, are to be added together. Thus, $a+b=c$, is read, a and b added, equals c. Again, $a+b+c=d+x$, is read, a, b, and c added, equals d and x added.

(5.) The symbol —, is called the sign of *subtraction*, or *minus;* and denotes that the quantity which is placed at the right of it is to be subtracted from the quantity on the left. Thus, $a-b=c$, is read, a diminished by b equals c.

(6.) When algebraic quantities are written without any sign prefixed, they are understood to be affected by the sign *plus*, and the quantities are said to be *positive* or *affirmative;* but quantities having the sign *minus* prefixed, are called *negative* quantities. Thus, a is the same as $+a$, b is the same as $+b$, and each is called a positive quantity; while $-a$, $-b$, are called negative quantities.

(7.) The symbol $\times$, is called the sign of *multiplication;* and denotes that the quantities between which it is placed, are to be multiplied together. Thus, $a \times b = c$, is read, a multiplied by b equals c. Multiplication is also represented by placing a *point* or *dot* between the factors, or terms to be multiplied. Thus, $a.b$ is the same as $a \times b$. Another method frequently employed, is, to unite the quantities in the form of a word. Thus, abc is the same as $a.b.c$, or $a \times b \times c$.

(8.) The symbol $\div$, is called the sign of *division;* and denotes that the quantity on the left of it, is to be divided by the quantity on the right. Thus, $a \div b = c$, is read, a divided by b equals c. Division is also indicated by placing the divisor under the dividend, with a horizontal line between them, like a vulgar fraction. Thus, $\frac{x}{y}$ is the same as $x \div y$.

(9.) When quantities are enclosed in a parenthesis, brace, or bracket, they are to be treated as a simple quantity. Thus, $(a+b)\div c$, indicates that the sum of a and b is to be divided by c. Again, each of the expressions, $(x-y)\div z$; $\{x-y\}\div z$; $[x-y]\div z$; is read, y subtracted from x, and the remainder divided by z. The same thing may be expressed by drawing a horizontal line, or bar, over the compound quantity, which line or bar is called a *vinculum*. Thus, $\overline{x+y}\times z$, denotes that the sum of x and y is to be multiplied by z.

(10.) When a quantity is added to itself several times, as $c+c+c+c$, we may write it but once, and prefix a number to show how many times its value has been repeated by this addition. Thus, $c+c+c$ is the same as $3c$; $d+d+d+d$ is the same as $4d$. The numbers thus prefixed to the quantities are called *coefficients*. Thus, in the expressions $3c$, $4d$, the coefficients of c and d are 3 and 4 respectively. A coefficient may consist, itself, of a letter. Thus, in the expression nx, n may be regarded as the coefficient of x; so also may x be considered as the coefficient of n.

When no coefficient is written, the quantity is regarded as having 1 for its coefficient. Thus, a is the same as $1a$, and xy is the same as $1xy$.

(11.) The continued product of a quantity into itself is, usually denoted by writing the quantity once, and placing a number over the quantity, a little to the right. Thus, $a\times a\times a$ is the same as a^3. The number thus placed over the quantity, is called the *exponent* of the quantity, and denotes the number of equal factors which

are to be multiplied together. Also, $a \times a \times a \times a$ is the same as a^4.

When a quantity is written without any exponent, it is understood that its exponent is 1. Thus, a is the same as a^1, and $(x+y)\times m$ is the same as $(x+y)^1 \times m^1$.

(12.) The *reciprocal* of a quantity is the value of a unit when divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a; also $\frac{1}{b}$ is the reciprocal of b.

(13.) An *Algebraic expression* is any combination of letters and numbers, formed by the aid of algebraic signs, in conformity with the foregoing definitions.

An algebraic expression composed of two or more terms connected by + or —, is called a *polynomial*. A polynomial composed of but two terms, is called a *binomial*; one composed of three terms, is called a *trinomial*.

Thus,

$$\left.\begin{array}{l} 3a + 5b \\ 7x^2 - 3y^3 \\ 3a - x^4 \end{array}\right\} \text{are binomials.}$$

$$\left.\begin{array}{l} 3a^2 + 2b - x \\ 4m + y + a \\ 5h - x + d \end{array}\right\} \text{are trinomials.}$$

ADDITION.

(14.) ADDITION, in Algebra, is finding the simplest expression for several algebraic quantities, connected by + or —.

Suppose we wish to find the sum of 3 apples, 7 apples, 4 apples, and 10 apples. From what we know of Arithmetic, we should proceed as in the following

OPERATION.

3 apples,
7 apples,
4 apples,
10 apples,

24 apples=sum of all.

If, instead of writing the word *apples*, we write only *a*, its initial letter, we shall have this second

OPERATION.

$3a$
$7a$
$4a$
$10a$

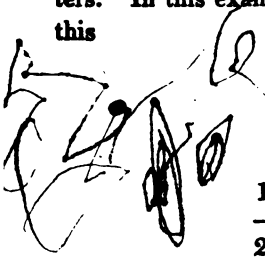
$24a$=sum of all the a's.

If a, instead of representing an apple, stands for any other thing, then would the sum of $3a$, $7a$, $4a$, and $10a$, be accurately represented by $24a$.

Frequently, in algebra, the quantities to be united or added, are all placed in the same horizontal line. The above quantities, when placed after this method, become:

$$3a+7a+4a+10a=24a.$$

(15.) We may remark that in algebra the results are, in general, true, whatever values are given to the letters. In this example, suppose $a=5$, and we shall have this



OPERATION.

$$\begin{aligned} 3a &= 3\times5 = 15 \\ 7a &= 7\times5 = 35 \\ 4a &= 4\times5 = 20 \\ 10a &= 10\times5 = 50 \\ \hline 24a &= 24\times5 = 120 \end{aligned}$$

Again, suppose $a=2$, and we have this second

OPERATION.

$$\begin{aligned} 3a &= 3\times2 = 6 \\ 7a &= 7\times2 = 14 \\ 4a &= 4\times2 = 8 \\ 10a &= 10\times2 = 20 \\ \hline 24a &= 24\times2 = 48 \end{aligned}$$

As a second example, suppose James to play 6 successive games at marbles. The first game he wins 6 marbles; the second game he loses 3 marbles; the third game he wins 10 marbles; the fourth game he loses 8 marbles; the fifth game he wins 4 marbles; the sixth game he loses 5 marbles. How many marbles will he now have, on the supposition that he had none before playing?

It is obvious that if we subtract the sum of his *losses* from the sum of his *gains*, we shall obtain the number which he finally has.

If we consider the gains as *positive* quantities, we ought to consider his losses as *negative* quantities.

The sum of his gains is found by this

OPERATION.

$$\begin{array}{r} 6m \\ 10m \\ 4m \\ \hline 20m = \text{sum of gains.} \end{array}$$

The sum of his losses is found by the following

OPERATION.

$$\begin{array}{r} -3m \\ -8m \\ -5m \\ \hline -16m = \text{sum of losses.} \end{array}$$

Hence, $20m-16m=4m=$his final number of marbles; that is, he had 4 marbles after playing the sixth game.

If m had represented \$100, instead of one marble, then at the end of the sixth game, James would have had $4m=\$400$.

(16.) From this we see that negative quantities are added in the same way as positive quantities, observing to prefix the — sign to the sum.

If we consider distances measured in a certain direction as positive, then will distances measured in an opposite direction be negative. Thus, calling north latitude positive, south latitude must be considered as negative.

Suppose a ship, in 20 degrees north latitude, to sail 15 degrees south, then 6 degrees north, then 30 degrees south, then 5 degrees north, and finally to go 10 degrees south. What latitude is she in?

Counting from the equator she is already 20 degrees north, to which adding the northings made, we have

$$20d+6d+5d=31d,$$

for the latitude in which she would have been in, provided she had made no southing. Where it may be remarked that we use d to represent one degree.

The sum of the southings is

$$-15d-30d-10d=-55d.$$

Uniting the sum of the northings with the sum of the southings, we get

$$31d-55d=-24d.$$

Now since the result is —, it shows that she is south of the equator, that is, she is in 24 degrees south latitude.

(17.) It must always be borne in mind that the office of the negative sign is to denote a state, condition, or effect, directly opposite to that denoted by the positive sign.

If the credits in a book account are considered as positive, the debts must be made negative.

If the degrees of a thermometer above zero are called +, then those below must be called —

We are now prepared to give a rule corresponding to

CASE I.

(18.) *When the quantities are alike but have unlike signs.*

RULE.

I. Find the sum of the coefficients of the positive quantities; and also the sum of the coefficients of the negative quantities.

II. Subtract the LESS *sum from the* GREATER.

III. Prefix the sign of the greater sum to the remainder, and annex the common letter or letters.

1. Find the sum of

$$3c-2c+5c-4c.$$

The sum of the coefficients of the positive terms is

$$3+5=8.$$

The sum of the coefficients of the negative terms is

$$-2-4=-6.$$

The difference is 2, to which, giving the sign of the greater, annexing the common letter, we have $2c$ for the sum sought.

In practice, we usually write all the different terms under each other, as in Arithmetic. Thus,

$$\begin{array}{r} 3c \\ -2c \\ 5c \\ -4c \\ \hline 2c=\text{sum sought.} \end{array}$$

Let this same example be wrought on the supposition that $c=10$.

$$\begin{array}{rrr} 3c = & 3\times 10 = & 30 \\ -2c = & -2\times 10 = & -20 \\ 5c = & 5\times 10 = & 50 \\ -4c = & -4\times 10 = & -40 \\ \hline 2c = & 2\times 10 = & 20 \end{array}$$

When $c=\frac{1}{2}$ it becomes

$$\begin{array}{rrr} 3c = & 3\times\frac{1}{2} = & 1\frac{1}{2} \\ -2c = & -2\times\frac{1}{2} = & -1 \\ 5c = & 5\times\frac{1}{2} = & 2\frac{1}{2} \\ -4c = & -4\times\frac{1}{2} = & -2 \\ \hline 2c = & 2\times\frac{1}{2} = & 1 \end{array}$$

2. What is the sum of

$$7ab-3ab+2ab+5ab-10ab\ ?$$

OPERATION.

$$\begin{array}{r} 7ab \\ -3ab \\ 2ab \\ 5ab \\ -10ab \\ \hline ab \end{array}$$

If the beginner finds the above too difficult, he may arrange the positive and negative terms in two distinct columns as in this second

OPERATION.

$$\begin{array}{rr} 7ab & -\ 3ab \\ 2ab & -\ 10ab \\ 5ab & \\ \hline 14ab & -\ 13ab=ab. \end{array}$$

3. What is the sum of $3ax-2ax+5ax-7ax+4ax$?

Ans. $3ax$.

4. What is the sum of $11bc-10bc+17bc-bc$?

Ans. $17bc$.

5. What is the sum of $49axy - 37axy - 10axy + 100axy - 6axy + 4axy$?

Ans. $100axy$

6. What is the sum of $6pq+3pq-7pq+8pq-pq$?
Ans. $9pq$.

7. What is the sum of $3mg-mg+6mg-8mg+10mg$?
Ans. $10mg$.

8. What is the value of $10h-h+6h-20h+8h$, when $h=5$?
Ans. $3h=3\times5=15$.

9. What is the value of $4x-10x+x-6x+5x$, when $x=4$?
Ans. $-6x=-6\times4=-24$.

10. What is the value of $3ad-ad+6ad-12ad$, when $a=2$, and $d=3$?
Ans. $-4ad=-4\times2\times3=-24$.

CASE II.

(19.) *When both quantities and signs are different.*

Suppose we wish to find the sum of $3a-5b+6a+2b-7a-b+4a+8b$.

We first proceed to find the sum of the a's by Case I.

Thus,

$$\begin{array}{r} 3a \\ 6a \\ -7a \\ 4a \\ \hline 6a=\text{the sum of the } a\text{'s.} \end{array}$$

In the same way we find the sum of the b's, as follows:

$$\begin{array}{r} -5b \\ 2b \\ -b \\ 8b \\ \hline 4b=\text{the sum of the } b\text{'s.} \end{array}$$

Connecting these two results with their proper signs, we get $6a+4b$ for the sum sought.

From this we see that examples under this Case may be wrought by the following

RULE.

I. Find the respective sums of like terms as in Case I.

II. Then write these sums one after another, with their proper signs.

EXAMPLES.

1. What is the sum of $3ax-2ab+4xy-2ax+3xy+7ab-2xy+6ax$?

$$\begin{array}{r} 3ax \\ -2ax \\ 6ax \\ \hline 7ax \end{array}$$

=sum of the terms containing ax.

$$\begin{array}{r} -2ab \\ 7ab \\ \hline 5ab \end{array}$$

=sum of the terms containing ab.

$$\begin{array}{r} 4xy \\ 3xy \\ -2xy \\ \hline 5xy \end{array}$$

=sum of the terms containing xy.

Therefore, $7ax+5ab+5xy$=the total sum.

2. What is the sum of $b-8x-y+8x+6y-4y-7b+3y-3x+10x-y+9b-x+3b+8b$?

By arranging the corresponding terms under each other, we obtain the following

OPERATION.

$$\begin{array}{r} b - 8x - y \\ -7b + 8x + 6y \\ 9b - 3x - 4y \\ 3b + 10x + 3y \\ 8b - x - y \\ \hline 14b + 6x + 3y \end{array}$$

3.

$$\begin{array}{r} 4a^2 + 5an \\ -3a^2 - 7an \\ 2a^2 - 3an \\ 5a^2 + 10an \\ \hline 8a^2 + 5an \end{array}$$

4.

$$\begin{array}{r} 4am - 3am^2 - 6ab \\ -7am + 4am^2 + ab \\ -8am - 10am^2 - 6ab \\ 1am + am^2 + 20ab \\ \hline -10am - 8am^2 + 9ab \end{array}$$

5.

$$\begin{array}{r} 2a^2x - 3ax^2 + 2ab \\ -7a^2x + 4ax^2 - 8ab \\ -6a^2x + 10ax^2 + 12ab \\ \hline -11a^2x + 11ax^2 + 6ab \end{array}$$

6.

$$\begin{array}{r} 3a^2b^3 - 7ab^4 + 5axy \\ -7a^2b^3 - 2ab^4 - axy \\ 8a^2b^3 + ab^4 - 7axy \\ \hline 4a^2b^3 - 8ab^4 - 3axy \end{array}$$

7. What is the sum of $3ah+6am-9xy+3ab-xy+4ah+10am-7xy-6ab+5xy+4ah-13am$?

Ans. $11ah+3am-12xy-3ab$.

8. What is the sum of $x^2 - y^3 + 4xy - 7x^2 + 8$ $-10xy+5x^2+2y^3+12xy$?

Ans. $-x^2+9y^3+6$

9. What is the sum of $3af-2g^2+7x-af+5g^2$ $10x+x-4af+3g^2-7x+h^4+x-3h^4$?

Ans. $-2af+6g^2-8x-2$

10. What is the sum of $3an^2-6a^3n+5a^2n^2-$ $+7an^2-6a^3n+7a^2n^2-10an^2+3a^3n+8an^2$?

Ans. $8an^2-10a^3n+12a^2$

11. What is the sum of $3ax+5bx+7cx-ax-2$ $-3cx+10ax-12bx+6bx-11cx+4ax+6bx-cx$?

Ans. $16ax+3bx-8$

12. What is the sum of $7r^2+3ry+y^2-r^2-4ry+6$ $-3r^2-7ry+5y^2+5r^2-ry+y^2-7ry$?

Ans. $8r^2-16ry+13$

13. What is the sum of $8m^2-mn+6n^2-m^2+3n$ $-6n^2+4mn-4m^2-5n^2+6m^2-6mn$?

Ans $9m^2-5n$

14. What is the sum of $16p^3+11p^2q-5pq^2-$ $+p^3-p^2q+10pq^2+q^3-17p^3+8pq^2-4p^2q$?

Ans. $6p^2q+13p$

15. What is the sum of $8amx-4xy+5y^2+9an$ $+10xy-11y^2+amx+5xy+17y^2-3amx$?

Ans. $15amx+11xy+11$

SUBTRACTION.

(20.) SUBTRACTION, in Algebra, is finding the simplest expression for the difference of two algebraic expressions.

If we wish to subtract b from a, we obviously obtain $a-b$, (Art. 5,) which is the same as the addition of a and $-b$.

Again, if we wish to subtract $b-c$ from a, we first subtract b from a and find, as above, $a-b$ for the result. Now, it is obvious, that we have subtracted too much by the quantity c, therefore, adding c to the above result, we finally obtain $a-b+c$, which is the same as the addition of a and $-b+c$.

From this, we conclude, that subtracting a quantity, is the same as adding it after the signs have been changed.

Hence, for the subtraction of algebraic quantities, we have this

RULE.

I. Write the terms to be subtracted, under the similar terms, if there are any, of those from which they are to be subtracted.

II. Conceive the signs of the terms of the polynomial which is to be subtracted, to be changed, and then proceed as in addition.

EXAMPLES.

1. From $4a+3b-2c$, subtract $a+2b+c$.

Actually changing the signs of the polynomial to be subtracted, and then placing it under the other polynomial, and proceeding as in addition, we have this

OPERATION.

$4a+3b-2c$
$-a-2b-c$

$3a+b-3c$ difference sought.

In practice we do not usually change the signs, but only conceive them to be changed, before performing our addition.

2.

From $32b+3a$
Take $5b+18a$

Remainder $27b-15a$

3.

From $6aby-3xy$
Take $-2aby+2xy$

Remainder $8aby-5xy$

4.

From $x^2+2xy+y^2$
Take $x^2-2xy+y^2$

Rem. $4xy$

5.

From $13x^2y-4xy^2+y^3$
Take $6x^2y+xy^2-y^3$

Rem. $7x^2y-5xy^2+2y^3$

6. From $17abc+b^3-3bc$, subtract $abc-6b^3-4bc$.
Ans. $16abc+7b^3+bc$.

7. From $11a^2x-13ax^3+17bx^2$, subtract $10a^2x-15ax^3-19bx^2$. *Ans.* $a^2x+2ax^3+36bx^2$.

8. From $4my-3nx+5mn$, subtract $3my-4nx-5mn$. *Ans.* $my+nx+10mn$.

9. From $9gh^2-7gy+5hm^3$, subtract $3gh^2+5gy+hm^3$. *Ans.* $6gh^2-12gy+4hm^3$.

10. From $11a^4x^3y+3ax-5ab+mg-3x^2y^2$, take $5xy+4a^4x^3y-6ax+9ab$.
Ans. $7a^4x^3y+9ax-14ab+mg-3x^2y^2-$

11. From $3a^2m - 6x^2y^3 + 2xy$, take $4a^2m + 6x^2y^3 + 5xy$. *Ans.* $-a^2m - 12x^2y^3 - 3xy$.

12. From $3a^2bc - 7axy + 3my + a$, take $a^2bc + 8axy + 6a - 4my$. *Ans.* $2a^2bc - 15axy + 7my - 5a$.

13. From $4ab^3 - 3a^3b + 5m^3 + 6n^2$, take $ab^3 + a^3b - m^3 + n^2$. *Ans.* $3ab^3 - 4a^3b + 6m^3 + 5n^2$.

(21.) We can express the subtraction of one polynomial from another, by writing the polynomial which is to be subtracted, after enclosing it in a parenthesis, immediately after the other polynomial from which it is to be subtracted, observing to place the negative sign before the parenthesis.

Thus, $ab - 6xy + 3am - (-3ab + xy - am)$, denotes that the polynomial enclosed in the parenthesis, is to be subtracted from the one which precedes it; and, since to perform subtraction, we must change all the signs of the terms to be subtracted, it follows that the parenthesis may be removed, provided we change the signs of all the terms which it encloses.

The above expression will thus become,

$$ab - 6xy + 3am + 3ab - xy + am,$$

which, reduced, gives

$$4ab - 7xy + 4am.$$

In the same way we have

$a^2b + xy - 7am - (mx + 6 - 13x^2)$, equivalent to

$$a^2b + xy - 7am - mx - 6 + 13x^2.$$

Also, $4axy - 3ay^2 - 7x^3 - (13a - my + 6)$, is the same as $4axy - 3ay^2 - 7x^3 - 13a + my - 6$.

It is also obvious that we may enclose any number of terms of a polynomial, within a parenthesis, with a nega-

tive sign before it, if we observe to change the signs of all the terms thus enclosed.

In this way, the polynomial

$$a^2b+xy-7am-mx-6+13x^2,$$

is made to take the following successive forms:

$$a^2b+xy-7am-(mx+6-13x^2)$$
$$a^2b+xy-(7am+mx+6-13x^2)$$
$$a^2b-(-xy+7am+mx+6-13x^2)$$
$$a^2b+xy-7am-mx-(6-13x^2).$$

MULTIPLICATION.

(22.) If we wish to multiply a by b, we must repeat a as many times as there are units in b, which, by (Art. 7,) is done by writing b immediately after a. Thus, a multiplied by b, is ab.

Again, if we wish to multiply a by $-b$, we observe that this is the same as to multiply $-b$ by a, hence, we must repeat $-b$, as many times as there are units in a. Repeating a minus quantity once, twice, thrice, or any number of times, can not change it to a positive quantity. Therefore, $-b$, multiplied by a, or, which is the same, a multiplied by $-b$, gives $-ab$.

Finally, if we wish to multiply $a-b$ by $c-d$, we first multiply it by c; thus, a multiplied by c, gives ac, and $-b$ multiplied by c, gives $-bc$, or the work may be written as follows:

$$\begin{array}{l} a-b \\ c \\ \hline ac-bc=a-b \text{ repeated } c \text{ times.} \end{array}$$

This result is evidently too great by the product of $a-b$ by d, since it was required to repeat $a-b$ as many times as there are units in c less d.

Repeating $a-b$ as many times as there are units in d, we have

$$\begin{array}{l} a-b \\ d \\ \hline ad-bd=a-b \text{ repeated } d \text{ times.} \end{array}$$

Subtracting this last result from the former, we have, (Art. 21,) $ac-bc-(ad-bd,)$ which becomes $ac-bc-ad+bd=a-b$ repeated $(c-d)$ times.

Hence, we see that $-b$, when multiplied by $-d$, produces in the product $+bd$.

(23.) *From all this, we discover that the product will have the sign +, when both factors have like signs, and the product will have the sign —, when the factors have contrary signs.*

(24.) If we wish to multiply $3a^2b$ by $4a^3b^2$, we observe that, (Art. 11,)

$$3a^2b = 3aab$$
$$4a^3b^2 = 4aaabb.$$

Hence, the product of $3a^2b$ by $4a^3b^2$, will be,

$$3aab \times 4aaabb = 12aaaaabbb = 12a^5b^3.$$

From which we discover, that the exponent of a, in the product, is equal to the sum of the exponents of a in

the factors; likewise, the exponent of b, in the product, is equal to the sum of the exponents of b in the factors.

Hence, the product of several letters having different exponents is equal to the product of all the letters having for exponents, the sums of their respective exponents in the factors.

CASE I.

(25.) From what has been said, we have for multiplying together two monomials, this

RULE.

I. Multiply together the coefficients, observing to prefix the sign + when both factors have like signs; and the sign — when they have contrary signs.

II. Write the letters one after another; if the same letter occur in both factors, add their exponents for a new exponent. It must be recollected that when no exponent is expressed, 1 *is understood to be the exponent,* (*Art.* 11.)

(26.) As in Arithmetic, it is obvious the product will be the same, in whatever order the letters are placed, but it will be found more convenient in practice, to have a uniform order for their arrangement. The order usually adopted, is, to place them alphabetically.

EXAMPLES.

1. Multiply $11ax^3y$ by $3a^4y^3$.

Ans. $33a^5x^3y^4$.

2. Multiply $7ab$ by $3ac$.

Ans. $21a^2bc$.

3. What is the product of $12xyz$ by $10x$?

Ans. $120\,x^2yz$.

4. What is the product of $8mn$ by $4m^2n^2$?

Ans. $32m^3n^3$.

5. What is the product of $-3a^2y$ by $6ag$?

Ans. $-18a^3gy$.

6. What is the product of $12ar$ by $-3abc$?

Ans. $-36a^2bcr$.

7. What is the product of $-6x^3y$ by $-7xy^3$?

Ans. $42x^4y^4$.

8. What is the product of $-10am^3n^4$ by -11?

Ans. $110am^3n^4$.

9. What is the product of $-\frac{1}{2}axy$ by $8ay$?

Ans. $-4a^2xy^2$.

10. Multiply $-\frac{1}{2}ax$ by $-\frac{1}{4}ay$.

Ans. $\frac{1}{8}a^2xy$.

11. Multiply $-\frac{1}{5}gm$ by $-\frac{1}{6}m^2$.

Ans. $\frac{1}{30}gm^3$.

12. Multiply $\frac{3}{8}ax$ by $\frac{1}{5}ax^2z$.

Ans. $\frac{3}{40}a^2x^3z$.

13. Multiply $7m^9n^8p^7$ by $6mn^2p^3$.

Ans. $42m^{10}n^{10}p^{10}$.

14. Multiply $3x$ by $-7xy^2$.

Ans. $-21x^2y^2$.

15. Multiply $-15abc$ by $-2a^2bc^3$.

Ans. $30a^3b^2c^4$.

16. Multiply $-\frac{6}{7}amg$ by $\frac{3}{5}axy$.

Ans. $-\frac{1}{3}\frac{8}{5}a^2mgxy$.

17. Multiply $\frac{3}{4}gh$ by $-\frac{5}{7}g^2h^3$.

Ans. $-\frac{1}{2}\frac{5}{8}g^3h^4$.

CASE II.

(27.) It is obvious that polynomials may be multiplied by the following

RULE.

I. Multiply all the terms of the multiplicand, successively by each term of the multiplier, and observe the same rules for the signs and exponents, as in Case I.

II. When there arise several partial products alike, they must be placed under each other and then added together in the total product.

(28.) The total product will be the same in whatever order we multiply by the terms of the multiplier, but for the sake of order, and uniformity, we begin with the left-hand term. By this means we are always extending our work towards the right, which is more natural and simple, than to commence on the right, and extend the work towards the left, as we are compelled to do in arithmetical multiplication.

EXAMPLES.

1. What is the product of $3a^2-6ax$ by $3a-m$?

OPERATION.

$3a^2-6ax$ multiplicand.
$3a-m$ multiplier.

$9a^3-18a^2x-3a^2m+6amx$=total product.

EXPLANATION.

Having placed the multiplier under the multiplicand, we commence with $3a$, the left-hand term of the multiplier, and multiply it into $3a^2$, the left-hand term of the multiplicand, and obtain the product $9a^3$, which we place for the left-hand term of the product. We then multi-

ply $3a$ into $-6ax$, the second term of the multiplicand, and obtain the product $-18a^2x$, which constitutes the second term of the product. Having multiplied all the terms of the multiplicand by $3a$, we next multiply by $-m$. Thus, $-m$ multiplied into $3a^2$, gives $-3a^2m$, which is the third term of the product, and $-m$ multiplied into $-6ax$, gives $6amx$, which is the last term of the product.

2. What is the product of $6x^2-3y^3$ by x^2-2y^3?

OPERATION.

$$\begin{array}{l} 6x^2-\ 3y^3 \text{ multiplicand.} \\ x^2-\ 2y^3 \text{ multiplier.} \\ \hline 6x^4-\ 3x^2y^3 \text{ product by } x^2. \\ \quad -12x^2y^3+6y^6 \text{ product by } -2y^3. \\ \hline 6x^4-15x^2y^3+6y^6 = \text{total product.} \end{array}$$

3. Multiply $a+b$ by $a+b$.

OPERATION.

$$\begin{array}{rl} & a+b \\ & a+b \\ \hline & a^2+ab \\ & \quad +ab+b^2 \\ \hline \textit{Ans.} & a^2+2ab+b^2. \end{array}$$

4. Multiply $a-b$ by $a-b$.

OPERATION.

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ \quad -ab + b^2 \\ \hline \text{Ans. } a^2 - 2ab + b^2. \end{array}$$

5. Multiply $a+b$ by $a-b$.

OPERATION.

$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ \quad -ab - b^2 \\ \hline \text{Ans. } a^2 \quad -b^2. \end{array}$$

6. What is the product of b^2m-3ay by $6x-3$?

Ans. $6b^2mx-18axy-3b^2m+9ay$.

7. What is the product of $a^6+a^4+a^2$ by a^2-1?

Ans. a^8-a^2.

8. Multiply a^2+ax+x^2 by a^2-ax+x^2.

Ans. $a^4+a^2x^2+x^4$.

9. What is the product of $7m+y$ by $7m-y$?

Ans. $49m^2-y^2$.

10. What is the product of $a+b+c$ by $a+b+c$?

Ans. $a^2+2ab+2ac+b^2+2bc+c^2$.

11. What is the product of $\frac{1}{2}a^2-3x$ by $\frac{1}{3}b-2y$?

Ans. $\frac{1}{6}a^2b-bx-a^2y+6xy$.

12. What is the product of $x^4-x^3+x^2-x+1$ by x^2+x-1? *Ans.* $x^6-x^4+x^3-x^2+2x-1$.

13. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$.

Ans. a^4-b^4.

14. What is the product of $2x^2+5x-7$ by $2x-3$?

Ans. $4x^3+4x^2-29x+21$.

DIVISION.

(29.) We know by the principles of Arithmetic, that if in division, we multiply the divisor into the quotient, the product will be the dividend.

Therefore, referring to what has been said under multiplication, (Art. 23,) we infer that when the dividend has the sign +, the divisor and quotient must have the same sign; but when the dividend has the sign —, then the divisor and quotient must have contrary signs.

(30.) *Hence, when the dividend and divisor have like signs, the quotient will have the sign +; and when the dividend and divisor have contrary signs, the quotient will have the sign —.*

We have also seen under multiplication, (Art. 24,) that the product of several letters of different powers is equal to the product of all the letters having for exponents the sum of their respective exponents.

(31.) *Hence, to divide any power of a letter by a different power of the same letter, it is obvious that the quotient will be a power of the same letter having for an exponent the excess of the exponent in the dividend above that of the divisor.*

(32.) If we divide $a^5 = aaaaa$, continually by a, we shall obtain the following results:

$a^5 \div a = a^{5-1} = a^4 = aaaa$

$a^4 \div a = a^{4-1} = a^3 = aaa$

$a^3 \div a = a^{3-1} = a^2 = aa$

$a^2 \div a = a^{2-1} = a^1 = a$

$a^1 \div a = a^{1-1} = a^0 = 1$ { since the divisor is equal to the dividend.

$a^0 \div a = a^{0-1} = a^{-1} = \frac{1}{a}$ = reciprocal of a, (Art. 12.)

$a^{-1} \div a = a^{-1-1} = a^{-2} = \frac{1}{aa} = \frac{1}{a^2}$ = reciprocal of a^2.

$a^{-2} \div a = a^{-2-1} = a^{-3} = \frac{1}{aaa} = \frac{1}{a^3}$ = reciprocal of a^3.

$a^{-3} \div a = a^{-3-1} = a^{-4} = \frac{1}{aaaa} = \frac{1}{a^4}$ = reciprocal of a^4.

$a^{-4} \div a = a^{-4-1} = a^{-5} = \frac{1}{aaaaa} = \frac{1}{a^5}$ = reciprocal of a^5.

(33.) *From the above scheme, we see that whenever the exponent of a quantity becomes* 0, *the value of the quantity is reduced to* 1.

(34.) *That whenever the exponent of a quantity is negative, the quantity is the reciprocal of what it would be were it positive.*

(35.) *Hence, changing the sign of the exponent of a quantity, is the same as taking the reciprocal of the quantity.*

CASE I.

(36.) From what has been said, we have, for dividing one monomial by another, this

RULE.

I. Divide the coefficient of the dividend by that of the divisor, observing to prefix to the quotient the sign +, when the signs of the dividend and divisor are alike; and the sign —, when they have contrary signs.

II. Subtract the exponents of the letters in the divisor from the exponents of the corresponding letters in the dividend; if letters occur in the divisor which do not in the dividend, they may (Art. 35,) *be written in the quotient by changing the signs of their exponents.*

(37.) It must be recollected here, and in all cases hereafter, that when the exponent of a letter is not written, 1 is always understood, (Art. 11,) and when the exponent becomes 0, the value of the power is 1. (Art. 33.)

EXAMPLES.

1. Divide $15a^3b$ by $5abxy^2$.

OPERATION.

$$\frac{15a^3b}{5abxy^2}=3a^2x^{-1}y^{-2}.$$

EXPLANATION.

We first divide 15 by 5, and obtain 3 for the quotient, which we consider as positive, since the signs of the dividend and divisor were alike, each being po-

tive. The exponent of a, in the divisor is 1 understood, (Art. 37,) which subtracted from 3, the exponent of a in the dividend, gives 2 for the exponent of a, in the quotient. The exponent of b, in the denominator, is the same as the exponent of b, in the numerator, each being 1, hence, the exponent of b, in the quotient, is 0, but when the exponent of a quantity becomes 0, its value is 1, (Art. 33.) So that b does not appear in the quotient. Since the letters x and y do not occur in the dividend, we write the x and y in the quotient with the signs of their exponents changed.

2. Divide $6a^4b^2$ by $3ab$.

Ans. $2a^3b$.

3. Divide $12x^4y^5z^6$ by $4xyz$.

Ans. $3x^3y^4z^5$.

4. Divide $10m^3xz^6$ by $5x$.

Ans. $2m^3z^6$.

5. Divide $-14a^3b^2$ by $7ab^2$.

Ans. $-2a^2$.

6. Divide $-16a^5b^4c^3$ by $-8ab^2c^3$.

Ans. $2a^4b^2$.

7. Divide $-13a^3y^4$ by $-26ay$.

Ans. $\frac{1}{2}a^2y^3$.

8. Divide $7l^3m^4n$ by $-3ln$.

Ans. $-\frac{7}{3}l^2m^4$.

9. Divide $8ab^2d^4$ by $4a^4bd$.

Ans. $2a^{-3}bd^3$.

10. Divide $20m^3n^2p$ by $10m^4n^3$.

Ans. $2m^{-1}n^{-1}p$.

11. Divide $35xyz$ by $7x^2y^3$.

Ans. $5x^{-1}y^{-2}z$.

12. Divide $-40x^3y$ by $4ab^2x$.

Ans. $-10a^{-1}b^{-2}x^2y$.

13. Divide $3xy$ by $-6x^2$.

Ans. $-\frac{1}{2}x^{-1}y$.

14. Divide $-4m^2y$ by $8m^4$.

Ans. $-\frac{1}{2}m^{-2}y$.

(38.) To divide one polynomial by another, we shall imitate the arithmetical method of *long division*. And in the arrangement of the work, we shall follow the *French* method of placing the divisor at the right of the dividend. Thus, to divide

$$a^3+a^2x+ab+bx \text{ by } a+x,$$

we proceed as follows:

OPERATION.

Dividend $=a^3+a^2x+ab+bx$ | $a+x=$divisor.

a^3+a^2x | $a^2+b=$quotient.

$ab+bx$

$ab+bx$

0

EXPLANATION.

Having placed the divisor at the right of the dividend, we seek how many times its left-hand term is contained in the left-hand term of the dividend, which we find to be a^2, which we place directly under the divisor, and then multiply the divisor by it, and subtract

the product from the dividend, then bringing down the remaining terms, we again seek how many times the left-hand term of the divisor is contained in the left-hand term of this remainder, which we find to be b, we then multiply the divisor by b, and again subtracting, there remains nothing, so that a^2+b is the complete quotient.

(39.) That the operation may be the most simple, it will be necessary to arrange both dividend and divisor according to the powers of some particular letter, commencing with the highest power.

CASE II.

(40.) To divide one polynomial by another, we have this

RULE.

I. Arrange the dividend and divisor with reference to a certain letter, then divide the first term on the left of the dividend by the first term on the left of the divisor, the result is the first term of the quotient; multiply the divisor by this term, and subtract the product from the dividend.

II. Then divide the first term of the remainder by the first term of the divisor, which gives the second term of the quotient; multiply the divisor by this second term; and subtract the product from the result after the first operation. Continue this process until we obtain 0 *for remainder, or when the division does not terminate, which is frequently the case, we can carry on the above process as far as we choose, and then place the last remainder over the divisor, forming a fraction, which must be added to the quotient*

EXAMPLES.

1. What is the quotient of $2a^2b+b^3+2ab^2+a^3$ divided by a^2+b^2+ab ?

Arranging the terms according to the powers of a, and proceeding agreeably to the above rule, we have this

OPERATION.

Dividend$=a^3+2a^2b+2ab^2+b^3$ | a^2+ab+b^2=divisor.

$a^3+a^2b+ab^2$ | $a+b$=quotient.

$a^2b+ab^2+b^3$

$a^2b+ab^2+b^3$

0

2. What is the quotient of $4x^3+4x^2-29x+23$ by $2x-3$?

OPERATION.

$4x^3+4x^2-29x+23$ | $2x-3$

$4x^3-6x^2$ | $2x^2+5x-7+\frac{2}{2x-3}$

$10x^2-29x$

$10x^2-15x$

$-14x+23$

$-14x+21$

2=remainder.

In this example, we find 2 for remainder, which, being placed over the divisor $2x-3$, gives the fraction $\frac{2}{2x-3}$ to be added to our quotient.

3. Divide $a^4+a^2z^2+z^4$ by a^2+az+z^2.

OPERATION.

$$\begin{array}{l|l} a^4+a^2z^2+z^4 & a^2+az+z^2 \\ a^4+a^3z\ +a^2z^2 & \overline{a^2-az+z^2} \\ \hline -a^3z\ +z^4 & \\ -a^3z\ -a^2z^2-az^3 & \\ \hline a^2z^2+az^3+z^4 & \\ a^2z^2+az^3+z^4 & \\ \hline 0 & \end{array}$$

4. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

Ans. $a^2-2ab+b^2$.

5. Divide $25a^2-10ax+x^2$ by $5a-x$.

Ans. $5a-x$.

6. Divide $6a^4-96$ by $3a-6$.

Ans. $2a^3+4a^2+8a+16$.

7. Divide $3x^4-26x^3y-14xy^3+37x^2y^2$ by $3x^2-5xy+2y^2$.

Ans. x^2-7xy.

8. Divide $x^2+2xy+y^2$ by $x+y$.

Ans. $x+y$.

9. Divide $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x^2-2xy+y^2$.

Ans. $x^2-2xy+y^2$.

10. Divide $64m^4n^6-25m^2n^8$ by $8m^2n^3+5mn^4$.

Ans. $8m^2n^3-5mn^4$.

CHAPTER II.

ALGEBRAIC FRACTIONS.

(41.) In our operations upon algebraic fractions, we shall follow the corresponding operations upon numerical fractions, so far as the nature of the subject will admit.

CASE I.

To reduce a monomial fraction to its lowest term, we have this

RULE.

I. Find the greatest common measure of the coefficients of the numerator and denominator. (See Arithmetic.)

II. Then, to this greatest common measure, annex the letters which are common to both numerator and denominator, give to these letters the lowest exponent which they have, whether in the numerator or denominator. The result will be the greatest common measure of both numerator and denominator.

III. Divide both numerator and denominator by this greatest common measure, (by rule under Art. 36,) *and the resulting fraction will be in the lowest terms.*

EXAMPLES.

1. Reduce $\frac{375a^3bxy}{15ab^2xy^3}$ to its lowest terms.

The greatest common measure of 375 and 15 is 15, to which annexing $abxy$, we have $15abxy$ for the greatest common measure of both numerator and denominator. Dividing the numerator by $15abxy$, we find

$$375a^3bxy \div 15abxy = 25a^2.$$

In the same way we find

$$15ab^2xy^3 \div 15abxy = by^2,$$

hence, we have

$$\frac{375a^3bxy}{15ab^2xy^3} = \frac{25a^2}{by^2},$$

which, by Rule under Art. 36, becomes

$$\frac{25a^2}{by^2} = 25a^2b^{-1}y^{-2}.$$

2. Reduce $\frac{42ax^3yz^5}{35xy^3z^3}$ to its lowest terms.

In this example, the greatest common measure of the numerator and denominator is $7xyz^3$, hence, dividing both numerator and denominator of our fraction by $7xyz^3$, we find

$$\frac{42ax^3yz^5}{35xy^3z^3} = \frac{6ax^2z^2}{5y^2},$$

which is in its lowest terms.

3. Reduce $\frac{-18mnx^2y^3}{72mx^3y^6}$ to its lowest terms.

Ans. $\frac{-}{4x}$

4. What is the simplest form of $\frac{13x^3}{26xy^4}$?

Ans. $\frac{x^2}{2y^4}$.

5. What is the simplest form of $\frac{108ab^3cd^7}{12abcd}$?

Ans. $9b^2d^6$.

6. What is the simplest form of $\frac{27abcd}{108a^3b^4m}$?

Ans. $\frac{cd}{4a^2b^3m}$.

7. What is the simplest form of $\frac{16xy^4}{32xy^6}$?

Ans. $\frac{1}{2y^2}$.

(42.) To obtain a general rule for reducing a fraction whose numerator, or denominator, or both, are polynomials, it would be necessary to show how to find the greatest common measure of two polynomials, a process which is too complicated and difficult for this place.

There are, however, many polynomial fractions, of which, the common measure of their respective numerators and denominators are at once obvious, and of course they are then readily reduced. We will illustrate this by a few

EXAMPLES.

1. Reduce $\frac{3xy+xz}{ax+bx}$ to its simplest form.

In this example, we see that x will divide both the numerator and denominator. Hence

$$\frac{3xy+xz}{ax+bx}=\frac{3y+z}{a+b}.$$

2. Reduce $\frac{ac+ay}{bc+by}$ to its simplest form.

In this example, the greatest measure of the numerator and denominator is obviously $c+y$. Hence,

$$\frac{ac+ay}{bc+by}=\frac{a}{b}.$$

3. Reduce $\frac{abx+ax}{abx-ax}$ to its simplest form.

Ans. $\frac{b+1}{b-1}$.

4. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x(x+b)(x-b)}{(x+b)(x+b)}$, to its simplest form.

Ans. $\frac{x^2-bx}{x+b}$.

CASE II.

(43.) To reduce a mixed quantity to the form of a fraction.

RULE.

Multiply the integral part by the denominator of the fraction, to which product add the numerator, and under the result place the given denominator.

EXAMPLES

1. Reduce $11x+\frac{x+y}{7x}$ to the form of a fraction.

In this example, the integral part is $11x$, which, multiplied by the denominator $7x$, gives $77x^2$, to which, adding the numerator $x+y$, we have $77x^2+x+y$ for the numerator of the fraction sought, under which, placing the denominator $7x$, we finally obtain

$\frac{77x^2+x+y}{7x}$, for the reduced form of $11x+\frac{x+y}{7x}$.

2. Reduce $x-\frac{bx+x^2}{m}$ to the form of a fraction.

Ans. $\frac{mx-bx-x^2}{m}$.

3. Reduce $y+3x-\frac{6}{3+a}$ to the form of a fraction.

Ans. $\frac{3y+9x+ay+3ax-6}{3+a}$.

4. Reduce $x-\frac{a^2-b^2}{x}$ to the form of a fraction.

Ans. $\frac{x^2-a^2+b^2}{x}$.

5. Reduce $3a^3-6+\frac{6-x^2}{7-y}$ to the form of a fraction.

Ans. $\frac{21a^3-36-3a^3y+6y-x^2}{7-y}$.

6. Reduce $9+\frac{3x^2-8c^4}{a-x^2}$ to the form of a fraction.

Ans. $\frac{9a-6x^2-8c^4}{a-x^2}$.

7. Reduce $a+b+\frac{4}{a-b}$ to the form of a fraction.

Ans. $\frac{a^2-b^2+4}{a-b}$.

8. Reduce $\frac{3m}{n+p}-x$ to the form of a fraction.

Ans. $\frac{3m-nx-px}{n+p}$.

9. Reduce $\frac{4+7a}{10}-5$ to the form of a fraction.

Ans. $\frac{7a-46}{10}$.

10. Reduce $\frac{4x}{3y+z}-8$ to the form of a fraction.

Ans. $\frac{4x-24y-8z}{3y+z}$.

11. Reduce $\frac{1-x}{1+x}+x-1$ to the form of a fraction.

Ans. $\frac{x^2-x}{x+1}$.

12. Reduce $\frac{4p-3q}{5}+p$ to the form of a fraction.

Ans. $\frac{9p-3q}{5}$.

CASE III.

(44.) To reduce a fraction to an integral, or mixed quantity.

RULE.

Divide the numerator by the denominator, the quotient will be the integral part; if there is a remainder, place it over the denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{9a-6x^2-8c^4}{a-x^2}$ to a mixed quantity.

Dividing the numerator by the denominator, we find this

FIRST OPERATION.

$$\begin{array}{l|l} 9a-6x^2-8c^4 & a-x^2 \\ 9a-9x^2 & 9=\text{integral part.} \\ \hline 3x^2-8c^4 = \text{numerator of fractional part.} & \end{array}$$

Therefore the quantity sought is $9+\frac{3x^2-8c^4}{a-x^2}$.

We will now change the order of the terms of the numerator and denominator by placing the x^2 first, we thus find this

SECOND OPERATION.

$$\begin{array}{l|l} -6x^2+9a-8c^4 & -x^2+a \\ -6x^2+6a & 6=\text{integral part.} \\ \hline 3a-8c^4=\text{numerator of fractional part.} & \end{array}$$

Therefore, the quantity sought is $6+\frac{3a-8c^4}{a-x^2}$.

These two results are equivalent, but under different forms.

2. Reduce $\frac{ax-x^2}{x}$ to an integral quantity.

Ans. $a-x$.

3. Reduce $\frac{6x^2-ax}{3x+1}$ to a mixed quantity.

Ans. $2x-\frac{2x+ax}{3x+1}$.

4. Reduce $\frac{m^3-y^3}{m-y}$ to an integral quantity.

Ans. m^2+my+y^2.

5. Reduce $\frac{20a^2-10a+6}{5a}$ to a mixed quantity.

Ans. $4a-2+\frac{6}{5a}$.

6. Reduce $\frac{9y^3-18y+8a^2y^2}{9y}$ to a mixed quantity.

Ans. $y^2-2+\frac{8a^2y}{9}$.

7. Reduce $\frac{14m^3-21n}{7m}$ to a mixed quantity.

Ans. $2m^2-\frac{3n}{m}$.

8. Reduce $\frac{12x^3-6x+4}{6x}$ to a mixed quantity.

Ans. $2x^2-1+\frac{2}{3x}$.

9. Reduce $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-2ab+b^2}$ to an integral quantity.

Ans. $a-b$.

10. Reduce $\frac{a^2-2ab+b^2}{a-b}$ to an integral quantity.

Ans. $a-b$.

11. Reduce $\frac{14m^2-7m+1}{7}$ to a mixed quantity.

Ans. $2m^2-m+\frac{1}{7}$.

12. Reduce $\frac{6x^2-3y^2+z^2}{3}$ to a mixed quantity.

Ans. $2x^2-y^2+\frac{z^2}{3}$.

CASE IV.

(45.) To reduce fractions to a common denominator.

RULE.

Multiply successively each numerator into all the denominators, except its own, for new numerators; and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{x}$, $\frac{b}{2}$, $\frac{c}{7a}$, to equivalent fractions having a common denominator.

$a \times 2 \times 7a = 14a^2$ = new numerator of first fraction.

$b \times x \times 7a = 7abx$ = new numerator of second fraction.

$c \times x \times 2 = 2cx$ = new numerator of third fraction.

and $x \times 2 \times 7a = 14ax$ = common denominator.

Therefore, $\frac{14a^2}{14ax}$; $\frac{7abx}{14ax}$; $\frac{2cx}{14ax}$; are the equivalent fractions sought.

2. Reduce $\frac{3m}{2a}$, $\frac{2b}{3x}$, and y, to fractions having a common denominator.

Ans. $\frac{9mx}{6ax}$; $\frac{4ab}{6ax}$; $\frac{6axy}{6ax}$.

3. Reduce $\frac{1}{2}$, $\frac{x^2}{3}$, $\frac{a^3+x^3}{a+x}$, to equivalent fractions having a common denominator.

Ans. $\frac{3a+3x}{6a+6x}$; $\frac{2ax^2+2x^3}{6a+6x}$; $\frac{6a^2+6x^3}{6a+6x}$.

4. Reduce $\frac{x}{3b}$, $\frac{6x^2}{5c}$, $\frac{a^2-x^2}{d}$, to fractions having a common denominator.

Ans. $\frac{5cdx}{15bcd}$; $\frac{18bdx^2}{15bcd}$; $\frac{15a^2bc-15bcx^3}{15bcd}$.

educe $\frac{a}{2}$, $\frac{b}{3}$, $\frac{c}{4}$, and $\frac{d}{6}$, to fractions having a
t denominator.

Ans. $\frac{72a}{144}$; $\frac{48b}{144}$; $\frac{36c}{144}$; $\frac{24d}{144}$.

educe $\frac{a}{2b}$, $\frac{b}{4c}$, $\frac{c}{6d}$, to fractions having a common
inator.

Ans. $\frac{24acd}{48bcd}$; $\frac{12b^2d}{48bcd}$; $\frac{8bc^2}{48bcd}$.

educe $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, to fractions having a common de-
or.

Ans. $\frac{bc}{abc}$; $\frac{ac}{abc}$; $\frac{ab}{abc}$.

educe $\frac{3a}{4}$, $\frac{5a}{3}$, $\frac{7a}{5}$, to fractions having a common
inator.

Ans. $\frac{45a}{60}$; $\frac{100a}{60}$; $\frac{84a}{60}$.

educe $\frac{6+a}{7}$, $\frac{5+b}{6}$, $\frac{c}{2}$, to fractions having a com-
nominator.

Ans. $\frac{72+12a}{84}$; $\frac{70+14b}{84}$; $\frac{42c}{84}$.

10. Reduce $\frac{6}{x+1}$, $\frac{4}{x-1}$, a, to fractions having a common denominator.

Ans. $\frac{6x-6}{x^2-1}$; $\frac{4x+4}{x^2-1}$; $\frac{ax^2-a}{x^2-1}$.

CASE V.

(46.) To add fractional quantities.

RULE.

Reduce the fractions to a common denominator; then add the numerators, and place their sum over the common denominator.

EXAMPLES.

1. What is the sum of $\frac{x}{3a}$, $\frac{1}{3}$, $\frac{y}{7}$?

These fractions, when reduced to a common denominator, become $\frac{21x}{63a}$, $\frac{21a}{63a}$, $\frac{9ay}{63a}$, adding their numerators we have $21x+21a+9ay$, placing this over the common denominator, we find

$$\frac{x}{3a}+\frac{1}{3}+\frac{y}{7}=\frac{21x+21a+9ay}{63a}=\frac{7x+7a+3ay}{21a}.$$

2. What is the sum of $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$?

Ans. $3x+\frac{23x}{45}$.

3. What is the sum of $\frac{2x}{3}, \frac{7x}{4}, \frac{2x+1}{5}$?

Ans. $2x+\frac{49x+12}{60}$.

4. What is the sum of $\frac{3x}{4}, \frac{4x}{5}, \frac{5x}{6}$?

Ans. $\frac{45x+48x+50x}{60}=2x+\frac{23x}{60}$.

5. What is the sum of $\frac{a+b}{2}, \frac{a-b}{2}$?

Ans. a.

6. What is the sum of $\frac{a^2+2ab+b^2}{4}, \frac{a^2-2ab+b^2}{4}$?

Ans. $\frac{a^2+b^2}{2}$.

7. What is the sum of $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$?

Ans. $\frac{bc+ac+ab}{abc}$.

8. What is the sum of $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \frac{x}{6}$?

Ans. $x+\frac{x}{4}$.

9. What is the sum of $\frac{a}{2}$, $\frac{b}{3}$, $\frac{c}{4}$?

Ans. $\frac{6a+4b+3c}{12}$.

10. What is the sum of $\frac{a}{a+b}$, $\frac{b}{a-b}$?

Ans. $\frac{a^2+b^2}{a^2-b^2}$.

11. What is the sum of $\frac{1}{x+y}$, $\frac{1}{x-y}$?

Ans. $\frac{2x}{x^2-y^2}$.

CASE VI.

(47.) To subtract one fraction from another.

RULE.

Reduce the fractions to a common denominator, then subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.

EXAMPLES.

1. From $\frac{3x+a}{4}$ subtract $\frac{2x-a}{3}$.

These fractions, when reduced to a common denominator, become $\frac{9x+3a}{12}$ and $\frac{8x-4a}{12}$. Subtracting t

numerators we have $9x+3a-(8x-4a)=x+7a$, placing this over the common denominator 12, we find

$$\frac{3x+a}{4}-\frac{2x-a}{3}=\frac{x+7a}{12}.$$

2. From $\frac{6m+y}{5}$ subtract $\frac{m+y}{4}$.

Ans. $\frac{19m-y}{20}$.

3. From $3y+\frac{y}{a}$ subtract $y-\frac{y-a}{c}$.

Ans. $2y+\frac{cy+ay-a^2}{ac}$.

4. From $\frac{x+y}{2}$ subtract $\frac{x-y}{2}$.

Ans. y.

5. From $\frac{x^2+2xy+y^2}{4xy}$ subtract $\frac{x^2-2xy+y^2}{4xy}$.

Ans. 1.

6. From $\frac{6-x}{2}$ subtract $a+\frac{3+y}{3}$.

Ans. $2-a-\frac{3x+2y}{6}$.

7. From $\frac{1}{x-y}$ subtract $\frac{1}{x+y}$.

Ans. $\frac{2y}{x^2-y^2}$.

8. From $\frac{1}{a}$ subtract $\frac{1}{b}$.

Ans. $\frac{b-a}{ab}$.

9. From $\frac{a+b}{2}$ subtract $\frac{a-b}{2}$.

Ans. b.

CASE VII.

(48.) To multiply fractional quantities together.

RULE.

If any of the quantities to be multiplied are mixed, they must, by Case II, be reduced to a fractional form; then multiply together all the numerators for a numerator, and all the denominators together for a denominator.

EXAMPLES.

1. Multiply $\frac{x+a}{2}$ by $\frac{x+b}{3}$.

The product of the numerators will be

$$(x+a)\times(x+b)=x^2+ax+bx+ab;$$

and the product of the denominators is $2\times3=6$.

Hence, $\frac{x+a}{2}\times\frac{x+b}{3}=\frac{x^2+ax+bx+ab}{6}$

2. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

3. What is the continued product of $\frac{3-x}{7}$, $\frac{4+x}{2}$, and $\frac{3}{7}$?

Ans. $\frac{36-3x-3x^2}{98}$.

4. What is the product of $\frac{a+b}{2}$ by $\frac{a-b}{2}$?

Ans. $\frac{a^2-b^2}{4}$.

5. What is the continued product of $\frac{2x}{m}$, $\frac{h-d}{x}$, $\frac{b}{c}$, and $\frac{1}{r-1}$?

Ans. $\frac{2bhx-2bdx}{cmrx-cmx}$.

6. What is the product of $y+\frac{y+1}{b}$ by $\frac{y-1}{2b}$?

Ans. $\frac{by^2-by+y^2-1}{2b^2}$.

7. What is the product of $\frac{1}{x+y}$ and $\frac{1}{x-y}$?

Ans. $\frac{1}{x^2-y^2}$.

8. What is the product of $3+\frac{x}{2}$ and $\frac{x}{y}$?

Ans. $\frac{6x+x^2}{2y}$.

9. What is the product of $\frac{x-y}{2}$, $\frac{2}{x^2-y^2}$, and $x+y$?

Ans. 1.

CASE VIII.

(49.) To divide one fraction by another.

RULE.

If there are any mixed quantities, reduce them to a fractional form, by Case II. Then invert the divisor, and multiply as in Case VII.

EXAMPLES.

1. Divide $\frac{3x+7}{4}$ by $\frac{4x-1}{5}$.

If we invert the divisor, and then multiply, we have $\frac{3x+7}{4}\times\frac{5}{4x-1}=\frac{15x+35}{16x-4}$ for the quotient.

2. Divide $\frac{x^2-y^2}{x}$ by $\frac{x^2+y^2}{y}$.

Ans. $\frac{x^2y-y^3}{x^3+xy^2}$.

3. Divide $\frac{7a^2y^{-3}}{3m^3}$ by $\frac{4ay^2}{5m}$.

Ans. $\frac{35a}{12m^3y^5}$.

4. Divide $\frac{y-b}{8cd}$ by $\frac{3cy}{4d}$.

Ans. $\frac{y-b}{6c^2y}$.

5. What is the quotient of $\frac{x}{x-1}$ divided by $\frac{x}{2}$?

Ans. $\frac{2}{x-1}$.

6. Divide $\frac{x^2-y^2}{2}$ by $\frac{x-y}{4}$.

Ans. $2x+2y$.

7. Divide $\frac{a}{3}$ by $\frac{b}{4}$.

Ans. $\frac{4a}{3b}$.

8. Divide $\frac{x^2-2xy+y^2}{a}$ by $\frac{x-y}{b}$.

Ans. $\frac{bx-by}{a}$.

9. Divide $\frac{3+x}{7}$ by $\frac{4-y}{5}$.

Ans. $\frac{15+5x}{28-7y}$.

10. Divide $\frac{7-y}{x}$ by $\frac{4+y}{y}$.

Ans. $\frac{7y-y^2}{4x+xy}$.

CHAPTER III.

SIMPLE EQUATIONS.

(50.) AN EQUATION is an expression of two equal quantities with the sign of equality placed between them.

The terms or quantities on the left-hand side of the sign of equality constitute the *first member* of the equation, those on the right constitute, the *second member.*

Thus, $x+2=a$, (1)

$\frac{x}{2}-1=b$, (2)

$3x+7=c$, (3)

are equations; the first is read, "x increased by 2 equals a."

The second is read, "one-half of x diminished by 1 equals b."

The third is read, "three times x increased by 7 equals c."

(51.) Nearly all the operations of algebra are performed by the aid of equations. The relations of a question or problem are first to be expressed by an equation containing known quantities as well as the unknown quantity. Afterwards we must make such

transformations upon this equation as to bring the unknown quantity by itself on one side of the equation, by which means it becomes known.

(52.) *An equation of the first degree, or a simple equation*, is one, in which the unknown quantity has no power above the first degree.

(53.) *A quadratic equation* is an equation of the second degree, that is, the unknown quantity is involved to the second power, and to no greater power.

(54.) An equation of the third, fourth, &c., degrees, is one which contains the unknown quantity to the third, fourth, &c., degrees; but to no superior degree.

(55) The following axioms will enable us to make many transformations upon the terms of an equation without destroying their equality.

AXIOMS.

I. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.

II. If equal quantities be subtracted from both members of an equation, the equality of the members will not be destroyed.

III. If both members of an equation be multiplied by the same quantity, the equality will not be destroyed.

IV. If both members of an equation be divided by the same quantity, the equality will not be destroyed.

CLEARING EQUATIONS OF FRACTIONS.

(56.) When some of the terms of an equation are fractional, it is necessary to so transform it as to cause the denominators to disappear, which process is called, *clearing of fractions.*

Let it be required to clear of fractions the following equation.

$$1+\frac{x}{2}+\frac{x}{3}-\frac{x}{6}=x. \qquad (1)$$

Now, by Axiom III, we can multiply all the terms of this equation by any number we please, without destroying the equality. If we multiply by a multiple of all the denominators, it is evident they will disappear.

If we choose the least multiple of the denominators as a multiplier, it is plain that the labor of multiplying will be the least possible.

Thus, in the above example, multiplying all the terms of both sides of the equation, by 6, which is the least multiple of 2, 3, and 6, we have

$$6+3x+2x-x=6x. \qquad (1)$$

This equation is now free of fractions.

(57.) Hence, to clear an equation of fractions, we deduce, from what has been said, this

RULE.

Multiply all the terms of the equation by any multiple of their denominators. If we choose the least common multiple of the denominators, for our multiplier, the terms of the fraction, when cleared, will be in their simplest form.

EXAMPLES.

1. Clear of fractions the equation $\frac{x-a}{5}=\frac{x+b}{2}-\frac{1}{7}$.

In this example, the least common multiple of the denominators 5, 2, and 7, is 70. Hence, multiplying all the terms of our equation by 70, we find

$$14x-14a=35x+35b-10,$$

for the equation when cleared of fractions.

2. Clear of fractions $\frac{x-2}{8}+\frac{x+a}{4}-\frac{x-b}{2}=x+\frac{x}{16}$.

Ans. $2x-4+4x+4a-8x+8b=16x+x$.

(58.) We must observe that when a fraction has the sign —, its value is required to be subtracted, so that, if it is written without the denominator, when cleared of fractions, all the signs of its numerator must be changed.

3. Clear of fractions $\frac{x-1}{2}+\frac{x+1}{3}-\frac{x-3}{4}=a+b-\frac{c}{7}$.

Ans. $42x-42+28x+28-21x+62=84a+84b-12c$.

4. Clear the equation, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=251$, of fractions.

Ans. $30x+20x+15x+12x+10x=15060$.

5. Clear the equation, $\frac{a^2}{x}+\frac{b}{m}+\frac{c}{d}=g$, of fractions.

Ans. $a^2dm+bdx+cmx=dgmx$.

6. Clear the equation, $\frac{x-1}{3}-\frac{x-a}{9}=\frac{x+b}{6}-4$, of fractions.

Ans. $6x-6-2x+2a=3x+3b-72$.

7. Clear the equation, $\frac{x-a}{b}+\frac{x-b}{c}=\frac{x-c}{d}$, of fractions.

Ans. $cdx-acd+bdx-b^2d=bcx-bc^2$.

8. Clear the equation, $\frac{x}{2}+\frac{a}{3}+\frac{b}{4}=\frac{c}{6}$, of fractions.

Ans. $6x+4a+3b=2c$.

9. Clear the equation, $\frac{x-1}{2}+\frac{x-2}{3}+\frac{x-3}{4}=\frac{x}{5}$, of fractions.

Ans. $30x-30+20x-40+15x-45=12x$.

10. Clear the equation, $\frac{a}{x}+\frac{3}{4}=\frac{5}{6}$, of fractions.

Ans. $12a+9x=10x$.

11. Clear the equation, $\frac{a}{b}+\frac{c}{x}+\frac{d}{x}=\frac{m}{x}$, of fractions.

Ans. $ax+bc+bd=bm$.

TRANSPOSITION OF THE TERMS OF AN EQUATION.

(59.) The next thing to be attended to, after clea the equation of fractions, is to transform it so that

the terms containing the unknown quantity may constitute one member of the equation.

If we take the equation

$$\frac{a}{x}-\frac{1}{2}+\frac{b}{3}=8, \qquad (1)$$

we have, when cleared of fractions,

$$6a-3x+2bx=48x. \qquad (2)$$

If we add to both members of this equation $3x-2bx$ (Axiom I,) it becomes

$$6a-3x+2bx+3x-2bx=48x+3x-2bx. \qquad (3)$$

All the terms of the left-hand member cancel each other, except $6a$.

Therefore, we have

$$6a=48x+3x-2bx, \qquad (4)$$

in which all the terms of the right-hand member contain x.

If we compare equation (4) with (2), we shall discover that the terms $-3x+2bx$, which are on the left side of equation (2), are on the right side of equation (4) with their signs changed.

Hence, we conclude that the terms of an equation may change sides, provided they change signs at the same time.

(60.) To transpose a term from one side of an equation to the other, we must observe this

RULE.

Any term may be transposed from one side of an equation to the other, by changing its sign.

EXAMPLES.

1. Clear the equation $\frac{x+6}{2}+26=\frac{5x}{4}+2$ of fractions, and transpose the terms so that all those containing x may constitute the right-hand member.

First, clearing the above equation of fractions, by Rule under Art. 57, we have

$$2x+12+104=5x+8.$$

Secondly, transposing $2x$ from the left member to the right member, and 8 from the right member to the left, we have

$$12+104-8=5x-2x, \text{ for the result required.}$$

2. Clear the equation $\frac{x}{2}\frac{a+x}{3}=7\frac{1}{2}$ of fractions, and transpose the terms.

Ans. $3x-2x=45+2a.$

3. Clear of fractions the equation $\frac{7x}{9}-3\frac{1}{3}+\frac{x}{6}=\frac{x}{9}+8$, and transpose the terms.

Ans. $14x+3x-2x=36+60.$

4. Clear of fractions and transpose the terms of the equation, $\frac{x}{a-b}-\frac{2+x}{a+b}=\frac{c}{a^2-b^2}$.

Ans. $ax+bx-ax+bx=c+2a-2b$.

(61.) We are now prepared to find the value of the unknown quantity. If we take the answer to the last example, it may be written thus,

$$(a+b-a+b)x=c+2a-2b.$$

Uniting the like terms within the parenthesis, it becomes

$$2bx=c+2a-2b.$$

Dividing both sides of this equation by $2b$, (Axiom IV,) we find

$$x=\frac{c+2a-2b}{2b}.$$

Hence, the value of x is now known, since it is equal to the expression

$$\frac{c+2a-2b}{2b}.$$

(62.) It is obvious that the terms of an equation may be so transposed as to cause all the terms to constitute but one member of the equation, and then the other member will be zero.

Thus, if $x+6-a=3-2x$,

then will $x+2x+6-3-a=0$.

If we so transpose the terms, of this last equation, as to make them constitute the right-hand member, we shall have

$$0 = -x - 2x - 6 + 3 + a.$$

In the higher parts of Analysis, equations are frequently reduced to this form, so that the positive and negative quantities cancel each other.

(63.) From what has been done, we discover that an equation of the first degree may be resolved by the following general

RULE.

I. If any of the terms of the equation are fractional, the equation must be cleared of fractions, by Rule under Art. 57.

II. The terms must then be so transposed that all those containing the unknown quantity may constitute one side or member of the equation, by Rule under Art. 60.

III. Then divide the algebraic sum of those terms on that side of the equation which are independent of the unknown quantity, by the algebraic sum of the coefficients of the terms containing the unknown quantity, and the quotient will be the value of the unknown quantity.

EXAMPLES.

1. In the equation $\frac{x}{3} + \frac{x}{4} = x - 10$, what is the value of x?

This cleared of fractions becomes

$$4x+3x=12x-120.$$

When the terms are transposed and united, we have

$$120=5x.$$

Dividing by 5, we get

$$24=x.$$

2. What is the value of x in $x-\frac{2x+1}{3}=\frac{x+3}{4}$?

Ans. $x=13$.

3. Given $\frac{21-3x}{3}-\frac{4x+6}{9}=6-\frac{5x+1}{4}$, to find x.

Ans. $x=3$

4. Find x from the equation $3ax+\frac{a}{2}-3=bx-a$.

Ans. $x=\frac{6-3a}{6a-2b}$.

5. Given $\frac{x-2}{4}+\frac{3x}{2}+\frac{15x}{2}=37$, to find x.

Ans. $x=6$.

6. Given $\frac{3cx}{a}-\frac{2bx}{m}-4=f$, to find x.

Ans. $x=\frac{afm+4am}{3cm-2ab}$.

7. Find x from the equation $\frac{8nx-b}{7}-\frac{3b}{2}=4-b-\frac{c}{2}$

Ans. $x=\frac{56+9b-7c}{16n}$.

8. Given $\frac{x-1}{2}+\frac{x+1}{2}=3x-12$, to find x.

Ans. $x=6$.

9. Given $x-\frac{3x-5}{13}+\frac{4x-2}{11}=x+1$, to find x.

Ans. $x=6$.

10. Given $\frac{x+7}{3}-3x+\frac{6x-2}{5}+3=x$, to find x.

Ans. $x=2$.

11. Given $\frac{3x-2}{7}+\frac{3x+2}{11}=x-1$, to find x.

Ans. $x=3$.

12. Given $\frac{x}{3}-\frac{x}{7}+x=11$, to find x.

Ans. $x=9\frac{6}{11}$.

13. Given $\frac{x}{2}\,\frac{x+2}{3}=\frac{x}{18}$, to find x.

Ans. $x=6$.

The true value of the unknown quantity, substituted for its representative, will cause the two members of the given equation to become identical.

In the last example, if we write 6 for x, the equation will become

$$\frac{6}{2}-\frac{6+2}{3}=\frac{6}{18} \text{ or, } 3-2\tfrac{2}{3}=\tfrac{1}{3},$$ where each member becomes $\frac{1}{3}$.

This method of verifying the solution of simple equations, should be used until the student becomes perfectly familiar with it.

14. Given $\frac{x-1}{4}+\frac{x+5}{10}=3-\frac{x}{5}$, to find x.

Ans. $x=5$.

This value of x causes each member of the equation to become 2.

15. Given $\frac{x-3}{4}-\frac{11-x}{2}=\frac{x+5}{2}-7$, to find x.

Ans. $x=7$.

This value of x causes each member of the equation to become -1.

16. Given $\frac{x-3}{5}+\frac{x-5}{7}=\frac{x-2}{35}$, to find x.

Ans. $x=4$.

This value of x causes each side of the equation to become $\frac{2}{21}$.

17. Given $\frac{4}{x+1}=\frac{6}{x+2}$, to find x.

Ans. $x=1$.

This value of x causes each member of the equation to become 2.

18. Given $\frac{x-a}{2}-\frac{x-b}{3}=\frac{x-c}{4}$, to find x.

Ans. $x=4b+3c-6a$.

QUESTIONS, THE SOLUTIONS OF WHICH REQUIRE EQUATIONS OF THE FIRST DEGREE.

(64.) In the solution of questions, by the aid of Algebra, the most difficult part is to obtain the proper equation which shall include all the necessary relations of the question. When once this equation of condition is properly found, the value of the unknown quantity is readily obtained by the Rule under Art. 63.

Suppose we wish to solve, by Algebra, the following question

1. What number is that, whose half increased by its third part and one more, shall equal itself?

If we suppose x to be the number sought, its half will be $\frac{x}{2}$ which, increased by its third part, becomes $\frac{x}{2}+\frac{x}{3}$, and this increased by one, becomes $\frac{x}{2}+\frac{x}{3}+1$, which by the question must equal the number.

Therefore, we have $\frac{x}{2}+\frac{x}{3}+1=x$ for the equation of condition.

Solving this by Rule under Art. 63, we have $x=6$.

VERIFICATION.

$$\frac{x}{2}=\tfrac{1}{2}\text{ of }6=3$$

$$\frac{x}{3}=\tfrac{1}{3}\text{ of }6=2$$

$$1\qquad\qquad=1$$

Therefore, $\frac{x}{2}+\frac{x}{3}+1=6$, which shows that 6 is truly the number sought.

Again, let us endeavor to solve this question:

2. What number is that whose third part exceeds its fourth part by 5?

Suppose x to be the number, then will its third part $=\frac{x}{3}$; its fourth part $=\frac{x}{4}$.

Therefore, the excess of its third part over its fourth part is expressed by $\frac{x}{3}-\frac{x}{4}$, which, by the question, must equal 5.

Hence, we have the following equation, $\frac{x}{3}-\frac{x}{4}=5$.

This solved, gives $x=60$; the third part of which is 20, and its fourth part is 15, so that its third part exceeds its fourth part by 5, hence, this is the number sought.

6. Divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, shall be respectively equal to each other.

If we denote the three parts by $2x$, $3x$, $4x$, it is plain that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, will be equal for all values of x.

Now by the question, the sum of these three parts must equal 36.

Therefore, $2x+3x+4x=36$.
Uniting terms, we have $9x=36$.
Dividing by 9, we obtain $x=4$.

Consequently, $2x=2\times4=8$, $3x=3\times4=12$, $4x=4\times4=16$ } the parts sought.

7. Two pieces of cloth are of the same price by the yard, but of different lengths, the one cost \$5, the other \$6$\frac{1}{2}$. If each piece had been 10 yards longer, their lengths would have been as 5 to 6. What was the length of each piece?

Since the price per yard was the same for both pieces, their lengths must have been to each other the same as the number of dollars which they cost, or as 5 to 6$\frac{1}{2}$, or, which is the same as 10 to 13.

Therefore, we will denote their lengths by $10x$ and $13x$.

These become, when increased by 10,

$$10x+10 \text{ and } 13x+10,$$

which, by the question, must be as 5 to 6.

Hence, $6(10x+10)=5(13x+10)$.

Expanding, $60x+60=65x+50$.

Transposing and uniting terms, we get $10=5x$, and $x=2$.

Therefore, $\left.\begin{matrix}10x=10\times 2=20\\ 13x=13\times 2=26\end{matrix}\right\}$ the lengths sought.

8. A person engaged a workman for 50 days. For each day that he labored he received 8 shillings, and for each day that he was idle he paid 5 shillings for his board. At the end of the 50 days, the account was settled, when the laborer received 10 pounds 18 shillings. Required the number of working days, and the number of idle days.

Let x equal the number of days he worked. Then will $50-x$ equal the number of days he was idle.

Also, $8x$ equals amount, in shilling, which he earned, and $(50-x)\times 5=250-5x$ equals amount of his board bill.

Therefore, he must receive $8x-(250-5x)=13x-250$, which, by the question, must equal £10 18*s*. $=218s$.

Hence, we have this condition

$$13x-250=218.$$

This readily gives $x=36$=number of days he worked, and $50-x=50-36=14$=number of idle days.

9. Divide the number 237 into two such parts, that the one may be contained in the other $1\frac{1}{2}$ times. What are these parts?

If x equals the smaller part, then the larger part will be $1\frac{1}{4}x=\frac{5x}{4}$.

Adding both parts together, we get

$$x+\frac{5x}{4}=237.$$

This solved, gives $x=105\frac{1}{3}$, and consequently

$$\frac{5x}{4}=131\frac{2}{3}.$$

So that the parts are $105\frac{1}{3}$ and $131\frac{2}{3}$.

Had we put $4x=$ the smaller part, then the larger part would have been $5x$.

Now taking the sum of the parts, we get

$$4x+5x=237.$$

This solved, gives $x=26\frac{1}{3}$.

Consequently, $4x=105\frac{1}{3}$ and $5x=131\frac{2}{3}$.

10. Three persons, A, B, and C, draw prizes in a lottery. A draws \$200; B draws as much as A, together with a third of what C draws; and C draws as much as A and B together. What is the amount of the prizes?

Let $x=$the number of dollars which C drew.

A drew \$200, and B drew $\$200+\frac{x}{3}$.

Since C's equals the sum of A's and B's, we have this condition

$$x=200+200+\frac{x}{3}.$$

Consequently, x=600=C's money.

Therefore, A's =$ 200
B's =$ 400
C's =$ 600

Sum of the prizes=$1200.

11. Two capitalists calculate their fortunes, and it appears that one is twice as rich as the other, and that together, they possess $38700. What is the capital of each ?

Ans. The one had $12900, the other $25800.

12. The sum of $2500 is to be divided between two brothers, so that one receives $4 as often as the other receives $1. How much does each receive ?

Ans. The one receives $500, the other $2000.

13. Two friends met a horse-dealer leading a horse, which they resolve to buy jointly. When they had agreed as to the price, they found that the one was able to pay only the fifth part of the price, and the other only the seventh part, which together amounted to $48. What was the price of the horse ?

Ans. $140.

14. A fortress has a garrison of 2600 men ; among whom there are nine times as many foot soldiers, and three times as many artillery soldiers as cavalry. How many are there of each kind ?

Ans. 200 cavalry, 600 artillery, and 1800 foot.

15. Find a number, such, that being multiplied by 7, and the product increased by 3, and this sum divided by 2, and 4 being subtracted from the quotient, we shall obtain 15.

Ans. 5.

16. One of my acquaintances is now 40 years old, his son is 9 years old. In how many years will this man, who is now more than 4 times as old as his son, be only twice as old as he is?

Ans. In 22 years.

17. A cook receives from his master $\$11\frac{1}{4}$, which he is to expend for the same number of chickens, ducks, and geese. Each chicken cost $\frac{1}{4}$ of dollar, each duck $\frac{3}{8}$ of a dollar, and each goose $\frac{1}{2}$ of a dollar. How many did he buy of each kind?

Ans. 10.

18. A gentleman spent $\frac{1}{10}$ of his fortune in Holland, $\frac{1}{3}$ in France, $\frac{3}{8}$ in England, and $\frac{1}{4}$ in Italy. He has now but $975 left. How much money had he at first?

Ans. $3000.

19. The sum of two numbers is fifty; but when the greater is multiplied by 6, and the smaller by 5, the sum of the two products is 276. What are the two numbers?

Ans. 26, and 24.

20. A person was desirous of giving 3 cents apiece to some beggars, but found he had not money enough by 8 cents; he therefore gave them 2 cents apiece, and

had 3 cents remaining. What was the number of beggars?

Ans. 11 beggars.

21. Two persons, A and B, lay out equal sums of money in trade: A gains \$126, and B loses \$87, and A's money is now double of B's. What did each lay out?

Ans. \$300.

22. The sum of \$1200 is to be divided between two persons, A and B, so that A's share shall be to B's share as 2 to 7. How much ought each to receive?

Ans. A \$266⅔, B \$933⅓.

23. Divide the number 46 into two unequal parts so that the greater divided by 7, and the less by 3, the quotients together may amount to 10. What are these parts?

Ans. 28 and 18.

24. In a company of 266 persons, consisting of officers, merchants and students, there were 4 times as many merchants, and twice as many officers as students. How many were there of each class?

Ans. 38 students, 152 merchants, and 76 officers.

25. A field of 864 square rods is to be divided among three farmers, A, B, C, so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much ought each to receive?

Ans. A 135, B 297, C 432 square rods.

26. Divide \$1520 among three persons, A, B, C, so that B may receive \$100 more than A; and C, \$270 more than B. How much ought each to receive?

Ans. A \$350, B \$450, C \$720.

27. A certain sum of money is to be divided among three persons, A, B, C, as follows: A shall receive \$3000 *less* than the half of it, B \$1000 *less* than the third part, and C is to receive \$800 *more* than the fourth part of the whole sum. What is the sum to be divided? and what does each receive?

Ans. The whole sum is \$38400. A receives \$16200, B \$11800, C \$10400.

28. A mason, 12 journeymen, and 4 assistants, receive together \$72 wages for a certain time. The mason receives \$1 daily, each journeyman \$$\frac{1}{2}$, and each assistant \$$\frac{1}{4}$. How many days must they have worked for this money?

Ans. 9 days.

29. A servant receives from his master \$40 wages, yearly, and a suit of livery. After he had served 5 months he asked for his discharge, and received for this time the livery, together with \$6$\frac{1}{2}$ in money. How much did the livery cost?

Ans. \$18.

30. Find a number such that $\frac{1}{3}$ thereof increased by $\frac{1}{4}$ of the same, shall be equal to $\frac{1}{7}$ of it if increased by 35.

Ans. 84.

31. A gentleman spends $\frac{4}{5}$ of his yearly income in board and lodging, $\frac{3}{4}$ of the remainder in clothes, and lays by £20 a year. What is his income?

Ans. £180.

32. A gentleman bought a chaise, horse, and harness, for $360. The horse cost twice as much as the harness, and the chaise cost twice as much as the harness and horse together. What was the price of each?

Ans. { The harness cost $40. The horse cost $80. The chaise cost $240.

33. If a reservoir can be exhausted by one engine in 7 hours, by another in 8 hours, and by a third in 9 hours, in what time will it be exhausted, if they are all worked together?

Ans. In $2\frac{1}{1}\frac{2}{9}\frac{3}{1}$ hours.

34. In an orchard of fruit trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ of them pears, $\frac{1}{8}$ of them peaches, 7 trees bear cherries, 3 plums, and 2 quinces. How many trees are there in the orchard?

Ans. 96 trees.

(66.) It would be a very pleasant exercise, and at the same time a very useful one, for the student to give analytical solutions to all the foregoing questions of simple equations. As an illustration of the kind of analysis required, the student is referred to Chapter XII, Higher Arithmetic.

EQUATIONS OF TWO OR MORE UNKNOWN QUANTITIES.

(67.) Suppose we have given the two equations.

$$x+y=19,$$
$$x-y=11,$$

to find the value of x and y.

If we take their sum, we shall have

$$2x=30.$$

Dividing by 2, we find

$$x=15.$$

Again, subtracting the second equation from the first, we have

$$2y=8.$$

Dividing by 2, we obtain

$$y=4.$$

2. We have given the two equations

$$\frac{x}{3}+\frac{y}{4}=8, \qquad (1)$$

$$\frac{x}{6}+\frac{y}{16}=3, \qquad (2)$$

to find the value of x and y.

We will first clear these equations of fractions, by multiplying the first by 12, and the second by 48, we thus obtain

$$4x+3y=96, \qquad (3)$$
$$8x+3y=144. \qquad (4)$$

Subtracting (3) from (4,) we have

$$4x=48.$$

Dividing by 4, we find

$$x=12.$$

If we multiply (3) by 2, it becomes

$$8x+6y=192. \qquad (5)$$

Now, subtracting (4) from (5,) we find

$$3y=48.$$

Dividing by 3, we find

$$y=16.$$

3. If we have given the two equations

$$2x-3y=4, \qquad (1)$$
$$8x-6y=40, \qquad (2)$$

to find x and y, we proceed as follows:

Dividing (2) by 2, it becomes

$$4x-3y=20. \qquad (3)$$

Subtracting (1) from (3,) we find

$$2x=16.$$

Therefore, $x=8.$

Multiplying (1) by 4, we have

$$8x-12y=16. \qquad (4)$$

Subtracting (4) from (2,) we get

$$6y=24.$$

Therefore, $y=4.$

ELIMINATION BY ADDITION AND SUBTRACTION.

(68.) From what has been done, we discover that an unknown quantity may be eliminated from two equations by the following

RULE.

Operate upon the two given equations, by multiplication or division, so that the coefficients of the quantity to be eliminated, may become the same in both equations; then add or subtract the two equations, as may be necessary, to cause these two terms to disappear.

EXAMPLES.

4. Given, to find x and y, the two equations

$$3x - y = 3, \quad (1)$$
$$y + 2x = 7. \quad (2)$$

If we add the two equations, we have

$$5x = 10.$$

Therefore, $$x = 2. \quad 3)$$

Again, multiplying (3) by 2, we get

$$2x = 4. \quad (4)$$

Subtracting (4) from (2,) we obtain

$$y = 3.$$

5. Given, to find x and y, the two equations

$$\frac{x}{2} + \frac{y}{3} = 6, \quad (1)$$

$$\frac{x}{3} + \frac{y}{2} = 6\tfrac{1}{2}. \quad (2)$$

Clearing these equations of fractions, by multiplying each by 6, they become

$$3x + 2y = 36, \quad (3)$$
$$2x + 3y = 39. \quad (4)$$

Multiplying (3) by 3, and (4) by 2, they become

$$9x+6y=108, \quad (5)$$
$$4x+6y=\ 78. \quad (6)$$

Subtracting (6) from (5,) we get

$$5x=30,$$

and $$x=6. \quad (7)$$

Multiplying (7) by 2, it becomes

$$2x=12. \quad (8)$$

Subtracting (8) from (4,) we find

$$3y=27.$$

Therefore, $$y=9.$$

6. Suppose we wish to find x, y, and z, from the three equations

$$5x-6y+4z=15, \quad (1)$$
$$7x+4y-3z=19, \quad (2)$$
$$2x+\ y+6z=46. \quad (3)$$

We will first eliminate y; for this purpose multiply (3) first by 4 and then by 6, and we shall obtain

$$8x+4y+24z=184, \quad (4)$$
$$12x+6y+36z=276. \quad (5)$$

Add (1) to (5,) and subtract (2) from (4,) and we have

$$17x+40z=291, \quad (6)$$
$$x+27z=165. \quad (7)$$

We have now the two equations (6) and (7,) and but two unknown quantities x and z.

Multiply (7) by 17, and it will become

$$17x+459z=2805. \quad (8)$$

Subtracting (6) from (8) we obtain

$$419z=2514. \qquad (9)$$

Dividing (9) by 419, we find

$$z=6. \qquad (10)$$

Multiplying (10) by 27, we find

$$27z=162. \qquad (11)$$

Subtracting (11) from (7,) we get

$$x=3. \qquad (12)$$

Multiplying (10) by 6, and (12) by 2, and then taking their sum, we find

$$6z+2x=42. \qquad (13)$$

Subtracting (13) from (3,) we get

$$y=4.$$

(69.) We will now repeat the solution of this last question, adopting a simple and easy method of indicating the successive steps in the operations.

The method which we propose to make use of, is to indicate by algebraic signs, the same operations upon the respective *numbers* of the different equations, as we wish to have performed upon the equations themselves.

Thus,

$(6)=(4)\times 3$ } shows, that equation (6) is obtained by multiplying equation (4) by 3.

$(10)=(7)+(1)$ } shows, that equation (10) is obtained by adding equations (7) and (1.)

$(11)=(6)-(3)$ } shows, that equation (11) is obtained by subtracting equation (3) from (6).

$(15)=(14)\div 3$ } shows, that equation (15) is obtained by dividing equation (14) by 3.

And so on for other combinations.

This kind of notation will become familiar by a little practice.

We will now resume our equations of last example.

Given $\left.\begin{array}{ll} 5x-6y+4z=15, & (1) \\ 7x+4y-3z=19, & (2) \\ 2x+\ y+6z=46, & (3) \end{array}\right\}$ to find x, y, and z.

$$\begin{array}{rl} 8x+\ 4y+\ 24z=\ 184, & (4)=\ (3)\times 4 \\ 12x+\ 6y+\ 36z=\ 276, & (5)=\ (3)\times 6 \\ 17x+\ 40z=\ 291, & (6)=\ (1)+(5) \\ x+\ 27z=\ 165, & (7)=\ (4)-(2) \\ 17x+459z=2805, & (8)=\ (7)\times 17 \\ 419z=2514, & (9)=\ (8)-(6) \\ z=6, & (10)=\ (9)\div 419 \\ 27z=162, & (11)=(10)\times 27 \\ =3, & (12)=\ (7)-(11) \\ 6z=36, & (13)=(10)\times 6 \\ 2x=6, & (14)=(12)\times 2 \\ 6z+2x=42, & (15)=(13)+(14) \\ y=4. & (16)=\ (3)-(15) \end{array}$$

Collecting equations (12,) (16,) and (10,) we ha

$$\textit{Ans.}\begin{cases} x=3, & (12)\\ y=4, & (16)\\ z=6. & (10)\end{cases}$$

ELIMINATION BY COMPARISON.

(70.) We may also eliminate one of the unk quantities of two equations, by the following proc

Take the two equations

$$5y-3x=-14, \qquad (1)$$
$$3y+4x=38. \qquad (2)$$

If we, for a moment, consider y as a known qua we may then, from each of these equations, find value of x by Rule under Art. 63.

We thus find

$$x=\frac{14+5y}{3}. \qquad (3)$$

$$x=\frac{38-3y}{4}. \qquad (4)$$

Putting these two values of x equal to each o we have

$$\frac{14+5y}{3}=\frac{38-3y}{4}. \qquad (5)$$

Clearing (5) of fractions, it becomes

$$56+20y=114-9y. \qquad (6)$$

Transposing and uniting terms, we find

$$29y=58.$$

Therefore, $$y=2.$$

This value of y substituted in either of the equations (3) or (4,) will give

$$x=8.$$

The above method of eliminating may be given as in the following

RULE.

I. Find, from each of the given equations, the value of one of the unknown quantities, by Rule under Art. **63,** *on the supposition that the other quantities are known.*

II. Then equate these different expressions of the value of the unknown, thus found, and we shall thus have a number of equations one less than were first given; and they will also contain a number of unknown quantities one less than at first.

III. Operating with these new equations, as was done with the given equations, we can again reduce their number one; and continuing this process, we shall finally have but one equation containing but one unknown quantity, which will then become known.

EXAMPLES.

1. Given $\left.\begin{array}{l} 7x+5y+2z=\ 79, \quad (1) \\ 8x+7y+9z=122, \quad (2) \\ \ \ x+4y+5z=\ 55, \quad (3) \end{array}\right\}$ to find x, y, and z.

By Rule under Art. 63, we find, by using (1) (2) and (3,)

$$x=\frac{79-5y-2z}{7}, \qquad (4)$$

$$x=\frac{122-7y-9z}{8}, \qquad (5)$$

$$x=55-4y-5z. \qquad (6)$$

Equating (4) and (6); and (5) and (6), we have

$$\frac{79-5y-2z}{7}=55-4y-5z, \qquad (7)$$

$$\frac{122-7y-9z}{8}=55-4y-5z. \qquad (8)$$

When cleared of fractions, (7) and (8) become

$$79-5y-2z=385-28y-35z,$$
$$122-7y-9z=440-32y-40z.$$

Transposing and uniting terms, we have

$$23y+33z=306, \qquad (9)$$
$$25y+31z=318. \qquad (10)$$

Equations (9) and (10) give

$$y=\frac{306-33z}{23}, \qquad (11)$$

$$y=\frac{318-31z}{25}. \qquad (12)$$

Equating (11) and (12), we have

$$\frac{306-33z}{23}=\frac{318-31z}{25}, \qquad (13)$$

which reduced, gives

$$z=3.$$

This value of z substituted in (11), gives

$$y=9.$$

And these values of z and y, substituted in (6), give

$$x=4.$$

2. Given $\left.\begin{matrix}\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62,\\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47,\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38,\end{matrix}\right\}$ to find x, y, and z.

These equations, when cleared of fractions, become

$$6x+\ 4y+\ 3z=\ 744, \qquad (1)$$
$$20x+15y+12z=2820, \qquad (2)$$
$$15x+12y+10z=2280. \qquad (3)$$

From (1), (2), and (3), we find

$$z=\frac{744-6x-4y}{3}, \qquad (4)$$

$$z=\frac{2820-20x-15y}{12}, \qquad (5)$$

$$z=\frac{2280-15x-12y}{10}. \qquad (6)$$

Equating (4) with (5); and (4) with (6), we have

$$\frac{744-6x-4y}{3}=\frac{2820-20x-15y}{12}, \qquad (7)$$

$$\frac{744-6x-4y}{3}=\frac{2280-15x-12y}{10}. \qquad (8)$$

Equations (7) and (8), when reduced, become

$$4x+\ y=156, \qquad (9)$$
$$15x+4y=600. \qquad (10)$$

Equations (9) and (10) give

$$y=156-4x, \qquad (11)$$

$$y=\frac{600-15x}{4}. \qquad (12)$$

Equating (11) and (12), we have

$$156-4x=\frac{600-15x}{4}. \qquad (13)$$

This reduced, gives

$$x=24.$$

Having found x, we readily find y and z to be

$$y=60\,;\ z=120.$$

ELIMINATION BY SUBSTITUTION.

(71.) There is still another method of elimination.

1. Suppose we have given the two equations

$$5x+2y=45, \qquad (1)$$

$$4x+\ y=33. \qquad (2)$$

From the first we find

$$y=\frac{45-5x}{2}. \qquad (3)$$

Substituting this value of y in (2), we have

$$4x+\frac{45-5x}{2}=33. \qquad (4)$$

Equation (4), when cleared of fractions, becomes

$$8x+45-5x=66. \qquad (5)$$

This gives

$$x=7.$$

Substituting this value of x in (3), we find

$$y=5.$$

2. Again, suppose we have given, to find x, y, and z, the three equations

$$2x+4y-3z=22, \quad (1)$$
$$4x-2y+5z=18, \quad (2)$$
$$6x+7y-\ z=63. \quad (3)$$

From equation (3,) we obtain

$$z=6x+7y-63. \quad (4)$$

Substituting this value of z, in (1) and (2), and they will become

$$2x+4y-3(6x+7y-63)=22, \quad (5)$$
$$4x-2y+5(6x+7y-63)=18. \quad (6)$$

Equations (5) and (6) become, after expanding, transposing, and uniting terms,

$$16x+17y=167, \quad (7)$$
$$34x+33y=333. \quad (8)$$

Equation (7) gives

$$x=\frac{167-17y}{16}. \quad (9)$$

This value of x, substituted in (8), gives

$$\frac{34(167-17y)}{16}+33y=333. \quad (10)$$

Equation (10), when solved as a simple equation of one unknown quantity, gives

$$y=7.$$

Substituting this value of y in (9), we find

$$x=3.$$

Using these values of x and y in (4), we obtain

$$z=4.$$

(72.) This method of eliminating may be comprehended in the following

RULE.

Having found the value of one of the unknown quantities, from either of the given equations, in terms of the other unknown quantities; substitute it for that unknown quantity in the remaining equations, and we shall thus obtain a new system of equations one less in number than those given. Operate with these new equations as with the first, and so continue until we find one single equation with but one unknown quantity, which will then become known.

EXAMPLES.

1. Given $\left\{\begin{array}{ll} x-w=50, & (1) \\ 3y-x=120, & (2) \\ 2z-y=120, & (3) \\ 3w-z=195, & (4) \end{array}\right\}$ to find w, x, y, and z.

From (1) we find

$$w=x-50. \qquad (5)$$

This value of w, substituted in (4), gives

$$3(x-50)-z=195, \text{ or } 3x-z=345. \qquad (6)$$

Equation (6) gives

$$z=3x-345. \qquad (7)$$

This value of z, substituted in (3), gives

$$2(3x-345)-y=120, \text{ or } 6x-y=810. \quad (8)$$

Equation (8) gives

$$y=6x-810. \quad (9)$$

This value of y, substituted in (2), gives

$$3(6x-810)-x=120, \quad (10)$$

or $$17x=2550. \quad (11)$$

Therefore, $$x=150.$$

This value of x, causes (9) to become

$$y=90.$$

Using the value of x in (7), we find

$$z=105.$$

Finally, using the value of x in (5), we find

$$w=100.$$

2. Given $\left\{\begin{array}{ll} x+\frac{1}{2}y=a, & (1) \\ y+\frac{1}{3}z=a, & (2) \\ z+\frac{1}{4}x=a, & (3) \end{array}\right\}$ to find x, y, and z.

Equation (3) gives

$$z=\frac{4a-x}{4}. \quad (4)$$

This value of z, substituted in (2,) we have

$$y+\frac{4a-x}{12}=a. \quad (5)$$

Clearing of fractions and uniting terms, (5) becomes

$$12y-x=8a. \quad (6)$$

From (6) we find

$$x=12y-8a. \qquad (7)$$

This value of x, substituted in (1), gives

$$12y-8a+\frac{y}{2}=a. \qquad (8)$$

Equation (8) gives

$$25y=18a. \qquad (9)$$

Therefore, $$y=\frac{18a}{25}.$$

This value of y, substituted in (7), gives

$$x=\frac{16a}{25}.$$

Substituting for x, in (4,) its value just found, we have

$$z=\frac{21a}{25}.$$

Hence, collecting values we have,

$$\left.\begin{array}{l} x=\frac{16}{25}a, \\ y=\frac{18}{25}a, \\ z=\frac{21}{25}a. \end{array}\right\}$$

We may observe that if a is any multiple of 25, the above value of x, y, and z, will be integers.

(73.) All equations of the first degree, containing any number of unknown quantities, can be solved by either of the Rules under Articles, 68, 70, and 72, or by a combination of the same.

The student must exercise his own judgment, as to *the choice* of the above Rules. In very many cases, it

will discover many short processes, which depend upon the particular equations given.

(74.) We will now solve a few equations, and shall endeavor to effect their solution in the simplest manner possible.

1. Given $\left\{\begin{matrix} 6x+5y=128, \\ 3x+4y=\ 88, \end{matrix}\right\}$ to find the values of x and y.

Adding the two equations, and dividing the sum by 9, we find

$$x+y=24. \qquad (1)$$

Multiplying (1) by 3, and subtracting the result from the second of the given equations, we have

$$y=16. \qquad (2)$$

Subtracting (2) from (1), we get

$$x=8.$$

2. Given $\left\{\begin{matrix} x+y=a, & (1) \\ y+z=b, & (2) \\ z+x=c, & (3) \end{matrix}\right\}$ to find x, y, and z.

Dividing the sum of these three equations by 2, we find

$$x+y+z=\frac{a+b+c}{2}. \qquad (4)$$

From (4) subtracting, successively, (2), (3), and (1), we find

$$\left.\begin{aligned} x&=\frac{a+c-b}{2},\\ y&=\frac{a-c+b}{2},\\ z&=\frac{-a+b+c}{2}. \end{aligned}\right\} \qquad (A)$$

Equations (1), (2), and (3), of this last example are so related that if in (1) we change x to y, y to z, and a to b, it will correspond with (2). Again, if in (2) we change y to z, z to x, and b to c, it will correspond with (3). Also, if in (3) we change z to x, x to y, and c to a, it will give (1), from which we first started. In each change we have advanced the letters one place lower in the alphabetical scale, observing that when we wish to change the last letters of the series, as z or c, we must change them respectively to x and a, the first of the series.

Since the above changes can be made with the primitive equations (1), (2), (3), without altering the conditions of the question, it follows that the same changes can be made in any of the equations derived from those. Thus, executing those changes in equations (A), we find that the first is changed into the second, the second into the third, and the third in turn is changed into the first.

3. Given $\left\{\begin{aligned} \frac{1}{x}+\frac{1}{y}&=a, \quad (1)\\ \frac{1}{y}+\frac{1}{z}&=b, \quad (2)\\ \frac{1}{z}+\frac{1}{x}&=c, \quad (3) \end{aligned}\right\}$ to find x, y, and z.

5. What is the square root of $-4a^4b^2$?

Ans. Impossible.

6. What is the cube root of $27a^3b^{12}$?

Ans. $3ab^4$.

7. What is the fourth root of $16a^{-3}bx^{-1}$?

Ans. $\pm 2a^{-\frac{3}{4}}b^{\frac{1}{4}}x^{-\frac{1}{4}} = \pm\dfrac{2b^{\frac{1}{4}}}{a^{\frac{3}{4}}x^{\frac{1}{4}}}$.

(80.) By comparing the operations of this rule with those of the rule under Art. 76, we see that involution and evolution of monomials may both be performed by this general

RULE.

Multiply the exponents of the respective letters by the exponent denoting the power or root.

EXAMPLES.

1. What is the cube root of the second power of $8a^3b^9$?

If we first raise $8a^3b^9$ to the second power it will become

$$(8a^3b^9)^2 = 64a^6b^{18}.$$

Extracting the third root, we find

$$(64a^6b^{18})^{\frac{1}{3}} = 4a^2b^6,$$

for the result required.

Again, first extracting the cube root of $8a^3b^9$, it becomes

$$(8a^3b^9)^{\frac{1}{3}} = 2ab^3.$$

Raising this to the second power, it becomes

$$(2ab^3)^2 = 4a^2b^6;$$

the same as before.

(81.) Hence the cube root of the square of a quantity is the same as the square of the cube root of the same quantity.

And, in general, *the nth root of the mth power of a quantity is the same as the mth power of the nth root of the same quantity.*

Therefore, $a^{\frac{4}{5}}$ may be read, the fourth power of the fifth root of a, or, the fifth root of the fourth power of a.

And in the same way, $(a+b)^{\frac{3}{2}}$ is read, the third power of the square root of the sum of a and b, or the square root of the third power of the sum of a and b.

2. What is the value of $(-3ab^2x^3)^{\frac{2}{3}}$?

Ans. $3^{\frac{2}{3}}a^{\frac{2}{3}}b^{\frac{4}{3}}x^2$.

3. What is the value of $(4a^{-2}b^4x)^{\frac{5}{2}}$?

Ans. $\pm 32a^{-5}b^{10}x^{\frac{5}{2}}$.

4. What is the value of $(9a^2b^4)^{\frac{3}{2}}$?

Ans. $\pm 27a^3b^6$.

5. What is the value of $(8a^6x)^{\frac{2}{3}}$?

Ans. $4a^4x^{\frac{2}{3}}$.

6. What is the value of $(64a^6b^6x^{12})^{\frac{3}{2}}$?

Ans. $\pm 512a^9b^9x^{18}$.

7. What is the value of $(64a^6b^6x^{12})^{\frac{2}{3}}$.

Ans. $16a^4b^4x^8$.

CASE II.

(82.) To extract any root of a polynomial, we have the following general

RULE.

I. Having arranged the polynomial according to the powers of some one of the letters, so that the highest power shall stand first, extract the required root of the first term, which will be the first term of the root sought.

II. Subtract the power of this first term of the root from the polynomial, and divide the first term of the remainder, by the first term of the root involved to the next inferior power, and multiplied by the number denoting the root; the quotient will be the second term of the root.

III. Subtract the power of the terms already found from the polynomial, and using the same divisor proceed as before.

This rule obviously verifies itself, since, whenever a new term is added to the root, the whole is raised to the given power, and the result is subtracted from the given polynomial. And when we thus find a power equal to the given polynomial, it is evident that the true root has been found.

1. What is the fifth root of

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\,?$$

OPERATION. ROOT.

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5(a+b.$$
$$a^5$$

$$5a^4) \quad 5a^4b$$

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

EXPLANATION.

We first found the fifth root of the first term a^5, to be a, which we placed to the right of the polynomial for the first term of the root. Raising a to the fifth power and subtracting it from the polynomial, we have $5a^4b$ for the first term of the remainder.

Since the number denoting the root is 5, we raise the first term of the root, a, to the fourth power, which thus becomes a^4, this multiplied by the number denoting the root, gives $5a^4$ for our divisor.

Now, dividing $5a^4b$ by $5a^4$, we get b, which we write for the second term of the root.

Involving this root to the fifth power by actual multiplication, as was done in Ex. 2, Art. 77, we have

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5,$$

which subtracted from the given polynomial leaves no remainder, so that we know that $a+b$ is the true root.

2. What is the square root of

$$4x^4-16x^3+24x^2-16x+4\,?$$

OPERATION. ROOT.

$$4x^4-16x^3+24x^2-16x+4(2x^2-4x+2.$$

$$(2x^2)^2=4x^4$$

$$4x^2) \quad -16x^3$$

$$(2x^2-4x)^2=4x^4-16x^3+16x^2$$

$$4x^2) \quad 8x^2$$

$$(2x^2-4x+2)^2=4x^4-16x^3+24x^2-16x+4$$

3. What is the square root of

$$16x^4+24x^3+89x^2+60x+100\ ?$$

Ans. $4x^2+3x+10$.

4. What is the cube root of

$$a^6+3a^5-3a^4-11a^3+6a^2+12a-8\ ?$$

Ans. a^2+a-2.

5. What is the sixth root of

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\ ?$$

Ans. $a-b$.

6. What is the fourth root of

$$a^4-4a^3b+6a^2b^2-4ab^3+b^4\ ?$$

Ans. $a-b$.

7. What is the seventh root of $m^7+7m^6n+21m^5n^2+35m^4n^3+35m^3n^4+21m^2n^5+7mn^6+n^7$?

Ans. $m+n$.

(83.) If we carefully observe the law by which a polynomial is raised to the second power, we shall, by reversing the process, be enabled to deduce a rule for the extraction of the *square root* of a polynomial, which

will be more simple than the above general rule, and of more interest, since the arithmetical rule is deduced from it.

By actual multiplication, we find

$$(a+b)^2=a^2+2ab+b^2,$$
$$(a+b+c)^2=a^2+2ab+b^2+2(a+b)c+c^2,$$
$$(a+b+c+d)^2$$
$$=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2,$$
$$(a+b+c+d+e)^2$$
$$=\left\{\begin{array}{l} a^2+2ab+b^2+2(a+b)c+c^2 \\ +2(a+b+c)d+d^2+2(a+b+c+d)e+e^2. \end{array}\right\}$$
$$\text{\&c} \qquad\qquad \text{\&c.}$$

From the above we discover, that

(84.) *The square of any polynomial is equal to the square of the first term, plus twice the first term into the second, plus the square of the second; plus twice the sum of the first two into the third, plus the square of the third; plus twice the sum of the first three into the fourth, plus the square of the fourth; and so on.*

(85.) Hence, the square root of a polynomial can be found by the following

RULE.

I *After arranging the polynomial according to the powers of some one of the letters, take the root of the first term for the first term of the required root, and subtract its square from the* polynomial.

II. Bring down the next two terms for a dividend. Divide it by twice the root just found, and add the quotient, both to the root and to the divisor. Multiply the divisor thus increased, into the term last placed in the root, and subtract the product from the dividend.

III. Bring down two or three additional terms and proceed as before.

EXAMPLES.

1. What is the square root of

$$a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2?$$

OPERATION. ROOT.

$$\begin{array}{lll}
 & & (a+b+c+d. \\
a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2 & & \\
a^2 & & \\
\hline
\quad 2ab+b^2 & & \\
\quad 2ab+b^2 & & \\
\hline
2(a+b)+c & 2(a+b)c+c^2 & \\
 & 2(a+b)c+c^2 & \\
\hline
2(a+b+c)+d & & 2(a+b+c)d+d^2 \\
 & & 2(a+b+c)d+d^2 \\
\hline
\end{array}$$

2. What is the square root of

$$4x^6+12x^5+5x^4-2x^3+7x^2-2x+1?$$

OPERATION. ROOT.

$(2x^3+3x^2-x+1.$

$4x^6+12x^5+5x^4-2x^3+7x^2-2x+1$

$4x^6$

$4x^3+3x^2$ $12x^5+5x^4$

$12x^5+9x^4$

$4x^3+6x^2-x$ $-4x^4-2x^3+7x^2$

$-4x^4-6x^3+x^2$

$4x^3+6x^2-2x+1$ $4x^3+6x^2-2x+1$

$4x^3+6x^2-2x+1$

0

3. What is the square root of

$$x^4-2x^2y^2-2x^2+y^4+2y^2+1?$$

Ans. x^2-y^2-1.

4. What is the square root of

$$9x^4y^4-30x^3y^3+25x^2y^2?$$

Ans. $3x^2y^2-5xy$.

5. What is the square root of

$$a^2+2ab-2ac+b^2-2bc+c^2$$

Ans. $a+b-c$.

6. What is the square root of

$$4m^2-36mn+81n^2?$$

Ans. $2m-9n$.

being useless, they have lent their aid in the solution of questions, which required the most refined and delicate analysis.

(103.) *Before closing this chapter, we will show the interpretation of the following symbols.*

$$\frac{0}{A}, \quad A \cdot \frac{A}{0}, \quad \frac{0}{0}.$$

We know from the nature of multiplication, that 0 multiplied by a finite quantity, that is, 0 repeated a finite number of times, must still remain equal to 0; hence, we have this condition

$$0 \times A = 0. \qquad (1)$$

Dividing both members of (1) by A, we find

$$0 = \frac{0}{A}. \qquad (2)$$

Therefore the symbol $\frac{0}{A}$, will always be equal to 0, as as long as A is a finite quantity.

(104.) Since the quotient arising from dividing one quantity by another, becomes greater in proportion as the divisor is diminished, it follows that when the divisor becomes less than any assignable quantity, then the quotient will exceed any assignable quantity. Hence, it is usual for mathematicians to say, that $\frac{A}{0}$ is the representation of an infinite quantity. The symbol employed to represent *infinity* is ∞, so that we have

$$\frac{A}{0} = \infty.$$

(105.) Dividing both members of (1) by 0, we find

$$A=\frac{0}{0}. \qquad (4)$$

This being true for all values of A, shows that $\frac{0}{0}$ is the symbol of an *indeterminate quantity*.

To illustrate this last symbol, we will take several

EXAMPLES.

1. What is the value of the fraction $\frac{x^2-a^2}{bx-ab}$, when $x=a$?

Substituting a for x, our fraction will become

$$\frac{x^2-a^2}{bx-ab}=\frac{a^2-a^2}{ab-ab}=\frac{0}{0}= \textit{an indeterminate quantity.}$$

If, before substituting a for x, we divide both numerator and denominator of the given fraction, by $x-a$, (Art. 42,) we find

$$\frac{x^2-a^2}{bx-ab}=\frac{x+a}{b}.$$

Now, substituting a for x, in this reduced form, we find

$$\frac{x+a}{b}=\frac{a+a}{b}=\frac{2a}{b}.$$

Therefore, $\frac{2a}{b}$ is the true value of $\frac{x^2-a^2}{bx-ab}$, when $x=a$.

2. What is the value of $\frac{x^2-ax}{x^2-2ax+a^2}$, when $x=a$?

Substituting a for x we find,

$$\frac{x^2-ax}{x^2-2ax+a^2}=\frac{a^2-a^2}{a^2-2a^2+a^2}=\frac{0}{0},$$

If we reduce this fraction by dividing both numerator and denominator by $x-a$, we find

$$\frac{x^2-ax}{x^2-2ax+a^2}=\frac{x}{x-a}.$$

Now, writing a for x in this reduced form, we find

$$\frac{x}{x-a}=\frac{a}{a-a}=\frac{a}{0}=\infty. \quad \text{(Art.10.)}$$

3. What is the value of $\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}$, when $x=a$?

When a is substituted for x, we have

$$\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}=\frac{a^3-3a^3+3a^3-a^3}{ab-ab}=\frac{0}{0}.$$

Reducing, by dividing numerator and denominator by $x-a$, we find

$$\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}=\frac{x^2-2ax+a^2}{b}.$$

Writing a for x, we have

$$\frac{x^2-2ax+a^2}{b}=\frac{a^2-2a^2+a^2}{b}=\frac{0}{b}=0. \quad \text{(Art. 103.)}$$

(106.) From the above, we conclude that whenever an algebraic fraction is reduced to the form $\frac{0}{0}$ there exists a factor common to both numerator and denominator.

If we reduce the fraction, by dividing both numerator and denominator by this common factor, it will become of one of the following forms:

$$\frac{A}{B} = \textit{a finite quantity.}$$

$$\frac{0}{A} = 0 = \textit{no value.}$$

$$\frac{A}{0} = \infty = \textit{an infinite quantity.}$$

CHAPTER V.

QUADRATIC EQUATIONS.

(107.) We have already (Art. 53,) defined a *quadratic equation*, to be an equation in which the unknown quantity does not exceed the second degree.

The most general form of a quadratic equation of one unknown quantity is

$$ax^2+bx=c. \qquad (1)$$

Dividing all the terms of (1) by a, (Axiom IV,) we find

$$x^2+\frac{b}{a}x=\frac{c}{a}, \qquad (2)$$

where, if we assume $A=\frac{b}{a}$, and $B=\frac{c}{a}$, we shall have

$$x^2+Ax=B. \qquad (3)$$

Equation (3) is as general a form for quadratics as equation (1).

In (3) A and B can have any values either positive or negative.

(108.) When $A=0$, equation (3) will become

$$x^2=B, \qquad (4)$$

which is called an *incomplete quadratic equation*, since one of the terms in the general forms (1) and (3) is wanting.

(109.) When $B=0$, equation (3) will become

$$x^2+Ax=0,$$

which divided by x is reduced to

$$x+A=0,$$

which is no longer a quadratic equation, but a simple equation.

(110.) If $A=0$ and $B=0$ at the same time, equation (3) will become

$$x^2=0,$$

which can only be satisfied by taking $x=0$.

INCOMPLETE QUADRATIC EQUATIONS.

(111.) We have just seen that the general form of an incomplete quadratic equation is

$$x^2=B. \qquad (1)$$

If we extract the square root of both members of this equation, we shall (Art. 79,) have

$$x=\pm\sqrt{B}. \qquad (a)$$

Equation (a) may be regarded as a general solution of incomplete quadratic equations.

(112.) To find the value of the unknown, when the equation which involves it leads to an incomplete quadratic equation, we have this

RULE.

II. Clear the equation of fractions by the same rule as for simple equations.

II. Then transpose and unite the like terms, if necessary, observing the rule under Art. 60, *and we shall thus obtain, after dividing by the coefficient of* x^2, *an equation of the form of* $x^2=B$. *Extracting the square root of both members, we shall find* $x=\pm\sqrt{B}$.

EXAMPLES.

1. Given $\frac{x^2+2}{19}+7=9$, to find the values of x.

This, when cleared of fractions, by multiplying by 19, becomes

$$x^2+2+133=171.$$

Transposing and uniting terms, we find $x^2=36$. If we compare this with our general form, we shall see that $B=36$. Extracting the square root, we have $x=\pm 6$, or as it may be better expressed, $x=6$, or $x=-6$.

We must be careful to interpret the double sign $\pm$, correctly, the meaning of which is, that the quantity before which it is placed may be either plus, or it may be minus. It does not mean that the quantity can be both plus and minus at the same time.

2. Given $\frac{3}{14x^2}+\frac{1}{2}=\frac{346}{686}$, to find the values of x.

This cleared of fractions, becomes

$$147+343x^2=346x^2.$$

Transposing and uniting terms $3x^2=147$.

Dividing by 3. $x^2=49$.

Extracting the square root, we find $x=\pm 7$.

3. Given $x^2-\frac{25x^2}{36}=44$, to find the values of x.

Ans. $x=\pm 12$.

4. Given $8+5x^2=\frac{x^2}{5}+4x^2+28$, to find the values of x.

Ans. $x=\pm 5$.

5. Given $2+\frac{x^2}{3}-7=\frac{x^2}{9}+13$, to find the values of x.

Ans. $x=\pm 9$.

6. Find the values of x from the equation

$$2x^2-35=80+\frac{15-x^2}{2}.$$

Ans. $x=\pm 7$.

7. Find the values of x from the equation

$$6+\frac{3x(x+1)}{5}=7+\frac{3x-2}{5}$$

Ans. $x=\pm 1$.

8. Find the values of x from the equation

$$\frac{x^2+7x+17}{35}=\frac{3+x}{5}.$$

Ans. $x=\pm 2$.

9. Find the values of x from the equation

$$\frac{x(x^2+1)}{20}=\frac{x}{2}.$$

Ans. $x=\pm 3$.

10. Find the values of x from the equation

$$\frac{x(x+3)}{9}-\frac{x+1}{3}=\frac{4}{9}+1.$$

Ans. $x=\pm 4$.

11. Given $\frac{945-171x^2}{43}=\frac{37-x^2}{3}$, to find the values of x.

Ans. $x=\pm 1$.

12. Given $\frac{x^2-8x+80}{2}=3x^2-4x$, to find the values of x.

113. If an equation involving one unknown quantity can be reduced to the form $x^n=N$, the value of x can be found by simply extracting the nth root of both members, thus

$$x=\sqrt[n]{N}.$$

114. Where it must be observed (Art. 79,) that when n is an *even* number, the value of x will be either plus or minus for all positive values of N, but for negative values of N the value of x will be impossible. When n is an *odd* number, the value of x will have the same sign that N has.

(115.) If the equation can be reduced to the form $x^{\frac{1}{m}}=N$, then x can be found by raising both members to the mth power, thus:

$$x=N^m.$$

Where x will be positive for all values of N, provided m is an *even* number, but when m is an *odd* number, then x will have the same sign as N.

(116.) Finally, when the equation can be reduced to the form

$$x^{\frac{n}{m}}=N.$$

We must first involve both members to the mth power, and then extract the nth root, or else we may first extract the nth root, and then involve to the mth power. (Art. 81.)

Thus, $$x=\sqrt[n]{N^m}.$$

EXAMPLES.

1. Given $\frac{\sqrt{x+28}}{\sqrt{x+4}}=\frac{\sqrt{x+38}}{\sqrt{x+6}}$ to find x.

This, when cleared of fractions, becomes

$$x+34\sqrt{x+168}=x+42\sqrt{x+152}.$$

Transposing and uniting terms, we have

$$8\sqrt{x}=16.$$

Dividing by 8. $$\sqrt{x}=2.$$

Raising to the second power. $$x=4.$$

2. Given $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$, to find x.

This equation, when cleared of fractions, by multiplying by $\sqrt{a+x}$, becomes

$$\sqrt{ax+x^2}+a+x=2a,$$

or, $$\sqrt{ax+x^2}=a-x.$$

Squaring both members.

$$ax+x^2=a^2-2ax+x^2,$$

or $$3ax=a^2,$$

or $$x=\frac{a}{3}.$$

3. Given $3+x^{\frac{2}{3}}=7$. to find the values of x.

Ans. $x=\pm 8$.

4. Given $(y^m-b)^{\frac{1}{2}}=a-d$, to find the values of y.

Ans. $y=\{(a-d)^2+b\}^{\frac{1}{m}}$.

5. Given $\sqrt{x-32}=16-\sqrt{x}$, to find the value of x.

Ans. $x=81$.

6. Given $(x+a)^{\frac{1}{2}}=\dfrac{a+b}{(x-a)^{\frac{1}{2}}}$, to find the values of x.

Ans. $x=\pm(2a^2+2ab+b^2)^{\frac{1}{2}}$.

7. Given $\dfrac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}=\dfrac{4}{x-1}$, to find the values of x.

If we multiply the numerator and denominator of the left-hand member by the numerator, it will become

$$(\sqrt{x}+\sqrt{x-1})^2=\frac{4}{x-1}.$$

Extracting the square root, we find

$$\sqrt{x}+\sqrt{x-1}=\frac{\pm 2}{\sqrt{x-1}}.$$

This readily gives

$$x=\tfrac{9}{5}, \text{ or } x=-\tfrac{1}{3}.$$

8. Given $(x^3-15)^{\frac{1}{2}}=7$, to find x.

Ans. $x=4$.

9. Given $(x^4-7)^{\frac{1}{2}}=3$, to find the values of x.

Ans. $x=\pm 2$.

10. Given $(x^{\frac{3}{4}}-1)^2=49$, to find x.

Ans. $x=16$.

11. Given $\frac{\sqrt{x}+1}{\sqrt{x}+5}=\frac{\sqrt{x}-3}{\sqrt{x}+4}$, to find x.

Ans. $x=\frac{361}{9}$.

COMPLETE QUADRATIC EQUATIONS.

(117.) We have already seen, that

$$ax^2+lx=c, \qquad (A)$$

is the most general form of a quadratic equation, where

$a=$ the coefficient of the first term,

$b=$ the coefficient of the second term,

$c=$ the term independent of x.

If we multiply the general quadratic equation (A) by $4a$, it will become

$$4a^2x^2+4alx=4ac. \qquad (1)$$

Adding b^2 to both members of (1), it becomes

$$4a^2x^2+4alx+b^2=b^2+4ac. \qquad (2)$$

If we extract the square root of both members of (2), we get

$$2ax+b=\pm\sqrt{b^2+4ac}. \qquad (3)$$

By transposition, (3) gives

$$2ax=-b\pm\sqrt{b^2+4ac}. \qquad (4)$$

Dividing (4) by $2a$, we get

$$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}. \qquad (B)$$

Formula (B) may be regarded as a general solution of all complete quadratic equations.

If we translate it into common language, we shall obtain a rule for solving complete quadratics.

Hence, to find the value of an unknown quantity, when given by a quadratic equation, we have this

RULE.

Having reduced the equation to the general form $ax^2+bx=c$, *we can find* x, *by taking the coefficient of the second term with its sign changed, plus or minus the square root of the square of the coefficient of the second term increased by four times the coefficient of the first term into the term independent of* x; *and the whole divided by twice the coefficient of the first term.*

EXAMPLES.

1. Given $4x-\frac{36-x}{x}=46$, to find the values of x.

This, when cleared of fractions, becomes

$$4x^2-36+x=46x.$$

Transposing and uniting terms, we have

$$4x^2-45x=36.$$

This compared with the general form

$$ax^2+bx=c,$$

gives $a=4$; $b=-45$; $c=36$.

The square of the coefficient of the second term is

$$(-45)^2=2025.$$

Four times the coefficient of the first term into the term independent of x, is

$$4\times4\times36=576.$$

Therefore, taking the square root of the square of the coefficient of the second term increased by four times the coefficient of the first term into the term independent of x, we get

$$\pm\sqrt{2025+576}=\pm\sqrt{2601}=\pm 51.$$

This added to the coefficient of the second term with the sign changed, gives

$$45\pm 51.$$

which must be divided by twice the coefficient of the first term. Hence,

$$x=\frac{45\pm 51}{8},$$

If we take the upper sign, we get

$$x=\frac{45+51}{8}=12.$$

If we take the lower sign, we find

$$x=\frac{45-51}{8}=-\frac{3}{4}.$$

Therefore, $$x=12, \text{ or } -\frac{3}{4}.$$

Either of which values of x, will verify the equation.

2. Given $\frac{3x-4}{x-4}=9-\frac{x-2}{2}$, to find the values of x.

This, when reduced to the general form, becomes

$$x^2-18x=-72.$$

Squaring 18, we get

$$(18)^2=324.$$

Four times the first coefficient multiplied into —72, gives

$$4\times-72=-288,$$

which added to 324, gives 36, the square root of which is ± 6. Therefore

$$x=\frac{18\pm 6}{2}=12 \text{ or } 6.$$

3. Given $\sqrt{3x-5}=\frac{\sqrt{7x^2+36x}}{x}$, to find the values of x.

Squaring both members, we have

$$3x-5=\frac{7x^2+36x}{x^2}.$$

This, cleared of fractions, becomes

$$3x^2-5x=7x+36.$$

Transposing and uniting terms, we have

$$3x^2-12x=36.$$

This, divided by 3, gives

$$x^2-4x=12.$$

Therefore, $x=\frac{4\pm\sqrt{(4)^2+4\times 12}}{2}=\frac{4\pm 8}{2}=6$, or -2.

4. Given $\frac{3}{x^2-3x}+\frac{3}{x^2+4x}=\frac{27}{8x}$, to find the vlaues of x.

This, by reduction, becomes

$$9x^2-7x=116.$$

Therefore, $x=\frac{7\pm\sqrt{7^2+4\times 9\times 116}}{18}=\frac{7\pm 65}{18}=4$, or $-3\frac{1}{9}$

5. Given $\frac{x^2+12}{2}+\frac{x}{2}=4x$, to find the values of x.

This reduced becomes

$$x^2-7x=-12.$$

Therefore, $x=\frac{7\pm\sqrt{7^2+4\times-12}}{2}=\frac{7\pm1}{2}=4$, or 3.

(118.) An equation of the form

$$ax^{2n}+bx^n=c, \qquad (A)$$

can be solved by the above rule, which indeed will agree with the form under consideration in the particular case of $n=1$.

If, in the above equation, we write y for x^n, and consequently y^2 for x^{2n}, it will become

$$ay^2+by=c,$$

which is precisely of the form of (A), Art. 117. Consequently,

$$y=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

Resubstituting x^n for y, we have

$$x^n=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

And, $$x=\left\{\frac{-b\pm\sqrt{b^2+4ac}}{2a}\right\}^{\frac{1}{n}}. \qquad (B)$$

This value of x must hold for all values of the constants n, a, b, and c, whether positive or negative, integral or fractional.

EXAMPLES.

1. Given $x^4+ax^2=b$, to find x.

This becomes $y^2+ay=b$, when for x^2 we write y.

Therefore, $$y=\frac{-a\pm\sqrt{a^2+4b}}{2}=x^2.$$

Hence, $$x=\pm\left\{\frac{-a\pm\sqrt{a^2+4b}}{2}\right\}^{\frac{1}{2}}.$$

2. Given $3x^{2n}-2x^n=8$, to find x.

$$x^n=\frac{2\pm10}{6}=2.$$

Therefore, $$x=\sqrt[n]{2}.$$

3. Given $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}$, to find x.

If, for $1+x-x^2$, we put y^2, our equation will become

$$2y^2-y=-\frac{1}{9},$$

or $$18y^2-9y=-1.$$

Therefore, $$y=\frac{9\pm3}{36}=\frac{1}{3}, \text{ or } \frac{1}{6}.$$

Hence, $$y^2=\frac{1}{9}, \text{ or } \frac{1}{36}.$$

Resubstituting $1+x-x^2$, for y^2, we have, when we take the first value of y^2,

$$1+x-x^2=\frac{1}{9},$$

or $$9x^2-9x=8.$$

Therefore, $$x=\frac{9\pm 3\sqrt{41}}{18}=\frac{1}{2}+\frac{1}{6}\sqrt{41}, \text{ or } \frac{1}{2}-\frac{1}{6}\sqrt{41}.$$

When we take the other value of y^2, we have

$$1+x-x^2=\frac{1}{36},$$

or $$36x^2-36x=35.$$

Therefore, $$x=\frac{36\pm 24\sqrt{11}}{72}=\frac{1}{2}+\frac{1}{3}\sqrt{11}, \text{ or } \frac{1}{2}-\frac{1}{3}\sqrt{11}.$$

Collecting these four values of x, we find

$$x=\tfrac{1}{2}+\tfrac{1}{6}\sqrt{41},$$
$$x=\tfrac{1}{2}-\tfrac{1}{6}\sqrt{41},$$
$$x=\tfrac{1}{2}+\tfrac{1}{3}\sqrt{11},$$
$$x=\tfrac{1}{2}-\tfrac{1}{3}\sqrt{11}.$$

4. Given $x^4-25x^2=-144$, to find the four values of x.

Let $x^2=y$, and the above equation will become

$$y^2-25y=-144.$$

Hence, $$y=16, \text{ or } y=9.$$

Consequently, $x=+4$, or -4; or else $x=+3$, or -3.

5. Given $x^4-7x^2=8$, to find the four values of x.

Assume $x^2=y$, and we find,

$$y^2-7y=8.$$

This solved gives

$$y=8, \text{ or } y=-1.$$

When $y=8$, we find

$$x=\pm\sqrt{8}.$$

But when $y=-1$, we find

$$x=\pm\sqrt{-1}.$$

These two last values are *impossible*.

6. Given $2x-7\sqrt{x}=99$, to find x.

If we let $\sqrt{x}=y$, we shall have

$$2y^2-7y=99.$$

Hence, $y=9, \text{ or } y=-\frac{11}{2}$.

If we take 9 for the value of y, we find $x=81$. But if we take $-\frac{11}{2}$ for the value of y, we find $x=\frac{121}{4}$.

7. Given $2x^2+\sqrt{2x^2+1}=11$, to find the four values of x.

Adding 1 to both members of the equation, we have

$$2x^2+1+\sqrt{2x^2+1}=12.$$

Assume $2x^2+1=y^2$,

and we obtain $y^2+y=12$.

This gives $y=3, \text{ or } y=-4$.

Hence we have these two equations,

$$2x^2+1=9,$$

$$2x^2+1=16.$$

The first of these gives

$$x=2, \text{ or } x=-2.$$

The second gives

$$x=\tfrac{1}{2}\sqrt{30}, \text{ or } x=-\tfrac{1}{2}\sqrt{30}.$$

8. Given $x^6-Sx^3=513$, to find one of the values of x.

Ans. $x=3$.

9. Given $x^4+4x^2=12$, to find two values of x.

Ans. $x=\pm\sqrt{2}$.

(119.) When, in the general form for quadratics $a=1$, the rule under Art. 117, is susceptible of considerable modification.

If we substitute 1 for a in formulas (A) and (B), Art. 117, they will become

$$x^2+bx=c. \qquad \text{(C)}$$

$$x=\frac{-b\pm\sqrt{b^2+4c}}{2}=-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2+c}. \qquad \text{(D)}$$

Therefore, when the coefficient of x^2 is a unit, the value of x can be found from formula (D).

All quadratic equations may be put under the form of (C), by dividing all the terms by the coefficient of x^2, so that the expression of (D) for x must be as general as that of (B), Art. 117.

Hence, for the solution of quadratic equations, we have this second

RULE.

Having reduced the equation to the form $x^2+bx=c$, *we can find* x, *by taking half the coefficient of the second term, with its sign changed, plus or minus the square root of the square of the half of the coefficient of the second term, increased by the term independent of* x.

EXAMPLES.

1. Given $x^2-10x=-24$, to find x.

In this example, half the coefficient of the second term is 5, which squared and added to -24, the term independent of x is 1. Extracting the square root of 1, we have ± 1.

Therefore, $x=5\pm 1=6$, or 4.

2. Given $\frac{x}{x+60}=\frac{7}{3x+5}$, to find x.

This, cleared of fractions, becomes

$$3x^2-5x=7x+420.$$

Transposing and uniting terms, we have

$$3x^2-12x=420.$$

Dividing by 3, we have

$$x^2-4x=140.$$

Therefore, $x=2\pm 12=14$, or -10.

3. Given $\frac{x+12}{x}=\frac{x}{x+12}+\frac{26}{5}$, to find x.

Ans. $x=3$, or -15.

4. Given $3x^6+42x^3=3321$, to find x.

Ans. $x=3$, or $(-41)^{\frac{1}{3}}$.

5. Given $x(x-1)-\frac{x^2}{5}=(5-x)\left(1+\frac{x}{5}\right)+1$, to find x.

Ans. $x=-2$, or $+3$.

6. Given $\frac{x^2-x}{2}+3x=8-\frac{x}{2}$, to find x.

Ans. $x=2$, or -8.

7. Given $x^2-8x=-17$, to find the values of x.

Ans. $x=4+\sqrt{-1}$, or $4-\sqrt{-1}$.

(120.) EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES, WHICH INVOLVE IN THEIR SOLUTION QUADRATIC EQUATIONS.

EXAMPLES.

1. Given $\left\{\begin{array}{ll} x+y=10, & (1) \\ xy=16, & (2) \end{array}\right\}$ to find x and y.

The square of (1) is

$$x^2+2xy+y^2=100. \qquad (3)$$

Subtracting 4 times the second from the (3), we get

$$x^2-2xy+y^2=36. \qquad (4)$$

The square root of (4) is

$$x-y=\pm 6. \qquad (5)$$

Half the sum of (1) and (5), gives

$$x=8, \text{ or } 2.$$

Half the difference of (1) and (5), gives

$$y=2, \text{ or } 8.$$

2. Given $\left\{\begin{array}{ll} x+y=a, & (1) \\ x^2+y^2=b, & (2) \end{array}\right\}$ to find x and y.

Squaring (1), we have

$$x^2+2xy+y^2=a^2. \qquad (3)$$

Subtracting (2) from (3), we get

$$2xy=a^2-b. \qquad (4)$$

Subtracting (4) from (2), we find

$$x^2-2xy+y^2=2b-a^2. \qquad (5)$$

Extracting the square root of (5), we get

$$x-y=\pm\sqrt{2b-a^2}. \qquad (6)$$

Taking half the sum of (1) and (6), we get

$$x=\frac{a}{2}\pm\frac{1}{2}\sqrt{2b-a^2}. \qquad (7)$$

Subtracting (7) from (1), we find

$$y=\frac{a}{2}\pm\frac{1}{2}\sqrt{2b-a^2}.$$

3. Given $\left\{\begin{array}{ll} x(y+z)=a, & (1) \\ y(z+x)=b, & (2) \\ z(x+y)=c, & (3) \end{array}\right\}$ to find x, y, and z.

If we take the sum of (1), (2), and (3), after expanding them, we shall have

$$2xy+2xz+2yz=a+b+c. \qquad (4)$$

If we subtract twice (3) from (4), we get

$$2xy=a+b-c. \qquad (5)$$

In a similar manner, by subtracting twice (2) from (4), we find

$$2xz=a-b+c. \qquad (6)$$

Subtracting twice (1) from (4), we find

$$2yz=-a+b+c. \qquad (7)$$

Dividing equations (5), (6), and (7), by 2, we find

$$xy=\frac{a+b-c}{2}. \qquad (8)$$

$$xz=\frac{a-b+c}{2}. \qquad (9)$$

$$yz=\frac{-a+b+c}{2}. \qquad (10)$$

Taking the continued product of (8), (9), and (10), we have

$$x^2y^2z^2=\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right) \qquad (11)$$

Dividing (11), successively by the squares of (10), (9), and (8), we have

$$x^2=\frac{(a+b-c)(a-b+c)}{2(-a+b+c)}. \qquad (12)$$

$$y^2=\frac{(a+b-c)(-a+b+c)}{2(a-b+c)}. \qquad (13)$$

$$z^2=\frac{(a-b+c)(-a+b+c)}{2(a+b-c)}. \qquad (14)$$

Extracting the square roots of (12), (13), and 14), we have

$$x=\pm\left\{\frac{(a+b-c)(a-b+c)}{2(-a+b-c)}\right\}^{\frac{1}{2}}. \qquad (15)$$

$$y=\pm\left\{\frac{(a+b-c)(-a+b+c}{2(a-b+c)}\right\}^{\frac{1}{2}}. \qquad (16)$$

$$z=\pm\left\{\frac{(a-b+c)(-a+b+c)}{2(a+b-c)}\right\}^{\frac{1}{2}}. \qquad (17)$$

4. Given $\left.\begin{matrix}x+\frac{1}{y}=a,\\ y+\frac{1}{x}=b,\end{matrix}\right\}$ to find x and y.

$$\textit{Ans.}\left\{\begin{matrix}x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{a}{b}}.\\ y=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-\frac{b}{a}}.\end{matrix}\right.$$

5. Given $\left.\begin{matrix}x^2+y^2=a,\\ xz=b,\\ yz=c,\end{matrix}\right\}$ to find x, y, and z.

$$\textit{Ans.}\left\{\begin{matrix}x=b\sqrt{\frac{a}{b^2+c^2}}.\\ y=c\sqrt{\frac{a}{b^2+c^2}}.\\ z=\sqrt{\frac{b^2+c^2}{a}}.\end{matrix}\right.$$

(121.) QUESTIONS WHICH REQUIRE FOR THEIR SOLUTION A KNOWLEDGE OF QUADRATIC EQUATIONS.

1. A widow possessed 13000 dollars, which she divided into two parts, and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Let x = the rate per cent. of the first part.

Let y = the rate per cent. of the second part.

Now, since the incomes of the two parts were equal, they must have been to each other reciprocally as x to y. Hence, if my denote the first part, then will mx denote the second part.

We shall then have

$$m(x+y)=13000.$$

Consequently, $$m=\frac{13000}{x+y}.$$

Therefore, $$\frac{13000y}{x+y}=\text{the first part.}$$

$$\frac{13000x}{x+y}=\text{the second part.}$$

The interest on these parts, at y and x per cent., respectively, is

$$\frac{130y^2}{x+y} \text{ and } \frac{130x^2}{x+y}.$$

Hence, by the conditions of the question, we have

$$\frac{130y^2}{x+y}=360. \qquad (1)$$

$$\frac{130x^2}{x+y}=490. \qquad (2)$$

Dividing (2) by (1), we get

$$\frac{x^2}{y^2}=\frac{49}{36}. \qquad (3)$$

Extracting the square root of (3), we have

$$\frac{x}{y}=\frac{7}{6}. \qquad (4)$$

Subtracting (1) from (2), we have

$$\frac{130(x^2-y^2)}{x+y}=130. \qquad (5)$$

Dividing both numerator and denominator, of the left-hand member of (5), by $x+y$, and also dividing both members by 130, we get

$$x-y=1. \qquad (6)$$

Dividing (6) by y, we find

$$\frac{x}{y}-1=\frac{1}{y}. \qquad (7)$$

Subtracting (7) from (4), we have

$$1=\frac{7}{y}-\frac{1}{y} \qquad (8)$$

Clearing (8) of fractions, we obtain

$$6y=7y-6 \qquad (9)$$

Therefore, $$y=6. \qquad (10)$$

Adding (10) and (6) we get $x=7$.

Therefore the rate per cent. of the first part was 7, and that of the second part was 6.

2. A certain capital is on interest at 4 per cent.; if we multiply the number of dollars in the capital, by the number of dollars in the interest for 5 months, we obtain \$117041$\frac{2}{3}$. What is the capital?

Ans. [redacted]

3. There are two numbers, one of which is greater than the other by 8, and whose product is 240. What numbers are they? *Ans.* 12 and 20.

4. The sum of two numbers is $=a$, their product $=b$. What numbers are they?

$$Ans. \frac{a+\sqrt{(a^2-4b)}}{2}, \frac{a-\sqrt{(a^2-4b)}}{2}.$$

5. It is required to find a number such, that if we multiply its third part by its fourth, and to the product add five times the number required, the sum exceeds the number 200 by as much as the number sought is less than 280. *Ans.* 48.

6. A person being asked his age, answered, "My mother was 20 years old when I was born, and her age, multiplied by mine, exceeds our united ages by 2500." What was his age? *Ans.* 42 years.

7. Determine the fortunes of three persons, A, B, C, from the following data: For every \$5 which A possesses B has \$9, and C \$10. Farther, if we multiply

A's money (expressed in dollars, and considered merely as a number) by B's; and B's money by C's, and add both products to the united fortunes of all three, we shall have 8832. How much had each?

Ans. A $40, B $72, C $80.

8. A person buys some pieces of cloth, at equal prices, for $60. Had he received three more pieces for the same sum, each piece would have cost him $1 less. How many pieces did he buy? *Ans.* 12 pieces.

9. Two travellers, A and B, set out at the same time, from two different places, C and D; A, from C to D; and B, from D to C. On the way they met, and it then appears that A had already gone 30 miles more than B, and, according to the rate at which they travel, A calculates that he can reach the place D in 4 days, and that B can arrive at the place C in 9 days. What is the distance between C and D? *Ans.* 150 miles.

10. Divide the number 60 into two such parts, that their product may be to the sum of their squares, in the ratio of 2 to 5. *Ans.* 20 and 40.

11. A grazier bought as many sheep as cost him $150, and after reserving 15 out of the number, he sold the remainder for $135, and gained $\$\frac{1}{4}$ a head. How many sheep did he buy? *Ans.* 75 sheep.

12. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted?

Ans. 24.

13. Two partners, A and B, gained $140 by trade; A's money was 3 months in trade, and his gain was $60 less than his stock; and B's money which was $50 more than A's, was in trade 5 months; what was A's stock?

Ans. $100.

14. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. What are the parts?

Ans. 10 and 14.

15. A company at a tavern had $60 to pay for their reckoning; but, before the bill was settled, two of them left the room, and then those who remained had $1 a-piece more to pay than before. How many were there in the company? *Ans.* 12.

16. There are two numbers whose difference is 15, and half their product is equal to the cube of the less number. What are those numbers?

Ans. 3 and 18.

17. A merchant bought a certain number of pieces of cloth, for $200, which he sold again at $10½ per piece, and gained by the bargain as much as one piece cost him. What was the number of pieces?

Ans. 20 pieces.

18. A and B together, agree to dig 100 rods of ditch for $100. That part of the ditch on which A was employed was more difficult of excavation than the part on which B was employed. It was therefore agreed that A should receive for each rod 25 cents more than B received for each rod which he dug. How many rods

must each dig, and at what prices, so that each may receive just $50?

Answer.

$$\begin{cases} \text{A must dig } \dfrac{400}{5+\sqrt{17}} \text{ rods at } \dfrac{5+\sqrt{17}}{8} \text{ dollars per rod.} \\ \text{B must dig } \dfrac{400}{3+\sqrt{17}} \text{ rods at } \dfrac{3+\sqrt{17}}{8} \text{ dollars per rod.} \end{cases}$$

PROPERTIES OF THE ROOTS OF QUADRATIC EQUATIONS.

(122.) We have seen that all quadratic equations can be reduced to this general form.

$$x^2+ax=b. \qquad (1)$$

This, when solved by the Rule under Art. 118, gives

$$x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b}. \qquad (2)$$

Therefore the two values of x are

$$-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}. \qquad (3)$$

$$-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}. \qquad (4)$$

(123.) Now, since $\frac{a^2}{4}=\left(\frac{a}{2}\right)^2$ is always positive for all real values of a, it follows that the sign of the expression $\frac{a^2}{4}+b$, depends upon the value of b.

(124.) When b is positive, or when b is negative and less than $\frac{a^2}{4}$, then will $\frac{a^2}{4}+b$ be positive, and consequently $\sqrt{\frac{a^2}{4}+b}$ will be real.

(125.) When b is negative and numerically greater than $\frac{a^2}{4}$, then $\frac{a^2}{4}+b$ will be negative, and consequently $\sqrt{\frac{a^2}{4}+b}$ will be *imaginary*.

CASE I.

When $\sqrt{\frac{a^2}{4}+b}$ *is real.*

1. If a is positive, and $\frac{a}{2}$ is numerically greater than $\sqrt{\frac{a^2}{4}+b}$, then will both values of x be *real* and *negative.*

2. When a is either positive or negative, and $\frac{a}{2}$ is numerically less than $\sqrt{\frac{a^2}{4}+b}$, then will both values of x be *real*, the one *positive*, and the other *negative.*

3. When a is negative and $\frac{a}{2}$ is numerically greater than $\sqrt{\frac{a^2}{4}+b}$, then both values of x will be *real* and *positive.*

CASE II.

When $\sqrt{\frac{a^2}{4}+b}$ *is imaginary,*

In this case both values of x are *imaginary* for all values of a.

(126.) When b is negative and numerically equal to $\frac{a^2}{4}$, then both values of x become $=-\frac{a}{2}$.

(127.) If we add together the two values of x, we have

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}\right)+\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}\right)=-a.$$

If we multiply them, we find

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}\right)\times\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}\right)=-b.$$

From which we see,

That the sum of the roots of the quadratic equation $x^2+ax=b$ *is equal to* $-a$.

And the product of the roots is equal to $-b$.

(128.) We have seen that every quadratic equation, when solved, gives two values for the unknown quantity. These values will both satisfy the algebraic conditions, and sometimes they will both satisfy the particular conditions of the problem, but in most cases but one value of the unknown is applicable to the problem; and the value to be used must be determined from the nature of the question.

We will illustrate this principle by the solution of some particular questions.

1. Find a number such that its square being subtracted from five times the number, shall give 6 for remainder.

Let $x =$ the number sought.

Then, by the conditions of the questions, we have

$$5x - x^2 = 6. \qquad (1)$$

Changing all the signs of (1), it becomes

$$x^2 - 5x = -6, \qquad (2)$$

which, when solved by the rule for quadratics, gives

$$x^2 = \frac{5 \pm 1}{2} = 3, \text{ or } 2.$$

Taking the first value, $x=3$, we find its square to be 9.

Five times this value of x is $5 \times 3 = 15$.

And $15 - 9 = 6$, therefore the number 3 satisfies the question.

The number 2 will satisfy it equally well, since its square $= 4$, which subtracted from five times $2 = 10$, gives for the remainder 6.

2. Find a number such that when added to 6, and the sum multiplied by the number, the product will equal the number diminished by 6.

Let $x =$ the number sought; then by the conditions of the question we have

$$(x+6)x = x - 6. \qquad (1)$$

Expanding and collecting terms, we find

$$x^2 + 5x = -6. \qquad (2)$$

This solved gives

$$x = \frac{-5 \pm 1}{2} = -3, \text{ or } -2.$$

Here, as in the last question, we find that both values *of x will satisfy* our question.

If we take the first value, $x=-3$, we find that the number -3 added to 6 gives 3, which multiplied by -3 gives -9; and this is the same as -3 diminished by 6.

If we take the second value, $x=-2$, we find that the number -2 added to 6 gives 4, which multiplied by -2 gives -8; and this is the same as -2 diminished by 6.

3. Find a number which subtracted from its square shall give 6 for remainder.

Let $x=$ the number, then we have

$$x^2-x=6.$$

This gives

$$x=\frac{1\pm5}{2}=3, \text{ or } -2.$$

If we take 3 for the number, its square is 9, from which subtracting 3, we have 6.

Again, taking -2 for the number, its square is 4, from which subtracting -2, we have 6.

So that both values of x satisfy the conditions of the question.

(129.) We will now add a couple of examples for the purpose of illustrating the case in which the roots are imaginary.

1. Find two numbers, whose sum is 8, and whose product is 17.

Let $x=$ the less number, then will $8-x=$ the greater number.

The product is $(8-x)x=8x-x^2$, which by the conditions of the question, is 17.

Therefore, we have this equation of condition,

$$x^2-8x=-17.$$

This, solved by the usual rules for quadratics, gives

$$x=4\pm\sqrt{-1}, \text{ for the less number,}$$

and $8-(4\pm\sqrt{-1})=4\mp\sqrt{-1}$, for the greater number.

Therefore, the numbers are $\begin{cases} 4\pm\sqrt{-1}, \\ 4\mp\sqrt{-1}, \end{cases}$

both of which are imaginary; we are therefore authorized to conclude that it is *impossible* to find two numbers whose sum is 8, and product 17.

We may also satisfy ourselves of this as follows: Since the sum of two numbers is 8, they must average just 4, hence the greater must exceed 4 just as much as the less falls short of 4. Therefore, any two numbers whose sum is 8, may be represented by

$$4+x,$$
$$4-x.$$

Taking their product, we have

$$(4+x)(4-x)=16-x^2.$$

Now, since x^2 is positive for all *real* values of x, it follows that the product $16-x^2$ is always less than 16; that is, *no two real numbers whose sum is* 8, *can be found such that their product can equal* 17.

If we put the expression for the product, which we have just found equal to 17, we shall have

$$16-x^2=17.$$

Consequently, $$x=\pm\sqrt{-1}.$$

And, $\left.\begin{matrix} 4+x=4\pm\sqrt{-1} \\ 4-x=4\mp\sqrt{-1} \end{matrix}\right\}$ the same values as found by the first method.

These values, although they are imaginary, will satisfy the algebraic conditions of the question; that is, their sum is

$$(4\pm\sqrt{-1})+(4\mp\sqrt{-1})=8,$$

and their product is

$$(4\pm\sqrt{-1})\times(4\mp\sqrt{-1})=17.$$

2. Find two numbers whose sum is 2, and sum of their reciprocals 1.

Denoting the numbers by x and y, we have the following relations:

$$\left.\begin{matrix} x+y=2, \\ \frac{1}{x}+\frac{1}{y}=1. \end{matrix}\right\} \qquad (1)$$

These, solved by the ordinary rules, give

$$\left.\begin{matrix} x=1\pm\sqrt{-1}, \\ y=1\mp\sqrt{-1}. \end{matrix}\right\} \qquad (2)$$

Both these values are imaginary, consequently the conditions of the question are absurd.

We may also show the impossibility of this question as follows: The sum being 2, the numbers may be denoted by

$$\left.\begin{matrix} 1+x, \\ 1-x. \end{matrix}\right\}$$

taking the sum of their reciprocals, we have

$$\frac{1}{1+x}+\frac{1}{1-x},$$

which, when reduced to a common denominator, becomes

$$\frac{2}{1-x^2}.$$

The denominator of this expression being always less than 1, for all real values of x, the expression must exceed 2. Therefore, *it is impossible to find two numbers whose sums shall equal* 2, *and the sum of their reciprocals equal* 1.

(130.) From what has been said, we conclude that, when in the course of the solution of an algebraic problem, we fall upon imaginary quantities, there must be conditions in the problem which are incompatible.

CHAPTER VI.

RATIO AND PROGRESSION.

(131.) By *Ratio* of two quantities, we mean their relation. When we compare quantities by seeing how much greater one is than another, we obtain *arithmetical ratio*. Thus, the arithmetical ratio of 6 to 4 is 2, since 6 exceeds 4 by 2; in the same way the arithmetical ratio of 11 to 7 is 4

In the relation $a-c=r$, (1)

r is the *arithmetical ratio* of a to c.

The first of the two terms which are compared is called the *antecedent;* the second is called the *consequent.* Thus referring to (1), we have

$a =$ *antecedent.*

$c =$ *consequent.*

$r =$ *ratio.*

From (1), we get by transposition

$$a=c+r, \quad (2)$$

$$c=a-r. \quad (3)$$

Equation (2) shows, *that in an arithmetical ratio the antecedent is equal to the consequent increased by the ratio.*

Equation (3) in like manner shows, *that the consequent is equal to the antecedent diminished by the ratio.*

(132.) When the arithmetical ratio of any two terms is the same as the ratio of any other two terms, the four terms together form an *arithmetical proportion.*

Thus if $a-c=r$; and $a'-c'=r$, then will

$$a-c=a'-c', \qquad (4)$$

which relation is an arithmetical proportion, and is read thus: *a is as much greater than c, as a' is greater than c'.*

Of the four quantities constituting an arithmetical proportion, the first and fourth are called the ***extremes***, the second and third are called the *means.*

The first and second, together, constitute the ***first couplet;*** the third and fourth constitute the second ***couplet.***

From equation (4), we get by transposing

$$a+c'=a'+c, \qquad (5)$$

which shows, *that the sum of the extremes, of an arithmetical proportion, is equal to the sum of the means.*

If $c=a'$, then (4) becomes

$$a-a'=a'-c', \qquad (6)$$

which changes (5) into

$$a+c'=2a'. \qquad (7)$$

So that if three terms constitute an arithmetical proportion, the sum of the extremes will equal twice the mean.

(133.) A series of quantities which increase or decrease by a constant difference form an *arithmetical progression.* When the series is increasing, it is called an *ascending progression;* when decreasing, it is called a *descending progression.*

Thus, of the two series

$$1, 3, 5, 7, 9, 11, \&c. \quad (8)$$

$$27, 23, 19, 15, 11, 7, \&c. \quad (9)$$

The first is an ascending progression, whose ratio or *common difference* is 2; the second is a descending progression, whose *common difference* is 4.

(134.) If $a =$ the first term of an ascending arithmetical progression, whose common difference $= d$, the successive terms will be

$$\left.\begin{array}{r} a = \text{first term}, \\ a+d = \text{second term}, \\ a+2d = \text{third term}, \\ a+3d = \text{fourth term}, \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a+(n-1)d = n\text{th term}. \end{array}\right\} \quad (10)$$

If we denote the last or nth term by l, we shall have

$$l = a+(n-1)d. \quad (11)$$

From (11) we readily deduce

$$a = l-(n-1)d, \quad (12)$$

$$d = \frac{l-a}{n-1}, \quad (13)$$

$$n = \frac{l-a}{d}+1. \quad (14)$$

When the progression is descending, we must write $-d$ for d in the above formulas.

Suppose, in an arithmetical progression, x to be a

term, which is preceded by q terms; and y to be a term which is followed by q terms; then by using (11) we have

$$x=a+qd, \tag{15}$$

$$y=l-qd. \tag{16}$$

Taking the sum of (15) and (16), we get

$$x+y=a+l. \tag{17}$$

That is, the sum of any two terms equi-distant from the extremes is equal to the sum of the extremes, so that the terms will average half the sum of the extremes; consequently, *the sum of all the terms equals half the sum of the extremes multiplied by the number of terms.*

Representing the sum of n terms by s, we have

$$s=\frac{a+l}{2}\times n. \tag{18}$$

From (18), we easily obtain

$$a=\frac{2s}{n}-l. \tag{19}$$

$$l=\frac{2s}{n}-a. \tag{20}$$

$$n=\frac{2s}{a+l}. \tag{21}$$

Any three of the quantities

a=first term,
d=common difference,
n=number of terms,
l=last term,
s=sum of all the terms,

being given, the remaining two can be found, which must give rise to 20 different formulas, as given in the following table for ARITHMETICAL PROGRESSION.

(135.) We have not deemed it necessary to exhibit the particular process of finding each distinct formula of the following table, since they were all derived from the two fundamental ones (11) and (18), by the usual operations of equations not exceeding the second degree.

No.	Given.	Required.	Formulas.
1	a, d, n	l	$l=a+(n-1)d$
2	a, d, s	l	$l=-\frac{1}{2}d\pm\sqrt{2ds+(a-\frac{1}{2}d)^2}$
3	a, n, s	l	$l=\frac{2s}{n}-a$
4	d, n, s	l	$l=\frac{s}{n}+\frac{(n-1)d}{2}$
5	a, d, n	s	$s=\frac{1}{2}n[2a+(n-1)d]$
6	a, d, l	s	$s=\frac{l+a}{2}+\frac{(l+a)(l-a)}{2d}$
7	a, n, l	s	$s=\frac{1}{2}n(a+l)$
8	d, n, l	s	$s=\frac{1}{2}n[2l-(n-1)d]$
9	a, n, l	d	$d=\frac{l-a}{n-1}$
10	a, n, s	d	$d=\frac{2s-2an}{n(n-1)}$
11	a, l, s	d	$d=\frac{(l+a)(l-a)}{2s-l-a}$
12	n, l, s	d	$d=\frac{2nl-2s}{n(n-1)}$

Table Continued.

No.	Given.	Required.	Formulas.
13	a, d, l	n	$n=\frac{l-a}{d}+1$
14	a, d, s	n	$n=\frac{d-2a}{2d}\pm\sqrt{\frac{2s}{d}+\left(\frac{2a-d}{2d}\right)^2}$
15	a, l, s	n	$n=\frac{2s}{l+a}$
16	d, l, s	n	$n=\frac{2l+d}{2d}\pm\sqrt{\left(\frac{2l+d}{2d}\right)^2-\frac{2s}{d}}$
17	d, n, l	a	$a=l-(n-1)d$
18	d, n, s	a	$\frac{s}{n}-\frac{(n-1)d}{2}$
19	d, l, s	a	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2ds}$
20	n, l, s	a	$a=\frac{2s}{n}-l$

EXAMPLES.

1. The first term of an arithmetical progression is 7, the common difference is $\frac{1}{4}$, and the number of terms is 16. What is the last term?

To solve this we take formula 1 from our table, which is

$$l=a+(n-1)d.$$

Substituting the above given values for a, d, and n, we find

$$l=7+\tfrac{1}{4}(16-1)=10\tfrac{3}{4}.$$

2. The first term of an arithmetical progression is $\frac{3}{4}$, the common difference is $\frac{1}{8}$, and the last term is $3\frac{1}{4}$. What is the number of terms?

In this example we take formula 13.

$$n=\frac{l-a}{d}+1;$$

which in this present case becomes

$$n=\frac{3\frac{7}{8}-\frac{3}{4}}{\frac{1}{8}}+1=26.$$

3. One hundred stones being placed on the ground in a straight line, at the distance of 2 yards from each other, how far will a person travel who shall bring them one by one to a basket, placed at 2 yards from the first stone?

In this example $a=4$; $d=4$; $n=100$, which values being substituted in formula 5, give

$$s=50(8+99\times 4)=20200 \text{ yards},$$

which, divided by 1760, the number of yards in one mile, we get

$$s=11 \text{ miles, } 840 \text{ yards.}$$

4. What is the sum of n terms of the progression

1, 3, 5, 7 9, ... ?

Ans. $s=n^2$.

5. What is the sum of n terms of the progression

1, 2, 3, 4, 5, ...?

Ans. $s=\frac{n(n+1)}{2}$.

6. A man buys 10 sheep, giving \$1 for the first, \$3 for the second, \$5 for the third, and so increasing in arithmetical progression. What will the last sheep cost at that rate? By formula 1. *Ans.* \$19.

7. A person travels 25 days, going 11 miles the first day, and 135 the last day; the miles which he travelled in the successive days form an arithmetical progression. How far did he go in the 25 days?

By formula 7. *Ans.* 1825 miles

8. A note becomes due in annual instalments, which are in arithmetical progression, whose common difference is 3; the first payment is 7 dollars, the last payment is 49 dollars. What is the number of instalments?

By formula 13. *Ans.* 15.

9. In a triangular field of corn, the number of hills in the successive rows are in arithmetical progression: in the first row there is but one hill, in the last row there are 81 hills; and the whole number of hills in the field is 1681. How many rows are there?

By formula 15. *Ans.* 41.

GEOMETRICAL RATIO.

(136.) When we compare quantities by seeing how many times greater one is than another, we obtain *geometrical ratio*. Thus, the geometrical ratio of 8 to 4 is 2, since 8 is 2 times as great as 4. Again, the geometrical ratio of 15 to 3 is 5.

In the relation, $$\frac{a}{c}=r, \qquad (1)$$

r is the geometrical ratio of a to c.

As in arithmetical ratio

a=*antecedent.*
c=*consequent.*
r=*ratio.*

From (1), we get by reduction

$$a = cr, \tag{2}$$

$$c = \frac{a}{r}. \tag{3}$$

Equation (2) shows, *that in a geometrical ratio the antecedent is equal to the consequent multiplied by the ratio.*

Equation (3) shows, *that the consequent is equal to the antecedent divided by the ratio.*

(137.) When the geometrical ratio of any two terms is the same as the ratio of any other two terms, the four terms together form a *geometrical proportion.*

Thus, if $\frac{a}{c} = r$; and $\frac{a'}{c'} = r$, then will

$$\frac{a}{c} = \frac{a'}{c'}, \tag{4}$$

which relation is a geometrical proportion, and is generally written thus:

$$a : c :: a' : c', \tag{5}$$

which is read as follows: *a is to c, as a' is to c'.*

Of the four quantities which constitute a geometrical proportion, as in arithmetical proportion, the first and fourth are called the *extremes*, the second and third are called the *means.*

The first and second constitute the *first couplet*; the third and fourth constitute the *second couplet.*

From equation (5), or its equivalent (4), we find

$$ac'=a'c, \qquad (6)$$

which shows, *that the product of the extremes, of a geometrical proportion, is equal to the product of the means.*

If $c=a'$, then (5) becomes

$$a : a' : : a' : c', \qquad (7)$$

which changes (6) into

$$ac'=a'^2, \qquad (8)$$

so that, if three terms constitute a geometrical proportion, the product of the extremes will equal the square of the mean.

(138.) Quantities are said to be in proportion *by inversion*, or *inversely*, when the consequents are taken as antecedents, and the antecedents as consequents.

From (5), or its equivalent (4), which is

$$\frac{a}{c}=\frac{a'}{c'}, \qquad (9)$$

we have by inverting both terms

$$\frac{c}{a}=\frac{c'}{a'}$$

Therefore, by Art. 137,

$$c : a : : c' : a'.$$

Which shows, that if four quantities are in proportion, they will be in proportion by inversion.

(139.) Quantities are in proportion by *alternation*, or *alternately*, when the antecedents form one of the couplets, and the consequents form the other.

Resuming (4),

$$\frac{a}{c}=\frac{a'}{c'}. \qquad (11)$$

Multiplying both terms of (11) by $\frac{c}{a'}$, it will become

$$\frac{a}{a'}=\frac{c}{c'}.$$

Therefore, by Art. 137,

$$a : a' : : c : c'. \qquad (12)$$

Which shows, that if four quantities are in proportion they will be so by alternation.

(140.) Quantities are in proportion by *composition*, when the sum of the antecedent and consequent is compared either with antecedent or consequent.

Resuming (4),

$$\frac{a}{c}=\frac{a'}{c'}. \qquad (13)$$

If to (13), we add the terms of the following equation, $\frac{c}{c}=\frac{c'}{c'}$, each of whose members is equal to unity, we have

$$\frac{a+c}{c}=\frac{a'+c'}{c'}.$$

Therefore, by Art. 137,

$$a+c : c : : a'+c' : c'. \qquad (14)$$

Which shows, that if four quantities are in proportion, they will be so by composition.

(141) Quantities are said to be in proportion by *division*, when the difference of the antecedent and consequent is compared with either antecedent or consequent.

If we subtract the equation $\frac{c}{c}=\frac{c'}{c'}$, each member of which is equal to 1, from equation (4), we find

$$\frac{a-c}{c}=\frac{a'-c'}{c'}. \qquad (15)$$

Therefore, by Art. 137, we have

$$a-c : c : : a'-c' : c'. \qquad (15)$$

Which shows, that if four quantities are in proportion they will be so by division.

Equation (4) is

$$\frac{a}{c}=\frac{a'}{c'}.$$

Raising each member to the nth power we have

$$\frac{a^n}{c^n}=\frac{a'^n}{c'^n}.$$

Therefore, by Art. 137, we have

$$a^n : c^n : : a'^n : c'^n. \qquad (16)$$

Which shows, that if four quantities are in proportion, like powers or roots of these quantities will also be in proportion.

If we have

$$\left.\begin{array}{l} a : c : : a' : c', \\ a : c : : a'' : c'', \\ a : c : : a''' : c''', \\ \quad \&c. \quad \&c. \end{array}\right\} \qquad (17)$$

We have by alternation

$$\left.\begin{array}{l} a : a' :: c : c', \\ a : a'' :: c : c'', \\ a : a''' :: c : c''', \\ \quad \&c. \quad \&c. \end{array}\right\}$$

Therefore, by inversion, Art. 138, we have

$$\left.\begin{array}{l} \frac{a'}{a} = \frac{c'}{c}, \\ \frac{a''}{a} = \frac{c''}{c}, \\ \frac{a'''}{a} = \frac{c'''}{c}, \\ \&c., \ \&c. \\ \text{We also have} \quad \frac{a}{a} = \frac{c}{c}. \end{array}\right\} \qquad (18)$$

Taking the sum of equations (18), we have

$$\frac{a+a'+a''+a'''+\&c.}{a} = \frac{c+c'+c''+c'''+\&c.}{c} \qquad (19)$$

Therefore, we have

$$a+a'+a''+a'''+\&c. : a :: c+c'+c''+c'''+\&c. : c.$$

Which shows, that if any number of quantities are proportional, the sum of all the antecedents will be to any one antecedent, as the sum of all the consequents is to its corresponding consequent.

(142.) If we have

$$a : c :: a' : c',$$
$$a'' : c'' :: a''' : c''',$$

then we find

$$\frac{a}{c}=\frac{a'}{c'}, \qquad (21)$$

$$\frac{a''}{c''}=\frac{a'''}{c'''}. \qquad (22)$$

Multiplying together the equations (21) and (22), we have

$$\frac{a\times a''}{c\times c''}=\frac{a'\times a'''}{c'\times c'''}. \qquad (23)$$

Therefore, by Art. 137, we have

$$a\times a'' : c\times c'' : : a'\times a''' : c'\times c'''. \qquad (24.)$$

Which shows, that if there be two sets of proportional quantities, the products of the corresponding terms will be proportional.

(143.) A series of quantities which increase or decrease by a constant multiplier, forms a *geometrical progression.* When the series is increasing, that is, when the constant multiplier exceeds a unit, it is called an *ascending progression;* when decreasing, or when the constant multiplier is less than a unit, then it is called a *descending progression.*

Thus, of the two series,

1, 3, 9, 27, 81, 243, &c, (25)

256, 128, 64, 32, 16, 8, &c. (26)

the first is an ascending progression, whose constant multiplier or *ratio* is 3; the second is a descending progression, whose ratio is $\frac{1}{2}$.

(144.) If a is the first term of a geometrical progression, whose ratio $=r$, the successive terms will be

$$\left.\begin{array}{l} a=\text{first term,} \\ ar=\text{second term,} \\ ar^2=\text{third term,} \\ ar^3=\text{fourth term,} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ ar^{n-1}=n\text{th term.} \end{array}\right\} \quad (27)$$

If we denote the last or nth term by l, we shall have

$$l=ar^{n-1}. \quad (28)$$

If we represent the sum of n terms of a geometrical progression by s, we shall have

$$s=a+ar+ar^2+ar^3+\ldots.+ar^{n-2}+ar^{n-1}. \quad (29)$$

Multiplying all the terms of (29) by the ratio r, we have

$$rs=ar+ar^2+ar^3+ar^4+\ldots.+ar^{n-1}+ar^n. \quad (30)$$

Subtracting (29) from (30), we get

$$(r-1)s=a(r^n-1). \quad (31.)$$

Therefore,

$$s=a\left\{\frac{r^n-1}{r-1}\right\}. \quad (32)$$

Any three of the quantities

a=first term,
r=ratio,
n=number of terms,
l=last term,
s=sum of all the terms,

being given, the remaining two can be found, which as in arithmetical progression, must give rise to 20 different formulas, as given in the following table for GEOMETRICAL PROGRESSION.

No.	Given.	Required.	Formulas.
1	a, r, n	l	$l=ar^{n-1}$
2	a, r, s		$l=\frac{a+(r-1)s}{r}$
3	a, n, s		$l(s-l)^{n-1}-a(s-a)^{n-1}=0$
4	r, n, s		$l=\frac{(r-1)sr^{n-1}}{r^n-1}$
5	a, r, n	s	$s=\frac{a(r^n-1)}{r-1}$
6	a, r, l		$s=\frac{rl-a}{r-1}$
7	a, n, l		$s=\frac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}$
8	r, n, l		$s=\frac{l(r^n-1)}{(r-1)r^{n-1}}$
9	r, n, l	a	$a=\frac{l}{r^{n-1}}$
10	r, n, s		$a=\frac{(r-1)s}{r^n-1}$
11	r, l, s		$a=rl-(r-1)s$
12	n, l, s		$a(s-a)^{n-1}-l(s-l)^{n-1}=0$

Table Continued.

No.	Given.	Required.	Formulas.
13	a, n, l	r	$r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$
14	a, n, s	r	$r^n-\frac{s}{a}r+\frac{s-a}{a}=0$
15	a, l, s	r	$r=\frac{s-a}{s-l}$
16	n, l, s	r	$r^n-\frac{s}{s-l}r^{n-1}+\frac{l}{s-l}=0$
17	a, r, l	n	$n=\frac{\log l-\log a}{\log r}+1$
18	a, r, s	n	$n=\frac{\log[a+(r-1)s]-\log a}{\log r}$
19	a, l, s	n	$n=\frac{\log l-\log a}{\log(s-a)-\log(s-l)}+1$
20	r, l, s	n	$n=\frac{\log l-\log[rl-(r-1)s]}{\log r}+1$

(145.) All the formulas of the above table are easily drawn from the conditions of (28) and (32), which conditions correspond with formulas (1) and (5), except the last four which involve logarithms.

EXAMPLES.

1. The first term of a geometrical progression is 5, the ratio 4, the number of terms is 9. What is the last term?

Formula (1), which is $l=ar^{n-1}$, gives

$$l=5\times 4^8=327680.$$

2. The first term of a geometrical progression is 4,

the ratio is 3, the number of terms is 10. What is the sum of all the terms?

Formula (5), which is $s=\frac{a(r^n-1)}{r-1}$, gives

$$s=\frac{4(3^{10}-1)}{2}=118096.$$

3. The last term of a geometrical progression is $106\frac{4}{5}\frac{0}{1}\frac{3}{2}$, the ratio is $\frac{3}{2}$, the number of terms 8. What is the first term?

Formula (9), which is $a=\frac{l}{r^{n-1}}$, gives

$$a=\frac{106\frac{4}{5}\frac{0}{1}\frac{3}{2}}{(\frac{3}{2})^7}=6\frac{1}{4}.$$

(146.) When the progression is descending, the ratio is less than one, and if we suppose the series extended to an infinite number of terms, the last term may be taken $l=0$, which causes formula 6 to become

$$s=\frac{a}{1-r}, \qquad (33)$$

Which shows that the sum of an infinite number of terms of a descending geometrical progression is equal to its first term, divided by one diminished by the ratio.

EXAMPLES.

1. What is the sum of the infinite progression

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\text{ \&c. ?}$$

In this example $a=1$, $r=\frac{1}{2}$, and (33) becomes

$$s=\frac{1}{1-\frac{1}{2}}=2.$$

2. What is the value of 0.33333 &c., or which is the same thing, of the infinite series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ &c. ?

Here $a=\frac{3}{10}$, $r=\frac{1}{10}$, and (33) gives

$$s=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{9}=\frac{1}{3}.$$

3. What is the value of 0.12121212 &c., or which is the same, of $\frac{12}{100}+\frac{12}{10000}+\frac{12}{1000000}+$ &c. ?

In this example $a=\frac{12}{100}$, $r=\frac{1}{100}$, and (33) gives

$$s=\frac{\frac{12}{100}}{1-\frac{1}{100}}=\frac{12}{99}=\frac{4}{33}.$$

4. What is the sum of the infinite series

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\text{ &c. ?}$$

Ans. $\frac{3}{2}$.

5. What is the sum of the infinite series

$$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}\text{ &c. ?}$$

Ans. $\frac{1}{4}$.

6. A person sowed a peck of wheat, and used the whole crop for seed the following year; the produce of the 2d year again for seed the 3d year, and so on. If, in the last year his crop is 1048576 pecks, how many pecks did he raise in all, allowing the increase to have been in a four-fold ratio ?

By formula 6. *Ans.* 1398101 pecks.

7. A king in India, named Sheran, wished, according to the Arabic author Asephad, that Sessa, the inventor of chess, should himself choose a reward. He requested the grains of wheat, which arise, when 1 is calculated for the first square of the board, 2 for the second

square, 4 for the third, and so on; reckoning for each of the 64 squares of the board, twice as many grains as for the preceding. When it was calculated, to the astonishment of the king, it was found to be an enormous number. What was it? By formula 5.

Ans. 18446744073709551615 grains.

HARMONICAL PROPORTION

(147.) Three quantities are in harmonical proportion, when the first has the same ratio to the third, as the difference between the first and second has to the difference between the second and third.

Four quantities are in harmonical proportion, when the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth.

Thus, if

$$a : c :: a-b : b-c, \qquad (2)$$

then will the three quantities a, b, c be in harmonical proportion. If

$$a : d :: a-b : c-d, \qquad (2)$$

then also will the four quantities a, b, c, and d be in harmonical proportion.

Multiplying means and extremes of (1), we have

$$ab-ac=ac-bc, \qquad (3)$$

which by transposition becomes

$$ab+bc=2ac. \qquad (4)$$

In a similar way equation (2) gives

$$ac+bd=2ad. \qquad (5)$$

Suppose a, b, c, d, e, &c., to be in harmonical progression; then from (4) we have

$$\left.\begin{array}{l} bc+ab=2ac \\ cd+bc=2bd \\ de+cd=2ce \\ \quad \&c. \end{array}\right\} \qquad (6)$$

Dividing the first of (6) by abc, the second by bcd, and the third by cde, &c., we find

$$\left.\begin{array}{l} \frac{1}{a}+\frac{1}{c}=\frac{2}{b} \\ \frac{1}{b}+\frac{1}{d}=\frac{2}{c} \\ \frac{1}{c}+\frac{1}{e}=\frac{2}{d} \\ \quad \&c. \end{array}\right\} \qquad (7)$$

From which we see that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., are in arithmetical progression.

Hence, the reciprocals of any number of terms in harmonical progression are in arithmetical progression; and conversely the reciprocals of the terms of any arithmetical progression must be in harmonical progression.

The reciprocals of the arithmetical series 1, 2, 3, 4, 5, 6, are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, whose numerators, when redu-

ced to a common denominator, are 60, 30, 20, 15, 12, 10, which by the above property must be in harmonical progression.

If six musical strings of equal tension and thickness, have their lengths in proportion to the above numbers, they will, when sounded together, produce more perfect harmony than could be produced by strings of different lengths; and hence we see the propriety of calling this kind of relation, *harmonical* or *musical* proportion.

CHAPTER VII.

PROBLEMS GENERALIZED.

(148.) Thus far, in most cases, we have made use of numerical values to represent the known quantities. Now it is obvious that if we use letters to represent these known quantities, our operations will be general, and of course applicable to all questions involving similar conditions.

Under this point of view, we see that Algebra furnishes us with concise and easy methods for determining general propositions and rules of operation.

To illustrate this, we will deduce some propositions involving two unknown quantities x and y.

Putting s = their sum, and d = their difference, we have

$$x+y=s. \qquad (1)$$

$$x-y=d. \qquad (2)$$

Adding (2) to (1), we get

$$2x=s+d. \qquad (3)$$

Subtracting (2) from (1), we get

$$2y=s-d. \qquad (4)$$

Dividing each member of (3) and (4) by 2, we obtain

$$x=\frac{s+d}{2}, \text{ or } x=\frac{s}{2}+\frac{d}{2}=\text{greater number.}$$

$$y=\frac{s-d}{2},\text{ or }y=\frac{s}{2}-\frac{d}{2}=\text{less number.}$$

Translating these results into common language, we have the following

PROPOSITION.

Half the difference of two quantities, added to half their sum, is equal to the greater.

Half the difference of two quantities, subtracted from half their sum, is equal to the less.

Again, if we take the squares of (1) and (2), we shall have

$$x^2+2xy+y^2=s^2. \quad (5)$$

$$x^2-2xy+y^2=d^2. \quad (6)$$

Subtracting (6) from (5), we find

$$4xy=s^2-d^2.$$

This result translated into common language gives this

PROPOSITION.

Four times the product of any two numbers, is equal to the square of their sum, diminished by the square of their difference.

Again, equations (5) and (6) by transposition give

$$x^2+y^2=s^2-2xy, \quad (7)$$

$$x^2+y^2=d^2+2xy. \quad (8)$$

Equations (7) and (8) are equivalent respectively to these two

PROPOSITIONS.

I. The sum of the squares of two quantities, is equal to the square of their sum, diminished by twice their product.

II. The sum of the squares of two quantities, is equal to the square of their difference, increased by twice their product.

Equations (5) and (6), when translated, yield these two

PROPOSITIONS.

I. The square of the sum of two quantities is equivalent to the square of the greater, plus twice the product of both, plus the square of the less.

II. The square of the difference of two quantities is equivalent to the square of the greater, minus twice the product of both, plus the square of the less.

(149.) From this we conclude that equations are only concise expressions for general propositions.

EXAMPLES.

1. Divide the number a into two such parts, that the first shall be to the second as m to n.

Let $\left.\begin{matrix} mx \\ nx \end{matrix}\right\}$ represent the two parts, which are obviously in the ratio of m to n.

Hence we have this condition,

$$mx+nx=a.$$

Dividing both members by $(m+n)$, we find

$$x=\frac{a}{m+n}.$$

Therefore, $$mx=\frac{ma}{m+n}=\frac{m}{m+n}a.$$

$$nx=\frac{na}{m+n}=\frac{n}{m+n}a.$$

2. Divide the number a into three parts, which shall be to each other respectively as m, n, and p.

Assume, $\left.\begin{matrix}mx\\nx\\pr\end{matrix}\right\}$ for the parts.

Taking the sum, we have

$$mx+nx+px=a,$$

and $$x=\frac{a}{m+n+p}.$$

Consequently,

$$mx=\frac{ma}{m+n+p}=\frac{m}{m+n+p}\times a.$$

$$nx=\frac{na}{m+n+p}=\frac{n}{m+n+p}\times a.$$

$$px=\frac{pa}{m+n+p}=\frac{p}{m+n+p}\times a.$$

From these examples we readily discover how to proceed for a greater number of parts.

(150.) The above leads to the following example.

3. To divide a number into three parts which shall be to each other in the ratio of three numbers.

RULE.

Form three fractions having for their numerators the respective numbers, and a common denominator equal to their sum. These fractions multiplied by the given number will give the respective parts sought.

As an example, let us solve this question:

4. A, B, and C, enter into partnership. A furnished \$180, B \$240, and C \$480. They gained \$300. What is each one's part of the gain?

$180+240+480=900=$ the common denominator.

$\frac{180}{900}=\frac{1}{5}$, $\frac{240}{900}=\frac{4}{15}$, $\frac{480}{900}=\frac{8}{15}$ } equal to the fractional parts of the gain which each receives.

Hence, A must have $\frac{1}{5}\times\$300=\$60$,
B " " $\frac{4}{15}\times\$300=\$80$,
C " " $\frac{8}{15}\times\$300=\$160$.

\$300 verification.

From this we discover that these general solutions lead immediately to the arithmetical rule for Fellowship.

(151.) We will now proceed to show how easy a thing it is to deduce the usual rules of simple interest by means of algebraic operations.

In our operations we shall use the following notation:

p=the principal.

r=the rate per cent., or the interest of \$1 for one year.

n=the number of years the principal is on interest.

a=the amount.

Then it is obvious that we shall have

rp=the interest of \$$p$ for one year at r per cent. (1)

nrp=the interest of \$$p$ for n years at r per cent. (2)

Hence we have this condition,

$$p+nrp=a. \qquad (3)$$

Dividing both members of (3) by $1+nr$, and we have

$$p=\frac{a}{1+nr}. \qquad (4)$$

Equation (3) also readily gives

$$r=\frac{a-p}{np}, \qquad (5)$$

$$n=\frac{a-p}{rp}. \qquad (6)$$

REMARK.—When the time is not an even number of years, we must reduce the months and days, if any, to the proper fractional part of a year. Thus, 3 years and 3 months, is the same as $3\frac{1}{4}$ years. 3 years 3 months and 3 days, is the same as $(3+\frac{1}{4}+\frac{1}{120})=3\frac{31}{120}$ years. And so on for other fractional parts of a year.

The rate per cent. is always a decimal.

Thus at 3 per cent. r becomes 0.03,
" 4 " " " 0.04,
" 5 " " " 0.05,
" 6 " " " 0.06,
" 7 " " " 0.07.

We will now translate these equations.

EQUATION (1).

The interest of a given sum for one year is equal to the principal multiplied by the rate per cent.

EQUATION (2).

The interest of a given sum for any given time is equal to the interest for one year multiplied by the number of years.

EQUATION (3).

The amount of a given sum for a given time at a given rate per cent. is equal to the interest added to the principal.

EQUATION (4).

The principal is equal to the amount divided by the amount of $1 *for the same time and at the same rate per cent.*

EQUATION (5).

The rate per cent. is equal to the interest of the given principal for the given time and the rate per cent., divided by the interest of the same principal for the same time at one per cent.

EQUATION (6).

The number of years is equal to the interest of the given principal for the given time and rate per cent., divided by the interest of the same principal at the same rate per cent. for one year.

5. A cistern can be filled by three pipes; the first can fill it in a hours, the second in b hours, the third in c hours. In what time will the cistern be filled when all three pipes are open at once?

Let x = the time sought.

Now since the first can fill it in a hours, it can fill $\frac{1}{a}$ part in one hour.

In the same way we see that the second can fill $\frac{1}{b}$ part in one hour.

The third can fill $\frac{1}{c}$ part in one hour.

All together will fill $(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})$ part in one hour.

But by supposition they can all fill $\frac{1}{x}$ part in one hour. Hence, we have this condition,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}. \qquad (1)$$

Clearing of fractions we get

$$bcx+acx+abx=abc. \qquad (2)$$

And $$x=\frac{abc}{bc+ac+ab}. \qquad (3)$$

Translating this expression, it becomes,

Divide the continued product of the respective times which each alone requires, by the sum of their products taken two at a time.

REMARK.—If one of the pipes, the first for example, instead of assisting the others to fill the cistern, was constantly drawing off from it, then all that would be necessary to do, so that formula (3) should be true, would be to change the sign of a. Making this change in (3), we get

$$x=\frac{-abc}{bc-ac-ab}=\frac{abc}{ac+ab-bc}.$$

This expression may under certain cases become negative. When $ac+ab$ is greater than bc, the value will be positive. But when $ac+ab$ is less than bc, it is negative. And should $ac+ab$ be equal to bc, then this expression will become infinite. These conditions may be expressed more concisely as follows:

When $a>\frac{bc}{b+c}$, the value of x is *positive*.

" $a<\frac{bc}{b+c}$, the value of x is *negative*.

" $a=\frac{bc}{b+c}$, the value of x is *infinite*.

We will add one more example, which will be useful to illustrate the *negative* sign.

6. A's age is a years, B's age is b years. When will be twice as old as B?

Let x = the time sought.

At the end of x years,

A will be $a+x$ years old.

B will be $b+x$ years old.

Hence, we must have this condition,

$$a+x=2(b+x). \qquad (1)$$

This reduced gives

$$x=a-2b. \qquad (2)$$

From this we see that A's age must, at this present time, exceed twice B's age, in order that x may be positive.

If A's age is now less than twice B's age, then x will be negative, and the time sought is not future, but past time.

As an illustration, suppose A is now 30 years old, and B is ten years old. In this case,

$$x=a-2b=30-20=10,$$

which shows that 10 years from now A will be twice as old as B.

If A is now 30 years old, and B is 20 years old, then

$$x=a-2b=30-40=-10,$$

which shows that 10 years ago A was twice as old as B was then.

ELIMINATION BY INDETERMINATE MULTIPLIERS.

(152.) Suppose we wish to find x and y from the equations

$$2x+3y=13, \qquad (1)$$
$$5x+4y=22. \qquad (2)$$

Multiplying (1) by m, we find

$$2mx+3my=13m. \qquad (3)$$

Adding (2) and (3), we have

$$(2m+5)x+(3m+4)y=13m+22. \quad (4)$$

Now, assume $3m+4=0$, which gives

$$m=-\tfrac{4}{3}. \quad (5)$$

This value of m causes equation (4) to become

$$x=\frac{13m+22}{2m+5}=2.$$

Again, if we had assumed $2m+5=0$, which would have given

$$m=-\tfrac{5}{2}, \quad (7)$$

then equation (4) would have become

$$y=\frac{13m+22}{3m+4}=3. \quad (8)$$

Now, returning to our former equations, we will subtract (2) from (3), we thus obtain

$$(2m-5)x+(3m-4)y=13m-22. \quad (9)$$

Assume $3m-4=0$, which gives

$$m=\tfrac{4}{3}. \quad (10)$$

This value of m causes (9) to become

$$x=\frac{13m-22}{2m-5}=2. \quad (11)$$

Again, assume $2m-5=0$, which gives

$$m=\tfrac{5}{2},$$

this causes (9) to become

$$y=\frac{13m-22}{3m-4}=3. \quad (13)$$

These values of x and y are the same as just found.

It is evident that had we multiplied (2) by m and then added, or subtracted the result from (1), we should then have found, in a similar manner, the same values for x and y.

(153.) We will now apply this method to the two literal equations,

$$X_1x+Y_1y=A_1, \qquad (1)$$
$$X_2x+Y_2y=A_2. \qquad (2)$$

In these equations the capital letters are supposed to be known, and their *subscript* numerals indicate the equation to which they belong. Thus,

X_2 is the coefficient of x in the second equation.

Y_1 is the coefficient of y in the first equation.

A_2 is the absolute term, or the term independent of x and y in the second equation.

Returning to our equations, we will multiply (1) by m and add the result to (2), we thus obtain

$$(X_1m+X_2)x+(Y_1m+Y_2)y=A_1m+A_2. \qquad (3)$$

Now, assume $Y_1m+Y_2=0$, which gives

$$m=-\frac{Y_2}{Y_1}. \qquad (4)$$

This causes (3) to become

$$(X_1m+X_2)x=A_1m+A_2, \qquad (5)$$

which gives immediately

$$x=\frac{A_1m+A_2}{X_1m+X_2}=\frac{A_1Y_2-A_2Y_1}{X_1Y_2-X_2Y_1}. \qquad (6)$$

Assume $X_1m+X_2=0$, which gives

$$m=-\frac{X_2}{X_1}. \tag{7}$$

This value of m causes (3) to become

$$y=\frac{A_1m+A_2}{Y_1m+Y_2}=\frac{A_2X_1-A_1X_2}{X_1Y_2-X_2Y_1}. \tag{8}$$

Hence, the values of x and y are

$$\left.\begin{aligned} x&=\frac{A_1Y_2-A_2Y_1}{X_1Y_2-X_2Y_1},\\ y&=\frac{A_2X_1-A_1X_2}{X_1Y_2-X_2Y_1}. \end{aligned}\right\} \tag{9}$$

These values of x and y may be considered as comprising the solution of all simple equations combining only two unknown quantities. If we wish to adapt this general solution to the equations

$$2x+3y=13,$$
$$5x+4y=22;$$

we must call

$$A_1=13;\ A_2=22.$$
$$X_1=2;\ X_2=5.$$
$$Y_1=3;\ Y_2=4.$$

These values substituted in (9), give

$$x=2;\ y=3.$$

(154.) As a still farther illustration of the method of elimination by means of indeterminate multipliers, we will proceed to the solution of three simultaneous simple equations, involving three unknown quantities x, y, and z.

And we will continue to make use of the notation by the assistance of subscript numbers.

Let the equations be as follows:

$$X_1x+Y_1y+Z_1z=A_1, \quad (1)$$
$$X_2x+Y_2y+Z_2z=A_2, \quad (2)$$
$$X_3x+Y_3y+Z_3z=A_3. \quad (3)$$

In these equations, as in those of the last example, the capital letters X, Y, Z, are the coefficients of their corresponding small letters. The small numerals placed at the base of these coefficients correspond to the particular equation to which they belong. Thus X_2 is the coefficient of x in the second equation; Y_3 is the coefficient of y in the third equation; Z_1 is the coefficient of z in the first equation, and so for the other coefficients. The letter A is used to denote the right-hand members of the equations, or the absolute terms; the subscript numbers in this case also denote the equation to which they belong

This kind of notation, by use of subscript numbers, is very natural and simple, and combines many advantages over the ordinary methods.

Having explained this method of notation, we will now proceed to the solution of our equations.

If we multiply (1) by m, and (2) by n, and then add the results, we shall obtain

$$\left.\begin{array}{l}(X_1m+X_2n)x+(Y_1m+Y_2n)y\\ +(Z_1m+Z_2n)z=A_1m+A_2n.\end{array}\right\} \quad (4)$$

From (4) subtracting (3) we find

$$\left.\begin{array}{l}(X_1m+X_2n-X_3)x+(Y_1m+Y_2n-Y_3)y\\ +(Z_1m+Z_2n-Z_3)z=A_1m+A_2n-A_3.\end{array}\right\} \quad (5)$$

In order to cause y and z to vanish from this equation, we will assume

$$Y_1m+Y_2n=Y_3, \quad (6)$$

$$Z_1m+Z_2n=Z_3. \quad (7)$$

This assumption causes (5) to become

$$(X_1m+X_2n-X_3)x=A_1m+A_2n-A_3. \quad (8)$$

Therefore,

$$x=\frac{A_1m+A_2n-A_3}{X_1m+X_2n-X_3}. \quad (9)$$

We must now find the values of m and n, by aid of conditions (6) and (7); for this purpose we will multiply (6) by p and from the result subtract (7). We thus obtain

$$(Y_1p-Z_1)m+(Y_2p-Z_2)n=Y_3p-Z_3. \quad (10)$$

Again, assume

$$Y_2p=Z_2. \quad (11)$$

Then equation (10) will give

$$m=\frac{Y_3p-Z_3}{Y_1p-Z_1}. \quad (12)$$

From the assumed condition (11) we find

$$p=\frac{Z_2}{Y_2}. \quad (13)$$

This value of p being substituted in (12) gives

$$m=\frac{Y_3Z_2-Y_2Z_3}{Y_1Z_2-Y_2Z_1}. \qquad (14)$$

If in equation (10) we assume

$$Y_1p=Z_1, \qquad (15)$$

then will it become

$$(Y_2p-Z_2)n=Y_3p-Z_3. \qquad (16)$$

Consequently,

$$n=\frac{Y_3p-Z_3}{Y_2p-Z_2}. \qquad (17)$$

Condition (15) gives

$$p=\frac{Z_1}{Y_1}. \qquad (18)$$

Substituting this value of p in (17), we find

$$n=\frac{Y_3Z_1-Y_1Z_3}{Y_2Z_1-Y_1Z_2},$$

or changing the signs of the numerator and denominator, we have

$$n=\frac{Y_1Z_3-Y_3Z_1}{Y_1Z_2-Y_2Z_1}. \qquad (19)$$

We have made this change in order that the values of m and n, given by equations (14) and (19), may have a common denominator.

Substituting these values of m and n in (9), we find, after a little reduction,

$$x=\frac{A_1Y_3Z_2-A_1Y_2Z_3+A_2Y_1Z_3-A_2Y_3Z_1+A_3Y_2Z_1-A_3Y_1Z_2}{X_1Y_3Z_2-X_1Y_2Z_3+X_2Y_1Z_3-X_2Y_3Z_1+X_3Y_2Z_1-X_3Y_1Z_2}. \qquad (20)$$

If we change the signs of all the terms of the numerator and denominator, and make a slight change in their arrangement we shall have

$$x=\frac{A_1Y_2Z_3+A_2Y_3Z_1+A_3Y_1Z_2-A_1Y_3Z_2-A_2Y_1Z_3-A_3Y_2Z_1}{X_1Y_2Z_3+X_2Y_3Z_1+X_3Y_1Z_2-X_1Y_3Z_2-X_2Y_1Z_3-X_3Y_2Z_1}. \qquad (21)$$

By a similar process we shall find the values of y and z, as below.

$$y=\frac{X_1A_2Z_3+X_2A_3Z_1+X_3A_1Z_2-X_1A_3Z_2-X_2A_1Z_3-X_3A_2Z_1}{X_1Y_2Z_3+X_2Y_3Z_1+X_3Y_1Z_2-X_1Y_3Z_2-X_2Y_1Z_3-X_3Y_2Z_1}. \qquad (22)$$

$$z=\frac{X_1Y_2A_3+X_2Y_3A_1+X_3Y_1A_2-X_1Y_3A_2-X_2Y_1A_3-X_3Y_2A_1}{X_1Y_2Z_3+X_2Y_3Z_1+X_3Y_1Z_2-X_1Y_3Z_2-X_2Y_1Z_3-X_3Y_2Z_1}. \qquad (23)$$

(155.) We will now proceed to point out some remarkable relations in the combinations of the subscript numbers, as given by equations (21), (22), and (23.)

I. The denominator, which is common to the three expressions, is composed of six distinct products, each consisting of three independent factors. Three of these products are positive, and three are negative.

II. The letters forming the different products of this common denominator being always arranged in alphabetical order X, Y, Z, we remark that the subscript numbers of the first product are 1, 2, 3. Now, if we add a unit to each of these numbers, observing that when the sum becomes 4 to substitute 1, we shall obtain 2, 3, 1, which are the subscript numbers of the second product. Again, increasing each of these by 1, observing as before, to write 1 when the sum becomes 4, we find 3, 1, 2, which are the subscript numbers of the third product. If we increase each of these last numbers by 1, observing the same law, we shall obtain 1, 2, 3, which are the subscript numbers belonging to the first product. A similar method of changing has already been noticed under Art. 74.

What we have said in regard to the subscript numbers of the positive products, applies equally well in respect to the negative products.

III. The numerator of the expression for x, may be derived from the common denominator by simply substituting A for X, observing to retain the same subscript numbers.

The numerator of the expression for y may be derived from the common denominator by substituting A for Y, observing to retain the same subscript numbers.

In the same way may the numerator of the expression for z be found by changing Z of the denominator into A, retaining the same subscript numbers.

(156.) We will now proceed to show how these expressions, for $x, y,$ and $z,$ can be obtained by a very simple and novel process, which is easily retained in the memory, and which is applicable to all simple equations involving only three unknown quantities.

Writing the coefficients and the absolute terms in the same order as they are now placed in equations (1), (2), (3), we have

$$\left.\begin{array}{llll} X_1 & Y_1 & Z_1 = & A_1, \\ X_2 & Y_2 & Z_2 = & A_2, \\ X_3 & Y_3 & Z_3 = & A_3. \end{array}\right\} \qquad (A)$$

Now, all the products of the common denominator can be found by multiplying together by threes, the coefficients which are found by passing obliquely from the left to the right, observing that if the products obtained by passing obliquely *downwards*, are taken positively, then those formed by passing obliquely *upwards* must be taken negatively, and conversely. This is in accordance with the property of the negative sign. (See Art. 17.) In the present case the products formed by passing obliquely *downwards*, are taken positive.

In this sort of checker-board movement, we must observe that when we run out at the bottom of any column, we must pass to the top of the same column; and when

we run out at the top, we must pass to the bottom of the same column.

This method is most readily performed upon the blackboard, by drawing oblique lines connecting the successive factors of the different products.

We will trace out this sort of oblique movement.

Commencing with X_1, we pass obliquely downwards to Y_2, and thence to Z_3, and thus obtain the positive product of $X_1Y_2Z_3$.

Commencing with X_2, we pass obliquely downwards to Y_3, and since we have now run out with the column of Z's at the bottom, we pass to Z_1, at the top of the column, and thus obtain the positive product $X_2Y_3Z_1$.

Again, commencing with X_3, we pass to Y_1, and thence obliquely downwards to Z_2, and find the positive product $X_3Y_1Z_2$.

Now, for the negative products we make similar movements obliquely upwards.

Thus, commencing with X_1, we pass to Y_3, and thence obliquely upwars to Z_2, and find the negative product $X_1Y_3Z_2$.

Commencing with X_2, we pass obliquely upwards to Y_1, and thence to Z_3, and find the negative product $X_2Y_1Z_3$.

Again, commencing with X_3, we pass obliquely upwards to Y_2, and thence to Z_1, and thus obtain the negative product $X_3Y_2Z_1$.

Having thus obtained the denominator which is common to the values of x, y, z; we may find the numerator of the value of x, by supposing the A's to take

the place of the X's, and then to repeat our checker-board movement. By changing the Y's into the A's, we shall find the numerator of the value of y; and by changing the Z's into A's, we shall find the numerator of the value of z.

(157.) We will now illustrate this method of solving simple equations containing only three unknown quantities, by a few examples.

1. Given $\left\{\begin{array}{l} 2x+3y+4z=16, \\ 3x+5y+7z=26, \\ 4x+2y+3z=19, \end{array}\right\}$ to find x, y, and z

We will first find the common denominator.

Positive Products.	Negative Products.
$2\times5\times3=30$	$2\times2\times7=-28$
$3\times2\times4=24$	$3\times3\times3=-27$
$4\times3\times7=84$	$4\times5\times4=-80$
138	-135
-135	

3=common denominator.

We have for the numerator of x the following operation:

Positive Products.	Negative Products.
$16\times5\times3=240$	$16\times2\times7=-224$
$26\times2\times4=208$	$26\times3\times3=-234$
$19\times3\times7=399$	$19\times5\times4=-380$
847	-838
-838	

9=numerator, for x.

To find the numerator for y, we have

Positive Products.	Negative Products.
$2\times26\times3=156$	$2\times19\times7=-266$
$3\times19\times4=228$	$3\times16\times3=-144$
$4\times16\times7=448$	$4\times26\times4=-416$
832	-826
-826	

6=numerator, for y.

To find the numerator for z, we have

Positive Products.	Negative Products.
$2\times5\times19=190$	$2\times2\times26=-104$
$3\times2\times16=96$	$3\times3\times19=-171$
$4\times3\times26=312$	$4\times5\times16=-320$
598	-595
-595	

3=numerator, for z.

Hence, $x = \frac{9}{3} = 3.$

$y = \frac{6}{3} = 2.$

$z = \frac{3}{3} = 1.$

When some of the coefficients are negative, we must observe the rule for the multiplication of signs.

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22, \\ 4x-2y+5z=18, \\ 6x+7y-\ z=63, \end{array}\right\}$ to find x, y, and z.

To find the common denominator, we have

Positive Products.				Negative Products.			
$2\times$	$-2\times$	$-1=$	4	$2\times$	$7\times$	$5=$	-70
$4\times$	$7\times$	$-3=$	-84	$4\times$	$4\times$	$-1=$	16
$6\times$	$4\times$	$5=$	120	$6\times$	$-2\times$	$-3=$	-36
			40				-90
			-90				
			-50	= common denominator.			

For the numerator of x, we have

Positive Products.				Negative Products.			
$22\times$	$-2\times$	$-1=$	44	$22\times$	$7\times$	$5=$	-770
$18\times$	$7\times$	$-3=$	-378	$18\times$	$4\times$	$-1=$	72
$63\times$	$4\times$	$5=$	1260	$63\times$	$-2\times$	$-3=$	-378
			926				-1076
			-1076				
			-150	= numerator, for x.			

Hence $x=\frac{-150}{-50}=3$.

Proceeding in a similar way we find the values of y and z.

3. Given $\left\{\begin{array}{l} x+\frac{1}{2}y=a, \\ y+\frac{1}{3}z=a, \\ z+\frac{1}{4}x=a, \end{array}\right\}$ to find x, y, and z.

We will arrange the coefficients, omitting the unknown quantities, observing also to write 0 for such terms as are wanting.

This arrangement being made, we have

$$\begin{array}{rrrcl} 1 & \frac{1}{2} & 0 & = & a \\ 0 & 1 & \frac{1}{3} & = & a \\ \frac{1}{4} & 0 & 1 & = & a \end{array}$$

Positive Products.	Negative Products.
$1\times1\times1=1$	$1\times0\times\frac{1}{3}=0$
$0\times0\times0=0$	$0\times\frac{1}{2}\times1=0$
$\frac{1}{4}\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{24}$	$\frac{1}{4}\times1\times0=0$
$\frac{24}{24}$ = common denominator.	0

For the numerator of x, we have

Positive Products.	Negative Products.
$a\times1\times1=a$	$a\times0\times\frac{1}{3}=0$
$a\times0\times0=0$	$a\times\frac{1}{2}\times1=-\frac{1}{2}a$
$a\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}a$	$a\times1\times0=0$
$\frac{7}{6}a$	$-\frac{1}{2}a$
$-\frac{1}{2}a$	

$\frac{2}{3}a$=numerator, for x.

Hence, $x=\frac{2}{3}a\div\frac{24}{24}=\frac{16}{15}a$.

By a similar process is the value of y and z found.

4. Given $\left\{\begin{array}{l} x+a(y+z)=m, \\ y+b(x+z)=n, \\ z+c(x+y)=p, \end{array}\right\}$ to find x, y, and z.

These coefficients, being properly arranged, give

$$\begin{array}{cccc} 1 & a & a & = & m \\ b & 1 & b & = & n \\ c & c & 1 & = & p \end{array}$$

Positive Products.	Negative Products.
$1\times1\times1=1$	$1\times c\times b=-bc$
$b\times c\times a=abc$	$b\times a\times1=-ab$
$c\times a\times b=abc$	$c\times1\times a=-ac$
$1+2abc$	$-ab-ac-bc$
$-ab-ac-bc$	

$1+2abc-ab-ac-bc$=common denominator.

For the numerator of x, we have

Positive Products	Negative Products.
$m\times1\times1=m$	$m\times c\times b=-bcm$
$n\times c\times a=acn$	$n\times a\times1=-an$
$p\times a\times b=abp$	$p\times1\times a=-ap$
$m+acn+abp$	$-bcm-an-ap$
$-bcm-an-ap$	

$m+acn+abp-bcm-an-ap$=numerator of x.

Hence, $$x=\frac{m+acn+abp-bcm-an-ap}{1+2abc-ab-ac-bc}.$$

In a similar way may the values of y and z be found.

This solution is much shorter than by the ordinary methods of elimination.

5. A, B, and C, owe together (a) \$2190, and none of them can alone pay this sum; but when they unite, it can be done in the following ways: first, by B's putting

$\frac{3}{7}$ of his property to all of A's; secondly, by C's putting $\frac{5}{9}$ of his property to all of B's; or by A's putting $\frac{2}{3}$ of his property to all of C's. How much was each worth?

Let x, y, and z, represent what A, B, and C, were respectively worth.

Then we shall have these conditions,

$$x+\tfrac{3}{7}y=a,$$
$$y+\tfrac{5}{9}z=a,$$
$$z+\tfrac{2}{3}x=a.$$

Clearing these of fractions, and arranging the coefficients, we have

$$\begin{array}{rrrcl} 7 & 3 & 0 & = & 7a \\ 0 & 9 & 5 & = & 9a \\ 2 & 0 & 3 & = & 3a \end{array}$$

Positive Products.	Negative Products.
$7\times9\times3=189$	$7\times0\times5=0$
$0\times0\times0=0$	$0\times3\times3=0$
$2\times3\times5=30$	$2\times9\times0=0$

219=common denominator. 0

For the numerator of x, we have

Positive Products.	Negative Products.
$7a\times9\times3=189a$	$7a\times0\times5=0$
$9a\times0\times0=0$	$9a\times3\times3=-81a$
$3a\times3\times5=45a$	$3a\times9\times0=0$
$234a$	$-81a$
$-81a$	

$153a$=numerator of x.

Hence, $x=\frac{153a}{219}=\frac{153\times 2190}{219}=1530.$

For the numerator of y, we find

Positive Products.	Negative Products.
$7\times 9a\times 3=189a$	$7\times 3a\times 5=-105a$
$0\times 3a\times 0=0$	$0\times 7a\times 3=0$
$2\times 7a\times 5=70a$	$2\times 9a\times 0=0$
$259a$	$-105a$.
$-105a$	
$154a$=numerator of y.	

Hence, $y=\frac{154a}{219}=1540.$

For the numerator of z, we have

Positive Products.	Negative Products.
$7\times 9\times 3a=189a$	$7\times 0\times 9a=0$
$0\times 0\times 7a=0$	$0\times 3\times 3a=0$
$2\times 3\times 9a=54a$	$2\times 9\times 7a=-126a$
$243a$	$-126a$
$-126a$	
$117a$=numerator of z.	

Hence, $z=\frac{117a}{219}=1170.$

Collecting the results, we find that

A was worth \$1530.
B " " \$1540.
C " " \$1170.

The student will find, after a little practice in this method, that it is much more simple than would at first sight seem.

Whenever some of the coefficients are zeros, as in the 3d and 5th examples, the work is much abridged, as in this case some of the products must become zero.

CHAPTER VIII.

MISCELLANEOUS SUBJECTS.

PROPERTIES OF NUMBERS.

(158.) Suppose the number N to be of the following form:

$$N=a_0+a_1r+a_2r^2+a_3r^3+\ldots\ldots+a_nr^n. \qquad (A)$$

If in this formula we suppose $r=10$, and each of the numbers $a_0, a_1, a_2, a_3, \ldots\ldots a_n$ to be less than $r=10$, then the above number will be expressed according to our decimal system of notation.

Thus the above formula, when adapted to the number 37854, gives

$$37854=4+5.10+8.10^2+7.10^3+3.10^4.$$

In this way all whole numbers may be represented by formula (A), where r denotes the base of the particular system of notation, and $a_0, a_1, a_2, a_3, \ldots. a_n$ are the respective digits, counting from the right-hand towards the left.

(159.) We will make use of the symbol S_d to denote the sum of all the digits; we will also represent the sum of the digits occupying the odd places, counting from the right-hand towards the left, by S_{odd}; and the sum of the even digits by S_{even}.

Our formula (A), is readily changed into

$N = S_d + a_1(r-1) + a_2(r^2-1) + a_3(r^3-1) \ldots . a_n(r^n-1).$

Now each of the expressions $r-1$; r^2-1 ; r^3-1 ; and r_n-1, is divisible by $r-1$. Hence, if the sum of the digits is divisible by $r-1$, then will the number be divisible by $r-1$. And whatever remainder is found by dividing S_d, the same remainder will be found by dividing N.

If we take the number 3458, expressed in the decimal scale, we find $S_d=20$, which divided by 9 will give 2 remainder, and this is the same remainder as found by dividing 3458 by 9.

If we take the number 3456, we find that $S_d=18$, is divisible by 9, and consequently 3456 is also divisible by 9. The same is true of the 24 numbers found by permuting these four digits, thus,

3456 ; 3465 ; 3546 ; 3564 ; 3645 ; 3654.
4356 ; 4365 ; 4536 ; 4563 ; 4635 ; 4653.
5346 ; 5364 ; 5436 ; 5463 ; 5634 ; 5643.
6345 ; 6354 ; 6435 ; 6453 ; 6534 ; 6543.

(160.) *Hence, in general, whatever remainder is found by dividing the sum of the digits of any number by* 9, *the same remainder will be found by dividing the number itself by* 9.

(161.) *If two numbers are composed of the same digits taken in a different order, their difference will always be divisible by* 9.

For the divisibility of any number by 9, we have seen, depends upon the divisibility of the sum of its digits. Hence the difference of two numbers composed of the same digits must of necessity be divisible by 9.

(162.) What has been said in reference to the digit 9, is applicable to the digit 3, since 3 is a divisor of 9.

(163.) Formula (A) can readily be made to assume this form :

$$N = S_{odd} - S_{even} + a_1(r+1) + a_2(r^2-1) + a_3(r^3+1) +, \text{ \&c.}$$

Now each of the expressions $r+1$; r^2-1; r^3+1; &c. is divisible by $r+1$.

Hence, in the decimal notation, we know that if $S_{odd} - S_{even}$ is divisible by 11, then will the number be divisible by 11.

Thus, in the number 968341, we have $S_{odd}=10$, $S_{even}=21$; consequently $S_{odd} - S_{even} = -11$. Therefore the number 968341 is divisible by 11.

(164.) From the generality of formula (A), we see that there are analogous properties for each system of notation.

(165.) Suppose we wish to transform any given number expressed in the common arithmetical scale, into another number of the same value, having for the radix r, and having a_0, a_1, a_2, a_3, &c. for digits; that is, let it be required to put N into the particular form as expressed by formula (A).

If we divide each member of (A) by r, we shall obtain the remainder a_0: that is, N divided by r will give the remainder a_0; dividing again by r, and we get the remainder a_1; again, dividing by r, we find a_2 for remainder, and so on. Hence we have this

RULE.

Divide the number by the base of the new scale, and write the remainder as the unit's digit sought; divide the quotient by the base again, and write the remainder as the digit next the unit's place; proceed in this way till a quotient is obtained less than the base, this quotient is the digit of the highest order in the number in its new form. Whenever there is no remainder, 0 is the corresponding digit.

EXAMPLES.

1. Represent the number 2931 in the scale whose base is 8.

Following the rule, we have this

OPERATION.

8	2931	
8	366	- - - - 3=first remainder=a_0,
8	45	- - - - 6=second remainder=a_1,
	a_4=5	- - - - 5=third remainder=a_3.

Hence, $2931=5.8^3+5.8^2+6.8+3$.

Now, if it be understood that the digits increase in an 8 fold ratio from the right-hand towards the left, it may be written 5563.

2. Convert 3714 into an equivalent number having 4 for the base.

OPERATION.

4	3714	
4	928	- - - - 2=first remainder=a_0.
4	232	- - - - 0=second remainder=a_1,
4	58	- - - - 0=third remainder=a_2,
4	14	- - - - 2=fourth remainder=a_3,
	a^5=3	- - - - 3=fifth remainder=a_4.

Hence, if we consider the digits as increasing in a four-fold ratio, our number will become

322002.

(166.) To reduce a number from any other scale into into the decimal, we have this

RULE.

Multiply the digit farthest to the left by the base of the scale in which the number is expressed, and add the next digit to the product; multiply the sum again by the base, and add the third digit; proceed in this way till the unit's digit is added, the result is the number in the decimal scale. This rule is evivdently the converse of the last rule.

EXAMPLE.

Let 3465 be a number expressed in the scale whose base is 9, it is required to express it in the decimal scale.

OPERATION.

$$\begin{array}{l} 3465 \\ 9 \\ \hline 31 \quad = 3\times 9+4, \\ 9 \\ \hline 285 \quad = 31\times 9+6, \\ 9 \\ \hline 2570 = 285\times 9+5. \end{array}$$

Hence, 2570 is the number sought.

CONVERSION OF REPETENDS INTO VULGAR FRACTIONS.

(167.) We will commence with *simple repetends.*

An we will denote the successive figures which constitute the repetend by $a_1, a_2, a_3, \ldots . a_n$.

The general form of a repetend will be

$$0.a_1a_2a_3\ldots.a_na_1a_2a_3\ldots.a_n \text{ \&c.}$$

If we denote this value by s, we shall have

$$s=0.a_1a_2a_3\ldots.a_na_1a_2a_3\ldots.a_n \text{ \&c.} \qquad (1)$$

In this case we have supposed the number of figures in the repetend to be n.

If we multiply (1) by 10^n, we shall have

$$s10^n=a_1a_2a_3\ldots.a_n.a_1a_2a_3\ldots.a_n \text{ \&c.} \qquad (2)$$

The right-hand member of (1) was multiplied by 10^n by removing the decimal point n places towards the right, we thus cause the decimal figures in (2) to occur after the *point* in precisely the same order as they occurred in (1).

Subtracting (1) from (2), observing that the decimal parts will cancel each other, we have

$$(10^n-1)s=a_1a_2a_3\ldots.a_n. \qquad (3)$$

Consequently, $$s=\frac{a_1a_2a_3\ldots.a_n}{10^n-1}. \qquad (A)$$

The denominator of formula (A) will consist of a succession of 9's, as many in number as there are figures in the repetend.

EXAMPLES.

1. What is the value of 0.3333 &c.?

$$Ans. =\frac{3}{9}=\frac{1}{3}.$$

2. What is the value of 0.123123123 &c.

$$Ans. s=\frac{123}{999}=\frac{41}{333}.$$

3. What is the value of 0.142857142857 &c.?

$$Ans.\ s=\frac{142857}{999999}=\tfrac{1}{7}.$$

(168.) When the repetend is compound.

Let

a=the repeating period.

n=the number of figures in a period.

b=the non-repeating number.

m=the number of decimal figures in b.

s=the value of the compound repetend.

Also, denote the successive figures which constitute the non-repeating part by $b_1, b_2, b_3, \ldots b_m$.

Then the general form of our *compound repetend* will be

$$s=0.b_1b_2b_3\ldots b_m a_1a_2a_3\ldots a_n a_1a_2a_3\ldots a_n \text{ &c.} \quad (1)$$

The value of the non-repeating part is evidently

$$\frac{b_1b_2b_3\ldots b_m}{10^m}. \quad (2)$$

Did the part which repeats commence immediately after the decimal point, then its value could be found at once by formula (A), (Art. 167). But there are m decimals before the repetend commences, consequently its value is $\frac{1}{10^m}$ part of the value obtained by formula (A). Hence for the repeating part we find

$$\frac{a_1a_2a_3\ldots a_n}{10^m(10^n-1)}.$$

Hence, the entire value of (1) is

$$s=\frac{b_1b_2b_3\ldots b_m}{10^m}+\frac{a_1a_2a_3\ldots a_n}{10^m(10^n-1)}.$$

or which is the same thing,

$$s=\frac{b(10^n-1)+a}{10^m(10^n-1)}. \qquad (B)$$

EXAMPLES.

1. What is the value of $0.017\dot{8}5\dot{7}14\dot{2}$?

In this example $n=6$; $m=3$; $a=857142$; $b=17$. These values, substituted in formula (B), give

$$s=\frac{17857125}{999999000}=\frac{1}{56}.$$

2. What is the value of $0.041\dot{6}$?

$$Ans.\ s=\frac{41\times 9+6}{9000}=\frac{375}{9000}=\frac{1}{24}.$$

INEQUALITIES.

(169.) When two quantities or expressions are equal, as the two members of an equation, this equality is denoted by the symbol $=$.

In a similar way the symbol $>$, placed between two quantities or expressions indicates that they are unequal. and the symbol is so placed as to open towards the greater quantity. Thus, $m>n$ shows that m is greater than n; $a<b$ shows that a is less than b.

In the theory of inequalities, *negative* quantities are considered as *less* than zero. And of two negative quantities, the one which has the greatest numerical value is considered the less.

Thus, $0>-4$, and $-5>-8$.

Nearly all the principles of equations will hold good in respect to inequalities, by considering the quantities

separated by the symbol $>$, as the two members of the inequality.

An inequality is said to continue in the *same sense*, when that member which was greater previous to a particular operation, remains so after the operation has been performed. And two inequalities are said to exist in the *same sense in regard to each other*, when the larger members both correspond with the left-hand members, or with the right-hand members. Thus, $a>b$ and $m>n$ exist in the same sense; so also do $x<y$ and $r<p$. But $a>b$ and $x<z$ are inequalities existing in opposite senses.

I. The same quantity may be added or subtracted from both members of an inequality, and the inequality will remain in the same sense as before.

Thus, if $6>2$, and we add 3 to both members, we shall have $6+3>2+3$; that is, $9>5$. Again, if $-3>-5$, and to both members we add 2, we shall have $-3+2>-5+2$; or, what is the same thing, $-1>-3$; again, adding 1 to both members of this last inequality, we have $0>-2$.

Hence, we may transpose any term of an inequality from one member to another, observing to change its *sign*.

Thus, suppose we have

$$3x+18>38-x,$$

we have by transposition,

$$3x+x>38-18,$$

or,

$$4x>20.$$

II. The corresponding members of two or more inequalities, existing in the same sense in respect to each other, may be added, and the resulting inequality will exist in the same sense as the given inequalities.

III. But if the corresponding members of two inequalities, existing in the same sense, be subtracted from each other, the resulting inequality will not always exist in the same sense as the given inequality.

In the two inequalities

$$16>15, \qquad (1)$$

$$14> 3, \qquad (2)$$

if we subtract the second from the first, we find

$$2<12. \qquad (3)$$

This resulting inequality exists in a contrary sense to the first and second.

IV. We can multiply or divide each member of an inequality by any positive quantity, without changing the sense of the inequality.

V. But, if each member be multiplied or divided by a negative quantity, the resulting inequality will take an opposite sense.

VI. If both members of an inequality are positive, they may be raised to the same power without changing the sense of the inequality.

VII. But, when both members are not positive, they may, when raised to the same power, have the sense of their inequality changed.

(170.) We will give a few examples involving the principles of inequalities.

EXAMPLES.

1. Find the limit of the value of x in the inequality

$$4x-2>10.$$

Ans. $x>3$.

2. Find the limit of x in the inequality

$$3x+2<12+x.$$

Ans. $x<5$.

3. Given $\left\{\begin{array}{l}\frac{ax}{m}+bx-ab>\frac{a^2}{m},\\ \frac{bx}{n}-ax+ab<\frac{b^2}{n},\end{array}\right\}$ to find the value of x.

Ans. $\left\{\begin{array}{l}x>a, \text{ and } x<b. \text{ That is, any}\\ \text{value between } a \text{ and } b \text{ will sat-}\\ \text{isfy the conditions.}\end{array}\right.$

4. A boy being asked how many apples he had, replied, "I have more than three score, and half my number diminished by 13 is less than a score." How many had he?

These conditions expressed in symbols, are

$$x>60,$$

$$\frac{x}{2}-13<20.$$

This second condition gives

$$x<66.$$

Hence, any number between 60 and 66, exclusive, will satisfy the above conditions.

PERMUTATIONS.

(171.) The different orders in which quantities may be arranged, are called their permutations.

In our investigations we shall use the same letter, with different *subscript* numbers, to denote the quantities to be permuted. Thus, let the n quantities to be permuted, be represented by $a_1, a_2, a_3, a_4, a_5, \ldots . a_n$.

(172.) We will now proceed to determine the expressions for the number of *permutations* of n different things, when taken one and one, two and two, three and three, $\ldots .r$ and r together, where r is any positive integer not greater than n.

The number of permutations of n things taken separately, or one by one, is evidently equal to the number of things, or to n.

The number of permutations of n things, taken two and two together, is $n(n-1)$.

For a_1 may be placed successively before $a_2, a_3, a_4, a_5, \ldots, a_n$, and thus form $(n-1)$ permutations taken two and two together; a_2 may be placed sucessively before $a_1, a_3, a_4, a_5, \ldots . a_n$, and thus form $(n-1)$ permutations different from the former, and the same may be done with each of the quantities $a_3, a_4, a_5, \ldots . a_n$, and we shall thus obtain $(n-1)$ permutations two and two together, repeated as many times as there are individual things, or n times. Hence, the total number of permutations of n things taken two and two is $n(n-1)$.

The number of permutations of n things taken three and three together, is equal to $n(n-1)(n-2)$.

For we have just shown that the number of permutations of n things two by two is $n(n-1)$, therefore the number of permutations of $a_2, a_3, a_4, a_5, \ldots a_n$, taken two and two, is found by diminishing n by a unit in the expression $n(n-1)$, which thus becomes $(n-1)(n-2)$. Now, writing a_1 before each of these permutations, we shall find $(n-1)(n-2)$ permutations of three by three, all commencing with a_1. It is obvious the same thing may be done with all the other letters, $a_2, a_3, a_4, a_5 \ldots a_n$. Therefore, the total number of permutations of n things taken three by three is $n(n-1)(n-2)$.

By a similar process we should be able to show that the number of permutations of n things taken four and four is

$$n(n-1)(n-2)(n-3).$$

(173.) *By this method of induction, we infer that the number of permutations of n things taken r and r together, will be given by the continued product of the natural numbers descending from n, until as many factors are used as there are things in each permutation.*

The general expression will be

$$n(n-1)(n-2)(n-3)\ldots.(n-r+1). \qquad (A)$$

If we suppose $r=n$, then all the different things will be found in each permutation, and formula (A) will become, where the order of its factors is reversed,

$$1\times2\times3\times4\times5\ldots.(n-1)n, \qquad (B)$$

which is the product of all the natural numbers as far as n.

EXAMPLES.

1. How many permutations can be made of six individual things, all being taken at a time?

Ans. 720.

2. In how many different ways may 15 persons sit at table?

Ans.

$$1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12\times13\times14\times15=1307674368000.$$

3. How many changes can be rung on 8 bells?

Ans. $1\times2\times3\times4\times5\times6\times7\times8=40320.$

(174.) Let us endeavor to find the number of permutations of n things, when any assigned number of them become identical.

Thus let us find an expression for the number of permutations of n things, r of which are identical with each other.

We have already found the number of permutations of n things all taken together, supposing them all different from each other, to be

$$n(n-1)\ldots.3.2.1.$$

If r of these quantities become identical, the permutations which arise by their interchange with each other, or from their *particular* permutations, which are $1\times2\times3\ldots.r$ in number, *for any assigned position of*

the other letters, are reduced to one. The number of permutations, therefore, when all the letters are different from each other, is $1\times2\times3\ldots.r$ times as great as when r of them become identical. Or, in other words,

$$\frac{n(n-1)\times\ldots.2\times1}{1\times2\times\ldots r},$$

is the expression for the number of permutations under the circumstances supposed.

If, in addition to r quantities, which become identical, there are s others, which, though different from the former, are still identical with each other, then there are $1\times2\times3\times\ldots.s$ permutations corresponding to their interchange with each other, which are reduced to one, for any given position of the other quantities. The expression for the number of permutations, under these circumstances, becomes

$$\frac{n(n-1)\times\ldots.2\times1}{1\times2\times\ldots.r\times1\times2\times\ldots.s}.$$

The same reasoning obviously applies to any number of classes of letters or things which become identical with each other, and consequently if, of n quantities, r_1 are of one kind, r_2 of another, r_3 of a third, and so on, as far as r_m of the mth class, then their whole number of permutations will be expressed by

$$\frac{n(n-1)(n-2)\times\ldots.3\times2\times1}{1\times2\ldots.r_1\times1\times2\ldots.r_2\times1\times2\ldots.r_3\times\ldots.1\times2\ldots.r_m}$$

(175.) We will illustrate this formula by a few

EXAMPLES.

1. Find the number of permutations (p) of the letters in the word *Algebra*.

In this case $n=7$, and the letter a appears twice; consequently,

$$p=\frac{7.6.5.4.3.2.1}{1.2}=2520.$$

2. Find the number of permutations (p) of the letters in the word *perseverance*.

In this case $n=12$, and the letter e appears four times, and r twice. Therefore,

$$p=\frac{12.11.10.9.8.7.6.5.4.3.2.1}{1.2.3.4\times 1.2}=9979200.$$

3. In how many different ways may three maple trees, five ash trees, and two elm trees, be set out in a single row? *Ans.* 2520.

COMBINATIONS.

(176.) By *combinations* of different letters or quantities, we mean the different collections which can be made of any assigned number of them, without reference to the order of their arrangement. Thus, a_1a_2, a_1a_3, and a_2a_3, are the only different combinations of the three letters a_1, a_2, a_3, taken two and two together, which form *six* different permutations. There is only *one* combination of the same three letters, taken all together, though they form *six* different permutations.

(177.) We will now proceed to determine the number of combinations of n things, taken r and r together, when r is less than n.

The number of combinations of n things, taken separately or one and one together, is clearly n.

The number of combinations of n things, taken two and two together, is $\frac{n(n-1)}{1.2}$.

For the number of permutations of n things, taken two and two together, is $n(n-1)$ (Art. 172); and there are two permutations (a_1a_2, a_2a_1) corresponding to one combination; the number of combinations will be found, therefore, by dividing the number of permutations by 2, or 1.2.

The number of combinations of n things taken three and three together, is

$$\frac{n(n-1)(n-2)}{1.2.3}.$$

For the number of permutations of n things, taken three and three together, is $n(n-1)(n-2)$, (Art. 172), and there are 1.2.3 permutations for one combination of three things. The number of combinations will therefore be found by dividing the number of permutations by 1.2.3.

By the same reasoning we might show that the number of combinations of n things taken r and r together, is

$$\frac{n(n-1)(n-2)\ldots.(n-r+1)}{1.2.3\ldots.r}. \qquad (D)$$

(178). We will illustrate this formula by a few simple

EXAMPLES.

1. How many lottery tickets, each having three numbers, can be formed out of 60 numbers?

$$Ans. \frac{60\times59\times58}{1\times2\times3}=34220.$$

2. How many diagonals can be drawn in a polygo of n sides?

It is evident that the whole number of lines which ca be drawn by joining the corners of the polygon is equ to the number of combinations of n things, taken tw and two, since a line can be drawn from any point to an other point.

Hence, the whole number of lines thus drawn is

$$\frac{n(n-1)}{1.2}.$$

But n of these lines must be required to form the side of the polygon. Therefore the number of diagonals is

$$\frac{n(n-1)}{1.2}-n=\frac{n(n-3)}{2}.$$

3. How many different triangles can be formed b joining the corners of a polygon of n sides, countin only such triangles as have their vertices correspondin with the corners of the polygon?

Now, as three points will determine the position of triangle, it is evident there must be as many triangle as the number of combinations of n things taken thre and three together, which is

$$\frac{n(n-1)(n-2)}{1.2.3}.$$

PROBABILITIES.

(179.) There is scarcely any subject which admits of a greater variety of instructive illustrations than the *Theory of Combinations*. For it is by the aid of this theory that many nice calculations are made in reference to chances and probabilities. We will for this reason make a few remarks, in this place, concerning the method of applying algebra to the calculation of probabilities.

(180.) If a expresses the number of favorable events or cases, and b the number of those which are unfavorable, the *chance* of the favorable event or of the required case existing, is expressed by

$$\frac{a}{a+b};$$

while the *chance* of an unfavorable event or of the required case not existing, is expressed by

$$\frac{b}{a+b}.$$

(181.) From this mode of representation, it will follow that *certainty*, which supposes all the events favorable, in the first case, when $b=0$; or all of them unfavorable, in the second case, when $a=0$, will be expressed by 1. The ratio, therefore, of the *chances* to *certainty*, or of the degree of probability to certainty, will be the ratio which the fraction, by which it is denoted, bears to unity, or the ratio of its numerator to its denominator.

(182.) The ratio of the chances of success to that of failure, or the ratio of the *odds for or against*, as expressed in popular language, will be that of a to b, or of b to a, which are the numerators of the fractions by which the respective chances are denoted.

(183.) The following are examples of the representation of simple chances:

1. What is the chance of throwing an ace with a single die?

There is only one face which can be uppermost; that is, there is but one *favorable* case, while there are five *unfavorable* cases. The *chance*, therefore, that this face is the *ace*, is $\frac{1}{6}$. The chance that this face is not the *ace* is $\frac{5}{6}$, for there are five out of six equally possible cases, which are favorable to this last case.

The chance that the face thrown is either an *ace* or a *deuce*, is $\frac{2}{6}$ or $\frac{1}{3}$. For there are here two favorable cases out of six which are equally likely to happen. The chance that it is neither an *ace* nor a *deuce* is $\frac{4}{6}=\frac{2}{3}$.

If the die had been a regular tetrahedron, whose faces were marked with the numbers 1, 2, 3, 4, the chance of its *resting upon an ace* would be $\frac{1}{4}$, the chance of its not doing so would be $\frac{3}{4}$.

The chance of drawing the ace of spades from a pack of 52 cards, is $\frac{1}{52}$. The chance of drawing any one of the four aces is $\frac{4}{52}=\frac{1}{13}$; for there are four favorable cases out of fifty-two which are both favorable and unfavorable, and all of them are equally likely to happen.

(184.) If 14 white and 6 black balls be thrown into an urn, the chance of drawing a white ball out of it, at

one trial, is $\frac{4}{6}=\frac{7}{12}$. The chance of failure, or of drawing a black ball, is $\frac{6}{12}=\frac{5}{12}$.

In the preceding, and in all other cases, the chances of success and failure of the *same* event are supplemental to each other, their sum being equal to 1, which is the measure of *certainty*. The knowledge of one, therefore, necessarily determines the other.

(185.) The following are some examples of *compound chances*:

To find the chance of throwing an ace twice in succession with a single die.

There are six cases which are equally likely to occur at the first throw, and the same number at the second. These may be combined or permuted together in $6\times 6=36$ different ways which are equally likely to happen, and only one of them is *favorable*. The chance is therefore $\frac{1}{36}$.

The chance of throwing *two aces* at one *contemporaneous* throw with two dice, is equally $\frac{1}{36}$: for the succession of time makes no difference whatever in the number of favorable and unfavorable permutations.

The chance of throwing an *ace* at the first throw, and a *deuce* at the second, is also $\frac{1}{36}$: for there is only one favorable permutation out of 36.

The chance of throwing an *ace* at one throw, and a *deuce* at the other, *without reference to their order of succession*, is $\frac{1}{18}$: for in this case there are two permutations forming one combination, (1, 2 and 2, 1), which are favorable to the hypothesis made, and two *only* out *of* the whole 36.

(186.) The chance of an event contingent upon other events, is the continued product of the chances of the separate events.

Let the several chances be

$$\frac{a_1}{a_1+b_1};\ \frac{a_2}{a_2+b_2};\ \frac{a_3}{a_3+b_3};\ \ldots\ \frac{a_n}{a_n+b_n};$$

where $a_1, a_2, a_3, \ldots a_n$ represent the numbers of cases which are favorable, and $b_1, b_2, b_3, \ldots b_n$ the numbers of cases which are unfavorable, to the particular hypothesis made in each separate event, whether of success or failure.

We will consider, in the first instance, the chance which is dependent upon the two separate chances

$$\frac{a_1}{a_1+b_1} \text{ and } \frac{a_2}{a_2+b_2}.$$

Every case in a_1+b_1 may be combined with every case in a_2+b_2, and thus form $(a_1+b_1)(a_2+b_2)$ combinations of cases, which are equally likely to happen.

The favorable cases in the first (a_1) may be combined severally with the favorable cases in the second (a_2), and thus form $a_1 \times a_2$ combinations of case favorable to the compound event.

The compound chance is therefore denoted by

$$\frac{a_1 \times a_2}{(a_1+b_1)(a_2+b_2)},$$

which is the product of the separate chances.

We will now consider the chance of the event contingent upon three other events, whose respective chances are

$$\frac{a_1}{a_1+b_1};\ \frac{a_2}{a_2+b_2};\ \frac{a_3}{a_3+b_3}.$$

The several combinations of all the cases in the two last chances, which are by the last case, $(a_1+b_1)(a_2+b_2)$ in number, may be severally combined with the a_3+b_3 different cases both favorable and unfavorable, of the third chance, and thus form $(a_1+b_1)(a_2+b_2)(a_3+b_3)$ combinations which are equally likely to happen.

The favorable cases in the two first chances, which are $a_1 \times a_2$ in number, may be combined severally with the a_3 favorable cases of the third chance, and thus form $a_1 \times a_2 \times a_3$ cases which are favorable to the compound event.

The chance, therefore, of the compound event is

$$\frac{a_1 \times a_2 \times a_3}{(a_1+b_1)(a_2+b_2)(a_3+b_3)},$$

which is the product of the simple chances.

It is obvious that the chance of a compound event depending upon a greater number of simple events may be found in the same way.

This is a most important proposition in the Doctrine of Chances, and makes the calculation of the chance of any compound event dependent upon the separate and simple chances of the several events, in their assigned order, upon which it is dependent.

(187.) We will add some cases of compound chances.

1. What is the chance of throwing an ace in the first *only* of two successive throws?

The first simple chance is $\frac{1}{6}$.

The second simple chance is $\frac{5}{6}$: for an ace must *not* be thrown the second time, and there are five *favorable* cases for its failure.

The compound chance is therefore $\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$.

2. What is the chance of drawing the four aces from a pack of cards in four successive trials?

The first simple chance is $\frac{4}{52}$.

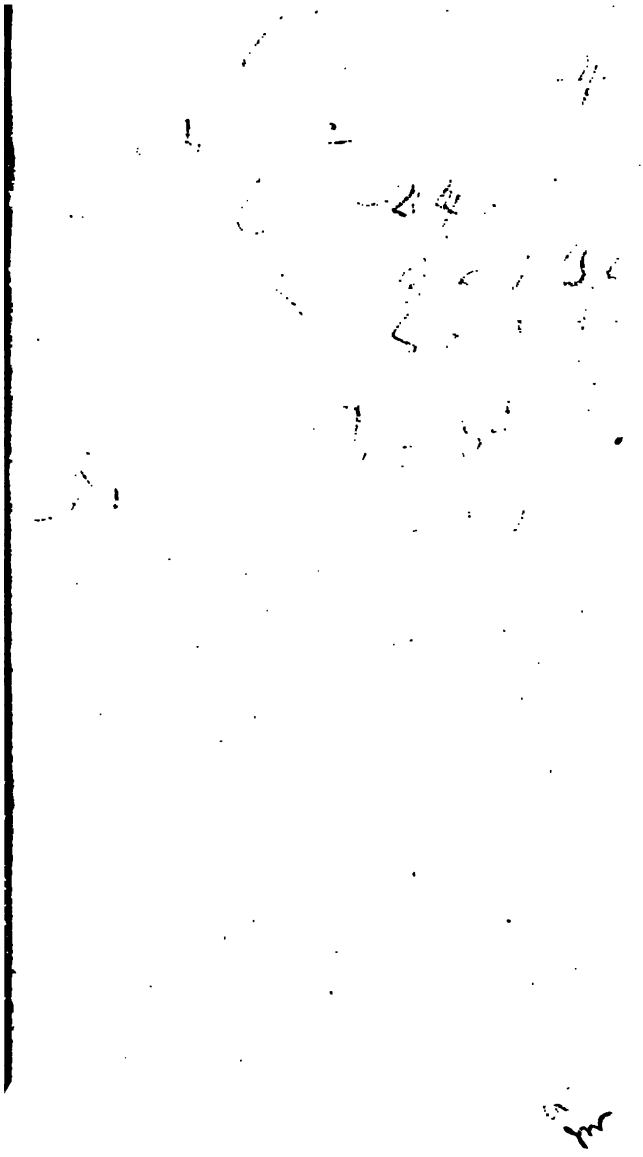
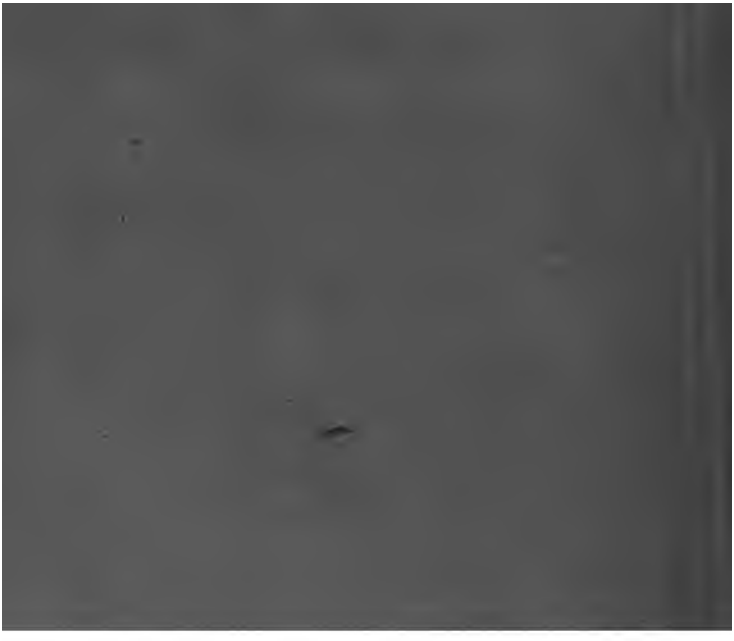
The second simple chance is $\frac{3}{51}$.

For if an ace be drawn the first time, there will remain only 3 aces and 51 cards.

The third simple chance is $\frac{2}{50}$.

For if two aces be drawn the two first times, there will remain only two aces and 50 cards.

The fourth simple chance is, by the above reasoning, $\frac{1}{49}$.

The compound chance required is, therefore,

$$\frac{4.3.2.1}{52.51.50.49} = \frac{1}{270725}.$$