Th. [illegible]ahes. [illegible]

Velocity shown to be a directed quantity and defined as the rate of change with reference to the [illegible] of any coördinate.

Treatment as analytic geometry with time for independent variable of:–

I.) Velocity in plane & space referred to rectangular fixed axes.

II.) Velocity in plane & space – – – [illegible] coördinates.

III.) Velocity in plane & space – oblique fixed axes.

IV.) Velocity in plane referred to any rotating [illegible] axes

V.) Change of origin for velocities.

Composition of rectangular & oblique linear velocities extended to angular velocities.

Instantaneous centre.

Acceleration in any direction defined as Limit $\Big\{ \dfrac{\text{Change of vel. in that direction}}{\text{Time occupied by change}} \Big\}$ ($t \to 0$)

Accelerations in plane and space along fixed rectangular axes.

Accelerations in plane [illegible] " [illegible] ordinates.

Change of origin [illegible]

The illustrative [illegible] [illegible] from [illegible]

In the "Dynamics" part of the course the differential equations

$$\frac{d^2x}{dt^2} = -ax \quad \& \quad \frac{d^2x}{dt^2} = -ax + b\frac{dx}{dt}$$

are often used and familiarity with them is desirable.

Serviceable too are:—

1 Method by trial and error for numerical [illegible]

[illegible] integration [illegible]

LESSONS ON RIGID DYNAMICS.

LESSONS

ON

RIGID DYNAMICS,

BY THE

REV. G. PIRIE, M.A.,

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PREFACE.

It will be generally acknowledged, I think, that there is no subject of Natural Philosophy, equal in importance to that familiarly known as Rigid Dynamics, of which the study is so exclusively restricted to the more advanced students of Mathematics. Yet this restriction cannot be said to be necessary, for the treatment of the subject involves none of the higher mathematical methods; and it must be allowed to be unfortunate, for the science of motion is the basis of Mechanical Engineering, and furnishes the explanation of many interesting terrestrial and cosmical phenomena.

This restriction of the study is chiefly due to the fact that, while the conceptions and reasoning peculiar to the subject are somewhat difficult, the explanations of its leading principles, given in the books commonly used by students, are for the most part very brief, and often, through brevity, obscure.

It is this deficiency of explanation which I have attempted to supply in the following little book. It is not my purpose to acquaint the student with the splendid generalizations of Lagrange and of more recent philosophers. For that the books in present use leave nothing to be desired. My aim is to render more general the study of this interesting science, by presenting as simple a view of its principles as is consistent with scientific accuracy, and to give a sound foundation to the student who is to proceed higher.

It is my hope that the book may be useful not only to students of Natural Philosophy, but also to engineers. Most of them possess a knowledge of the principles of Mechanics, of the method of Co-ordinate Geometry, and of the Integral Calculus; and that is all that is here required.

The principle on which this science is based has been so long connected with the name of D'Alembert that it would hardly be recognised under any other. Nevertheless there is no doubt that Euler has more claim to its authorship, inasmuch as he first used it. D'Alembert admits this, but says that Euler gave no proof. I believe D'Alembert's real merit to be, that his explanation was exactly suited to clear away the difficulties which were perplexing men's minds.

The works to which I am principally indebted are:— Thomson and Tait's *Natural Philosophy;* Routh's *Rigid Dynamics;* Resal's *Cinématique Pure;* Rankine's *Machinery and Millwork;* Walton's *Mechanical Problems;* Whewell's *History of the Inductive Sciences;* Willis' *Principles of Mechanism;* Müller's *Lehrbuch der kosmischen Physik;* Montucla's *Histoire des Mathématiques;* D'Alembert's *Traité de Dynamique,* and Euler's *Mechanik.*

My thanks are due to Dr Campion, of Queens' College, for many valuable suggestions which he has made; and to several of my pupils for their frank statement of their difficulties.

G. PIRIE.

QUEENS' COLLEGE, CAMBRIDGE.
December, 1874.

CONTENTS.

GEOMETRY OF MOTION.

I.

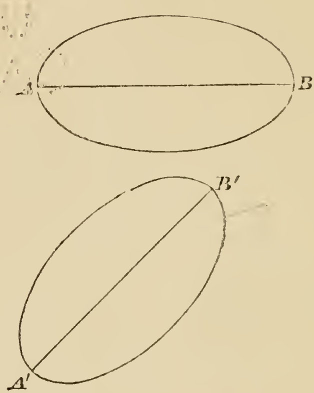
1. A RIGID body is an assemblage of particles such that the distance between each pair is unchangeable. The movements of such a body are very different from those of a set of independent points. Its fixed connections introduce a common movement. Any straight line or any plane of particles in the body must remain always a straight line or plane. If all such planes remain parallel to themselves, the motion is one of translation. But if any such plane makes an angle with its former position the motion is rotational. And the velocity of rotation—angular velocity—is measured by the rate at which the plane is describing angles.

Thus the connecting rod of the driving wheels of a goods' locomotive has only a translational motion;—so also (approximately) the axis of the earth in its yearly motion round the sun. In a well-thrown quoit the motions are combined.

2. From this definition of rotation it follows that a point cannot rotate. It may revolve about another point, but it contains no lines nor planes which can describe angles. For rotation there must be an extended system. A point in motion may be said to be revolving about any point whatever situated in the line through it at right angles to its direction of motion, for it is moving at the moment in a circle with the point as centre. But the body of which this is a point may not be rotating. For rotation it is necessary that the different points of the body should be at the moment revolving about the same axis.

Suppose a man to move round a column viewing its parts in succession. In this case he is also rotating. Were he to move without rotation, he must work round the column sometimes forwards, sometimes sidewards, sometimes backwards, but always facing the same point of the compass.

3. From the definition it follows also that rotation is directional, i.e. it takes place not so much about an axis or point



as about a direction or in a plane. The body AB has rotated in passing to the position $A'B'$; but the amount of the rotation is measured by the angle between the straight lines AB and $A'B'$. It matters not to the angular velocity of a carriage-wheel whether it is rolling along a level road or up a hill, or whether the wheel, being raised from the ground, is whirled round its own axle. Or on a larger scale, whether a ship rounds a promontory or swings with her anchor fixed through the same angle is indifferent to the amount and direction of the rotation performed. There is indeed in general an axis round which the body may, in a stricter sense, be said to rotate, for every point of the body moves in a circle about it. This rotation is the more easily imagined, but it ought not to be allowed to expel the idea of the other. It is a pity there are not separate words to distinguish them. We will in future speak of them as rotation round a point or axis, and rotation round a direction or in a plane.

4. The method which mathematicians adopt in treating of simultaneous motions is to consider them one after the other. A velocity is the describing of a certain length or angle in a certain time. Properties of small linear and angular displacements are properties of linear and angular velocities. If, therefore, a body is subject to two independent motions, as rotations about two axes or a rotation and a translation, it is considered to obey them in turn each for a very short time. A rifle bullet moving towards the target and rotating all the time is supposed to approach the target without turning, through an infinitely small space, and then to turn round through an infinitely small angle, much like a man descending a spiral staircase. This is not the actual motion, any more than a polygon is a curve, but it differs as little as we please from the real motion, and it clears our ideas and enables us to apply mathematical methods to the problems.

If a body is solicited to two different motions by two simultaneous causes, it will in reality follow neither; but it may be supposed to have followed both.

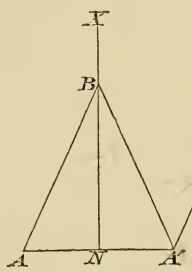
Thus the very extravagant idea of some of the earliest writers on projectiles that a cannon-ball went straight until it had exhausted the force of projection and then fell down straight under gravity, had in it, notwithstanding its grievous confusion of force with velocity, a germ of truth, (viz.) that the causes of motion must be considered separately.

A skater describing circles, the nut of a screw, a crank rod one end of which moves in a straight line while the other describes a circle, the arms of a common form of reaping machine which rotate about an inclined axis while carried forwards by the machine, and hundreds of other familiar cases, supply examples of translation combined with rotation.

Examples of combined rotations are seen in the screw or paddles of a steamer, which are rotating about a horizontal axis while the steamer may be moving round a curve and thus rotating about the vertical; or in the common gyroscopic toy, where a metal ring rotates about a diameter of a circle, and is borne along also by the rotation of this circle about the vertical; or in the sails of a windmill, which may have rotations about their own axis, about the vertical (if the wind veers) and with the whole body of the windmill round the polar axis of the earth.

5. *Motions in one plane and in parallel planes.*

Let A be a point of a rigid body which is moved to A'. Bisect AA' perpendi ularly by the straight line NX. On

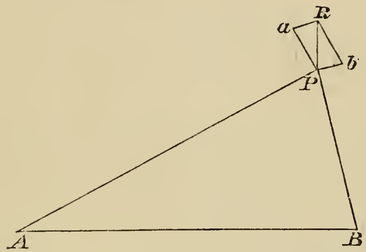


NX take any point B. We may suppose that the point A has been moved to A' by the body having been caused to rotate about an axis through B at right angles to the plane of the paper. In this case the line of particles AB has taken up the position $A'B$. Now cause the body to rotate about an axis through A' perpendicular to the plane of the paper through an equal and opposite angle. The line $A'B$ takes up the position $A'B'$ parallel to AB, whence we infer that a displacement of translation is equivalent to two equal and opposite displacements of rotation about two parallel axes. If these displacements are small, AA' is at right angles to AB, and the proposition becomes that two equal and opposite angular velocities about two parallel axes are equivalent to a translational velocity in a direction at right angles to the plane of the axes.

It will be convenient to denote such axes perpendicular to the plane of the paper by the point where they cut this plane. Thus we might have spoken of A' and B as axes.

This important proposition may be reduced to the parallelogram of linear velocities.

Let P be any point of a body which has simultaneous equal and opposite rotations round A and B. The velocities of P

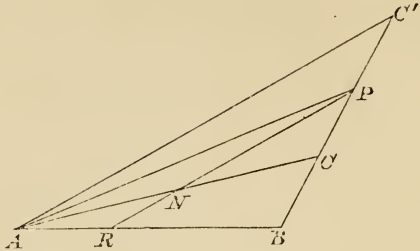


due to these are represented by Pa and Pb perpendicular and proportional to PA and PB respectively. The resultant velocity is represented by the diagonal PR of the parallelogram on Pa, Pb. But this parallelogram is similar to that whose sides are PA, PB. Hence the velocity of P is proportional and perpendicular to AB. As this holds for every point of the body, the whole is being translated at right angles to AB, and with velocity proportional to it.

6. From the definition of rotation it is clear that two equal and opposite rotations cannot produce a rotation; for they turn a straight line in the body through equal but opposite angles.

For the same reason the angular velocity of a body rotating about two parallel axes is the sum or difference of their separate angular velocities.

Let P be any point of a body which has angular velocities in the same direction round A and B. And take on PB a point C such that PB and PC are proportional to the angular velocities round A and B respectively. Then the linear velocities of P will be at right angles to PA and PB, and proportional to $PA \, . \, PB$ and $PB \, . \, PC$ respectively, i. e. to PA



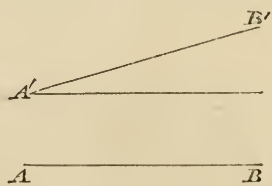
and PC. Hence, joining AC and bisecting it in N, the resultant velocity of P is at right angles to PN and is measured by twice PN. P may therefore be taken to be revolving about any point in PN. Let PN produced cut AB in R; we can shew that R is the same point wherever P is, and therefore the whole body is rotating about R. The angular velocity has been settled independently.

Produce CP to C', making PC' equal to PC. Join AC'. PNR is parallel to AC', and therefore

$$AR : RB :: C'P : PB;$$

or R divides AB inversely as the angular velocities round A and B.

7. Any displacement whatever may be given to a body by a translation and a rotation about an axis.

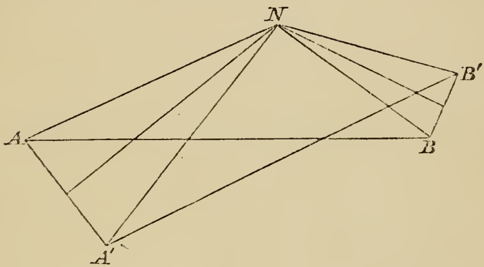


For to bring AB to $A'B'$ it is only necessary to translate the body till a point A reaches its new position A', and then to rotate the body about A'.

Thus in general any motion of a body may be composed of a rotation round an arbitrary point and a translation.

And this point may generally be so chosen that the movement of translation shall not be required. In other words, there is one point which is unaffected by the change of position. To find this point. Bisect AA' and BB' perpendicularly, and let the bisectors meet in N.

Join NA, NA', NB, NB'.



Then NA is equal to NA' and NB to NB'.

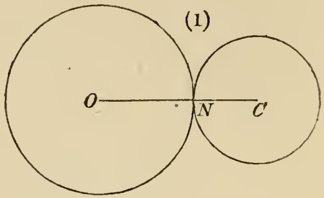
If, then, we can shew that the angle ANA' is equal to the angle BNB', we shall have proved that when the body is rotated about N, so as to move A to A', B is brought to B', and so for any other point. For N is a point of the body, since the triangles ANB and $A'NB'$ are equal in all respects.

Now the triangles ANB, $A'NB'$, have all their sides equal each to each; therefore the angles ANB and $A'NB'$ are equal. Take away the common angle $A'NB$, and the remainder ANA' is equal to the remainder BNB'.

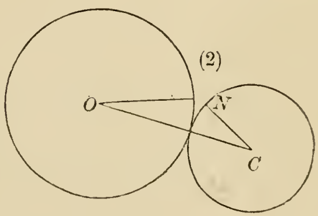
Hence any motion of a rigid body, except one of translation, is one of rotation round some axis; and this is called the instantaneous axis.

If the body be a plane one, and be moving in its plane, this axis cuts the plane in the instantaneous centre. To find its position it is only necessary to take two points whose directions of motion are known, and to draw perpendiculars from them to these directions. Their intersection gives the centre required.

8. For the illustration of these principles consider the motion of a circular hoop C, rolling on another cylinder O, which is fixed. The rolling of the hoop from position (1) to

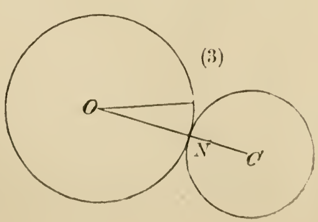


an infinitely near position (2) may be considered as taking place in one of three principal ways.



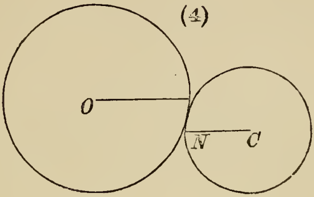
First, by two rotations.

The whole hoop may rotate through a small angle round O, thus coming into position (3), and then a rotation round C will bring the hoop into position (2).



Secondly:

The hoop may be translated into position (4), and then brought by a rotation round C to position (2).



Thirdly:

By one rotation round the instantaneous centre. This point must lie in OC; for OC is at right angles to the direction of motion of C. To get another line on which it lies, consider the motions of points very near to N. These are moving away from or towards O. Hence N must be the instantaneous centre. That the body is at the moment rotating round N will be better seen by looking on the circles as many-sided polygons of equal sides. Each angular point becomes in turn the centre of rotation. But the axis is continually changing, and if the question considered be one of change of velocity, the motion must not be considered as if it were round a fixed point at N.

If Ω be the angular velocity of revolution of C about O, measured by the angle at O, the velocity of rotation of the hoop round C in the first method will be measured by the angle at C; and since the arcs of the two circles which have been in contact are equal, it will be

$$\Omega \frac{ON}{CN}.$$

If we combine these by Art. 6, they give a resultant rotation round N.

In the second method the linear velocity is measured by the distance the centre has moved, i.e. by $OC\,.\,\Omega$, and the angular velocity by the angle between CN (fig. 4) and CN (fig. 2), i.e. by ($\angle\, O + \angle\, C$) (fig. 2).

It is therefore
$$\Omega + \Omega\,.\,\frac{ON}{CN},$$

or
$$\Omega\,.\,\frac{OC}{CN}.$$

To reduce this to either of the others, consider the linear velocity $OC\,.\,\Omega$ as the resultant of two equal and opposite angular velocities.

The single angular velocity in the third method is the same as in the second, but it is about N. For the whole rotation must be measured by the same angle whatever be the axis. Or we may see it thus. Calling it ω, the linear velocity of C due to it is $\omega\, CN$, and this must be the same as that found by the last method, viz. $\Omega\,.\,OC$.

EXAMPLES.

1. If two points are rigidly connected, their velocities in the direction of the straight line joining them are equal.

2. A mirror rotates about a vertical axis with an angular velocity ω, and a ray of light falls on it from a distant fixed point on the horizon. What is the angular velocity of the reflected ray?

3. Express a velocity of 100 revolutions a minute in units of angular velocity.

4. Compare the velocity of rotation of the earth with the mean angular velocity of revolution of its centre.

5. If v be the linear velocity in Art. 5, which is equivalent to the angular velocities ω, $-\omega$ about A and B, shew that $v = AB\,.\,\omega$.

6. Where is the instantaneous centre when a ladder is slipping down in a vertical plane between a wall and the ground?

7. The paddle-wheel of a steamboat is rotating with velocity ω, and the vessel is moving with velocity v; where is the instantaneous axis of the paddle-wheel?

8. Prove that any motion of a rigid body of which the points move in parallel planes may be represented by supposing a right cylinder fixed in the body to roll on a right cylinder fixed in space.

9. What are these cylinders in the case of question 7?

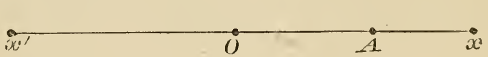
10. If a straight rod be moving in any manner in a plane, the directions of motion of all its points will in general touch a parabola.

11. AP, BQ are two arms moveable round the fixed centres A, B; and the points P and Q are connected by a link (rod) PQ; shew that the angular velocities of the arms AP, BQ are inversely proportional to the segments into which the link, or its direction produced, divides AB.

II.

GEOMETRY OF MOTION.

1. We now come to the case of simultaneous rotations about axes inclined to one another. The motions of points are no longer in one plane or in parallel planes. It will be necessary to represent the axes themselves. The way in which an angular velocity is geometrically represented is as follows: take the axis xx'; on it take a point O; let Ox be the direction which is considered positive. Place a watch at O with its face towards x. A rotation whose direction coincides with the direction of motion of its hands is considered positive. It is measured by the line OA, which contains as many units



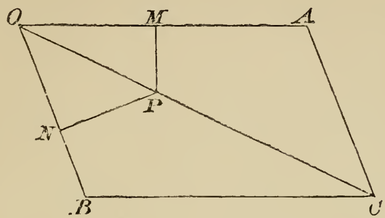
of length as the angular velocity contains units of angular velocity. An angular velocity in the opposite direction is represented by a straight line measured along Ox'.

With this convention a positive angular velocity round Ox—one of a rectangular system of axes as usually drawn—will tend from Oy to Oz; one round Oy from Oz to Ox; one round Oz from Ox to Oy.

The basis of this subject is the proposition called the parallelogram of angular velocities, which is:

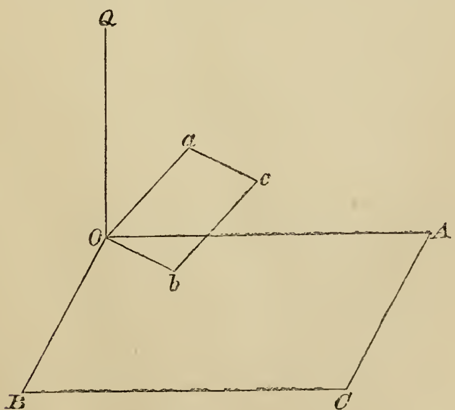
If a body have simultaneous angular velocities about two inclined intersecting axes, and if these be represented by the adjacent sides of a parallelogram, then shall the resultant angular velocity be about the diagonal which passes through their intersection and proportional to it in magnitude.

Let a body be rotating simultaneously about OA and OB with velocities proportional to OA and OB.



Then, first, the points on OC are at rest. For, taking such a point P and drawing PM, PN at right angles to OA and OB, P's linear velocity due to its rotation round OA is upwards from the plane of the paper, and proportional to $OA \,.\, PM$; while that due to rotation round OB is downwards and proportional to $OB \,.\, PN$. As these products are twice the areas of the triangles OPA, OPB respectively, they are equal. As they are opposite the point P is at rest. The body is therefore rotating about OC.

To settle its angular velocity about OC; draw a perpendicular to the plane AOB through O, and on this take a point Q.



The angular velocities of the body round OA, OB will be proportional to the linear velocities of Q perpendicular to these, i.e. Oa, Ob, which are drawn in the plane of AOB, at right angles to OA, OB and proportional to them respectively.

The resultant angular velocity about OC will be represented by the resultant linear velocity of Q, i.e. by Oc, the diagonal of the parallelogram Oa, Ob. But the parallelograms OA, OB and Oa, Ob are similar, the latter being turned round through a right angle. Therefore the diagonal Oc is proportional and perpendicular to OC. And the resultant angular velocity of the body is proportional to OC.

2. Angular velocities, then, are quantities which obey the parallelogram law, and all its consequences will hold good for them. A body rotating with velocity ω about any axis may be considered to have a component angular velocity $\omega \cos \alpha$ about any other axis inclined to the former at an angle α. There will be a parallelopiped of angular velocities; and in general the analogy between angular velocities and forces in Statics is complete.

We will take for the illustration of this the pendulum experiment by Foucault, by which the rotation of the earth is rendered visible.

Draw a circle representing a section of the earth through its polar axis NS. Let O be the centre, and A any place on its surface.

In this experiment, a pendulum is set swinging in any vertical plane at A. We assume that wherever the point of suspension may be, the plane in which the pendulum swings will remain parallel to itself. If the earth were rotating about OA, the effect of this would be that the plane of the pendulum would be left behind by the earth, and would appear to an observer, unconscious of the earth's motion, to follow the sun. Now this is in part what happens. The earth does not indeed rotate about OA; its rotation is about NS; but this is equivalent to one about OA proportional to $\cos NOA$, and one about a perpendicular to OA

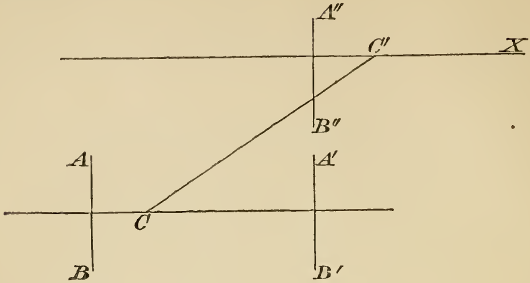
proportional to sin NOA. This latter is what carries the building and the whole apparatus eastward. It does not affect the present question. But the other rotation—that about OA—causes the plane of the pendulum to follow that of the sun with an angular velocity, which is to that of the earth as cos NOA to 1.

This experiment requires the greatest care for its exhibition. If the pendulum move in even the most elongated oval, instead of swinging in a plane, the axis of this oval will rotate from a very different cause, viz. the resistance of the air. When Foucault exhibited his experiment in Paris to the French 'savants,' he used a heavy ball hung from the roof of the Observatory, and set it off by burning a thread which held the ball out of the position of rest.

3. If a rigid body has one point fixed, there is at any moment a straight line of points at rest. In other words, any displacement of a rigid body, one point of which is fixed, may be effected by a single rotation about some axis through that point. The proof is the same as that by which we shewed that any displacement of a plane body in its plane can be given by a rotation round one point; if instead of a plane we consider a sphere in the body with the fixed point as centre. The points, then, represent straight lines through the centre; the straight lines in the figure become arcs of great circles and represent planes passing through the centre; but the reasoning is precisely the same.

Any displacement may therefore be given to a rigid body by translating it so that a chosen point comes into its new position, and then making it rotate round some axis through that point. The direction of this axis and the angular displacement remain the same whatever point be chosen. The direction and amount of translation may change, but the translation cannot affect the angular movement.

The point may be chosen so that the direction of translation is that round which the rotation takes place. For let C' be the new position of C, and let $C'X$ be the axis of rotation. Let AB represent a plane in the body perpendicular to this axis. Let $A''B'$ be its final position. This is parallel to AB; for neither the translation nor the rotation about $C'X$ affects its direction.



If, then, we first translate the whole body along a parallel to $C'X$ until AB comes to $A'B'$ in the same plane with $A''B''$, we shall be able by one rotation about a parallel to $C'X$ to bring $A'B'$ to $A''B''$, i.e. the direction of translation will be the axis of rotation.

Hence every small motion is reducible to that of a screw in its nut. And all points of any rigid body are at the same moment moving in coaxial helices. If the pitch of the screw be zero the motion will be one of rotation simply, if it be infinite it will be a translation.

It is of course not always equally easy to see what these axes and directions are. In the case of a rifle bullet, for example, the motion is already reduced. In the general case the first point is to find out the series of planes which remain parallel, or—what is the same thing—to find the direction of rotation. Thus suppose we are considering the motion of the earth at any instant. This consists of a rotation round its polar axis and a revolution of its centre in the plane of the ecliptic round the sun. And suppose we wish to reduce it to the screw motion. We observe that the planes which remain parallel are those parallel to the equator. Hence the axis of the screw is perpendicular to the equator. To find the actual position of this axis we must consider *all velocities* projected on a plane parallel to the equator. Then the motion is similar to that of the hoop in Lesson I. Art. 8, which rotates about its centre while the centre revolves about a fixed point.

Let Ω be the velocity of rotation of the earth, V the component velocity of its centre in the plane of the equator, R the distance between two axes each perpendicular to the equator, through the centres of the sun and earth respectively. By Lesson I. Art. 8, V and Ω are equivalent to two angular velocities, $\frac{V}{R}$ about the axis through the sun, and $\Omega - \frac{V}{R}$ about the axis through the centre of the earth. And these are equivalent to an angular velocity round a parallel axis in the plane of the others, distant $R - \frac{V}{\Omega}$ from that through the sun. This last is therefore the axis of rotation.

4. An extremely elegant geometrical conception of the motion of a body round a fixed point was introduced by Poinsot. Any such motion may be completely represented by imagining a cone fixed in the moving body to roll on a cone fixed in space. For every body with one point fixed is rotating about a certain axis. As the motion changes, this axis takes up different positions, and describes a cone whose vertex is the fixed point. Now by reasoning exactly similar to that of Lesson I. Art. 8, any cone rolling on another with the same vertex has for its instantaneous axis its line of contact with the other. This axis therefore describes a cone whose vertex is the fixed point. But this is precisely the motion to be represented. The rate of rotation will depend on the dimensions of the rolling cone.

As an example of this take the case of a top spinning with angular velocity ω about its axis OC, while that axis is rotating with angular velocity Ω about the vertical OV, to which it is inclined at an angle α. By the parallelogram of angular velocities the resultant axis is OR—between OC and OV—inclined to OC at an angle given by

$$\frac{\sin ROC}{\sin(\alpha - ROC)} = \frac{\Omega}{\omega}.$$

Hence the motion is the same as if an imaginary right circular cone in the top, whose axis was the axis of the top, and whose semivertical angle was ROC, were to roll on the cone in space whose axis was OV and semivertical angle ROV. As ω, the angular velocity of the top about OC, is in general large compared with Ω, the angular velocity of OC about OV, the angle ROC will be small. If a series of concentric coloured circles round C be drawn on the head of the top, that which corresponds to R will be the only pure colour seen as the top spins.

EXAMPLES.

1. Prove that the proposition (Art. 1) holds for angular accelerations.

2. If a ship is rolling and pitching with equal angular velocities, what is her actual motion?

3. Two circular discs can turn about fixed perpendicular intersecting axes. If the axes be so placed that the circumference of one of the discs (which is rough) presses against the plane of the other, and if the former disc be caused to rotate about its axis with given angular velocity, find the angular velocity of the other.

4. A heavy cylindrical crushing stone rolls on a horizontal table round a vertical axis. Represent its motion (1) as two rotations, (2) as a single rotation.

5. Two bevil-wheels, with fixed axes, roll together; prove that the ratio of their angular velocities is that of the cosecants of their semivertical angles.

6. Prove that if a motion be reduced, as in Art. 3, to that of a screw in its nut, the translational velocity will be less than if it had been reduced in any other manner.

7. Prove that the rotation is not altered however the motion is reduced.

III.

D'ALEMBERT'S PRINCIPLE.

1. THE ideas of force and matter would seem to be equally fundamental. One cannot be conceived except as acting on or being acted on by the other. Force is 'that which changes or tends to change the state of rest or motion of matter;' or as Newton's first law of motion might be expressed, 'without force a body can experience no change either in the quantity or direction of its velocity.'

The second law is that the force in any direction is proportional to the quantity of motion it produces in that direction.

Two lumps of matter (masses) are *defined* to be equal, when the same force acting during equal times on both generates in them equal velocities. Two forces are *defined* to be equal, when acting on the same mass for equal times they generate in it equal velocities. Then it is *found by experiment*, that double forces acting on the same mass for equal times generate double velocities; and in general that the whole force in any time is proportional to the product of the mass moved and the velocity generated. And this product is called the quantity of motion or momentum. The force *at* any moment is measured by the rate of change of quantity of motion, i.e. by the product of the mass and the rate of change of the velocity.

2. Such are the laws by which the motions of a single particle are determined. Newton's third law, promulgated

in 1687,—that action and reaction between connected bodies whether at rest or in motion are equal and opposite—gives the means of determining the motion of a system of particles. But this was not at first appreciated. The student will find in the second volume of Whewell's *History of the Inductive Sciences*, or in Walton's *Mechanical Problems*, an interesting sketch of the errors into which mathematicians fell, and the difficulties they overcame before arriving at that principle first correctly stated by D'Alembert in 1742.

The first Rigid Dynamics problem which was solved on correct principles had been proposed by Mersenne in 1646. It was 'to find the centre of oscillation,' or the length of the simple pendulum which swings in the same time as any given rigid body swinging about a horizontal axis. This was solved by Huyghens in 1673 by the help of the correct principle, that if a pendulum in the shape of a rigid rod loaded with any weights make part of an oscillation, and if then the weights be disengaged from the constraining rod and reflected upwards with the velocities acquired, the centre of gravity will rise to the same height as it came from. But the main difficulty in the transition from a particle to a system still remained, viz. what effect motion impressed on one part of a rigid body has on another part.

In 1686 James Bernoulli gave expression to this difficulty by proposing to physicists the following query: "Given m, m' two equal bodies attached to an inflexible straight rod, which is capable of motion in a vertical plane about one end which is fixed; let r, r' denote the distances of m, m' from this end; v, v' their velocities for any position of the straight line in its descent from an assigned position; u, u' the velocities they would have acquired in descending the same arcs unconnectedly. Then through the connection m has lost $u-v$, and m' has gained $v'-u'$. (Query) Whether the relation (similar to that of forces and arms in a lever) $u-v : v'-u' :: r' : r$ be the correct expression of the circumstances of the motion?"

The nearness of this to the true expression—which is, that u, v, u', v' must be the velocities acquired in an infinitely small time—illustrates strikingly the groping of mathematicians for the principle which was to be the pillar supporting this science.

Bernoulli's query was shortly afterwards correctly answered by the Marquis de l'Hopital.

In 1716 a solution of Mersenne's problem was given by Hermann, founded, (says Whewell), "on the *statical equivalence* of the solicitations of gravity and the vicarious solicitations which correspond to the actual motion of each part, or, as it has been expressed by more modern writers, the equilibrium of the impressed and effective forces."

In 1736 Euler published his *Mechanics*, in which, while recognizing the correctness of the solutions of individual problems by other philosophers, he complains that a new geometrical solution is required for each separate problem, and therefore tries to reduce their methods to analysis. Thereafter the impulse given by D'Alembert in 1742 caused this science to spring almost at once into the maturity of the present day. Before the death of Euler the solution of the problems of the subject was pushed as far as the knowledge of differential equations would allow.

3. The following is a translation of D'Alembert's own statement of his principle. It will be better understood, if it be borne in mind that forces may be measured by the quantities of motion they give or would give were they allowed; in fact, that for purposes of reasoning force and motion produced are convertible. Also that a principle proved for any number of rigidly connected particles is proved for a rigid body.

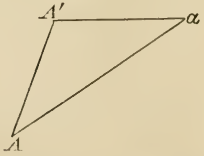
"*Given a system of bodies related to one another in any manner whatever; and suppose that on each of these bodies a particular motion is impressed, which it cannot follow on account of the constraint of the other bodies; to find the motion which each body will take.*

"*Solution.* Let A, B, C, &c. be the bodies which compose the system, and suppose that the movements* a, b, c, &c. have become impressed on them, which they are forced by their mutual actions to change into the movements a', b', c', &c. It is clear that we can regard the movement a impressed on A as composed of the movement a' which it has taken, and of another α; that in the same way we may consider the movements b, c, &c. as composed of the movements b', β; c', γ, &c.; whence it follows that the movements of the bodies A, B, C, &c., among one another would have been the same if, instead of giving them the impulses a, b, c, &c., we had given them the double impulses a', α; b', β; c', γ; &c.

"Now by the supposition the bodies A, B, C, &c., have of themselves taken the movements a', b', c', &c., hence the movements α, β, γ, &c., must be such that they do not derange anything in the movements a', b', c', &c., that is to say, that had the bodies only received the movements α, β, γ, &c., these movements must have destroyed one another and the system have remained at rest.

"Whence results the following principle for finding the motion of several bodies which act on one another. Decompose the movements a, b, c, &c., impressed on each body, each into two others a', α; b', β; c', γ, &c., which are such that, had the bodies only received the movements a', b', c', &c., they might have kept these movements without interfering with one another; and had they only been subject to the movements α, β, γ, &c., the system would have remained at rest. It is clear that a', b', c', &c. will be the motions which these bodies will take."

4. Thus if the impressed forces are such as would make a certain body acquire the velocity represented by Aa, while



its actual velocity is AA', $A'a$ represents a velocity which the

* a on A, b on B, c on C, &c.

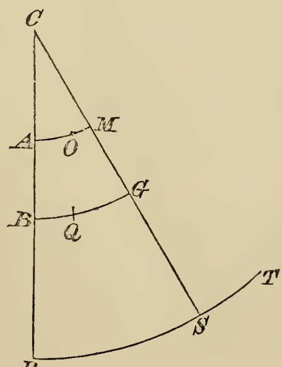
body is invited to take but does not. Aa being proportional to the impressed force, AA' is proportional to that part of it which is effective in producing motion, while $A'a$ is proportional to the force of constraint which that body exercises on those around it, and aA' is proportional to the force of constraint which they exercise on it.

We may look on this triangle in three ways: (1) that the impressed force is the resultant of the part which has gone to cause motion, and of the part gone to balance the force of constraint; (2) that the impressed forces and the force of constraint on the body have as their result the motion produced; or (3) that the force of constraint on the body balances the impressed force and gives the motion.

Any one of these forces is the resultant of the other two, or any one reversed is in equilibrium with the other two.

So much for one body or element of mass. When the rigid system is considered, of which this body is a part, the forces of constraint are in equilibrium among themselves, and therefore the remaining two sets, the impressed and reversed effective forces, are in equilibrium.

5. The first problem to which D'Alembert applies his principle is—to find the velocity of a rod CR, fixed at C, and



loaded with any number* of bodies A, B, R; supposing that these bodies, if the rod did not hinder them, would have

* A, B, R here and elsewhere denote not only the positions but the masses.

described in equal times the infinitely short lines AO, BQ, RT perpendicular to the rod.

The whole difficulty consists in finding the length RS traversed by one of the bodies R in the time in which it would, unconstrained, have traversed RT; for then the velocities BG, AM of the other bodies will be known. Now consider the impressed velocities RT, BQ, AO as composed of the velocities RS, ST; $BG, -QG$; $AM, -OM$. By our principle the lever CAR would have remained at rest if the bodies R, B, A had only received the velocities ST, $-QG$, $-OM$. Hence

$$A\,.\,MO\,.\,AC + B\,.\,GQ\,.\,BC = R\,.\,ST\,.\,CR$$

(since $A\,.\,MO$ is proportional to the force which produces the motion MO in the body A); that is

$$A\,.\,AC\,(AM - AO) + B\,.\,BC\,(BG - BQ) = R\,.\,CR\,(RT - RS).$$

Now $$AM : RS :: AC : CR,$$

and $$BG : RS :: BC : CR.$$

Substituting these values of AM and BG, there results a simple equation for RS.

D'Alembert's solution, given above, is geometrical. The analytical expressions and methods (introduced by Euler) are so much more convenient and powerful, that they have been universally adopted. Our future proceedings will consist in finding convenient expressions for the effective forces, and then solving problems by any statical method.

The reasoning by which D'Alembert's principle is established is obviously applicable to any system of bodies however connected; as, for instance, to fluids. The science of fluid motion does not branch off until we come to introduce the condition of rigidity in finding expressions for the resultant of the effective forces.

6. The impressed or external forces are the cause of the motion and of all the other forces. Which are the impressed forces will appear by the particular system which happens to

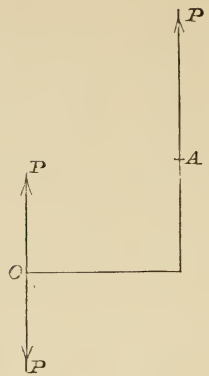
be under consideration. The same force may be external to one system and internal to another.

The constraining force on A (Art. 5) is internal when the whole system $CABR$ is under consideration; but did we wish to find this force of constraint, CA would be considered as a system in motion, and the action of BR on A would be one of the external forces. The pressure between the foot of a man and the deck of a ship on which he is, is external to the ship and also to him, and is the cause of his own forward motion and of a slight backward motion of the ship; but if the man and ship be looked on as parts of one system the pressure is internal, and the man may dash himself as violently as he pleases against any part of the vessel without quickening the voyage for himself and his fellow-passengers.

It is most important that the student should have in every problem a clear idea as to the system which he is considering.

7. Before the time of Poinsot (early in the present century), mechanicians had no better idea of the effect of a system of forces than that it could be reduced to two or more forces acting at separate points. The action of a door on its hinges was taken as consisting of forces acting at the different hinges. This has disadvantages. When the hinges are more than two, the forces cannot be found by the formulæ for rigid bodies. And when problems of motion are considered, it becomes very inconvenient to have no cause for rotation other than the moment of a force, which is different round different axes.

This defect is supplied by Poinsot's theory of couples. He was led to it by considering what could be the resultant of two parallel and equal forces acting in opposite directions. In this theory the force P acting at A is equivalent to the parallel force P at any point O and to the couple (PP), which is reducible to no single resultant. The effect of the force P at A on a given body must be the same whatever point O we may choose, for it will not be altered by our looking at it; but in certain cases, the most convenient position of O will be suggested. Thus O might be a fixed point round which the body could turn. In that case, P at O will be a pressure, and the couple will make



the body rotate. If it be objected that the couple, to cause rotation about O, should be $\left(\frac{P}{2}, \frac{P}{2}\right)$, with an arm double OA; the answer is, that rotation is not about a line or point, but about a direction or in a plane; and that these two couples are in fact exactly equivalent. Of course a single force need not always be resolved into a force and a couple. If A were such a point that a force acting there translated the whole body, it would not simplify our conceptions, but the reverse, to look on it as a force causing partly translation, partly rotation. In this theory the united actions of all the hinges of a door would be a single force and a single couple. The couple is the same with respect to all parallel axes, but varies in magnitude as the line of action of the force changes.

Couples exist uncombined in nature in the case of magnets. There is no pressure on the point of support of a magnet due to the earth's magnetic action; for that consists of two equal and opposite forces acting on the North and South poles respectively.

8. Forces are also divided in Kinetics into impulsive and

accelerating forces, i. e. into blows and finite forces. The impulse, generating a finite momentum instantaneously, is the simpler in theory. Time is no element in the calculation. The other forces require time to develop a finite quantity of motion, and their effect in an infinitely small time is as nothing compared with that of the class of blows. They are called finite forces. Attractions, tensions of elastic strings, pressures of gases, are examples of this class. There is probably in nature no perfect impulse which takes absolutely no time in its action; but it is usual to consider as such all forces which produce an appreciable change of motion in an inappreciable time, as explosions and impacts.

A blow is measured by the momentum or quantity of motion it generates. A constant finite force is measured by the momentum it generates in a unit of time; a variable one by the quantity of motion it would generate in the unit of time if it had throughout that time the same magnitude as at the moment of consideration, or, in other words, by the rate of generation of momentum, or the momentum generated in an infinitely short time (during which it may be supposed constant) divided by this time. The total force during a finite time, or the force *at* a moment, is comparable in effect with a blow; but not the force during an infinitely small time.

The same laws of motion apply to both classes. Momentum generated is the measure of both. Hence the figures and reasoning of D'Alembert are applicable to both.

The dynamics of impulses introduces only algebraical equations; that of finite forces depends on differential equations. The equations of either can be deduced from those of the other. A finite force may be looked on as the limit of a series of small impulses; an impulse as the limit of the total of a very large finite force acting during a very short time.

The student must be cautioned against regarding a sudden change or annihilation of a finite force as an impulse. If a cricket-ball is struck by a bat, it moves off with a measurable velocity. If it is let fall, it begins to move with an infinitely small velocity. What is finite is the acceleration. Again, if it is moving in one direction and is struck by a bat, it is sent off at once in a different direction; but a ball rolling off a table moves at first horizontally. But although the direction of motion experiences no immediate change, the curvature of the path does; for there is a sudden accession of downward force, and therefore of downward acceleration. If a body be supported by two strings, and if one be cut, the tension of the other will be instantaneously but not impulsively changed. No finite change of a finite force can convert it into an impulse.

9. The effective force is that component of the impressed force which is effective in causing motion. It is found, not from the impressed force, but from its being the force necessary to produce the actual motion, for the same motion must always be caused by the same force. It is the force which is required for producing the deviation from the straight line and the change of velocity. If a particle is revolving with constant velocity round a fixed axis, the effective force is the centipetal force. If a heavy body falls without rotation, the whole force of gravity is effective. But if it is swinging about a horizontal axis, the weight goes partly to balance the pressure on the axis.

If the motion is known, the force requisite to produce it is easily found. But the problem of this science is inverse. It is: "Given the forces, find the motion." Now the method of treating any inverse problem is to solve it as if it were a direct problem, and thus get equations for the unknown quantities. If the question is one of an impulse, we suppose an element δm to have the components of its velocity along the axes changed from u, v, w to u', v', w'. The change of momentum is then $\delta m\,(u'-u)$, $\delta m\,(v'-v)$, $\delta m\,(w'-w)$, and these are the measures of the components of the effective force.

If the motion is accelerated, the rate of increase of momentum is $\delta m\frac{du}{dt}$, $\delta m\frac{dv}{dt}$, $\delta m\frac{dw}{dt}$; and these are the measures of the components.

If the co-ordinates of the element δm be x, y, z, the velocities u, v, w are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and the accelerations are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$. We will in general work with the velocities, because the connection between impulsive and accelerating forces is brought out; but if in any problem the position of the system be required, it will be necessary to use the co-ordinates. (It will be as well to mention here that for purposes of abbreviation we will use the fluxional notation. Thus $\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$; so that $u = \dot{x}$ and $\dot{u} = \ddot{x}$. As t is always the independent variable, this can cause no ambiguity.)

There may be effective forces when there are no impressed forces, as in the case of a circular ring set rotating about a vertical axis and then left to itself. The effectives are then in equilibrium among themselves. They are in that case the centripetal forces.

It is convenient to suppose the velocity and acceleration to increase, and therefore to suppose that the effective forces act in the direction towards which the co-ordinates or angles are positive. The result shews by its sign whether the velocity is in that direction or the other, and whether the acceleration is a retardation or not.

Example. *A carriage-wheel, whose radius is* a, *is rolling with constant velocity* v *along a road. What is the force which gives its motion to a particle* (δm) *of mud on it just passing the top of the wheel?*

The question is: What is the acceleration of this particle? Now this in any direction is equal to its value relatively to any point, (say) the centre, together with the absolute acceleration of this point. The acceleration of the centre is zero. And the only acceleration of δm relatively to it is the centripetal one $\omega^2 a$ (ω being the angular velocity). Since the point of contact with the road is at rest, $\omega = \frac{v}{a}$. Therefore the effective force acts towards the centre and is $\delta m \cdot \frac{v^2}{a}$.

If the particle were now to leave the wheel and to move freely with an acceleration β which would be vertically downwards, the force necessary to give this would be $\delta m \,.\, \beta$. Whence we may infer that the force of adhesion to the wheel is $\delta m \left(\frac{v^2}{a} - \beta\right)$ downwards.

10. The internal forces, or forces of constraint, or the lost forces,—as some have called them, in contrast to the effective forces—which are in equilibrium among themselves, can exist only in a system of particles. They are stresses, or couples which we call tendencies to break, or any molecular forces in a rigid body; or they may be pressures, tensions, attractions between the bodies of a system. They vanish not collectively only, but separately, when the impressed forces on the particles of the system are entirely effective in producing motion. Thus there are no internal pressures between a number of bricks falling in a block without rotation.

To find these forces at any point, we must consider a system to which they are external, i.e. we must reduce the system to one bounded by the point at which they act.

11. If then a system be acted on by impressed forces $X_1Y_1Z_1$, $X_2Y_2Z_2$, &c. acting at various points, and if $\delta_1 m$, $\delta_2 m$, &c. be elements of mass at points whose co-ordinates are $x_1y_1z_1$, $x_2y_2z_2$, &c. and whose velocities are $u_1v_1w_1$, $u_2v_2w_2$, &c.; the system of forces $X_1Y_1Z_1$, $X_2Y_2Z_2$, &c., and the system

$$-\delta_1 m \frac{du_1}{dt}, \quad -\delta_1 m \frac{dv_1}{dt}, \quad -\delta_1 m \frac{dw_1}{dt},$$

$$-\delta_2 m \frac{du_2}{dt}, \quad -\delta_2 m \frac{dv_2}{dt}, \quad -\delta_2 m \frac{dw_2}{dt},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

acting at $x_1y_1z_1$, $x_2y_2z_2$, ... all taken together are in equilibrium.

If the impressed forces are impulsive, they are in equilibrium with

$$-\delta_1 m (u_1' - u_1), \quad -\delta_1 m (v_1' - v_1), \quad -\delta_1 m (w_1' - w_1),$$
$$-\delta_2 m (u_2' - u_2), \quad -\delta_2 m (v_2' - v_2), \quad -\delta_2 m (w_2' - w_2).$$

....

12. It will illustrate the application of D'Alembert's principle if we solve the following problem.

A *and* B *are two masses connected by a rigid massless framework. They receive impulses which would, if they were unconnected, generate in them given velocities* Aa, Bb. *What velocities and directions will they take?*

The force of constraint must act along AB, for an impulse along that line is that which neither body can obey. If then we measure from the points a and b, aA' and bB', to represent the velocities caused by the constraint, they will be parallel to AB and opposite in direction, and

$$A \, . \, aA' = B \, . \, bB' \text{ ....(1)},$$

for these represent the forces of constraint. If, further,

$$A'B' = AB \text{ ....(2)},$$

AA' and BB' represent the velocities of A and B.

The process above indicated is a reverse one, and although in the above case the positions of A' and B' can be found by a geometrical construction, in general an equation is required.

Such is the geometrical solution.

The analytical one is derived from the condition—which is equivalent to (1)—that the forces

$$A \, . \, Aa, \quad B \, . \, Bb, \quad -A \, . \, AA', \quad -B \, . \, BB'$$

are in equilibrium.

Writing down the statical equations to which this gives rise, and the geometrical equation that the components of AA' and BB' along AB are equal—equivalent to (2)—we can determine AA', BB', and their directions.

13. Let us suppose that the lever in D'Alembert's example (Art. 5) is supported at an angle α with the vertical and then let go, and that it is required to find the *initial* accelerations of A, B, R.

The accelerations will afterwards be partly along the rod and partly at right angles to it, but as each body begins to move in a perpendicular to the rod, the initial acceleration will be in that direction. If the angular velocity be called ω, the accelerations will be

$$AC\frac{d\omega}{dt}, \quad BC\frac{d\omega}{dt}, \quad RC\frac{d\omega}{dt}.$$

The forces causing these must be

$$A.AC\frac{d\omega}{dt}, \quad B.BC\frac{d\omega}{dt}, \quad R.RC\frac{d\omega}{dt},$$

acting at A, B, R perpendicularly to $CABR$. These reversed are in equilibrium with the weights Ag, Bg, Rg, and the pressure on the axis C.

Taking moments about C,

$$(A.AC^2 + B.BC^2 + R.RC^2)\frac{d\omega}{dt}$$
$$= g.\sin\alpha.(A.AC + B.BC + R.RC),$$

which determines the initial angular acceleration.

This solution should be carefully compared with that in Art. 5.

14. To illustrate this subject farther we will answer the following question.

In the case of the compound pendulum (a rigid body swinging about a horizontal axis), find the force which acts on any infinitesimal part of the whole mass to balance its weight and to give it its acceleration. Hence shew that if an infinitely small mass be hung by an infinitely short cord from a point (x, y) *of the pendulum, the inclination of the cord to* x *will be*

$$\tan^{-1}\frac{-g\sin\theta - \dot{\omega}x + \omega^2 y}{g\cos\theta + \omega y + \omega^2 x},$$

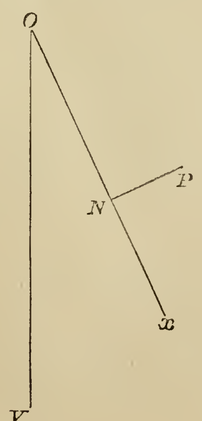
the axis of x *being the line through the point of support and a fixed point in the pendulum, and* ω, $\dot{\omega}$ *being used to denote* $\frac{d\theta}{dt}$, $\frac{d^2\theta}{dt^2}$ *respectively.*

It will save repetition if we mention that we shall in future adhere to these meanings of the symbols and also use δm for an element of mass, so that $\Sigma\delta m$ is the mass of the whole body, r for the distance of δm from an axis of rotation, θ for the angle between a fixed line in the body and a fixed line in space.

That $\frac{d\theta}{dt}$ is the same for all lines fixed in the body may be seen by considering one whose vectorial angle is $\theta + \alpha$. This angle α is quite independent of the time or motion, being the angle between two lines or planes of particles. And $\frac{d\theta}{dt}$ is therefore equal to $\frac{d(\theta+\alpha)}{dt}$. $\frac{d\theta}{dt}$ is in fact the angular velocity of the body.

We are here asked to find the constraining force on any element δm situated at a point x, y, for that is the resultant of the reversed impressed forces and the effective forces; in other words, it balances the weight and gives the acceleration.

Let O be the point of intersection of the plane of the



paper by the axis of suspension, OV the vertical, Ox a convenient line fixed in the pendulum; P the position of δm on the positive side of Ox from OV; PN a perpendicular on Ox. Then ON is x; PN, y.

The impressed force is $\delta m \,.\, g$, parallel to OV. Reversing this it is equivalent to

$$-\delta m\, g \cos\theta \text{ parallel to } Ox,$$
$$\delta m\, g \sin\theta \quad ,, \quad ,, \quad Oy.$$

The accelerations of P are $\omega^2 r$ from P to O; and $\dot\omega r$ at right angles to OP and away from OV. These are resolvable into $-\omega^2 x$ along and $-\omega^2 y$ perpendicular to Ox,

$$-\dot\omega y \quad ,, \quad ,, \quad \dot\omega x \quad ,, \quad ,, \quad ,, \quad Ox.$$

Hence the effective forces on P are

$$\left.\begin{array}{l}\delta m\,(-\omega^2 x - \dot\omega y) \text{ along and} \\ \delta m\,(-\omega^2 y + \dot\omega x) \text{ perpendicular to}\end{array}\right\} Ox.$$

The required constraining force is then the resultant of $\delta m\,(-g\cos\theta - \omega^2 x - \dot\omega y)$ along and $\delta m\,(g\sin\theta - \omega^2 y + \dot\omega x)$ perpendicular to Ox.

In answer to the second part, we observe that if the element be connected with its neighbours only by a short cord, the tension of that cord is the constraining force whose value we have just found. The direction of the cord will therefore be the direction of the resultant found above, or will be inclined to Ox at an angle whose tangent is

$$\frac{-g\sin\theta + \omega^2 y - \dot\omega x}{g\cos\theta + \omega^2 x + \dot\omega y}.$$

EXAMPLES.

1. A and B are two masses connected rigidly. If A receives a blow which is fitted to impress a velocity AC, and if it actually takes the velocity AC'; shew that CC' is parallel to AB.

2. A ball A is in motion. A blow is given which would (were it at rest) impress a velocity AB on it. It moves instead with velocity AB'. With what velocity and in what direction was it moving before?

3. A cricket-ball is rotating with velocity ω round the direction of the line of wickets; on touching the ground it is acted on by an impulsive couple, which would have given it if at rest an angular velocity ω' about a horizontal line perpendicular to this direction. Round what direction will it actually rotate?

4. Does the grooving of a rifle increase or diminish the force of recoil?

5. A blow and a constant finite force acting for half a second produce in the same mass the same velocity. Prove that the measure of the force is double that of the blow.

6. If F be a finite force which acting during a time τ causes the same momentum as a blow P, prove that

$$P = \int_0^\tau F dt.$$

7. How must a particle move that the effective forces may vanish?

8. A carriage-wheel is rolling with given velocity and acceleration along a road. Find the force which gives its motion to a particle of mud passing the top of the wheel.

9. A circular ring of mass m and radius r is rotating with constant velocity about its centre. It can bear without breaking a stretching force T; prove that the angular velocity must not exceed $\left\{\frac{2\pi T}{mr}\right\}^{\frac{1}{2}}$.

IV.

REDUCTION OF THE EXPRESSIONS FOR THE EFFECTIVE FORCES.

1. Any system of forces is equivalent to a single resultant force acting at any point and to a resultant couple. This must be the case with the effective forces of any moving system. These consist, if the change of motion be sudden, of a force on each element of mass δm whose components are

$$\delta m\,(u' - u),\ \delta m\,(v' - v),\ \delta m\,(w' - w)\,;$$

if the change be gradual, of

$$\delta m\frac{du}{dt},\ dm\frac{dv}{dt},\ dm\frac{dw}{dt}.$$

If x, y, z be the co-ordinates of δm, referred to any origin, and Σ denote summation over the whole system, the resultant force will be the resultant of

$$\Sigma\,\delta m\,(u' - u),\ \ \Sigma\,\delta m\,(v' - v),\ \ \Sigma\,\delta m\,(w' - w)$$

acting at the origin. If the motion be accelerated, of

$$\Sigma\,\delta m\frac{du}{dt},\ \Sigma\delta m\frac{dv}{dt},\ \Sigma dm\frac{dw}{dt},$$

and the couple will be the resultant of

$$\Sigma\delta m\,\{y\,(w' - w) - z\,(v' - v)\},\ \Sigma\delta m\,\{z\,(u' - u) - x\,(w' - w)\},$$

$$\Sigma\delta m\,\{x\,(v' - v) - y\,(u' - u)\}\,;$$

or, if the motion be accelerated, of

$$\Sigma\delta m\left(y\frac{dw}{dt}-z\frac{dv}{dt}\right),\ \Sigma\delta m\left(z\frac{du}{dt}-x\frac{dw}{dt}\right),\ \Sigma\delta m\left(x\frac{dv}{dt}-y\frac{du}{dt}\right),$$

round the directions Ox, Oy, Oz.

The symbol Σ introduced above has clearly the following properties:

(1) that if a be a quantity which is the same for every point,

$$\Sigma\,(\delta m\,.\,a)=a\,.\,\Sigma\,\delta m;$$

(2) that $\Sigma\,(\delta m\,U)\pm\Sigma\,(\delta m\,V)=\Sigma\,\delta m\,(U\pm V)$,

in which U and V are any functions of x, y, z or their differential coefficients;

(3) that $\dfrac{d}{dt}.\,\Sigma\,(\delta m\,V)=\Sigma\left(\delta m\dfrac{dV}{dt}\right)$.

We proceed to put the above expressions into forms more suitable for practical purposes.

2. Let m_1, m_2 ... be any masses; x_1, x_2 ... their distances from a fixed plane.

If $m_1x_1+m_2x_2+\ldots=(m_1+m_2+\ldots)\bar{x}$, $\bar{x}$ may be said to be the average distance from the plane of all the masses. The point determined by this and similar equations for $\bar{y}$ and $\bar{z}$ is called the Centre of Mass; a name which was first used by Daniel Bernoulli. As masses are proportional to their weights at the same locality, this point coincides with the centre of gravity.

If we differentiate the above equation with respect to the time, we have $m_1u_1+m_2u_2+\ldots=(m_1+m_2+\ldots)\,\bar{u}$, whence the proposition; that the sum of the linear momenta of any masses in a given direction is equal to the momentum in that direction of their united masses moving with the velocity of the centre of mass. From this property the point is called the Centre of Inertia.

Differentiating once more we have

$$m_1 \frac{du_1}{dt} + m_2 \frac{du_2}{dt} + \ldots = (m_1 + m_2 + \ldots) \frac{d\bar{u}}{dt},$$

and so for the other component accelerations.

Hence the single resultant of the effective forces is that force which would be necessary to move the whole mass collected at the centre of inertia with the motion of the centre of inertia.

These equations may be written

$$\Sigma (mx) = \bar{x} \Sigma m,$$

$$\Sigma (mu) = \bar{u} \Sigma m,$$

$$\Sigma \left(m \frac{du}{dt} \right) = \frac{d\bar{u}}{dt} \Sigma m.$$

This point will in future be denoted by the letter G. It is obvious that if G lie on the plane yz,

$$m_1 x_1 + m_2 x_2 + \ldots\ldots = 0.$$

If it be moving in or parallel to the plane of yz,

$$m_1 u_1 + m_2 u_2 + \ldots\ldots = 0,$$

and so for higher differentials.

3. Let any fixed point O be taken as origin of coordinates. Let the co-ordinates of any point P referred to O be x, y, z; while those of P referred to parallel axes through G are ξ, η, ζ, and those of G with respect to O are $\bar{x}, \bar{y}, \bar{z}$.

Then $$x = \bar{x} + \xi, \quad y = \bar{y} + \eta, \quad z = \bar{z} + \zeta.$$

If $$m_1 + m_2 + \ldots\ldots = M,$$

we have by definition

$$m_1 x_1 + m_2 x_2 + \ldots\ldots = M\bar{x},$$

and $$m_1 \xi_1 + m_2 \xi_2 + \ldots\ldots = 0,$$

whence $$m_1 \frac{d\xi_1}{dt} + m_2 \frac{d\xi_2}{dt} + \ldots\ldots = 0,$$

and so for higher differentials and for the other co-ordinates.

Now consider terms of the form $my\frac{dw}{dt}$, i.e. $my\frac{d^2z}{dt^2}$.

$$m_1y_1\frac{d^2z_1}{dt^2}+m_2y_2\frac{d^2z_2}{dt^2}+\ldots\ldots=m_1(\bar{y}+\eta_1)\left(\frac{d^2\bar{z}}{dt^2}+\frac{d^2\zeta_1}{dt^2}\right)$$

$$+m_2(\bar{y}+\eta_2)\left(\frac{d^2z}{dt^2}+\frac{d^2\zeta_2}{dt^2}\right)$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\bar{y}\frac{d^2\bar{z}}{dt^2}(m_1+m_2+\ldots\ldots\quad)$$

$$+m_1\eta_1\frac{d^2\zeta_1}{dt^2}+m_2\eta_2\frac{d^2\zeta_2}{dt^2}+\ldots\ldots\ldots\ldots$$

$$+\bar{y}\left(m_1\frac{d^2\zeta_1}{dt^2}+m_2\frac{d^2\zeta_2}{dt^2}+\ldots\ldots\quad\right)$$

$$+\frac{d^2\bar{z}}{dt^2}(m_1\eta_1+m_2\eta_2+\ldots\ldots\quad).$$

The last two expressions vanish and there remains

$$m_1y_1\frac{d^2z_1}{dt^2}+m_2y_2\frac{d^2z_2}{dt^2}+\ldots\ldots=M\bar{y}\frac{d^2\bar{z}}{dt^2}$$

$$+m_1\eta_1\frac{d^2\zeta_1}{dt^2}+m_2\eta_2\frac{d^2\zeta_2}{dt^2}+\ldots\ldots$$

or

$$\Sigma\delta my\frac{d^2z}{dt^2}=M\bar{y}\frac{d^2\bar{z}}{dt^2}+\Sigma\delta m\eta\frac{d^2\zeta}{dt^2}.$$

By similar reasoning

$$\Sigma\delta m\, z\frac{d^2y}{dt^2}=M\bar{z}\frac{d^2\bar{y}}{dt^2}+\Sigma\delta m\ \zeta\frac{d^2\eta}{dt^2}.$$

Hence

$$\Sigma\delta m\left(z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}\right)=M\left(\bar{z}\frac{d^2\bar{y}}{dt^2}-\bar{y}\frac{d^2\bar{z}}{dt^2}\right)$$

$$+\Sigma\delta m\left(\zeta\frac{d^2\eta}{dt^2}-\eta\frac{d^2\zeta}{dt^2}\right).$$

The left-hand expression is the moment of all the effective forces about Ox, whence we have the important result: the moment of the effective forces on a system about any axis is equal to their moment about a parallel axis through the centre of inertia taken as if this axis were fixed, together with the moment of the effective force on the whole mass supposed collected at the centre of inertia and moving with it, about the original axis.

Now suppose that G is passing through O at the moment whose circumstances we are considering.

Then $\bar{y}=0$, $\bar{z}=0$,

and $$\Sigma\delta m\left(z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}\right)=\Sigma\delta m\left(\zeta\frac{d^2\eta}{dt^2}-\eta\frac{d^2\zeta}{dt^2}\right).$$

The interpretation of this is that the moment of the effective forces about an axis through the centre of inertia is the same as if that point were fixed.

Precisely similar results hold in the case of the following functions; (all of them of the greatest importance).

$$\Sigma\delta m\, yz,\ \Sigma\delta m\, zx,\ \Sigma\delta m\, xy,$$

thus $$\Sigma\delta m\, xy = M\bar{x}\bar{y}+\Sigma\delta m\,\xi\eta;$$

$$\Sigma\delta m\,(y^2+z^2),\ \Sigma\delta m\,(z^2+x^2),\ \Sigma\delta m\,(x^2+y^2),$$

thus

$$\Sigma\delta m\,(y^2+z^2)=M(\bar{y}^2+\bar{z}^2)+\Sigma\delta m\,(\eta^2+\zeta^2);$$

$$\Sigma\delta m\left(\frac{dx}{dt}\right)^2,\ \Sigma\delta m\left(\frac{dy}{dt}\right)^2,\ \Sigma\delta m\left(\frac{dz}{dt}\right)^2,$$

or their sums,

thus $$\Sigma\delta m\left\{\left(\frac{dz}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2\right\}=M\left\{\left(\frac{d\bar{z}}{dt}\right)^2+\left(\frac{d\bar{x}}{dt}\right)^2\right\}$$
$$+\Sigma\delta m\left\{\left(\frac{d\zeta}{dt}\right)^2+\left(\frac{d\xi}{dt}\right)^2\right\};$$

$$\Sigma\delta m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right),\ \Sigma\delta m\left(z\frac{dx}{dt}-z\frac{dx}{dt}\right),\ \Sigma\delta m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right),$$

thus $$\Sigma\delta m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=M\left(\bar{x}\frac{d\bar{y}}{dt}-\bar{y}\frac{d\bar{x}}{dt}\right)+\Sigma\delta m\left(\xi\frac{d\eta}{dt}-\eta\frac{d\xi}{dt}\right);$$

$$\Sigma\delta m\frac{d^2x}{dt^2},\quad \Sigma\delta m\frac{d^2y}{dt^2},\quad \Sigma\delta m\frac{d^2z}{dt^2},$$

thus $$\Sigma\delta m\frac{d^2x}{dt^2}=M\frac{d^2\bar{x}}{dt^2}+\Sigma\delta m\frac{d^2\xi}{dt^2}.$$

We leave the working out of these to the student.

These equalities must be carefully distinguished from the somewhat similar equations which arise from the properties of relative velocities and accelerations. If G were *any point whatever* it would be true that

$$\frac{dx}{dt}=\frac{d\bar{x}}{dt}+\frac{d\xi}{dt},$$

$$\frac{d^2x}{dt^2}=\frac{d^2\bar{x}}{dt^2}+\frac{d^2\xi}{dt^2}.$$

But these are purely geometrical. There is in them no mention of mass. Now the equations which we have proved above are physical. They are true only when G is the centre of inertia.

4. It will be noticed that the effective forces would reduce to a force acting at any point and a couple; but the centre of inertia is that point at which the resultant effective force would produce the actual motion of the point on the whole mass collected there; and it is also the point which may be taken as fixed while we consider the rotation.

It will, therefore, in general be convenient to choose our origin, so that the centre of inertia is just passing through it. But—it may be objected—our co-ordinate axes are fixed; velocities and accelerations must be measured by reference to fixed points and lines. How then can we choose our origin to be coincident with a moving point? The answer is: Our origin is fixed, and the centre of inertia is passing through it at the moment under consideration. The equations are found from the consideration of the motion at an instant chosen to embrace its most general conditions: and provided we find the acceleration or velocity rightly at the moment we consider, it matters not where the origin may be, after or before. If, however, the mind of the student insists on contemplating the motion in successive intervals, the proceedings of the origin (as we must imagine them) can easily be represented. Suppose the centre of mass a material visible point, and suppose it illuminated by a series of electric sparks (which do not remain on the eye, and therefore shew a moving body as if it were fixed), then the centre of mass will be seen by the light of the successive sparks standing still in its different positions. The student has already come across the same difficulty in the case of the accelerations of a point measured along and perpendicular to a revolving line.

5. So far our conclusions are true for any system of constant mass; for systems of free particles, for strings, for fluids. There remains to introduce the condition of rigidity, i. e. to reduce for one rigid body the couples

$$\Sigma\delta m\,\{\xi\,(v'-v)-\eta\,(u'-u)\},$$

and

$$\Sigma\delta m\left(\xi\frac{dv}{dt}-\eta\,\frac{du}{dt}\right),$$

to a form directly connected with rotation.

In the general case this is beyond the limits proposed in these introductory Lessons. Whatever problems involving three dimensions we attempt shall be considered with the help of the unreduced expressions.

We suppose then henceforth that the rotation is altogether about the direction of Oz.

The following Lemma will be of service. If x, y be the rectangular, and r, θ the polar co-ordinates of a point,

$$x\frac{dy}{dt}-y\,\frac{dx}{dt}=r^2\,\frac{d\theta}{dt}.$$

For $$x\frac{dy}{dt} - y\frac{dx}{dt} = x^2\frac{d}{dt}\left(\frac{y}{x}\right)$$

$$= r^2\cos^2\theta\frac{d}{dt}(\tan\theta)$$

$$= r^2\frac{d\theta}{dt}.$$

Now let r and θ be the radius vector and the vectorial angle of an element of mass of a rigid body, which can only rotate round a fixed axis. $\frac{d\theta}{dt}$ is the rate of change of the angle between a line of particles in the body and a line fixed in space. Therefore $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ are the same for every element, being in fact the angular velocity and angular acceleration of the body. Denote the former by ω. Also during the motion and for the same element δm, r does not change.

Hence $$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\omega,$$

differentiating $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = r^2\frac{d\omega}{dt}.$$

Hence $$\Sigma\delta m\left(x\frac{dv}{dt} - y\frac{du}{dt}\right) = \frac{d\omega}{dt}\Sigma\delta m r^2.$$

If the change of motion be sudden,

$$\Sigma\delta m\{x(v' - v) - y(u' - u)\} = (\omega' - \omega)\Sigma\delta m r^2.$$

These are the moments of the effective forces round the fixed axis. The single resultant through the axis is the same as through any other point, and is therefore the resultant effective force of the whole mass collected at the centre of inertia.

If there is no fixed axis the centre of inertia is taken as the point, the motion of which and the motion round which determine the circumstances. The resultant effective force is that of the whole mass collected there and moving with its motion. And the motion round it is the same as if it were a fixed point. The forces therefore reduce to

$$M(\bar{u}' - \bar{u}),\ M(\bar{v}' - \bar{v})$$

acting at the centre of inertia, and a couple $(\omega' - \omega)\,\Sigma\delta m r^2$; or, if the motion be accelerated, to

$$M\frac{d\bar{u}}{dt},\ M\frac{d\bar{v}}{dt},$$

and a couple $\frac{d\omega}{dt}\,\Sigma\delta m\, r^2$, ($r$ being the distance of δm from the centre of inertia).

If the system consists of rigid bodies not rigidly connected these forces may be reduced to one at the common centre of mass, but the couples must be taken for the separate bodies.

6. The expression $\Sigma\delta m\,(xv - yu)$ or $\Sigma\delta m\left(r^2\frac{d\theta}{dt}\right)$ is called the *angular momentum* about the origin. It has sometimes been called the *moment of the momentum.* These names might with advantage be kept separate.

If there is a fixed axis of rotation,

$$\Sigma\delta m\left(r^2\frac{d\theta}{dt}\right) = \omega\,\Sigma\delta m\, r^2.$$

If not,

$$\Sigma\delta m\,(xv - yu) = M(\bar{x}\bar{v} - \bar{y}\bar{u}) + \omega\,\Sigma\delta m\, r^2,$$

in which r is the distance of the element δm from the axis through the centre of inertia. Let the term "moment of momentum" be kept for the whole mass collected at the centre of inertia, i.e. for the expression $M(\bar{x}\bar{v} - \bar{y}\bar{u})$, and let the term $\omega\Sigma\delta m\, r^2$ be called the "quantity of rotational motion." Round a fixed axis or round the centre of inertia, angular momentum is then the same as quantity of rotation.

It is important to notice that the quantity of rotational motion of an element is measured by the square of the distance from the axis. When Newton attacked the problem of precession, he proved that if a rotating ring communicated motion to a mass attached to it, the whole quantity of motion would remain the same. This is right; but Newton measured the quantity of motion by the sum of the linear motions of the elements, which is wrong. In Rigid Dynamics we introduce a new kind of motion—rotation—caused by another kind of force (viz. a couple). Now a couple is measured by the moment of a simple force; quantity of rotation therefore is measured by the moment of a momentum.

Imagine a fly-wheel whose mass is m and radius a, rotating about its centre with velocity ω. Every point is moving with linear velocity $a\omega$. And therefore in one sense (excluding the idea of direction) the whole quantity of linear momentum is $ma\omega$. The impulse which applied at a point on the wheel will stop the motion is $ma\omega$. There is sometimes an advantage in considering the motion thus; but our knowledge of the Geometry of motion indicates distinctly that the only complete way of treating problems of motion will be to consider a body as animated by a directional translation and a rotation. The whole linear momentum, in any direction, of the wheel mentioned above is zero; the angular momentum is $ma^2\omega$. In this view the force which stops the motion is a couple, and there is also a single force,—a pressure on the axle.

7. Our present results applied to those of Lesson III. enable us to assert that the impressed forces are in equilibrum with the forces $M\frac{d\bar{u}}{dt}$ and $M\frac{d\bar{v}}{dt}$ acting at the centre of inertia, reversed, and the couple $\frac{d\omega}{dt}\Sigma\delta m\, r^2$ reversed. If the forces are impulsive they are in equilibrium with $M(\bar{u}'-\bar{u})$, $M(\bar{v}'-\bar{v})$ and $(\omega'-\omega)\Sigma\delta m\, r^2$ all reversed.

8. The reduction of the expressions for the effective forces is now theoretically complete. They have been shewn to be equivalent, when the motion is continuous, to a resultant force $M\frac{d\bar{u}}{dt}$, $M\frac{d\bar{v}}{dt}$ acting at the centre of inertia and to a

resultant couple. But practically the solutions of problems may be much simplified by a proper choice of co-ordinates. Thus if we use the above expressions or those in x, y when the motion is referable to a fixed point, it is obvious that we shall have for each problem to work through those differentiations which shew that $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ parallel to the axes are equivalent to $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ along and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ at right angles to the radius vector.

It might be convenient to employ the expressions for the accelerations along the tangent and normal to the path of the centre of inertia. Or, yet again, if the motion of G is best defined by reference to a point A which is itself moving, we can use the proposition that the acceleration (or velocity) of G in any direction is equal to that of G relatively to A (supposing A fixed) in that direction, together with the acceleration (or velocity) resolved in the same direction.

In general in every analytical solution of a motion of a rigid system equations are required connecting the velocities of different parts. These are called the geometrical equations, and may often be simplified or reduced in number by a proper choice of variables.

9. *A system consisting of two masses* A *and* B *rigidly connected by a straight massless wire is moving without rotation with velocity* V. *A point* O *of the wire between* A *and the centre of inertia suddenly becomes fixed, and the system proceeds to rotate about* O *with angular velocity* ω. *It is required to find the resultant impulsive forces which must have caused this change of motion.*

The momentum of translation has suffered a change,

$$B \, . \, OB \, . \, \omega - A \, . \, OA \, . \, \omega - (A + B)\, V.$$

This therefore is the measure of the force which acting at G at right angles to AB would cause the change.

The angular momentum about G has been changed from zero to

$$(A \, . \, AG^2 + B \, . \, BG^2)\,\omega.$$

This expression is therefore the measure of the couple which must have caused the change.

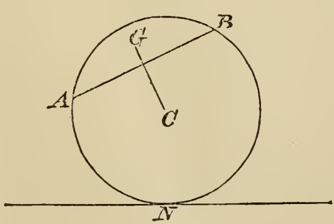
10. *Given that a circular hoop of radius* a *is rolling with uniform velocity* v *along a road. What are the resultant effective forces* (1) *on the whole,* (2) *on a given part?*

(1) As the centre of inertia is moving uniformly in a straight line, the resultant force is zero.

To find the angular velocity (ω), consider the motion of the point in contact with the ground.

This is carried forwards with velocity v by the motion of translation. It is carried backwards by the rotation round the centre with velocity $a\omega$, and its resultant velocity is zero; for it is the instantaneous centre. Hence $v = a\omega$, or the rotation is also uniform. Hence the couple effective in producing rotation is zero also.

(2) Let m be the mass and G the centre of inertia of the part AB, the effective forces on which are to be found.



The acceleration of G in any direction is equal to that of C in that direction, together with that of G relatively to C measured in the same direction. Now C moves uniformly and G moves uniformly about C. Hence the only acceleration of G is towards C and is $\omega^2 \, . \, CG$. The resultant effective force on AB is therefore $m\omega^2 OG$, and acts at G towards C.

The moment of the effective couple is $\frac{d\omega}{dt}\Sigma\delta m r^2$, where r is the distance of δm from G. As ω is constant this vanishes.

EXAMPLES.

1. Prove that the centres of mass and inertia necessarily coincide; but that the centre of gravity does not coincide with these unless the weights of the different parts of the body may be supposed to act in parallel lines.

2. A grindstone is rotating about its axle. Shew that its angular momentum is the same round all axes parallel to this.

3. A cannon-ball whose mass is 30 lbs. is fired with a velocity of 1000 feet per second. What is its momentum at the moment of discharge?

What is the moment of its momentum about a point 10 feet immediately above the mouth of the gun?

4. How is the centre of inertia of a rigid body moving when the resultant effective force is zero?

5. Find a general expression for all functions of co-ordinates and differential coefficients for which the properties of Art. (3) are true.

6. Find the effective forces for the systems in Art. 10, supposing the hoop to be rolling with given acceleration $\dot{v}$.

7. Two uniform rods OA, AB, of lengths $2a$, $2b$ are hinged at A—O being fixed—and they rotate in one plane with angular velocities ω, ω'. Prove that the force which causes the motion of translation of AB is the resultant of the following:

$$\text{mass } AB\,.\left(\omega'^2\,.\,b + 2\omega^2 a\cos\phi - 2\frac{d\omega}{dt}\,a\sin\phi\right)\text{ along } BA,$$

and mass $AB\left(\frac{d\omega'}{dt}\,.\,b + 2\frac{d\omega}{dt}\,a\cos\phi + 2\omega^2 a\sin\phi\right)$, at right angles to BA; ϕ being the angle between AB and OA produced.

V.

FIRST APPLICATIONS.

1. We have now proved that the forces on each element of a rigid body, which are effective in causing the motion, are together equivalent to that force which, acting on the whole mass collected at the centre of inertia, would cause the actual motion of that point; and, to a couple which, were the centre of inertia fixed, would cause the actual motion of rotation of the body. The impressed forces are also reducible to a single force at a point and to a couple. The reversed effective forces are in equilibrium with the impressed forces. Putting these together, we infer (1) that the motion of the centre of inertia is the same as if the whole mass were collected there, and the impressed forces acted at that point each in its own direction; (2) that the effect of the impressed forces to cause rotation is the same as if the centre of inertia were fixed.

These principles are fruitful of important consequences. From (1):

If there be no impressed force or no resultant impressed force, the centre of inertia must either remain at rest or move on with velocity unchanged in magnitude or direction. When a shell explodes in the air before striking, the forces of explosion are all internal; and the centre of inertia of all the fragments moves on in the same curve as if nothing had occurred. A plank sliding on smooth ice will move so that its centre of inertia will describe a straight line with constant velocity. Supposing the solar system to be so far from the stars that their attraction may be neglected; the centre of inertia of the sun and planets must be at rest or moving uniformly in a straight line. If there are no external forces in a particular direction, or if their resultant is at right angles to that direction, then the velocity of the centre of inertia or the linear momentum of the system in that direction remains constant. The impressed forces acting on a chain-shot are vertical. Hence the horizontal velocity of its centre of inertia is constant.

In an old and instructive problem it is supposed that a man is placed upon a perfectly smooth hard sheet of ice, so that skates avail him nothing. How is he to get off? The only external forces are his weight and the support of the surface. These acting down and up cannot help him along. His centre of mass will not move by any action internal to his person. He must get external force. Let him throw away something that he has about him. This becomes a body external to himself, and its reaction gives him an impulse backwards. Or the projected body may be looked on as still part of his system; in which view the common centre of mass remains at rest, but as the body moves in one direction, his centre of mass must move in the other.

The resistance of water is very small to a boat moving slowly through it; and, accordingly, every one has noticed that on his moving to the end of a boat, in order to get out, the boat, if not previously fastened to the shore, moves back. The common centre of mass, however, of himself and the boat has not moved. Nor will it move although he springs out, until by the pressure of his feet on the land he introduces an external force.

Again, whatever the impressed forces may be, or wherever they may act, they produce the same effect on the motion of the centre of inertia as if they acted there.

If a ship be pulled to shore by a rope of given tension, it matters nothing to the motion of the centre of inertia at what point the rope is attached. A billiard-ball struck horizontally will move off equally quickly wherever it is struck, provided the force of the blow be the same. Every one has noticed how a table-napkin ring, squirred out between the finger and the table, keeps rotating, but soon stops moving away and comes back. The only impressed force is the friction of the table. And this, though acting at the rim, first stops the velocity of the centre and then causes it to acquire a velocity in the opposite direction. And this effect of the force is the same as if it acted at the centre.

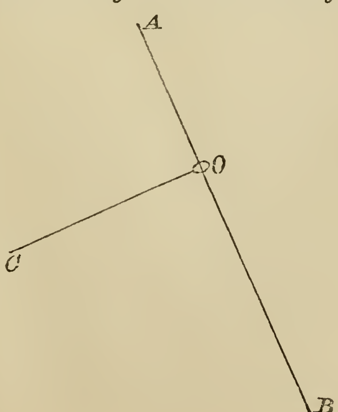
2. From (2):

If a body at rest, or moving without rotation, be acted on by a blow or force whose line of action passes through the centre of inertia, it will move on without rotation: if in such a case the motion of the centre of inertia be stopped, the whole motion will be stopped.

To pull a boat to land without making it rotate, pull it by this point. A sportsman once told the author that a certain salmon he hooked gave him much trouble. He tried as usual to get the fish's head down stream, but could not. When at last the fish was landed, he found that he had hooked it by the belly. The scientific expression of this is, that he had been pulling at the centre of inertia, and consequently had been unable to turn the fish round.

Example.

AB *is a smooth fixed inclined straight wire.* CO *is a rod furnished with a ring* O *at one end by which it hangs*



from the wire. CO *is taken, and being held at right angles to* AB *is then let go. What will be the motion?*

There are only two impressed forces, the weight and the pressure. Both these act through the centre of inertia. Hence they have no moment about it, and there will be no rotation. Thus the rod CO will slide down, keeping always perpendicular to AB.

3. The resultant effective couple

$$\Sigma\delta m\left(x\frac{dv}{dt}-y\frac{du}{dt}\right)$$

is the rate of change of the angular momentum $\Sigma\delta m\,(xv-yu)$. This couple reversed is equal to the moment of the impressed forces about the point taken as origin. Hence angular momentum plays the same part in rotation that linear momentum does in translation. If there are no impressed forces, the angular momentum about any axis is constant. If the resultant of the impressed forces has no moment about a certain axis, as by passing through it or acting parallel to it, the whole angular momentum of the system about that axis remains constant.

In the general case, the angular momentum of any body about any axis is equal to the moment of the momentum of the mass collected at the centre of inertia about that axis, viz. $m\,(\bar{x}\bar{v}-\bar{y}\bar{u})$, together with the angular momentum about a parallel axis through the centre of inertia, $\omega\,.\,\Sigma\delta mr^2$.

In the solar system—there being no external force—the sum of the moments of the momenta of the sun and planets about any axis, together with the sum of their angular momenta about parallel axes through their centres, is constant. When an iceberg becomes detached from near the pole where its motion round the polar axis of the earth is small, and floats to near the equator where its motion is large, the angular momentum of the earth must have diminished as much as the moment of the momentum of the iceberg has increased. (The angular momentum of the iceberg about an axis in itself is neglected.) When Don Quixote was lifted up by the windmill and became a part of its system, the only impressed force external to both being an impulse on the axle of the mill, the angular momentum of the sails about the axle must have diminished as much as his increased.

If a number of particles moving freely becomes rigidly connected, this does not change their united angular momentum. If a rotating body contains liquid, and this liquid solidifies, the angular momentum remains unchanged.

It has been mentioned that rotation takes place not so much about a point as about a direction. If a watch that is going be finely enough poised horizontally on a point, it will be seen to make small oscillations. These are due to the oscillations of the balance-wheel inside it. The angular momentum of the whole watch is at every moment zero. Hence the angular momentum of the balance-wheel, together with its moment of momentum, is equal and opposite to the angular momentum of the rest of the watch. Spin a top on a plate, and float the plate in water. When the top has come to rest, its angular momentum must have been communicated to the floating system.

The angular momentum about a fixed axis or one through the centre of inertia of a rigid body is

$$\omega \Sigma \delta m r^2.$$

Hence we are enabled to assert that if a body is rotating about a fixed direction under the action of no impressed forces, or of impressed forces whose resultant passes through the fixed axis or through the centre of inertia, the angular velocity remains constant.

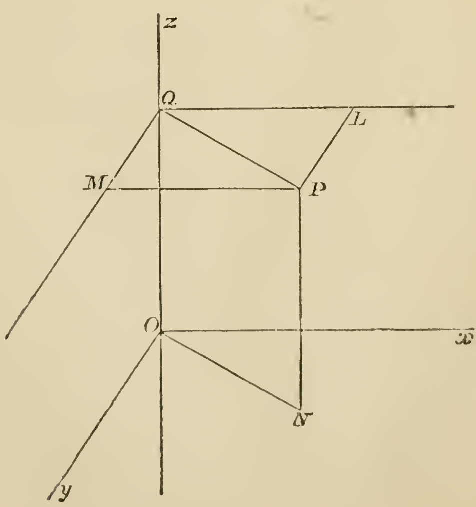
A grindstone set rotating and left to itself, or any heavy body thrown up into the air and rotating in the plane of projection, or an awkward man upon a slide, are examples of this.

4. It has been stated before, that the tendencies of forces and couples are to cause translation and rotation respectively. We now see that it would be more correct to say that they cause translation of the centre of inertia and rotation round it. A couple does not necessarily cause rotation about the direction of its own axis. We shall soon see when this is the case. All we can at present say is that the impressed couple

is in equilibrium with the reversed effective couples necessary to produce the motion. We have previously seen the complete analogies which subsist between the theory of forces and that of rotations. This analogy must not be confused with the physical connection mentioned above, which is between couples and rotations.

In 1777, shortly after our science had been established on its present basis, an English mathematician of some eminence, called Landen, wrote a treatise, in which he professed himself dissatisfied with the principles and conclusions of Euler and D'Alembert; and in which he gave a new theory of motion of his own. This theory gave rise to a good deal of discussion in England. When carefully examined it is found to depend on the principle, that any force whose effect in causing rotation in a body is the same as that of a given force may be substituted for it. This is fallacious; for it takes no account of the action of a force in causing translation. Since that time there has been no doubt thrown on the accuracy of our principles.

5. *A body is rotating with no impressed forces and with constant angular velocity* ω *about the axis* Oz, *which is fixed.*



What pressures has the axis to sustain in consequence of the motion?

Suppose the axis acted on by forces X, Y at O (there will evidently be none along Oz), and by a couple whose components are G_x, G_y about the directions Ox, Oy.

Consider the motion of an element of mass δm at P. $PMQL$ is a plane parallel to xy. PM, PL, PN are the co-ordinates of P.

The acceleration of P is $\omega^2 . PQ$ along PQ. This is equivalent to $-\omega^2 . PM$ along Ox; $-\omega^2 . PL$ along Oy.

The effective force at P is therefore

$$-\delta m\, \omega^2 PM \text{ along } Ox,$$

and

$$-\delta m\, \omega^2 PL \text{ along } Oy.$$

These reversed and taken all over the body are in equilibrium with X, Y and G_x, G_y.

Hence

$$X + \omega^2 \Sigma \delta m PM = 0,$$

$$Y + \omega^2 \Sigma \delta m PL = 0,$$

$$G_x - \omega^2 \Sigma \delta m PL . PN = 0,$$

$$G_y + \omega^2 \Sigma \delta m PM . PN = 0.$$

It must be remembered that rotation is positive from x to y, from y to z, and from z to x.

These equations give the pressures on the axis.

Suppose the whole of Oz set free except the point O, and that we wished to know whether the forces of the motion would permit the body to continue rotating about Oz. This will be the case if G_x and G_y are zero. Hence

$$\Sigma \delta m PL . PM \text{ and } \Sigma \delta m PM . PN$$

must both be zero.

If these conditions hold, Oz is called an axis of permanent rotation, or a principal axis of the body.

If the whole axis be set free, and the body still continue to rotate about it, X and Y must be zero.

Hence

$$\Sigma \delta m \, PM \text{ and } \Sigma \delta m \, PL$$

must be zero. This means that the axis Oz must pass through the centre of mass.

It is easy to see how the axis may suffer a breaking couple, but no single pressure. Take a rod and attach it rigidly to a fixed axis at its centre of mass. Let this axis be inclined to it. If it be now made to rotate it will endeavour to get into a position perpendicular to the axis, but if it were set free the centre of mass would not move.

6. *A raft of any form, whose mass is* M *and whose centre of inertia is* C, *is at rest on the surface of still water. A man whose mass is* m, *who is standing at a given point* P *of the raft, commences to move with velocity* v *relatively to the raft in a direction at right angles to* CP. *What motion of the raft will this cause?*

We take the man to be a moving point, and the water not to hinder the motion of the raft.

The common centre of mass of the raft and man remains at rest: in other words, there is no resultant linear momentum.

The velocity of C must then be in a direction opposite to that of the man. Let V be this velocity of C. Let Ω be the angular velocity with which the raft begins to move. Then the absolute velocity of the man is $v - V - CP \, . \, \Omega$; the momenta are MV and $m(v - V - CP \, . \, \Omega)$, and

$$-MV + m(v - V - CP \, . \, \Omega) = 0.$$

Secondly; as there are no horizontal external forces, the whole angular momentum about any vertical axis remains unchanged.

Consider the angular momentum round C. The angular momentum of the raft is

$$\Omega \Sigma \delta M r^2,$$

in which r is the distance of an element δM from C.

The moment of momentum of the man round C is $m(V + \Omega . CP)\, CP$ due to the motion of the raft, and $-mv . CP$ due to his own on the raft.

Hence

$$m(V - v + \Omega . CP)\, CP + \Omega \Sigma \delta M r^2 = 0,$$

which gives Ω in terms of known quantities.

These two equations give V and Ω in terms of v and known quantities.

It is indifferent whether the unknown quantities V and Ω are assumed to be in the direction in which from previous considerations we know them to be, or in some assumed positive direction.

7. We can now write down the equations of motion for any system moving under any forces in one plane or in parallel planes.

Let there be one rigid body acted on by impressed forces reducible to forces X, Y along the axes, and whose moment is L round the centre of inertia; then (IV. 6) if the forces are impulsive,

$$M(\bar{u}' - \bar{u}) = X, \quad M(\bar{v}' - \bar{v}) = Y,$$

$$(\omega' - \omega)\Sigma \delta m r^2 = L.$$

If they are finite,

$$M\frac{d\bar{u}}{dt} = X, \quad M\frac{d\bar{v}}{dt} = Y,$$

$$\frac{d\omega}{dt}\Sigma \delta m r^2 = L.$$

If we desire to take moments about another point than the centre of inertia; let the co-ordinates of the centre of inertia with respect to it be $\bar{x}$, $\bar{y}$, and let the moment of the forces about it be L'; then, the other quantities remaining as before, the third equation will become

$$(\omega' - \omega)\,\Sigma\delta mr^2 + M\{\bar{x}(\bar{v}' - \bar{v}) - \bar{y}\,(u' - \bar{u})\} = L',$$

or

$$\frac{d\omega}{dt}\Sigma\delta mr^2 + M\left(\bar{x}\frac{d\bar{v}}{dt} - \bar{y}\frac{d\bar{u}}{dt}\right) = L'.$$

If there are a number of bodies in the system similar equations hold for each or for any number taken together.

EXAMPLES.

1. A person, who has been standing on smooth ice, falls down. In what line does his centre of inertia move?

2. The earth in cooling contracts. Does this make the day longer or shorter?

3. A man walks across the deck of a small yacht. Has the yacht rotated? Has its centre moved?

4. Two smooth spheres rest one above another on a smooth horizontal plane. If the equilibrium be disturbed, in what line will their common centre of gravity move?

5. A straight uniform rod can slide with its ends on two smooth fixed straight wires placed at right angles to one another in a horizontal plane. It is set moving. Prove that its angular velocity will remain constant.

6. How does it appear that linear and angular momenta obey the parallelogram law?

7. A rigid body attached to a string is allowed to fall until the string becomes tight. Shew that if it fall so that there is no immediate rotation, there will be no subsequent rotation.

8. A man is being weighed in one scale of a large balance. If he jump straight up, what will be the effect on the machine? and what will be the result when he meets the scale again?

9. Explain how it is that a boy in a swing can increase his arc of swing by crouching when at the lowest point.

10. If a rigid body previously at rest be set in motion by a single blow; prove that after moving for any time it can be again reduced to rest by an equal and opposite blow acting in the same line.

11. The centre of gyration of a rigid body capable of revolving about a fixed axis is the point at which the whole mass must be collected, that the angular velocity communicated by a given couple may be the same as before. If k be the distance of this point from the axis, prove that

$$mk^2 = \Sigma \delta m r^2.$$

12. A man is placed in a canoe without a paddle or any means of touching the water. Can he work round the head of the canoe?

VI.

MOMENTS AND PRODUCTS OF INERTIA.

1. We have had occasion to remark that the quantity $\Sigma\delta m\, r^2$, in which r is the distance of an element δm from an axis, and the summation extends over the whole of a rigid body, will be of constant occurrence in problems of rotation. It is called the moment of inertia of the body about the axis. We have also found that expressions of the form

$$\Sigma\delta m\, yz,\ \Sigma\delta m\, zx,\ \Sigma\delta m\, xy,$$

are of common though less frequent occurrence. These have been called products of inertia.

It will save us much repetition if we investigate once for all some of the properties of these moments and products, and their values for the most common axes of the most common bodies.

The values of moments and products of inertia must depend ultimately on summation or integration for the various elements of the body; but after this has been accomplished for the simplest axes possible, they can be found without summation for any other axes.

2. *The moment of inertia of a uniform rod of mass* m *and length* 2a *about an axis through its middle point at right angles to it.*

Suppose the rod a line of particles. Let the distance of one of these from the middle point be r, and its mass $\mu\delta r$, so that $\mu 2a$ is m.

The moment of inertia is $\Sigma\mu\delta r . r^2$. Supposing δr to diminish indefinitely this becomes $\mu\int r^2 dr$, the limits of r being $-a$ and $+a$. The value of this integral is $\frac{\mu . 2a^3}{3}$. But $\mu . 2a = m$, hence the required moment of inertia is $m . \frac{a^2}{3}$.

It is clear from the nature of the expression $\Sigma\delta mr^2$ that every moment of inertia will be the product of the mass and a square. It is usually written mk^2. k is called the "radius of gyration." It is the distance of the centre of gyration from the axis. (See Less. V. Ex. 11.) The value of k in the above example is $\frac{a}{\sqrt{3}}$.

The product of inertia of the rod about two axes x, y through the middle point and in the same plane with it, of which x makes an angle α with the rod, is

$$\Sigma\mu\delta r . xy, \text{ i.e. } \Sigma\mu\delta r . r\sin\alpha . r\cos\alpha.$$

In the limit

$$\mu \sin\alpha\cos\alpha \int_{-a}^{+a} r^2\,dr,$$

or

$$\mu\sin\alpha\cos\alpha . \frac{2a^3}{3},$$

but $\mu . 2a = m$, therefore the product of inertia is

$$m . \sin\alpha . \cos\alpha . \frac{a^2}{3}.$$

3. *The moment of inertia of a uniform circular plate of mass* m *and radius* a *about an axis through its centre perpendicular to its plane.*

Imagine the circle composed of narrow concentric rings. Let the radius of one be r, its breadth δr, its mass $\mu . 2\pi r\delta r$, so that $\mu . \pi a^2 = m$.

The moment of inertia of this ring is $\mu\, 2\pi r\delta r . r^2$, for all points are equidistant from the centre. Therefore the moment of the whole is $\int_0^a \mu 2\pi r^3\,dr$, i.e. $\mu\pi \cdot \frac{a}{2}$. Now $\mu\pi a^2 = m$; therefore the moment of inertia is $m\frac{a^2}{2}$, and the radius of gyration is $\frac{a}{\sqrt{2}}$.

The product of inertia of the circle about this perpendicular and a diameter is plainly zero; for the co-ordinate perpendicular to the circle of every element vanishes.

The product of inertia about any two axes in its plane is also zero. For each of the axes divides the body symmetrically. Hence for every element δm at x, y whose product is $\delta m \,.\, xy$, there is an element δm at $-x, y$ for which the product is $-\delta m\, xy$; and the sum will consequently vanish.

4. The moments of inertia of any body about the axes of co-ordinates are

$$\Sigma\delta m\,(y^2+z^2),\ \ \Sigma\delta m\,(z^2+x^2),\ \ \Sigma\delta m\,(x^2+y^2).$$

In the case of a body altogether in one plane (x, y), one co-ordinate (z) is always zero. Hence the moment of inertia of such a body about an axis perpendicular to its plane is equal to the sum of the moments of inertia about two axes at right angles to one another in its plane. Hence we can infer that the moment of inertia of a circular plate about a diameter is $m\frac{a^2}{4}$.

The products of inertia of such a body about axes, one of which is perpendicular to it, are zero.

5. *The moment of inertia of a uniform right circular cone bounded by a plane perpendicular to the axis about its axis.*

Imagine the cone made up of a great number of equally thin circular disks.

Let the distance of one of these from the vertex be x, its thickness δx. If the semivertical angle be α, its radius will be

$x \tan \alpha$. Let its mass be $\mu\pi x^2 \tan^2 \alpha \,.\, \delta x$. Its moment of inertia about the axis of the cone is mass $\times \frac{x^2}{2} \tan^2 \alpha$,

or $$\mu x^2 \tan^2 \alpha\pi\delta x \,.\, \frac{x^2}{2} \tan^2 \alpha.$$

The moment of inertia of the whole cone will be the sum of all these, each indefinitely diminished, or

$$\frac{\mu \tan^4 \alpha\pi}{2} \int_0^h x^4\, dx = \frac{\mu\pi \tan^4 \alpha}{10} \,.\, h^5,$$

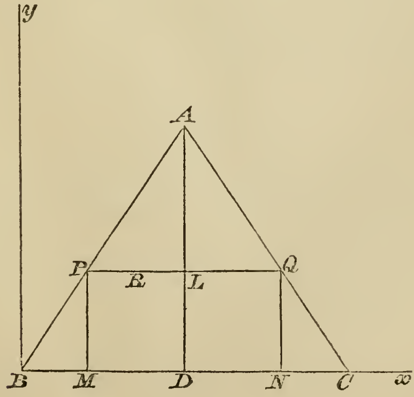
where h is the cone's height.

Now the mass of the cone is $\int_0^h \mu\pi x^2 \tan^2 \alpha dx = \frac{1}{3} \mu \tan^2 \alpha\pi h^3$.

Whence $$k = h \tan \alpha \sqrt{\frac{3}{10}}.$$

6. *The product of inertia of an isosceles triangular plate* ABC *about the base* BC, *and a line through* B *at right angles to* BC.

Take BC as the axis of x and the perpendicular as that of y. Draw AD to the middle point of BC. Let $AD = h$. Imagine the triangle made up of infinitely narrow strips parallel to BC, of thickness δy, one of which is PQ, cutting AD in L. From P, Q draw PM, QN at right angles to BC.



Consider an element of mass $\mu\delta x \,.\, \delta y$ at R a point on PQ. Its product of inertia is $\mu\delta x \,.\, \delta y \,.\, x \,.\, y$. That of PQ is $\mu\delta y \,.\, y \int x\, dx$, the limits being BM and BN.

It is therefore

$$\frac{\mu}{2}\delta y \,.\, y\,(BN^2 - BM^2)$$

$$= \frac{\mu}{2}\delta y \,.\, y\,(BN + BM) \,.\, PQ$$

$$= 2\mu\delta y \,.\, y \,.\, BD \,.\, PL.$$

Now $\quad PL = (h - y)\tan\dfrac{A}{2},\;\; BD = h\tan\dfrac{A}{2};$

therefore the whole product of inertia

$$= 2\mu \,.\, h\tan^2\frac{A}{2}\int_0^h y\,(h - y)\,dy$$

$$= \frac{\mu}{3}h^4\tan^2\frac{A}{2}.$$

But the mass of the triangular plate $= \mu h^2\tan\dfrac{A}{2}$;

therefore the product of inertia

$$= m \,.\, \frac{h^2\tan\dfrac{A}{2}}{3}.$$

This is easily seen to be the same as the product of inertia about the same axes of two masses, each equal to $\dfrac{m}{3}$, placed at the middle points of AB, AC.

There will be found in the first chapter of Routh's *Rigid Dynamics* an instructive proof that the moments and products of inertia of a triangular plate about any axes whatever are the same as those of three particles, each of mass equal to one-third of the mass of the plate, placed at the middle points of the sides.

7. When $\Sigma\delta m\,xz = 0$ and $\Sigma\delta m\,yz = 0$, the axis of z is a principal axis (Lesson V. Art. 5). When all three products of inertia vanish we have a set of three principal axes. But $\Sigma\delta m\,xz$ and $\Sigma\delta m\,yz$ may vanish without $\Sigma\delta m\,xy$ vanishing; and any one, as $\Sigma\delta m\,xy$, may vanish without either x or y being a principal axis. If indeed the body is altogether in the plane of xy, two axes for which $\Sigma\delta m\,xy$ vanishes are principal axes, for we know that $\Sigma\delta m\,xz = 0$ and $\Sigma\delta m\,yz = 0$.

In many cases the position of a principal axis can be seen at once. Thus at every point in a plane body one principal axis is the perpendicular to the plane. Again, if in any uniform body a straight line can be drawn with respect to which the body is exactly symmetrical, this must be a principal axis at every point in its length. Any diameter of a uniform circle or sphere is a principal axis at any point in its line; but the diagonal of a rectangular plate is not for this reason a principal axis at its middle point; for every straight line drawn perpendicular to it is not equally divided by it.

8. Table of the squares of the radii of gyration for several bodies (supposed uniform) of frequent occurrence.

1. For a rod of length $2a$ about a perpendicular through the centre of mass, $k^2 = \frac{a^2}{3}$.

2. For a rectangular plate, one of whose sides is of length $2a$, about a line through the centre parallel to the other, $k^2 = \frac{a^2}{3}$.

3. For an elliptic plate about either axis,

$$k^2 = \frac{\text{square of other semiaxis}}{4}.$$

4. For an ellipsoid about any of the three axes,

$$k^2 = \frac{\text{sum of squares of the other semiaxes}}{5}.$$

These include as particular cases squares, circles, and spheres.

EXAMPLES.

1. Prove that

(α) For a straight rod of length $2a$ about an axis passing through one end and making an angle β with the rod

$$k = \frac{2}{\sqrt{3}} . a \sin \beta.$$

(β) For any arc of a circle of radius a about an axis through the centre perpendicular to its plane $k = a$.

(γ) For a right circular cylinder of radius a about its axis $k = \frac{a}{\sqrt{2}}$.

(δ) For a straight rod about an axis parallel to itself at distance c, $k = c$.

2. The moments of inertia of a uniform elliptic plate of semiaxes a, b about its major and minor axes are

$$m . \frac{b^2}{4}, \quad m \frac{a^2}{4}.$$

That about a perpendicular to its plane through its centre is

$$m \frac{a^2 + b^2}{4}.$$

3. The moment of inertia of a sphere of radius a about a diameter is $\frac{2}{5} ma^2$.

4. The moment of inertia of a uniform rectangular plate whose sides are a and b about the side a is $m\frac{b^2}{3}$.

5. For what bodies and for what axes is the moment of inertia zero?

6. The moment of inertia of a uniform cube whose length of edge is $2a$ about a line through the centre parallel to an edge is $m.\frac{2a^2}{3}$.

7. What are the principal axes of a uniform rectangular plate at the middle point of one side?

8. Investigate the product of inertia of a right-angled triangle about the sides containing the right angle.

9. What is the moment of inertia of a shell, whose mass is m and which is bounded by concentric spherical surfaces whose radii are a and A, about an axis through its centre?

10. The centre of a uniform parallelopiped is fixed. About what axes will it rotate permanently when no forces act on it but those whose resultant passes through the centre?

11. Prove that the moment of inertia of a uniform ellipsoid whose semiaxes are a, b, c about the axis a is

$$m.\frac{b^2+c^2}{5}.$$

12. The density of a cylinder of mass m and radius a varies as the n^{th} power of the distance from the axis. Prove that the moment of inertia about the axis is

$$ma^2.\frac{n+2}{n+4}.$$

13. Prove that the quantity of rotation in a cylinder of radius a rotating with velocity ω about its axis is

$$\omega \,.\, m \frac{a^2}{2}.$$

14. Prove that the quantity of rotation in a rotating sphere of radius a is $\omega \,.\, m \,.\, \frac{2}{5} a^2$.

15. A uniform rectangular plate is rotating with constant velocity about an axis in its plane through the centre. If the centre be set free and the plate keep rotating about the same axis, which is that axis?

16. A grindstone of mass m and radius a is rotating with velocity ω, and is stopped at a constant rate in t seconds. What is the rate of loss of angular momentum?

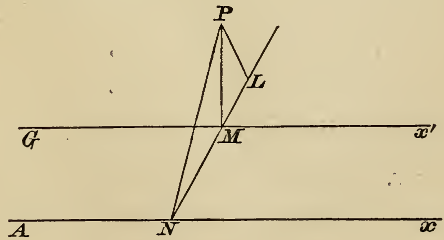
VII.

MOMENTS AND PRODUCTS OF INERTIA.

1. WHEN the moments and products have been found about any axes through the centre of mass of a body, we can find them at once about any other parallel axes by the theorem of Lesson IV. Art. 2.

The following is a geometrical proof that the moment of inertia about any axis is equal to that about a parallel axis through the centre of mass, together with the moment of inertia of the whole mass at the centre of mass about the former axis.

Let Ax and Gx' be parallel axes, the latter through the centre of mass. And let P denote any element of the body.



Let a plane through P at right angles to the axes cut them in N and M; join these and draw PL perpendicular to NM.

The moment of inertia about Ax is $\Sigma\delta m\, PN^2$. Now by Euc. II. 12,

$$PN^2 = PM^2 + MN^2 + 2MN \cdot ML.$$

Hence $$\Sigma\delta m\, PN^2 = \Sigma\delta m\, PM^2 + m \cdot MN^2 + 2MN \cdot \Sigma\delta m\, ML,$$

since MN is the same for all elements, being the distance between the parallel axes.

Also $\Sigma\delta m\, ML$ is zero, since Gx passes through the centre of mass;

$$\therefore\ \Sigma\delta m\, PN^2 = \Sigma\delta m\, PM^2 + m \cdot MN^2.$$

2. If then k is the radius of gyration about an axis through the centre of mass, and a the distance of a parallel axis, the moment of inertia about this latter is $m(k^2 + a^2)$. Call this A. If there be another axis in the same plane at a distance b from the centre of mass, the moment of inertia about it is $m(k^2 + b^2)$. Call this B. Then

$$A - ma^2 = B - mb^2,$$

and thus knowing the positions of two parallel axes relatively to the centre of mass, and the moment of inertia about either, we can find that about the other.

3. If a, b, c are the co-ordinates of the centre of mass, and ξ, η, ζ the co-ordinates of any element δm relatively to it,

$$\Sigma\delta m\, xy = \Sigma\delta m\, \xi\eta + mab,$$

and $$\Sigma\delta m xz = \Sigma\delta m\, \xi\zeta + mac.$$

If the centre of mass lies on the axis of x,

$$b = 0,\ c = 0;\ \therefore\ \Sigma\delta m\, xy = \Sigma\delta m\, \xi\eta, \text{ and } \Sigma\delta m\, xz = \Sigma\delta m\, \xi\zeta.$$

If this line is a principal axis at the centre of mass,

$$\Sigma\delta m\, \xi\eta = 0, \text{ and } \Sigma\delta m\, \xi\zeta = 0;$$

therefore $$\Sigma\delta m\, xy = 0, \text{ and } \Sigma\delta m\, xz = 0,$$

or it is a principal axis at every point in its length.

4. The moment of inertia of a sphere of radius a about a diameter has been found to be $\frac{2}{5} m \,.\, a^2$.

Hence about a tangent it is

$$\frac{2}{5} ma^2 + ma^2 = \frac{7}{5} ma^2.$$

5. *To find the product of inertia of a rectangular plate about two sides.*

The straight lines drawn through the centre parallel to the sides are clearly principal axes. Hence the required product is equal to the product of inertia of the whole mass collected at the centre about the sides. It is the product of the mass and one-fourth the area of the rectangle.

EXAMPLES.

1. The radius of gyration of a circular arc about a line through its middle point at right angles to its plane is radius $\times \sqrt{2}$.

2. That of a circular plate about a tangent is

$$\text{radius} \times \sqrt{\frac{5}{4}}.$$

3. Find the moment of inertia of a right cone about a line through the vertex perpendicular to the axis.

4. The moment of inertia of a parallelogram whose sides are $2a$ and $2b$ in length about an axis through the centre at right angles to its plane is $\frac{m}{3}(a^2 + b^2)$.

5. What are the principal axes at a corner of a square?

6. Prove that for an elliptic area about a line through a focus parallel to the minor axis

$$k^2 = \frac{5a^2 - 4b^2}{4}.$$

7. Prove that for a circular arc of radius r, which subtends an angle 2α at the centre about an axis through its centre of gravity perpendicular to its plane,

$$k^2 = r^2\left(1 - \frac{\sin^2\alpha}{\alpha^2}\right).$$

VIII.

MOMENTS AND PRODUCTS OF INERTIA.

1. HAVING now found the moments and products of inertia of a body about all axes parallel to a given set about which they are known, we proceed to complete the subject by finding them about sets of axes inclined to these. We will take first the case of axes in one plane.

2. *At any point of a given rigid body and in any plane there are always two axes at right angles, such that the product of inertia about them is zero.*

Let O be the point, Ox, Oy any perpendicular axes in the plane. Let the moments of inertia about Ox, Oy be denoted by a, b, and the product $\Sigma\delta m\, xy$ by f. Let Ox', Oy' be another set of axes in the plane inclined to the others at an angle α. i.e. $\angle\, xOx' = \angle\, yOy' = \alpha$.

We will endeavour to choose α so that $\Sigma\delta m\, x'y' = 0$.

If the polar co-ordinates of δm be r, θ,

$$x = r\cos\theta,\ \ y = r\sin\theta,$$

$$x' = r\cos(\theta - \alpha),\ \ y' = r\sin(\theta - \alpha),$$

$$\Sigma\delta m\, x'y' = \frac{1}{2}\Sigma\delta m\, r^2 \sin 2(\theta - \alpha)$$

$$= \cos 2\alpha\, \Sigma\delta m\, r^2 \sin\theta\cos\theta$$

$$- \frac{\sin 2\alpha}{2}\Sigma\delta m\, r^2 (\cos^2\theta - \sin^2\theta).$$

If this vanishes

$$\tan 2\alpha = \frac{2\,\Sigma\delta m\, r^2 \sin\theta\cos\theta}{\Sigma\delta m\,(r^2\cos^2\theta - r^2\sin^2\theta)}$$

$$= \frac{2\,\Sigma\delta m\, xy}{\Sigma\delta m\,(x^2 - y^2)}.$$

If the distance of δm from the plane xOy be z,

$$a = \Sigma\delta m\,(y^2 + z^2),$$

$$b = \Sigma\delta m\,(z^2 + x^2),$$

and

$$\tan 2\alpha = \frac{2f}{b - a}.$$

As this equation has always a solution, there is always a set of axes having the required property.

3. Farther, *for each axis of that set, the moment of inertia is a maximum or minimum.*

For let α be variable and find the value for which

$\Sigma\delta m\,(y'^2 + z^2)$, or $\Sigma\delta m\, y'^2$ is greatest or least,

$$\Sigma\delta m\, y'^2 = \Sigma\delta m\, r^2 \sin^2(\theta - \alpha).$$

Differentiating with respect to α;

$$2\Sigma\delta m\, r^2 . \sin(\theta - \alpha)\cos(\theta - \alpha) = 0,$$

but we have seen that this is equivalent to

$$\Sigma\delta m\, x'y' = 0.$$

4. Farther, *the moment of inertia about any line* Ox′ *inclined at an angle* α *to* Ox *is*

$$a\cos^2\alpha + b\sin^2\alpha - 2f\sin\alpha\cos\alpha.$$

For $\Sigma\delta m\,(y'^2 + z^2)$

$$= \Sigma\delta m\, r^2\sin^2(\theta - \alpha) + \Sigma\delta m\, z^2$$

$$= \cos^2\alpha\,\Sigma\delta m\, r^2\sin^2\theta - 2\cos\alpha\sin\alpha\,\Sigma\delta m\, r^2\sin\theta\cos\theta$$
$$+ \sin^2\alpha\,\Sigma\delta m\, r^2\cos^2\theta + (\cos^2\alpha + \sin^2\alpha)\,\Sigma\delta m\, z^2$$

$$= \cos^2\alpha\,\Sigma\delta m\,(y^2 + z^2) + \sin^2\alpha\,\Sigma\delta m\,(x^2 + z^2)$$
$$- 2\cos\alpha\sin\alpha\,\Sigma\delta m\, xy$$

$$= a\cos^2\alpha + b\sin^2\alpha - 2f\sin\alpha\cos\alpha.$$

If $f=0$, or if Ox, Oy are principal axes and the moments of inertia about them are A and B, this becomes

$$A\cos^2\alpha + B\sin^2\alpha.$$

5. *The moment of inertia of a rectangular plate about a diagonal.*

If the lengths of the sides are $2a$, $2b$, the moments of inertia about two lines through the centre parallel to the sides are

$$m\frac{b^2}{3},\quad m\frac{a^2}{3}.$$

And $$\tan\alpha = \frac{b}{a},$$

whence $$\sin^2\alpha = \frac{b^2}{a^2+b^2},$$

$$\cos^2\alpha = \frac{a^2}{a^2+b^2};$$

therefore the moment required is

$$m\frac{b^2}{3}\cdot\frac{a^2}{a^2+b^2} + m\frac{a^2}{3}\cdot\frac{b^2}{a^2+b^2}$$

$$=\frac{2m}{3}\frac{a^2b^2}{a^2+b^2}.$$

6. *To find the position of the principal axes of a uniform rectangular plate* ABCD *at the point* A.

The moment of inertia about AB is $m\frac{AD^2}{3}$, that about AD is $m\frac{AB^2}{3}$.

The product about AB, AD is $\frac{m}{4}.AB.AD$.

Hence, substituting in Art. 2, if α be the angle which one of the required axes makes with AB,

$$\tan 2\alpha = \frac{\frac{1}{2}.AB.AD}{\frac{AB^2}{3} - \frac{AD^2}{3}}$$

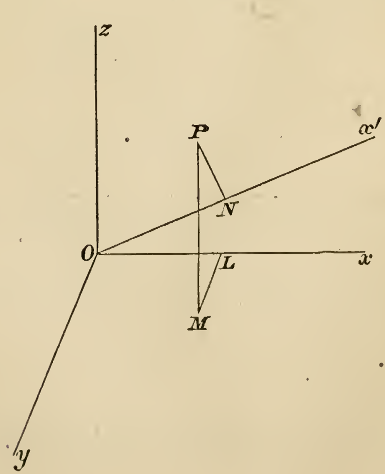
$$= \frac{3}{2}\frac{AB.AD}{AB^2 - AD^2}.$$

7. *There are at every point of a rigid body three axes at right angles to one another, for which the products of inertia vanish.*

We might apply the same method as before, but an indirect method is here more simple.

Given the moments of inertia a, b, c about three axes Ox, Oy, Oz at right angles; and the products d, e, f; ($\Sigma\delta m\, yz$, $\Sigma\delta m\, zx$, $\Sigma\delta m\, xy$ respectively) let us find the moment of inertia about a line Ox' inclined at angles α, β, γ to Ox, Oy, Oz.

Let OL, LM, MP be the co-ordinates x, y, z of an ele-



ment of mass δm at a point P. From P let fall PN perpendicular to Ox'.

Projecting $OLMP$ on OM,

$$ON = OL \cos\alpha + LM \cos\beta + MP \cos\gamma,$$

and $$PN^2 = OP^2 - ON^2;$$

$$\therefore\ PN^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$$
$$= x^2(1-\cos^2\alpha) + y^2(1-\cos^2\beta) + z^2(1-\cos^2\gamma)$$
$$- 2yz\cos\beta\cos\gamma - 2zx\cos\gamma\cos\alpha - 2xy\cos\alpha\cos\beta.$$

But $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$

$$\therefore\ PN^2 = x^2(\cos^2\beta + \cos^2\gamma) + y^2(\cos^2\gamma + \cos^2\alpha)$$
$$+ z^2(\cos^2\alpha + \cos^2\beta)$$
$$- 2yz\cos\beta\cos\gamma - 2zx\cos\gamma\cos\alpha - 2xy\cos\alpha\cos\beta$$
$$= \cos^2\alpha\,(y^2+z^2) + \cos^2\beta\,(z^2+x^2) + \cos^2\gamma\,(x^2+y^2)$$
$$- 2yz\cos\beta\cos\gamma - 2zx\cos\gamma\cos\alpha - 2xy\cos\alpha\cos\beta;$$

$$\therefore\ \Sigma\delta m\, PN^2 = a\cos^2\alpha + b\cos^2\beta + c\cos^2\gamma$$
$$- 2d\cos\beta\cos\gamma - 2e\cos\gamma\cos\alpha - 2f\cos\alpha\cos\beta.$$

To represent this geometrically take a point Q on ON.

Let its distance from O be ρ, and its co-ordinates ξ, η, ζ;

then $$\xi = \rho\cos\alpha,\quad \eta = \rho\cos\beta,\quad \zeta = \rho\cos\gamma,$$

and

$$\Sigma\delta m PN^2 = \frac{a\xi^2 + b\eta^2 + c\zeta^2 - 2d\eta\zeta - 2e\zeta\xi - 2f\xi\eta}{\rho^2}.$$

Now the equation

$$a\xi^2 + b\eta^2 + c\zeta^2 - 2d\eta\zeta - 2e\zeta\xi - 2f\xi\eta = 1$$

denotes an ellipsoid whose centre is at O; for a, b, c are necessarily positive. If then Q is a point on this

$$\Sigma\delta m PN^2 = \frac{1}{\rho^2},$$

or the moment of inertia about any line through O, is measured by the square of the reciprocal of the radius vector of this ellipsoid which coincides with the line.

This is called the momental ellipsoid. It has no physical existence, but is an artifice to bring under the methods of geometry the properties of moments of inertia. The momental ellipsoid has a definite form for every point of a rigid body.

If this ellipsoid be referred to another set of axes, and its equation become

$$a'\xi^2 + b'\eta^2 + c'\zeta^2 - 2d'\eta\zeta - 2e'\zeta\xi - 2f'\xi\eta = 1,$$

the coefficients a', b', c' will be the moments of inertia about the new axes, and d', e', f' will be the products.

Now every ellipsoid has three axes, to which if it is referred its equation takes the form

$$A\xi^2 + B\eta^2 + C\zeta^2 = 1.$$

With respect to these axes, the products of inertia vanish.

8. Hence we see that the moment of inertia about one of the principal axes is the greatest, and about another the least possible. It was from this property that Euler, who first thoroughly investigated the subject, gave them the name.

It is now clear, that for all questions depending only on moments and products of inertia, any body may be replaced by its momental ellipsoid. And farther, that any two systems which have the same momental ellipsoid at a point, are about that point kinetically identical.

If the moments of inertia of a body about three axes at right angles through a point are equal, the ellipsoid becomes a sphere. They are therefore equal about all axes, and every axis is a principal axis. The body is then said to be kinetically symmetrical with respect to that point. Thus a cube is kinetically symmetrical about its centre.

The following question is of some interest.

9. *Under what circumstances is there a point in a body such that the moments of inertia about all axes through it are equal?*

If there is such a point, all sets of axes through it are principal axes.

Let the co-ordinates of the point referred to the principal axes at the centre of mass be a, b, c. Then the products of inertia of the body about the parallel axes through the point are

$$m\,.\,bc,\ m\,.\,ca,\ m\,.\,ab,$$

for those about the axes through the centre of mass are zero.

If all axes at the point are to be principal axes, these must be so;

$$\therefore\ bc = 0,\ ca = 0,\ ab = 0,$$

equations which require that two of a, b, c should be zero.

Let $b = 0$, $c = 0$, then the point required lies on the axis of x,—one of the set of principal axes at the centre of mass.

But further, it is necessary that the moments of inertia about these axes should be equal. Let A, B, C be the moments of inertia about the axes through the centre of mass. Then those about the parallel axes through the point required are

$$A,\ B + ma^2,\ C + ma^2.$$

If these are to be equal, we have

$$B = C \text{ and } A - B = ma^2.$$

Hence our condition is, that two of the principal moments at the centre of mass should be equal. In other words, the momental ellipsoid at the centre of mass must be a spheroid. And then the point lies on the unequal axis at a distance from the centre equal to

$$\pm\left\{\frac{A - B}{m}\right\}^{\frac{1}{2}}.$$

EXAMPLES.

1. Given A, B the moments of inertia of a body about two principal axes Ox, Oy, prove that the product of inertia about axes Ox', Oy' in the same plane, inclined to the former set at an angle α, is

$$\frac{\sin 2\alpha}{2}(A - B).$$

2. Prove that any two of the principal moments of inertia are together greater than the third.

3. No ellipsoid except a sphere can be its own momental ellipsoid at its centre.

4. Every elliptic plate is similar to the section of its momental ellipsoid made by its own plane.

5. If a, b, c are the semiaxes of the momental ellipsoid of a rigid body in order of magnitude, shew that

$$c \text{ is greater than } \frac{ab}{\sqrt{a^2 + b^2}}.$$

6. Given the angular velocity of a body which is rotating about a fixed point; about what axis must it be rotating, so that the angular momentum shall be greatest?

7. Two systems of equal mass have the same principal axes, and the same moments of inertia about them at some one point; prove that they have the same principal axes at any point, and the same moments of inertia about any axis.

8. Prove that there can always be found three points, one on each of the three principal axes of any system at any point, such that the moments and products of inertia of

three suitable equal masses collected at them, are equal to the moments and products of inertia of the system about any axes whatever through that point.

9. If these equal masses be each one-third of the mass of the system (m), shew that the distances of the three points along the axes are

$$\left\{\frac{3}{2}\frac{(B+C-A)}{m}\right\}^{\frac{1}{2}}, \quad \left\{\frac{3}{2}\frac{(C+A-B)}{m}\right\}^{\frac{1}{2}}, \quad \left\{\frac{3}{2}\frac{(A+B-C)}{m}\right\}^{\frac{1}{2}}.$$

10. In a triangular plate ABC, D is the middle point of BC, and E the foot of the perpendicular let fall from A on BC. Shew that the middle point of DE is the point at which BC is a principal axis.

11. Shew that the difference of the moments of inertia of a body round two axes in a given plane which are equally inclined to a fixed line in the same plane, is proportional to the sine of the angle between those axes.

IX.

CASES OF MOTION WITHOUT ROTATION.

1. THE complete solution of a problem of motion would involve the finding of the position of the system at a given time, of the velocities at a given time or in a given position, and of the values of any previously unknown forces, such as pressures or frictions which may act on the system. If the forces are impulsive, only velocities and forces can be required; for the position is unaltered during the impulse, and to follow the subsequent changes belongs to a separate problem of the other kind. Questions of impulsive motion can then always be solved, for the changes of velocity and the forces appear as unknown quantities in equations which are in general simple algebraical equations. But if the forces are of the kind called finite, the equations of motion are differential equations of the second order as regards co-ordinates of position. In some simple cases these can be completely solved and the requirements of the above solution satisfied; but in more complicated cases we can get no farther than a first integral, that is, an algebraical equation giving the velocities. In such a case to find the position at a given time is impossible. Our demands must be limited by what we can get; and the words "to find the motion" have come to mean, "to find the velocities of the system in any position."

We will therefore in general use velocities and their first differential coefficients in the expressions for the effective forces; but if, in any case, the co-ordinates of position must enter, we can use their first and second differential coefficients to express velocities and accelerations.

The unknown external forces can usually be found, for they depend on accelerations and velocities which by the solution have been made to depend on the co-ordinates of position.

An important class of problems has to do with finding the stresses or internal forces at a given point of a body. The action of one part of a body on another is threefold. It may consist of a longitudinal stress, normal to the plane of separation, a transverse or shearing stress, tangential to the plane of separation, and a bending or breaking couple. These are found by considering them as forces external to one part of the body. As in the case of the external pressures, the motion of the whole must first have been investigated.

The first difficulty of a problem in Rigid Dynamics is overcome when by the reversing of the effective forces the whole is reduced to a system in equilibrium. Any method which is available to find the forces or the position of equilibrium of a system is equally available here to find the unknown forces or the state of motion of a system.

In a problem of any complexity it is an assistance to draw two diagrams; one kinematical, representing the changes in the velocities or the accelerations; and the other dynamical, representing the resultant reversed effective forces, and the impressed forces. This is done in Art. 4 of the present Lesson.

2. *A uniform rod* AB *of mass* m *and length* 2a *is let fall in a horizontal position. After falling through a height* h *it is brought to rest by its ends striking two fixed supports at the same level. What will be the breaking couple at a point* P?

The stress couple or, as we may call it, the bending or breaking couple in a system in equilibrium is equal and opposite to the moment of all the other forces acting on either of the parts of the body which are separated at the point under consideration.

The rod in falling through h acquires a velocity $\sqrt{2gh}$.

This is stopped suddenly by two equal pressures. Hence each of these is $\frac{m\sqrt{2gh}}{2}$.

Consider the part AP. It is in equilibrium under the action of BP, the stopping blow $\frac{m}{2}\sqrt{2gh}$ acting upwards at A, and the reversed effective force acting at the middle point of AP. This is the force which has changed the momentum from mass $AP \times \sqrt{2gh}$ to zero. It is therefore

$$-m\frac{AP}{AB}.\sqrt{2gh}, \text{ acting downwards.}$$

Reversed it becomes $m\frac{AP}{AB} \times \sqrt{2gh}$, acting downwards.

The action of BP on AP may be composed of a force and a couple. Taking moments about P we shall avoid the force, and the couple $= \frac{m}{2}.\sqrt{2gh}.AP - m\frac{AP}{AB}.\sqrt{2gh}.\frac{AP}{2}$

$$= \frac{m}{2}\sqrt{2gh}.\frac{AP.PB}{AB}.$$

3. *A straight rod* AB *of mass* m *hangs from a fixed point* O *by an elastic string (natural length* a, *modulus of elasticity* λ), *which is fastened to the end* A. *It is pulled down and then let go. Find the longitudinal stress at any point* P, *when the point* A *is at a distance* x *below* O.

The part BP is in equilibrium under the stress of AP upwards, its own weight downwards and the reversed effective force at its centre of inertia. To know the effective force we must first know the acceleration. Thus we must first consider the whole rod AB.

This is in equilibrium under the tension, its own weight and its own resultant effective force reversed.

Let the downward velocity of the centre of inertia be v.

The effective force is then $m\frac{dv}{dt}$ or $mv\frac{dv}{dx}$ acting downwards. Then the forces mg downwards and $mv\frac{dv}{dx}$ and T (the tension) acting upwards are in equilibrium;

$$\therefore\ mv\frac{dv}{dx} = -T + mg, \text{ and } T = \lambda\frac{x-a}{a};$$

$$\therefore\ mv\frac{dv}{dx} = -\lambda\frac{x-a}{a} + mg.$$

Were the velocity required we should integrate this. As it is we can return to the consideration of BP. The acceleration of every point on the rod is the same. Hence the effective force on BP is

$$\text{mass } BP \times v\,.\,\frac{dv}{dx} \text{ downwards,}$$

or

$$m\,.\,\frac{BP}{AB}\,.\,v\frac{dv}{dx} \text{ acting downwards.}$$

BP is in equilibrium under this reversed and the stress, and $m\,.\,\frac{BP}{AB}\,.\,g$ acting downwards.

Hence the stress is

$$m\,.\,\frac{BP}{AB}g - m\frac{BP}{AB}\,.\,v\frac{dv}{dx},$$

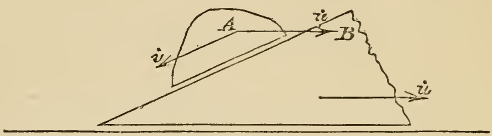
but $mv\frac{dv}{dx}$ has been found above. Substituting;

$$\text{stress at } P = \lambda\,.\,\frac{x-a}{a}\,.\,\frac{BP}{AB}.$$

4. *A wedge* B *whose angle is* α *and whose faces are smooth, rests with one of them in contact with a horizontal table. A rigid body* A *with a plane face is placed on the other with the plane face in contact with it. What will be the velocity of each at any subsequent time?*

Taking both bodies as one system, the impressed forces are all vertical. Hence the common centre of inertia will have no horizontal velocity; or—what is equivalent—there will be no horizontal momentum of the system.

If u be the velocity of the wedge, and v be that of A relatively to the wedge, i.e. down the incline, the horizontal velocity of A is $u - v\cos\alpha$.



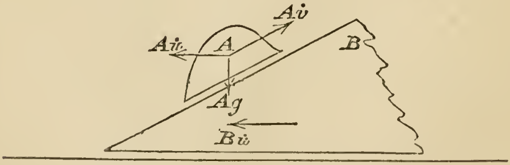
Hence $$A\,(u - v\cos\alpha) + Bu = 0 \text{ ....(1).}$$

Next consider A. Its effective forces are

$$A\,\frac{du}{dt} \text{ horizontally, } A\,\frac{dv}{dt} \text{ down the incline,}$$

and its impressed forces are its weight and the pressure which acts at right angles to the incline. Therefore reversing the effectives and resolving along the incline,

$$A\,\frac{dv}{dt} - A\,\frac{du}{dt}\cos\alpha = Ag\,.\,\sin\alpha \text{.........(2).}$$



These two equations suffice to determine u and v at any time.

The investigation may be continued by finding the pressure between B and A, also by finding the actual positions of A and B in terms of the time.

EXAMPLES.

1. A railway train is going along a level with constant velocity. The friction of the rails is for each carriage one-hundredth part of the pressure. What is the tension of the couplings of the last carriage if its mass is 2 tons?

2. Find this tension; supposing the mass of the carriage to be m, the coefficient of friction μ, and the train to be running down an incline of one in h, with an acceleration β.

3. Prove that the transverse stress at P, in the system of Art. 2, is

$$\frac{m}{2} \cdot \sqrt{2gh}\,\frac{AB - 2 \cdot AP}{AB}.$$

4. A man is placed on a long boat which rests on the surface of still water. Shew that if he could walk with absolutely constant velocity along it, there would be no horizontal force between his feet and the boat except when starting and stopping.

5. A system consisting of two uniform rods AC, CB, rigidly connected at right angles at C, falls without rotation in a vertical plane, and strikes a smooth horizontal plane at B; if there is no rotation produced by the impact, shew that the inclination of BC to the horizon is

$$\tan^{-1}\frac{BC(2 \cdot AC + BC)}{AC^2}.$$

Find in that case the impulsive breaking couple at C.

6. Two small balls each of mass m, are placed at the ends of a diameter within a circular tube of mass $4m$ lying on a smooth horizontal table, and the balls are connected by elastic strings within the tube which are stretched to twice their natural length. One of the strings suddenly breaks. Prove that when the other resumes its natural length, the

centre of the tube is moving with a velocity $\left(\frac{\pi a \epsilon}{60m}\right)^{\frac{1}{2}}$, where a denotes the radius of the circle, and ϵ the original tension of the string.

Find all the circumstances of the problem when the balls meet.

7. A wedge B, whose angle is β, is laid on an inclined plane whose inclination is $\alpha + \beta$, with its edge toward the upper part of the plane. The wedge being at rest on the plane, a body M is projected along its upper surface from the base with a velocity due to a height $a \tan \beta$, where a is the length of the upper side of the wedge; the coefficient of friction for either surface of the wedge is $\tan \beta$, and the motion takes place in one vertical plane. Shew that during the motion of the body on the wedge, the wedge will not slip provided the ratio of M to B is greater than that of $\tan \alpha$ to $\tan \beta$.

Shew also that the body comes to rest before reaching the edge of the wedge, and that the wedge will immediately begin to slide down the incline, but that the body will not slide on the wedge.

X.

FIXED CENTRES.

1. It will be remembered that the effective forces on a rigid body of mass m, whose centre of inertia is G, rotating about a fixed axis O, and having a radius of gyration k about an axis through G parallel to the fixed axis, were reduced, if the change of motion was sudden, to a force $mOG(\omega' - \omega)$ acting at G at right angles to OG, and a couple $mk^2(\omega' - \omega)$, or, which is equivalent, to a force $mOG(\omega' - \omega)$ at O and a couple $m(k^2 + OG^2)(\omega' - \omega)$; if the motion was accelerated they were reduced to forces at G, $mOG\omega^2$ along GO and $mOG\frac{d\omega}{dt}$ at right angles to GO, and to a couple $mk^2\frac{d\omega}{dt}$, or, what is the same thing, to a force at G, $mOG\omega^2$ along GO, a force $mOG\frac{d\omega}{dt}$ at O, and a couple $m(k^2 + OG^2)\frac{d\omega}{dt}$.

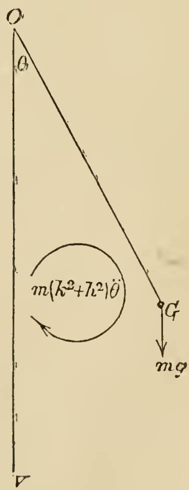
In our diagrams we will represent couples by arcs of circles. The positions of these may be any whatever, for the effect of a couple is absolute. mk^2 will always be employed to denote the moment of inertia about the centre of inertia.

2. The problem of determining the law under which a heavy body swings about a horizontal axis is one of the most important in the history of science.

A simple pendulum is a thing of theory; our accurate knowledge of the acceleration of gravity depends therefore on our understanding the rigid (or compound) pendulum. We have seen that it was the first problem to which D'Alembert applied his principle.

The name of the problem in those days was the 'centre of oscillation.' It was required to find if there were a point at which the whole mass of the body might be concentrated, so as to form a simple pendulum whose law of oscillation was the same.

Let the plane of the paper be that in which G the centre of inertia swings. Let O be the intersection of the axis with that plane. Take OG as the line in the pendulum by which its rotation is measured. Let θ be the angle which OG makes with the vertical. Let k be the radius of gyration about an axis through G parallel to the fixed axis; let OG be called h, and let m be the mass of the body.



The only impressed force which has a moment about O is mg, the weight acting at G. The angular acceleration is $\frac{d^2\theta}{dt^2}$, and the rate of increase of the angular momentum $m(k^2+h^2)\frac{d^2\theta}{dt^2}$ is the measure of the effective couple.

Reversing this, it must be equal and opposite to the moment of mg round O.

Therefore $\quad m(k^2+h^2)\dfrac{d^2\theta}{dt^2}+mgh\sin\theta=0,$

$$\frac{d^2\theta}{dt^2}=-\frac{gh}{h^2+k^2}\cdot\sin\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Multiplying by $2\dfrac{d\theta}{dt}$ and integrating

$$\left(\frac{d\theta}{dt}\right)^2=C+\frac{2gh}{h^2+k^2}\cos\theta.$$

If the pendulum began to move when θ was equal to α,

$$C=-\frac{2gh}{h^2+k^2}\cos\alpha,$$

and
$$\left(\frac{d\theta}{dt}\right)^2=\frac{2gh}{h^2+k^2}(\cos\theta-\cos\alpha)\;\ldots\ldots\ldots\ldots (2).$$

This equation cannot in general be integrated farther. It is therefore not possible to find the position in terms of the time. Equation (2) enables us, what is very important, to find how far the pendulum will go with a given initial angular velocity; for α gives the position of instantaneous rest. If then the angular velocity is Ω when θ is zero,

$$\Omega^2=\frac{2gh}{k^2+h^2}\cdot(1-\cos\alpha).$$

If $\quad \alpha=\pi,\quad \Omega=2\sqrt{\dfrac{gh}{k^2+h^2}},$

this is the least angular velocity at the lowest point which will send the body right round.

If Ω is less than this, the motion will be one of oscillation about the lowest position.

3. The most important case is when the angle of oscillation is very small. Then $\frac{\sin\theta}{\theta}$ differs infinitely little from unity, and the equation (1) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{h^2+k^2}\theta.$$

The solution of this (Lesson XV.) is

$$\theta = A\cos\left(\sqrt{\frac{gh}{h^2+k^2}}\,.\,t + B\right),$$

and indicates an oscillation, called a simple harmonic motion, whose complete period is $2\pi\sqrt{\frac{h^2+k^2}{gh}}$.

If l be the length of a simple pendulum its equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\,.\,\theta.$$

If the times of oscillation of these are equal,

$$l = \frac{h^2+k^2}{h} = h + \frac{k^2}{h}.$$

If a length OO' equal to this be measured along OG, O' is the centre of oscillation. It is clear that $OG\,.\,O'G = k^2$. Hence O and O' are convertible. If the pendulum be hung up by O', O will be the centre of oscillation.

In the above work the assumption has been made that the solution of

$$\frac{d^2\theta}{dt^2} = a\theta + b\theta^2 + c\theta^3 + \ldots$$

differs infinitely little from that of

$$\frac{d^2\theta}{dt^2} = a\theta,$$

when θ becomes infinitely small.

The time of oscillation of a compound pendulum depends on $h + \frac{k^2}{h}$. In calculating the value of g from pendulum experiments, the main advantage is that the time of one oscillation can be very accurately measured. The difficulties are the determination of h and k. The point G cannot be got at, and as every body is more or less irregular and variable in density, k cannot be calculated with sufficient accuracy. These quantities must therefore be determined from experiments. Bessel observed the times of oscillation about different axes, the distances between which were very accurately known. Captain Kater employed the property of convertibility.

4. Another interesting application of the present problem is the old way of measuring the velocity of a bullet or cannon-ball. The ball was fired into a mass called Robins' ballistic pendulum; which was thereby set off with a certain angular velocity about its axis. The angle through which the mass ascended was found by the length of a piece of tape which was fastened to a point in the pendulum, and came through a slit immediately below the axis.

Hence by equation (2) the initial angular velocity was calculated. But, the mass of the cannon-ball being m, its velocity V, the common initial angular velocity of it and the mass (M) ω, the distance of its passage below the axis p, the angular momentum of the bullet about the axis before impact was mVp, and after it sticks in the mass, $mp^2\omega$. Now the angular momentum gained by the mass must be equal to the moment of the momentum about the axis lost by the bullet. For, considering the two as parts of one system, there is no external force that has any moment round the axis.

Hence $$mVp - mp^2\omega = M(k^2 + h^2)\,\omega.$$

From this V can be found.

Hutton used to suspend his cannon as a pendulum, and measure the angle through which it was raised by the discharge.

5. *A plane body at rest has a fixed point in it. It is struck by a blow in its own plane. How must this act that there may be no pressure on the fixed point?*

Let O be the fixed point, G the centre of inertia, mk^2 the moment of inertia about G, ω the angular velocity produced, P the blow, p the distance of its line of action from O.

The velocity of G is changed from zero to $OG \cdot \omega$, which requires a force $mOG \cdot \omega$ at G. And the angular momentum about G has been changed from zero to $mk^2\omega$, which requires a corresponding couple.

If there is no action on the fixed point, P is in equilibrium with a force $-m \cdot OG \cdot \omega$, and the couple $-mk^2\omega$. Hence P must act at right angles to OG, and

$$P = m \cdot OG \cdot \omega, \quad P(p - OG) = mk^2\omega,$$

whence
$$p = OG + \frac{k^2}{OG}.$$

The name 'centre of percussion' has been given to the point of action of P when there is no pressure on the axis. If there be no fixed point and if the blow act at this point, the point O will be the centre of spontaneous rotation.

6. *A body is moveable about a fixed point* O. *It begins to move with given angular acceleration* $\frac{d\omega}{dt}$ *about the line* Oz. *What couple has produced this motion?*

Let the magnitude of the couple be G, and let its axis make angles α, β, γ with the axes Ox, Oy, Oz.

The impressed forces are the couple G, and the action of O. We will take moments about axes through O, and so avoid this latter.

If x, y, z are the co-ordinates of an element δm, and r its distance from Oz; its accelerations are $\omega^2 . r$ towards Oz and $\frac{d\omega}{dt} r$ parallel to the plane xy, and perpendicular to r. Now in the very beginning of the motion ω is infinitely small, but $\frac{d\omega}{dt}$ is finite. Therefore the latter is the only acceleration that requires an effective force. This effective force $dm \frac{d\omega}{dt} r$ is equivalent to two, viz. $-\delta m \frac{d\omega}{dt} y$ parallel to Ox, and $dm \frac{d\omega}{dt} x$ parallel to Oy. (See fig. v. 5.)

Reversing these for all the elements, and taking moments about the axes,

$$G . \cos \alpha - \frac{d\omega}{dt} . \Sigma \delta mxz = 0,$$

$$G \cos \beta - \frac{d\omega}{dt} \Sigma \delta myz = 0,$$

$$G \cos \gamma - \frac{d\omega}{dt} \Sigma \delta mr^2 = 0.$$

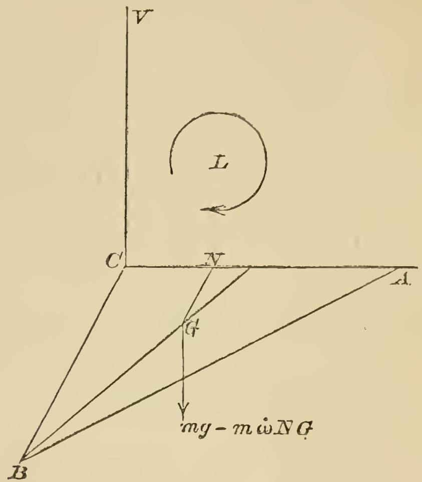
Whence the cosines of the angles which the axis of the required couple makes with the axes are proportional to the products and moment of inertia e, d, c (Lesson VIII.); and the moment of the couple is

$$\frac{d\omega}{dt} (e^2 + d^2 + c^2)^{\frac{1}{2}}.$$

The student of solid geometry will have no difficulty in proving that this axis is the radius of the momental ellipsoid of the body at O, diametral to the plane of the couple.

Similar reasoning holds in the case of an impulsive couple. So that in general a couple will tend to cause rotation round its own axis only when this is parallel to one of a set of principal axes.

7. *A uniform triangular plate* ACB *right-angled at* C, *is rotating about* CA *as a fixed horizontal axis. Find the wrench couple in the vertical plane through the axis when the plate comes to be horizontal in its descent.*



Let the angular velocity in that position be ω. Let G be the centre of gravity, GN a perpendicular to CA. Let m be the mass, and let a and b be the lengths of CB, CA. The moment of inertia about CA is, by Lesson VI, $m\frac{a^2}{6}$.

Hence the value of the angular acceleration $\frac{d\omega}{dt}$ in this position is, by Art. 2, $\frac{2g}{a}$.

The impressed forces are the weight and the actions of the axis. We are only concerned with that one of them which is a couple in the plane VCA. Call it L. The effective forces are $m\omega^2 NG$ along GN at G, and $m\frac{d\omega}{dt}NG$ vertically downwards at G, of which only the latter has a moment about BC; and a couple round an axis through G parallel to CA, which has no moment about BC.

Reversing these and taking moments about CB,

$$L + mg \,.\, CN - m \,.\, NG \frac{d\omega}{dt} \,.\, CN = 0,$$

and
$$CN = \frac{b}{3},$$

whence
$$L = -\frac{mgb}{9}.$$

8. *A uniform rod* OA, *of mass* m *and length* 2a, *swings as a pendulum about* O. *Find the components of stress at a point* P *distant* 2b *from* A, *when the rod has reached a position inclined at an angle* θ *to the vertical.*

We know from Art. 2 that the angular acceleration $\frac{d^2\theta}{dt^2}$ is equal to

$$-\frac{3g \sin\theta}{4a},$$

and that the square of the angular velocity $\left(\frac{d\theta}{dt}\right)^2$ is

$$\frac{3g}{2a}(\cos\theta - \cos\alpha),$$

in which α is the value of θ when the velocity is exhausted. Let us consider the part PA. The impressed forces are $m\frac{b}{a}.g$ acting at G, its centre of inertia; and the required action of OP, which we will take to be (1) a longitudinal stress T along PO, (2) a transverse stress S at right angles to PA; and a couple L.

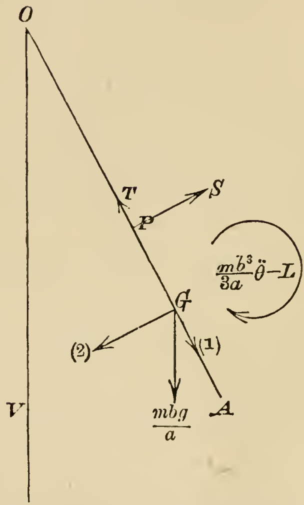
The accelerations of G are $OG \,.\left(\frac{d\theta}{dt}\right)^2$ along GO, and $OG \,.\frac{d^2\theta}{dt^2}$ at right angles to GO. Hence the reversed effective forces are

$$m\frac{b}{a} \,.\, (2a - b)\left(\frac{d\theta}{dt}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$m\frac{b}{a}.(2a-b)\frac{d^2\theta}{dt^2} \text{ .... } (2),$$

acting at G along and perpendicular to OG; and a couple

$$m\frac{b}{a}\cdot\frac{b^2}{3}\frac{d^2\theta}{dt^2}.$$



Resolving for PA along and perpendicular to PA,

$$T - m\frac{b}{a}g\cos\theta - \frac{mb}{a}(2a-b)\left(\frac{d\theta}{dt}\right)^2 = 0,$$

$$S - m\frac{b}{a}g\sin\theta - \frac{mb}{a}(2a-b)\frac{d^2\theta}{dt^2} = 0.$$

Taking moments about P,

$$L - m\frac{b^2}{a}g\sin\theta - \frac{mb^3}{3a}\frac{d^2\theta}{dt^2} - \frac{mb^2}{a}(2a-b)\frac{d^2\theta}{dt^2} = 0.$$

Substituting the values of

$$\frac{d^2\theta}{dt^2} \text{ and } \left(\frac{d\theta}{dt}\right)^2,$$

we have

$$T=\frac{mbg}{a}\left\{\cos\theta+\frac{(2a-b)}{a}\frac{3}{2}(\cos\theta-\cos\alpha)\right\},$$

$$S=\frac{mb}{a}g\sin\theta\left\{1-\frac{(2a-b)3}{4a}\right\}$$

$$=\frac{mb}{4a}g\sin\theta\left(\frac{3b}{a}-2\right),$$

$$L=\frac{mb^2}{2a}g\sin\theta\left(\frac{b}{a}-1\right).$$

EXAMPLES.

1. Explain why it is easier to support a long rod in a vertical position on the tip of the finger than a short one.

2. A uniform rod of length $2a$ can rotate about one end. It is allowed to fall from its position of unstable equilibrium. Prove that its angular velocity when it is horizontal is

$$\sqrt{\frac{3}{2}\cdot\frac{g}{a}}.$$

3. A bar magnet is suspended in such a manner that it can oscillate in a horizontal plane. Shew that its moment of inertia about the line of suspension can be determined by observing the times of oscillation with and without small pieces of lead attached to it.

4. A perfectly rough cube is placed on a horizontal plate. Determine the initial motion of the cube if the plate is made to begin to move with a given velocity, in a direction at right angles to one of the faces of the cube.

5. A rectangular uniform plate, moveable in every direction about its centre which is supported, rests horizontally; if a heavy adhesive particle be placed at one corner, prove that the rectangle will begin to move about that diagonal which does not pass through that corner.

6. Mersenne and others, in seeking for the centre of oscillation, assumed that it was the same as the centre of percussion. Why is this guess right?

7. A slender circular ring is cracked at one point, and is made to revolve in its own plane about the opposite point, with a constant angular velocity ω; shew that the tendency to break when greatest is measured by a couple whose moment is

$$\frac{ma^2\omega^2}{2\pi} \cdot \frac{\sin^2\phi}{\cos\phi},$$

where m is the mass, a the radius of the ring, and $a\phi$ is the arc contained between the fixed point and the point where the tendency to break is greatest.

8. A uniform semicircular arc of mass m and radius a is fixed at its ends to two points in the same vertical line, and is rotating with constant angular velocity ω. Prove that the horizontal pressure on the upper end is

$$\frac{m}{\pi}(g - \omega^2 a).$$

9. If the semicircle in the previous question had an angular acceleration $\dot{\omega}$, shew that there would be another horizontal pressure at the upper support at right angles to its plane and of magnitude

$$\frac{ma\dot{\omega}}{\pi}.$$

10. A uniform rod of length $2a$ is suspended by a point in its length distant h from the centre of gravity. Prove that the time of oscillation is a minimum when

$$h = \frac{a}{\sqrt{3}}.$$

11. A uniform rod of mass m makes small oscillations in a vertical plane inside of a smooth right circular cylinder of radius r, which is fixed with its axis horizontal. If the rod subtend an angle 2α at the centre of the cylinder, prove that the length of the simple equivalent pendulum is

$$r \cdot \frac{1 + 2\cos^2\alpha}{3\cos\alpha}.$$

12. A physical line fixed perpendicularly to an axis in its own plane revolves about that axis with a given velocity; find the pressure on the axis. Supposing the axis to be set free, about what axis will the line now move?

13. If a uniform rod capable of turning about one fixed extremity be struck by a given impulse at any point, find the point at which the tendency to snap in two is greatest.

14. A straight heavy rod oscillates about one end in a vertical plane, coming to rest in a horizontal position; shew that if ϕ be the angle included between the direction of the rod and the direction of the pressure upon the axis at any time, and θ be the inclination of the rod to the horizon at the same time, $\tan\phi \tan\theta = \frac{1}{10}$.

XI.

MOTION OF ONE BODY.

1. When the forces have been found which are in equilibrium with one another, we employ any statical method to obtain the equations of motion. Thus we resolve the forces along two directions at right angles, and take moments about a point. The resolutions present no difficulty. Taking moments round an axis means that we express the condition that the forces shall have no effect in producing rotation round this axis. The effect of a couple is the same about whatever point in its plane the motion be considered; in other words, a couple has always a certain absolute rotational effect. The effect of a force not passing through the axis is the moment of the force about the axis.

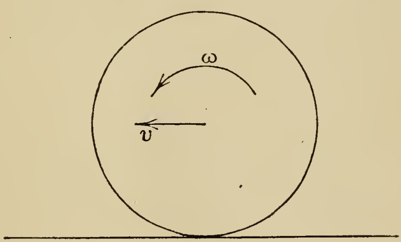
Now in the problems which are about to engage our attention, we shall find couples and moments mixed up. There will be effective forces at the centre of inertia, and an effective couple. There will be impressed forces acting at various points, and occasionally an impressed couple—for example, a tendency to break. Taking moments about any point will be expressing the condition that the algebraical sum of the couples together with the algebraical sum of the moments about the point of the unreduced forces is zero.

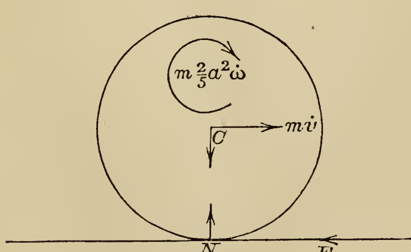
Or, if it seems convenient, each impressed force may be reduced to a parallel force acting at the point round which moments are to be taken and a couple. Taking moments will then be equating to zero the algebraical sum of all the couples.

Forces may be resolved in any convenient direction, and moments may be taken, not only about the centre of inertia, but about any convenient point. This direction and this point are fixed. There is no such thing in Dynamics as resolving along a moving direction, or as taking moments round a moving point. Force is absolute in its action. The point round which we take moments may be defined by a certain point of the moving system passing through it at the time, but it is itself fixed.

(mk^2 will always be used to denote the moment of inertia about an axis through the centre of inertia.)

2. *A sphere of mass* m, *and radius* a, *is set rotating with velocity* Ω *about a horizontal axis. It is then laid gently down on a horizontal plane whose coefficient of roughness is* μ, *and at the same time let go. Find the motion.*





The first question is, Will it roll or slide? Let v be the velocity of the centre, and ω the angular velocity.

The effective force is $m\frac{dv}{dt}$ parallel to the plane. The couple is $m\frac{2}{5}a^2\frac{d\omega}{dt}$.

The impressed forces are the friction F horizontally, the weight mg downwards, and the pressure of the plane R upwards. Reversing the effective forces, the equations of equilibrium become

$$m\frac{dv}{dt} = F,\ mg - R = 0,$$

$$m\frac{2}{5}a^2 . \frac{d\omega}{dt} = -Fa.$$

F as it tends to prevent the motion of the point of contact acts onwards. Hence it increases v and diminishes ω. Now v is at first zero, and ω is Ω, therefore v is at first less than $a\omega$, and there is slipping. During this motion,

$$F = \mu R = \mu mg.$$

Hence $$\frac{dv}{dt} = mg,\ v = \mu gt,$$

and the distance travelled in time t is $\frac{\mu gt^2}{2}$.

Also $$\frac{d\omega}{dt} = -\frac{5}{2}\frac{\mu g}{a};$$

$$\therefore\quad \Omega - \omega = \frac{5}{2}\frac{\mu gt}{a}.$$

When v becomes equal to $a\omega$, the problem changes. Complete rolling begins. F is no longer equal to μR, but instead, we have a geometrical equation $v = a\omega$,

whence $$\frac{dv}{dt} = a\frac{d\omega}{dt}.$$

The dynamical equations are the same.

From them $$\frac{dv}{dt} = -\frac{2}{5} a \frac{d\omega}{dt}.$$

Hence $\frac{dv}{dt}$ and $\frac{d\omega}{dt}$ are both zero. F is also zero; and the sphere rolls on with constant velocity. The above reasoning holds good for all circular bodies like wheels or barrels, for the only difference would be in the moment of inertia. That the constant velocity is never in practice attained, is due to the imperfect rigidity of the body, and to imperfect flatness and roughness of the plane.

It should be observed that, to determine the rolling motion, only one dynamical equation is necessary. For taking moments about N,

$$m \frac{2}{5} a^2 \frac{d\omega}{dt} + m \frac{dv}{dt} . a = 0,$$

an equation which results from the elimination of F from the former two.

3. *A uniform rigid circular hoop (mass* m, *radius* a), *cracked completely at one point* C, *is rolling on a rough horizontal plane. Find the breaking couple at the point* A *opposite* C, *when the diameter through* C *is inclined to the horizon at an angle* α.

The angular velocity is (by the last example) constant. Let this be ω. Consider the upper part of the hoop from C to A. Let G be its centre of inertia. Join G to O the centre of the circle. The part which we are considering is in equilibrium under its weight at G, the action of the other part at A, and the reversed effectives. The action at A is composite. Let the couple be called L. The effective forces are equivalent to a single force $\frac{m}{2} \omega^2 . OG$ from G to O, and a couple $\frac{m}{2} k^2 \frac{d\omega}{dt}$, in which k is the radius of gyration about G (Lesson IV., Art. 10). The couple vanishes because ω is constant. Reversing the force and taking moments about A,

$$L+\frac{m}{2}g\,.\,(a\cos\alpha+OG\sin\alpha)-\frac{m}{2}\omega^2\,.\,OG\,.\,a=0,$$

an equation for L.

4. *A square board whose mass is* m, *and whose length of side is* 2a, *is rotating freely about one diagonal with angular velocity* ω. *One end of the other diagonal is suddenly fixed. What will be the subsequent motion?*

It will clearly be a rotation about an axis through the fixed point parallel to the former axis of rotation. Let the new angular velocity be ω'. As the only impressed force is the impulse at the point which becomes fixed, the angular momentum about that point remains unchanged. Now the moment of inertia about a line through the centre parallel to either side is $m\frac{a^2}{3}$. This therefore is also the moment of inertia about the diagonal. Hence $m\frac{a^2}{3}\omega$ was the quantity of rotation, and was the angular momentum about any axis parallel to the diagonal. The angular momentum has become

$$m\left(\frac{a^2}{3}+2a^2\right)\omega'.$$

Equating these, $$\frac{\omega'}{\omega}=\frac{1}{7}.$$

Consider the question another way.

Let v' be the velocity of the centre after the impact. Let P be the force of the blow. The momentum of translation has been changed from zero to mv'. And the angular momentum about the centre of inertia has been changed by

$$m\frac{a^2}{3}(\omega'-\omega).$$

Hence P at the fixed point, $-mv'$ at the centre, and the couple $-m\frac{a^2}{3}(\omega' - \omega)$, are in equilibrium;

whence $$mv' = P,$$

$$m\frac{a^2}{3}(\omega' - \omega) + Pa\sqrt{2} = 0.$$

Also since the centre moves round the end of a diagonal,

$$v' = a\sqrt{2}\,.\,\omega'.$$

Eliminating v' and P we have the same result as before.

5. *A rough imperfectly elastic hoop is projected horizontally straight forwards from a man's hand, an underhand twist being given it so as to make it rotate about a horizontal axis. Prove that on striking the ground it will rebound vertically into the air, if the coefficient* e *of frictional elasticity be given by the equation* $\frac{1+e}{1-e} = \frac{v}{a\omega}$; a *being the radius of the hoop, and* v, ω *the linear and angular velocities of projection.*

Prove that if the hoop be perfectly elastic and the coefficient of frictional elasticity be $\frac{1}{3}$, *the hoop will rebound into the thrower's hand if* $a\omega = 2v$.

In this problem the vertical velocity does not come under consideration until the second part. The horizontal velocity and the angular velocity remain constant until the ground is touched. Then the point touching the ground gets stopped by friction. When its velocity is zero the friction has exhausted its force, but the friction of restitution begins, and finally, when the contact with the ground ceases, the horizontal velocity of the centre is known to be destroyed. There are therefore two periods and two motions.

At the end of the first period v is changed into an unknown velocity V, and ω into an unknown velocity Ω. But the velocity of N is zero;

$$\therefore \; V + a\Omega = 0 \text{ ....} (1).$$

If F be the friction, the reversed effective forces $m(V - v)$ and $mk^2(\Omega - \omega)$ in equilibrium with this give

$$F + m(V - v) = 0 \text{ ....} (2),$$

$$Fa + mk^2(\Omega - \omega) = 0 \text{ ....} (3).$$

In the second period the friction is Fe; the velocity V is changed to zero, and Ω to an unknown velocity ω'.

The equations will therefore be

$$Fe + m(O - V) = 0 \text{ ....} (4),$$

$$Fe \, . \, a + mk^2(\omega' - \Omega) = 0 \text{ ....} (5).$$

No. (5) is for our purpose useless; but from (1), (2), (3), (4) we deduce

$$\frac{1+e}{1-e} = \frac{v}{a\omega}, \text{ since } k = a.$$

The rule for elimination in all such cases is: by means of the geometrical relations find the value of F or other unknown quantities; then substitute these values and find the final motion.

In the second part the hoop is supposed perfectly elastic; its centre will then rebound with the vertical velocity with which it came down. If the horizontal velocity be exactly reversed, it will bound back the same way as it came. Hence equations (4) and (5) become

$$Fe + m(-v - V) = 0 \text{ ....} (6),$$

$$Fe \, . \, a + mk^2(\omega' - \Omega) = 0.$$

Putting $e = \frac{1}{3}$, $k = a$, and using (1), (2), (3), (6) we have the required result.

6. *A sphere of mass* m *and radius* a, *whose centre of inertia* G *is distant* b *from its centre of figure* C, *is placed upon a perfectly rough table. Find the time of an oscillation when the oscillations are very small.*

Here C is geometrically, G kinetically important.

Let the angle between CG and the vertical be θ. Then $\dot{\theta}$ measures the angular velocity, $\ddot{\theta}$ the angular acceleration of the body. Let v be the velocity of C. This is horizontal. Let it be measured in the direction corresponding to an increase of θ. Let mk^2 be the moment of inertia of the sphere about a horizontal axis through G.

The impressed forces are the weight, friction, and pressure.

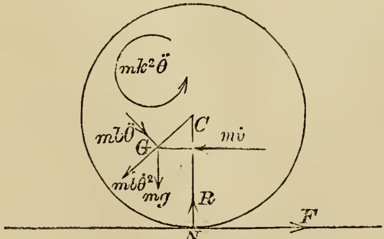
To get the effective forces. The acceleration of G is composed (1) of that relative to C, i.e. $\dot{\theta}^2 b$ towards C, and $\ddot{\theta} b$ at right angles to CG, and (2) of that of C, i.e. $\dot{v}$ horizontally.

The effective forces are therefore reducible to

$$m\dot{\theta}^2 b,\ m\ddot{\theta} b,\ m\dot{v} \text{ at } G,$$

and a couple $$mk^2\,\ddot{\theta}.$$

Reversed they form with the impressed forces the system in the figure.



Now as the motion is always very small, $\dot{\theta}$ may be neglected.

Taking moments about N to avoid F and R, we have

$$m\dot{v}(a - b\cos\theta) + mb\ddot{\theta}(b - a\cos\theta) + mk^2\ddot{\theta} + mgb\sin\theta = 0.$$

Also since there is perfect rolling,

$$v = a\dot{\theta} \text{ and } \dot{v} = a\ddot{\theta}.$$

And ultimately $\cos\theta = 1$, $\sin\theta = \theta$.

Hence $$(a^2 + k^2 + b^2 - 2ab)\ddot{\theta} = -bg\theta,$$

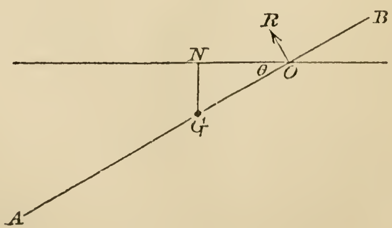
an equation which gives the time of a small oscillation.

7. We have seen that in cases of motion whose differential equations do not admit of complete solution, the time of oscillation can be found when the motion is small. In the same way when the motion does not remain small, initial circumstances of motion can be found; such as the values of unknown forces, the direction of motion of a point, or the curvature of its path at the commencement of the motion. Such problems are not of much physical interest. They are valuable as giving examples of successive approximation.

To find the initial value of a force is usually very simple.

A uniform rod AB, *of mass* m, *and length* 2a, *rests horizontally, being partly supported by a smooth peg* O. *On all support but that of* O *being withdrawn, find the pressure at* O *and approximate to the initial motion.*

Let $OG = r$, and let θ be the angle AB makes with the horizon. Let R be the pressure.



The general equations of motion are

$$\left.\begin{aligned} m\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} &= mg\sin\theta, \\ \frac{m}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= mg\cos\theta - R, \\ m\frac{a^2}{3}\cdot\frac{d^2\theta}{dt^2} &= R\,.\,r \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

On eliminating R, an equation presents itself which is once integrable and leads to

$$\left(\frac{dr}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\left(r^2 + \frac{a^2}{3}\right) = 2gr\sin\theta.$$

But no other equation involving velocities only can be got, and therefore R cannot in the general case be found since it depends on

$$\frac{dr}{dt} \text{ and } \frac{d\theta}{dt}.$$

But the initial value of R can be found. Putting for θ, $\frac{dr}{dt}$, $\frac{d\theta}{dt}$ zero, and for r its initial value r_0, we have initially

$$\left.\begin{aligned} \frac{d^2r}{dt^2} &= 0, \\ mr_0\frac{d^2\theta}{dt^2} &= mg - R, \\ m\frac{a^2}{3}\frac{d^2\theta}{dt^2} &= R\,.\,r_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2):$$

from which

$$R = \frac{mga^2}{a^2 + 3r_0^2} \text{ and } \frac{d^2\theta}{dt^2} = \frac{3gr_0}{a^2 + 3r_0}.$$

It is probably needless to remark that equations (2) are not differential equations, and cannot be integrated.

Let us now endeavour to find the initial radius of curvature of the path of G. This involves a closer approximation to the initial motion than was necessary in finding R. Quantities which were neglected in that operation must not be neglected.

G begins to move downwards. Hence the normal is horizontal. If ON, GN are x and y, the radius of curvature is initially $\frac{y^2}{2x}$. As this assumes the form $\frac{0}{0}$ it must be evaluated. Differentiating numerator and denominator and putting x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ zero,

$$\rho = \frac{3\left(\frac{d^2y_0}{dt^2}\right)^2}{\frac{d^4x_0}{dt^4}}.$$

Now either by differentiating

$$x = r\cos\theta,\ y = r\sin\theta,$$

or from the properties of relative accelerations,

$$\frac{d^2y}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\sin\theta + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\cos\theta,$$

$$\frac{d^2x}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\cos\theta - \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\sin\theta,$$

whence
$$\frac{d^2y_0}{dt^2} = r_0\frac{d^2\theta_0}{dt^2}.$$

In differentiating to find $\frac{d^4x_0}{dt^4}$, we may in the last differentiation neglect all vanishing terms, and in the next-last all squares and products of vanishing terms,

$$\frac{d^3x}{dt^3} = \cos\theta\left(\frac{d^3r}{dt^3} - 2r\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}\right)$$

$$-r\cdot\frac{d^2\theta}{dt^2}\cdot\cos\theta\frac{d\theta}{dt} - \sin\theta\left(r\frac{d^3\theta}{dt^3}\right),$$

$$\frac{d^4x_0}{dt^4} = \frac{d^4r_0}{dt^4} - 3r_0\left(\frac{d^2\theta_0}{dt^2}\right)^2;$$

$$\therefore\ \rho = \frac{3r_0^{\ 2}\left(\frac{d^2\theta_0}{dt^2}\right)^2}{\frac{d^4r_0}{dt^4} - 3r_0\left(\frac{d^2\theta_0}{dt^2}\right)^2}.$$

Now $$\frac{d^2\theta_0}{dt^2} = \frac{3g\,.\,r_0}{a^2 + 3r_0^{\ 2}},$$

and differentiating the equation

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = g\sin\theta,$$

$$\frac{d^3r}{dt^3} - 2r\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = g\cos\theta\frac{d\theta}{dt},$$

$$\frac{d^4r_0}{dt^4} - 2r_0\left(\frac{d^2\theta_0}{dt^2}\right)^2 = g\frac{d^2\theta_0}{dt^2};$$

$$\therefore\ \rho = \frac{3r_0^{\ 2}}{\frac{g}{\frac{d^2\theta_0}{dt^2}} - r_0} = \frac{3r_0^{\ 2}}{\frac{a^2 + 3r_0^{\ 2}}{3r_0} - r_0} = \frac{9r_0^{\ 3}}{a^2}.$$

To find the initial radius of curvature of a point when the system is set off with a finite velocity is simple enough; for

$$\rho = \frac{\text{sq. of velocity of point}}{\text{normal acceleration}},$$

and the normal acceleration is known when the impressed forces are known.

If, however, as in the above problem, the system starts from rest, the value of ρ assumes the form $\frac{0}{0}$, and when the direction of the normal is not known the evaluation is tedious.

PROBLEMS FOR SOLUTION.

1. A uniform rod AB is whirled away on the surface of smooth ice. Prove that the longitudinal stress at a given point P is constant throughout the motion, and proportional to $AP.PB$.

2. A uniform rod has a ring at one end, by which it slides on a smooth straight horizontal wire. If it starts from rest in any position in the vertical plane with the wire, find the motion and the supporting pressure of the wire, and prove that the other end of the rod moves in an ellipse.

3. A wheel in the form of a cylinder of radius R and thickness A has an axle of radius r and length a cut out of the same piece, the axes and centres of gravity being coincident. The whole is suspended with the axis horizontal by three vertical strings, one of which is coiled round the wheel and the other two round the axle at equal distances on either side of the wheel; prove that if the first string be drawn up or let down in any way the tensions of the other two will not be altered provided

$$\frac{a}{A} = \frac{R^3(R-2r)}{r^3(2R-r)} + 1.$$

4. A cylinder unrolls itself from a vertical string, the other end of which is fixed. Prove that the motion is uniformly accelerated.

5. A cube is rotating with angular velocity ω about a diagonal, when one of its edges which does not meet that diagonal suddenly becomes fixed; shew that the angular velocity about this edge as axis will be $\frac{\omega}{4\sqrt{3}}$.

6. The ends of a rod of length $2a$ are constrained to move on the smooth arc of a vertical circle of radius c. If the rod be displaced from its position of unstable equilibrium, find the breaking couple at any point in any position.

7. A uniform rigid bar, suspended at one end by a thread, rests on a perfectly smooth horizontal plane at a given angle with it; if the thread be cut, shew that the contact with the plane will be unbroken during the motion.

8. A circular ring hangs in a vertical plane on two pegs. If one be removed, prove that, P_1, P_2 being the instantaneous pressures on the other peg calculated on the supposition that the ring is (1) smooth, (2) rough,

$$P_1^2 : P_2^2 :: 1 : 1 + \frac{\tan^2 \alpha}{4},$$

where α is the angle which the line drawn from the centre to the peg makes with the vertical.

9. A uniform inelastic rod falls without rotation inclined at any angle to the horizon, and hits a smooth fixed peg at a distance from its upper end equal to one-third of its length. Prove that the lower end begins to descend vertically.

10. A rod of length $2a$ has a ring at one end which slides upon a smooth fixed horizontal rod. The former being initially vertical, an angular velocity ω is impressed upon it about the ring in the vertical plane containing the fixed rod; prove that the greatest angle it will make with the vertical is

$$2 \sin^{-1} \sqrt{\frac{a\omega^2}{12g}}.$$

11. A uniform inelastic rod, inclined at an angle θ with the vertical, falls without rotation and strikes a smooth hard horizontal plane. Shew that its centre of gravity immediately moves with a velocity

$$V . \frac{3 \sin^2 \theta}{1 + 3 \sin^2 \theta},$$

V being its previous velocity.

12. An inelastic ball of given radius is dropped from the window of a carriage travelling uniformly along a level road upon the wheel, which it hits at the highest point; determine the subsequent motion of the ball relatively to the carriage, the rim of the wheel being perfectly rough.

13. The end of a uniform rod of weight W can slide by a smooth ring on a vertical rod; the other end sliding on a smooth horizontal plane. The rod descends from a position inclined β to the horizon. Shew that the rod does not leave the plane during the descent, and that its minimum pressure on it is $\frac{W}{4}\cos^2\beta$.

14. A triangular lamina ABC is suspended horizontally by vertical strings attached to its angular points. If the strings at B and C be simultaneously cut, shew that there will be no instantaneous change of tension in the string at A, provided $AD = CD \, . \cos ADC$, D being the middle point of BC.

15. An imperfectly elastic sphere descending vertically comes in contact with a fixed rough point, the impact taking place at a point angularly distant α from the lowest point, and the coefficient of elasticity being e. Shew that it will commence moving off horizontally after the impact if

$$\tan^2\alpha = \frac{7e}{5}.$$

XII.

PROBLEMS.

A system of rigid bodies.

1. In these problems the expressions for the effective forces are written down for each body of the system separately. The equations of motion are always easily written down in whatever co-ordinates the changes of velocity are expressed. But their solution and the geometrical equations are much simplified by a judicious choice of variables.

As a general rule it is best to take co-ordinates which are all independent of one another. We get by this means the least possible number of variables, and so avoid having to differentiate geometrical equations.

Suppose for example that two spheres, A and B, were placed A above B on a plane and were disturbed, and that it was required to find the motion so long as they were in contact. We might take the co-ordinates of the centre of B as x_1, y_1, those of A as x_2, y_2, and denote their angles of rotation by other symbols. But it is clear that x_2, y_2 are not independent of x_1, y_1, but connected by the relation that the distance between the centres is constant. Hence it would be better to denote the co-ordinates of A's centre relatively to B's by r, θ. Then r is constant and no other geometrical equation is needed. The properties of relative accelerations enable us to express at once the absolute acceleration of A's centre in any direction.

It is not necessary nor even expedient, in drawing the necessary diagrams and finding the effective forces, to consider in what direction velocities actually are or actually increase. Measure them in the positive direction of the co-ordinates, and the result will shew by its sign in which direction they are, and in which they increase.

Forces of connection which are independent of one another and also of the position of the system (as rolling frictions and normal pressures, but not sliding frictions nor tensions of elastic strings), may be avoided when the motion only is required by grouping the various systems, so as to make these forces internal, and by resolving and taking moments in suitable ways. The equation arrived at by considering a whole system, is just the equation which would have been arrived at, had each body been separately considered and the mutual actions eliminated.

2. *A bullet of mass* m, *moving with velocity* V, *strikes perpendicularly at the centre a uniform rectangular door of mass* M *and breadth* 2a. *If the bullet sticks, find the angular velocity of the door.*

Let this be ω, and let the measure of the blow on the door be P. Then the blow suffered by the bullet is $-P$.

Consider (1) the bullet. This has its velocity changed from V to $a\omega$. The force necessary to do this is $m(a\omega - V)$. This reversed is in equilibrium with $-P$;

$$\therefore \quad P + m(a\omega - V) = 0.$$

Consider (2) the door.

The moment of P round the line of hinges generates a quantity of angular momentum $M\frac{4a^2}{3}\omega$. Hence the couples Pa and $-M\frac{4a^2}{3}\omega$ are in equilibrium;

$$\therefore \; P - M\frac{4a}{3}\omega = 0.$$

Eliminating P, we have

$$3m(a\omega - V) + 4aM\omega = 0;$$

$$\therefore \omega = \frac{3mV}{a(3m+4M)}.$$

For example, $m = 1$ ounce, $M = 200$ lbs., $2a = 3$ feet, $V = 500$ feet per second,

$$\omega = \frac{1500}{\frac{3}{2}(3 + 4 \,.\, 200 \,.\, 16)} = \frac{1000}{12803} = \cdot 078 \text{ nearly.}$$

As the unit of angular velocity is, when the unit of circular measure is described in one second, the door describes

$$\cdot 078 \text{ of } 57^\circ . 29,$$

or about $4\frac{1}{2}^\circ$ per second.

(α) Solve the problem by the principle that the whole angular momentum about the line of hinges is not changed by the impact.

(β) Find the resultant impulsive pressure on the line of hinges, by taking moments for the door about the centre, and then substituting the value of ω.

$$\left(\text{Answer } \frac{mMV}{3m+4M}\right).$$

(γ) Find this pressure by considering the two together, and resolving at right angles to the plane of the door.

3. *A uniform cylinder of mass* M *and radius* a, *has a hollow of any form in it filled with fluid of the same density and of mass* m. *The cylinder being allowed to roll down a perfectly rough plane inclined at an angle* α *to the horizon with its axis horizontal, find the motion.*

Here the fluid is supposed not to rotate. As far as translation is concerned the whole is one, but the moment of inertia must be calculated for the solid part alone.

Take a section of the cylinder at right angles to its axis. Let C be the centre, which is also the common centre of inertia of the solid and fluid. Let v be the velocity of C, and ω the angular velocity of the cylinder. Then $v = a\omega$ and

$$\frac{dv}{dt} = a\frac{d\omega}{dt}.$$

The effective forces of translation are reducible to one,

$$(M+m)\frac{dv}{dt},$$

acting at C parallel to the incline. The couple is due to the motion of the solid part alone, and is therefore equivalent to $MK^2\frac{d\omega}{dt}$, MK^2 being the moment of inertia about the centre of inertia of the solid.

To calculate this, call this point G; and suppose the whole cylinder solid. Its moment of inertia round C would be $\frac{1}{2}(M+m)\,a^2$. But this is made up of MK^2 round G, and of the moment of inertia of the part which in our problem is replaced by fluid. Let its centre of gravity be G', and its moment of inertia about it mk^2. Then

$$\frac{1}{2}(M+m)a^2 = M(K^2+CG^2)+m\,(k^2+CG'^2).$$

Suppose MK^2 found from this equation.

The impressed forces are the weight $(M+m)\,g$ at C, and the friction and pressure at the point of contact.

Reversing the effectives and taking moments about this point;

$$(M+m)\,a\frac{dv}{dt}+MK^2\frac{d\omega}{dt}-(M+m)\,ga\,.\sin\alpha=0.$$

Since $\frac{dv}{dt}=a\frac{d\omega}{dt}$, the motion is uniformly accelerated.

(α) Prove that the cylinder will roll down faster than if it had been solid.

(β) Calculate MK^2, supposing the hollow a concentric cylinder of radius b.

$$\text{Ans.}\quad M.\frac{a^2+b^2}{2}.$$

(γ) Prove that the friction acts up the plane and is constant.

$$\left\{\text{It is}\quad (M+m)\left(g\sin\alpha-\frac{dv}{dt}\right)\right\}.$$

(δ) Prove that the normal pressure is constant. Supposing the liquid to be in an excentric cavity, would it tend to cause the cylinder to jump off the plane?

Would it have this tendency if it were of different density from the solid?

4. *A smooth circular tube* (*mass* M) *has a particle* (*mass* m) *inside of it, and is set in motion in any manner with every point touching a smooth inclined plane. Prove that the particle will move with constant velocity round the centre of the tube, and that if* ω *be this angular velocity and* a *the radius of the tube, the pressure of the particle on the tube is constant and equal to*

$$\frac{\mathrm{Mma}\omega^2}{\mathrm{M+m}}.$$

The impressed forces on the tube are its own weight and the normal pressure of the particle. These have no moment round the centre, and therefore the angular velocity of the tube remains constant. Let ω be the angular velocity of the particle round the centre of (not relatively to) the tube. Let $\dot{u}$, $\dot{v}$ be the accelerations of the centre in and at right angles to the direction of the particle at any moment. Also let λg, μg be the component accelerations in these directions of the force of gravity on a unit mass.

Then the accelerations of the particle are

$$\dot{u} - a\omega^2 \text{ and } \dot{v} + a\dot{\omega}.$$

Let R be the normal pressure, inwards on the particle, outwards on the tube. Then the tube would be in equilibrium under

$$M\lambda g,\ R \text{ and } M\dot{u} \text{ reversed,}$$

and $$M\mu g, \quad \text{and } M\dot{v} \quad \ldots\ldots$$

The particle would be in equilibrium under

$$m\lambda g,\ -R \text{ and } m(\dot{u} - a\omega^2) \text{ reversed,}$$

and $$m\mu g, \quad \text{and } m(\dot{v} + a\dot{\omega}) \quad \ldots\ldots$$

Resolving $$\mu g - \dot{v} = 0,$$

$$\mu g - \dot{v} - a\dot{\omega} = 0;$$

whence $\dot{\omega} = 0$, or the particle moves round the tube with constant velocity.

Resolving again, $$M\lambda g + R - M\dot{u} = 0,$$

and $$m\lambda g - R - m(\dot{u} - a\omega^2) = 0.$$

Whence, $$R\left(\frac{1}{M} + \frac{1}{m}\right) = a\omega^2.$$

(α) Supposing Ω the angular velocity of the tube, what will be the time in which the particle will come round to the same point of the tube again?

(β) Shew from first principles that gravity has no effect in altering the relative motion of the tube and particle, or the mutual pressure.

(γ) In what path does the common centre of inertia move?

(δ) About what point is the whole angular momentum constant?

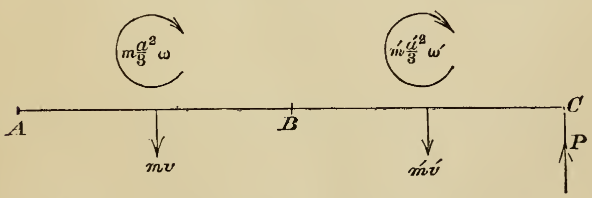
(ϵ) Prove by means of this principle that the angular velocity of the particle is constant, assuming that the angular velocity of the tube is constant.

5. *Two uniform rods* AB, BC, *of masses* m, m′, *and lengths* 2a, 2a′, *are connected by a joint at* B, *and are lying in a straight line. A blow* P *is struck at* C, *in a direction perpendicular to* ABC. *With what velocities will the system begin to move?*

The immediate effect of P upon BC will be to make its centre move off in a direction parallel to that of P, and to make it rotate. Hence the point B must begin to move at right angles to BC. Hence the action at B between the rods will at first be in this direction, and in general there will be no motion nor force at first along ABC.

Let v, v' be the velocities with which the centres of inertia of AB, BC move off. Let ω, ω' be the initial angular velocities measured in any but the same direction. The impressed forces are P and the action at B. This latter we will avoid. The effective forces are mv and $m'v'$ at the centres of inertia, and couples $m\frac{a^2}{3}\omega$, $m'\frac{a'^2}{3}\omega'$ on AB, BC respectively. Reversing these, as in the figure, and resolving for the whole system:

$$mv + m'v' - P = 0 \text{ .... } (1).$$



Taking moments for AB round B,

$$mv \,.\, a - m\frac{a^2}{3}\omega = 0 \text{ .... } (2).$$

Taking moments for BC round B,

$$P \,.\, 2a' - m'v' \,.\, a' - m'\frac{a'^2}{3}\omega' = 0 \text{ .... } (3).$$

Also, since B is common to AB and BC,

$$v' - a'\omega' = v + a\omega \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

These are four simple equations to determine the unknown quantities v, v', ω, ω'.

These are probably the most convenient equations. There are many others equivalent to the above, which might have been chosen. Thus, taking moments for the whole about C,

$$m\frac{a^2}{3}\omega - mv(2a' + a) + m'\frac{a'^2}{3}\omega' - m'v'a = 0.$$

Supposing the action of BC on AC to be a force X at right angles to AB; we have, on resolving for AB at right angles to AB,

$$X - mv = 0.$$

Considering BC, and resolving,

$$P - X - m'v' = 0,$$

which two are equivalent to (1).

Taking moments for AB about its centre of inertia,

$$X - m\frac{a^2}{3}\omega = 0,$$

and so on.

(α) Justify, by writing down equations of motion, the assumption that there is no force nor motion along ABC.

(β) Prove that AB begins to turn about a point one-third of its length from A.

(γ) Solve this problem by the statical method of virtual velocities. (See Art. 7.)

6. *A little squirrel clings to a thin rough hoop, the plane of which is vertical, and which is rolling along a perfectly rough horizontal plane. The squirrel makes a point of keeping at a constant height above the plane, and selects his place*

on the hoop so as to travel from a position of instantaneous rest the greatest possible distance in a given time. Prove that, m *being the mass of the squirrel and* M *that of the hoop, the inclination of the squirrel's distance from the centre of the hoop to the vertical is equal to*

$$\cos^{-1}\frac{m}{m+2M}.$$

Here the impressed forces are the weights, the friction, and the pressure of the ground. There are also the forces by which the squirrel clings to the hoop; but we will take both as one system, and so these will be internal.

The linear accelerations of the squirrel and hoop are the same. Let this be $\dot{v}$. If the angular acceleration of the hoop be $\dot{\omega}$, the effective forces are

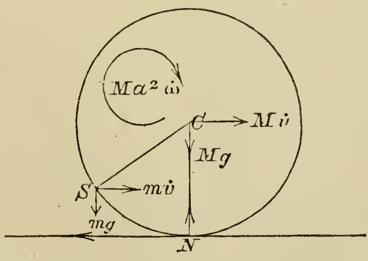
$$m\dot{v} \text{ at } S, \quad M\dot{v} \text{ at } C,$$

and a couple $Ma^2\dot{\omega}$,

(a being the radius of the hoop).

Reversing these, and taking moments for the whole system about N,

$$mg\,a\sin C - m\dot{v}\,(a - a\cos C) - M\dot{v}\,a - Ma^2\dot{\omega} = 0.$$



Also, since there is perfect rolling,

$$\dot{v} = a\dot{\omega};$$

$$\therefore\ \dot{v}\,\{m(1-\cos C) + 2M\} = mg\sin C.$$

Hence the acceleration is constant while the squirrel keeps to the same place. And therefore the greatest possible distance will correspond to the greatest possible acceleration.

Now $\dot{v}$ is to be made a maximum by the variation of C. Hence, differentiating

$$\frac{\sin C}{2M + m - m\cos C}$$

with respect to C, and equating the result to zero, we have

$$\cos C = \frac{m}{m + 2M}.$$

We will write down some equations of motion by which the problem might have been solved. Suppose F and R the friction and pressure at N. Let T and Q be the tangential and normal forces of the hoop on the squirrel, and reversed of the squirrel on the hoop.

Considering the squirrel, we have, by resolving along the radius CS and perpendicular to it,

$$mg\cos C - Q - m\dot{v}\sin C = 0,$$

$$-mg\sin C + T - m\dot{v}\cos C = 0.$$

Consider the hoop. Resolving horizontally and vertically, and taking moments about C,

$$F + Q\sin C - T\cos C - M\dot{v} = 0,$$

$$R - Mg - Q\cos C - T\sin C = 0,$$

$$Fa - Ta + Ma^2\dot{\omega} = 0.$$

It is evident that on eliminating F from the last and first of this last set we shall have the equation of moments for the hoop alone about N; and that on eliminating F, Q, T we shall have the single equation in the former solution.

It is also clear that F, R, T and Q can be found in terms of the accelerations from the above equations.

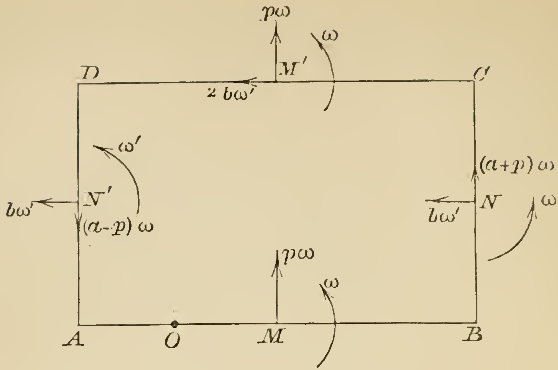
In general, by writing down the equations of motion for each body and the connecting equations, we obtain sufficient equations to determine the motion and the forces of connection.

7. *A rectangle is formed of four uniform rods of lengths* 2a *and* 2b, *which are connected by hinges at their ends. The rectangle is revolving about its centre on a smooth horizontal plane with an angular velocity* n, *when a point in one of the sides of length* 2a *suddenly becomes fixed. Shew that the angular velocity of the side of length* 2b *becomes immediately*

$$\frac{3a+b}{6a+4b} \cdot n.$$

It was mentioned that any convenient statical method might be applied to the solution of problems on motion. We shall for this problem use the method of virtual velocities. The virtual moment of a force is the force multiplied by the displacement in its direction of the point of application. The virtual moment of a couple is the sum of the virtual moments of the two parallel forces which compose it. This is easily seen to be the product of the measure of the couple and the angular displacement of the body on which it acts.

The rectangle has been revolving about the centre with velocity n; hence the middle points of the rods have been moving with velocities bn, an, and the rods have been rotating about their centres with angular velocity n. The point O becomes fixed. Opposite sides, as AB and CD, will still remain parallel; and, since they must make equal angles with any direction, their angular velocities must be equal. Let them be ω. Let the angular velocity of BC and AD be ω'. Then the velocities may be all expressed in terms of ω and ω'. For let M, N, M', N' be the middle points of the sides, and let OM be called p. Then the velocity of B is $(a+p)\,\omega$. That of N is the same as that of B in the direction of BN, and is $b\omega'$ in the perpendicular direction. And so on, as represented.



Now subtracting from these the previous velocities, we have the changes in the velocities. These will be:

for AB, $p\omega,\ -bn,\ \omega - n;$

for BC, $(a+p)\omega - an,\ b\omega',\ \omega' - n;$

for CD, $p\omega,\ 2b\omega' - bn,\ \omega - n;$

for DA, $(a-p)\omega - an,\ b\omega',\ \omega' - n.$

If we multiply the changes in the linear velocities by the masses, which are proportional to $2a$, $2b$, and the changes in the angular velocities by $2a \cdot \frac{a^2}{3}$, $2b \cdot \frac{b^2}{3}$, and reverse them, we shall have a system in equilibrium with the one impressed force, the impulse at O.

These reversed forces will then be (as to magnitude):

on AB,

$2ap\omega$, $2abn$, at M and a couple, $2a \cdot \frac{a^2}{3} \cdot (\omega - n)$;

on BC,

$2(a+p)b\omega - 2abn$, $2b^2\omega'$, at N and a couple, $2b \cdot \frac{b^2}{3} \cdot (\omega' - n)$;

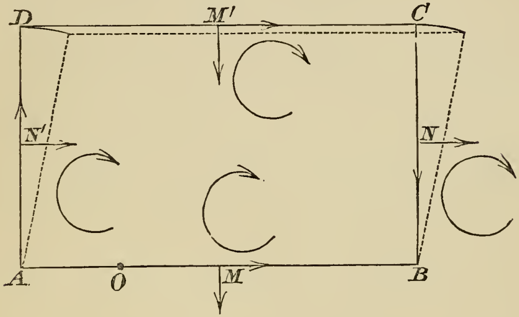
on CD,

$2ap\omega$, $4ab\omega' - 2abn$, at M' and a couple, $2a \cdot \frac{a^2}{3} \cdot (\omega - n)$;

on DA,

$2(a-p)b\omega - 2abn$, $2b^2\omega'$, at N' and a couple, $2b \cdot \frac{b^2}{3}(\omega' - n)$;

and will act in the directions indicated.



Now the problem proposed is to find ω'. Let us suppose a small angular displacement $\delta\theta$ given to the system in its own plane, keeping AB fixed. The angular displacement of DC will be zero. The displacements of N and N' will be $b\delta\theta$, and that of M' will be $2b\delta\theta$. The virtual moments of the couples on AB and DC vanish.

The virtual moment of the forces at M also vanish, and that of the forces parallel to CB. And the equation of virtual moments becomes

$$2b^2\omega' \cdot b\delta\theta + (4ab\omega' - 2abn)\,2b\delta\theta + 2b^2\omega' \cdot b\delta\theta + \frac{4b^3}{3}(\omega' - n)\,\delta\theta + \frac{4b^3}{3}(\omega' - n)\,\delta\theta = 0;$$

whence

$$\omega'\left(b + 4a + b + \frac{1}{3}b + \frac{1}{3}b\right) = n\left(2a + \frac{1}{3}b + \frac{1}{3}b\right);$$

or
$$\omega'(8b + 12a) = n(6a + 2b),$$

or
$$\omega' = n \cdot \frac{3a+b}{6a+4b}.$$

(α) What displacement must be given to obtain an equation involving only ω?

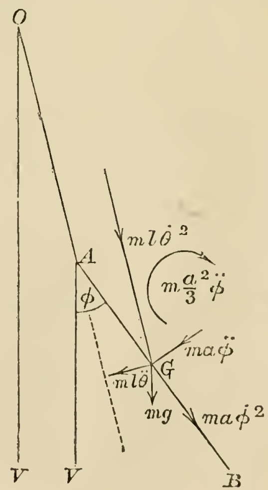
(β) Write down the equation obtained by keeping the point O and a point on DC opposite to O fixed, and giving an angular displacement about these.

(γ) Prove by giving the whole a displacement parallel to AB, that the component of the impulse at O along AB is

$$4b(a+b)\,\omega'.$$

8. *A pendulum consists of a uniform rod* AB, *of mass* m *and length* 2a, *attached by a string of length* l *at* A *to a fixed point* O. *It makes small oscillations in one plane. Find their law.*

Let G be the centre of inertia of AB; and denote by θ and ϕ the angles which OA and AB make with the vertical.



The accelerations of A are

$$l\left(\frac{d\theta}{dt}\right)^2 \text{ along } AO,$$

$$\text{and } l\frac{d^2\theta}{dt^2} \text{ perpendicular to } OA;$$

therefore the accelerations of G are those in these directions, together with the accelerations relatively to A, which are

$$a\left(\frac{d\phi}{dt}\right)^2 \text{ along } GA, \text{ and } a\frac{d^2\phi}{dt^2} \text{ perpendicular to } AB.$$

The angular acceleration is $\dfrac{d^2\phi}{dt^2}$.

The impressed forces are the weight mg at G, and the tension of OA. The reversed effective forces will act as in the figure. We can get two equations without the unknown tension.

1st. Take moments about A,

$$m\,.\,l\left(\frac{d\theta}{dt}\right)^2.\,a\,.\sin(\phi-\theta)+ml\frac{d^2\theta}{dt^2}a\cos(\phi-\theta)$$

$$+ma\frac{d^2\phi}{dt^2}.\,a+mk^2\frac{d^2\phi}{dt^2}+mga\sin\phi=0.$$

2nd. Resolve perpendicular to OA,

$$ml\frac{d^2\theta}{dt^2}-ma\left(\frac{d\phi}{dt}\right)^2\sin(\phi-\theta)$$

$$+ma\frac{d^2\phi}{dt^2}.\cos(\phi-\theta)+mg\sin\theta=0.$$

We can solve these equations if we assume that the oscillations are small, so that the velocities vanish while the accelerations do not. Making $\dfrac{d\phi}{dt}$, $\dfrac{d\theta}{dt}$ zero; and putting for $\cos\phi$, $\cos\theta$ the value 1, for $\sin\phi$, $\sin\theta$ the values ϕ, θ,

$$\left.\begin{aligned} al\frac{d^2\theta}{dt^2}+\frac{d^2\phi}{dt^2}(a^2+k^2) &= -ag\phi, \\ l\frac{d^2\theta}{dt^2}+a\frac{d^2\phi}{dt^2} &= -g\theta. \end{aligned}\right\}$$

A solution of these is clearly (Lesson XV.)

$$\theta=A\sin(nt+B),$$

$$\phi=A'\sin(nt+B).$$

To find n we have

$$\left.\begin{array}{l} -aln^2A-(a^2+k^2)\,n^2A'=-agA' \\ -ln^2A-an^2A'=-gA \end{array}\right\}.$$

Whence $\qquad a^2ln^4=\{ag-(a^2+k^2)\,n^2\}(g-ln^2).$

If the roots of this be $n_1^{\,2}$ and $n_2^{\,2}$, and the values of the ratio $\dfrac{A'}{A}$ corresponding to these be μ_1 and μ_2, the solution will be

$$\theta=A_1\sin(n_1t+B_1)+A_2\sin(n_2t+B_2),$$
$$\phi=\mu_1A_1\sin(n_1t+B_1)+\mu_2A_2\sin(n_2t+B_2).$$

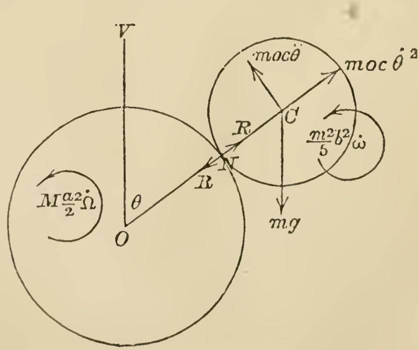
B_1, B_2 are to be determined by the circumstances of the motion.

These equations indicate a double oscillation. The two parts are independent and co-existent.

(α) Obtain these differential equations by taking x, y as the vertical and horizontal co-ordinates of G, and $\ddot{x}$, $\ddot{y}$ as its accelerations, and then differentiating the geometrical equations.

(β) Obtain them by the method of virtual velocities.

9. *A cylinder of mass* M *and radius* a *can rotate about its axis which is horizontal and fixed. A rough sphere of mass* m *and radius* b *is placed on its top and disturbed from*



its position of unstable equilibrium. Find in what position it will leave the cylinder.

Let θ be the angle the line of centres makes with the vertical. Let ω and Ω denote the angular velocities of the sphere and cylinder respectively. The accelerations of C are

$$(a+b)\,\dot{\theta}^2, \text{ and } (a+b)\,\ddot{\theta},$$

along and perpendicular to CO respectively. The angular accelerations will be $\dot{\omega}$ and $\dot{\Omega}$ respectively. Thus the effective forces reversed will form a system as in the figure.

The impressed forces are the weights at C and O, the pressure of the axle at O, and the mutual friction and normal pressure (R) at M. We will avoid the action at O and the friction. The pressure R is necessary; for the instant when it vanishes is the instant of separation.

Considering the sphere and resolving along OC,

$$m\,(a+b)\,\dot{\theta}^2 + R - mg\cos\theta = 0\text{............}(1);$$

taking moments about N,

$$m\,(a+b)\,\ddot{\theta}b + m\,\frac{2}{5}\,b^2\,\dot{\omega} - mgb\sin\theta = 0 \text{ ......}(2).$$

Considering both the sphere and cylinder, and taking moments about O,

$$m\,(a+b)^2\,\ddot{\theta} + m\,\frac{2}{5}\,b^2\,\dot{\omega} + M\,\frac{a^2}{2}\,\dot{\Omega} - mg\,(a+b)\sin\theta = 0\text{....}(3).$$

We have also a geometrical equation since there is no sliding and the points of the two bodies at N are moving together;

$$(a+b)\,\dot{\theta} - b\omega = a\Omega\text{..................}(4).$$

From the last three equations $\dot{\theta}$ has to be found, and R may then be found by the first. Differentiating (4), and substituting for $\dot{\Omega}$ in (3), we have

$$\ddot{\theta}\left\{m\,(a+b)^2 + \frac{Ma}{2}\,(a+b)\right\} + \dot{\omega}\left\{\frac{2}{5}\,mb^2 - \frac{abM}{2}\right\} = mg\,(a+b)\sin\theta;$$

eliminating $\dot{\omega}$ between this and (2),

$$\ddot{\theta}\,(a+b)\,(7M+4m)=g\sin\theta\,(4m+5M),$$

whence multiplying by $2\dot{\theta}$ and integrating,

$$\dot{\theta}^2\,(a+b)\,(7M+4m)=C-2g\cos\theta\,(5M+4m).$$

Now at the beginning of the motion $\dot{\theta}$ was zero, and $\cos\theta$ was 1;

$$\therefore\ C=2g\,(5M+4m),$$

$$\dot{\theta}^2=\frac{2g\,(1-\cos\theta)}{(a+b)}\cdot\frac{5M+4m}{7M+4m}.$$

Substituting this in (1) and making R zero, the equation which gives θ when the bodies separate is

$$2g\,(1-\cos\theta)\frac{5M+4m}{7M+4m}=g\cos\theta,$$

whence $$\cos\theta=\frac{10M+8m}{17M+12m}.$$

(α) Calling F the friction at N, acting at right angles to OC, but upwards on the sphere and downwards on the cylinder; prove that

$$F=\frac{Ma}{2}\dot{\Omega},$$

$$F-mg\sin\theta+m\,.\,(a+b)\,\ddot{\theta}=0,$$

$$F=\frac{2m}{5}b\dot{\omega}.$$

(β) What forces would be introduced if we resolved for the whole system at right angles to OC?

(γ) What forces would be introduced if we resolved for the cylinder along ON?

10. *A string without weight is coiled round a rough horizontal solid cylinder, of which the mass is* M *and radius* a, *and which is capable of turning about its axis. To the free end of the string is attached a chain of mass* m *and length* l; *if the chain be gathered up close and then let go, prove*

that, if θ be the angle through which the cylinder has turned after a time t, *before the chain is fully stretched,*

$$Ma\theta = \frac{m}{l}\left(\frac{gt^2}{2} - a\theta\right)^2.$$

This problem illustrates the fact that an infinite number of infinitely small blows has the effect of a continuous force. The uncoiled part of the chain and the cylinder are moving with finite acceleration, and the coil is falling like a free heavy body. But at every instant a link passes from the coil to the straight part, and so has its velocity instantaneously changed by a finite amount. There is therefore an impulsive tension on the uncoiled part as often as a link is added to it. This impulsive tension is the opposite of that which changes the motion of the link, and if the whole be taken as one system it will be internal and disappear.

The whole increase of angular momentum about the axis of the cylinder is caused by the weight mg.

But the rate of change of angular momentum is the measure of the rotational effect of a force. The statement of this is what is meant by taking moments about a certain axis. Hence the rate of increase of angular momentum about the axis is mga.

The rate of increase of the angular momentum of the cylinder is

$$\frac{Ma^2}{2} \cdot \frac{d^2\theta}{dt^2}.$$

Let z be the length of the uncoiled part of the chain at any time. Then its velocity is $a\frac{d\theta}{dt}$, and the rate of increase of the moment of its momentum about the axis is

$$\frac{d}{dt}\left(\frac{mz}{l} a^2 \frac{d\theta}{dt}\right),$$

for z is variable as well as θ.

The coil moves with velocity gt, and its mass is $m\frac{l-z}{l}$. Hence the rate of change of its moment of momentum is

$$\frac{d}{dt}\left(m\frac{l-z}{l} \cdot gta^2\right).$$

The sum of all these is mga,

$$\frac{Ma^2}{2}\frac{d^2\theta}{dt^2} + \frac{ma}{l}\frac{d}{dt}\left(z\frac{d\theta}{dt}\right) + \frac{m}{l}ga\frac{d}{at}(\overline{l-zt}) = mga.$$

Integrating once

$$\frac{d\theta}{dt} \cdot \frac{Ma}{2} + \frac{ma}{l}z\frac{d\theta}{dt} + \frac{m}{l}(l-z)\,gt = \frac{mg\,.\,t}{a} \;\ldots\ldots\; (1).$$

Now since the coil is moving with velocity gt, and the straight part with velocity $a\frac{d\theta}{dt}$, the rate of uncoiling, $\frac{dz}{dt}$, is equal to $gt - a\frac{d\theta}{dt}$, and therefore $z = \frac{1}{2}gt^2 - a\theta$, since z, θ and t all begin from zero together.

From (1), $$\frac{d\theta}{dt}\frac{Mal}{2m} = z\left(gt - a\frac{d\theta}{dt}\right),$$

but $$gt - a\frac{d\theta}{dt} = \frac{dz}{dt};$$

$$\therefore\;\frac{d\theta}{dt}\frac{Mal}{m} = 2z\,.\,\frac{dz}{dt}.$$

Integrating $$Ma\theta = \frac{m}{l}\,.\,z^2.$$

(α) Shew that the impulsive tension caused by the uncoiling of a link δz is $\frac{m\delta z}{l}\left(gt - a\frac{d\theta}{dt}\right)$.

(β) Shew that the finite tension due to the weight of the straight part is $\frac{mMzg}{Ml+2mz}$.

(γ) Compare these.

PROBLEMS.

1. Two infinitely rough wheels revolving uniformly in the same plane are suddenly brought into contact and their axes are kept fixed; determine what changes are made in the angular velocities.

2. Two particles of any elasticities and of masses m and m', joined by an elastic string, are placed in a vertical line; the string is stretched and they are simultaneously let go. Prove that whenever m comes to rest, m' is moving with a velocity $gt\dfrac{m+m'}{m'}$.

3. A string has two particles m and m' attached to its ends. The mass m' lies on a smooth horizontal table, and m is held so that the string is horizontal with a length a beyond the edge of the table. If m be let drop, prove that the initial radius of curvature of its path will be $a\left(1+\dfrac{m}{m'}\right)$.

4. A little animal, the mass of which is m, is resting on the middle point of a thin uniform bar, the mass of which is m' and the length $2a$, the ends of the bar being attached by small rings to two smooth fixed rods at right angles to each other in a horizontal plane. Supposing the animal to start off along the bar with a relative velocity V, prove that, θ being the inclination of the bar to either rod, the angular velocity initially impressed upon the bar will be

$$\frac{3m}{3m+4m'}\cdot\frac{V\sin 2\theta}{a}.$$

5. A thin hollow smooth ring (mass M and radius a), of which the plane is vertical, and which contains a bead of mass m, is placed upon a smooth horizontal plane. Prove that the bead, having been placed near the lowest point of the ring, will oscillate synchronously with a pendulum, the length of which is $\dfrac{Ma}{m+M}$.

6. Two equal heavy spheres, one solid, the other hollow, and the hollow filled with fluid, are rotating with the same angular velocity about a horizontal axis, and are laid side by side on a rough horizontal plane, the coefficient of friction for both being μ; if the interior radius of the sphere be one-half of the exterior, and the density of the fluid be equal to that of the solid, find the distance between them at any time, supposing that they move in parallel lines. (*Result.* Let Ω be the initial angular velocity, a the radius of each. Then the distance which the solid sphere has covered is $\frac{\mu g t^2}{2}$ before perfect rolling begins, and $\mu g t_1\left(t - \frac{t_1}{2}\right)$ afterwards, where $t_1 = \frac{2}{7} \cdot \frac{a\Omega}{\mu g}$. The same results hold for the other, only

$$t_1 = \frac{111}{31} \cdot \frac{a\Omega}{\mu g}.)$$

7. A cylinder rolls down the rough upper face of a wedge which is capable of moving on a smooth horizontal table; prove that the accelerations are uniform.

8. An iceberg floats without change of volume from latitude λ_1 to latitude λ_2. Shew that the angular velocity of the earth is diminished (very nearly) by the fraction

$$\frac{5}{2}\frac{m}{M}(\cos^2\lambda_2 - \cos^2\lambda_1)$$

of itself, m and M being the masses, and the earth supposed spherical and homogeneous.

9. A heavy circular disk is rotating in a horizontal plane about its centre, which is fixed. An insect walks from the centre with constant velocity along a certain radius, and then flies away. Determine the whole motion.

10. A loaded cannon is suspended from a fixed horizontal axis, and rests with its axis horizontal and perpendicular to the fixed axis, the supporting ropes being equally inclined to the vertical; if v be the initial velocity of the ball, whose

mass is $\frac{1}{n}$th of the mass of the cannon, and h the distance between the axis of the cannon and the axis of support, shew that when it is fired off, the tension of each rope is immediately changed in the ratio

$$v^2 + n^2gh : n(n+1)gh.$$

(The moment of inertia of the cannon about its centre is neglected.) If a cannon be supported in a gun-boat in the manner described, what would be the effect of firing it off?

11. A rod whose centre is fixed is rotating uniformly in a vertical plane. A perfectly elastic ball of equal mass is dropped from a height equal to one-fourth of the circumference of the circle described by the end of the rod, and strikes it when horizontal at one extremity. After eight revolutions of the rod the ball again strikes it; prove that the rod was horizontal when the ball was dropped.

12. An imperfectly elastic ball is let fall upon a smooth hoop, of which the mass is equal to that of the ball, and which is suspended from a point in its circumference about which it is capable of moving freely in a vertical plane; prove that, if e be the modulus of elasticity, and α the inclination to the vertical of the radius passing through the point at which the ball must strike the hoop in order that it may rebound horizontally, $\tan^2\alpha = \frac{2e}{3}$.

13. A square formed of equal and similar uniform rods, jointed freely at the ends, is revolving with constant velocity about its middle point. Shew that if one of the angular points suddenly becomes fixed while the four joints remain free, the angular velocity of each rod will be at once diminished in the ratio 5 : 2.

14. Four equal uniform rods, jointed at their ends, when falling freely as a square with one diagonal vertical, are caught by means of a light hook at the middle point of one of the lower rods. Prove that that rod will be brought to rest by the impact, and will remain at rest during the rest of the motion.

15. $ABCD$ is a uniform heavy chain whose length equals $3l$, which is fastened to a peg at A, hangs down to a distance l, and passes over a smooth peg at C, which is very near A. If the chain be slightly disturbed so that its end D descends, prove that the impulsive pressures at A and C at the moment when it has run entirely down are $m\,.\sqrt{2lg}$ and $2m\sqrt{2lg}$, m being the mass of the chain.

16. A mass M attached to the end A of a chain AC, is placed (with the chain) on a horizontal plane in such wise that a portion AB of the chain forms a straight line, the remaining portion BC being heaped up at B: the mass M is then set in motion in the direction B to A with a given velocity, and so moves in a straight line, dragging the chain; determine the motion.

17. Two uniform rods OA, AB, of lengths $2a$, $2b$, and of masses proportional to their lengths, are jointed together at A, and are rotating round the fixed hinge O in the same straight line, and with equal angular velocities, when the outer AB comes against a fixed obstacle P. If the position of this be such as to reduce both rods to rest, prove that

$$AP = 2b^2 \cdot \frac{3a + 2b}{2a^2 + 6ab + 3b^2}.$$

18. Three equal particles, each repelling with a force varying as the distance, are at rest at the corners of an equilateral triangle, being connected by three fine inextensible strings, which form the sides. If one of the strings be cut, shew that the tension of each of the other two is instantaneously increased by one-fifth of its previous amount.

19. CP, AP are two equal uniform heavy beams, connected by a free hinge at P. The beam CP turns freely in a vertical plane about a fixed horizontal axis through C, while A slides freely on a vertical groove of which C is the highest point; prove that if the system make small oscillations about its position of equilibrium, the length of the simple isochronous pendulum is $\frac{CP}{3}$.

20. A uniform string hangs at rest over a smooth peg. Half the string on one side is cut off. Shew that the pressure on the peg is instantaneously reduced by one-third.

21. A smooth sphere M is on a horizontal plane, and another sphere m resting on it is just disturbed from its position of unstable equilibrium. The spheres being supposed homogeneous, shew that, whatever their radii or weights, the upper sphere will leave the lower before the line of centres is $\cos^{-1}\frac{2}{3}$ from the vertical.

XIII.

ENERGY.

1. We have now to consider a method into which the element of 'time' does not directly enter; in which force is considered not as generating a certain acceleration, but as pulling through a certain space; in which position and velocity are therefore the language, but never time nor acceleration.

The question which of these two was the proper expression of the effect of force caused a controversy very memorable in the annals of mathematics. We can now see that both sides were right. All our methods hitherto have been based on the former. We now turn to the latter.

2. When a force P drags its point of application through a small space, of which δp is the measure in the direction of the force, it is said to have done work $P\delta p$. And if the point has been forced back δp against P, work $P\delta p$ has been done against the force. Thus in this system the effect of a force is work, in the other it was momentum.

The work of a couple L, which moves a body through an angle $\delta\theta$, is $L\delta\theta$.

The accumulated work of a force P is clearly $\int Pdp$, the limits being the values of p in the extreme positions. Thus the work done by gravity on a stone of mass m, moving from a height h_1 to a height h_2, is $mg\,(h_1 - h_2)$; and this is the same by whatever path, constrained or free, the stone has reached the lower level.

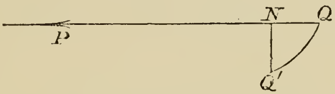
The work done by a radial force R, in pulling a body from a distance r_1 from the centre of force to a distance r_2, is

$$\int_{r_2}^{r_1} Rdr.$$

And in general the work done by any force, whose components are X, Y, Z, in bringing its point of application from x_1, y_1, z_1 to x_2, y_2, z_2, is

$$\int(Xdx + Ydy + Zdz),$$

the limits being given by these points. For if P be the force of which X, Y, Z are components, and if QQ' be an element of the arc (δs) traversed by the point of application,



$$P.\delta p = P.QN = P \cos NQQ'.QQ'.$$

But the sum of the components of any forces in any direction is equal to the component of their resultant in that direction; therefore

$$P \cos NQQ' = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds},$$

whence, in the limit,

$$Pdp = Xdx + Ydy + Zdz.$$

To measure work we must have some constant and easily accessible force. Take the force of gravity acting on a pound of matter at a given locality. Then the work expended in raising this pound to the height of one foot will be the unit of work. And any quantity of work will be measured by the number of feet to which it would suffice to raise the pound.

3. The work done by an impulse I, which causes the velocity of its point of application to change from u to u', is

$$\frac{I}{2}(u + u').$$

For suppose I the limit of a very great constant finite force P, acting during a very short time τ. Let α be the acceleration of the point of action of P resolved in its own direction. The space described by a point moving with finite acceleration α is $\frac{v^2 - u^2}{2\alpha}$, where $v - u$ is the increase of velocity.

This holds good also *in the beginning* of every accelerated motion whatever be its law.

Hence the work done, $P\delta p$,

$$= P \, . \, \frac{u'^2 - u^2}{2\alpha} \, .$$

But
$$u' - u = \alpha \, . \, \tau \, ;$$

therefore
$$\text{work done} = P\tau \, . \, \frac{u' + u}{2} \, .$$

And when τ is infinitely small,

$$I = P\tau,$$

whence the formula.

A very general proof of this is to be found in the 308th section of the first volume of Thomson and Tait's *Natural Philosophy*.

From this we see that if a ball strikes perpendicularly a fixed hard surface and rebounds with equal velocity there is no work done.

4. When a body or system possesses the power of doing work it is said to have energy. Thus a moving cannon-ball could force back a resisting body through a certain space before exhausting its own motion; this energy is called kinetic. Or the water in a mill-dam could do work before falling to a lower level; it has energy which is called potential, and which is due to a position of advantage relatively to a force. A mass of gunpowder has in it a store of the energy of chemical affinity. Steam in the boiler of an engine has the energy of heat; a man not utterly prostrate possesses a certain amount of another form of energy; and so on.

It is not often that the whole of any form of energy possessed by a body can be made available for work. A stone at the top of a tower has energy due to its height above the bottom. At the bottom its available potential energy is exhausted, but it is clear that if there were a pit at hand the stone could do additional work before getting to the bottom of that. Its total energy due to the attraction of the earth would never be exhausted till it had reached the centre.

The amount which cannot be made available was called by Clausius the Entropy. It has been proposed to give this name to the amount which can be made available. As the usage in English works has come to be different from that in foreign works, we will avoid the word altogether.

5. There are two general principles to which this method has led.

Imagine a system A which possesses a certain store of energy, and which can be completely isolated so that no energy can pass out or in. Let it be connected in any way with a system B possessing similar properties, and let no energy pass except between these. Then the total energy of all the different forms in A and B remains the same as before. This is the law of the *conservation of energy*.

But the available energy is now less than when the systems were separate. For that depends on the excess of energy which one possessed over the other; and by the passage of energy between them the amount possessed by the one is lessened and by the other increased. The two systems are in fact brought more nearly to a level. Now, as no system in nature can be completely isolated and made energy-tight, every such system as A is always communicating its energy to those around it, as B, which possess less. And the available energy of all is being lessened. This is the law of the *dissipation of energy*.

These laws are proved by experiment; it is found that energy which disappears in one form reappears in some other, and to an equal amount. But for those forms of which we can take cognizance in this science (viz.), the energy of motion and of position, the conservation law can be deduced from the laws of motion.

6. The potential energy of a system relatively to a force is measured by the work which it can do against the force,

or by minus the work which the force can do upon it before it reaches the position of zero force. It is therefore for one force

$$-\int(Xdx + Ydy + Zdz),$$

the limits being given by the actual position and the position of zero force; for a number of forces it is

$$-\Sigma\int(Xdx + Ydy + Zdz).$$

It is almost needless to remark that the origin from which these co-ordinates are measured is a matter of convenience.

The kinetic energy of a particle of mass m, moving with velocity v, is the work it can do before being stopped. This may be measured against any force. Let us choose gravity. A particle of mass m projected upwards with velocity v will move through a space $\frac{v^2}{2g}$ before stopping. It has thus pushed back the point of application of the force mg through a space $\frac{v^2}{2g}$. Hence the work it has done is $mg \times \frac{v^2}{2g}$ or $\frac{mv^2}{2}$. This may of course be expressed in any co-ordinates, as

$$\frac{m}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\},$$

or in one plane $$\frac{m}{2}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\}.$$

The kinetic energy of a moving rigid body is

$$\frac{1}{2}\Sigma\delta m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}.$$

7. To establish the law of the conservation of energy, we have then to shew that the gain or loss of kinetic energy while the system passes from one configuration to another is equal to the loss or gain of potential energy.

Let x, y, z be the co-ordinates of an element δm of the system; u, v, w its component velocities; X, Y, Z the components of the impressed forces. Then these are in equilibrium with the reversed effective forces. These last are

$$-\delta m\frac{du}{dt},\quad -\delta m\frac{dv}{dt},\quad -\delta m\frac{dw}{dt}.$$

Now
$$\frac{du}{dt}=\frac{du}{dx}\frac{dx}{dt}=u\frac{du}{dx}=\frac{1}{2}\frac{du^2}{dx}.$$

Thus the reversed effective forces may be expressed

$$-\frac{\delta m}{2}\frac{d(u^2)}{dx},\quad -\frac{\delta m}{2}\frac{d(v^2)}{dy},\quad -\frac{\delta m}{2}\frac{d(w^2)}{dz}.$$

Hence, by the principle of virtual velocities, if the point x, y, z receive a displacement whose projections parallel to the axes are $\delta x, \delta y, \delta z$,

$$\Sigma\left\{\left(\frac{\delta m}{2}\frac{d(u^2)}{dx}-X\right)\delta x+\left(\frac{\delta m}{2}\frac{dv^2}{dy}-Y\right)\delta y\right.$$
$$\left.+\left(\frac{\delta m}{2}\frac{dw^2}{dz}-Z\right)\delta z\right\}=0.$$

Now suppose (1) that the displacements $\delta x, \delta y, \delta z$ are consistent with the geometrical relations of the system; for example, that they involve no breaking of connections. Then in this equation X, Y, Z come to represent the impressed forces whose virtual moments do not vanish.

Of all possible displacements one must be the displacement due to the actual motion. If we suppose this to be the one given to the system,

$$\delta x,\ \delta y,\ \delta z \text{ are } \frac{dx}{ds}.\delta s,\ \frac{dy}{ds}.\delta s,\ \frac{dz}{ds}.\delta s,$$

in which δs is the element of arc described by x, y, z. Whence the equation

$$\Sigma\frac{\delta m}{2}\left(\frac{du^2}{dx}.\delta x+\frac{dv^2}{dy}.\delta y+\frac{dw^2}{dz}.\delta z\right)$$
$$=\Sigma(X\delta x+Y\delta y+Z\delta z)$$

becomes

$$\frac{1}{2}\Sigma\delta m\left(\frac{du^2}{ds}+\frac{dv^2}{ds}+\frac{dw^2}{ds}\right)=\Sigma\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right).$$

Now suppose (2) that the forces X, Y, Z are functions of the co-ordinates alone. Then this equation is integrable, and gives rise to

$$\frac{1}{2}\Sigma\delta m\,(u^2+v^2+w^2)=C+\Sigma\int(Xdx+Ydy+Zdz).$$

But the left-hand side of this is the kinetic energy, and

$$\Sigma\int(Xdx+Ydy+Zdz)$$

is the work done by the forces on the system, (i.e.) the loss of potential energy. The equation may therefore be expressed:

the sum of the kinetic and potential energies is constant.

On the above supposition as to the forces,

$$\int(Xdx+Ydy+Zdz)$$

will be a function of the limiting values of the co-ordinates. Hence the work done by these forces is independent of the paths pursued by the points of the body: such a system of forces is called "Conservative."

8. The supposition (2) which we have made as to the nature of the impressed forces limits the cases of motion to which the principle, as expressed by the above equation, can be applied. It excludes all cases into which sliding friction enters, for this, depending on the direction of motion, is not a conservative force. All effects of animal force, such as a constraint of one part of the system to move according to some given law, and all explosions or impacts of imperfectly elastic bodies, are excluded.

The nature of the forces thus excluded is easily seen from the energy stand-point. The only forms of energy which our present science can deal with, are the kinetic energy of motion, and the potential energy of position. That energy is conserved must mean to us that the sum of the energies in these two forms is constant. If then, in any case of motion, energy passes into or comes from any other form, as when heat is generated by sliding friction, or when energy is imported into the system through an animal or an explosion; or when sound or other vibrations are produced; in such cases, although the conservation principle still holds true, the energy passes into forms which do not come within the scope of Rigid Dynamics, and we can only use the principle when we can make allowance for the amount of the energy which has thus changed form.

The student who is familiar with the application of virtual velocities to the theory of equilibrium, knows that the forces which do not appear in that equation when the displacement given is consistent with the geometrical relations of the system, are (1) all internal pressures and tensions between rigidly connected parts, (2) external forces (as normal pressures) whose point of application experiences no displacement in the direction of the force. There is to be added to these rolling friction, whose point of application being the instantaneous centre experiences no displacement.

9. The expression for the kinetic energy of a body or system moving in one plane, viz. $\frac{1}{2}\Sigma\delta m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}$ falls under the rule of Lesson IV, Art. 3. The kinetic energy of a rigid body is therefore equal to that of the whole mass supposed collected at its centre of inertia, and moving with it, together with the similar function relative to that point; i.e. to the kinetic energy of translation together with that of rotation.

Now the velocity of every point relatively to the centre of inertia is $r\omega$, where ω is the angular velocity, and r the distance of the point from the centre of inertia. If then V is the velocity of the centre of inertia, the kinetic energy of the body is

$$\frac{1}{2}MV^2+\frac{1}{2}\omega^2\Sigma\delta mr^2,$$

$$\text{or } \frac{1}{2}MV^2+\frac{M}{2}k^2\omega^2.$$

If the motion is round a fixed axis about which the radius of gyration is K, the kinetic energy is clearly

$$\frac{1}{2}MK^2\omega^2.$$

10. *A rough sphere of mass* m *and radius* a, *rolls in a vertical plane inside of a fixed horizontal cylinder of radius* b. *Find the motion.*

Let the velocity of the centre be v; the angular velocity ω; the angle which the radius to the point of contact makes with the vertical θ. Then if the centre be at a height y above its lowest point, its potential energy may be measured by mgy; its kinetic energy is $\frac{m}{2}\left(v^2 + \frac{2}{5}a^2\omega^2\right)$, whence

$$v^2 + \frac{2}{5}a^2\omega^2 + 2gy = \text{constant}.$$

Now the velocity of the point of contact is zero;

$$\therefore\ v - a\omega = 0,$$

also $$v = (b-a)\frac{d\theta}{dt},\ y = (b-a)(1-\cos\theta);$$

therefore the equation of energy becomes

$$\frac{7}{5}(b-a)^2\left(\frac{d\theta}{dt}\right)^2 + 2g(b-a)(1-\cos\theta) = \text{constant}.$$

To determine the constant, suppose the sphere projected from the lowest point with velocity V. Then when $\theta = 0$,

$$(b-a)\frac{d\theta}{dt} = V;$$

whence the constant is $\frac{7}{5}V^2$; therefore the motion is given by

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{V^2}{(b-a)^2} - \frac{10g}{7}\frac{1-\cos\theta}{b-a}.$$

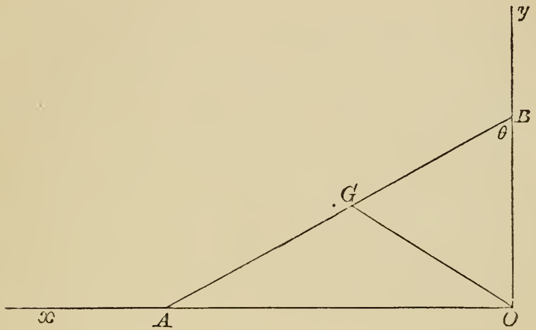
To find the time of a small oscillation, differentiate; then

$$\frac{d^2\theta}{dt^2} = -\frac{5}{7}g\frac{\sin\theta}{b-a},$$

whence the period of oscillation is $2\pi\sqrt{\frac{7(b-a)}{5g}}$.

11. *A uniform ladder of mass* m, *slips down between a smooth wall and a smooth horizontal plane always keeping in a vertical plane perpendicular to their intersection. Find the motion.*

Let the planes be those of y and x, and let θ be the angle which the ladder makes with the vertical at any time.



Then the angular velocity is $\frac{d\theta}{dt}$ and the linear velocity of the centre of inertia is $OG \cdot \frac{d\theta}{dt}$, since G keeps at a constant distance from O.

Hence the kinetic energy is

$$\frac{m}{2}(OG^2 + k^2)\left(\frac{d\theta}{dt}\right)^2.$$

The potential energy is due to the height of G above Ox, and is therefore

$$mg \cdot OG \cos\theta;$$

therefore

$$\frac{1}{2}(OG^2 + k^2)\left(\frac{d\theta}{dt}\right)^2 + g \cdot OG \cos\theta = \text{constant}.$$

If the motion began when AB was vertical, this constant is $g \cdot OG$.

Whence $$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g \cdot OG \cdot (1 - \cos\theta)}{OG^2 + k^2},$$

or $$\frac{d\theta}{dt} = \pm 2\sqrt{\frac{g \cdot OG}{OG^2 + k^2}} \cdot \sin\frac{\theta}{2},$$

which can be again integrated, and θ found in terms of t.

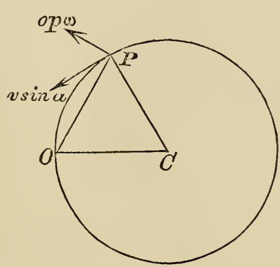
12. If the object of our investigation is to find, not the whole motion, but the motion in some particular position or the position of instantaneous rest, this method has a great advantage over earlier methods. In these, integration is almost always necessary. The general problem is worked out and the particular case deduced. Here the particular case is the easier. Thus, suppose that in the above problem it were required to find the angular velocity of the ladder just before it reaches the ground. The kinetic energy is zero at first, and the potential energy at last. Therefore the kinetic energy at last, $\frac{1}{2}m(OG^2 + k^2)\omega^2$, must be equal to the potential energy at first, (viz.) $mg \cdot \frac{1}{2}AB$.

13. *A particle is attached to the circumference of a massive cylinder, and starting from the end of a horizontal diameter pulls up another particle hanging at the end of a string wound round the cylinder, by making the cylinder rotate about its axis, which is horizontal. Prove that if the former particle first reaches the lowest point the ratio of the masses of the particles is* $\frac{\pi}{2}$.

Here, both at first and at last the kinetic energy is zero. Hence the potential energy gained by one particle is equal to that lost by the other. But the cylinder turns through a right angle; therefore the particle attached to it descends a depth equal to the radius; while the other ascends a height equal to one-fourth of the circumference. And these spaces are inversely as the masses of the particles. Therefore the masses are in the ratio of the arc of the quadrant to the radius.

14. *A uniform tube in the form of a common helix (screw) of mass* M *can move round a vertical axis coincident with one of the generating lines of its own cylinder. A particle of mass* m *is dropped in at the top. Find its velocity and the angular velocity of the tube when it has reached any position.*

There are two principles which will give two equations of motion. 1st, That the angular momentum of the whole about the axis of rotation must remain zero; and, 2nd, That the kinetic energy at the end must be equal to the potential energy lost by the descent. Let the radius of the cylinder be a. Let the inclination of the curve of the screw to the vertical be α; let ω be the angular velocity of the tube at any moment, v the velocity of the particle relatively to the tube. Then its vertical velocity is $v \cos \alpha$, and its horizontal velocity is compounded of the velocity of the point of the helix at which it is, which is $OP \,.\, \omega$ at right angles to OP; and of $v \sin \alpha$ relatively to the helix at right angles to CP.



The figure represents a horizontal section of the cylinder of the screw, through the particle. O is the axis of revolution, C that of the screw, P the particle.

Let $\angle POC$ be called θ. Then $\angle OPC = \frac{\pi}{2} - \frac{\theta}{2}$, and

$$OP = 2a \,.\, \sin \frac{\theta}{2}.$$

The moment of the momentum of P about the vertical axis through O is $mv \sin \alpha \, (a - a \cos \theta) + m \,.\, OP^2 \,.\, \omega$.

That of the tube is $M 2a^2\omega$.

Hence, by the first principle,

$$mv \sin\alpha \,.\, a(1-\cos\theta) + m4a^2 \sin^2\frac{\theta}{2} \,.\, \omega + M 2a^2\omega = 0,$$

for initially the system was at rest.

Again, the square of the velocity of P is

$$v^2\cos^2\alpha + v^2\sin^2\alpha + OP^2\omega^2 + 2v\sin\alpha \,.\, OP\omega \,.\, \sin\frac{\theta}{2}.$$

The moment of inertia of the tube about the axis through O is $2Ma^2$. Its kinetic energy is therefore $Ma^2\omega^2$.

The total kinetic energy is

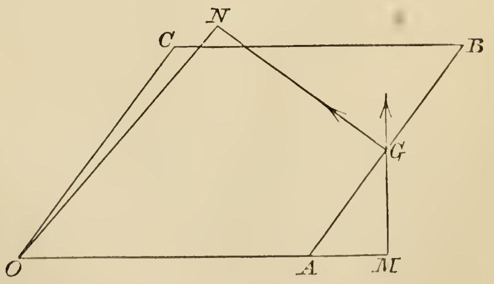
$$Ma^2\omega^2 + \frac{m}{2}\left(v^2 + 4a^2\sin^2\frac{\theta}{2}\,\omega^2 + 4a\sin^2\frac{\theta}{2}\, v\sin\alpha \,.\, \omega\right).$$

This must be equal to the loss of potential energy, which if the height fallen through by the particle be h, will be mgh.

These two equations suffice to give v and ω.

15. *A square formed of four similar uniform rods jointed freely at their ends, is laid upon a smooth horizontal table, one of its angular points being fixed; if angular velocities Ω, Ω' be communicated to the two sides containing this angle, shew that the greatest value of the angle* (2ϕ) *between them is given by the equation*

$$\cos 2\phi = -\frac{5}{6}\frac{(\Omega-\Omega')^2}{\Omega^2+\Omega'^2}.$$



Let O be the fixed point. The angular velocities of OA and BC will be equal since these must remain parallel. Let this be ω. And let that of OC and AB be ω'.

We will use for the solution of this the facts that the whole angular momentum round O remains unchanged and that the kinetic energy remains constant.

The masses of the rods are equal, and will divide out of each term in the equation. We may therefore take each as unity. Let the length of each be $2a$.

The angular momentum of OA is $\frac{4a^2}{3}\omega$.

The angular momentum of AB about O is equal to the angular momentum about its centre of inertia G, which is $\frac{a^2}{3}\omega'$, together with the moment of momentum of its mass, concentrated at G, about O. Draw GM and GN perpendicular to AO and AB. Now the velocity of G is compounded of the velocity of A, $2a\omega$ along MG, and of its own relative velocity, $a\omega'$ along GN. And the moment of the resultant momentum about any point is equal to the sum of the moments of the separate momenta. It is therefore

$$2a\omega \,.\, OM + a\omega' \,.\, ON,$$

or
$$2a\omega\,(2a + a\cos 2\phi) + a\omega'\,(a + 2a\cos 2\phi).$$

Hence the whole angular momentum about O is

$$\frac{4a^2}{3}(\omega + \omega') + \frac{a^2}{3}(\omega + \omega') + (\omega + \omega')\,2a^2(2 + \cos 2\phi),$$
$$+ (\omega + \omega')\,a^2(1 + 2\cos 2\phi).$$

Now this is always equal to its initial value. But initially ω was Ω, ω' was Ω' and 2ϕ was a right angle. In the final circumstances which the question contemplates, the angle 2ϕ has a maximum or minimum value. Hence ω and ω' are equal.

Equating the values of the angular momentum in these states

$$2\omega\left(\frac{5a^2}{3} + 5a^2 + 4a^2\cos 2\phi\right) = (\Omega + \Omega')\left(\frac{5a^2}{3} + 5a^2\right),$$

or
$$\omega = \frac{5(\Omega + \Omega')}{10 + 6\cos 2\phi} \quad\text{.....................(1).}$$

Consider next the kinetic energy.

That of OA is $\frac{4a^2}{3}\omega^2$; that of AB is

$$\frac{a^2}{3}\omega'^2 + 4a^2\omega^2 + a^2\omega'^2 + 4a^2\omega\omega' \cos 2\phi.$$

The whole will be

$$(\omega^2 + \omega'^2)\left\{\frac{5a^2}{3} + 5a^2\right\} + 8a^2\omega\,\omega' \cos 2\phi.$$

The initial and final values of these also are equal. Whence

$$2\omega^2 . \frac{20}{3} + 8\omega^2 \cos 2\phi = (\Omega^2 + \Omega'^2)\frac{20}{3}.$$

Whence $$\omega^2 = \frac{5(\Omega^2+\Omega'^2)}{10 + 6\cos 2\phi} \quad \text{.....................(2).}$$

Eliminating ω from (1) and (2) we have the required result.

16. The following general proposition*, which is due to Clausius, is of such simple demonstration and of such value in molecular theories, that it ought to be better known in England than it is.

Let x, y, z be the coordinates of an element δm of a system not necessarily rigid.

Then $$\frac{d}{dt}\Sigma\delta m x^2 = 2\Sigma\delta m x \frac{dx}{dt},$$

and $$\frac{d^2}{dt^2}\Sigma\delta m x^2 = 2\Sigma\delta m x \frac{d^2x}{dt^2} + 2\Sigma\delta m \left(\frac{dx}{dt}\right)^2.$$

Now $$\Sigma\delta m x \frac{d^2x}{dt^2} = \Sigma\delta m \frac{d^2x}{dt^2} . x,$$

and $\delta m \frac{d^2x}{dt^2}$ is the component in the direction of the axis of x, of the whole force, whether external or constraining, which acts on δm. Call this X.

* I am indebted to Professor Maxwell for my introduction to this subject.

Then $$\frac{d^2}{dt^2}\Sigma\delta mx^2 = 2\Sigma\delta mXx + 2\Sigma\delta m\left(\frac{dx}{dt}\right)^2,$$

so $$\frac{d^2}{dt^2}\Sigma\delta my^2 = 2\Sigma\delta mYy + 2\Sigma\delta m\left(\frac{dy}{dt}\right)^2,$$

and $$\frac{d^2}{dt^2}\Sigma\delta mz^2 = 2\Sigma\delta mZz + 2\Sigma\delta m\left(\frac{dz}{dt}\right)^2.$$

Now let A, B, C be the moments of inertia of the system about the axes.

Then $$\Sigma\delta m(x^2+y^2+z^2) = \frac{1}{2}(A+B+C).$$

Adding the above equations we have

$$\frac{1}{4}\frac{d^2}{dt^2}(A+B+C) = \Sigma\delta m(Xx+Yy+Zz) + \Sigma\delta m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}.$$

Now suppose the system to be such that $A+B+C$ does not vary during the interval of time δt. This will be the case if the system be at rest: and 1st, if the system be a rigid body rotating about certain fixed axes; 2nd, if the system be a homogeneous fluid whose particles are in motion but which always occupies the same portion of space, for then any element δm which moves away from a point is replaced by another equal element; 3rd, if the system consist of a number of molecules each moving with a periodic motion whose period is much smaller than δt; for then the different values of x, y, z for any molecule will recur many times in the course of δt; and in various other cases of motion.

All these kinds of motion are called "stationary." For such we have

$$-\Sigma(Xx+Yy+Zz) = \Sigma\delta m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}.$$

The right-hand expression is twice the kinetic energy of the system; that on the left hand Clausius proposes to call the *virial* function. The virial equation holds whenever

$$\frac{d^2}{dt^2}(A+B+C) = 0.$$

If the system is at rest, then the virial function vanishes because X, Y, Z separately vanish for every particle.

The above equation expresses that the virial function is equal to twice the kinetic energy. The part of this function which depends on internal forces admits of being simplified.

17. Suppose the forces between any two particles at a distance r to be R. Let it be considered positive when it is repulsive. Then there are two terms in the above sum which arise from the action between two particles at x_1, y_1, z_1, and x_2, y_2, z_2 respectively. These terms are

$$Xx_1 - Xx_2,$$

$$\text{or} \quad -R\frac{x_2 - x_1}{r} \cdot x_1 + R\frac{x_2 - x_1}{r} x_2 \quad \text{or} \quad R\frac{(x_2 - x_1)^2}{r}.$$

Now $\qquad r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$

Whence $\qquad \Sigma\,(Xx + Yy + Zz) = + \Sigma Rr.$

If then the forces in any system are partly external, symbolized by X, Y, Z; and partly internal actions symbolized by R, the equation becomes

$$\frac{1}{2}\Sigma\,(Xx + Yy + Zz) + \frac{1}{2}\Sigma Rr = \text{kinetic energy}.$$

The proposition will hold good for any direction and for any number of particles. If the action R between the particles is attractive the term ΣRr is negative.

18. This equation is evidently of very general application. It gives, through ΣRr, a measure of the internal forces in bodies, whatever the nature of these forces may be, and whether the particles of the body are at rest or in motion. Its most important application hitherto has been to the kinetic theory of gases. This is beyond our limits; but we will shew how to apply it in one or two very simple cases.

Suppose a thin ring of mass m and radius a, composed of a number of particles, to be rotating with angular velocity

ω in its plane about its centre. Let R be the tension. Let δs be the arc between two adjacent particles. Then by the equation

$$\Sigma R\delta s = ma^2\omega^2.$$

But R is the same for all points of the ring,

and $$\Sigma \delta s = 2a\pi;$$

$$\therefore R = \frac{ma\omega^2}{2\pi}.$$

These equations can easily be verified by other methods.

Again, suppose a network of light cords in equilibrium under any external forces X, Y, Z acting at points x, y, z, and let T be the tension along a cord of length r, then

$$\Sigma (Xx + Yy + Zz) = \Sigma Tr.$$

EXAMPLES.

1. Prove that a flywheel of radius a rotating with velocity ω has in it energy enough to raise a mass equal to its own to a height $\frac{a^2\omega^2}{2g}$.

2. A cannon-ball of mass M raises by its recoil a mass M' to a height of h feet. If the mass of the cannon-ball is m, shew that its velocity of projection is

$$\left\{\frac{MM'}{m^2} 2gh\right\}^{\frac{1}{2}}.$$

3. A weight is attached to an elastic string which is fastened to a point. Apply the principle of energy to determine its motion when it falls from rest, the string being initially vertical and unstretched.

4. A nut slides smoothly on its screw. If this be placed in a vertical position and the nut be allowed to

run down, prove that its angular velocity when it has descended a space h, will be

$$\left\{\frac{2gh}{k^2 + a^2 \tan^2 \alpha}\right\}^{\frac{1}{2}},$$

in which a is the radius of the screw-cylinder, α is the inclination of its tangent to the horizon, and k is the radius of gyration of the nut about the axis.

5. A thin uniform smooth tube of length $2a$ is balancing horizontally about its middle point which is fixed; a uniform rod whose mass is $\frac{1}{n}$th of that of the tube and whose length is $2a'$, is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube; prove that if v is the velocity of projection of the rod, the angular velocity of the tube and rod when their middle points coincide is

$$\left\{\frac{3v^2}{a'^2 + na^2}\right\}^{\frac{1}{2}}.$$

6. A circular ring is free to move on a smooth horizontal plane on which it lies; and a uniform rod has its extremities connected with and moveable on the smooth arc of the ring; the system being set in motion on the plane, shew that the angular velocity of the rod is constant; and describe the paths of the centres of the rod and ring.

7. A narrow smooth semicircular tube is fixed in a vertical plane with its vertex upwards, and a heavy flexible string passing through it hangs at rest; shew that if the string be cut at one of the ends of the tube, the velocity which the longer portion of the string will have attained when it is just leaving the tube will be

$$(ag)^{\frac{1}{2}}\left\{2\pi - \frac{a}{l}(\pi^2 - 4)\right\}^{\frac{1}{2}};$$

l being the length of the longer portion, and a the radius of the tube.

8. A particle is suspended so as to oscillate in a cycloid whose vertex is at the lowest point; if it begin to move from a point distant a from the lowest point measured along the curve, and the medium in which it moves give a small resistance kv^2 to the acceleration, prove that before it next comes to rest energy will have been dissipated, which is $\frac{8ka}{3}$ of its original value.

9. A fine circular tube carrying within it a heavy particle is set revolving about a vertical diameter. Shew that the difference of the squares of the absolute velocities of the particle at any two given points of the tube, equidistant from the axis, is the same for all initial velocities of the particle and the tube.

10. A rough cylinder of radius a loaded so that its centre of gravity is at a distance h from its axis is placed on a board of n times its mass which can move on a smooth horizontal plane. Find the time of a small oscillation, and prove that if l be the length of the simple equivalent pendulum

$$lh = k^2 + \frac{n}{n+1}(a-h)^2,$$

where k is the radius of gyration of the cylinder about a horizontal axis through its centre of gravity.

11. A mass M of fluid is running round a circular channel of radius a, with velocity u; another equal mass is running round a channel of radius b, with velocity v; the radius of the one channel is made to increase and the other to diminish till each has the original value of the other. Shew that the work required to produce the change is

$$\frac{1}{2}\left(\frac{v^2}{a^2} - \frac{u^2}{b^2}\right)(b^2 - a^2)M.$$

12. A smooth thin tube in the shape of a quadrant of a circle, of radius a, is fixed in a vertical plane with its lowest radius horizontal. A heavy uniform inextensible

string, of length $\frac{\pi a}{2}$, is held wholly within the tube and then let go. Find the velocity during the subsequent motion.

13. A uniform imperfectly elastic beam, of length $2a$ moving parallel to itself impinges on a fixed obstacle. Prove that the kinetic energy after will be to that before impact as $3c^2 + e^2a^2$ to $3c^2 + a^2$; c being the distance from the middle point to the point of impact, and e the modulus of elasticity.

14. A plane body is struck by a blow in its own plane. Prove that the work done by the blow will be greater if the body be free than if a point of the body were fixed.

15. Which of the systems described in the Problems at the end of Lessons XI. and XII. are conservative?

16. If A, B, C be the moments of inertia round three axes at right angles of a uniform cylinder rolling with constant velocity along a plane; prove that

$$\frac{d^2}{dt^2}(A + B + C) \text{ is constant.}$$

17. Verify the virial equation in the case of a uniform chain hanging in the common catenary.

XIV.

PRECESSIONAL MOTION.

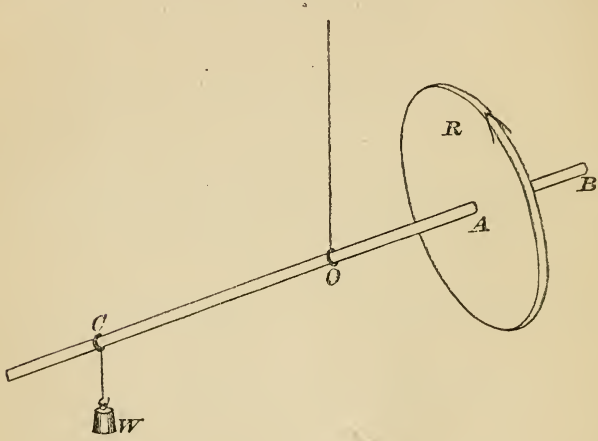
1. We have, in these introductory Lessons, avoided the more complicated phenomena of motion. But there is one class of motions of such paradoxical appearance and such important nature that we will state the phenomena and give a general explanation of their cause.

Every one knows that a rapidly spinning top not only rotates about its axis but with its axis about the vertical.

The explanation of the seasons depends on the fact that the earth, while rotating and revolving, keeps its axis always parallel to itself. But when its directions in successive years are compared with one another, they are found not to remain parallel but to move in a cone, pointing to different fixed stars in the course of ages, and to take nearly 26,000 years to return to the same direction. This is called the precession of the equinoxes.

2. The whole of these phenomena can be illustrated by one piece of apparatus, called Fessel's.

Let R be a metal disc or ring which can rotate freely about the rod BAC at the point A, but cannot move along it. O is a point at which the rod is supported either by a pivot or by a long cord. Let there be means for suspending from the other side a weight W, of such size that it can balance R or overbalance it, or be overbalanced by it. Now let R be set rapidly spinning in the direction of the arrow head, and the system afterwards left to itself. And (1) let W be so placed that it exactly balanced R when all was at rest. Then it is found that the axis COB continues to point constantly in the same direction. In fact, if the workmanship is so good that R will spin for some considerable time, the apparatus may be used, like Foucoult's pendulum, to give a visible proof of the earth's daily rotation. For, as the earth rotates, the axis COB whose direction is absolutely fixed in space appears to rotate slowly backwards.



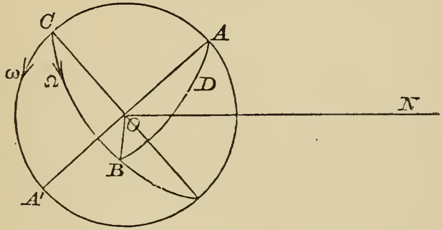
This is the case of a coin thrown up into the air and made to rotate in its own plane. The only force is that of gravity which acts through its centre of inertia. Hence it will continue to rotate about the same direction unless it is disturbed. (2) Let R overbalance W, as in the case of the common top or the gyroscopic toy, in which indeed everything on the left of O is wanting. Then it is found that the axis COB rotates about the vertical, so that B approaches and C recedes from us. (3) Let W overbalance R when at rest. Then it is found that in the motion COB rotates about the vertical, so that B recedes and C approaches.

From this experiment it is clear that the motion of the axis is caused by the couple due to gravity acting round a horizontal axis.

In the case of the earth the couple is the attraction of the sun on the protuberant parts of the earth. Were the earth a sphere there would be no precession.

The explanation of these phenomena, which we will now give, is confessedly imperfect, but on its own suppositions it is satisfactory. It is applied to the complete explanation of precession and to accurate calculation in Airy's Mathematical Tracts.

3. Imagine a sphere rotating freely about a diameter AOA' with angular velocity Ω, and struck by an impulsive

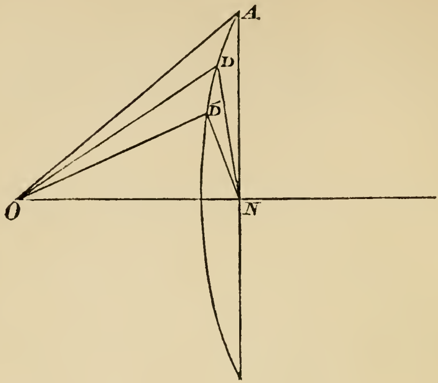


couple which would, if the sphere were at rest, generate an angular velocity ω about a perpendicular axis OB. Then the resultant angular velocity will be $\sqrt{\Omega^2+\omega^2}$, and the sphere will rotate about OD where $\tan AOD = \frac{\omega}{\Omega}$.

Now suppose that ω is small compared with Ω, i.e. either that the sphere is rotating rapidly or that the impulsive couple is small. Then $\left(\frac{\omega}{\Omega}\right)^2$ may be neglected in comparison with $\frac{\omega}{\Omega}$, and thus we may say that the velocity of rotation will not be altered by the blow, whereas the axis will.

(That OD will lie on the same side of OA as OB will be seen by considering the resultant velocity of C.)

Now let another impulsive couple act on the sphere tending to cause rotation about an axis at right angles to OD, not in the plane AOB, but so that the planes of both the couples pass through one line ON. Then the body will begin to



rotate about the direction OD', DD' being at right angles to the plane ODN; and the angular velocity will be unchanged. And if a number of such impulses act one after the other, their planes all containing ON, the axis will proceed to describe a pyramid (not necessarily re-entering) in space with O as vertex. But if the couples be equal and at equal intervals, the pyramid will be regular and re-entering. And again, if these impulses are numerous and small, so as to approach the case of an accelerating couple, the pyramid becomes a right circular cone, with O as vertex and ON as axis.

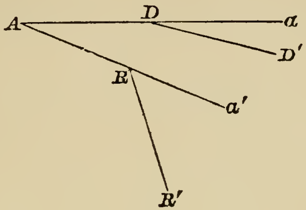
Suppose that the impulse takes place at intervals of time δt. Then in each interval the axis describes an angle equal to AOD, i. e. to $\frac{\omega}{\Omega}$. Now to complete a revolution it must cover the whole cone. Let the semivertical angle be α. To complete a revolution the axis must describe $2\pi \sin \alpha$. And this it will do in time

$$2\pi \sin \alpha . \frac{\Omega \delta t}{\omega}.$$

4. Such are the proceedings of the axis of rotation in space. If the end of each axis on the sphere were marked

with chalk, how would the successive marks appear when the sphere was stopped?

Suppose the sphere to be beginning to rotate about A. Let D, D' be the positions in space of the ends of the next



two axes; and let R, R' be the points of the sphere which are to coincide with them when they become axes. Let the lines in the figure represent planes through the centre of the sphere. Then RAD is the angle through which the sphere must turn while it rotates about A; and $R'Ra'$ must be equal to RAD and $D'Da$ together. Now AD, DD', &c. form an equiangular pyramid. Hence, as the sphere rotates with constant velocity, and the impulses take place at equal intervals of time, the locus of R—the successive chalk marks—will be the angular points of an equiangular polygon. If the impulses be all equal the polygon will also be equilateral. In the limiting case the marks will trace out a circle on the sphere.

Let its radius be ρ. Then $AD = AR$.

And $$AR = \rho \,.\, \angle R'Ra' \text{ in the limit}$$
$$= \rho\,(\Omega + \omega)\,\delta t.$$

And since $\frac{\omega}{\Omega}$ ultimately vanishes
$$AR = \rho\Omega\delta t.$$

Again, $$AD \text{ (fig. Art. 3)} = AO \angle AOD = R\frac{\omega}{\Omega},$$

in which R denotes the radius of the sphere;

therefore $$\rho\Omega\delta t = R\frac{\omega}{\Omega} \text{ and } \rho = R\frac{\omega}{\Omega^2\delta t}.$$

The time in which the instantaneous axis passes through all these positions in the sphere is $\frac{2\pi}{\Omega}$, i.e. the same as one complete revolution of the sphere. This latter circle is therefore a very small one, and very quickly described compared with that which its centre describes in space.

5. We have considered a sphere, in order that every axis might be a permanent axis of rotation. For if an axis of rotation is not a principal one the forces introduced by the motion tend themselves to alter the axis. Happily in the important case of motion this supposition is not far wrong. The earth is itself very nearly a sphere, and Ω is so large compared with ω that the circle described on the surface by the locus of the end of the real axis of rotation is exceedingly small (a few feet in radius), and may for all purposes be neglected. The momental ellipsoid of a top is not so nearly a sphere. It is an oblate spheroid. But here again the rotation is so fast that the true axis is never far from the axis of figure, and the centrifugal or other effective forces of motion have never any effect that would interfere with the above reasoning. If the rotation is not fast the wobbling which sets in shews that the axis of rotation is far from the axis of figure.

We see, then, that a top or the sphere described above is not in general rotating about its axis of figure, but about a not principal axis very close to it. This goes through its various positions in the course of a single revolution of the body, and the axis which is the mean of all these describes in space a cone of finite size.

EXAMPLES.

1. A perfectly balanced gyroscope is rotating with given angular velocity; supposing it to be acted on by a small constant couple in a vertical plane through its axis, find the precession.

2. Prove that the finite couple corresponding to ω in Art. 3 is $\frac{A\omega}{\delta t}$, A being the moment of inertia of the sphere about an axis through the centre.

3. Assuming that the polar axis of the earth changes its direction by 20·5″ every year, and that the angle (α) between the poles of the ecliptic and equator is $23\frac{1}{2}$ degrees; find the time which the polar axis takes to complete a revolution in space.

4. Find the radius of the circle in which the axis of rotation cuts the surface of the earth. (Radius of earth is nearly 4000 miles.)

5. What time does it take to describe this circle?

6. A top of mass m, whose centre of gravity is distant h from its vertex, and whose radius of gyration about an axis through the vertex at right angles to its axis of figure is k, is rotating with its axis inclined at α to the vertical. Prove that

$$\frac{\omega}{\delta t} = \frac{gh \sin \alpha}{k^2}.$$

7. Assuming that the rotation will be always about an axis very near the axis of figure, prove that if the axis of the top is observed to make a complete revolution about the vertical in time T, the angular velocity of the top is

$$\frac{ghT}{2\pi k^2}.$$

XV.

DIFFERENTIAL EQUATIONS.

1. The complete analytical solution of a physical problem depends in general on that of a differential equation. And each physical science depends in general on equations of a particular type. It is therefore necessary for the student of a department like the present to have a working acquaintance with the class of equations peculiar to it. Happily this is easily acquired. A little practice, without systematic study of a treatise on Differential Equations, is all that is necessary. The solution of Differential Equations being a reverse process is to some extent guess work. A knowledge of the nature of the result seldom fails to suggest its form.

In the following brief notes on differential equations, which have been put together chiefly for reference, the student must not expect a regular exposition of the subject, but must be content to assume some of the principles and solutions.

2. Many of the differential equations which present themselves to the student of Rigid Dynamics are not capable of complete solution. Those which are soluble, are mostly of the class called "linear with constant coefficients"; i.e. in which the differential coefficients and the dependent variable appear only in the first power, and with constant coefficients. Here are some of the most common.

$$(1) \qquad \frac{dx}{dt} + ax = bf(t).$$

$$(2) \qquad \frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = cf(t).$$

$$(3)\qquad \left.\begin{aligned} \frac{d^2x}{dt^2} + a\frac{dy}{dt} + bx = 0 \\ \frac{d^2y}{dt^2} + a'\frac{dx}{dt} + b'y = 0 \end{aligned}\right\}.$$

The order of a differential equation is the order of the highest differential coefficient which occurs in it.

The complete solution of a differential equation must contain as many arbitrary constants as the number which expresses the order of the equation. For a differential equation is formed by eliminating the constants in an ordinary equation from the results of successive differentiations.

We can see that the solution of a physical problem requires such constants; for, taking the case of pendulum motion, all such bodies move after one law, but the velocity or position of any one at any time will depend on the velocity and position at starting. Hence in physical problems the arbitrary constants are determined from the circumstances of the motion being known in some one position or at some one time.

Thus equations of the form (1) contain one constant, of (2) two constants, and of (3) two each. Linear differential equations have farther these properties.

(1) If the order is n, the addition of n independent values of x, each with an arbitrary constant, gives the complete solution. For it satisfies all the conditions of a complete solution.

(2) That part of the solution which contains the arbitrary constants is the same for all equations having the same terms involving the dependent variable and its differential coefficients. Thus the solution of (2) is found from that of

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0,$$

by adding any particular value of x (without arbitrary constants) which makes

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx \text{ equal to } cf(t).$$

Lastly, every particular solution of a differential equation corresponds to a possible motion or other physical condition.

3. *First form.*

$$\frac{dx}{dt} + ax = bf(t).$$

Multiplying by e^{at}

$$e^{at}\frac{dx}{dt} + ae^{at}x = be^{at}f(t).$$

Integrating

$$xe^{at} = A + b\int e^{at}f(t)dt \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The following are important cases:

$$(\alpha) \quad \frac{dx}{dt} + ax = 0.$$

The solution is $x = Ae^{-at}$.

$$(\beta) \quad \frac{dx}{dt} + ax = b \sin nt.$$

The part of the solution involving the constant is $x = Ae^{-at}$. For the second part we remark that $x = p \sin nt + q \cos nt$ will satisfy the equation provided p and q be properly determined. To secure this, differentiate and substitute; then

$$(pn + qa) \cos nt + (pa - qn) \sin nt = b \sin nt.$$

If these are identical,

$$\left.\begin{array}{l} pn + qa = 0 \\ pa - qn = b \end{array}\right\};$$

whence the solution is

$$x = Ae^{-at} + \frac{b}{n^2 + a^2}(a \sin nt - n \cos nt).$$

The solution might have been obtained from (1) by remembering that $\int e^{at} \sin nt\, dt$ is of the form

$$e^{at}(p \sin nt + q \cos nt).$$

The first part (Ae^{-at}) of the solution indicates a gradually diminishing or (if a be negative) increasing motion; the second indicates a motion of oscillation superimposed on this.

4. *Second form.*

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = cf(t).$$

Consider first

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0;$$

it is clear that $x = Ae^{mt}$ will satisfy it if $m^2 + am + b = 0$. Let m_1, m_2 be the roots of this quadratic. Then the solution is

$$x = Ae^{m_1t} + Be^{m_2t}.$$

If the roots of the quadratic are impossible, and of the form $\alpha \pm \beta\sqrt{-1}$,

the solution is $x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$.

For the particular value that must be added when $cf(t)$ is not zero, we must in general depend on happy thoughts. The rules are too long to be given here for all the cases.

The following are important:

(α) $$\frac{d^2x}{dt^2} - n^2x = 0.$$

Solution. $x = Ae^{nt} + Be^{-nt}$.

(β) $$\frac{d^2x}{dt^2} + n^2x = 0.$$

Solution. $x = A\cos(nt + B)$ or $A\sin nt + B\cos nt$.

(γ) $\frac{d^2x}{dt^2} + n^2x = a$ can be expressed

$$\frac{d^2}{dt^2}\left(x - \frac{a}{n^2}\right) + n^2\left(x - \frac{a}{n^2}\right) = 0.$$

(δ) $$\frac{d^2x}{dt^2} + n^2x = a\sin mt.$$

The solution of

$$\frac{d^2x}{dt^2} + n^2x = 0 \text{ is } x = Ae^{nt} + Be^{-nt}.$$

And it is clear that $x = p \sin mt$ will make

$$\frac{d^2x}{dt^2} + n^2x = a \sin mt,$$

if p have the right value. To determine p, differentiate $p \sin mt$ twice and substitute. Then $-pm^2 + n^2p = a$; whence $p = \frac{a}{n^2 - m^2}$, and the complete solution is

$$x = Ae^{nt} + Be^{-nt} + \frac{a}{n^2 - m^2} \sin mt.$$

The equation (β) indicates a motion of oscillation whose central point is the origin, and whose period is $\frac{2\pi}{n}$; (γ) indicates a similar motion with the central point at a distance $\frac{a}{n^2}$ from the origin.

5. Equations of the form (3) are in general the expression of co-existent oscillations, and the coefficients will be such as give a solution in sines and cosines.

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + a\frac{dy}{dt} + bx = 0 \\ \frac{d^2y}{dt^2} + a'\frac{dx}{dt} + b'y = 0 \end{aligned}\right\}.$$

Assume $x = A \sin(nt + B)$,

$$y = A' \cos(nt + B).$$

It is evident that these will satisfy the equations, if A, A' be suitably determined. To do this, differentiate and substitute

$$\left.\begin{aligned} -An^2 - A'na + bA = 0 \\ -A'n^2 + a'nA + b'A' = 0 \end{aligned}\right\}.$$

From these the ratio $\frac{A'}{A}$ is found, and also a quadratic for n^2. Let the values of n^2 be n_1^2, n_2^2. Suppose them positive. And suppose $A' = \mu A$.

Then $x = A_1 \sin(n_1 t + B_1) + A_2 \sin(n_2 t + B_2)$,

$$y = \mu A_1 \cos(n_1 t + B_1) + \mu A_2 \cos(n_2 t + B_2).$$

In this n_1, n_2 and μ have definite values; A_1, A_2, B_1, B_2 depend on the initial circumstances. As there are here four constants no extension of generality is gained by taking the roots $-n_1$, $-n_2$ of the biquadratic for n. The solution indicates that there are two independent oscillations going on together in x, and two also in y. If the values of n^2 are not positive, x and y will involve exponentials.

6. When the Differential Equations, at which we arrive by eliminating unknown forces, are not completely integrable, they can frequently be integrated once.

The equation

$$\frac{d^2\theta}{dt^2} + f(\theta)\left(\frac{d\theta}{dt}\right)^2 = \phi(\theta)$$

is of constant occurrence, especially in problems of sliding friction. Assume $\left(\frac{d\theta}{dt}\right)^2 = 2z$;

then $$\frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} = \frac{dz}{dt},$$

and $$\frac{d^2\theta}{dt^2} = \frac{dz}{d\theta}.$$

The equation may therefore be written

$$\frac{dz}{d\theta} + 2f(\theta) z = \phi(\theta).$$

Multiplying by $e^{2\int f(\theta)d\theta}$, both sides become perfect integrals.

Integrating, $ze^{2\int f(\theta)d\theta} = A + \int \phi(\theta) e^{2\int f(\theta)d\theta} . d\theta.$

EXAMPLES.

1. What kind of motion is indicated by the equation $x = ct \sin nt$?

2. Of what differential equation is $x = at + Ae^{mt}$ the solution?

Solve the following equations:

3. $$\frac{dx}{dt} + ax = be^t.$$

4. $$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 4x = 0.$$

5. $$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 4x = t.$$

6. $$\frac{d^2x}{dt^2} + n^2x = a \cos mt.$$

7. $$\left.\begin{aligned} \frac{dx}{dt} + ay &= 0 \\ \frac{dy}{dt} + ax &= 0 \end{aligned}\right\}.$$

8. $$\left.\begin{aligned} \frac{dx}{dt} + ay &= 0 \\ \frac{dy}{dt} + ax &= c \sin mt \end{aligned}\right\}.$$

Prove that the equations

9. $$\left.\begin{aligned} \frac{d^2x}{dt^2} - a\frac{dy}{dt} &= 0 \\ \frac{d^2y}{dt^2} + a\frac{dx}{dt} + y &= 0 \end{aligned}\right\}$$

represent an oscillatory motion.

MISCELLANEOUS PROBLEMS.

1. A rigid body in which A, B, C are three points, moves so that these come into the positions a, b, c. Aa, Bb, Cc being very small spaces given in magnitude and direction, find the motion of translation and of rotation of the body.

2. A flywheel is driven by a piston acting on a crank alternately up and down with a force P; find the limits between which the velocity varies.

3. If a mass is animated by simultaneous velocities, its moment of momentum about any axis is equal to the sum of the moments of the separate momenta about that axis. How does this appear?

4. A lamina rotating in its plane about its centre of inertia is suddenly brought to rest by sticking a two-pronged fork into it. Shew (1) that the impulses on the prongs are equal, (2) that they are of the same magnitude wherever the fork is stuck in.

5. A wheel of which an axle projecting on each side forms a part, is supported in a vertical plane by having the axle on each side resting on a pair of friction wheels each of which is just like the first wheel and is similarly supported, and so on indefinitely; compare the inertia of the whole system in relation to a rotation of the first wheel with that of the first wheel alone.

6. A pendulum performs small oscillations in a medium of which the resistance varies as the square of the velocity; given the number of oscillations in which the arc of oscillation is reduced one half, compare the original resistance with the weight of the pendulum.

7. Shew that when the centre of gravity of any system of material particles in motion passes through a point of contrary flexure, the momentum of the system is in general a maximum or minimum, and the resultant of the effective forces is zero.

8. Two points B, C of a circular ring moveable in its own plane about its centre are connected with a fixed point A by elastic strings the natural length of each of which is equal to the shortest distance c, between A and the ring. Supposing the ring turned through any angle and let go, calculate the motion and shew that the time of a small oscillation is $\pi\left(\frac{mc}{2\lambda}\right)^{\frac{1}{2}}$, where m is the mass of the ring, and λ the modulus of elasticity of the string.

9. A uniform bar of length $2a$ is suspended horizontally by two parallel strings each of length l attached at distances c from the middle point. It receives a small angular disturbance so that it oscillates about a vertical axis. Prove that it makes small oscillations in the same time as a simple pendulum of length $\frac{4al}{3c}$.

10. Prove that if any straight line (taken to be the axis of z) is a principal axis at some point (not necessarily the origin) $\frac{\Sigma\delta myz}{\bar{y}} = \frac{\Sigma\delta mxz}{\bar{x}}$.

11. A uniform rod of length $2a$ has its ends on two straight lines meeting at right angles in a point O, and makes an angle θ with one of them. Every point of the rod is attracted to the point O with a force $\frac{\mu}{(\text{distance})^2}$. Prove that if ρ be the distance of an element of mass $\nu\delta\rho$ from the middle point of the rod

$$\Sigma\int(Xdx + Ydy + Zdz) = 2\mu\nu\alpha\int_{-a}^{+a}\int\frac{\rho\sin 2\theta d\theta d\rho}{(\rho^2 + a^2 - 2a\rho\cos 2\theta)^{\frac{3}{2}}}.$$

12. A uniform revolving rod the centre of gravity of which is initially at rest, moves in a plane under the action of a constant force in the direction of its length; prove that

the square of the radius of curvature of the path of the rod's centre of gravity varies as the versed-sine of the angle through which the rod has revolved at the end of any time from the beginning of the motion.

13. A particle is attached by a string to the end of a rod n times as long as the string, which rotates in a given manner about the other end; the whole motion taking place in a horizontal plane. If θ be the inclination of the rod and string, and ω the angular velocity of the rod at the time t, prove that

$$\frac{d^2\theta}{dt^2} + \frac{d\omega}{dt} + n\left(\frac{d\omega}{dt}\cos\theta + \omega^2 \sin\theta\right) = 0.$$

14. A uniform circular disc, whose upper surface is imperfectly rough, rests on a smooth horizontal table. A particle is tied by a stretched inextensible string to the centre and then projected along the disc at right angles to the string. Prove that the particle will come to rest on the disc before the string becomes slack.

15. A Catharine wheel is constructed by rolling a thin casing of powder several times round the circumference of a circular disc of radius a. If the wheel burn for a time T, and the powder be fired off with relative velocity V along the circumference, shew that the angle turned through by the wheel will be

$$\frac{VT}{a}\left\{1 - c\log\left(1 + \frac{1}{c}\right)\right\},$$

where $2c$ is the ratio of the masses of the disc and powder. The casing is supposed so thin that the distance of all the powder from the centre of the disc is a.

SUGGESTIONS AND RESULTS FOR THE EXAMPLES AND PROBLEMS.

I. Page 10.

(3) The unit of angular velocity is when the unit of circular measure is described in one second.

(4) The earth rotates once in 23 hours 56 minutes (nearly).

(7) At a distance $\frac{v}{\omega}$ from the axis of the paddle-wheel.

(8) The reasoning is given in Lesson II. Art. 4.

(9) The cylinder fixed in space becomes a plane.

(10) The focus is at the instantaneous centre.

(11) The velocities of P and Q are proportional to their distance from the intersection of AP and BQ.

II. Page 18.

(3) The points of contact are moving with the same velocity.

(4) It has one rotation round the vertical axis and another round the line of contact with the table.

(6) and (7) These appear from the identity of the laws for the composition of forces, and those for the composition of angular velocities.

III. Page 34.

(2) With a velocity represented by BB'.

(3) Round a horizontal axis inclined to the line of wickets at $\tan^{-1}\frac{\omega'}{\omega}$.

(4) Diminishes.

(8) If $\dot{v}$, $\dot{\omega}$ be the acceleration of the centre and the angular acceleration, the force required will be that found in the Example Art. 9, and also a horizontal force $\delta m(\dot{v}+a\dot{\omega})$.

IV. Page 48.

(2) The centre of inertia is fixed.

(3) 30,000; 300,000.

(4) With constant velocity in a straight line.

(5) Any function of the forms

$$\Sigma\delta m\frac{d^n x}{dt^n},\ \Sigma\delta m\left(\frac{d^n x}{dt^n}\right)^2,\ \Sigma\delta my\frac{d^n x}{dt^n},$$

or the sum of such terms.

(6) On the whole hoop; the effective force is horizontal, and is mass of hoop $\times\,\dot{v}$; the couple is mass $.\,a^2.\,\dot{\omega}$. On the part AB; the force acts at G, and is compounded of $m\,.\,CG\,.\,\omega^2$ along GC, $m\dot{v}$ horizontally, and $m\,.\,CG\,.\,\dot{\omega}$ at right angles to GC. The couple is $\dot{\omega}$ (moment of inertia about G) and $\dot{v}=a\dot{\omega}$.

(7) Calling the centre of inertia of AB, G; the acceleration of G relatively to A along GA is $\omega'^2.\,b$. That of A is compounded of $\omega^2.\,2a$ along AO and $\frac{d\omega}{\delta t}.\,2a$ at right angles to AO. Hence the acceleration of A measured along BA is

$\omega^2 \,.\, 2a \cos\phi - \frac{d\omega}{dt} \,.\, 2a \sin\phi$. The whole acceleration of G along BA is

$$\omega'^2 b + \omega^2 2a \cos\phi - \frac{d\omega}{dt} \,.\, 2a \sin\phi.$$

V. Page 58.

(2) The angular momentum remains constant while the moment of inertia diminishes.

(4) Vertically downwards.

(5) The angular momentum about the instantaneous centre remains constant.

(7) The direction of the string must pass through the centre of inertia.

(8) (Second part). There has been no external force, and therefore all will come to rest again.

(9) He increases the angular momentum of projection.

(10) The angular and linear momenta do not alter in the interval between the blows.

(12) Yes. By swinging his arm round in a horizontal plane.

VI. Page 66.

(5) The only two bodies are a particle and a straight rod.

(7) The side, its perpendicular bisector and the normal to the plane.

(8) See Art. 6.

(9) $\frac{2}{5} \cdot \frac{A^5 - a^5}{A^3 - a^3}$.

(10) An axis parallel to an edge.

(16) $\frac{ma^2\omega}{2t}$.

VII. Page 71.

(3) $\frac{3mh^2}{20}(4 + \tan^2 \alpha)$.

VIII. Page 80.

(6) The moment of inertia must be greatest.

IX. Page 87.

(1) A force equal to the weight of $44\frac{4}{5}$ lbs.

(2) Neglecting the square of $\frac{1}{h}$, the tension is

$$m\left(\beta - \frac{g}{h} \pm \mu g\right).$$

(5) If V be the velocity just before striking, and the angle just found be called α, the couple will be

$$\text{mass } AC \times \frac{V \sin \alpha \,.\, AC}{2}.$$

X. Page 99.

(1) Because the long rod takes longer to fall.

(4) Let V be the velocity of the plate, and $2a$ the edge of the cube. Then the cube will begin to rise up about its edge with angular velocity $\frac{3V}{8a}$.

(5) See Art. 6.

(6) The distance of each from the fixed axis depends on the moment of the resultant effective couple.

(13) If O be the fixed point, OA the rod, P the point where the blow is struck, Q the required point, G the middle point of QA,

$$OA^3 = 3 \,.\, OP \,.\, AQ \,.\, OG.$$

XI. Page 114.

(2) The length of the rod being $2a$, and the angle it makes with the vertical at first being α; the angular velocity when the angle with the vertical becomes θ is the square root of $\frac{6g}{a} \cdot \frac{\cos\theta - \cos\alpha}{1 + 3\sin^2\theta}$. The centre of gravity moves in a vertical straight line.

(5) The angular momentum about that edge is unchanged.

(6) First find the value of the pressure at one end. The motion is the same as that of a compound pendulum.

(7) Prove that the pressure never vanishes.

(12) The ball starts forwards with four-sevenths of the velocity of the centre of the wheel. It remains in contact with the rim and rolls off backwards, leaving the wheel when its direction from the centre makes an angle with the vertical whose cosine is $\frac{1}{17}\left\{10 + \frac{16}{7}\frac{V^2}{g(r+R)}\right\}$, V being the velocity of the wheel and r, R the radii of the ball and wheel.

XII. Page 137.

(1) Consider each as acted on by an impulsive friction, and take moments for each about its centre.

(3) The external force is $(m + m')g$.

(4) The angular momentum about the instantaneous centre of the bar remains zero.

(12) At the moment of greatest compression the velocity of the ball and of the point it touches resolved along the normal are equal.

(17) Displace the system, keeping O and P fixed. Or take moments for AB about A and for the whole system about O.

(19) Use the method of virtual velocities and give the natural displacement.

(22) There is no rotation.

XIII. Page 159.

(3) Supposing m the mass, a the unstretched length of the string, x the distance from the point of suspension, and taking the point of suspension as the origin, the potential energy due to the tension is $\int_a^x T dx$, and that due to gravity is $-mgx$.

Whence $\frac{1}{2}mv^2 + \lambda \int \frac{x-a}{a} dx - mgx = -mga$, λ being the modulus of elasticity.

(6) The common centre of inertia will move in a straight line with uniform velocity; and the centres of the ring and rod will describe circles relatively to it.

(11) The angular momentum of each remains constant, and the work goes to increase or diminish the kinetic energy.

(12) Observe that the velocities of all points of the string are at any moment equal.

XIV. Page 168.

(1) If L be the couple and A the moment of inertia; the axis will rotate about the vertical in time $\frac{2\pi \sin \alpha \Omega A}{L}$.

(3) The true time is 25,868 years. Take $\sin 23\frac{1}{2}^\circ$ to be ·41.

(4) Nearly four feet.

(5) One day.

XV. Page 176.

(1) Oscillations of an increasing magnitude and constant period about the same central point.

(2) $\frac{dx}{dt} - mx = a(1 - mt)$.

(3) $x = Ae^{-at} + (b - a)e^{t}$.

(4) $x = Ae^{t} + Be^{4t}$

(5) The additional terms must be of the form $p + qt$.

(6) $x = A \sin nt + B \cos nt + \frac{a}{n^2 - m^2} \cos mt$.

(8) Differentiating the second and substituting for $\frac{dx}{dt}$ from the first, we find

$$\frac{d^2y}{dt^2} - a^2 \frac{dy}{dt} = c \sin mt,$$

the solution of which is $y = A + Be^{a^2 t} + p \sin mt + q \cos mt$, in which A and B are arbitrary while p and q can be found by differentiating and substituting.

(9) Solve as in Art. 5.

MISCELLANEOUS PROBLEMS.

(1) If a direction can be found along which the components of Aa, Bb, Cc are all equal this will be the direction of translation. To this end, from any point O draw Oa', Ob', Oc' parallel and proportional to them, and on these as diameters describe spheres. If these intersect in p, Op is the required direction.

(2) The variation of kinetic energy is equal to the variation of the potential energy.

(3) It follows from the parallelogram law.

(5) The angular velocities diminish in geometrical progression.

(10) Suppose it is a principal axis at a distance c from the origin. Using the symbols of Lesson VIII. Art. 2, we must have

$$\Sigma dmzr \cos(\theta - \alpha) = 0, \quad \Sigma dmzr \sin(\theta - \alpha) = 0.$$

(11) $X\delta x + Y\delta y + Z\delta z$ for one element is $\frac{\mu}{r^2} . \delta r$; and the variation of r is due to the displacement of the whole body.

(13) Resolve at right angles to the string, and introduce the expressions for relative acceleration.

(15) The whole angular momentum is constant.

THE END.

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