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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

ELEVENTH SESSION, 1892-93.

First Meeting, November 11th, 1892.

J. ALISON, Esq., M.A., F.R.S.E., in the Chair.

For this Session the following Office-bearers were elected :—

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Vice-President—Professor KNOTT, D.Sc., F.R.S.E.

Secretary—Mr JOHN B. CLARK, M.A., F.R.S.E.

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Messrs J. W. BUTTERS, JOHN M'COWAN, D.Sc.; WILLIAM PEDDIE, D.Sc., F.R.S.E.; CHARLES TWEEDIE, M.A., B.Sc.; and WM. WALLACE, M.A.

Note on the Solution of Partial Differential Equations by the Method of Reciprocation.

By JOHN M'COWAN, D.Sc.

In a paper "On the Theory of Long Waves" recently published,* I drew attention to a point of some interest in the theory of the solution of partial differential equations by what is usually termed the method of reciprocation. As the subject was apart from the main object of the paper, however, I there noticed the peculiarity but briefly, and so I have thought that it might be of some use to enter into the matter here somewhat more fully. On first adverting to the point, it was my opinion that it could hardly have escaped the notice of those who had made much application of the method of solution by reciprocation, though I had not myself seen any reference made to it; but as I have not had my attention drawn to any such notice since the publication of the paper last March, it seems probable that it has not previously been discussed.

The method of reciprocation, due, I believe, to Legendre, is so well known that the following brief description may here suffice. By changing the variables in the given equation from x, y, and z—employing the usual notation—to p, q and $px+qy-z$ respectively, say x', y' and z' for the sake of symmetry in the relations, a new equation in x', y', z', p', q', r', s', t' called the reciprocal equation, is obtained, which may be easier of solution than the original equation. Between the equations the following reciprocal relations hold, viz., $xx'+yy'=z+z'$, $x'=p$, $y'=q$, $x=p'$, $y=q'$, and others between the differential co-efficients of the second order, which need not be written down. The equations are thus reciprocal to one another, and from the solution of one that of the other may be obtained by means of the preceding relations.

To these reciprocal relations a geometrical interpretation may be given, which has the advantage of showing very clearly the nature of certain limitations to which reference is about to be made. If x, y, z be regarded as the co-ordinates of a point on a surface S, then x', y', z' are the co-ordinates of a point on a surface S′ which is the polar reciprocal of the surface S with respect to the paraboloid of revolution $x^2+y^2=2z$, and conversely.

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* *Philosophical Magazine*, March 1892.

Consider first the process of reciprocation from the purely analytical point of view, without reference to its geometrical significance. In forming the reciprocal equation p and q (written x' and y') are taken as independent variables, and therefore it is tacitly assumed that p and q are independent. There must therefore be cases in which the method will fail to give, legitimately at least, the full solution of an equation: in some cases, as, for example, that discussed in the paper on waves, already referred to, the most important class of solutions is that in which p is a function of q. It is true that when the *general* solution of an equation can be obtained by this method, those cases in which p and q are not independent may be derived from it by considering them as limiting cases—though it may not always be easy to do so—and the process might be justified in the same way as in other limiting cases; but it must be remembered that it is by no means the same thing to prove that a theorem holds true up to a limit, and to prove that it holds at the limit.

It is seldom, however, that the general solution of even the reciprocal equation can be obtained in finite terms: generally only a series of particular solutions can be obtained, or, if the reciprocal equation is linear, a general solution made up from these. From such particular solutions no solutions of the original equation in which $p = f(q)$ can be obtained, and from a general solution made up from them such a solution could in general only be derived with great difficulty. Hence, even though a logically rigorous method be not insisted on, the method will generally fail in practice to give this important class of solutions.

The nature of the restriction is made very clear by taking the geometrical point of view. If a surface S is the locus of points whose co-ordinates are so related as to give a solution of the original equation, S may for brevity be said to give a solution of the equation: then its reciprocal S′ gives a solution of the reciprocal equation. Now if $p = f(q)$, S will be a developable surface, and its polar reciprocal S′ will degenerate into a line, the locus of the poles of its tangent planes; and conversely. But if S′ is a line its co-ordinates cannot well be spoken of as satisfying any partial differential equation, for p' q' r' s' t' are essentially indeterminate. Thus, then, it is clear that solutions of the type $p = f(q)$ can only be got by the method of reciprocation by considering line solutions of the reciprocal equation: such lines can only, of course, be regarded as giving solutions of an equation when regarded as limiting forms of surfaces which give solutions.

The general conclusion following from these considerations may therefore be thus stated :—The method of solution of partial differential equations by reciprocation can only be regarded as theoretically complete when the deduction of solutions of the type which gives $p=f(q)$ as limiting cases of the general solution is fully justified : but in practice, even though the theoretical completeness of the method of reciprocation be assumed, it should be supplemented by an independent investigation of solutions of the limiting type $p=f(q)$. It should be noticed that Poisson has given a method for obtaining solutions of this type of a certain class of equations : a class to which, it should further be remarked, the method of reciprocation is generally very applicable.

In exemplification of the foregoing discussion, it is desirable to add one or two examples.

Take first the frequently occurring equation

$$q^2r - 2pqs + p^2t = 0 \ldots \quad \ldots \quad \ldots \quad (1).$$

The transformed or reciprocal equation is therefore

$$x'^2r' + 2x'y's' + y'^2t' = 0 \quad \ldots \quad \ldots \quad (2),$$

of which the well known solution is

$$z' = x'f_1(y'/x') + f_2(y'/x') \quad \ldots \quad \ldots \quad (3).$$

This gives

$$x = p' = \frac{\delta z'}{\delta x'} = f_1 - \frac{y'}{x'}f'_1 - \frac{y'}{x'^2}f'_2 \ldots \quad (4).$$

$$y = q' = \frac{\delta z'}{\delta y'} = f'_1 + \frac{1}{x'}f'_2 \quad \ldots \quad \ldots \quad (5)$$

$$\therefore \quad z = xx' + yy' - z' = -f_2(y'/x') \quad \ldots \quad (6).$$

But (4) and (5) give

$$x + yy'/x' = f_1 \quad \ldots \quad \ldots \quad \ldots \quad (7),$$

which, by means of (6) gives, changing the arbitrary functions,

$$x + y\mathrm{F}_1(z) = \mathrm{F}_2(z) \ldots \quad \ldots \quad \ldots \quad (8),$$

which is therefore the general solution of the given equation (1), but may be more symmetrically written

$$z = x\phi(z) + y\psi(z) \ldots \quad \ldots \quad \ldots \quad (9).$$

Next seek solutions of the type $p=f(q)$, by Poisson's method.

Regarding p as a function of q in (1), it may be written

$$q^2dp^2 - 2pqdp.dq + p^2dq^2 = 0 \quad \ldots \quad (10),$$

whence

$$qdp - pdq = 0 \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\therefore \quad Ap = q \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\therefore \quad z = F(x + Ay) \quad \ldots \quad \ldots \quad \ldots \quad (13),$$

the solution sought, and which it may be noted is what the general solution (8) above reduces to when $F_1(z)$ is taken $=$ A, (or $\psi = A\phi$ in (9)).

This example has been taken to show how the special solution may be deduced as a limiting case of the general solution, when that is known; and to show also how easily it may be got by a direct process. It is to be carefully noted that it cannot be derived from anything less than a general solution; it makes y'/x' in (7) constant, and so is not derivable from any true determinate solution of the reciprocal equation (2).

Consider next an equation whose general solution cannot be obtained in finite terms; for example,

$$r - p^2t = 0 \quad \ldots \quad \ldots \quad \ldots \quad (14).$$

Seek first by Poisson's method the solutions in which q and p are not independent: on this hypothesis (14) gives

$$dp^2 = p^2dq^2 = 0 \quad \ldots \quad \ldots \quad (15)$$

$$\therefore \quad dp \pm p\,dq = 0 \quad \ldots \quad \ldots \quad (16)$$

$$\therefore \quad q = \pm \log mp. \quad \ldots \quad \ldots \quad (17)$$

$$\therefore \quad \left.\begin{array}{lll} z = ax & \pm y\log ma & - f(a) \\ 0 = x & \pm y/a & - f'(a) \end{array}\right\} \quad \ldots \quad (18).$$

Thus there are two sets of such solutions, got by taking the upper or the lower signs throughout in (18), in which a may be eliminated between the pair of equations, or retained as a variable parameter, as may be most convenient.

These two sets of special solutions have thus been almost immediately obtained by recognising directly the possibility of their existence. Consider next what can be done towards the solution of (14) by the method of reciprocation which may be said to ignore such solutions.

The reciprocal equation is

$$x'^2r' - t' = 0 \quad \ldots \quad \ldots \quad \ldots \quad (19).$$

This cannot be solved in finite terms, but

$$z' = Cx'^m\epsilon^{ny} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

is obviously a particular solution, provided that

$$m^2 = n(n-1) \quad \ldots \quad \ldots \quad \ldots \quad (21).$$

Hence the general solution of (19) is given by

$$z' = \Sigma C_m x'^m \epsilon^{ny'}, \quad \ldots \quad \ldots \quad (22),$$

in which all values of m and n may be taken, subject to the condition (21). To get the general (or any) solution of (14) from this, the reciprocal relations between the equations have to be used.

Thus

$$x = p' = \Sigma m C_m x'^{m-1} \epsilon^{ny'} \quad \ldots \quad \ldots \quad (23)$$

$$y = q' = \Sigma n \, C_m x'^m \, \epsilon^{ny'} \quad \ldots \quad \ldots \quad (24)$$

$$z = xx' + yy' - z' = \Sigma C_m(m + ny' - 1)x'^m \epsilon^{ny'} \quad \ldots \quad (25).$$

These three equations (22), (24), (25), the condition (21) being understood, may be regarded as together forming the general solution of (14), x' and y' being regarded as variable parameters, which may in special cases be eliminated between these three equations if found desirable.

Now in some limiting form these three equations will presumably reduce so as to give the special solution (18), but it is not easy to see in what particular manner. From (18), in fact, it follows in virtue of the reciprocal relations, that

$$x' = p = \quad a$$

$$y' = q = \pm \log ma$$

$$z' = xx' + yy' - z = f(a)$$

whence

$$y' = \pm \log mx' = F(z') \quad \ldots \quad \ldots \quad (26),$$

showing that the solutions of (19) which give by reciprocation those of (18), are given by the lines determined by (26). It is evident, however, that (26) can only in a very guarded sense be regarded as giving a solution of (19) at all: it corresponds to none of the particular solutions of the type (20), and it is not easy to see how to get at it as a limiting form of (22).

Thus, finally, we see that it is practically impossible to arrive at any of the important solutions included in (18), either in their general form or as particular cases of it, by the method of reciprocation. In the case considered, regarding (14) as referring to an important problem in the propagation of sound, it may be said that the case missed by reciprocation is physically the most important: it is needless to multiply examples, but further instances may be found in the equations of wave motion which are discussed in the paper already cited.

On the use of Dimensional Equations.

By W. PEDDIE, D.Sc.

[Abstract.]

The second law of motion may be expressed as a dimensional equation in the form

$$f = m\frac{l}{t^2} \quad \ldots \quad \ldots \quad \ldots \quad (1),$$

where the meanings of the quantities are obvious.

If we cut out the factor m from each side, we may write this in the usual form,

$$\ddot{x} = a\frac{x}{t^2} \quad \ldots \quad \ldots \quad \ldots \quad (2).$$

The general solution is

$$x = \mathrm{A}t^n + \mathrm{B}t^{-m},$$

where $$m(m+1) = n(n-1).$$

Taking one term only, we get

$$\ddot{x} = n(n-1)\mathrm{A}t^{n-2} = n(n-1)\mathrm{A}^{\frac{2}{n}}x^{\frac{n-2}{n}} \quad \ldots \quad \ldots \quad (3),$$

so that the limited problem corresponds to powers of the distance as the law of acceleration.

We then have in (2) $a = n(n-1)$, and so, when the law of force is given in terms of the distance, we can use (2) and (3) to get an expression for t.

Second Meeting, December 9th, 1892.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

On certain results involving Areal and Trilinear co-ordinates.

By Professor A. H. ANGLIN.

We propose to obtain certain results involving areal and trilinear co-ordinates, by a uniform method of changing to Cartesian co-ordinates with two sides of the triangle of reference as axes.

Taking ABC as the triangle of reference, change to Cartesian co-ordinates with CA and CB as axes. Then, if $\bar{x}$, $\bar{y}$ denote the Cartesian, x, y, z the areal, and α, β, γ the trilinear co-ordinates of any point, we have at once

$$\bar{x} = bx, \qquad \bar{y} = ay\,;$$

and

$$\bar{x} = \alpha/\sin C, \quad \bar{y} = \beta/\sin C.$$

1. *To find the distance between two points.*

If r be the distance between two points whose areal co-ordinates are given, substituting in the usual expression in oblique Cartesians for the square of the distance, we get

$$r^2 = b^2(x_1 - x_2)^2 + a^2(y_1 - y_2)^2 + 2ab\,(x_1 - x_2)(y_1 - y_2)\cos C.$$

$$\therefore \quad -r^2 = c^2(x_1 - x_2)(y_1 - y_2) - a^2(y_1 - y_2)(x_1 - x_2 + y_1 - y_2)$$
$$- b^2(x_1 - x_2)(x_1 - x_2 + y_1 - y_2).$$

Thus, by the invariable relation $x + y + z = 1$ in areals, we have

$$-r^2 = a^2(y_1 - y_2)(z_1 - z_2) + b^2(z_1 - z_2)(x_1 - x_2) + c^2(x_1 - x_2)(y_1 - y_2) \ldots \quad (1).$$

The corresponding in trilinears may be obtained independently in like manner, or deduced from the foregoing, the result being

$$-r^2 = \{(\beta_1 - \beta_2)(\gamma_1 - \gamma_2)\sin A + \ldots\}/\sin A \sin B \sin C \quad \ldots \quad (1)'$$

The distance between two points may also be expressed in other interesting forms.

Since $$(x_1-x_2)+(y_1-y_2)=-(z_1-z_2),$$
we have $$2(x_1-x_2)(y_1-y_2)=(z_1-z_2)^2-(x_1-x_2)^2-(y_1-y_2)^2.$$

Substituting in

$$r^2=b^2(x_1-x_2)^2+a^2(y_1-y_2)^2+(a^2+b^2-c^2)(x_1-x_2)(y_1-y_2),$$

we shall get

$$2r^2=(b^2+c^2-a^2)(x_1-x_2)^2+\text{two similar expressions},$$

or

$$r^2=bc\cos A(x_1-x_2)^2+ca\cos B(y_1-y_2)^2+ab\cos C(z_1-z_2)^2 \quad \ldots \quad (2).$$

The corresponding in trilinears will be found to be

$$r^2=\frac{\sin 2A(\alpha_1-\alpha_2)^2+\sin 2B(\beta_1-\beta_2)^2+\sin 2C(\gamma_1-\gamma_2)^2}{2\sin A\sin B\sin C} \quad \ldots \quad (2)'.$$

Further, since

$$x_1-x_2=(x_1y_2-x_2y_1)-(z_1x_2-z_2x_1)=Z-Y \text{ suppose},$$

with like equivalents for y_1-y_2 and z_1-z_2, substituting in (1) and (2) we shall get the additional forms

$$\begin{aligned} r^2&=a^2(X-Y)(X-Z)+b^2(Y-Z)(Y-X)+c^2(Z-X)(Z-Y) \quad \ldots \quad (3)\\ &=bc.\cos A(Y-Z)^2+ca\cos B(Z-X)^2+ab\cos C(X-Y)^2\\ &=a^2X^2+b^2Y^2+c^2Z^2-2bcYZ\cos A-2caZX\cos B-2abXY\cos C,\end{aligned}$$

which are sometimes useful.

2. *To find the perpendicular distance of a point from a straight line.*

Let the equation to the line in areal co-ordinates be

$$lx+my+nz=0,$$

and x', y', z' the co-ordinates of the point.

Reducing the problem to the oblique Cartesian system, we have to find the perpendicular from the point (bx', ay') on the line

$$a(l-n)\bar{x}+b(m-n)\bar{y}+nab=0.$$

Now the perpendicular from (x', y') on the line $Ax+By+C=0$ in oblique Cartesians being

$$(Ax'+By'+C)\sin\omega/\sqrt{A^2+B^2-2AB\cos\omega},$$

the required perpendicular

$$= \frac{ab\sin C\{(l-n)x'+(m-n)y'+n\}}{\surd\{a^2(l-n)^2+b^2(m-n)^2-2ab(l-n)(m-n)\cos C\}}$$

$$= \qquad 2(lx'+my'+nz')/d,$$

where

$$d^2 = a^2l^2+b^2m^2+c^2n^2-2bcmn\cos A - \ldots$$
$$= (l-m)(l-n)a^2+(m-n)(m-l)b^2+(n-l)(n-m)c^2.$$

[The corresponding expression in Trilinears may be deduced from this, or obtained independently, as follows :—

If the point be (α', β', γ'). and the line $l\alpha+m\beta+n\gamma=0$, changing to Cartesians, we seek the perpendicular from the point ($\alpha'\operatorname{cosec}C$, $\beta'\operatorname{cosec}C$) on the line

$$(cl-an)\bar{x}+(cm-bn)\bar{y}+2n\Delta\operatorname{cosec}C=0,$$

which

$$= \frac{(cl-an)\alpha'+(cm-bn)\beta'+2n\Delta}{\surd\{(cl-an)^2+(cm-bn)^2-2(cl-an)(cm-bn)\cos C\}}$$

$$= \qquad (l\alpha'+m\beta'+n\gamma')/d,$$

where

$$d^2 = l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C.]$$

3. *The perpendicular from a given point on the line joining two other given points may be noticed.*

The equation to the line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) being

$$\Sigma(y_1z_2-y_2z_1)x+(z_1x_2-z_2x_1)y+(x_1y_2-x_2y_1)z=0,$$

the perpendicular on it from the point (x_3, y_3, z_3) becomes

$$2\Delta\{(y_1z_2-y_2z_1)x_3+\ \ldots\ \}/d,$$

where

$$d^2=(X-Y)(X-Z)a^2+(Y-Z)(Y-X)b^2+(Z-X)(Z-Y)c^2.$$

Thus, by reference to the third expression for the distance between two points, we see that the perpendicular is

$$2\Delta(x_1y_2z_3)/d,$$

where d is the distance between the points (x_1, y_1, z_1), (x_2, y_2, z_2).

In Trilinears, the corresponding expression will be found to be

$$abc(\alpha_1\beta_2\gamma_3)/4\Delta^2 d.$$

These results also follow directly from the consideration that, in Cartesians, the perpendicular is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y & 1 \end{vmatrix} \frac{\sin\omega}{d};$$

and on transformation to the other systems we readily obtain the above expressions.

4. *To find the area of a triangle in terms of the co-ordinates of its angular points.*

(1) Independently of the corresponding expression in Cartesians.

If Δ' denote the area, since twice area = side × perpendicular, we have by the foregoing

$$2\Delta' = 2\Delta(x_1 y_2 z_3) \cdot \frac{d}{d}$$

$$\therefore \quad \Delta' = \Delta(x_1 y_2 z_3),$$

involving the areal co-ordinates; and

$$\Delta' = abc(\alpha_1\beta_2\gamma_3)/8\Delta^2,$$

involving the trilinear co-ordinates of the points.

(2) Directly from the expression in Cartesians.

We have

$$2\Delta' = \begin{vmatrix} \bar{x}_1 & \bar{y}_1 & 1 \\ \bar{x}_2 & \bar{y}_3 & 1 \\ \bar{x}_3 & \bar{y}_3 & 1 \end{vmatrix} \sin C = ab\sin C \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= 2\Delta \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ since } x + y + z = 1.$$

Thus $$\Delta' = \Delta(x_1 y_2 z_3).$$

[The corresponding in Trilinears may be deduced from this, or obtained independently thus :—

$$2\Delta' = \begin{vmatrix} \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \beta_2 & 1 \\ \alpha_3 & \beta_3 & 1 \end{vmatrix} \frac{\sin C}{\sin^2 C}, \text{ by direct substitution ;}$$

$$\therefore \quad 2\Delta'\sin C\cdot 2\Delta = \begin{vmatrix} \alpha_1, & \beta_1 & a\alpha_1 + b\beta_1 + c\gamma_1 \\ \alpha_2, & \beta_2, & a\alpha_2 + b\beta_2 + c\gamma_2 \\ \alpha_3, & \beta_3, & a\alpha_3 + b\beta_3 + c\gamma_3 \end{vmatrix}, \text{ since } a\alpha + b\beta + c\gamma = 2\Delta$$

$$= c \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Thus $$\Delta' = abc(\alpha_1\beta_2\gamma_3)/8\Delta^2.]$$

5. *To find the condition that two lines may be at right angles.*

In Trilinears, if the equations to the lines be of the form $l\alpha + m\beta + n\gamma = 0$, on changing to Cartesians they will be of the form

$$(cl - an)\bar{x} + (cm - bn)\bar{y} + 2n\Delta\operatorname{cosec}C = 0.$$

Hence, the condition that two lines in oblique Cartesians may be at right angles, becomes

$$(cl - an)(cl' - an') + (cm - bn)(cm' - bn')$$
$$- \{(cl - an)(cm' - bn') + (cl' - an')(cm - bn)\}\cos C = 0.$$

Now the co-efficient of $ll' + mm' + nn'$ is c^2, while that of $mn' + m'n$ is $-(bc - ac\cos C)$, that is $-c^2\cos A$; and those of $nl' + n'l$ and $lm' + l'm$ are $-c^2\cos B$ and $-c^2\cos C$ respectively.

Hence the condition becomes

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B$$
$$- (lm' + l'm)\cos C = 0.$$

[The corresponding for areals may be obtained in like manner, or deduced from the foregoing by writing la, mb, nc for l, m, n respectively ; and is

$$a^2ll' + \ldots - (mn' + m'n)bc\cos A - \ldots = 0.]$$

6. *To find the condition that two lines may be parallel.*

The equation $lx+my+nz=0$ in areals becomes

$$a(l-n)\bar{x}+b(m-n)\bar{y}+nab=0$$

in Cartesians; and the condition for parallelism of two lines is therefore

$$(l-n)(m'-n')-(l'-n')(m-n)=0,$$

that is,

$$mn'-m'n+nl'-n'l+lm'-l'm=0,$$

or

$$\begin{vmatrix} l, & m & n \\ l' & m' & n' \\ 1 & 1 & 1 \end{vmatrix}=0;$$

the corresponding in trilinears being

$$\begin{vmatrix} l, & m & n \\ l' & m' & n' \\ a & b & c \end{vmatrix}=0.$$

7. *To find the angle between two lines.*

If ϕ be the angle between two lines in oblique Cartesians whose equations are of the form $Ax+By+C=0$,

$$\tan\phi=\frac{(AB'-A'B)\sin\omega}{AA'+BB'-(AB'+A'B)\cos\omega}$$

Expressing the trilinear equations in Cartesians and substituting, we get

$$\tan\phi=\frac{c\{a(mn'-m'n)+\ \dots\}\sin C}{c^2\{ll'+\ \dots\ -(mn'+m'n)\cos A-\ \dots\}}$$

$$=\frac{(mn'-m'n)\sin A+\ \dots}{ll'+\ \dots\ -(mn'+m'n)\cos A\ -\dots}$$

In areals this becomes

$$\tan\phi=\frac{2\Delta(mn'-m'n+\ \dots)}{a^2ll'+\ \dots\ -(mn'+m'n)bc\cos A\ -\dots}$$

[The expressions for $\sin\phi$ may be worth noticing; and may be deduced from those for $\tan\phi$, or obtained independently, thus :—

In oblique Cartesians

$$\sin\phi=\frac{(AB'-A'B)\sin\omega}{\sqrt{A^2+B^2-2AB\cos\omega}.\ \sqrt{A'^2+B'^2-2A'B'\cos\omega}}$$

which for areals becomes

$$\frac{2\Delta(mn' - m'n + \dots)}{\sqrt{(l-m)(l-n)a^2 + \dots} \cdot \sqrt{(l'-m')(l'-n')a^2 + \dots}}$$

and for trilinears

$$\frac{(mn' - m'n)\sin A + \dots}{\sqrt{l^2 + \dots - 2mn\cos A - \dots}\ \sqrt{l'^2 + \dots - 2m'n'\cos A - \dots}}\ ]$$

8. We will now consider the general equation of the second degree, and obtain certain results by the same method.

To find the conditions that the equation may represent a circle.

Let the equation in areals be

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0.$$

Changing to Cartesians, and removing z by the relation $x + y + z = 1$, this becomes

$$(2v' - w - u)\frac{\bar{x}^2}{b^2} + (2u' - v - w)\frac{\bar{y}^2}{a^2} + 2(u' + v' - w - w')\frac{\bar{x}\bar{y}}{ab}$$
$$+ \dots = 0.$$

Now the general equation

$$Ax^2 + 2Bxy + Cy^2 + \dots = 0$$

in oblique Cartesians will represent a circle if $A = C = B\sec\omega$; hence the required conditions are

$$\frac{2u' - v - w}{a^2} = \frac{2v' - w - u}{b^2} = \frac{u' + v' - w - w'}{ab\cos C},$$

each of which ratios

$$= \frac{2w' - u - v}{a^2 + b^2 - 2ab\cos C} = \frac{2w' - u - v}{c^2}.$$

[The corresponding for trilinears may be likewise obtained, or at once deduced, when we get

$$2bcu' - c^2v - b^2w = 2cav' - a^2w - c^2u = 2abw' - b^2u - a^2v.]$$

9. *To find the condition that the equation may represent an ellipse, parabola, or hyperbola.*

The above equation in Cartesians will represent these curves respectively according as $B^2 - AC$ is negative, zero, or positive. If we take the general equation in trilinears

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

and change to Cartesians, removing γ by the relation $a\alpha + b\beta + c\gamma = 2\Delta$, it becomes

$$(a^2w + c^2u - 2cav')x^2 + (c^2v + b^2w - 2bcu')y^2 \\ + 2(abw + c^2w' - cau' - bcv')xy + \dots = 0.$$

Thus the required condition is that

$$-(abw + c^2w' - cau' - bcv')^2 + (a^2w + c^2u - 2cav')(c^2v + b^2w - 2bcu'),$$

or, with a known notation,

$$Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab$$

is positive, zero, or negative; or that

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c & 0 \end{vmatrix} \text{ is negative, zero, or positive.}$$

The corresponding in areals may be obtained in a similar way, or deduced from the preceding by putting $a = b = c = 1$, when the condition is that

$$(2u' - v - w)(2v' - w - u) - (u' + v' - w - w')^2,$$

or,

$$U + V + W + 2U' + 2V' + 2W'$$

is positive, zero, or negative; or that

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \text{ is negative, zero, or positive.}$$

The condition that the equation in areals may represent a parabola can also be expressed under another interesting form.

Since

$$2(u' + v' - w - w') = (2u' - v - w) + (2v' - w - u) - (2w' - u - v),$$

the expression

$$4(u' + v' - w - w')^2 - 4(2u' - v - w)(2v' - w - u)$$

is equal to the product of the four expressions

$$\sqrt{2u' - v - w} \pm \sqrt{2v' - w - u} \pm \sqrt{2w' - u - v}.$$

Hence the equation represents a parabola if any one of these expressions is zero.

10. *To find the condition that the equation may represent a rectangular hyperbola.*

The equation $Ax^2 + 2Bxy + Cy^2 + \ldots = 0$ in oblique Cartesians will represent a rectangular hyperbola if the lines $Ax^2 + 2Bxy + Cy^2 = 0$ are at right angles, the condition for which is that

$$A + C - 2B\cos\omega = 0.$$

Hence, for trilinears, the required condition is that

$$a^2w + c^2u - 2cav' + c^2v + b^2w - 2bcu' - 2(abw + c^2w' - cau' - bcv')\cos C = 0,$$

$$\therefore \quad c^2(u + v + w) - 2c(b - a\cos C)u' - \ldots \quad = 0,$$

that is,

$$u + v + w - 2u'\cos A - 2v'\cos B - 2w'\cos C = 0.$$

[For areals, the condition is that

$$a^2u + b^2v + c^2w - 2bc\cos A . u' - \ldots = 0,$$

or

$$a^2(u + u' - v' - w') + b^2(v + v' - w' - u') + c^2(w + w' - u' - v') = 0.]$$

11. *To find expressions for the product, and sum of squares of the semi-axes, when the equation represents a central conic.*

If the general Cartesian equation $a'x^2 + 2hxy + b'y^2 + 2gx + 2fy + c = 0$ become $Ax^2 + By^2 + C = 0$ when the conic is referred to its principal axes, the product of the semi-axes is $C/\sqrt{AB}$, and the sum of their squares is $-C(A+B)/AB$; and if the original axes be oblique these are respectively equal to

$$D\sin\omega/(a'b' - h^2)^{\frac{3}{2}} \text{ and } -D(a' + b' - 2h\cos\omega)/(a'b' - h^2)^2,$$

$$\text{where } D = \text{the discriminant} \begin{vmatrix} a' & h & g \\ h & b' & f \\ g & f & c \end{vmatrix}.$$

Transforming the general equation

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0$$

in areals to the Cartesian system, it becomes

$$(w+u-2v')\frac{x^2}{b^2}+(v+w-2u')\frac{y^2}{a^2}+2(w+w'-u-v')\frac{xy}{ab}$$
$$+2(v'-w)\frac{x}{b}+2(u'-w)\frac{y}{a}+w=0\,;$$

whence, substituting, we shall get

$$a^2b^2D=\begin{vmatrix} u & w' & v' \\ w' & v' & u' \\ v' & u' & w \end{vmatrix}\equiv H,$$

$$a^2b^2(a'b'-h^2)=-\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}\equiv -K$$

and $$a^2b^2(a'+b'-2h\cos\omega)$$
$$=(u+u'-v'-w')a^2+(v+v'-w'-u')b^2+(w+w'-u'-v')c^2\equiv I.$$

Thus, the product of the semi-axes $= 2\Delta H/(-K)^{\frac{3}{2}}$,
and the sum of their squares $= -HI/K^2$.

[Proceeding in like manner with the general equation

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

in trilinears, we shall get

$$D=a^2b^2c^4.H,$$

$$a'b'-h^2=-c^2\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}=-c^2K' \text{ suppose},$$

$$a'+b'-2h\cos\omega=c^2(u+v+w-2u'\cos A-2v'\cos B-2w'\cos C)$$
$$=c^2.I' \text{ suppose.}$$

Thus, the product of the semi-axes $= 2abc\Delta H/(-K')^{\frac{3}{2}}$,
and the sum of their squares $= -a^2b^2c^2HI'/K'^2$.]

12. *Particular forms of the general equation.*

The general equation in areals represents :—

Two straight lines if $H=0$;
an ellipse, parabola, or hyperbola according as $K <=> 0$;
a rectangular hyperbola if $I=0$;
while in trilinears the corresponding conditions are

$$H=0,\ K <=> 0,\ I'=0 \text{ respectively.}$$

We append the values of these functions for three particular forms of the general equation.

(1) For the circumscribed conic in areals, $lyz+mzx+nxy=0$

$$4H=lmn;\ 4K=l^2+m^2+n^2-2mn-2nl-2lm;$$
$$2I=a^2(l-m-n)+b^2(m-n-l)+c^2(n-l-m),$$

or
$$-I=lbc\cos A+mca\cos B+nab\cos C;$$

and the condition for a parabola is equivalent to

$$\sqrt{l}\pm\sqrt{m}\pm\sqrt{n}=0.$$

For the same conic in trilinears, $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$,

$$4H=lmn;\ 4K'=al^2+b^2m^2+c^2n^2-2bcmn-2canl-2ablm;$$
$$-I'=l\cos A+m\cos B+n\cos C;$$

and condition for a parabola becomes

$$\sqrt{al}\pm\sqrt{bm}\pm\sqrt{cn}=0.$$

(2) For the inscribed conic, or conics touching sides of triangle of reference in areals, $\sqrt{lx}\pm\sqrt{my}\pm\sqrt{nz}=0$.

$$H=-4l^2m^2n^2;\ K=-4lmn(l+m+n);$$
$$I=(l+m+n)(a^2l+b^2m+c^2n)-a^2mn-b^2nl-c^2lm.$$

For the same conics in trilinears, $\sqrt{l\alpha}\pm\sqrt{m\beta}\pm\sqrt{n\gamma}=0$

$$H=-4l^2m^2n^2;\ K=-4lmn(bcl+cam+abn);$$
$$I=l^2+m^2+n^2+mn+nl+lm.$$

(3) For the conic with respect to which the triangle of reference is self-conjugate, the equation to which in areals is $lx^2+my^2+nz^2=0$,

$$H=lmn;\ -K=mn+nl+lm,$$
$$I=la^2+mb^2+nc^2;$$

and for the same conic in trilinears $l\alpha^2+m\beta^2+n\gamma^2=0$

$$H=lmn;\ K'=mna^2+nlb^2+lmc^2;$$
$$I'=l+m+n.$$

History of the Nine-point Circle.

By J. S. MACKAY.

The earliest author to whom the discovery of the nine-point circle has been attributed is Euler, but no one has ever given a reference to any passage in Euler's writings where the characteristic property of this circle is either stated or implied. The attribution to Euler is simply a mistake, and the origin of the mistake may, I think, be explained. It is worth while doing this, in order that subsequent investigators may be spared the labour and chagrin of a fruitless search through Euler's numerous writings.

Catalan in his *Théorèmes et Problèmes de Géométrie Élémentaire*, 5th ed. p. 126 (1872), or 6th ed. p. 170 (1879), says that the learned Terquem attributed the theorem of the nine-point circle to Euler, and refers to the *Nouvelles Annales de Mathématiques*, I. 196 (1842). If the first volume of the *Nouvelles Annales* be consulted, it will be found that Terquem has two articles on the rectilineal triangle. The first (pp. 79–87) is entitled *Considérations sur le triangle rectiligne, d'après Euler;* the second (pp. 196–200) has the same title, but *d'après Euler* is omitted. In the first article Terquem mentions that Euler discovered certain properties * of the triangle, and refers to the place where they are to be found (*Novi Commentarii Academiae . . . Petropolitanae*, xi. 114, 1765). He says he thinks it useful to reproduce them with some developments, and this is exactly what he does, for the first article is a synopsis of Euler's results, and the second article, which begins with the property of the nine-point circle, contains the developments.

Who, then, is the discoverer of the nine-point circle?

The fact is that there have been several independent discoverers of it, English, French, German, Swiss. Their researches will be treated of in the order of publication; and it will conduce to brevity of statement if the following notation be laid down:

ABC	is the fundamental triangle
I, I_1, I_2, I_3	are the incentre and the three excentres
O, M	are the circumcentre and the nine-point centre
X, Y, Z	are the feet of the perpendiculars from A, B, C on the opposite sides.

* An abstract of Euler's paper will be found in the *Proceedings of the Edinburgh Mathematical Society*, IV. 51-55 (1886).

In Leybourn's *Mathematical Repository*, new series, I. 18 (pagination of the Questions to be answered) Benjamin Bevan proposed in 1804 the following:

"*In a plane triangle let O_0 be the centre of a circle passing through I_1, I_2, I_3, then will $OO_0 = OI$ and be in the same right line, and*

$$O_0I_1 = O_0I_2 = O_0I_3 = 2R$$

or the diameter of the circumscribing circle."

When it is remembered that triangle $I_1I_2I_3$ has I for orthocentre and ABC for orthic triangle, and that the circumcircle of ABC is the nine-point circle of $I_1I_2I_3$, it will be seen that Bevan's theorem establishes the conclusions:

(1) *That the nine-point centre bisects the distance between the circumcentre and the orthocentre.*

(2) *That the radius of the nine-point circle is half the radius of the circumcircle.*

The proof of Bevan's theorem given in the *Mathematical Repository*, Vol. I., Part I., p. 143, is by John Butterworth of Haggate.

In the *Gentleman's Mathematical Companion for the year* 1807 (which was published in 1806) John Butterworth proposes the question:

When the base and vertical angle are given, what is the locus of the centre of the circle passing through the three centres of the circles touching one side and the prolongation of the other two sides of a plane triangle?

In the *Mathematical Companion* for 1808 (pp. 132–3) two solutions of the question are given, the first by the proposer, and the second by John Whitley. Whitley's solution shows that the circumcircle of ABC goes through seven points connected with the triangle $I_1I_2I_3$, namely, the feet of its perpendiculars, the mid points of two of its sides, and the mid points of two of the segments intercepted between the orthocentre and the vertices. It is evident from the tenour of his proof that he must have been aware that the circumcircle of ABC passed through the other two points which make up the well-known nine, but for the purpose he had in hand he did not happen to require them, and they are consequently not mentioned.

Butterworth's solution of his own question shows that the circumcircle of ABC bisects the lines II_1, II_2, II_3, and that the circumcentre O bisects IO_0.

The next step in the history of the nine-point circle is the discussion not of the relations between the circumcircles of $I_1I_2I_3$ and ABC, but of those between the circumcircles of ABC and XYZ.

The nine points are explicitly mentioned in Gergonne's *Annales de Mathématiques* in an article by Brianchon and Poncelet. The article appeared on the 1st January 1821 in vol. xi., and the theorem establishing the characteristic property of the circle in question occurs at p. 215. As this article is reprinted by Poncelet in his *Applications d'Analyse et de Géométrie*, II. 504–516 (1864) I infer that it owes its origin rather to Poncelet than to Brianchon. Poncelet does not draw attention to the easy inference that the radius of the nine-point circle is half the radius of the circumcircle, nor to the position of the nine-point centre. This is natural enough, for the title of his article is *Recherches sur la détermination d'une hyperbole équilatère, au moyen de quatre conditions données*, and the existence of the nine-point circle is noticed incidentally. It may be worth mentioning that in Poncelet's demonstration there occurs the theorem which Mr R. F. Davis discovered some years ago and applied to the triplicate-ratio and Taylor circles, namely:

If on each side of a triangle ABC a pair of points are taken so that any two pairs are concyclic, then all the six points are concyclic.

Karl Wilhelm Feuerbach's *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks* appeared at Nürnberg in 1822. In § 56 of it occurs the theorem that "the circle which goes through the feet of the perpendiculars of a triangle meets the sides at their mid points," but nothing is said of the other three points. The radius of the circle, $\frac{1}{2}$R, is mentioned, and the position of the centre is given, midway between the orthocentre and the circumcentre. In § 57 Feuerbach proves that the circle which goes through the feet of the perpendiculars of a triangle touches the incircle and the excircles, and this is the first enunciation of that interesting property of the nine-point circle. The proof consists in showing that the distance between the nine-point centre and the incentre is equal to $\frac{1}{2}R - r$.

In the *Philosophical Magazine*, II. 29–31 (1827) T. S. Davies proves the characteristic property of the nine-point circle. At the outset of his article, which is entitled *Symmetrical Properties of Plane Triangles*, and dated Janr. 15th, 1827, he says: "The following properties . . . do not appear to have been noticed by mathematicians." Davies was very scrupulous in giving to his predecessors the credit of their discoveries; hence he is another discoverer of the nine-point circle. In the fourth of his propositions and in the corollaries thereto, Davies, besides mentioning the length of the nine-point radius and the position of the nine-point centre, remarks that the centroid is also situated on the line which contains the orthocentre, the nine-point centre, and the circumcentre.

In Gergonne's *Annales de Mathématiques*, xix. 37–64 there is an article* by Steiner entitled *Développement d'une série de théorèmes relatifs aux sections coniques.* The article appeared in 1828, and in the course of it Steiner shows, among other things, that the nine-point circle property is only a particular case of a more general theorem. He remarks also that the centroid is on the line joining the orthocentre and the circumcentre (attributing this property to Carnot; it was discovered by Euler); he states that the four points, the circumcentre, the centroid, the nine-point centre, and the orthocentre, form a harmonic range, and that the orthocentre and the centroid are the centres of similitude of the nine-point circle and the circumcircle; and lastly he adds, without proof, the statement that the nine-point circle touches the incircle and the excircles.

In a long note to § 12 of his tractate† *Die geometrischen Constructionen, ausgeführt mittelst der geraden Linie und eines festen Kreises*, which appeared in 1833, Steiner discusses the nine-point circle and the circumcircle in connection with their centres of similitude, and he enunciates the theorem that twelve points associated with the triangle lie on one and the same circle. At the end of the note Steiner states that when he announced the theorem that the nine-point circle touched the incircle and the excircles he was not aware that it had been previously made known by Feuerbach.

Hitherto the circle had received no special name. The designation "nine-point circle" (*le cercle des neuf points*) was bestowed on it in 1842 by Terquem, one of the editors of the *Nouvelles Annales*; see that journal, Vol. I., p. 198. It has also been called the six-points circle, the twelve-points circle, the n-point circle, Feuer-

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* Republished in Steiner's *Gesammelte Werke*, I. 191-210 (1881).

† Republished in Steiner's *Gesammelte Werke* I. 489-492.

bach's circle, Euler's circle, Terquem's circle, *il circolo medioscritto*, the medioscribed circle, the mid circle, the circum-midcircle.

There are other demonstrations of the characteristic property of the nine-point circle quite distinct from those given in the articles already spoken of. One by Terquem will be found in *Nouvelles Annales*, I. 196 (1842); a second by C. Adams in *Die Lehre von den Transversalen*, p. 37 (1843), and a third, also by Adams, in *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, pp. 14–16 (1846); a fourth by T. T. Wilkinson in the *Lady's and Gentleman's Diary* for 1855, p. 67; a fifth by William Godward in *Mathematical Questions from the Educational Times*, VII. 86–7 (1867). Besides these there are several other proofs not essentially different. Those who are curious in such matters may refer to proofs by Rev. Joseph Wolstenholme in the *Quarterly Journal of Mathematics*, II. 138–9 (1858); by W. H. B[esant] in the *Oxford, Cambridge and Dublin Messenger of Mathematics*, III. 222–3 (1866); by Desboves in *Questions de Géométrie Élémentaire*, 2nd ed., pp. 146–7 (1875); by Captain Mennesson in *Nouvelle Correspondance Mathématique*, IV. 241–2 (1878); by Rev. John Wilson in *Proceedings of the Edinburgh Mathematical Society*, VI. 38–40 (1888).

Of the theorem that the nine-point circle touches the incircle and the excircles the first two published proofs, namely, Feuerbach's in 1822 and Terquem's in 1842, were analytical. Steiner merely enunciated the theorem. The first geometrical proof appeared in the *Nouvelles Annales*, IX. 401–3 (1850) in an article entitled *Note sur le triangle rectiligne* by J. Mention. The next appeared in the *Lady's and Gentleman's Diary*. In the *Diary* for 1853, p. 77, W. H. Levy proposes the question:

In any plane triangle the sum of the four distances from the point of bisection of the line joining the centre of the circumscribing circle and the point of intersection of the perpendiculars from the opposite angles upon the sides, to the centres of the inscribed and escribed circles is equal to three times the diameter of the circumscribing circle.

A geometrical solution is given in the *Diary* for 1854, p. 56, where the following results are established:

$$MI = \tfrac{1}{2}R - r$$

$$MI_1 = \tfrac{1}{2}R + r_1 \quad MI_2 = \tfrac{1}{2}R + r_2 \quad MI_3 = \tfrac{1}{2}R + r_3;$$

and hence

$$\Sigma(MI) = 2R + (r_1 + r_2 + r_3 - r) = 3D.$$

Notwithstanding Davies's article in the *Philosophical Magazine* in 1827, the nine-point circle seems to have been almost unknown in the United Kingdom; and hence it is not so curious as it might appear at first sight that among the 23 mathematicians who answered Mr Levy's question only one should have drawn any further inference from it.

In the same number of the *Diary* on p. 72, T. T. Wilkinson of Burnley, who was a friend of Davies, and was acquainted* with his paper in the *Philosophical Magazine*, proposes the question:

Let ABC be any triangle; AD, BE, CF the perpendiculars drawn from A, B, C to the opposite sides, mutually intersecting at P: then the circle described through D, E, F, the feet of the perpendiculars will be tangential to the sixteen inscribed and escribed circles of the triangles ABC, APB, BPC, and CPA.

The solutions of Wilkinson's question given in the *Diary* for 1855, pp. 67–9, all depend on the four results established in the *Diary* for 1854.

In the *Diary* for 1857, pp. 86–9, John Joshua Robinson enunciates and proves the theorem:

The circle described through the middle points of the sides of any triangle is tangential to several infinite systems of circles inscribed and escribed to triangles drawn according to a given law.

Or the theorem may be expressed more fully:

If the radical centres of the inscribed and escribed circles of any triangle be taken, and circles be inscribed and escribed to the triangles formed by joining these radical centres, and the radical centres of the latter system of circles be again taken and circles inscribed and escribed to the triangles thus formed, and so on ad infinitum, *the infinite number of circles thus formed, as well as the original system of inscribed and escribed circles, always touch the circle drawn through the middle points of the first triangle.*

The editor of the *Diary* in a note to Robinson's theorem mentions that Wilkinson first announced his beautiful theorem under this general form of enunciation.

In the *Diary* for 1858, pp. 86–7, Wilkinson has a short article

* I am in possession of a collection of printed mathematical papers which belonged to Wilkinson. The paper of Davies's referred to is imperfect, but is completed in Wilkinson's handwriting.

entitled *Notae Geometricae*, from which the following extracts are taken.

"In the extended solution to which allusion is made [Robinson's solution] I noticed that the property became porismatic when *any three points* are taken for the feet of the perpendiculars, and the triangles thence resulting are constructed according to an almost obvious law. All the triangles thus formed are evidently those of *least* perimeter to the *primitives* obtained by bisecting the angles formed by joining the feet of the perpendiculars in each instance, and hence connect themselves immediately with the many beautiful and curious properties known to result from this view of the subject. . . .

"But the property admits of a more general enunciation . . . by projecting the system upon a plane . . . ; for the perpendiculars then become lines drawn conjugate to the opposite sides, the contacts are preserved, and the circles become conics similar and similarly placed. Hence if ABC be a triangle inscribed in a conic, and if through each vertex there be drawn a transversal respectively conjugate to the opposite side, then

(1) These three transversals will intersect in the same point O.

(2) The middle points of the lines OA, OB, OC, the middle points of the sides AB, BC, CA, and the points *a*, *b*, *c*, where the transversals meet the sides, are nine points situated in a second conic, similar and similarly placed with respect to the first.

(3) This second conic is also tangential to the sixteen conics inscribed and escribed to the triangles AOB, BOC, COA, and similar and similarly placed with respect to the two first conics.

In this connection, reference may be made to a paper of Professor Eugenio Beltrami (read 12th March 1863), *Intorno alle coniche di nove Punti e ad alcune quistioni che ne dipendono*, printed in the *Memorie della Accademia delle Scienze dell'Istituto di Bologna*, 2nd series, Vol. II., pp. 361–395 (1862); and also to a paper by Schröter (dated October 1867), *Erweiterung einiger bekannten Eigenschaften des ebenen Dreiecks*, in Crelle's *Journal* LXVIII. 203–234 (1868).

In an article,* dated July 17, 1860, the Rev. George Salmon called attention to Feuerbach's theorem. He says:

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* *Quarterly Journal of Mathematics*, IV. 152-4 (1861).

"The following elementary theorems may interest some of the readers of the *Quarterly Journal.*

(1) The distance between the point of intersection of the perpendiculars of a triangle and the centre of the circumscribing circle is given by the equation

$$D^2 = R^2 - 8R^2 \cos A \cos B \cos C.$$

(2) The distance between the point of intersection of perpendiculars and the centre of the inscribed circle is given by the equation

$$d^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

Hence $$D^2 - 2d^2 = R^2 - 4r^2.$$

(3) It follows that if the two circles be fixed, the locus of the intersection of perpendiculars is a circle whose radius is $R - 2r$, and whose centre is found by producing the line joining the centres to a distance equal to itself, and so that the centre of the inscribed circle may lie in the middle.

From the preceding theorems Dr Hart, to whom I had happened to mention them, drew the following inferences:

Consider the circle passing through the middle points of the sides of the triangle; its radius $= \frac{1}{2}R$, and its centre is the middle point of the line joining the centre of the circumscribing circle to the intersection of perpendiculars. The line then joining this middle point to the centre of the inscribed circle is the line joining the middle points of the sides of a triangle whose base has been proved to be $R - 2r$. Its length is therefore $\frac{1}{2}R - r$.

(4) Hence, when the inscribed and circumscribing circles are given, the locus of the centre of the circle passing through the middle points of sides is a circle having for its centre the centre of the inscribed circle.

Further: the distance between the centres of the inscribed circle and of the circle through the middle points of sides has been proved to be exactly the difference between their radii; and the same argument applies to any of the four circles which touch the three sides of the given triangle; hence

(5) The circle which passes through the middle points of the sides of a triangle touches the four circles which touch the three sides.

This theorem was new both to Dr Hart and myself,* but I have lately learned from a friend that it belongs to M. Terquem, who has given it in his *Annales*, T. I., p. 196. Dr Hart has since made other more direct demonstrations of it, and Sir William R. Hamilton, to whom he mentioned the theorem, was sufficiently interested by it to take the trouble of investigating it algebraically, when he obtained very simple constructions for the common point and common tangent of the two circles."

Dr Salmon then sums up Sir William Hamilton's results in three theorems, which he states and proves.

In the same volume of the *Quarterly Journal* (pp. 245–252), Mr John Casey has an article, dated Nov. 27, 1860, in which he proves not only Feuerbach's theorem, but also, without knowing the results that had been previously published in the *Diary*, extends the contact to an indefinite number of circles. See Casey's *Sequel to Euclid*, 6th ed., pp. 105–6 (1892), where a proof is also given of the following extension (due to Dr Hart) of Feuerbach's theorem :

If the three sides of a plane triangle be replaced by three circles, then the circles touching these, which correspond to the inscribed and escribed circles of a plane triangle, are all touched by another circle.

See also Lachlan's *Elementary Treatise on Modern Pure Geometry*, pp. 251–7 (1893).

There are other demonstrations of Feuerbach's theorem besides those already spoken of. The following references may be given.

Mr J. M'Dowell in the *Quarterly Journal of Pure and Applied Mathematics*, V. 269–271 (1862).

Mr W. F. Walker in the same periodical, VIII. 47–50 (1867).

Herr J. Lappe in Crelle's *Journal*, LXXI. 387–392 (1870).

Herr Binder in 1872 communicated to Dr Richard Baltzer a proof which will be found in the latter's *Elemente der Mathematik*, II. 92–3 (1883).

Mr J. P. Taylor in the *Quarterly Journal*, XIII. 197 (1875).

Mr E. M. Langley in 1876 discovered the proof given in the *Harpur Euclid*, p. 489 (1890).

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* Might it not be that Dr Salmon had forgotten it and rediscovered it? This conjecture is made because Mr J. J. Robinson begins an article in the *Diary* for 1858, p. 88, by saying : "My best thanks are due to the Rev. George Salmon of Trinity College, Dublin, for having called my attention to two errors which somehow have crept into my former paper," that is, the paper in which Feuerbach's theorem was extended.

M. Chadu of Bordeaux in *Nouvelle Correspondance Mathématique*, V. 230–2 (1879).

Mr W. F. M'Michael in the *Messenger of Mathematics*, XI. 77–8 (1882).

Mr Morgan Jenkins in *Mathematical Questions from the Educational Times*, XXXI [illegible] in the same volume, [illegible]

Mr William Ha[illegible] communicated to me a proof which will be found in the *Proceedings of the Edinburgh Mathematical Society*, V. 102–3 (1887). In a letter, dated May 6, 1888, the Rev. G. Richardson somewhat abbreviates Mr Harvey's proof.

In the *Messenger of Mathematics*, XIII. 116–120 (1884), Mr C. Leudesdorf has an article, dated Nov. 7, 1883, and entitled "Proofs of Feuerbach's Theorem." He there discusses several of the published proofs, and shows how some of them may be simplified.

Mr Samuel Roberts in the *Messenger*, XVII. 57–60 (1887).

In Milne's *Companion to the Weekly Problem Papers*, pp. 187–8 (1888), will be found a proof by Mr R. F. Davis; another, by Mr W. S. M'Cay, occurs in M'Lelland's *Treatise on the Geometry of the Circle*, p. 183 (1891); and still another, due to Professor Purser, in Nixon's *Euclid Revised*, pp. 350–1, or in Lachlan's *Elementary Treatise on Modern Pure Geometry*, pp. 206–7 (1893).

FIRST DEMONSTRATION (Whitley, 1807).

When the base and vertical angle are given, what is the locus of the centre of the circle passing through the three centres of the circles touching one side and the prolongation of the other two sides of a plane triangle?

FIGURE 1.

Let ABC be a plane triangle, AVCW'UBU' the circumscribing circle, and I_1, I_2, I_3 the centres of the circles specified in the question.

Then by known properties the lines joining the angles A, B, C of the triangle and the centres I_1, I_2, I_3 respectively will bisect those angles, and meet in I the centre of the inscribed circle. Also the lines joining I_1, I_2, I_3 will pass through the angular points A, B, C of the triangle, and be perpendicular to AI_1, BI_2, CI_3; and if AI_1, BI_2 meet the circumscribing circle BU'CU again in U and V, and I_2I_3, I_1I_2 meet it also in U', W' respectively, then will

$$IU = I_1U, \quad IV = I_2V, \quad I_2U' = I_3U', \quad I_1W' = I_2W'.$$

Draw $U'O_0$, $W'O_0$ perpendicular to I_2I_3, I_1I_2 and their intersection O_0 will evidently be the centre of the circle passing through I_1, I_2, I_3. The rest being drawn as per figure, it is obvious that $W'VU'O_0$ is a parallelogram, and also that UU' is perpendicular to BC and a diameter of the circle $BU'CU$.

Now the base BC and vertical angle BAC being given, EF will be given, as will also

$$UC = UI = W'V = O_0U';$$

therefore O_0U' is given, and the point U' being given, the locus of O_0 is consequently a given circle of which U' is the centre and radius O_0U' equal to UC.

SECOND DEMONSTRATION (Poncelet, 1821).

The circle which passes through the feet of the perpendiculars let fall from the vertices of any triangle on the opposite sides passes also through the mid points of these three sides as well as through the mid points of the distances which separate the vertices from the point of concourse of the perpendiculars.

FIGURE 2.

Let X, Y, Z be the feet of the perpendiculars let fall from the vertices of the triangle ABC on the opposite sides, and let A', B', C' be the mid points of these sides.

The right-angled triangles CBZ and ABX being similar, we have

$$BC : BZ = AB : BX;$$

whence, since A' and C' are the mid points of BC and AB,

$$BA' \cdot BX = BC' \cdot BZ;$$

that is to say, the four points A', X, C', Z belong to one and the same circumference.

It could be proved in a similar way that the four points A', X, B', Y are on one circle, as well as the four points B', Y, C', Z.

Now, if it were possible that the three circles in question were not one and the same circle, it would be necessary that the directions of the chords which are common to them two and two should meet in a single point; but these chords are precisely the sides of the triangle ABC, which cannot meet in the same point; therefore it is equally impossible to suppose that the three circles are different; therefore they are one and the same circle.

Let now U, V, W be the mid points of the distances HA, HB, HC which separate H, the point of concourse of the perpendiculars of the triangle ABC, from the respective vertices.

The right-angled triangles CHX, CBZ being similar, we have

$$CH : CX = CB : CZ;$$

whence, since the points W and A′ are the mid points of the distances CH and CB

$$CW \cdot CZ = CX \cdot CA';$$

that is to say, the circle which passes through A′, X, Z passes also through W.

It could be proved in the same way that this circle passes through the two other points, U, V; therefore it passes at the same time through the nine points X, Y, Z, A′, B′, C′, U, V, W.

Poncelet's proof may be somewhat simplified in the following manner.*

FIGURE 2

Because	B, Z, Y, C	are concyclic;
therefore		$AB \cdot AZ = AC \cdot AY$;
therefore		$AC' \cdot AZ = AB' \cdot AY$;
therefore	B′, C′, Y, Z	are concyclic.
Because	A, Z, H, Y	are concyclic;
therefore		$BA \cdot BZ = BH \cdot BY$;
therefore		$BC' \cdot BZ = BV \cdot BY$;
therefore	C′, Y, Z, V	are concyclic,
that is,	V is on the circle through B′, C′, Y, Z.	

Similarly W is on the circle through B′, C′, Y, Z.

that is B′, C′, Y, Z, V, W are concyclic.

Hence C′, A′, Z, X, W, U are concyclic.

Now there are three points C′, Z, W common to these two sets of six points;

therefore the two sets of six points lie all on one circle; and the nine points A′, B′, C′, X, Y, Z, U, V, W are concyclic.

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* One of the principal features of this simplification has been given by Mr R. D. Bohannan in the *Annals of Mathematics*, I. 112 (1884).

Note.—When the diagram of a triangle and its nine-point circle has to be constructed, it will be found convenient to begin by describing the nine-point circle; then to choose three points on its circumference for the vertices of the median triangle A'B'C'; then through A', B', C' to draw parallels to B'C', C'A', A'B'. These parallels will intersect the circle again at the feet of the perpendiculars X, Y, Z, and will intersect each other at the vertices of the fundamental triangle ABC.

THIRD DEMONSTRATION (Feuerbach, 1822).

The radius of the circle circumscribed about the triangle XYZ, which is made by the feet of the perpendiculars in the triangle ABC, is half as large as the radius of the circle circumscribed about the triangle ABC, and its centre M bisects the distance between O, the centre of the circle ABC, and H the point of intersection of the perpendiculars.

FIGURE 3.

Join OA, OH, and draw OA' perpendicular to BC. Through A' and M the mid point of OH draw A'M and produce it to meet the perpendicular AX in U.

Because $MH = MO$
$\angle HMU = \angle OMA'$
$\angle MHU = \angle MOA'$
therefore $HU = OA'$ and $UM = A'M$.

But because $OA' =$ half AH
therefore $HU =$ half AH, and $AU = OA'$.

Again, because AU is parallel to OA',
therefore $UA' = AO$ and $A'M =$ half AO.

From M draw MD perpendicular to BC, and join MX.

Then because the straight lines HX, MD, OA' are all parallel to each other $HM : OM = XD : A'D$;
and since $HM = OM$, therefore $XD = A'D$,
and consequently $MA' = MX$.

And since $MA' =$ half AO, therefore $MX =$ half AO.

Similarly $MY =$ half BO, $MZ =$ half CO.

Now since $AO = BO = CO$
therefore $MX = MY = MZ$

and M, the mid point of OH, is the centre of the circle circumscribed about the triangle XYZ.

Feuerbach's proof may be simplified as follows.*

FIGURE 3.

Let ABC be a triangle, H its orthocentre, O its circumcentre.

Draw OA′ perpendicular to BC and therefore bisecting BC. Bisect AH at U; join OA, OH, A′U.

Because AH is twice OA′, therefore AU = OA′;
therefore AOA′U is a parallelogram, and A′U = OA = R.

Because UH is equal and parallel to OA′,
therefore A′U and OH bisect each other at M.

Now since M is the mid point of the hypotenuse of the right-angled triangle A′XU, the circle described with M (the mid point of OH) as centre and radius equal to half A′U, that is, equal to $\frac{1}{2}$R, will pass through the three points A′, X, U.

Hence also the same circle will pass through B′, Y, V, and C′, Z, W.

FOURTH DEMONSTRATION (Davies, 1827).

PROPOSITION I.

Let ABC be any plane triangle, and let perpendiculars AX, BY, CZ be demitted from each angle upon its opposite side, and prolonged to meet the circumscribing circle in R, S, T; then if the triangle RST be formed, its angles will be bisected by the said perpendiculars.

FIGURE 4.

For, since the lines CZ, BY are perpendicular to the lines AB, AC, and the angle BAC common to the two triangles BYA, CZA, the angle ABY is equal to the angle ACZ. Hence they stand on equal arcs AS, AT. But the angles ARS, ART stand on the same two arcs, and therefore they are equal; or the angle SRT is bisected by AR.

In the same manner the angles RST, STR are proved to be bisected by BS, CT respectively.

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* This simplification is given in Dr Th. Spieker's *Lehrbuch der ebenen Geometrie*, 15th ed., p. 216, or § 220 (1881).

Cor. 1. The angles of the triangle XYZ are also bisected by the same perpendiculars.

For each side of this triangle is manifestly parallel to a corresponding side of RST.

Cor. 2. Each of the triangles AZY, BXZ, CYX is similar to the original triangle.

For the angles AXZ, ACZ being equal, their complements BXZ, BAC are equal. In like manner it may be shown that BZX is equal to ACB; and therefore the triangle BXZ is similar to ABC. And so of the others.

Or this corollary may be thus deduced:

Because BZC, BYC are right angles, a circle will pass through B, Z, Y, C; and therefore the angles AZY, AYZ are equal respectively to ACB, ABC. And so of the others.

Proposition II.

A circle described through the feet of the perpendiculars, X, Y, Z will also bisect the sides of the triangle.

Figure 5.

For, let the circle cut HA in U, HB in V, HC in W; BC in A′, CA in B′, and AB in C′.

Also, join UA′, VB′, and WC′.

Then, since A′XU, B′YV, C′ZW are right angles, the lines A′U, B′V, C′W are diameters of the circle XYZ. They also pass through the middles of the arcs YZ, ZX, XY; and are, consequently, perpendicular to the middles of the chords YZ, ZX, XY which respectively subtend those arcs. But YZ is also a chord of the semicircle BZYC; and as UA′ is a chord perpendicular to the middle of it, it passes through the centre of the semicircle, and therefore bisects the diameter BC. Hence A′ is the middle of BC.

In the same manner it may be proved that CA, AB are bisected in B′ and C′.

Proposition III.

Let H be the point of intersection of the perpendiculars AX, BY, CZ; then the distance of H from each of the angles A, B, C is bisected by the circle XYZ.

Figure 5.

For join C′V.

Then VC′ZY is a quadrilateral in a circle, and the angle BC′V is equa to the opposite angle ZYV. But ZAYH is also a quadrilateral in a circle, and therefore the angle ZYH is equal to ZAH. Hence the angle BC'V is equal to ZAH, or C'V is parallel to AH. Consequently we have

$$BV : BH = BC' : BA = 1 : 2;$$

or BH is bisected in the point V.

In like manner it appears that U and W are the middles of AH and CH.

Cor. Let O be the centre of the circumscribing circle, and the perpendiculars OA', OB', OC' drawn; we shall have AH equal to twice the perpendiculars OA', BH to twice OB', and CH to twice OC'.

For by the above demonstration A'V, C'V are parallel to CH and AH respectively, and consequently to C'O and A'O respectively; whence C'V is equal to OA'. But C'V is half AH, or AH is equal to twice C'V, or to twice OA'.

The same reasoning applies to the other stated equalities.

Proposition IV.

Let M be the centre of the circle XYZ, O that of the circle ABC, and H the intersection of the perpendiculars; these three points M, O, H are in one straight line.

Figure 5.

For, since OA' is parallel and equal to HU, the lines A'U, OH bisect each other in their point of intersection, or OH passes through the middle of UA', the diameter of the circle XYZ, and therefore through its centre M.

Cor. 1. The centre M of the circle XYZ is midway between H and O.

Cor. 2. It is known that the centre of gravity of the triangle is also in HO. Whence *four* important points belonging to the triangle are in one line.

Cor. 3. The diameter of the circle XYZ is half the diameter of the circle circumscribing the triangle ABC.

For, join A'B', B'C', C'A'.

Then this triangle is similar to the triangle ABC, and has half its linear dimensions. Hence the diameter of a circle about A'B'C' (viz., the circle XYZ by Proposition III.) is half the diameter of that about ABC.

Davies's proof may be somewhat simplified in the following manner.

FIGURE 5.

Let AX, BY, CZ be the perpendiculars of the triangle ABC intersecting in H. Join YZ, ZX, XY; and let the circle circumscribed about XYZ cut BC at A′ and AX at U.

Then A′U is a diameter of the circle XYZ.

But since AX bisects the angle YXZ,
therefore U is the mid point of the arc YZ;
therefore A′U bisects perpendicularly the chord YZ.

Now the circle BZYC, whose diameter is BC, has also YZ for a chord;
therefore A′U passes through the centre of the circle BZYC,
that is, A′ is the mid point of BC.

Hence also the circle XYZ passes through the mid points of CA and AB; that is, the circle through the feet of the perpendiculars of a triangle bisects the sides of the triangle.

Now X, Y, Z are also the feet of the perpendiculars of the triangles HCB, CHA, BAH;
therefore the circle XYZ bisects the sides HA, HB, HC.

FIFTH DEMONSTRATION (Steiner, 1828).

If from any point O in the plane of the triangle ABC there be drawn OA′, OB′, OC′ respectively perpendicular to BC, CA, AB, then

$$AB'^2 + BC'^2 + CA'^2 = BA'^2 + CB'^2 + AC'^2;$$

and this is the necessary and sufficient condition that the perpendiculars to BC, CA, AB at the points A′, B′, C′ should be concurrent.

FIGURE 6.

Through A′, B′, C′ the feet of the three perpendiculars let there be described a circle whose centre is M and which cuts the sides of the triangle again at X, Y, Z.

Join OM, and produce it to H so that MH is equal to MO.

Because the perpendiculars from M on the three sides of the triangle would pass through the mid points of the intercepted chords A′X, B′Y, C′Z, it follows that the perpendiculars to the three sides at the points X, Y, Z are concurrent at the point H. Hence the theorem:

If from any point O in the plane of a triangle ABC there be drawn OA′, OB′, OC′ respectively perpendicular to BC, CA, AB, and if through A′, B′, C′ the feet of these perpendiculars there be described a circle whose centre is M and which cuts the sides again at X, Y, Z, the perpendiculars to the sides at the last three points will be concurrent at a point H such that M is the mid point of OH.

Join B′C′, YZ, OA, HA.

Then $\angle AB'C' = \angle AZY$,

because they stand on the same arc YC′.

But on account of the cyclic quadrilaterals B′OC′A, ZHYA

$$\angle AB'C' = \angle AOC' \text{ and } \angle AZY = \angle AHY;$$

therefore $\angle AOC' = \angle AHY$.

But $\angle B'OC' = \angle YHZ$,

because each is supplementary to $\angle A$;

therefore $\angle AOB' = \angle AHZ$, by subtraction;

therefore $\angle OAB' = \angle HAZ$,

because these angles are complementary to the former two.

But $\angle OAB' = \angle OC'B'$,

on account of the cyclic quadrilateral B′OC′A;

therefore $\angle OC'B' = \angle HAZ$.

Now since C′O is perpendicular to AZ,

therefore C′B′ is perpendicular to AH;

and the same is the case with C′A′ and BH, with A′B′ and CH.

Let a be the mid point of the chord B′C′;
then the straight line Ma will be perpendicular to B′C′, and consequently parallel to HA.

For the same reasons if b and c be the mid points of C′A′ and A′B′ respectively, the straight lines Mb and Mc will be respectively perpendicular to C′A′ and A′B′.

Hence the theorem:

If from any point O in the plane of a triangle ABC there be drawn OA′, OB′, OC′ respectively perpendicular to BC, CA, AB, and if from the vertices of the triangle other perpendiculars be drawn to B′C′, C′A′, A′B′ the sides of the triangle A′B′C′, these last three perpendiculars *will be concurrent at one point H. And further, if from this last point there be drawn to the sides of the triangle ABC the perpendiculars HX, HY, HZ, the six points A′, B′, C′, X, Y, Z will belong to one circle having its centre M at the mid point of OH.*

From the preceding there is easily deduced the solution of the problem:

Straight lines OA, OB, OC being drawn from any point O in the plane of a triangle ABC to its three vertices, to inscribe in this triangle another triangle XYZ whose three sides YZ, ZX, XY may be respectively perpendicular to these straight lines.

It has been proved that $\angle OAC = \angle HAB$,

and since the same relation should hold good for the three vertices of the triangle ABC, therefore

$$\angle OAC = \angle HAB, \quad \angle OBA = \angle HBC, \quad \angle OCB = \angle HCA.$$

Hence the theorem:

Through any point O in the plane of a triangle ABC let there be drawn to its vertices the straight lines OA, OB, OC; if through the same vertices there be drawn three new straight lines making with the sides AB, BC, CA angles respectively equal to the angles OAC, OBA, OCB these last three straight lines will be concurrent at a point H; and if from the points O, H perpendiculars OA′, OB′, OC′, HX, HY, HZ be drawn to BC, CA, AB, their feet A′, B′, C′, X, Y, Z will belong all six to one circle whose centre M is the mid point of OH.

Among various particular cases we shall call attention only to the following:

Suppose the point O to be the centre of the circle circumscribed about the triangle ABC, the feet A′, B′, C′ of the perpendiculars OA′, OB′, OC′ will be the mid points of BC, CA, AB, and consequently the straight lines B′C′, C′A′, A′B′ will be respectively parallel to the sides BC, CA, AB. Now since AH is perpendicular to B′C′, it will be perpendicular also to BC, and consequently the point H will be the point of concourse of the perpendiculars drawn from the vertices of the triangle ABC to the opposite sides. Hence the theorem:

The mid points A′, B′, C′ of the sides of a triangle ABC and X, Y, Z the feet of the perpendiculars drawn from the vertices to the *opposite sides are six points situated on the circumference of a circle whose centre M is the mid point of the straight line which joins O the circumscribed centre to H the point of concourse of the perpendiculars of the triangle ABC. Further, the three radii OA, OB, OC are respectively perpendicular to the sides YZ, ZX, XY of the triangle XYZ; and finally these radii are so situated that the angles OAB, OBC, OCA are respectively equal to the angles HAC, HBA, HCB, or XAC, YBA, ZCB.*

On the straight line OH there exists a fourth point G (*Carnot*), the intersection of the straight lines AA′, BB′, CC′ which join the vertices of the triangle ABC to the mid points of its opposite sides, and these four points O, G, M, H are situated harmonically, that is to say, so that

$$GM : GO = HM : HO$$

which is the same as

$$1 : 2 = 3 : 6.$$

Besides, the points H, G are the centres of similitude of the two circles which have their centres at M and O; therefore the circle which has its centre at M passes through the middle of the straight lines HA, HB, HC; and the points X, Y, Z are respectively the mid points of the straight lines HR, HS, HT, the prolongations of HX, HY, HZ to the circumference of the circle whose centre is O.*

The circle whose centre is M possesses, in particular, this property well worthy of remark: it touches each of the four circles inscribed and escribed to the triangle ABC.

This demonstration of Steiner's contains some of the fundamental propositions relating to the subject of Isogonals.

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* Hence this theorem is easily inferred:

If on the circumference of the circle whose centre is O, four points A, B, C, D be taken arbitrarily, these four points will be, three and three, the vertices of four inscribed triangles to which will correspond four H points, four M points, and four G points. Now, these four points of each kind will belong to one circle whose radius will be equal to that of the given circle for the four H points, half of this radius for the four M points, and one third of it for the four G points. Besides, the centres of these three new circles will be with the point O harmonically situated on one straight line, as are the four points H, M, G, O; in such a way that the centre O will be the common centre of similitude of these three new circles.

For example:

(1) $\left.\begin{matrix} OA, HA \\ OB, HB \\ OC, HC \end{matrix}\right\}$ are isogonals with respect to $\left\{\begin{matrix} \angle A \\ \angle B \\ \angle C \end{matrix}\right.$

(2) O, H are isogonals with respect to ABC.

(3) If O, H be isogonals with respect to ABC, the mid point of OH is equidistant from the feet of the perpendiculars drawn from O, H to the sides of ABC. Or, in other words,

The projections on the sides of a triangle of two isogonal points furnish six concyclic points.

(4) If O, H be isogonals with respect to ABC, the sides of the pedal triangle corresponding to O are perpendicular to HA, HB, HC; and the sides of the pedal triangle corresponding to H are perpendicular to OA, OB, OC.

(5) If three lines drawn from the vertices of a triangle be concurrent, their isogonals with respect to the angles of the triangle are also concurrent.

(6) Since the radius of the circumcircle drawn to any vertex is isogonal to the perpendicular from that vertex to the opposite side, therefore the three perpendiculars of a triangle are concurrent.

SIXTH DEMONSTRATION (Terquem, 1842).

FIGURE 7.

Let ABC be a triangle, A′, B′, C′ the mid points of the sides, X, Y, Z the feet of the perpendiculars which intersect at H, and U, V, W the mid points of AH, BH, CH.

Join B′C′, C′A′, A′B′, C′X.

Then $$C'X = \tfrac{1}{2}AB = A'B';$$

therefore B′, C′, A′, X are the four vertices of a trapezium having equal diagonals.

This trapezium is therefore inscriptible in a circle; and therefore the three feet of the perpendiculars and the three mid points of the sides are concyclic.

If B′U be joined, it will be parallel to CZ, and therefore perpendicular to A′B′ which is parallel to AB.

Similarly C′U is perpendicular to A′C′;
therefore the three mid points and U are the vertices of a quadrilateral inscriptible in a circle;
and therefore the nine points mentioned are on a circumference whose radius is $\frac{1}{2}$R.

SEVENTH DEMONSTRATION (Adams, 1843).

The circle described through the feet of the perpendiculars of a given triangle passes through the mid points of the sides.

FIGURE 5.

Let the circle described through X, Y, Z the feet of the perpendiculars cut BC, CA, AB at A′, B′, C′.

Then, by Carnot's theorem

$$\frac{CA}{BA}\cdot\frac{AB'}{CB'}\cdot\frac{BC'}{AC'}=\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ},$$
$$= -1,$$

since AX, BY, CZ are concurrent.

Now $\angle AC'B' = \angle AYZ$,
since B′, C′, Z, Y are concyclic;
and $\angle AYZ = \angle ABC$,
since B, Z, Y, C are concyclic;
therefore $\angle AC'B' = \angle ABC$.

Hence C′B′ is parallel to BC;

therefore $$\frac{AB'}{CB'}=\frac{AC'}{BC'};$$

therefore $$\frac{AB'}{CB'}\cdot\frac{BC'}{AC'}= 1;$$

therefore $$\frac{CA'}{BA'}= -1,$$

that is, A′ is the mid point of BC.

Similarly B′, C′ are the mid points of CA, AB.

[Adams does not mention the other three points through which the circle passes. They are mid points of sides of the triangles HCB, CHA, BAH, and X, Y, Z are the feet of the perpendiculars of these triangles.]

EIGHTH DEMONSTRATION (Adams, 1846).

FIGURE 9.

Let ABC be a triangle, I, I_1, I_2, I_3 the incentre and excentres; then I is the orthocentre of the triangle $I_1I_2I_3$ and ABC is the orthic triangle.

About the triangle ABC circumscribe a circle, and let it meet AI_1 at U and I_2I_3 at U′. Join UU′, and draw ID, I_1D_1, I_2D_2, I_3D_3 perpendicular to BC.

Because AU and AU′ bisect adjacent angles at A,
therefore ∠ UAU′ is right;
therefore UU′ is a diameter of the circle ABC.

And because the arc BU = the arc CU,
therefore UU′ passes through A′, the mid point of BC
and is perpendicular to BC.

Again since $BD_2 = s = CD_3$,
therefore D_2 and D_3 are equidistant from A.

And since $BD = s_2 = CD_1$,
therefore D and D_1 are equidistant from A′.

Lastly, since A′ is the mid point of D_2D_3,
and since I_2D_2, I_3D_3 and U′A′ are parallel,
therefore U′ is the mid point of I_2I_3.

And since A′ is the mid point of DD_1,
and since ID, I_1D_1 and UA′ are parallel,
therefore U is the mid point of II_1.

That is, the circle ABC passes through the mid points of I_2I_3 and II_1.

Hence also it passes through the mid points of I_3I_1, II_2, and I_1I_2, II_3; in other words, it is the nine-point circle of the triangle $I_1I_2I_3$.

NINTH DEMONSTRATION (Wilkinson, 1855).

FIGURE 8.

Let ABC be the triangle; X, Y, Z the feet of the perpendiculars; A′, B′, C′ the bisections of the sides; U, V, W the bisections of the lines AH, BH, CH.

Then taking the four points U, C′, A′, W,
we have UC′ and WA′ each parallel and equal to $\frac{1}{2}$BH
also UW and C′A′ each parallel and equal to $\frac{1}{2}$AC

hence UWA′C′ is a rectangle,
and the four points lie in a circle upon UA′ or WC′ as diameter.

Similarly UVA′B′ is a rectangle,
and the four points U, V, A′, B′ are in a circle upon UA′ or VB′ as diameter.

But ∠ UXA′ = ∠ VYB′ = ∠ WZC′ = a right angle;
hence the points X, Y, Z lie in the same circle as the points U, A′, V, B′, W, C′.

TENTH DEMONSTRATION (Godward, 1878).

FIGURE 10.

If the following lemma be assumed:

Let H be any point within or without a circle whose centre is O, and let AR be any chord passing through H, then the locus of the mid point of HA or HR is a circle whose centre is the mid point of HO, and whose radius is half the radius of the given circle,
the characteristic property of the nine-point circle follows at once.

$$\text{For}\quad \left.\begin{array}{l} X,\ Y,\ Z \\ A',\ B',\ C' \\ U,\ V,\ W \end{array}\right\} \text{ are the mid points of } \left\{\begin{array}{l} HR,\ HS,\ HT \\ HL_1,\ HL_2,\ HL_3 \\ HA,\ HB,\ HC. \end{array}\right.$$

FEUERBACH'S THEOREM.

The nine-point circle of a triangle touches the incircle and the three excircles.

FIRST DEMONSTRATION (J. Mention, 1850).

The late M. Richard having asked me several times for a geometrical demonstration of the contact of the nine-point circle, here is the mode* in which I arrived at it, some time ago.

I propose to draw, through the middle of one of the sides BC, a circle tangent to the system (r, r_1) or (r_2, r_3).

Let N be the foot of the interior bisector of angle A; this point is the internal centre of similitude of the system (r, r_1). Let D, D_1 be the points where (r), (r_1) touch BC; A′ the middle point of BC, and X the foot of the perpendicular on BC from A.

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* M. Mention's notation has been slightly changed.

This proposition is easily proved

$ND \cdot ND_1 = NA' \cdot NX$ (*Nouvelles Annales*, III. 496);

thus the required circle passes through the foot of the perpendicular.

Now I choose the middle of the side BC because it is a point on the radical axis of each of the systems (r, r_1), (r_2, r_3), and I am brought to this special question :

To find the position of a circle tangent to two given circles and passing through a point on their radical axis.

This position is fixed very clearly by making use of a solution, as elegant as it is little known, given for the general case by M. Cauchy* when he was a pupil of the Polytechnic School, which leads to the following theorem :

A′ is a point on the radical axis of two circles O, O′; B, B′ the points of contact of one of the common tangents. The points C, C′, where the lines A′B, A′B′ cut the circles, are the points where the circle tangent to O, O′ and passing through A′ touches the circles.

The centre of the circle is situated on the perpendicular let fall from A′ on the common tangent; and if δ denotes the distance of A′ from this tangent, its radius is equal to $t^2/2\delta$, where t is the common length of the tangents drawn from A′ to the two circles.

Hence denoting by tang (r, r_1) the common tangent to the circles (r), (r_1), and coming back to the original triangle, the perpendicular let fall from A′ on tang (r, r_1) contains the centre of the circle passing through A′ tangentially to the system (r, r_1).

But tang (r, r_1) is perpendicular to the radius of the circumscribed circle issuing from the vertex A ; therefore this perpendicular is a radius of the nine-point circle.

That is more than enough to establish the identity of the required circle and the nine-point circle.

SECOND DEMONSTRATION (1854).

The nine-point circle of a triangle touches the incircle.

FIGURE 11.

Let ABC be a triangle, H the orthocentre, I the incentre, O the circumcentre, and M, the mid point of HO, the nine-point centre.

Join IM, and draw ID, ML perpendicular to BC.

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* *Correspondance sur l'École Polytechnique*, I. 193 (1804).

Through O draw UU′ a diameter of the circumcircle perpendicular to BC;
from A draw AK perpendicular to UU′.

Join AU, AU′, AO, MA′.

Then UU′ bisects BC and the arc BUC;
therefore AU bisects ∠ BAC, and passes through I.

Now

$$\begin{aligned} OK &= AX - OA', \\ &= AH + HX - OA' \\ &= 2OA' + HX - OA', \\ &= OA' + HX, \\ &= 2ML. \end{aligned}$$

But the triangles AKU′, IDN, having their sides mutually perpendicular, are similar;

therefore $U'K : AK = ND : ID$;

therefore $U'K : A'X = ND : ID$;

therefore $ID \cdot U'K = A'X \cdot ND$,

$$= A'D \cdot DX.$$

Hence if from M a perpendicular be drawn to ID,

$$\begin{aligned} MI^2 &= (ID - ML)^2 + (A'D - A'L)^2, \\ &= (ID - \tfrac{1}{2}OK)^2 + (A'D - \tfrac{1}{2}A'X)^2, \\ &= ID^2 - ID \cdot OK + \tfrac{1}{4}OK^2 + A'D^2 - A'D \cdot A'X + \tfrac{1}{4}A'X^2, \\ &= \tfrac{1}{4}OK^2 + \tfrac{1}{4}A'X^2 + ID^2 - ID \cdot OK - A'D(A'X - A'D), \\ &= \tfrac{1}{4}OA^2 + ID^2 - ID \cdot OK - A'D \cdot DX, \\ &= \tfrac{1}{4}OA^2 + ID^2 - ID \cdot OK - ID \cdot U'K, \\ &= \tfrac{1}{4}OA^2 + ID^2 - ID \cdot OU', \\ &= \tfrac{1}{4}R^2 + r^2 - Rr, \\ &= (\tfrac{1}{2}R - r)^2; \end{aligned}$$

or, $MI = \tfrac{1}{2}R - r$.

Lastly, since the distance between the centres of the nine-point and inscribed circles is equal to the difference of their radii, therefore the nine-point circle touches the incircle.

The nine-point circle of a triangle touches the three excircles.

FIGURE 12.

Let ABC be a triangle, H the orthocentre, I_1 an excentre, O the circumcentre, and M, the mid point of HO, the nine-point centre.

Join I_1M, and draw I_1D_1, ML perpendicular to BC.

Through O draw UU′ a diameter of the circumcircle perpendicular to BC;
from A draw AK perpendicular to UU′.

Join AU, AU′, AO, MA′.

Then UU′ bisects BC and the arc BUC;
therefore AU bisects ∠ BAC, and passes through I_1.

$$
\begin{aligned}
\text{Now} \qquad OK &= AX - OA', \\
&= AH + HX - OA', \\
&= 2OA' + HX - OA', \\
&= OA' + HX, \\
&= 2ML.
\end{aligned}
$$

But the triangles AKU′, I_1D_1N, having their sides mutually perpendicular, are similar;

$$
\begin{aligned}
\text{therefore} \qquad U'K : AK &= ND_1 : I_1D_1; \\
\text{therefore} \qquad U'K : A'X &= ND_1 : I_1D_1; \\
\text{therefore} \qquad I_1D_1 \cdot U'K &= A'X \cdot ND_1, \\
&= A'D_1 \cdot D_1X.
\end{aligned}
$$

Hence if from M a perpendicular be drawn to I_1D_1 produced,

$$
\begin{aligned}
MI_1^2 &= (I_1D_1 + ML)^2 + (A'D_1 + A'L)^2, \\
&= (I_1D_1 + \tfrac{1}{2}OK)^2 + (A'D_1 + \tfrac{1}{2}A'X)^2, \\
&= I_1D_1^2 + I_1D_1 \cdot OK + \tfrac{1}{4}OK^2 + A'D_1^2 + A'D_1 \cdot A'X + \tfrac{1}{4}A'X^2, \\
&= \tfrac{1}{4}OK^2 + \tfrac{1}{4}A'X^2 + I_1D_1^2 + I_1D_1 \cdot OK + A'D_1(A'X + A'D_1), \\
&= \tfrac{1}{4}OA^2 + I_1D_1^2 + I_1D_1 \cdot OK + A'D_1 \cdot D_1X, \\
&= \tfrac{1}{4}OA^2 + I_1D_1^2 + I_1D_1 \cdot OK + I_1D_1 \cdot U'K, \\
&= \tfrac{1}{4}OA^2 + I_1D_1^2 + I_1D_1 \cdot OU', \\
&= \tfrac{1}{4}R^2 + r_1^2 + Rr_1, \\
&= (\tfrac{1}{2}R + r_1)^2;
\end{aligned}
$$

or, $MI_1 = \tfrac{1}{2}R + r_1$.

Lastly, since the distance between the centres of the nine-point and any one of the escribed circles is equal to the sum of their radii, therefore the nine-point circle touches all the excircles.

THIRD DEMONSTRATION (J. M'Dowell, 1862).

FIGURE 13.

Let ABC be a triangle, H, I, O, the orthocentre, the incentre and the circumcentre;
let AX, ID, OA′ be perpendicular to BC.

Bisect AH in U; join A′U, OH intersecting in M;
and join I with A′, M, U.

Produce AI to N, and AX to L making UL equal to UA′, and join A′L.

Then A′ is the mid point of BC;
therefore A′U is a diameter of the nine-point circle, and equal to R.

But OH passes through the centre of the nine-point circle;
therefore M is the nine-point centre,

and $\quad MA' = MU = \frac{1}{2}R.$

Again $\quad \angle A'UX = \angle OAX = C - B;$

therefore $\quad \angle A'LX$ is the complement of $\frac{1}{2}(C-B)$;

therefore $\quad \angle LA'X = \frac{1}{2}(C-B) = NID.$

Hence the triangles NID, LA′X are similar;*

$$\begin{aligned}
\text{therefore} \quad & A'X : XL = ID : ND; \\
\text{therefore} \quad & ID \cdot XL = A'X \cdot ND; \\
\text{therefore} \quad & r \cdot XL = A'D \cdot DX; \\
\text{therefore} \quad & r \cdot XL + r \cdot UX = A'D \cdot DX + r \cdot UX; \\
\text{therefore} \quad & Rr = A'D \cdot DX + r \cdot UX.
\end{aligned}$$

Now if from I a perpendicular be drawn to UX,

$$\begin{aligned}
A'D^2 + IU^2 &= A'D^2 + DX^2 + (UX - r)^2, \\
&= A'X^2 - 2A'D \cdot DX + UX^2 - 2r \cdot UX + r^2, \\
&= A'X^2 + UX^2 - 2(A'D \cdot DX + r \cdot UX) + r^2, \\
&= R^2 - 2Rr + r^2;
\end{aligned}$$

$$\begin{aligned}
\text{therefore} \quad IA'^2 + IU^2 &= R^2 - 2Rr + 2r^2. \\
\text{But} \quad IA'^2 + IU^2 &= 2MA'^2 + 2MI^2, \\
&= \tfrac{1}{2}R^2 + 2MI^2; \\
\text{therefore} \quad \tfrac{1}{2}R^2 + 2MI^2 &= R^2 - 2Rr + 2r^2; \\
\text{therefore} \quad MI^2 &= \tfrac{1}{4}R^2 - Rr + r^2, \\
&= (\tfrac{1}{2}R - r)^2; \\
\text{or,} \quad MI &= \tfrac{1}{2}R - r.
\end{aligned}$$

Lastly, since the distance between the centres of the nine-point and inscribed circles is equal to the difference of their radii, therefore the nine-point circle touches the inscribed circle.

Suppose now I_1 to be the centre of an escribed circle, and r_1 its radius; by changing the sign of r in $MI = \frac{1}{2}R - r$

we have $\quad MI_1 = \frac{1}{2}R + r_1;$

---

* I have supplied the proof that the triangles NID, LA′X are equiangular, and have omitted the proof that $A'X \cdot ND = A'D \cdot DX$. It should be added that Mr M'Dowell does not use the terms orthocentre, incentre, circumcentre or nine-point circle.

therefore the circle through middle points of sides and the escribed circle touch one another externally. Hence the theorem is proved, but as this last principle, viz., the change of r into $-r_1$ is not recognised by Euclid, I shall proceed to give a legitimate geometrical proof that the circle with centre M and radius $\frac{1}{2}$R also touches the three escribed circles.

First, I may remark that only *one* circle can be described through the points A′ and X touching the inscribed circle.

For suppose DX less than A′D, produce A′X through X to a point Y such that $A'Y \cdot YX = YD^2$; the tangents from Y to the inscribed circle give the points of contact of the required circle with the inscribed, but one of these points is D, and the circle through A′, X and D is therefore infinite; thus only *one finite circle* can be described through the points A′ and X to touch the inscribed circle, This circle is therefore the one with centre M and radius $\frac{1}{2}$R.

Take $A'D_1 = A'D$, then D_1 is the point of contact of the circle escribed to BC, and N is clearly a centre of similitude of this escribed circle and the inscribed circle.

By a known geometrical property

$$A'X \cdot A'N = A'D^2;$$

therefore taking away $A'N^2$ from these equals we have

$$A'N \cdot NX = D_1N \cdot ND;$$

therefore also by another known geometrical property the circle through A′ and touching the inscribed circle and the circle escribed to BC must also pass through X; but by what has just been proved, this is the circle with centre M and radius $\frac{1}{2}$R.

FOURTH DEMONSTRATION (Binder, 1872).

FIGURE 14.

If U be the mid point of the arc BC, then AU bisects not only $\angle$ BAC but also $\angle$ OAX,

because $\angle OAU = \angle AUO = \angle UAX.$

Let the circle with centre U and radius UC cut AU at I and I_1, then I and I_1 are the centres of the incircle and first excircle of ABC.

These circles touch BC at D and D_1 and A′ is the mid point of DD_1.

Now $\angle BCU = \angle BAU = \angle UAC$;
therefore triangle UCN is similar to UAC,

and $$UN \cdot UA = UC^2,$$
$$= UI^2;$$
and hence $$A'N \cdot A'X = A'D^2.$$

If the radius IK of the incircle be drawn in the direction MA′, then $\angle KID = \angle OAX$, and the triangles KIN, DIN are equal and similar.

Join A′K and let it meet the incircle at T;

then $$A'K \cdot AT = A'D^2,$$
$$= A'N \cdot A'X;$$

therefore the points K, T, N, X are concyclic.

Hence $$2\angle A'TX \text{ or } 2\angle KTX = 2\angle KNX \text{ or } 2\angle KND$$
$$= 2\angle KID$$
$$= \angle A'MX;$$

therefore T lies on Feuerbach's circle.

From the similarity of the isosceles triangles A′MT and KIT, the points M, I, T lie on a straight line, and Feuerbach's circle is touched internally by the incircle.

In like manner Feuerbach's circle is touched [illegible] circle whose centre is I_1, because A′ [illegible]

FIFTH DEMONSTRATION ([illegible] Taylor, 1875).

FIGURE 15.

Let A′, C′ be the middle points of BC, AB; AN the bisector of A; AX perpendicular on BC; I centre of inscribed circle; D its point of contact with BC; D_1 the point of contact of escribed circle; A′U diameter of nine-point circle.

It is easy to prove that $A'X \cdot A'N = A'D^2$ (see M'Dowell's *Exercises on Euclid*, Art. 86). Hence if A′ be centre, and $A'D^2$ constant of inversion, the inscribed circle will invert into itself, as will also the escribed touching at D_1, since $A'D_1 = A'D$;
while the nine-point will invert into a straight line perpendicular to A′U, making therefore with BC an angle

$$= A'UX = A'C'X = BC'X - BC'A'$$
$$= 2BAX - BAC = BAX - CAX = C - B.$$

Now NS the tangent from N to inscribed circle which also touches the escribed circle makes with BC an angle

$$= DIS = 2DIN = 2XAN = C - B.$$

Therefore this line is the inverse of the nine-point circle. And as it touches the inverse of the inscribed circle and the inverse of the escribed circle, the nine-point circle touches the inscribed and the escribed circles.

If T be the point of contact of the inscribed and nine-point circles, the tangent to the inscribed circle at that point can readily be proved to touch the nine-point circle without using the theory of inversion.

[This is done in Johnson's *Treatise on Trigonometry*, p. 139 (1889).]

SIXTH DEMONSTRATION (E. M. Langley, 1876).

FIGURE 16.

Lemma.

A′ is any fixed point; B_1C_1 a fixed straight line touching a fixed **circle at P**; K is any other point on B_1C_1.

If along A′K there be taken A′B′ such that

A′K·A′B′ = square of tangent from D to fixed circle

... another fixed circle touching the first and passing through..

Let A′P cut the first circle again in Q, and let QR be the tangent at Q.

[R must always be taken on the opposite side of QA′ to B′, when the circle is on the opposite side of B_1C_1 to A′, and always on the same side of QA′ as B′ when the circle is on the same side of B_1C_1 as A′.]

Then $A'K \cdot A'B' = A'P \cdot A'Q$,

therefore B′, K, P, Q are concyclic;

therefore $\angle A'B'Q = \angle A'PK = \angle RQP$;

therefore B′ lies on a fixed circle through A′ and Q touching RQ, and therefore the first fixed circle at Q.

The nine-point circle of a triangle touches the incircle and the excircles.

Consider the incircle and that excircle which touches BC between B and C.

The sides of the triangle are three of the common tangents to these two circles. Let the fourth common tangent B_1NC_1 be drawn cutting AC, CB, BA in B_1, N, C_1. Let A′, B′, C′ be mid points of BC, CA, AB.

Then evidently $AB_1 = AB$, $AC_1 = AC$,
and AN is the internal bisector of $\angle BAC$;
therefore AN bisects CC_1 at right angles;
therefore A′B′ passes through W, the point where AN cuts CC_1.

If D, D_1 be the points of contact with BC,

$$A'D_1 = A'D = \tfrac{1}{2}(AB - AC) = \tfrac{1}{2}BC_1 = A'W.$$

Let A′B′, A′C′ cut B_1C_1 in K, L.

Then $\quad A'K : A'W = BC_1 : BA = A'W : A'B'$;

therefore $\quad A'K \cdot A'B' = A'W^2 = A'D^2 = A'D_1^2$.

Similarly $\quad A'L \cdot A'C' = A'D^2 = A'D_1^2$;

therefore B′, C′ lie on the circle through A′ touching the incircle and the first excircle.

If "external" be written for "internal" and

$$AB + AC \text{ for } AB - AC$$

the preceding investigation applies to the remaining excircles.

SEVENTH DEMONSTRATION (Chadu, 1879).

FIGURE 17.

ABC is a triangle; D_2, D_3 are the points of contact with BC of the excircles I_2, I_3; D_2', D_3' are the points of contact of these circles with the other exterior common tangent; N′ is the point of intersection of D_2D_3 and $D_2'D_3'$; A′ is the mid point of BC.

1°. The perpendicular AO drawn from A to $D_2'D_3'$ passes through the circumcentre of the triangle ABC.*

2°. If U be the point of intersection of the straight lines AX, A′K respectively perpendicular to BC, $D_2'D_3'$,
the circle described on A′U as diameter is the nine-point circle of the triangle ABC.

That being premised, we have

$$A'U \cdot A'K = A'X \cdot A'N'.$$

* Because BC $D_2'D_3'$ are antiparallel, $\angle BAO$ is equal to the complement of $\angle ACB$.

But N′ being the point of intersection of the side BC and the bisector of the exterior angle A of triangle ABC,

$$A'X\cdot A'N' = \left(\frac{AB+AC}{2}\right)^2 = \frac{D_2D_3^2}{4} = A'D_2^2.$$

If $A'D_2'$ cut the circumference I_2 at L_2

$$A'L_2\cdot A'D_2' = A'D_2^2 = A'U\cdot A'K;$$

therefore $\angle UL_2D_2'$ is right.

And since the lines I_2D_2', A′K are parallel, the circumference described on A′U as diameter is tangent at L_2 to the circumference I_2.

In the same way, the point L_3 where $A'D_3'$ cuts the circumference I_3 is the point of contact of this circumference with the nine-point circle.

In the same way again, the nine-point circle touches the incircle I and the excircle I_1.

Let D, D_1 be the points of contact with BC of the circles I, I_1; D′, D_1' the points of contact of these circles with the other interior common tangent; N the point of intersection of DD_1 and $D'D_1'$; K′ the point of intersection of A′U and $D'D_1'$.

We have $$A'U\cdot A'K = A'X\cdot A'N.$$

But N being the point of intersection of the side BC and the bisector of $\angle$ A,

$$A'X\cdot A'N = \left(\frac{AB-AC}{2}\right)^2 = \frac{DD_1^2}{4} = A'D^2.$$

If A′D′ cut the circumference I at L,

$$A'L\cdot A'D' = A'D^2 = A'U\cdot A'K';$$

therefore $\angle ULD'$ is right.

And since the lines ID′, A′K′ are parallel, the circumference described on A′U as diameter is tangent at L to the circumference I.

In the same way, the point L_1 where $A'D_1'$ cuts the circumference I_1 is the point of contact of this circumference with the nine-point circle.

EIGHTH DEMONSTRATION (W. Harvey, 1883).

FIGURE 18.

Of the triangle ABC, O is the circumcentre, H the orthocentre, and A′ is the mid point of BC.

OA′ produced bisects the arc BC in U ; I the incentre lies on AU and is so situated that $AI \cdot IU = 2Rr$; also $\angle XAU = \angle AUO = \angle OAU$.

M, the centre of the nine-point circle, bisects the distance HO, and the circumference passes through A′, X, and K the mid point of AH. Hence M bisects both A′K and HO, and $OA' = HK = AK$; therefore A′K is parallel to OA.

MLP is a radius of the nine-point circle, bisecting the chord XA′ in L and the arc XA′ in P ; ID is a radius of the incircle.

Since the arc XPA′ is bisected at P,

$$\text{therefore} \quad \angle XA'P = \text{half } \angle XKA', \quad = \text{half } \angle XAO, \quad = \angle XAU \text{ or } \angle AUO.$$

Hence if through I we draw a straight line (not shown in the figure) parallel to BC to meet AX and OU, the segments of this line are respectively equal to XD and A′D,
and we have by similar triangles

$$XD : IA = LP : PA',$$
$$A'D : IU = LP : PA';$$
$$\text{therefore} \quad XD \cdot A'D : AI \cdot IU = LP^2 : A'P^2.$$
$$\text{But} \quad AI \cdot IU = 2Rr, \text{ and } A'P^2 = R \cdot LP;$$
$$\text{therefore} \quad XD \cdot A'D = 2r \cdot LP;$$
$$\text{hence} \quad A'L^2 - LD^2 = XD \cdot A'D = 2r \cdot LP.$$
$$\text{Now} \quad IM^2 = (ML - r)^2 + LD^2,$$
$$= (ML - r)^2 + A'L^2 - 2r \cdot LP,$$
$$= ML^2 + A'L^2 - 2r(ML + LP) + r^2,$$
$$= \tfrac{1}{4}R^2 - rR + r^2;$$
$$\text{therefore} \quad IM = \tfrac{1}{2}R - r,$$

and the tangency of the circles is evident.

NINTH DEMONSTRATION (G. Richardson, 1888).

FIGURE 19.

In triangle ABC, H, O, I, M are orthocentre, circumcentre incentre, nine-point centre. UU′, QQ′ are diameters of the circumcircle and nine-point circle perpendicular to BC. I is situated on

AU, and QQ′ bisects the chord A′X at L and the arc at Q. Through I a parallel to BC is drawn meeting UU′ at S and AH at T; the rest of the construction is obvious from the figure.

Since C′ is the mid point of the hypotenuse of ABX,

therefore $\angle XC'B = 180° - 2B.$

Now $\angle A'C'B = A$;

therefore $\angle XC'A' = 180° - 2B - A$;

therefore $\angle XQ'Q = 90° - B - \frac{1}{2}A$

$= \angle BAX - \angle BAI$

$= \angle IAX.$

Hence triangles QQ′X, QXL, IAT, IUS are similar;

therefore $$\frac{A'D}{BU} = \frac{IS}{IU} = \frac{QX}{QQ'} = \frac{QX}{R}$$

$$\frac{DX}{AI} = \frac{IT}{AI} = \frac{QL}{QX}.$$

But $$\frac{BU}{2R} = \frac{BU}{UU'} = \frac{IF}{AI} = \frac{r}{AI};$$

therefore $$\frac{A'D}{BU}\cdot\frac{DX}{AI}\cdot\frac{BU}{2R} = \frac{QX}{R}\cdot\frac{QL}{QX}\cdot\frac{r}{AI};$$

therefore $A'D \cdot DX = 2r \cdot QL$;

therefore $\frac{1}{4}R^2 - MD^2 = 2r \cdot QL.$

Now $MD^2 + ID^2 - MI^2 = 2ID \cdot ML$;

therefore, by addition,

$$\frac{1}{4}R + r^2 - MI^2 = 2r \cdot MQ = Rr;$$

therefore $MI^2 = (\frac{1}{2}R - r)^2.$

ADDITIONAL PROPERTIES.

(1) The nine-point circle of ABC is the nine-point circle of HCB, CHA, BAH, for it passes through the mid points of their sides; hence the circumcircles of these triangles are equal.

In naming these triangles the order of the letters is such that X is the foot of the perpendicular from the first named vertex, Y the foot of that from the second, and Z the foot of that from the third. This is a matter of much more importance than appears at first sight.

(2) If P be any point on the circumcircle of ABC, and H the

orthocentre, the locus of the mid point of PH when P moves along the circumference is the nine-point circle.

See Godward's demonstration of the characteristic property of the nine-point circle.

(3) The nine-point circle bisects all straight lines drawn from the orthocentre H to the circumcircle ABC; hence the nine-point circle bisects all straight lines drawn from A to the circle HCB, from B to the circle CHA, from C to the circle BAH.

(4) If straight lines be drawn from the incentre or any one of the excentres of a triangle to the circumference of the circle passing through the other three centres, they will be bisected by the circumcircle.

(5) Since the nine-point circle touches the incircle and the excircles of ABC, it touches also the incircle and the excircles of HCB, CHA, BAH.

(6) If A′, B′, C′, the mid points of the sides of ABC, be taken as the feet of perpendiculars of a second triangle $A_1B_1C_1$, the nine-point circle of ABC will be the nine-point circle of $A_1B_1C_1$, and hence will touch another set of 16 circles.

Again take the mid points of the sides of $A_1B_1C_1$ and make them the feet of perpendiculars of a third triangle $A_2B_2C_3$. Another set of 16 circles will thus be obtained which are all touched by the nine-point circle And this process may be carried indefinitely far.

It will be found that these successive triangles $A_1B_1C_1$, $A_2B_2C_2$, and so on, approximate more and more to an equilateral triangle; and consequently that the nine-point circle of ABC will not only be the nine-point circle of the limiting triangle, but also the incircle of it.

(7) Instead of taking the mid points of the sides of ABC and making them the feet of the perpendiculars of a second triangle, take the feet of the perpendiculars of ABC and make them the mid points of the sides of a second triangle. There is thus obtained another set of 16 circles all touched by the nine-point circle; and this process also may be carried indefinitely far.

(8) Thirdly take the U, V, W points and make them either the

mid points of the sides, or the feet of the perpendiculars, of a second triangle, and other sets of circles are obtained all touched by the nine-point circle.

(9) Lastly, take a circle whose centre is O, radius R, and any point H inside it. It will be seen that H may be the orthocentre of an indefinite number of triangles inscribed in ABC. The nine-point circles of these triangles are all equal since their radii are $\frac{1}{2}$R, and their centres are at the mid point of OH; hence this indefinite number of triangles have all the same nine-point circle, and their incircles and excircles are all touched by it.

(10) If through the vertices of ABC straight lines be drawn parallel to the opposite sides, a new triangle $A_1B_1C_1$ is formed, and the nine-point circle of ABC touches the nine-point circles of the triangles A_1BC, B_1CA, C_1AB at the mid points of BC, CA, AB.

(11) If the perpendicular AX of ABC be produced to A_1 so that A_1X is equal to AX, and if through A_1 there be drawn A_1B_1 parallel to AB, and A_1C_1 parallel to AC, and these parallels meet BC at B_1, C_1, the nine-point circles of ABC, $A_1B_1C_1$ touch each other at X.

(12) If through A′, B′, C′, and U, V, W two sets of three lines are drawn parallel to the external bisectors of the angles A, B, C respectively, two new triangles will be formed having the same nine-point circle as ABC.

(13) If I′ denote the centre of any one of the excircles of the triangles ABC, HCB, CHA, BAH, the nine-point circle of ABC touches the common tangent of the circle I′ and the circle described on MI′ as diameter.

(14) If H be the orthocentre of ABC, and on HA, HB, HC three points U′, V′, W′ be taken such that

$$HU' = HA/n, \quad HV' = HB/n, \quad HW' = HC/n;$$

and on HA′, HB′, HC′ three other points A″, B″, C″ such that $HA'' = 2HA'/n$, $HB'' = 2HB'/n$, $HC'' = 2HC'/n$, then

(*a*) The lines U′A″, V′B″, W′C″ intersect on the line HO in a point M′ such that $HM' = HO/n$.

(*b*) The six points U′, V′, W′, A″, B″, C″ lie on a circle whose centre coincides with M′, and whose radius is R/n.

(15) If A, B, C, D, E be five points on a circle, the consecutive intersections of the nine-point circles of the triangles ABC, BCD, CDE, DEA, EAB lie on another circle whose radius is one half that of the first.

(16) In triangle ABC the circles described on AG, B′C′, BC as diameters are coaxal. If G′ be a point on A′H such that A′G′ = ⅓A′H, and A″ a point on AA′ produced such that AA″ = 2AA′, the nine-point circle of the triangle is coaxal with the circles described on AG′ and HA″ as diameters.*

(17) If OA′ be produced to A_1 so that $A'A_1 = OA'$, and similar constructions be made with OB′, OC′, a new triangle $A_1B_1C_1$ is obtained of which O is the orthocentre, H the circumcentre, and the nine-point circle coincides with the nine-point circle of ABC.†

(18) If from a point P on the circumcircle of a triangle, whose orthocentre is H, perpendiculars PD, PE are drawn to two of the sides, then HP, DE intersect on the nine-point circle.‡

(19) From the ends of a diameter of a given circle perpendiculars are drawn on the sides of an inscribed triangle; the two Wallace lines thus obtained intersect at right angles on the nine-point circle of the triangle.§

(20) Let AA′, BB′, CC′ be the medians of triangle ABC, intersecting in G; let AX, BY, CZ be the perpendiculars from the vertices on the opposite sides intersecting in H; and let B′C′, YZ intersect in U; C′A′, ZX in V; and A′B′, XY in W. Then AU,

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* (10)-(16).—Mr J. Griffiths in *Mathematical Questions from the Educational Times*, II. 69 (1864); III. 102, 76 (1865); IV. 60 (1865); V. 72 (1866); VII. 57-8, 76 (1867).

† (17) T. T. Wilkinson in *Mathematical Questions from the Educational Times*, VI. 25 (1866).

‡ (18) Dr C. Taylor in *Mathematical Questions from the Educational Times*, XVII. 92 (1872).

§ (19) Mr R. Tucker in *Mathematical Questions from the Educational Times* III. 58 (1865).

BV, CW will all be perpendicular to GH; and the triangle UVW will circumscribe the triangle ABC.

Let N, P, Q be the feet of the interior bisectors of the angles A, B, C, and N′, P′, Q′ the feet of the exterior bisectors; then the six straight lines UN, VP, WQ, UN′, VP′, WQ′ pass three and three through four points which are the points of contact of the nine-point circle with the inscribed and escribed circles.*

Geometrical Note.

By R. TUCKER, M.A.

If in a triangle ABC, points are taken on the sides such that

$$BP : CP = CQ : AQ = AR : BR = m : n = CP' : BP'$$
$$= AQ' : CQ' = BR' : AR'$$

then the radical axis of the circles PQR, P′Q′R′ passes through the centroid and "S." points of ABC; and if QR, Q′R′ cut in 1, RP, R′P′ in 2, PQ, P′Q′ in 3, then the equation to the circle 123 is

$$abc\Sigma\alpha\beta\gamma = mn\Sigma a\alpha.\Sigma a\alpha\{-mna^2 + (m^2 + mn + n^2)(b^2 + c^2)\}.$$

FIGURE 20.

The points P, Q, R are given by

$$(0,\ nc,\ mb),\ (mc,\ 0,\ na),\ (nb,\ ma,\ 0),$$

i.e., P, in trilinear co-ordinates, is (0, nc sinA, mb sinA), etc.;
and P′, Q′, R′ by

$$(0,\ mc,\ nb),\ (nc,\ 0,\ ma),\ (mb,\ na,\ 0).$$

It is hence evident that the pairs of triangles are concentroidal with each other and with ABC.

It is also evident that PQ′, P′Q are parallel to AB, and so on; also that P′Q, PR′ intersect on the median through A; and so on.

The triangle $PQR = (m^2 - mn + n^2)\Delta =$ the triangle P′Q′R′.

The equation to the circle PQR is

$$(m^2 - mn + n^2)abc.\Sigma(\alpha\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha.\ -mna^2 + m^2b^2 + n^2c^2),$$

and to P′Q′R′ is

$$(m^2 - mn + n^2)abc.\Sigma(\alpha\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha.\ -mna^2 + n^2b^2 + m^2c^2).$$

* (20) Rev. W. A. Whitworth in *Mathematical Questions from the Educational Times*, X. 51 (1868).

The radical axis of these circles is, therefore,

$$\Sigma(a\alpha . b^2 - c^2) = 0, \text{ hence } \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

The radical axis of either of the circles and of the circumcircle is of the form $l^2\mathrm{P} - l\mathrm{Q} + \mathrm{R} = 0$, where P, Q, R are linear functions of α, β, γ; and the envelope of each of these axes is the conic

$$(a^3\alpha + b^3\beta + c^3\gamma)^2 = 4(ab^2\alpha + bc^2\beta + ca^2\gamma)(ac^2\alpha + ba^2\beta + cb^2\gamma) \quad \ldots \quad (a).$$

The tangents in (a) intersect in the point $a\alpha/(a^4 - b^2c^2) = \ldots = \ldots$.

The radical centre of the three circles is

$$a\alpha/[a^4 - b^2c^2 + mnk(k - 3a^2)] = \ldots = \ldots;$$

where $$k \equiv a^2 + b^2 + c^2.$$

The equations to QR, Q′R′ are

$$\left.\begin{array}{l} -mna\alpha + n^2 b\beta + m^2 c\gamma = 0 \\ -mna\alpha + m^2 b\beta + n^2 c\gamma = 0 \end{array}\right\} \quad \ldots \quad \ldots \quad (b);$$

and 1, their point of intersection, is on the median through A, and is given by

$$a\alpha/(m^2 + n^2) = b\beta/(mn) = c\gamma/mn.$$

Similarly 2, 3 are

$$a\alpha/mn = b\beta/(m^2 + n^2) = c\gamma/mn,$$
$$a\alpha/mn = b\beta/mn = c\gamma/(m^2 + n^2).$$

The above lines (b) envelope the parabola $a^2\alpha^2 = 4bc\beta\gamma$, and so on.

The triangle 123 is readily found to be

$$= (m^2 - mn + n^2)^2\Delta.$$

The circle 123 has its equation

$$abc\,\Sigma(a\beta\gamma) = mn\Sigma(a\alpha).\Sigma\{a\alpha. - mna^2 + (m^2 + mn + n^2)(b^2 + c^2)\} \quad \ldots \quad (2)$$

The radical axis of this circle and the circumcircle can be written

$$(1 - mn)k\Sigma(a\alpha) = \Sigma(a^3\alpha),$$

hence it is a straight line parallel to the chord of contact of the conic (a).

The lines PR, P′Q′, … intersect in 4, 5, 6, given by

$$a\alpha/(mn - n^2) = b\beta/m^2 = c\gamma/m^2, \ldots,$$

showing that these points are also on the medians, as is evident from the symmetry of the figure.

The lines PR′, P′Q, ... intersect in p, q, r, where p is given by

$$a\alpha/(m-n)=b\beta/n=c\gamma/n.$$

The conic through PP′QQ′RR′ has for its equation

$$mn(a\alpha+b\beta+c\gamma)^2=bc\beta\gamma+ca\gamma\alpha+ab\alpha\beta \quad \ldots \quad (4),$$

which, in the figure, is an ellipse, concentric, similar and similarly situated with the minimum circum-ellipse of ABC.

The polar of A, with regard to (4), is

$$2amn\alpha-(m^2+n^2)(b\beta+c\gamma)=0,$$

therefore it is parallel to BC, and cuts AC in J (say); so that $AJ=2mn.AC$. The triangle formed by the three polars (for A, B, C) is

$$=4(m^2-mn+n^2)^2\Delta.$$

The tangents to the conic at P, P′ are given by

$$a\alpha(m^2+n^2)+b\beta m(m-n)-c\gamma n\,(m-n)=0,$$
$$a\alpha(m^2+n^2)-b\beta n\,(m-n)+c\gamma m(m-n)=0,$$

and intersect, on the median through A, in the point

$$\frac{a\alpha}{-(m-n)^2}=\frac{b\beta}{m^2+n^2}=\frac{c\gamma}{m^2+n^2};$$

and the triangle formed by this and the corresponding points equals the above triangle.

To find the "S." points of PQR, P′Q′R′, assume the sides of these triangles to be p, q, r; p', q', r'; then

$$\left.\begin{aligned}p^2&=m^2c^2+n^2b^2-2mnbc\cos A\\ p'^2&=m^2b^2+n^2c^2-2mnbc\cos A\end{aligned}\right\}\text{ etc.}=\text{etc.};$$

$$\therefore \quad \Sigma(p^2)=(m^2-mn+n^2)(a^2+b^2+c^2)=K\ \text{(suppose)}=\Sigma(p'^2).$$

The "S." lines through Q, R, respectively, are

$$\begin{vmatrix}\alpha & \beta & \gamma\\ nbcr^2 & ca(mr^2+np^2) & mabp^2\\ mc & o & na\end{vmatrix},\quad \begin{vmatrix}\alpha & \beta & \gamma\\ mbcq^2 & ncap^2 & (mp^2+nq^2)ab\\ nb & ma & o\end{vmatrix},$$

i.e.,

$$-na\alpha(mr^2+np^2)+(n^2r^2-m^2p^2)b\beta+mc\gamma(mr^2+np^2)=0,$$
$$ma\alpha(mp^2+nq^2)-(mp^2+nq^2)nb\beta-c\gamma(m^2q^2-n^2p^2)=0;$$

whence we get, for the "S." point of $PQR(K_1)$,

$$\frac{a\alpha}{mq^2+nr^2}=\frac{b\beta}{mr^2+np^2}=\frac{c\gamma}{mp^2+nq^2}=\frac{2\Delta}{K}.$$

Similarly, for the "S." point of $P'Q'R'(K_2)$, we have

$$\frac{a\alpha}{nq'^2+mr'^2}=\frac{b\beta}{nr'^2+mp'^2}=\frac{c\gamma}{np'^2+mq'^2}=\frac{2\Delta}{K}.$$

The triangle 123 is directly in perspective with ABC, and has the centroid of the triangles for centre of perspective; hence we can readily obtain the co-ordinates of the principal points.

For (1) the "S." point

$$a\alpha/[a^2+mn(b^2+c^2+2a^2)]=\ldots=\ldots;$$

(2*a*) the positive "B." point

$$a\alpha/[(m^2+n^2)c^2a^2+mnb\,(c^2+a^2)]=\ldots=\ldots;$$

(2*b*) the negative "B." point

$$a\alpha/[(m^2+n^2)a^2b^2+mnc^2(a^2+b^2)]=\ldots=\ldots;$$

(3) the in-centre

$$a\alpha/[a(m^2+n^2)+(b+c)mn]=\ldots=\ldots;$$

(4) the orthocentre

$$\alpha/[(m^2+n^2)\cos B\cos C+mn\cos A]=\ldots=\ldots;$$

(5) the circumcentre

$$\alpha/[(m^2+n^2)\cos A+mn\cos(B-C)]=\ldots=\ldots.$$

It is readily seen that the lines (AP, BQ), (AP', BQ') intersect on the conic $c^2\gamma^2=ab\alpha\beta$, which touches CA, CB at A and B, and passes through the centroid.

The co-ordinates of the centre are

$$\{\tfrac{1}{3}(2c\sin B),\ \tfrac{1}{3}(2c\sin A),\ \tfrac{1}{3}(-a\sin B)\};$$

like results hold for the other points of intersection.

[The preceding Note consists of a solution of Questions 11599 and 11670 of the *Educational Times*, and is published in vol. lviii. (pp. 119–123) of the "Reprint" from that journal. It is given here with the editor's kind consent. Part also of Question 11599 was proposed by Prof. Neuberg as Question 787 of *Mathesis*. In the number for January 1893, Prof. Neuberg points out that (*a*) *supra* is a conic touching the Brocardians of the Lemoine-line, where they meet the reciprocal of that line.]

Note of Newton's Theorem of Symmetric Functions.

By C. TWEEDIE, M.A., B.Sc.

It can be shown as follows that Newton's Theorem can be derived from elementary considerations without making use of the idea of an equation and its roots.

Let F.S. denote indiscriminately any function of $a_1, a_2 \ldots a_r$ which can be expressed in terms of Σa_1, $\Sigma a_1 a_2$, $\Sigma a_1 a_2 a_3$, etc.

Then $\Sigma a_1 = \Sigma a_1 = \text{F.S.}$

$$\text{F.S.} = (\Sigma a_1)^2 = \Sigma a_1^2 + 2\Sigma a_1 a_2 = \Sigma a_1^2 + \text{F.S.}$$

$\therefore$ $\Sigma a_1^2 = \text{F.S.}$

$$(\Sigma a_1)^3 = (\Sigma a_1^2 + \text{F.S.})\Sigma a_1$$
$$= \Sigma a_1^3 + \Sigma a_1^2 a_2 + \text{F.S.}$$

and $\therefore$ *if* $\Sigma a_1^2 a_2 = \text{F.S.}$ we have

$$\Sigma a_1^3 = \text{F.S.}$$

and $(\Sigma a_1)^3 = \Sigma a_1^3 + \text{F.S.}$

Similarly $(\Sigma a_1)^4 = \Sigma a_1^4 + \Sigma a_1^3 a_2 + \text{F.S.}$
$= \Sigma a_1^4 + \text{F.S.}$ if $\Sigma a_1^3 a_2$ be F.S., and so on.

... ...

Assume this is true up to Σa_1^r, to show it is true for Σa_1^{r+1}.

We have $(\Sigma a_1)^{r-1} = \text{F.S.} = \Sigma a_1^{r-1} + \Sigma a_1^{r-2} a_2 + \text{F.S.}$

$$(\Sigma a_1)^r = \quad = \Sigma a_1^r + \Sigma a_1^{r-1} a_2 + \text{F.S.}$$
$$= \Sigma a_1^r + \text{F.S. by assumption,}$$

while $\Sigma a_1^{r-2} a_2$; $\Sigma a_1^{r-1} a_2$ are F.S.

Now $(\Sigma a_1)^{r+1} = \text{F.S.} \quad = (\Sigma a_1)^r \Sigma a_1 = \Sigma a_1^{r+1} + \Sigma a_1^r a_2 + \text{F.S.}$
if then we can show $\Sigma a_1^r a_2 = \text{F.S.}$ so will also Σa_1^{r+1}.

Now
$$(\Sigma a_1)^{r+1} = \Sigma a_1 (\Sigma a_1)^r = \Sigma a_1 \{\Sigma a_1^r + \Sigma a_1^{r-1} a_2 + \text{F.S.}\}$$
$$= \Sigma a_1^{r+1} + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2^2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$
$$= (\Sigma a_1^{r+1} + \Sigma a_1^{r-1} a_2^2) + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$
$$= \Sigma a_1^{r-1} \quad \Sigma a_1^2 \quad + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$
$$= \text{F.S.} \quad + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$

$$\therefore \quad 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 = \text{F.S.} \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

Again, $(\Sigma a_1^{r-1})\Sigma a_1 a_2 = \text{F.S.}$

$$i.e., \ \Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 = \text{F.S.} \quad \ldots \quad \ldots \quad (2).$$

From (1) and (2) by subtraction

$$\Sigma a_1^r a_2 = \text{F.S.} - \text{F.S.} = \text{F.S.}$$

$$\therefore \quad \Sigma a_1^{r+1} = (\Sigma a_1)^{r+1} - \text{F.S.} - \text{F.S.}$$
$$= \text{F.S.}$$

Third Meeting, January 13, 1893.

C. G. KNOTT, Esq., D.Sc., F.R.S.E., in the Chair.

The Quaternion and its Depreciators.

By Prof. C. G. KNOTT, D.Sc., F.R.S.E.

Of late years there has arisen a clique of vector analysts who refuse to admit the quaternion to the glorious company of vectors. There are others again who take exception to some of Hamilton's most fundamental principles, and make corrections as they deem them, which logically revolutionise the whole basis of the calculus.

These rebellious ones do not agree at all amongst themselves; but their disloyal sentiments may be conveniently discussed under three headings.

First, there is the broad question as to the value of the quaternion as a fundamental geometrical conception.

Second, there is the question of notation.

Third, there is the question of the sign of the square of a vector when quaternion expressions are to be transformed into ordinary algebraic expressions.

In discussing these points, I shall give what seems to me to be the most natural geometrical approach to the calculus of quaternions. The position of the innovators will thus be better understood.

I. THE QUATERNION AS A GEOMETRICAL CONCEPTION.

(FIGURES 21, 22, 23).

In the preface to the third edition of his *Quaternions*, Professor Tait speaks of Professor Willard Gibbs as one of the retarders of quaternion progress, and of his system of notation as "a sort of hermaphrodite monster compounded of the notations of Hamilton and Grassmann" Professor Gibbs, in a letter published in *Nature*, April 2, 1891, virtually admits both impeachments. For he proceeds to give reasons for his antagonistic attitude, first, to Quaternions as an algebra of vectors, and, second, to Hamilton's notation. His objection to Hamilton's selective system of notation is based upon the dogma that the quaternion product cannot claim a fundamental place in a system of vector analysis. In support of this contention, Professor Gibbs presents a broad argument from geometry, which he thinks he strengthens by a reference to trigonometrical usage. He says :—

"It will hardly be denied that sines and cosines play the leading part in trigonometry. Now the notations $V\alpha\beta$ and $S\alpha\beta$ represent the sine and cosine of the angle included between α and β, combined in each case with certain other simple notions. But the sine and cosine combined with these auxiliary notions are incomparably more amenable to analytical transformation than the simple sine and cosine of trigonometry, exactly as, etc., etc."

What does this argument amount to? Certainly no quaternionist ever denied the importance of the sine and the cosine in trigonometry; and Hamilton was unquestionably the first to show forth the analytical power of the functions $S\alpha\beta$ and $V\alpha\beta$. But because these functions are so incomparably more amenable to analytical transformation than their trigonometrical ghosts, are we to infer that they are necessarily superior to or more fundamental than *anything else?* Yet that is the remarkable logic we are treated to.

Mr Heaviside, in his series of articles on "Electromagnetic Theory," published in the *Electrician*, seems to be referring to this argument when he says :—"The justification for the treatment of scalar and vector products as fundamental ideas in vector algebra is to be found in the distributive property they possess." *A fortiori*, the justification for the treatment of the quaternion product as a fundamental idea in vector algebra is to be found in the distributive and associative property it possesses.

Moreover, as Professor Macfarlane points out, the angle itself is of greater fundamental importance than its sine or cosine. So, on the principle of answering a wise man according to his wisdom, I say :—

It will hardly be denied that angles and their functions play the leading part in trigonometry. Now the notation $\alpha\beta^{-1}$ represents the angle included between α and β combined with certain other simple notions. But the angle combined with these auxiliary notions is incomparably more amenable to analytical transformation than the simple angle of trigonometry, and so on—which statement proves just as much and just as little as the great original itself.

But the real argument advanced by Professor Gibbs is as follows :—

"$V\alpha\beta$ represents in magnitude the area of the parallelogram determined by the sides α and β, and in direction the normal to the plane of the parallelogram. $S\gamma V\alpha\beta$ represents the volume of the parallelepiped determined by the edges $\alpha\beta\gamma$. These conceptions are the very foundations of geometry.* . . . I do not know of anything which can be urged in favour of the quaternion product as a *fundamental* notion in vector analysis, which does not appear trivial or artificial in comparison with the above considerations. The same is true of the quaternionic quotient and of the quaternion in general."

"These conceptions"—what conceptions? It can hardly be the conceptions of vector and scalar products of vectors, for these are altogether of the nineteenth century, whereas geometry is of all centuries. It must then be simply the conceptions of the parallelogram as the typical area, and of the parallelepiped as the typical volume. But to speak of these conceptions, and these conceptions *only*—as must be understood if the argument means anything—to speak of these as the very foundations of geometry is surely a misuse of terms, to put it most mildly. Is not the inclination of two lines as fundamental a conception as either of these? Indeed, underlying all the recognised theorems of parallelograms and parallelepipeds there is the great axiom of parallel lines. That lies at the foundation of geometry, if anything so lies.

To appreciate the real character of this argument, let us consider the meaning and purpose of a vector analysis. Having formed the conception of a vector, we have next to find what relations exist between any two vectors. We have to compare one with another; and this we may do by taking either their difference or their ratio. The geometry of displacements and velocities suggests the well-known addition theorem

$$\alpha + \delta = \beta,$$

in which, by adding the vector δ, we pass from the vector α to the vector β. But this method, which is always given first as the simplest, does not seem to me to be more fundamental geometrically than the other method which gives us the quaternion. When we wish to compare fully two lengths, a and b, we divide the one by the other. We form the quotient a/b, and this quotient is defined as the factor which changes b into a. Now a vector is a directed

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* The part omitted here is the part already given about the sines and cosines.

length. By an obvious generalisation, therefore, we compare two vectors by taking their quotient α/β, and by defining this quotient as the factor which changes the vector β into the vector α. This is the germ out of which the whole of vector analysis naturally grows. A more fundamental conception it is impossible to make. Yet Gibbs calls it trivial and artificial! Far more fundamental—we are told—are the conceptions of a vector-bounded area and of a vector-bounded volume, whose bounding vectors may have an infinity of values. Again, a vector is an embodiment of direction; and to know how to change a direction is surely demanded of a vector analyst from the very beginning. But a change of direction is an angular displacement—that is, a versor. Or take the case of a body strained homogeneously. The vector joining any pair of points changes by a process which is a combination of stretching and turning. A simpler description cannot be imagined. It is completely symbolised by the quaternion with its tensor and versor factors. And *this*, we are taught, is trivial and artificial! On the contrary, so fundamental and natural is the conception of the quotient of two vectors that it can be made intelligible to any one. We all unconsciously perform the operation when estimating the time that must be allowed to catch a train.

There is a certain superficial plausibility in the argument that the quaternion product of two vectors is *in that form* less suggestive of geometric significance than the scalar and vector parts taken severally. But when Professor Gibbs says that "the same is true of the quaternionic quotient," he invites the severest criticism. For not only is the quaternion quotient, as a geometrical conception, more fundamental and direct than its own scalar and vector parts; but, if *simplicity* of conception be a guide, it is infinitely more fundamental than even the much-lauded vector and scalar products. To a quaternionist, however, the product $\alpha\beta$ is fundamentally as intelligible as the quotient; for it is simply α/β^{-1}.

Professor Gibbs would have us base the whole of vector analysis on the two geometrical ideas embodied in the formulae $V\alpha\beta$ and $-S\gamma V\alpha\beta$. These are defined, and from the definitions, combined with recognised geometrical truths, the calculus is developed. Clifford, in his *Dynamic*, starts in this very way; and such a method may have an apparent advantage in introducing an otherwise ignorant student rapidly to the merits of a concise and expressive notation. It is "spoon meat," as Mr MacAulay puts it in his recent letter to *Nature* (December 15, 1892). But the average student will probably make little real progress along these lines. He will probably fail to grasp the unity of the calculus as developed from its broad quaternion basis. His faith—his credulity indeed—is severely tested from the very outset. Certain geometric conceptions are put forward and represented by a symbolism of a distinctly arbitrary character. For example, $V\alpha\beta$ does not really mean the area of the parallelogram determined by the vectors α and β, but is a mode of representing that area by a vector line perpendicular to its plane.* And, again, the transition from the parallelepiped $S\gamma V\alpha\beta$ to the uniplanar projection $S\gamma\delta$ cannot but seem to be a piece of legerdemain, involving the transformation of an area into a line. The method requires indeed a succession of definitions, and a careful geometrical discussion of the properties of the quantities so defined. In quaternions, however, the whole is a beautiful and compact development from the fundamental conception of the factor (α/β) which changes β into α. Corresponding to every such quaternion, there is another quaternion known as the conjugate, which will turn β into a particular vector α', equal in length to α, but lying equally inclined to β on the opposite side of it. In short, $\alpha\beta\alpha'$ lie in one plane, and α' is, so to speak, the reflection of α in β (regarded as a mirror).

The geometry is of the very simplest. Suppose, for example, that the quaternion α/β does not change the length of β, but simply its direction—in other words, that it is a versor merely.

Call it q, and its conjugate Kq. Then if $\overline{OB}$, $\overline{OA}$, $\overline{OA'}$ (Fig. 21) are β, α, α' respectively, we get at once

$$q.\overline{OB} = \overline{OA} \quad \text{or} \quad q\beta = \alpha$$
$$Kq.\overline{OB} = \overline{OA}' \quad \text{or} \quad Kq\beta = \alpha'.$$

Hence
$$(q + Kq)\beta = \alpha + \alpha' = \overline{OC} = \beta \times 2\cos\theta,$$

and
$$(q - Kq)\beta = \alpha - \alpha' = \overline{OD} = \overline{OB}'.2\sin\theta.$$

But OB′ is simply OB turned through a right angle. Hence if

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* This is very clearly brought out in O'Brien's system of vector analysis, briefly described further on.

we take i to represent the quadrantal versor, having the same axis as q, $\overline{\mathrm{OB}'} = i\beta$.

Thus we get

$$q + \mathrm{K}q = 2\cos\theta$$
$$q - \mathrm{K}q = 2i\sin\theta.$$

That is, the sum of a quaternion and its conjugate is a scalar quantity; while the difference is a quadrantal quaternion, changing the length of the vector on which it acts in the ratio of 1 to $2\sin\theta$.

The quadrantal quaternion is evidently of great importance, and it has a property of peculiar value.

Thus let ii' be any two given quadrantal versors, and let them be represented by double arrow-headed unit lines in the directions of their axis, as shown in Figure 22.

Take the unit vector at right angles to both. Then assuming the distributive law, we have

$$(i + i')\beta = i\beta + i'\beta$$
$$= \alpha \;\; + \alpha'.$$

But if we construct on i and i' a parallelogram like that which gives us the resultant vector $\alpha + \alpha'$, we get for its diagonal a directed line parallel to the axis of the quaternion which will turn β into $\alpha + \alpha'$. Not only so, but the length of this diagonal has the same ratio to the length of $\alpha + \alpha'$, which the length of i (or i') has to the length of β. We may therefore regard this diagonal as representing the quadrantal quaternion $(i + i')$. The conclusion is that quadrantal versors and (by an easy extension) quadrantal quaternions are compounded just like vectors. Since, so far, no definition of a vector acting on another vector has been given, we may (if no inconsistency arises) identify quadrantal quaternions and vectors. It is this identification which so wonderfully simplifies the calculus, and yet in no way destroys its generality. We shall refer to this later on.

Meanwhile the point to be noted is that, with this identification of quadrantal quaternion and vector, we conclude that

$$q + \mathrm{K}q = 2\mathrm{S}q, \text{ a scalar.}$$
$$q - \mathrm{K}q = 2\mathrm{V}q, \text{ a vector,}$$

where the meanings of $\mathrm{S}q$ and $\mathrm{V}q$ are easily detected. If q is a versor, $\mathrm{S}q$ is the *cosine* of the angle through which q turns a vector perpendicular to its axis; and $\mathrm{V}q$ is the vector (or quadrantal quaternion) measured along this axis, and of length equal to the *sine* of the same angle. By a simple extension, if a and b are the lengths (or *tensors*) of α and β, we find

$$\frac{\alpha}{\beta}=\mathrm{S}\frac{\alpha}{\beta}+\mathrm{V}\frac{\alpha}{\beta},$$

where $$\mathrm{S}\frac{\alpha}{\beta}=\frac{a}{b}\cos\theta \text{ and } \mathrm{V}\frac{\alpha}{\beta}=i\frac{a}{b}\sin\theta.$$

And now consider the result of operating by two quadrantal versors in succession. Let $i'i''$ be these versors (Fig. 23). Draw the planes perpendicular to them, and let γ be a vector along the line of intersection. Take β perpendicular to γ and i', so that $i'\beta=\gamma$.

Then $$i''(i'\beta)=i''\gamma=\alpha,$$

or, assuming the associative law, we get

$$i''i'.\beta=\alpha.$$

Hence $i''i'$ is the quaternion α/β, which (as is obvious from the figure) has its axis perpendicular to i' and i'', and turns β through an angle equal to the complement of the angle between i' and i''. Consequently we find, θ being the angle between i' and i'',

$$\mathrm{S}i''i=\mathrm{S}\frac{\alpha}{\beta}=-\cos\theta$$

$$\mathrm{V}i''i=\mathrm{V}\frac{\alpha}{\beta}=i\sin\theta.$$

From this we readily see that, with the identification of vectors and quadrantal quaternions,

$$\alpha\beta=\mathrm{S}\alpha\beta+\mathrm{V}\alpha\beta,$$

and $$\mathrm{S}\alpha\beta=-ab\cos\theta$$

$$\mathrm{V}\alpha\beta=i.ab\sin\theta,$$

where i is a unit vector perpendicular to the plane $\alpha\beta$. Thus the geometric meanings of $\mathrm{S}\alpha\beta$ and $\mathrm{V}\alpha\beta$ grow naturally out of the original conception of the quaternion quotient of two vectors, taken in conjunction with the identification of vectors and quadrantal quaternions, and with the assumption that the distributive and associative laws hold.

In any vector analysis which begins by separately defining the parts of the complete quaternion product, there is a want of cohesion from the very beginning, and there is nothing that can be compared with the beauty and solidarity of the quaternion calculus.

Take, by way of comparison, the symbolic algebra of the Rev. M. O'Brien, to which the systems affected by Gibbs, Heaviside, and Macfarlane have a strong family likeness. O'Brien, at that time Professor of Astronomy and Natural Philosophy in King's College, London, published his most important paper in the *Philosophical Transactions* (1852). He begins by defining what he calls the longitudinal and lateral translations of the vector β with reference to the vector α. These are symbolised as products in the form $\alpha \times \beta$ and $\alpha.\beta$—the reason being because they are distributive. It is readily seen that $\alpha \times \beta$ is the product of the lengths of α and β into the cosine of the angle between them; in fact, Hamilton's $-S\alpha\beta$, Grassmann's "inner" product, and Gibbs's "direct" product $(\alpha.\beta)$. In $\alpha.\beta$ O'Brien recognises the area of the parallelogram, of which α and β are the sides. In developing his system, he finds that the line perpendicular to the plane containing these two vectors is of fundamental importance. He calls it the directrix, and uses for it the symbol D. Thus $D\alpha.\beta$ corresponds geometrically to Hamilton's $V\alpha\beta$. It will be noticed that O'Brien keeps quite distinct the conception of the product $\alpha.\beta$, and that of its directrix $D\alpha.\beta$. From the definitions it follows that $\alpha.\alpha$ is zero, so that $\alpha \times \alpha$ may be written α^2 without any fear of ambiguity. Then α^2 is assumed to be the square of the length of α. It is abundantly evident that O'Brien's vector in multiplication is not intended to have any versor characteristic. He sees that the square of every unit vector must be the same, and confessedly *assumes* it to be $+1$, pointing out, however, that if he could see any reason for making it -1, his system would be the same as Hamilton's. We shall return to this further on.

Meanwhile, take another of the arguments accumulated by Professor Gibbs in favour of the non-quaternionic basis of vector analysis. He writes:—

"How much more deeply rooted in the nature of things are the functions $S\alpha\beta$ and $V\alpha\beta$ than any which depend on the definition of a quaternion will appear in a strong light if we try to extend our formulae to space of four or more dimensions. It will not be claimed that the notions of quaternions will apply to such a space, . . . But vectors exist in such a space, and there must be a vector analysis for such a space. The notions of geometrical addition and the scalar product are evidently applicable to such a space. As we cannot define the direction of a vector in space of four or more dimensions by the condition of perpendicularity to two given vectors, the definition of $V\alpha\beta$, as given above, will not apply *totidem verbis* to space of four or more dimensions. But a little change in the definition, which would make no essential difference in three dimensions, would enable us to apply the idea at once to space of any number of dimensions."

To elucidate the "nature of *things*" by an appeal to the fourth dimension—to solve the Irish Question by a discussion of social life in Mars—it is a grand conception, worthy of the scorner of the trivial and artificial quaternion of three dimensions. But is it not the glory of quaternions that it is so pre-eminently a tri-dimensional calculus? Geometers who look forward to a four dimensional existence may think their time in three dimensions best employed by confining their attention only to such mathematical methods as *seem* to be applicable to the higher space. But he lives best who works best in the particular environment of the moment. The man who fasts a whole week in prospect of a feast of unique magnificence is hardly rational. And note that Professor Gibbs has to make "a little change in the definition" of $V\alpha\beta$, ere he can make it serviceable in his evanescing vision of four dimensional space. Even his own vector analysis does not apply at once; and, with the admission of the necessity of change, the argument loses all point.

"There must be a vector analysis in such a space"—true, and there must be in space of n dimensions an M-in-one corresponding to the 4-in-one in 3-dimensional space. Moreover, the geometrical significance of a quaternion, as the factor that changes one vector into another, must have its analogue in space of four or higher dimensions. For if there be vectors, there must be modes of changing one into another.

In further pursuit of his end, Professor Gibbs draws a comparison between the quaternion and the linear and vector function, which latter he regards as quite enough for all purposes. He asserts that "nothing is more simple than the definition of a linear vector function, while the definition of a quaternion is far from simple." Observe, it is the simplicity of the *definition* that is here spoken of; but a definition will appear simple or the reverse, according to the degree of previous knowledge possessed. I question very much that a vector function of a vector is an easy conception to make on the part of one who is just entering upon the study of a vector analysis. It is only by a study of its properties, geometrical and dynamical, that the linear vector function becomes intelligible. Not until the thing symbolised is got a hold of by the mind can the definition of the symbol convey any adequate meaning. But, on the other hand, if the conception of a vector be realised at all, the further conception of the geometric meaning of the quotient and product of two vectors is a very simple step indeed. A simpler can hardly be imagined.

When Professor Gibbs speaks of the definition of a quaternion being far from simple, he probably has in mind the truth that a quaternion is expressible as the sum of a scalar and a vector. Mr Heaviside says: "The quaternion is regarded as a complex of scalar and vector." The pure analyst may think of it so; but the physicist should think of it in its purely geometrical significance as made up of tensor and versor. Its property of being decomposable into scalar and vector parts with geometric meanings, at first sight so distinct from its own fundamental characteristic, is an absolutely invaluable one. The quaternion includes within itself the conception of a rotation, a stretching, a vector area, and a projection. You may choose whichever part or parts may serve your purpose for the moment—they are all there uniquely determined when the quaternion is given. There truly is a king of quantities. "Upon earth there is not his like."

Still another argument, advanced in all seriousness by both Gibbs and Heaviside, is that even the avowed quaternionist comparatively rarely uses the quaternion, but is constantly manipulating his scalar and vector products. Now, it is true that the symbols S and V throng the pages of Hamilton and Tait; but the expression $V\alpha\beta$ does not hide the truth that $\alpha\beta$ is a quaternion. It rather displays it. By way of illustration, let us apply the Gibbs-Heaviside argument to trigonometry. In any treatment of this subject, the quantities $\sin\theta$ and $\cos\theta$ occur a hundred times at least for once that θ occurs singly. Is the angle, then, of no fundamental importance in trigonometry? There is more than an apparent analogy here. For just as *sin* and *cos* are selective symbols operating on θ, so are V and S selective symbols operating on q.

II. Comparison of Notations.

Professor Gibbs, having to his own satisfaction got rid of the "trivial and artificial" quaternion, is, for consistency's sake, obliged to object to the selective system of notation. This is not, however, the ostensible ground on which he recommends the adoption of a notation in which vector and scalar products of two vectors are indicated by symbols inserted between the quantities. This he regards as the natural mode of representation. Consequently he suggests $\alpha \times \beta$ to represent what he calls the "skew" product, and $\alpha . \beta$ to represent what he calls the "direct" product.* The skew product is Hamilton's vector product, which is certainly an infinitely more suitable name, even from Gibb's own limited point of view. The "direct" product—a most inappropriate name, it seems to me—is the product of the lengths of the vectors into the cosine of the angle between them, and corresponds to Hamilton's $-S\alpha\beta$. It is obvious that, though there may be a saving of labour in writing $\alpha . \beta$ instead of $S\alpha\beta$, no such advantage attaches to $\alpha \times \beta$ as compared with $V\alpha\beta$. But it is when more than two vectors have to be joined together that the inferiority of the suggested notation becomes painfully evident. Thus the expression $S\alpha\beta\gamma$ must be written $-\alpha . \beta \times \gamma$, which is less compact and less symmetrical than Hamilton's form. Again, the expression $VV\alpha\beta V\gamma\delta$ must be written $(\alpha \times \beta) \times (\gamma \times \delta)$, where the brackets are all-essential. The quantity $V\alpha\beta\gamma$ cannot be expressed by Gibbs at all in simple form, but has to be given in the expanded form

$$-\alpha . \beta \times \gamma + \alpha \times (\beta \times \gamma).$$

Such an expression as $V\alpha . \beta V\gamma\delta$ can only be displayed in the extraordinary form

$$-\alpha\beta . \gamma \times \delta + \alpha \times (\beta \times (\gamma \times \delta)).$$

It is occasionally necessary to use brackets in somewhat complex quaternion formulae, although in general a separating "dot" suffices to prevent ambiguity. But, in Gibbs's system, brackets have to be introduced just as soon as we begin to pass to the simplest formulae involving three vectors. The *cross* and *dot* are, in short, quite unequal to the task of distinguishing vector and scalar quantities.

Heaviside, in his notation, retains Hamilton's V, but drops the S, so that where no initial V exists, the product is taken to be the scalar product. Thus he would write $S\alpha\beta\gamma$ in the form $-\alpha V\beta\gamma$, in which, it appears to me, the symmetry of the expression is, to a large extent, lost, and in which there is no gain in compactness. The possibility of cyclically permuting $\alpha\beta\gamma$ without altering the value of $S\alpha\beta\gamma$ is by no means so evident in Heaviside's form.

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* These are O'Brien's very symbols, but used with the meanings interchanged.

One of the peculiar merits of Hamilton's notation is the way in which vector quantities stand out in relief among quantities of a different character. Small Greek letters are in general used for vectors; small Roman letters for scalars. The selective symbols V, S, T, U, K are evident at a glance, and we know what a quantity is before we have to inquire narrowly into its constitution. Not so with Gibbs's notation, in which any really complex expression becomes bewildering in its dots, crosses, and brackets. Heaviside has to a large extent destroyed the perspicuity of Hamilton's notation by employing capitals for the frequently occurring single quantities, so that the very important symbol V is not conspicuous. He distinguishes vectors from scalars by using heavy type. This distinguishes them sufficiently, no doubt, in print; but vector analysis is a thing *to be used*, and it is hopeless to write, easily and rapidly, capital letters and thick-lined capital letters with pencil, pen, or chalk. His own suggestion of a suffix notation to be used in manuscript is an unconscious condemnation of his whole system. A good notation in vector analysis requires these three things: (1) rapidity and ease in *writing* the frequently recurring quantities; (2) a distinction, evident at a glance, between vectors and scalars; and (3), as important as any, the vector and scalar parts of products thrown out in clear relief. It is abundantly evident that, in these respects, Hamilton's notation easily holds its own.

Apart altogether from the comparison that has just been made, there is, I think, a fundamental objection to a notation like O'Brien's and Gibbs's. It is that, corresponding to either product, there is no process by which a generalised quotient can be formed by taking one of the members over to the other side of the equation. Thus the equation

$$\alpha \times \beta = \gamma$$

suggests by its very form that there ought to be a transformation like

$$\alpha = \gamma \div \beta.$$

But there is no such, for obvious geometrical reasons. In other words, given γ and β, α is not determined. This simply shows, of course, that $\alpha \times \beta$ has no claim whatever to being regarded as a complete or generalised product. Exactly the same is true of $\alpha . \beta$. Now in quaternions we have $\alpha\beta = q$, where any one is determined uniquely when the other two are given. We are able at once to write $\alpha = q\beta^{-1}$ or $\beta = \alpha^{-1}q$. But in the equations

$$\mathrm{V}\alpha\beta = \gamma \text{ and } \mathrm{S}\alpha\beta = a$$

there is no suggestion of the possibility of taking β or a to the other side as a kind of divisor. By the very law of their being, S and V are selective symbols, and (like *sin*, *cos*, *log*, etc.) operate on the whole quantity $a\beta$. But in Gibbs's notation we have two quantities having all the appearance of ordinary products, to which, however, the familiar transformations which are suggested by their form are inapplicable. Such a restriction is surely inexpedient, especially when the desired end can be attained by a less objectionable and infinitely more perspicuous notation such as Hamilton has provided.

III. The Versorial Character of Vectors.

The identification of quadrantal quaternions and vectors has already been described as constituting one of the most important simplifications effected in the calculus. If a quadrantal quaternion operate *twice* on the same vector perpendicular to its axis, it will turn that vector through *two* right angles, and change the length of the vector in the ratio of a^2 to 1, where a is the tensor or stretching part of the quaternion. In symbols, if a is the quadrantal quaternion *or vector*, and β the perpendicular vector acted on, we get

$$aa\beta = -a^2\beta,$$

because the direction of β is simply inverted; or $a^2 = -a^2$. In words, the square of a vector is equal to *minus* the square of its length. If i, j, k are unit vectors, then $i^2 = j^2 = k^2 = -1$. This negative sign, which O'Brien puzzled over long ago, is a stumbling-block and rock of offence to both Mr Heaviside and Dr Macfarlane. It reappears whenever the quantity $Sa\beta$ is transformed into its value in ordinary algebraic quantities. Heaviside apparently was the first to kick against this peculiarity of quaternions. In his earlier papers he used the symbolism of quaternions because of its expressive compactness; and having found it irksome to be continually changing signs of scalar products, when he had occasion to transform these into ordinary algebraic symbols, he determined to take the scalar product as *plus* the product of the tensors into the cosine of the angle between the vectors. This O'Brien touch seems *so far* to have led to no confusion. Heaviside's formulae are quasi-quaternionic, and are a considerable simplification on the corresponding Cartesian expressions. But as the change involves the very fundamental one of making i^2, j^2, k^2 each *plus* unity, it is certain that the system is not quaternions. What, then, is it? To what, if fully developed, would it lead us? Macfarlane completely answers this question. In his pamphlet, *The Algebra of Physics*, he works out very fully O'Brien's and Heaviside's vector analysis, and obtains a system very similar up to a certain point to Hamilton's quaternions, but departing widely therefrom in certain of its higher developments. It is much more complicated, YET NO MORE GENERAL. When Dr Heaviside has realised the complication which is the logical outcome of his imagined simplification, we trust he will return into the paths of quaternionic rectitude. In his recent paper on the *Forces, Stresses, and Fluxes in the Electromagnetic Field* (Phil. Trans. 1892), he writes that his system "is simply the elements of quaternions without the quaternions, with the notation simplified to the uttermost, and with the very inconvenient *minus* sign before scalar products done away with." As we shall see presently, the first nine words of this sentence are fundamentally inconsistent with the last twelve.

Let us consider, first, what is common to quaternions, and to the system advocated by Heaviside and Macfarlane. It is well known that quaternions may be built up analytically upon the properties of i, j, k, three unit vectors (or right versors), at right angles to one another. Now Heaviside and Macfarlane admit the relations

$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik,$$

which also hold in quaternions. Gibbs, it may be noted, does not use the complete product at all, but writes his relations thus:

$$i \times j = k = -j \times i, \text{ etc, ; } i.i = 1, \text{ etc.}$$

O'Brien and his unconscious followers, however, boldly put $i^2 = j^2 = k^2 = +1$, thereby clashing at once with quaternions.

Taking, then, what is common to the two, namely, the set of equations represented by $ij = k = -ji$, let us consider the product of the three vectors, i, $i+j$, j, the values of i^2, j^2, k^2 being meanwhile left undetermined.

Then by one mode of association,

$$i(i+j)j = i(ij + j^2) = ik + ij^2 = -j + ij^2,$$

and by another mode of association,

$$i(i+j)j = (i^2 + ij)j = i^2 j + kj = i^2 j - i.$$

Here the distributive law is assumed. Now if these quantities are to be the same, that is, if the associative law is also to hold, we must have $$i^2 = j^2 = -1.$$

If we use $+1$, we get opposite vectors, and the associative law does not hold in vector products. The above, of course, is a very simple case. In the completely general case in this rival system, the products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are different quantities, giving the same scalar part, but quite different vector parts. It is surprising that this aspect of the question should have escaped O'Brien.

Let us represent vectors in Heaviside's and Macfarlane's system by Roman letters *abcd* ..., and corresponding Hamiltonian vectors by $\alpha\beta\gamma\delta$ .... Then it is easy to see that, since the scalar part of the product *ab* is equal to $-\mathrm{S}\alpha\beta$,

$$ab = -\mathrm{S}\alpha\beta + \mathrm{V}\alpha\beta = -\mathrm{K}\alpha\beta = -\beta\alpha,$$

and it may be shown that

$$(ab)c = -\gamma\alpha\beta$$
$$a(bc) = -\beta\gamma\alpha.$$

Now in quaternions we get in general *six* different quantities by permutations of α, β, γ; and at first sight it might seem that this new vector algebra gives *twelve* different products, since each arrangement such as *abc* gives two products by different associations. But inquiry soon shows that this is not so; for although there are two quantities got by different associations of any given arrangement, each quantity so obtained is reproduced in a particular association of some other particular arrangement. We easily see, in fact that $$\alpha\beta\gamma = -(bc)a = -c(ab)$$
$$\gamma\alpha\beta = -(ab)c = -b(ca),$$

and so on. It is evident that the O'Brien system gives us absolutely nothing more than is given by quaternions, but simply adds complexity. In quaternions we get all possible products by permutation *only*; in this other system we get the same number of quantities, partly by association, partly by permutation. The complexities of the system are still more pronounced when we pass to products of four or more vectors. Macfarlane glories in his five products obtained by different associations, namely,

$$((ab)c)d, \quad (a(bc))d, \quad (ab)(cd), \quad a((bc)d), \quad a(b(cd)).$$

But then we find that each one of these is reproduced in four other associations of particular arrangements. For example,

$$(ab)(cd) = ((ad)b)c = (d(ba))c = b((dc)a) = b(c(ad)).$$

All this hopeless confusion is the result of putting i^2, j^2, k^2 each equal to unity. Well may we be grateful to Hamilton for having given us an associative vector algebra of the utmost generality. A most interesting discussion of this very point is given in §§ 50–56 of the Preface to Hamilton's *Lectures on Quaternions*. It is there shown, from general considerations of the symmetry of space, that, when the rules for the multiplication of vectors are made to differ as little as possible from the usual rules for the multiplication of numbers in algebra, the result is the quaternion system of vector analysis, the *commutative law only* being departed from. These sections should be carefully considered by all would-be innovators.

The question naturally arises—What meaning are we to attach to the equations $ij = k$, $jk = i$, etc? Heaviside and Macfarlane seem to regard i and j as mutually perpendicular vectors, which, by their product, give a third vector perpendicular to both.* In quaternions the meaning is obvious, for i is the versor which, acting on j, turns it into k. Moreover, Professor Gibbs, on page 6 of his pamphlet, explicitly enunciates the same principle when he says that "the effect of the skew [*i.e.*, vector] multiplication by a [any unit vector] upon vectors in a plane perpendicular to a is simply to rotate them all 90° in that plane." To which, by way of commentary, we may quote the following from Heaviside:

"In a given equation" [in quaternions, that is], "one vector may be a vector, and another a quaternion. Or the same vector, in one and the same equation, may be a vector in one place and a quaternion (versor or turner) in another. This amalgamation of the vectorial and quaternionic functions is very puzzling. You never know how things may turn out."

Puzzling?—then should Mr Heaviside find his own system as puzzling as any. For when he writes the vector product

$$ij = k,$$

he is simply acting on j by i, or on i by j, and turning it through a

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* O'Brien seems to be much more consistent here, for his product $a.\beta$ is the area, and he uses $\mathrm{D}a.\beta$ as the symbolism for the quantity $\mathrm{V}a\beta$. Where Heaviside and Macfarlane cease to be O'Brienites, they become inconsistent.

right angle. It is impossible to get rid of this versorial effect of a vector. It stares you in the face from the very beginning. It is the only rational way of impressing the meaning of the equations.

Leaving Gibbs and Heaviside to harmonise, if possible, their differences, I shall here call attention briefly to one distinction between Hamilton's quaternions and Grassmann's *Ausdehnungslehre.* In the *Ausdehnungslehre* of 1862, Grassmann explains the meaning of his units e_1, e_2, e_3. . . . The *essential* feature of these is, that $e_1e_2 = -e_2e_1$, $e_1e_3 = -e_3e_1$, and so on for any pairs. Since the units are supposed to be of the same kind, it follows that $e_1e_1 = -e_1e_1$ also, an equation which cannot be true unless e_1^2 vanishes. Similarly the squares of all the units vanish. Grassmann also suggests that another algebra is given if we assume e_1^2, e_2^2, e_3^2 . . . to be each equal to $+1$, and all the products to be zero.

It is evident that the whole mode of looking at the question is fundamentally different in the two cases; and that it is impossible to identify Grassmann's units with Hamilton's i, j, k. Grassmann's "outer" and "inner" products in the *Ausdehnungslehre* of 1844 correspond to Hamilton's $V\alpha\beta$ and $-S\alpha\beta$; but there is no doubt that Grassmann failed to see that these quantities could be combined by subtraction, so as to give a new quantity having a very simple geometrical meaning, namely the quaternion of Hamilton.

IV. General Conclusions.

The general conclusions at which we have arrived may be summarised briefly as follows:

(1) The quaternion quotient is as fundamental a geometrical conception as the vector sum, the vector product, and the scalar product of two vectors, so that Professor Gibbs's argument, which is based upon the assertion that it is certainly not so, is void and meaningless.

(2) Whatever demerits may exist in Hamilton's own notation, there has not as yet been suggested anything that can be regarded as an improvement. The changes introduced by Gibbs and Heaviside destroy some of the most perspicuous and symmetrical features of the quaternion notation. Leaving out of account a few very exceptional cases, these suggested notations cannot for a moment compare with Hamilton's in clearness, compactness, and facility for manipulation.

(3) In the original conception of a vector (as involved in the addition theorem, for example), there is nothing inconsistent with its versorial character in multiplication. The truth is, that many physical quantities, which are symbolised by vectors, are essentially rotational. It is not merely displacement, or velocity, or acceleration that is so symbolised. Moments of velocities and forces, rotations themselves, vortex axes, and a whole host of similar quantities in electricity and magnetism, are either simple vectors or localised vectors. Or again, it is universally admitted that a displacement may be regarded as a rotation about an infinitely distant axis. Every vector in space may be regarded as a vector arc upon a spherical surface of infinite radius. But a vector arc on a spherical surface is a versor. On what physical ground, then, can any one object to a vector having a versorial quality? Indeed, notwithstanding all assertions to the contrary, Heaviside and Macfarlane really use the vector as a versorial operator; for what other meaning can be attached to the equation $ij = k$? Gibbs, as we said, explicitly uses the vector as a versor. The versorial character of a vector being thus admitted, there is no sufficient reason for regarding the square of a vector as other than *minus* the square of its length.

(4) The vector algebra, which is built upon the *assumption* that $i^2 = j^2 = k^2 = +1$ is non-associative in its products. And yet, notwithstanding this *appearance* of greater generality, it gives us absolutely no new thing. Its non-associative character is partly balanced by the fact that its non-commutativeness is incomplete. It simply muddles what is beautifully clear in quaternions.

In this paper I have limited myself to the consideration of the fundamental differences that exist between quaternions and the systems advocated by Gibbs, Heaviside, and Macfarlane. To complete the discussion, however, it would be necessary to review the systems in themselves as they have been developed. Of the three, Professor Gibbs has given us the most consistent system in his pamphlet, *The Elements of Vector Analysis*. In a paper communicated to the Royal Society of Edinburgh, I have entered at some length into a criticism of the contents of this pamphlet.

I show that Professor Gibbs, although ostensibly excluding the quaternion, introduces it in a covert way in his treatment of the linear and vector function. Not only so, but in certain volume surface and line integrals he uses *the quaternion product itself*, thereby perjuring his whole position, as described in his letter to *Nature*. Then there is his treatment of the quantities and operations that cluster round the quaternion operator ∇. By their tinkering processes, Gibbs, Heaviside, and Macfarlane all reduce this beautiful operator to a mere make-believe, which, in the simpler applications, appears to have all the essential attributes of the true ∇, but utterly fails when higher things are demanded of it.

It is a fair question—What has induced these scientific writers to take up their antagonistic attitude to the quaternion calculus? Heaviside and Macfarlane confess that their grievance is the *minus* sign. It is marvellous—indeed, almost ludicrous—to have mathematicians take fright at such a very simple matter. To the beginner, perhaps, who is constantly translating the quaternion quantities into ordinary analytical form, the necessity of changing the sign before scalar quantities is at first a little irksome. But, with a very little experience, the irksomeness quite vanishes away. It is no more formidable than re-arranging the terms of an equation by shifting them to different sides. Possibly, however, this preliminary peculiarity may have deterred many from continuing their study of quaternions. Heaviside, with inimitable assurance, thinks his system is what the physicist wants. An algebra non-associative in its products! When once the physicist has realised the full meaning of this, he will surely take courage, and tackle the quaternion analysis in earnest.

Gibbs, however, although he uses a symbolism for $-S\alpha\beta$, and thereby appears to side with Heaviside, nowhere confesses to have been repulsed by the "unnatural" and "inconvenient" *minus* sign. Why, then, does he object to Hamilton's system? His ostensible reasons, as given in the first letter to *Nature*, have been shown to be based on a complete misapprehension. Evidently he has not taken the trouble to get into the spirit of quaternions—and this, I believe, to be the true explanation of the apathy amongst physicists towards quaternion analysis—or (if we may judge from his second letter to *Nature*) he has so convinced himself as to the all-efficiency of Grassmann's methods, that he is determined to bar out the great thing in Hamilton's system which is lacking in Grassmann's. With what success, or non-success rather, he manages this, is shown in my paper communicated to the *Royal Society of Edinburgh*.

On some Loci connected with Conics.

By A. J. PRESSLAND, M.A.

1. For a proof of the following theorem due to Frégier, see Salmon's Conic Sections, p. 175, or Gergonne's *Annales*, VI. 231 (1816).

"If two straight lines at right angles be drawn through any point on a conic, the line joining their other points of section will pass through a fixed point on the normal."

If the conic be $x^2/a^2 + y^2/b^2 = 1$ ... ... A,

and the point be $a\cos\theta$, $b\sin\theta$ then the lines $x = a\cos\theta$, $y = b\sin\theta$ will be a pair of lines through the point at right angles. The chord joining their other points of section is

$$x/a\cos\theta + y/b\sin\theta = 0,$$

which intersects the normal in the point

$$x = \frac{a^2 - b^2}{a^2 + b^2} a\cos\theta, \quad y = -\frac{a^2 - b^2}{a^2 + b^2} b\sin\theta,$$

the locus of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 \quad \ldots \quad \ldots \quad \ldots \quad \text{B.}$$

If we take this conic as the original, the corresponding locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^4 \quad \ldots \quad \ldots \quad \ldots \quad \text{C.}$$

Now the polar of $a\cos\theta$, $b\sin\theta$ with respect to B is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2,$$

which is a tangent to C.

We could in this manner obtain an infinite series of concentric similar and similarly placed ellipses, such that of any three consecutive ones the external one would be the polar reciprocal of the internal one with respect to the middle one.

It can be shown that this is true of concentric similar and similarly situated ellipses whose major axes are in Geometrical Progression.

The same theorem holds for the Hyperbola.

If the parabola $y^2=4ax$ be taken, the lines $x=am^2$, $y=2am$ will be a pair of lines at right angles through the point am^2, $2am$. The line joining their other points of section is $y=-2am$, which intersects the normal in the point $4a+am^2$, $-2am$, the locus of which is

$$y^2=4a(x-4a),$$

a parabola having the same latus rectum and axis as the original one.

If we find the corresponding locus for this parabola we obtain

$$y^2=4a(x-8a).$$

Now the polar of $(am^2, 2am)$ with respect to $y^2=4a(x-4a)$ is

$$my=x-8a+am^2,$$

which is a tangent to $\qquad y^2=4a(x-8a)$.

It can be shown that this theorem is true of a series of parabolas having the same latus rectum and axis and similarly placed, provided that their vertices are distributed at equal intervals along the axis.

One example of the last series is found as an answer to the following question :—

If parallel chords of a parabola be drawn, the locus of the point of intersection of the tangents at their extremities is a straight line, and the locus of the point of intersection of the normals at their extremities another straight line. To find the locus of the point of intersection of these two lines.

If $y=2am$ be the diameter on which the tangents intersect the normals will intersect on the normal at $4am^2$, $-4am$, which is

$$-8am(x-4am^2)+4a(y+4am)=0,$$

and the locus reduces to $\qquad y^2=4a(x-3a)$.

This can be shown geometrically to be the locus of the point of intersection of normals at right angles.

2. If from a point h, k three normals be drawn to the parabola $y^2=4ax$ the equation $m^3-m(h-2a)/a-k/a=0$ determines their intersection with the curve.

If tangents be drawn at these points the orthocentre of the triangle so formed is

$$-a,\ a\{m_1+m_2+m_3+m_1m_2m_3\},$$

or $\qquad -a \quad k.$

That is the intersection of the diameter through h, k with the directrix determines the orthocentre.

3. In the central conic it is geometrically proved that (Fig. 26)

$$PG.PF = BC^2, \ Pg.PF = AC^2.$$

Now from the circles NGFK and $ngFk$

$\angle FGK = \angle FNK = \angle FnP$, since P, N, C, F, n are concyclic.
$= \angle Fgk$.

Hence GK is parallel to gk and the circles FGNK, $gFnk$ touch at F.

If CL be perpendicular to TPt the points C, n, L, P, N, F are concyclic, and the centre of the corresponding circle is the mid point of CP.

If this point be O, then $\angle OFP = \angle OPF = \angle CNF$.

Therefore OF touches the circle NGFK, and therefore $nFgk$.

As $On = OF = ON$, On and ON are also tangents. But NOn is a straight line. Therefore nON is the other common tangent to the two circles.

These two circles are therefore cut by $CnLPNF$ orthogonally.

This theorem is quite independent of the ellipse except for the direction of TPt. If this be taken at pleasure, the theorem still holds, and from it a number of geometrical derivatives can be obtained.

(i.) The circle about F, N, K passes through G. This can be extended to similar circles derivable from the cyclic hexagon $CnLPNF$ whose opposite sides are parallel. Thus the point of intersection of Pn and CL is on the circle ntL.

(ii.) Six circles of the series FGNK are obtainable, each of which touches the two adjacent ones.

(iii.) Two sets of three circles corresponding to $knFg$ are obtainable. The circles of each set touch each other and four circles of group (ii.).

(iv.) O is the radical centre of all these circles.

Taking the pair of circles FGNK, $Fgkn$ we shall get another circle, the image of PNC in their axis, and on it a point P′ the image of P.

If now CB, CA and P be fixed, the locus of F will be a circle on CP as diameter, and the locus of P′ a cardioid with P as pole.

Fourth Meeting, February 10, 1893.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

Note on Attraction.

By Professor TAIT.

It is well known (see *Thomson and Tait,* §§ 517, 518) that a spherical shell, whose surface-density is inversely as the cube of the distance from an external point, as well as a solid sphere whose density is inversely as the fifth power of the distance from an external point, are centrobaric. The centre of gravity is, in each case, the "image" of the external point.

To show that these express the same physical truth, we may of course recur to the method of electric images from which they were derived. But we may even more easily prove it by a direct process, for it is obviously only necessary to show that a thin shell, both of whose surfaces give the *same* image of an external point, has everywhere its thickness proportional to the square of the distance from that point.

Call O the object, and I the image, point; and draw any radius-vector IPQ, meeting the respective surfaces of the shell in P and Q. Then, ultimately,

$$\mathrm{OQ} - \mathrm{OP} = \mathrm{QP}\cos\mathrm{OPI},$$

or, in the usual notation,

$$\delta\left(\frac{r}{e}\right) = \delta r\cos\mathrm{OPI},$$

whence (introducing the new factor r)

$$r^2\frac{\delta e}{e^2} = \delta r\left(\frac{r}{e} - r\cos\mathrm{OPI}\right) = \delta r\mathrm{OI}\cos\mathrm{IOP}.$$

But IOP is equal to the angle between IP and the normal at P, so that the thickness of the shell at P is

$$\delta r\cos\mathrm{IOP} = \frac{r^2\delta e}{\mathrm{OI}.e^2}.$$

Notes: (1) On a Geometrical Problem.
(2) On an Algebraical Equation of Professor Cayley's.

By Professor STEGGALL.

On the Fundamental Principles of Quaternions and other Vector Analyses.

By Dr WILLIAM PEDDIE.

When a student of mathematics commences the study of a subject which involves the assimilation of what are, to him, fundamentally new ideas, his progress is, as a rule, slow at first. And, even after he has become accustomed to these ideas, he may still require a long course of laborious practice, before he can attain to that mastery of the method which would enable him to use it as a powerful aid to research. Thus students, familiar with geometrical methods, when first commencing the study of Cartesian analysis, require much practice before they can call up mentally the geometrical figure corresponding to a given equation. And, the more general the new method is, the greater is the difficulty felt to be. So, in Hamilton's system of quaternions, the difficulty of assimilation is greater than it is in the Cartesian analysis. And it seems as if it were for this reason that, in recent years, attempts have been made, by men of known mathematical ability, to smooth the paths.

Practically, all these attempts consist in using, instead of Hamilton's, another system of quaternions, cut up into parts; the parts of that system being used because they are imagined to be superior to the corresponding parts of Hamilton's system in respect of naturalness. Subsequently, I shall say somewhat regarding the reasonableness (or unreasonableness) of this claim; but, whatever conclusion be accepted on this point, M'Auley's appeal to the spoon-feeders, to "provide spoon-meat of the same *kind* as the other physicians" (*Nature*, Dec. 15, 1892), is most appropriate.

Some of the strictures recently passed on quaternions refer rather to the way in which the subject is presented in the standard treatises than to quaternions themselves. Heaviside (*Electrician*, Nov. 18, 1892) refers to three special "sticking-points" in Tait's treatise. One of these is the investigation of Hamilton's cubic in Chap. V. There is no special difficulty in the investigation, though the process may have been difficult enough to *discover*; and, curiously enough, when one turns with great expectations to Heaviside's alternative process, it is with genuine disappointment that it is found to be, *step-for-step*, Hamilton's with $c'\text{V}mn$ written instead of $\text{V}\lambda\mu$, and with other corresponding surface changes.

Another special "sticking-point" is said to be in Chap. IV., "where the reader may be puzzled to find out why the usual simple notion of differentials is departed from, although the departure is said to be obligatory." Surely the fact that, in this chapter, the usual notion is freely used, should produce reflection rather than misconception.

Chief of all the "sticking-points" is "the fundamental Chap. II., wherein the rules for the multiplication of vectors are made to depend upon the difficult mathematics of spherical conics, combined with versors, quaternions, and metaphysics." It is somewhat puzzling to find Heaviside speaking of the mathematics of spherical conics—at least so far as they are used there—as difficult. The "metaphysics" evidently refers to Hamilton's speculation, which Tait takes care to call a *quasi*-metaphysical speculation. His conclusions are the necessary logical results of his postulates, which, in so far as they refer to the nature of space, express the results of experience, and cannot be called metaphysical. One of the chief merits of this chapter, from a student's point of view, lies in the wealth of alternative proofs which it contains. Doubtless, by assuming the fundamental rules of vector multiplication, the identification of unit vectors and quadrantal versors might have been more directly made. But it seems as if Heaviside had failed to notice that Tait's method shows that such sweeping assumptions are unnecessary—that *partial* assumption of certain of the rules only is needed.

Another bone of contention is the minus sign which appears in the square of a vector or in a scalar product. Gibbs says (*Nature*, April 2, 1891), "When we come to functions having an analogy to multiplication, the product of the length of two vectors and the cosine of the angle which they include, from any point of view except that of the quaternionist, seems more simple than the same quantity taken negatively." Macfarlane (*Proc. Amer. Ass.*, 1891) says, that "a student of physics finds a difficulty in the principle of quaternions which makes the square of a vector negative"; and

Heaviside (*Electrician*, Dec. 9, 1892) writes, that "the vector having to submit to the quaternion, leads to the extraordinary result that the square of every vector is a *negative* scalar. This is merely because it is true for quadrantal versors, and the vector has to follow suit. The reciprocal of a vector, too, goes the wrong way, merely to accomodate versors and quaternions."

Now this point raises the whole question of the value of quaternions as such. Given that the quaternion is useless, or nearly so, in itself; and that scalar and vector products are only of use separately; no one will quarrel greatly with the advocates of the positive sign. Heaviside remarks that the physicist "is very much concerned with vectors, but not at all, or at any rate scarcely at all, with quaternions"; that "if the usual investigations of physical mathematics involved quaternions, then the physicist would no doubt have to use them. But they do not. If you translate physical investigations into vectorial language, you do not get quaternions; you get vector algebra instead." Gibbs remarks, that "the question arises whether the quaternionic product can claim a prominent and fundamental place in the system of vector analysis. It certainly does not hold any such place among the fundamental geometrical conceptions as the geometrical sum, the scalar product, or the vector product. The geometrical sum $\alpha + \beta$ represents the third side of a triangle as determined by the sides α and β. $V\alpha\beta$ represents in magnitude the area of the parallelogram determined by the sides α and β, and, in direction, the normal to the plane of the parallelogram. $S\gamma V\alpha\beta$ represents the volume of the parallelopiped, determined by the edges α, β, and γ. These conceptions are the very foundations of geometry." "I do not know of anything which can be urged in favour of the quaternionic product of two vectors as a *fundamental* notion in vector analysis, which does not appear trivial or artificial in comparison with the above considerations. The same is true of the quaternionic quotient, and of the quaternion in general."

Whatever be the case with regard to the mathematician, the statement that the physicist is scarcely, if at all, concerned with quaternions has surely been made without sufficient reflection. We may observe the velocities of a planet at two distinct instants, and merely describe the facts: or we may ask *how* the one became the other. The answer may be given in two ways—either by stating what vector quantity *added to* the one gives the other, or by stating what quantity *acting upon* the one gives the other. The former corresponds to the methods of pure vector analysis; the latter to those of quaternions, involving turning and lengthening. Both methods are of importance to the physicist. He sometimes wishes to consider the external addition to the changing quantity; sometimes to consider the internal changes as such. And one might just as rationally assert that he has to do only with quaternions as that he has to do only with vectors, since it is easy to use a notation expressing a vector in terms of a quaternion. The so-called vector analyst really uses a notation which expresses a quaternion in terms of vectors, and so his analysis simply bristles with quaternions: the quantity is none the less a quaternion because he chooses to shut his eyes to the fact, or at least not to use it as such. When he deals with $V\alpha\beta$, the quantity $\alpha\beta$ is the thing which turns the unit vector β into the unit vector $+\alpha$, or into $-\alpha$, according as we define the square of a unit vector to be equal to positive, or to negative unity, respectively—if we operate from right to left. But the vector analyst refuses to take advantage of what is in his power. Surely Heaviside would not have spoken of "wrong ways" if he had observed that, while in the quaternionic system $\alpha\beta$ turns β into $-\alpha$ and α/β turns β into $+\alpha$; in a similar system, in which the square of a unit vector is positive unity, $\alpha\beta$ would simply do what the quaternionic α/β does: and, if no fancied metaphysical necessity made the analyst regard the reciprocal of a direction as identical with the direction itself, α/β would do what the quaternionic $\alpha\beta$ does. The one method is as "natural" as the other. The choice of one must be ruled by expediency; the test of expediency being chiefly generality and applicability.

I believe most distinctly that students will prefer quaternion methods to those by which it is proposed to supersede them. The former develope a system naturally without any assumptions beyond those made fundamentally. In the latter, new definitions take the place of connecting links—as in the case of a working hypothesis which does not work well. An almost endless series of examples might be given of the singular inapplicability of the non-quaternionic systems to physical and other problems. Macfarlane is practically the only recent writer on the subject who does not arbitrarily exclude the quaternion from his system, which differs from ordinary quaternions in that the square of a unit vector is positive unity, and that he chooses to operate from left to right.

The quaternionic aspect of his system may be seen thus. Let a be any vector whatsoever, and let i be any unit vector. In Macfarlane's system aii represents a vector got by rotating a rigid-body-wise through two right angles round the axis of i: the corresponding vector in quaternions is $-iai$ or iai^{-1}. In Macfarlane's system $-i(ai)$ [or $i \cdot ai^{-1}$, if he did not fancy that the direction reciprocal to a given direction *should* be that direction itself] is the vector $-a$: in quaternions this is aii. Certain results have been interchanged, and that is all.

It might not have appeared *a priori* that this was all, for in this system a restriction, which holds in quaternions, disappears. The associative law does not apply, and in this respect the new algebra might have been more general; for, as Kelland points out in the Preface to Kelland and Tait's *Introduction to Quaternions*, generality is attained by the removal of restrictions. In arithmetic, the treatment of fractions was impossible until multiplication ceased to be regarded as a series of additions; and algebra became possible when negative quantities were recognised. But, in algebra, the commutative law holds. Quaternions—the self-contained algebraic system most suitable to tri-dimensional space—became possible when it was denied. But, in quaternions, the associative law holds. It may be that, in some system free from this restriction, greater generality will be reached. But the essential identity, pair by pair, of the results of the two systems under consideration, precludes the idea in this case. And so, the new system being no more general than quaternions, and being distinctly less workable (for no one will maintain that a non-associative algebra is so workable as an associative algebra), expediency decides in favour of quaternions.

Macfarlane asks, "What reason do writers on quaternions give for taking $xx' + yy' + zz'$ negatively in the case of the product of two vectors?" and asserts that "the true reason for taking the expression negatively is to satisfy the rule of association." This is not so: for it is easy to prove that we may take the square of a unit vector as positive unity, and yet get the associative law; provided only that we take $ij = \sqrt{-1}k$, etc., where i, j, and k, are unit rectangular vectors, and ij or $\sqrt{-1}k$ is the quadrantal versor whose axis is k. But, in this case, the product of an even number of vectors is a linear function of the three unit rectangular versors, while the product of an odd number is linear in i, j, k. Thus odd and even products are fundamentally distinct, and simplicity is lost.

Another point, in regard to which quaternions have been attacked, is that of applicability to space of n-dimensions. Hyde (*Directional Calculus*, Preface), speaking of Grassmann's method, says, "It seems scarcely possible that any method can be devised, comparable with this, for investigating n-dimensional space;" and Macfarlane asserts that "the method of Hamilton appears to be restricted to space of three dimensions." Gibbs speaks more strongly. "As a contribution to analysis in general, I suppose that there is no question that Grassmann's system is of indefinitely greater extension [than Hamilton's] having no limitation to any particular number of dimensions" (*Nature*, May 28, 1891). "How much more deeply noted in the nature of things are the functions $S\alpha\beta$ and $V\alpha\beta$ than any which depend on the definition of a quaternion, will appear in a strong light if we try to extend our formulæ to space of four or more dimensions. It will not be claimed that the notions of quaternions will apply to such a space, except, indeed, in such a limited and artificial manner as to rob them of their value as a system of geometrical algebra. But vectors exist in such a space, and there must be a vector analysis for such a space. The notions of geometrical addition and the scalar product are evidently applicable in such a space. As we cannot define the direction of a vector, in space of four or more dimensions, by the condition of perpendicularity to two given vectors, the definition of $V\alpha\beta$, as given above, will not apply *totidem verbis* to space of four or more dimensions. But a little change in the definition, which would make no essential difference in three dimensions, would enable us to apply the idea at once to space of any number of dimensions" (*Nature*, April 2, 1891).

Fortunately, the "strong light" of which Gibbs speaks shines the other way. The notions of quaternions *are* applicable to space of four or any number of dimensions. The general system should give a definition of $V\alpha\beta$, perfectly definite in space of any dimensions, and reducing to the usual one when the dimensions are limited to three. And it does.

The problem is to find a general system involving quantities $i, j, k, l, \ldots \ldots$, which represent unit rectangular vectors in cyclical order, and obey the laws $i^2=j^2=\ldots\ldots=-1$; $ij=-ji$, ...; and also the associative law. And the system must reduce to quaternions when only three of these vectors exist.

This problem has been worked out by Clifford in a paper *On the Classification of Geometric Algebras.* He makes the above assumptions, and then seeks to find what assumption must be made analogous to the Hamilton law $ijk = -1$. The following method of procedure is perhaps more in accordance with Hamiltonian ideas.

In three dimensions, the product of two unit rectangular vectors is the remaining rectangular unit vector. Assume generally that the product of $n-1$ such units, in cyclical order, is a vector quantity representable by the remaining rectangular vector; so that

$$ijk\ldots\ldots m = -\psi_n n,$$

where $-\psi_n$ is the operator which transforms n into $ijk\ldots\ldots m$, and we get

$$ijk\ldots\ldots n = \psi_n;$$

and, if we put $\psi_n\psi_n^{-1} = \psi_n^{-1}\psi_n = 1$, we get

$$1 = \psi_n^{-1} ijk\ldots\ldots n.$$

Also $n\psi_n = nijk\ldots\ldots n = \pm ijk\ldots\ldots m$, according as n is even or odd; that is, $n\psi_n = \pm\psi_n n$ according as n is odd or even. And $n = \pm\psi_n n\psi_n^{-1}$ according as n is odd or even. In this way we see that, quite generally, ψ_n and ψ_n^{-1} are symbols commutative with vectors if n is odd, but non-commutative if n is even.

Whatever be the sign of ψ_n^2, the sign of ψ_{n+1}^2 must be similar or dissimilar according as n is odd or even: for ψ_{n+1}^2 is reduced to $\pm\psi_n^2$ by n interchanges of the vector $(n+1)$ with other vectors, together with the substitution of -1 for $(n+1)^2$. It follows that, in the case of even values *alone*, ψ^2 is positive and negative unity alternately; and the same rule holds in the case of odd values alone. In the special case of three dimensions, $\psi^2 = -1$, from which all other cases may be deduced.

In an odd space of n-dimensions, we may put

$$\psi_n = \omega^{\frac{n+1}{2}},$$

where ω is a quantity whose square is negative unity. And, in an even space of n-dimensions, we may put

$$\psi_n = \omega^{\frac{n}{2}},$$

where ω^2 is also negative unity.

Except in the cases in which n is divisible by 4, we may suppose ω to be the imaginary of ordinary algebra; in these cases

ψ would be positive or negative unity, so that the more general symbol should be retained because of the non-commutative nature of ψ in spaces of even dimensions.

In particular, in two dimensions, $ij = \sqrt{-1}$. Hence we get $j = -i\sqrt{-1} = \sqrt{-1}i$, and $i = -\sqrt{-1}j$. If $a = xi + yj$, the operator $\sqrt{-1}(\equiv\psi_2)$ gives $\sqrt{-1}a = xj - yi$. In this case there is no need to retain the symbols i, j; for $a = xi + yj = (x + y\sqrt{-1})i$, and i denotes a given direction, so that a may be completely denoted by $x + y\sqrt{-1}$. It appears, therefore, that complex algebra is a special case of this generalised quaternionic system. Ordinary arithmetic may be regarded as the special case $\psi_0 = 1$.

Thus, in respect of generality, as well as of simplicity, the quaternionic method has the advantage.

In four dimensions, from $ijkl = \psi$, we get $ijk = -\psi l$, $jkl = \psi i$, $kli = -\psi j$, $lij = \psi k$. It does not follow that the space is non-symmetrical, or that, as the condition of symmetry, we should have $ijk = -\psi l$, $jkl = -\psi i$, etc. For we have seen that, in the symmetrical two-dimensional space, we have $i = -\sqrt{-1}j$, $j = \sqrt{-1}i$, *not* $j = -\sqrt{-1}i$, as a necessary condition for symmetry.

In any space $V\alpha\beta$ represents a directed area in the plane of α, β. In three dimensions, it happens to be representable by a linear vector.

Fifth Meeting, March 10, 1893.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

Early History of the Symmedian Point.

By J. S. MACKAY, M.A., LL.D.

In 1873, at the Lyons meeting of the French Association for the Advancement of the Sciences, Monsieur Emile Lemoine called attention to a particular point within a plane triangle which he called the centre of antiparallel medians. Since that time the properties of this remarkable point and of the lines and circles connected with it have been investigated by various writers, foremost among whom is Monsieur Lemoine himself. The results obtained by them are so numerous (indeed every month adds to their number) and so widely scattered through the mathematical periodicals of the world that it would be a task of considerable magnitude to make even an undigested collection of them. It is the purpose of the present paper to state those properties of the point which had been discovered previously to 1873. A short sketch of some of them will be found at the end of a memoir read by Monsieur Lemoine at the Grenoble meeting (1885) of the French Association, and in a memoir by Monsieur Emile Vigarié at the Paris meeting (1889) of the same Association. The references given by Dr Emmerich in his *Die Brocardschen Gebilde* (1891) are very valuable. It is a pity they are not more explicit.

If ABC be a triangle, AA′ the median from A, then AR the image of AA′ in the bisector of angle A is called the symmedian from A. It is not difficult to prove that AA′ bisects all parallels to BC, and that AR bisects all antiparallels to BC. Hence Monsieur Lemoine proposed* to call AR an antiparallel median. This name however has been replaced by symmedian (*symédiane* abbreviated from *symétrique de la médiane*) a happy coinage † of Monsieur Maurice d'Ocagne.

Since the three medians and the three symmedians are isogonally conjugate with respect to the three angles of the triangle, those theorems which have been established regarding isogonally conjugate lines in general can at once be applied to the particular case of medians and symmedians.

The point of concurrency of the three symmedians, which it is usual to denote by K, has received various names such as minimum-point, ‡ Grebe's point, § Lemoine's point. ‖ The designation symmedian point, suggested ¶ by Mr Tucker, seems preferable to all of these.

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* *Nouvelles Annales de Mathématiques*, 2nd series, XII. 364 (1873).

† *Nouvelles Annales de Mathématiques*, 3rd series, II. 451 (1883).

‡ Dr E. W. Grebe in Grunert's *Archiv der Mathematik*, IX. 251 (1847).

§ Dr A. Emmerich's *Die Brocardschen Gebilde*, p. 37 (1891).

‖ Prof. J. Neuberg's *Mémoire sur le Tétraèdre*, p. 3 (1884).

¶ *Educational Times*, XXXVII. 211 (1884).

The first mention of the symmedian point that I have found is in Leybourn's *Mathematical Repository*, old series, III. 71, where the following question is proposed* for demonstration by "Yanto."

If K be the point in a triangle from which perpendiculars are drawn to the sides of the triangle so that the sum of their squares is the least possible; twice the area of the triangle is a mean proportional between the sum of the squares of the sides of the triangle and the sum of the squares of the above-mentioned perpendiculars.

The second mention of K is in Leybourn's *Mathematical Repository*, new series, Vol. I. Part I. pp. 26-7.

Question 12, proposed by James Cunliffe, Bolton, is:

It is required to determine the locus of a point, from whence, if perpendiculars be drawn to three straight lines given by position, the sum of the squares of the said perpendiculars may be equal to a given magnitude.

In the solution of this question—the locus is an ellipse—given by Mr J. I. it is shown that if K be taken such that KL, KM, KN (perpendicular to BC, CA, AB) are proportional to BC, CA, AB, then $KL^2 + KM^2 + KN^2$ is a minimum, and that AK produced divides BC into segments which are proportional to AB^2 and AC^2.

Seeing that solutions of the first 30 questions proposed in the *Mathematical Repository* were to be in the hands of the editor by the first day of February 1804, it may be assumed that Mr J. I.'s solution was published in that year. I have some grounds (which need not be stated here) for conjecturing that Mr J. I. was James Ivory, known for his theorem regarding the attractions of ellipsoids on external and internal particles.

Ivory's theorem that the distances of K from the sides are directly proportional to the sides taken along with the well-known theorem that the distances of the centroid G from the sides are inversely proportional to the sides, establishes the theorem that G and K are inverse points with respect to the triangle.

In Leybourn's *Mathematical Repository*, new series, Vol. I. Part II. p. 19 (1806), Ivory proves the theorem:

If P and Q be two points taken on a pair of lines isogonal with

* I am not quite certain at what date, for my copy of Vol. III. is imperfect. But at p. 80 a letter is printed, dated March 1st, 1802, and at p. 83 another dated Sept. 8, 1802. It may therefore be presumed that the question was published in 1803.

respect to angle BAC, the distances of P from AB and AC are inversely proportional to those of Q from AB and AC.

The converse of this theorem, taken with what immediately precedes, might easily suggest that the lines drawn from A to G and K (hitherto known only by its minimum property) were isogonal with respect to angle BAC; but Ivory makes no explicit mention of the fact.

The other theorem given by Ivory, namely, that AK produced divides BC into segments, which are proportional to AB^2 and AC^2, is easily seen to be a particular case of a theorem regarding isogonals which was known to the ancient Greeks.* The theorem is:

If ABC be a triangle, and if AP, AQ be isogonal with respect to A, and meet BC in P and Q, then

$$BP \cdot BQ : CQ \cdot CP = AB^2 : AC^2.$$

It may be worth mentioning that Pappus proves also that if

$$BP \cdot BQ : CQ \cdot CP > AB^2 : AC^2$$

then $$\text{angle } BAP > \text{angle } CAQ.$$

Lhuilier in his *Élémens d'Analyse*, pp. 296-8 (1809), states and proves the theorem of "Yanto," shows that the distances of any point in a symmedian from the adjacent sides are proportional to those sides, that the segments into which a symmedian divides the opposite side are proportional to the squares of the adjacent sides, and adds:

"This doctrine can be extended to any polygons and even to polyhedrons. I shall content myself, for example, with determining that point in space from which, if perpendiculars be let fall on the faces of a tetrahedron, the sum of their squares is a minimum, and with determining that minimum."

He then proves that

(1) The perpendiculars drawn from this minimum-point are directly proportional to the faces on which they fall.

(2) The perpendicular on any face is a fourth proportional to the sum of the squares of the four faces, to the square of this face, and to the altitude of the tetrahedron which corresponds to this face.

(3) Thrice the volume of a tetrahedron is a mean proportional

* See Pappus's *Mathematical Collection*, VI., 12. The same theorem differently stated is more than once proved in Book VII, among the lemmas which Pappus gives for Apollonius's treatise on *Determinate Section*.

between the sum of the squares of the four faces and the sum of the squares of the perpendiculars let fall on them from the minimum point.

In this connection reference may be made to Professor J. Neuberg's *Mémoire sur le Tétraèdre* (1884).

The fourth discoverer of the point K is L. C. Schulz von Strasznicki. C. F. A. Jacobi says that Schulz published a pamphlet in 1827 with the title "Das gradlinige Dreieck und die dreiseitige Pyramide nach allen Analogien dargestellt." This pamphlet I have not seen. About the same time Schulz published in Baumgaertner and D'Ettingshausen's *Zeitschrift für Physik und Mathematik*, I. 396, II. 530, two articles, the first on the plane triangle and the second on the tetrahedron. Probably these two articles and the pamphlet are the same thing. In the first article he proves the following results : *

(1) If K (defined by its minimum property) be joined to the vertices, the fundamental triangle will be divided into three other triangles whose areas will be as the squares of the sides of the fundamental triangle on which they rest.

(2) The straight lines drawn through each vertex and through K will divide the opposite sides into two segments proportional to the squares of the adjacent sides; hence a simple geometrical construction for finding K.

(3) The same straight lines will divide each of the angles of the triangle into two partial angles whose sines will be as the adjacent sides.

(4) If the point K is replaced by the centroid G, the sines of the partial angles will be as the reciprocals of the adjacent sides.

(5) If the point K is replaced by the circumcentre O, the cosines of the partial angles will be directly as the adjacent sides.

(6) If the point K is replaced by the orthocentre H, the cosines of the partial angles will be inversely as the adjacent sides.

(7) Generally, if the angles of a triangle be divided in such a manner that for each of them the sines of the partial angles may be to each other directly or inversely as any powers or functions of the

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* This account of Schulz's articles is taken from Férussac's *Bulletin des Sciences Mathématiques*, VIII. 2 (1827).

adjacent sides the three straight lines will be concurrent; and if each side be divided into segments which are to each other as functions of the adjacent sides, and each point of section be joined to the opposite vertex, the three straight lines will be concurrent.

Steiner in a paper published* in Gergonne's *Annales de Mathématiques* XIX. 37-64 (1828) states and proves some of the fundamental theorems relating to isogonally conjugate points and lines. Thus

(1) The orthogonal projections on the sides of a triangle of two isogonally conjugate points furnish six concyclic points.

(2) If P, Q be isogonally conjugate points with respect to ABC, the sides of the pedal triangle corresponding to P are perpendicular to QA, QB, QC; and the sides of the pedal triangle corresponding to Q are perpendicular to PA, PB, PC.

(3) If three lines drawn from the vertices of a triangle be concurrent, their isogonal conjugates with respect to the angles of the triangle are also concurrent.

(4) Every point in the interior of a triangle may be considered as one of the foci of an ellipse inscribed in the triangle.

(5) The feet of the perpendiculars let fall from the foci of an ellipse on its tangents are all situated on the same circle having the major axis of this ellipse for diameter.

(6) If an angle be circumscribed to an ellipse the straight lines drawn from the two foci to the vertex of that angle are isogonal with respect to it.

(7) The rectangle under the perpendiculars let fall from the two foci of an ellipse on any one of its tangents is constant and consequently equal to the square of the semiaxis minor of the ellipse.

In C. Adams's *Die Lehre von den Transversalen*, pp. 79-80 (1843) the following theorem is proved:

Let D, E, F be the points of contact of the incircle with the sides of ABC, and Γ be the point at which AD, BE, CF are concurrent. If through Γ parallels be drawn to the sides of triangle DEF, these parallels will cut the sides of DEF in six concyclic points.†

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* Republished in Steiner's *Gesammelte Werke*, I. 191-210 (1881).

† See the following paper on *Adams's Hexagons and Circles.*

It is now known that Γ is the symmedian point of DEF; hence this six-point circle of Adams is the first Lemoine circle of DEF, or as Mr Tucker has called it, the triplicate-ratio circle.

Adams shows also that the centre of his six-point circle is the mid point of ΓI, where I is the incentre of ABC and consequently the circumcentre of DEF.

It will conduce to brevity of statement if the following definitions and notation be laid down.

If AR, BS, CT be the symmedians of ABC, then AR′, BS′, CT′ their harmonic conjugates with respect to the sides of ABC may be called the external symmedians,* or the exsymmedians of ABC. The points R, R′ are situated on BC, S, S′ on CA, T, T′ on AB. Let the exsymmedians intersect each other at K_1, K_2, K_3, and let AK_1 meet the circumcircle ABC whose centre is O at D. The mid point of BC is A′.

The following properties occur in C. Adams's *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, pp. 1-5 (1846).

(1) The theorem quoted from Pappus VI., 12.

(2) The corollary $BR : CR = AB^2 : AC^2$.

(3) The tangents to the circumcircle at the vertices coincide with the exsymmedians of the triangle.

(4) The symmedian from any vertex and the exsymmedians from the two other vertices are concurrent.

(5) DR, DR′ are the symmedian and exsymmedian of triangle BCD drawn from D.

(6) BR, BK_1 are the symmedian and exsymmedian of triangle ABD drawn from B.

Similarly for CR, CK, and triangle ACD.

(7) $AR'^2 + BK_1^2 = K_1R'^2$.

(8) OR is perpendicular to K_1R'.

(9) AR′ is a mean proportional between A′R′ and RR′.

In this connection it may be worth mentioning that Pappus in his *Mathematical Collection*, VII., 119, gives the following theorem as a lemma for one of the propositions in Apollonius's *Loci Plani*:

* Monsieur Clément Thiry in *Le Troisième Livre de Géométrie*, p. 42 (1887).

If $AB^2 : AC^2 = BR' : CR'$
then $BR' \cdot CR' = AR'^2$.

Dr E. W. Grebe of Cassel in Grunert's *Archiv der Mathematik*, IX, 250-9 (1847) discusses the point K and gives it the name minimum-point. He indicates two constructions for finding K.

(1) On the sides of ABC let squares X, Y, Z be described either all outwardly to the triangle or all inwardly. Produce the sides of the squares Y, Z opposite to AC, AB to meet in A'; the sides of the squares Z, X opposite to BA, BC to meet in B'; the sides of the squares X, Y opposite to CB, CA to meet in C'. Then A'A, B'B, C'C will be concurrent at K which will be the minimum-point not only of ABC but of A'B'C'.

(2) Find the isogonally conjugate point to G the centroid.

Denote by L, M, N the projections of K on BC, CA, AB.

(3) Various expressions for $KL^2 + KM^2 + KN^2$.

(4) Expressions for the segments BL, CL, CM, AM, AN, BN in terms of the sides a, b, c, and in terms of the sides and angles.

(5) Expressions for AK, BK, CK in terms of the sides, and in terms of the sides and the three medians.

(6) Expressions for MN, NL, LM in terms of the sides and area of ABC, and in terms of the sides, area, and medians of ABC.

(7) K is the centroid of LMN.

Grebe shows that if the square on the side AB be described inwardly to the triangle and the other two squares outwardly, an analogous point, K_3, is obtained, and he gives three sets of expressions for its distances from BC, CA, AB.

The next mention of K is in the *Nouvelles Annales*, 1st series, VII. 407-9 and 454 (1848). The theorem is thus stated:

If through each angle of a triangle a straight line is drawn which cuts the opposite sides into two segments proportional to the squares of the adjacent sides the three straight lines are concurrent at a point such that the sum of the squares of its distances from the sides of the triangle is a minimum.

The theorem was communicated by Captain Hossard to M. Poudra who gave a geometrical solution in the course of which it is seen that the perpendiculars from K on the sides are proportional to those sides and that K is the centroid of the triangle LMN. At the end of Captain Hossard's analytical solution it is added that the square of the distance AK is

$$\frac{b^2c^2(b^2+c^2+2bc\cos A)}{(a^2+b^2+c^2)^2}$$

an expression almost identical with that given by Grebe.

C. F. A. Jacobi in his *Die Entfernungsörter geradliniger Dreiecke*, pp. 12-13 (1851) draws attention to isogonal points (*Gegenpunkte* he calls them), and proves that if K be the point isogonal to G then K is the centroid of the triangle whose vertices are the projections of K on the sides of ABC, and the sum of the squares of the distances of K from the sides of ABC is a minimum. He adds that a Viennese mathematician L. C. Schulz von Strasznicki gave another proof by the help of co-ordinate geometry and the differential calculus.

Monsieur Catalan in Lafremoire's *Théorèmes et Problèmes de Géométrie Élémentaire*, 2nd ed., p. 161 (1852) proves that if K be the minimum point of ABC it is the centroid of the triangle LMN.

In Schlömilch's *Uebungsbuch zum Studium der höheren Analysis*, I. § 33 (1860) there is enunciated the theorem

The three straight lines uhich join the mid points of the sides of a triangle to the mid points of the perpendiculars on them from the vertices are concurrent.

Dr Emmerich says that the identity of this point of concurrency with the symmedian point was made evident by Wetzig.

Dr Franz Wetzig in Crelle's *Journal* LXII. 349-361 (1863) gives five or six properties of the symmedian point, but adds nothing to what had previously been known. The symmedians he calls minimum-axes, and remarks that they are analogous to the medians. He returns however to the subject four years later.

In *Mathematical Questions from the Educational Times*, III. 30-1 (1865) Mr W. J. Miller points out that the straight lines joining the three excentres I_1, I_2, I_3, of a triangle to the mid points of the sides are concurrent at a point such that the sum of the squares of the perpendiculars drawn therefrom on the sides of the triangle $I_1I_2I_3$ is a minimum, and these perpendiculars are, moreover, proportional to the sides on which they fall.

In the *Lady's and Gentleman's Diary* for 1865, pp. 89–90, Mr Stephen Watson proposes two questions for solution. The first is:

Show that three rectangles can be inscribed in any triangle, so that they may severally have a side coincident in direction with the respective sides of the triangle, and their diagonals all intersecting in the same point. Also show that one circle will circumscribe all the three rectangles, and find its radius.

The common centre of these three rectangles is the symmedian point, and the circle circumscribing them is Lemoine's second circle.

The radius of the circle, given in Mr Watson's solution published the year following, is equal to

$$\frac{abc}{a^2+b^2+c^2}$$

The second is:

Through each two of the angles of a triangle ABC any circles are described cutting the sides again in D, E ; F, G ; H, I ; and at each of those pairs of points tangents are drawn to the circles, meeting in P, Q, R. Show that the loci of P, Q, R are conics passing respectively through the angles of the triangle, and intersecting the two contiguous sides, in each case, in two points D′, E′ ; F′, G′ ; H′, I′. Also show that the tangents to those conics at the angles, and the lines D′E′, F′G′, H′I′ all pass through one point.

This point is the symmedian point, and is identified by Mr Watson with the centre of the three rectangles in the previous question.

In the *Nouvelles Annales de Mathématiques*, 2nd series, IV. 403-4 (1865) Monsieur J. J. A. Mathieu mentions as inverse points with respect to triangle ABC the centroid G and the point of intersection of AK_1, BK_2, CK_3. This point of intersection, he states, has for polar the straight line which passes through the points of intersection of each side with the tangent to the circumcircle drawn through the opposite vertex.

Let I, I_1, I_2, I_3 be the incentre and excentres of ABC,
Γ, Γ_1, Γ_2, Γ_3 the Gergonne points,
and Γ', Γ_1', Γ_2', Γ_3' the points complementary to Γ, etc.;
then $\Gamma\Gamma'$, $\Gamma_1\Gamma_1'$, $\Gamma_2\Gamma_2'$, $\Gamma_3\Gamma_3'$
are concurrent at G the centroid of ABC,
and $I\Gamma'$, $I_1\Gamma_1'$, $I_2\Gamma_2'$, $I_3\Gamma_3'$
are concurrent at K the symmedian point of ABC.

The preceding theorem was enunciated by Mr William Godward in the *Lady's and Gentleman's Diary* for 1866, p. 72, and a solution by trilinear coordinates appeared in the same periodical the following year. In connection with this subject it may be worth while to compare *Lady's and Gentleman's Diary* for 1865, pp. 63-5, and *Mathematical Questions from the Educational Times*, II. 86-8 (1865).

In the *Diary* for 1867, p. 71, Mr Thomas Milbourn enunciates the theorem,

If δ be the diameter of the circle remarked by Mr Stephen Watson, that is, the second Lemoine circle, and d the diameter of the circumcircle, then

$$\frac{1}{\delta^2}+\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

In Schlömilch's *Zeitschrift für Mathematik*, XII. 281-301 (1867) Dr Wetzig communicates a considerable number of properties relating not only to K but to K_1, K_2, K_3 which he calls harmonically associated (*harmonisch zugeordneten*) to K with respect to ABC. Thus

(1) If XYZ be the orthic triangle of ABC its sides are parallel to those of $K_1K_2K_3$.

(2) AK_1, BK_2, CK_3 meet at K and bisect the sides of XYZ.

(3) K is the centre of a conic which touches the sides of ABC at X, Y, Z.

(4) On the medians of ABC are situated the symmedian points of the triangles AYZ, BZX, CXY.

(5) AK_1, BK_2, CK_3 meet BC, CA, AB, at R, S, T. Perpendiculars RR′, SS′, TT′ to BC, CA, AB divide the sides of XYZ in the same proportions as the sides of ABC.

(6) A, K, R, K_1 is a harmonic range.

(7) XR′ goes through K.

(8) The symmedian point of AYZ is situated on the perpendicular from K to BC.

(9) $KYZ : KZX : KXY = KBC : KCA : KAB$
$= BC^2 : CA^2 : AB^2.$

(10) In the point systems A, B, C, ..., X, Y, Z, ...
K corresponds to itself and the circumcentre of the first system corresponds to the orthocentre in the second.

(11) The points K_1, K_2, K_3, of the first system lie with the corresponding points of the second on the perpendicular bisectors of BC, CA, AB and are equally distant from BC, CA, AB.

(12) $K_2A : K_3A = CX : BX$, etc.

(13) $K_2K_3 \cdot K_3K_1 \cdot K_1K_2 : BC \cdot CA \cdot AB = 2$ circle $K_1K_2K_3$: circle ABC

(14) $AA' \cdot BB' \cdot CC' : AK_1 \cdot BK_2 \cdot CK_3 = R : 4$ radius of circle $K_1K_2K_3$
$= AK \cdot BK \cdot CK : KK_1 \cdot KK_2 \cdot KK_3$

(15) $a \cdot AK : b \cdot BK : c \cdot CK = AA' : BB' : CC'$

(16) $$a^2 \frac{AK_1}{KK_1} = b^2 \frac{BK_2}{KK_2} = c^2 \frac{CK_3}{KK_3} = \frac{a^2+b^2+c^2}{2}$$

(17) $$\frac{KK_1}{AK_1} : \frac{KK_2}{BK_2} : \frac{KK_3}{CK_3} = a^2 : b^2 : c^2$$

(18) Then follow expressions for the distances from BC, CA, AB of K, K_1, K_2, K_3.

If Σ, Σ_1, Σ_2, Σ_3, denote the sum of the squares of these respective distances

$$\frac{1}{\Sigma} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}$$

(19) If k_1', k_1'', k_1''' denote the distances of K_1 from BC, CA, AB
$k_1'k_1''k_1''' : k_2'k_2''k_2''' : k_3'k_3''k_3''' = a^3 : b^3 : c^3$.

(20) K is the centroid of triangle LMN, and the sides of LMN are proportional to the medians of ABC.

(21) MN is perpendicular to AA′, and the angles of LMN are equal to the angles which the corresponding medians make with one another.

(22) If k_1, k_2, k_3 denote the distances of K from BC, CA, AB
$\frac{1}{3}LMN : ABC = k_1^2 + k_2^2 + k_3^2 : a^2 + b^2 + c^2$

(23) Corresponding property for triangle $L_1M_1N_1$

(24) $$\frac{\frac{1}{3}LMN}{\Sigma} = \frac{L_1M_1N_1}{\Sigma_1} = \text{etc.} = \frac{\Delta}{a^2+b^2+c^2}.$$

A construction for determining K is given in Schömilch's *Zeitschrift*, XVI. 168 (1871) from a communication by Const. Harkema in Petersburg.

The Elements of Quaternions *(First Paper).*

By Dr WILLIAM PEDDIE.

In this paper the laws of addition and subtraction of vectors were considered, and examples of their extreme usefulness in geometrical applications were given.

Adams's Hexagons and Circles.

By J. S. MACKAY, M.A., LL.D.

FIGURE 24.

In triangle ABC, AD, BE, CF are concurrent at O; through O parallels are drawn to EF, FD, DE, meeting the sides of ABC in L, M, P, Q, S, T, and the sides of DEF in L′, M′, P′, Q′, S′, T′. The two hexagons LMPQST, L′M′P′Q′S′T′ thus formed have the following properties:

(1) The sides L′M′, P′Q′, S′T′ of the latter are parallel to the sides of ABC.

The complete quadrilateral AFOEBC has its diagonal AO cut harmonically by EF and BC;

therefore A, U, O, D is a harmonic range,

and E·A U O D is a harmonic pencil.

Now OP′EQ′ is a parallelogram;

therefore P′Q′ is bisected by EO;

therefore P′Q′ is parallel to that ray of the harmonic pencil which is conjugate to EO, namely EA.

In like manner S′T′ is parallel to AB, and L′M′ to BC.

(2) The sides QS, TL, MP of the former are parallel to the sides of DEF.

Since PEQ′P′ and QEP′Q′ are parallelograms,

therefore PE = QE.

Similarly TF = SF;

therefore PE : QE = TF : SF.

Now PT is parallel to EF;

therefore QS is parallel to EF.

In like manner TL is parallel to FD, and MP to DE.

(3) The two hexagons are similar, and the first is four times the second.

This follows from the fact that the hexagons are made up of similar and similarly situated triangles, the ratio of whose homologous sides is that of 2 : 1.

(4) If D, E, F be the points of contact of the incircle with BC, CA, AB, the two hexagons obtained as before are inscriptible in circles; the radius of the greater circle is double the radius of the less; the centre of the greater circle is the centre of the incircle; and if Γ be the point of concurrency of AD, BE, CF the centre of the less circle is the mid point of ΓI.

FIGURE 25

Angle M L′Q′ = CDE = CED = EP′Q′;

therefore L′, M′, P′, Q′ are concyclic.

Similarly P′, Q′, S′, T′ are concyclic,

and S′, T′, L′, M′ are concyclic;

hence, by a theorem of Poncelet's, all the six points are concyclic.*

That the six points, L, M, P, Q, S, T are concyclic, follows from the preceding.

Since LM, PQ, ST, are chords of the greater circle, and they are bisected at D, E, F, therefore the centre of the greater circle is found by drawing perpendiculars to BC, CA, AB at D, E, F. But these perpendiculars are concurrent at the incentre.

Since Γ is the centre of similitude of the two hexagons, it is the external centre of similitude of the two circles circumscribed about them; and since I is the centre of the greater circle, the centre of the smaller circle must lie on ΓI. But because the radius of the greater circle is double the radius of the less, the centre of the less circle must be at I′ such that

$$I\Gamma : I'\Gamma = 2 : 1.$$

The preceding properties have been taken from C. Adams's *Die Lehre von den Transversalen*, pp. 77-80 (1843). The following considerations may be added:

(5) If, instead of the points of contact of the incircle D, E, F, there be taken the points of contact of the excircles D_1, E_1, F_1, or D_2, E_2, F_2 or D_3, E_3, F_3 analogous properties will be obtained.

Hence the existence of three other pairs of circles.

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* Adams's proof of this is somewhat different.

(6) It is known that Γ is the symmedian point of the triangle DEF; hence the circle L′M′P′Q′S′T′ is the triplicate-ratio, or first Lemoine, circle of DEF.

Γ is frequently called the Gergonne point of triangle ABC.

(7) If the points of concurrency of the triads

$$\left.\begin{array}{l} AD_1,\ BE_1,\ CF_1 \\ AD_2,\ BE_2,\ CF_2 \\ AD_3,\ BE_3,\ CF_3 \end{array}\right\} \quad \text{be} \quad \left\{\begin{array}{l} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{array}\right.$$

then $\Gamma_1 \quad \Gamma_2 \quad \Gamma_3$ are the symmedian points of the triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$; and three of the circles referred to in (5) will be Lemoine circles of these triangles.

Γ_1, Γ_2, Γ_3 are frequently called the adjoint Gergonne points of triangle ABC.

(8) Let the tangents to the circumcircle ABC at the points A, B, C meet each other at K_1, K_2, K_3; then, if triangle ABC be acute-angled, the circle ABC will be the incircle of triangle $K_1K_2K_3$, and if triangle ABC be obtuse-angled, the circle ABC will be an excircle of triangle $K_1K_2K_3$. Hence the relation in which triangle ABC stands to $K_1K_2K_3$ will, if ABC be acute-angled, be that in which triangle DEF stands to ABC, and so on. The two systems of points therefore

$$\begin{array}{l} D,\ E,\ F,\ I,\ \Gamma,\ A,\ B,\ C, \ldots \\ A,\ B,\ C,\ O,\ K,\ K_1,\ K_2,\ K_3 \ldots \end{array}$$

correspond.

(9) As the straight lines through Γ parallel to the sides of DEF cut the sides of ABC in six concyclic points, so the straight lines through K parallel to the sides of ABC will cut the sides of $K_1K_2K_3$ in six concyclic points. Hence the existence of another circle connected with the triangle.

Since I, the circumcentre of DEF, is the centre of the circle LMPQST, therefore O, the circumcentre of ABC, will be the centre of this new circle.

(10) If in the second system of points referred to in (8) those points be taken which correspond to $\Gamma_1, \Gamma_2, \Gamma_3$ in the first system, three other sets of six concyclic points will be obtained.

Sixth Meeting, April 14, 1893.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

Action at a distance, and the transmission of stress by isotropic elastic solid media.

By C. CHREE, M.A.

INTRODUCTION.

§ 1. The mutual action of two electrified bodies was regarded by Maxwell as transmitted by a medium. According to him the stress in the medium* consists of a "tension like a rope" along the lines of electrical force whose intensity per unit of area is $R^2/8\pi$, where R is the resultant electric intensity, and of a pressure numerically equal to this in all orthogonal directions. Maxwell's remarks are somewhat vague but his notation is strongly suggestive of an elastic solid medium. It has, however, been pointed out by Minchin† that Maxwell's stress system would not in an ordinary elastic solid give origin to strains consistent with the "equations of compatibility" which the theory of elastic solids supplies. Considerable interest still attaches to the theory of an elastic solid medium propagating stresses equivalent to the action between distant bodies of forces varying inversely as the square of the distance. For in the first place, it has been pointed out that the stress system given by Maxwell does not constitute a unique solution‡ of his equations; and, in the second place, it has been suggested that some medium must exist for the transmission of gravitational forces. The statical problem of the propagation of gravitational forces by an isotropic elastic medium has been treated by Minchin.§ His treatment how-

* *Electricity and Magnetism*, 3rd edition, Art. 106.

† *Treatise on Statics*, vol. II., 3rd edition, pp. 451-3.

‡ Minchin l.c., or Maxwell's *Electricity and Magnetism*, 3rd edition, Art. 110 footnote.

§ Minchin l.c., pp. 454-8.

ever neglects a certain surface condition. I have thus thought it worth while to consider the problem independently, employing the ordinary surface conditions. The first part of the paper is devoted more especially to the electrostatic problem, but the elastic solid problem is essentially the same throughout.

§ 2. The ordinary "action at a distance" theory regards a charge of electricity as a very thin superficial layer which repels a charge of the same sign and attracts one of opposite sign with a force varying inversely as the square of the distance. In interpreting this as an elastic solid problem the most obvious plan is to regard the layer as a thin shell of elastic material containing and surrounded by other elastic material, the layer being the only material exercising what we may term "gravitational forces". Supposing initially there are no gravitational forces and no strains anywhere in the medium, the endowment of the layer with the gravitational forces gives origin to a system of stresses required to keep the medium in equilibrium. These stresses reversed in sign would be those which a theory such as Maxwell's would substitute for the action at a distance of the thin layer. In the electrostatic problem the layer must be supposed extremely thin while at the same time the surface density is finite. In order to avoid the risk of unduly limiting the problem the layer is regarded here as of a different elastic material from that either inside or outside it. It is assumed, however, that the material of the layer is not wholly incompressible but satisfies the ordinary elastic solid equations, and that its elastic constants are neither infinitely great nor infinitely small compared to those of an adjacent medium. The assumption is also made that all the media are isotropic.

§ 3. In all the cases treated here the applied forces, and so the strains and stresses, are functions only of the distance r from a fixed point. The displacement u at every point is along the radius vector, and the dilatation Δ is given by

$$\Delta = \frac{du}{dr} + \frac{2u}{r} \qquad \dots \quad \dots \quad \dots \qquad (1).$$

In an isotropic medium whose density is ρ and elastic constants m, n, in the notation of Thomson and Tait's *Natural Philosophy*, the bodily equations of equilibrium reduce in such a case to the one equation

$$(m+n)\frac{d\Delta}{dr}+\rho\frac{dV}{dr}=0 \quad \ldots \quad \ldots \quad \ldots \quad (2),$$

where V, supposed a function of r only, is the potential of the bodily forces.

In the most general case considered here V is of the type

$$Vr^2+V'/r,$$

where V and V′ are constants, and the complete solution of (2) is

$$\Delta=A-\frac{\rho}{m+n}(Vr^2+V'r^{-1}) \quad \ldots \quad \ldots \quad (3),$$

where A is an arbitrary constant.

Substituting for Δ in (1) we deduce as the complete value of u

$$u=\tfrac{1}{3}Ar+Br^{-2}-\frac{\rho}{m+n}\{\tfrac{1}{5}Vr^3+\tfrac{1}{2}V'\} \quad \ldots \quad \ldots \quad (4),$$

where B is a second arbitrary constant.

The stress system consists of a principal stress along r and two other principal stresses perpendicular to r. The latter two are equal and may be supposed to act along any two mutually orthogonal directions in the plane perpendicular to r. Employing the notation introduced by Professor Pearson,* we shall denote the stress along the radius by $\widehat{rr}$, and employ $\widehat{\theta\theta}$ for the stress in any perpendicular direction, or what we may call the *transverse* stress.

The relations between the stresses and strains are

$$\left.\begin{aligned}\widehat{rr}&=(m-n)\Delta+2n\frac{du}{dr},\\ \widehat{\theta\theta}&=(m-n)\Delta+2n\frac{u}{r}\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5).$$

The ordinary three surface conditions satisfied by the stresses reduce to one, viz:

$$\widehat{rr}=\text{radial surface force per unit of surface} \quad \ldots \quad (6).$$

If the surface be "free", or acted on by no forces, then $\widehat{rr}$ must vanish over it. At a common surface of two media $\widehat{rr}$ must be continuous. This condition appears to be considered unnecessary by Prof. Minchin. In place of it he omits what is equivalent to the constant A in (4), on the ground that the corresponding term contributes

* Todhunter and Pearson's "*History of Elasticity*", Vol. I., p. 321.

nothing to the "gravitative action" on the element of the medium (l.c., p. 454). A further obvious condition at a common surface of two media is the continuity of the displacements, in this case of the radial displacement.

§ 4. When we attempt to picture to ourselves the state of matters close to the interface of two different media we encounter a difficulty which has occurred to several writers. Regarding the media as composed of molecules, the molecules of one of the media when close to the interface may be acted on by the molecules of the other medium, even supposing there is no mixing of the media. Thus it seems not unlikely there may be a narrow debateable ground wherein the relations between stress and strain show a gradual transition from the equations that hold inside the one medium to those that hold inside the other. The thickness of this transition zone must probably be a very small quantity, and in ordinary elastic solid problems its existence or nonexistence may be of little importance. In such applications, however, as to a hypothetical electrostatic medium, in which the gravitating layer is supposed extremely thin, the possibility of such a "modified action" ought to be present to the mind of the reader. The modified action, if appreciable, might affect the entire nature of the solution, so far at least as concerns the strains and stresses in the layer itself. While such a possibility may affect our attitude towards the solution it does not justify our dispensing with elastic solid surface conditions while applying elastic solid internal equations. As Professor Minchin employs the same elastic constants for space outside and inside the gravitating body there would appear no reason for supposing any modified action in the cases he treats, and thus his neglect of the continuity in the value of the radial stress must have some other explanation. This neglect leads Professor Minchin to the conclusion that "*the stress of the ether is discontinuous at the surface of the body*" (l.c., p. 455). This may be true though it presents serious difficulties, but it does not flow from the ordinary elastic solid theory.

§ 5. Before passing to our special problems we may employ the fundamental equations already given to show, in an elementary way, that Maxwell's stresses can not exist in any ordinary isotropic elastic medium.

The electrostatic force R in the air outside a spherical surface, over which a charge Q is uniformly distributed, is given by $R=Q/r^2$, and so Maxwell's radial stress would be $Q^2/(8\pi r^4)$. But supposing no bodily forces to act we find from (3), (4), and (5),

$$\widehat{rr}=(m-\tfrac{1}{3}n)A-4nr^{-3}B.$$

If the medium extend to infinity A vanishes, but in any case the term involving the negative power of r depends on r^{-3} and not on r^{-4} as Maxwell's theory requires.

Electrostatic Medium, Single Layer.

§ 6. Our first problem deals with three isotropic media, the surfaces separating which are concentric spheres. The inmost material is a core of radius e whose elastic constants are m, n; while the outmost material extends from $r=c$ to $r=\infty$ and has elastic constants m_2, n_2. Between these is a layer of material whose elastic constants are m_1, n_1 and density ρ, the particles of which *repel* one another with a force varying inversely as the square of the distance. The media are supposed in an unstrained state before the gravitational force commences to act, and our object is to find the strains and stresses in the state of final equilibrium under the action of the gravitational forces.

No bodily forces exist except in the layer where they answer to a potential

$$V=\tfrac{2}{3}\pi\rho(r^2+2e^3r^{-1}) \quad \dots \quad \dots \quad \dots \quad (7).$$

The expressions for the dilatation and displacement may be derived from (3) and (4). Thus, employing A, A_1, B_1, B_2 as arbitrary constants, we have

in the core

$$\left.\begin{aligned}\Delta&=A,\\ u&=\tfrac{1}{3}rA\end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (8),$$

in the layer

$$\left.\begin{aligned}\Delta_1&=-\tfrac{2}{3}\pi\frac{\rho^2}{m_1+n_1}(r^2+2e^3r^{-1}),\\ u_1&=-\tfrac{2}{3}\pi\frac{\rho^2}{m_1+n_1}(\tfrac{1}{5}r^3+e^3)+\tfrac{1}{3}rA_1+r^{-2}B_1\end{aligned}\right\} \quad \dots \quad (9),$$

outside the layer

$$\left.\begin{aligned}\Delta_2&=0,\\ u_2&=r^{-2}B_2\end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (10).$$

The suffixes $_1$, $_2$ distinguish quantities referring to the layer and external medium respectively from those referring to the core.

The strains must not be infinite at the origin and should vanish at infinity, so no negative powers of r are admitted in (8) and no positive powers in (10). From (5) we find for the radial stresses in the three media

$$\widehat{rr} = (m - \tfrac{1}{3}n)\mathrm{A} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11),$$

$$\widehat{rr}_1 = -\tfrac{2}{3}\pi\frac{\rho^2}{m_1+n_1}\{\tfrac{1}{5}r^2(5m_1+n_1)+2r^{-1}e^3(m_1-n_1)\}$$
$$+(m_1-\tfrac{1}{3}n_1)\mathrm{A}_1 - 4r^{-3}n_1\mathrm{B}_1, \quad \ldots \quad (12),$$

$$\widehat{rr}_2 = -4r^{-3}n_2\mathrm{B}_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13).$$

The surface conditions are the continuity of the displacement and radial stress. The equations embodying these conditions are easily written down, and their solution may be effected without serious difficulty on the lines adopted in treating the more complicated problem of § 8. It is thus sufficient to record the results we require without giving the algebraical work. Suppose then for shortness that

$$\frac{c^3}{e^3}(3m-n+4n_1)(3m_1-n_1+4n_2)$$
$$-4(n_1-n_2)\{3m_1-n_1-(3m-n)\}=\mathrm{D} \quad \ldots \quad (14),$$

and merely record the values of A and B_2, viz.,

$$\mathrm{A}.\mathrm{D} = 2\pi\rho^2(c-e)^2[c^2e^{-3}(c+2e)(3m_1-n_1+4n_2)$$
$$+\tfrac{4}{5}(n_1-n_2)e^{-3}(2c^3+4c^2e+6ce^2+3e^3)] \quad \ldots \quad (15),$$

$$\mathrm{B}_2.\mathrm{D} = 2\pi\rho^2(c-e)^2[\tfrac{1}{5}c^3e^{-3}(2c^3+4c^2e+6ce^2+3e^3)(3m-n+4n_1)$$
$$+c^2(c+2e)\{3m_1-n_1-(3m-n)\}] \quad \ldots \quad (16).$$

§ 7. In the case which presents an analogy to the electrostatic problem $(c-e)/e$ is very small. For it, retaining only lowest powers of $(c-e)/e$, we find

$$\left.\begin{aligned}\mathrm{A} &= 6\pi\rho^2(c-e)^2/(3m-n+4n_2),\\ \mathrm{B}_2 &= 2\pi\rho^2(c-e)^2e^3/(3m-n+4n_2)\end{aligned}\right\} \quad \ldots \quad (17).$$

Now suppose that, however small $c-e$ may be,

$$\rho(c-e)=\sigma,$$

where σ is finite. Then putting

$$m-\tfrac{1}{3}n=k \quad \ldots \quad \ldots \quad \ldots \quad (18),$$

so that k is the *bulk modulus* in the core, and substituting from (17) in (8), (10), and (5) we find

in the core

$$\left.\begin{aligned} u/r = \frac{du}{dr} &= 2\pi\sigma^2/(3k+4n_2), \\ \widehat{rr} = \widehat{\theta\theta} &= 6\pi\sigma^2 k/(3k+4n_2) \end{aligned}\right\} \quad \dots \quad (19),$$

and outside the layer

$$\frac{u_2}{r} = -\frac{1}{2}\frac{du_2}{dr} = \frac{\widehat{rr_2}}{-4n_2} = \frac{\widehat{\theta\theta_2}}{2n_2} = \frac{2\pi\sigma^2}{3k+4n_2}\left(\frac{e}{r}\right)^3 \quad \dots \quad (20).$$

The resultant per unit of surface of the radial forces exerted by the two media on the intervening layer being F, measured inwards, we have to the present degree of approximation, for all values of k or n_2,

$$F \equiv (\widehat{rr})_e - (\widehat{rr_2})_c = 2\pi\sigma^2 \quad \dots \quad \dots \quad (21).*$$

The stresses in the medium on Maxwell's theory ought, as already explained, to be numerically equal but of opposite sign to those just found. Thus the action of the elastic medium is seen by (21) to supply the well known value for the electric force exerted on itself by a charged surface. The real fact is that both the transverse and radial stresses in the layer are only of the same order of magnitude as the stresses outside it, and so, to the present degree of approximation, F alone must suffice to balance the mutual repulsion existing between the elements of the layer. Thus (21) ought to be regarded rather as a partial verification of the accuracy of our work than as affording any support to the theory of an elastic medium.

While, as we have just seen, there is a difference between the values of the radial stresses at the *two* surfaces of the thin layer, *no discontinuity* such as Professor Minchin's treatment leads to *is found at either surface.* We shall not examine the stresses in the layer at present, but shall do so in treating the gravitational problem, and shall then show how the radial stress varies in a continuous way throughout the entire thickness.

It should be noticed that, to the present degree of approximation, the strains and stresses in the core and outside the layer are independent of the magnitude of the elastic constants in the layer,

* The suffixes e, c outside the brackets indicate the radii of the surfaces where the respective stresses are measured.

provided these constants be as originally assumed, neither very great nor very small compared to those in the other media.

The radial stress outside the layer is numerically double the transverse stress, and not equal to it as in Maxwell's theory. In the core the principal stresses are all equal and their values are everywhere the same. Not only are there in general stresses in the core but their magnitude depends partly on the external medium. Conversely by (20) the stresses in the external medium are partly dependent on the elastic properties of the core. These results are strikingly different from those observed in electrostatics, where the electric force, and so Maxwell's stress, vanishes inside the charged surface, and where the force outside does not depend on the internal dielectric. The only obvious way of getting rid of these discrepancies is to assume k/n_2 negligible.

Supposing the same media inside and outside the layer, this would require the medium to offer very great resistance to torsion but very small resistance to change of volume. Such properties, so far as my knowledge goes, have never been observed in actual experiments. The hypothesis is thus a very extreme one, but the value it supplies for the stresses outside the layer, viz.

$$\left.\begin{aligned} \widehat{rr}_2 &= -2\pi\sigma^2(e/r)^3, \\ \widehat{\theta\theta}_2 &= \pi\sigma^2(e/r)^3 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (22),$$

are so simple as to merit attention. These with their signs reversed bear a certain resemblance to Maxwell's stresses, whose values for a surface density σ are

$$\text{radial tension} = \text{transverse pressure} = 2\pi\sigma^2(e/r)^4 \quad \ldots \quad (23),$$

but the law of force is of course different.

Electrostatic Medium, Two Layers.

§ 8. In the electrostatic problem lines of force run from a charged surface to an oppositely charged. Thus there may seem a radical difference between the elastic problem last treated and that of a charged spherical surface. A closer approach to the conditions of the electrical problem would seem to be the elastic problem of two thin layers with properties such as those of the single layer of our last problem.

Let us suppose then that a spherical layer whose surfaces are of radii OE $= e$ and OC $= c$ has a density $+\rho_1$, while an outer layer whose surfaces are of radii OB $= b$ and OA $= a$ has a density $-\rho_2$, and let the densities elsewhere be negligible. We shall for simplicity suppose the elastic constants of both layers to be m_1, n_1, while everywhere else the elastic constants are m, n. It is assumed that m_1, n_1 are neither very great nor very small compared to m, n, so that the ordinary surface conditions may apply. The medium outside the outer layer extends to infinity. Positive matter is supposed to repel positive and attract negative and conversely.

In the inner layer at a distance r from the centre the bodily force is directed outwards and answers to the potential

$$V_1 = \tfrac{2}{3}\pi\rho_1 (r^2 + 2e^3 r^{-1}) \quad \ldots \quad \ldots \quad (24),$$

while in the outer layer the resultant outwardly directed bodily force answers to the potential

$$V_2 = \tfrac{2}{3}\pi\{2\rho_1(c^3 - e^3)r^{-1} + \rho_2(r^2 + 2b^3 r^{-1})\} \quad \ldots \quad (25).$$

The solution of the bodily equations in terms of arbitrary constants is as follows, quantities referring to the several media being distinguished by suffixes,

from centre O to E

$$\left.\begin{aligned} \Delta &= A, \\ u &= \tfrac{1}{3} r A, \\ \widehat{rr} &= (m - \tfrac{1}{3}n)A \end{aligned}\right\} \quad \ldots \quad \ldots \quad (26),$$

from E to C

$$\left.\begin{aligned} \Delta_1 &= -\tfrac{2}{3}\pi \frac{\rho_1^2}{m_1 + n_1}(r^2 + 2e^3 r^{-1}) + A_1, \\ u_1 &= -\tfrac{2}{3}\pi \frac{\rho_1^2}{m_1 + n_1}(\tfrac{1}{5}r^3 + e^3) + \tfrac{1}{3} r A_1 + r^{-2}B_1, \\ \widehat{rr}_1 &= -\tfrac{2}{3}\pi \frac{\rho_1^2}{m_1 + n_1}\{\tfrac{1}{5}r^2(5m_1 + n_1) + 2e^3 r^{-1}(m_1 - n_1)\} \\ &\qquad + (m_1 - \tfrac{1}{3}n_1)A_1 - 4r^{-3} n_1 B_1 \end{aligned}\right\} \quad \ldots \quad (27),$$

from C to B

$$\left.\begin{aligned} \Delta_2 &= A_2 \quad , \\ u_2 &= \tfrac{1}{3} r A_2 + r^{-2}B_2, \\ \widehat{rr}_2 &= (m - \tfrac{1}{3}n)A_2 - 4r^{-3} n B_2 \end{aligned}\right\} \quad \ldots \quad (28),$$

from B to A

$$\left.\begin{aligned}\Delta_3 &= -\tfrac{2}{3}\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(r^2+2b^3r^{-1})+2\rho_1(c^3-e^3)r^{-1}\}+A_3,\\ u_3 &= -\tfrac{2}{3}\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(\tfrac{1}{5}r^3+b^3)+\rho_1(c^3-e^3)\}+\tfrac{1}{3}rA_3+r^{-2}B_3,\\ \widehat{rr}_3 &= -\tfrac{2}{3}\pi\frac{\rho_2}{m_1+n_1}[\rho_2\{\tfrac{1}{5}r^2(5m_1+n_1)+2b^3r^{-1}(m_1-n_1)\}\\ &\qquad +2\rho_1(c^3-e^3)r^{-1}(m_1-n_1)]+(m_1-\tfrac{1}{3}n_1)A_3\\ &\qquad -4r^{-3}n_1B_3\end{aligned}\right\} \quad \ldots \quad (29),$$

outside A

$$\left.\begin{aligned}\Delta_4 &= 0,\\ u_4 &= r^{-2}B_4,\\ \widehat{rr}_4 &= -4r^{-3}nB_4\end{aligned}\right\} \quad \ldots \quad \ldots \quad (30).$$

The continuity of the radial displacement and stress at the common surfaces leads to the following equations :

over $r=e$

$$A = -2\pi\frac{\rho_1^2}{m_1+n_1}(\tfrac{1}{5}e^2+e^2)+A_1+3e^{-3}B_1 \quad \ldots \quad (31),$$

$$(3m-n)A = -2\pi\frac{\rho_1^2}{m_1+n_1}\{\tfrac{1}{5}e^2(5m_1+n_1)+2e^2(m_1-n_1)\} + (3m_1-n_1)A_1-12e^{-3}n_1B_1 \quad \ldots \quad (32),$$

over $r=c$

$$A_2+3c^{-3}B_2 = -2\pi\frac{\rho_1^2}{m_1+n_1}(\tfrac{1}{5}c^2+e^3c^{-1})+A_1+3c^{-3}B_1 \quad \ldots \quad (33),$$

$$(3m-n)A_2-12c^{-3}nB_2 = -2\pi\frac{\rho_1^2}{m_1+n_1}\{\tfrac{1}{5}c^2(5m_1+n_1)+2e^3c^{-1}(m_1-n_1)\} + (3m_1-n_1)A_1-12c^{-3}n_1B_1 \quad \ldots \quad \ldots \quad (34),$$

over $r=b$

$$A_2+3b^{-3}B_2 = -2\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(\tfrac{1}{5}b^2+b^2)+\rho_1(c^3-e^3)b^{-1}\} + A_3+3b^{-3}B_3 \quad \ldots \quad \ldots \quad (35),$$

$$(3m-n)A_2-12b^{-3}nB_2 = -2\pi\frac{\rho_2}{m_1+n_1}[\rho_2\{\tfrac{1}{5}b^2(5m_1+n_1) + 2b^2(m_1-n_1)\}+2\rho_1(c^3-e^3)b^{-1}(m_1-n_1)] + (3m_1-n_1)A_3-12b^{-3}n_1B_3 \quad \ldots \quad \ldots \quad (36),$$

over $r = a$

$$3a^{-3}B_4 = -2\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(\tfrac{1}{5}a^2 + b^3a^{-1}) + \rho_1(c^3 - e^3)a^{-1}\} \\ + A_3 + 3a^{-3}B_3 \quad \ldots \quad \ldots \quad (37),$$

$$-12a^{-3}nB_4 = -2\pi\frac{\rho_2}{m_1+n_1}[\rho_2\{\tfrac{1}{5}a^2(5m_1+n_1) + 2b^3a^{-1}(m_1-n_1)\} \\ + 2\rho_1(c^3-e^3)a^{-1}(m_1-n_1)] + (3m_1-n_1)A_3 - 12a^{-3}n_1B_3 \quad \ldots \quad (38).$$

The above eight equations suffice to determine the eight arbitrary constants of the solution.

From (31) and (32)

$$(3m - n + 4n_1)A = -6\pi\rho_1^2e^2 + 3(m_1+n_1)A_1 \quad \ldots \quad \ldots \quad (39),$$

$$\{3m_1 - n_1 - (3m-n)\}A = -\tfrac{6}{5}\pi\rho_1^2e^2 + 9e^{-3}(m_1+n_1)B_1 \quad \ldots \quad (40);$$

from (33) and (34)

$$(3m - n + 4n_1)A_2 + 12c^{-3}(n_1 - n)B_2 = -2\pi\rho_1^2(c^3 + 2e^3)c^{-1} \\ + 3(m_1+n_1)A_1 \quad \ldots \quad \ldots \quad \ldots \quad (41),$$

$$\{3m_1 - n_1 - (3m-n)\}A_2 + 3c^{-3}(3m_1 - n_1 + 4n)B_2 \\ = \tfrac{2}{5}\pi\rho_1^2(2c^3 - 5e^3)c^{-1} + 9c^{-3}(m_1+n_1)B_1 \quad \ldots \quad (42);$$

from (35) and (36)

$$(3m - n + 4n_1)A_2 + 12b^{-3}(n_1 - n)B_2 \\ = -2\pi\rho_2\{3\rho_2b^2 + 2\rho_1(c^3 - e^3)b^{-1}\} + 3(m_1+n_1)A_3 \quad \ldots \quad (43),$$

$$\{3m_1 - n_1 - (3m-n)\}A_2 + 3b^{-3}(3m_1 - n_1 + 4n)B_2 \\ = -2\pi\rho_2\{\tfrac{3}{5}\rho_2b^2 + \rho_1(c^3 - e^3)b^{-1}\} + 9b^{-3}(m_1+n_1)B_3 \quad \ldots \quad (44);$$

from (37) and (38)

$$12a^{-3}(n_1 - n)B_4 = -2\pi\rho_2\{\rho_2(a^2 + 2b^3a^{-1}) + 2\rho_1(c^3 - e^3)a^{-1}\} \\ + 3(m_1+n_1)A_3 \quad \ldots \quad \ldots \quad \ldots \quad (45),$$

$$3a^{-3}(3m_1 - n_1 + 4n)B_4 = 2\pi\rho_2a^{-1}\{\tfrac{1}{5}\rho_2(2a^3 - 5b^3) - \rho_1(c^3 - e^3)\} \\ + 9a^{-3}(m_1+n_1)B_3 \quad \ldots \quad \ldots \quad \ldots \quad (46);$$

from (39) and (41) eliminating A_1

$$(3m - n + 4n_1)(A_2 - A) + 12c^{-3}(n_1 - n)B_2 \\ = -2\pi\rho_1^2c^{-1}(c - e)^2(c + 2e) \quad \ldots \quad \ldots \quad (47);$$

from (40) and (42) eliminating B_1

$$\{3m_1 - n_1 - (3m - n)\}(c^3A_2 - e^3A) + 3(3m_1 - n_1 + 4n)B_2$$
$$= \tfrac{2}{5}\pi\rho_1^2(c - e)^2(2c^3 + 4c^2e + 6ce^2 + 3e^3) \quad \dots \quad \dots \quad (48);$$

from (43) and (45) eliminating A_3

$$(3m - n + 4n_1)A_2 + 12(n_1 - n)(b^{-3}B_2 - a^{-3}B_4)$$
$$= 2\pi\rho_2\frac{(a - b)}{a}\{\rho_2(a - b)(a + 2b) - 2\rho_1 b^{-1}(c - e)(c^2 + ce + e^2)\} \quad \dots \quad (49);$$

from (44) and (46) eliminating B_3

$$b^3\{3m_1 - n_1 - (3m - n)\}A_2 + 3(3m_1 - n_1 + 4n)(B_2 - B_4)$$
$$= 2\pi\rho_2(a - b)\{-\tfrac{1}{5}\rho_2(a - b)(2a^3 + 4a^2b + 6ab^2 + 3b^3)$$
$$+ \rho_1(c - e)(a + b)(c^2 + ce + e^2)\} \quad \dots \quad \dots \quad \dots \quad (50).$$

The equations (47)–(50) are true whatever be the thickness of the layers, and the determination from them of A, A_2, B_2, B_4 presents no difficulty apart from the length of the expressions. When these four constants are determined the other four, A_1, B_1, A_3, B_3 may easily be found by means of (39), (40), (45), (46).

§ 9. For the electrostatic problem we shall confine our attention to the case when the layers are very thin, *i,e.*, when $(c - e)/e$ and $(a - b)/a$ are very small. For this case putting

$$\rho_1(c - e) = \sigma_1, \ \rho_2(a - b) = \sigma_2 \quad \dots \quad \dots \quad (51),$$

we easily find from (47)-(50), retaining only lowest powers of σ_1 and σ_2,

$$A = \frac{2\pi}{m + n}\{\sigma_1^2 - 2\sigma_1\sigma_2(c/b)^2 + \sigma_2^2\} \quad \dots \quad \dots \quad (52),$$

$$A_2 = \frac{2\pi}{m + n}\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2\} \quad \dots \quad \dots \quad \dots \quad (53),$$

$$B_2 = \tfrac{2}{3}\frac{\pi}{m + n}\sigma_1^2c^3 \quad \dots \quad \dots \quad \dots \quad \dots \quad (54),$$

$$B_4 = \tfrac{2}{3}\frac{\pi}{m + n}\{\sigma_1^2c^3 - 2\sigma_1\sigma_2c^2b + \sigma_2^2b^3\} \quad \dots \quad \dots \quad (55).$$

Substituting these values, and denoting the bulk modulus outside the layers by k as before, we find:

from O to E

$$\frac{u}{r} = \frac{1}{3k}\widehat{rr} = \tfrac{2}{3}\frac{\pi}{m + n}\{\sigma_1^2 - 2\sigma_1\sigma_2(c/b)^2 + \sigma_2^2\} \quad \dots \quad (56),$$

from C to B

$$\left.\begin{aligned} u_2 &= \tfrac{2}{3}\frac{\pi}{m+n}\left[r\{\sigma_2 - 2\sigma_1\sigma_2(c/b)^2\} + r^{-2}c^3\sigma_1^2\right], \\ \frac{du_2}{dr} &= \tfrac{2}{3}\frac{\pi}{m+n}\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2 - 2\sigma_1^2(c/r)^3\}, \\ \widehat{rr}_2 &= \frac{2\pi}{m+n}\left[k\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2\} - \tfrac{4}{3}n(c/r)^3\right] \end{aligned}\right\} \quad \ldots \quad (57),$$

outside A

$$\frac{u_4}{r} = -\tfrac{1}{2}\frac{du_4}{dr} = -\frac{\widehat{rr}_4}{4n} = \tfrac{2}{3}\frac{\pi}{m+n}r^{-3}\{\sigma_1^2c^3 - 2\sigma_1\sigma_2c^2b + \sigma_2^2b^3\} \quad \ldots \quad (58).$$

All these expressions are independent of the elastic constants of the material in the thin layers.

The inwardly directed resultant of the forces exerted by the two adjacent media on the inner layer per unit of surface is to the present degree of approximation

$$(\widehat{rr})_c - (\widehat{rr}_2)_c = 2\pi\sigma_1^2 \quad \ldots \quad \ldots \quad (59)$$

while that on the outer layer is

$$(\widehat{rr}_2)_b - (\widehat{rr}_4)_a = 2\pi\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2\} \quad \ldots \quad (60).$$

These values supply a partial verification of the accuracy of our work, for the transverse stress in either layer is only of the same order of magnitude as the radial stress, and to the present degree of approximation the resultant action of the adjacent media must balance the gravitational force.

This is obviously true of the inner layer on which no gravitational force is exerted by the other. Again, the outer layer exerts on itself a force $2\pi\sigma_2^2$ outwards per unit of surface, while the inner layer contributes a force $(4\pi\sigma_1c^2)\sigma_2/b^2$ inwards.

§ 10. In the problem analogous to the electrostatic problem, where the charges on the two surfaces are equal and opposite, we are to put

$$4\pi c^2\sigma_1 = 4\pi b^2\sigma_2 = Q \quad \ldots \quad \ldots \quad (61),$$

so that Q answers to the charge on the positively electrified surface.

Making this substitution and using (18), we get

from O to E

$$\frac{u}{r} = \frac{\widehat{rr}}{3k} = \frac{Q^2}{8\pi}\,\frac{b^4 - c^4}{b^4c^4}/(3k + 4n) \quad \ldots \quad \ldots \quad (62),$$

from C to B

$$\left.\begin{aligned} u_2 &= \frac{Q^2}{8\pi}\,\frac{1}{3k+4n}\left(-\frac{r}{b^4}+\frac{1}{cr^2}\right), \\ \widehat{rr}_2 &= -\frac{Q^2}{8\pi}\,\frac{1}{3k+4n}\left(\frac{3k}{b^4}+\frac{4n}{cr^3}\right), \\ \widehat{\theta\theta}_2 &= -\frac{Q^2}{8\pi}\,\frac{1}{3k+4n}\left(\frac{3k}{b^4}-\frac{2n}{cr^3}\right) \end{aligned}\right\} \quad \ldots \quad (63),$$

outside A

$$\frac{u_4}{r} = -\frac{\widehat{rr}_4}{4n} = \frac{\widehat{\theta\theta}_4}{2n} = \frac{Q^2}{8\pi}\,\frac{1}{3k+4n}\,\frac{b-c}{bcr^3} \quad \ldots \quad (64).$$

In this case the right hand side of (60) equals $-2\pi\sigma_2^2$; or the resultant of the actions of the adjacent media on the outer layer is numerically the same as if the inner layer did not exist, but is directed outwards.

The strains and stresses in the core will not vanish even approximately unless either $(b-c)/c$ be very small, *i.e.*, the layers very close together, or else k/n be negligible. While outside the outer layer the strains and stresses will be negligible only if $(b-c)/c$ be very small.

In the medium between the two layers the radial stress is always a pressure, but the transverse stress may be a pressure or a tension according to circumstances. It will be everywhere a tension if

$$n > \tfrac{3}{2}k(c/b) \quad \ldots \quad \ldots \quad \ldots \quad (65),$$

and everywhere a pressure if

$$n < \tfrac{3}{2}k(c/b)^4 \quad \ldots \quad \ldots \quad \ldots \quad (66).$$

The former case includes that in which k/n is negligible. When this is so we have between the layers

$$-\widehat{rr}_2 = 2\widehat{\theta\theta}_2 = Q^2/(8\pi cr^3) \quad \ldots \quad \ldots \quad (67),$$

outside the outer layer

$$-\widehat{rr}_4 = 2\widehat{\theta\theta}_4 = Q^2(b-c)/(8\pi bcr^3) \quad \ldots \quad (68).$$

For the strain and stress in a medium propagating electrostatic action the signs of all the above expressions are to be reversed. According to Maxwell's theory the stresses should vanish except between the two charged surfaces, and there we should have a radial tension $Q^2/(8\pi r^4)$ and an equal pressure in all orthogonal directions. The nearest approach to coincidence with his theory is thus when the distance apart of the two layers is very small compared to the radius of either surface.

For the case when the radius of the outer surface becomes infinite, while the total distribution over it remains numerically equal to that on the inner surface, we put $(c/b) = 0$ in (62) and (63) while regarding r as finite. The results so obtained agree with those already found in (19) and (20) for a single layer when $n_2 = n$, so that the existence of the layer at infinity is of no consequence.

Gravitational Medium, Single Layer.

§ 11. We now pass to the gravitational problem and consider first a single layer of radii e, c, density ρ, and elastic constants m_1, n_1, containing and surrounded by elastic media. To shorten the expressions we shall suppose the external and internal media the same and possessed of elastic constants m, n. In this case the layer being self-attractive we must change the sign of all terms containing ρ^2 or σ^2 in equations (9)–(23). Putting

$$\frac{c^3}{e^3}(3m - n + 4n_1)(3m_1 - n_1 + 4n) - 4(n_1 - n)\{3m_1 - n_1 - (3m - n)\} = \mathrm{D} \quad \ldots \quad (69),$$

we easily find from the surface conditions

$$\mathrm{A.D} = -2\pi\rho^2(c-e)^2 e^{-3}[c^2(c+2e)(3m_1 - n_1 + 4n) + \tfrac{4}{5}(n_1 - n)(2c^3 + 4c^2e + 6ce^2 + 3e^3)] \quad \ldots \quad (70),$$

$$\mathrm{B_2.D} = -\tfrac{2}{3}\pi\rho^2(c-e)^2[c^2(c+2e)\{3m_1 - n_1 - (3m-n)\} + \tfrac{1}{5}(c/e)^3(2c^3 + 4c^2e + 6ce^2 + 3e^3)(3m - n + 4n_1)] \quad \ldots \quad (71),$$

$$\mathrm{A_1.D} = -2\pi\rho^2 e^{-3}\left[c^2(c^3 + 2e^3)(3m - n + 4n_1) + \tfrac{4}{5}\frac{n_1 - n}{m_1 + n_1}\left\{3e^5\Big(2(3m-n) - 5m_1 + 3n_1\Big) - c^2(c^3 + 5e^3)(3m - n + 4n_1)\right\}\right] \quad \ldots \quad \ldots \quad \ldots \quad (72),$$

$$\mathrm{B_1.D} = -\tfrac{2}{3}\pi\rho^2c^2\left[\left\{c^3 + 2e^3 - \tfrac{4}{5}(c^3 + 5e^3)\frac{n_1 - n}{m_1 + n_1}\right\}\{3m_1 - n_1 - (3m - n)\} + \tfrac{3}{5}ce^2\,\frac{3m_1 - n_1 + 4n}{m_1 + n_1}\{2(3m - n) - 5m_1 + 3n_1\}\right] \quad \ldots \quad (73).$$

Substituting the value of A in (8) and that of B_2 in (10) we have the displacements in the core and outside the layer. Again substituting for A_1 and B_1 in (9) and changing the sign of ρ^2 in the particular solutions we have the displacement in the thin layer. The solution so obtained is in all respects complete.

Assuming the bulk modulus and rigidity positive quantities, we may easily prove that D is essentially positive and A essentially negative. Thus by (5) and (8) the three principal stresses at every point in the core consist of three equal pressures of constant value. If $3m_1 - n_1 > 3m - n$ the value of B_2 is essentially negative, but if $(3m - n)/(3m_1 - n_1)$ be large and $c - e$ be not very small B_2 may be positive. We conclude from (5) and (10) that outside the layer the radial stress is always opposite in sign to and numerically double of the transverse stress; the radial stress is necessarily a tension if the layer have a larger bulk modulus,—*i.e.*, is less compressible—than the other medium, but if the layer be considerably more compressible than the other medium, and be neither of unusually great rigidity nor extremely thin, the radial stress may be a pressure.

§ 12. To enter into details in the general case would involve dealing with very cumbrous algebraical expressions. I shall thus consider only a special case, which sufficiently illustrates the nature of the results. Thus let

$$n_1 = n,$$

or suppose that the layer has the same rigidity as the other medium and differs from it only in compressibility. In this case we find:

in the core

$$u/r = \widehat{rr}/(3m - n) = -\tfrac{2}{9}\pi\frac{\rho^2}{m+n}(c-e)^2\frac{c+2e}{c} \qquad \dots (74),$$

in the layer

$$u_1 = \tfrac{2}{3}\pi\frac{\rho^2}{m_1+n}\Big[\tfrac{1}{5}r^3 + e^3 - \tfrac{1}{3}(r/c)(c^3+2e^3)$$
$$- \frac{e^3}{cr^2}\,\frac{1}{m+n}\{\tfrac{1}{5}ce^2(m+n) + \tfrac{1}{3}(c-e)^2(c+2e)(m_1-m)\}\Big] \dots (75),$$

$$\widehat{rr}_1 = \tfrac{2}{9}\frac{\pi\rho^2}{m_1+n}\left[(c-r)\left(\frac{2e^3}{cr} - c - r\right)(3m_1 - n) + \tfrac{4}{5}\frac{(r-e)^2}{r^3}(2r^3\right.$$
$$\left. + 4r^2e + 6re^2 + 3e^3)n + 4\frac{e^3}{cr^3}(c-e)^2(c+2e)\frac{n(m_1-m)}{m+n}\right] \dots (76),$$

outside the layer

$$\frac{u_2}{r} = -\frac{\widehat{rr_2}}{4n} = -\tfrac{2}{9}\frac{\pi\rho^2}{m_1+n}\,\frac{(c-e)^2}{r^3}\left\{\tfrac{1}{5}(2c^3+4c^2e+6ce^2+3e^3)\right.$$

$$\left.+\frac{e^3(c+2e)}{c}\,\frac{m_1-m}{m+n}\right\}\quad\ldots(77).$$

From (76) we find for the radial stresses at the inner and outer surfaces of the layer the respective values

$$(\widehat{rr_1})_e = -\tfrac{2}{9}\pi\rho^2(c-e)^2\frac{c+2e}{c}\,\frac{3m-n}{m+n}\quad\ldots\quad\ldots\quad\ldots\quad(78),$$

$$(\widehat{rr_1})_b = \tfrac{8}{9}\pi\rho^2\frac{n}{m_1+n}\,\frac{(c-e)^2}{c^3}\left\{\tfrac{1}{5}(2c^3+4c^2e+6ce^2+3e^3)\right.$$

$$\left.+e^3\frac{(c+2e)}{c}\,\frac{m_1-m}{m+n}\right\}\quad\ldots\ (79).$$

Comparing (74) with (78) and (77) with (79) we see that our solution gives, as it ought, complete continuity in the value of the radial stress.

§ 13. To examine in detail the solution for a layer of any thickness would occupy too much space. When the layer is very thin the first approximations to the displacements, strains, and stresses in the core and outside the layer are given by (19) and (20), when $-\sigma^2$ is replaced by $(\rho h)^2$. The results so obtained apply whether the media outside and inside the layer are the same or not, and so are in one way more general than the results (74)–(77). Their degree of approximation is not however sufficiently close to show the variation of the strains and stresses throughout the layer. This variation may be satisfactorily illustrated by the special case, $n_1=n$, so we shall confine our attention to it. Thus taking (75) let

$$c-e=h,\ r-e=\xi,$$

so that h is the thickness of the layer and ξ the distance of a point in it from the inner surface, and expand in powers of the small quantities h/e and ξ/e to any required degree of approximation. For our present purpose we may content ourselves with

$$u_1 = -\tfrac{4}{3}\pi\rho^2\frac{h^2e}{m+n}\left\{1-\tfrac{2}{3}\frac{h}{e}-2\frac{\xi}{e}\,\frac{m_1-m}{m_1+n}+\frac{\xi}{e}\left(1-\frac{\xi^2}{h^2}\right)\frac{m+n}{m_1+n}\right\}\ldots(80).$$

As first approximations we find for the principal strains and stresses throughout the layer:

$$\left.\begin{aligned}
u_1/r &= -\tfrac{2}{3}\pi\rho^2\frac{h^2}{m+n},\\
\frac{du_1}{dr} \equiv \frac{du_1}{d\xi} &= -\tfrac{2}{3}\pi\rho^2\frac{h^2}{m+n}\left\{1-3\frac{m_1-m}{m_1+n}-3\frac{\xi^2}{h^2}\,\frac{m+n}{m_1+n}\right\},\\
\widehat{rr}_1 &= 2\pi\rho^2\left\{\xi^2-\tfrac{1}{3}h^2\frac{3m-n}{m+n}\right\},\\
\widehat{\theta\theta}_1 &= -\tfrac{4}{3}\pi\rho^2h^2\frac{n}{m+n}\left\{1+\tfrac{3}{2}\left(1-\frac{\xi^2}{h^2}\right)\frac{(m_1-n)(m+n)}{n(m_1+n)}\right\}
\end{aligned}\right\} \quad (81).$$

Thus the transverse strain is always a compression and if $m_1 > n$, as appears the case in all satisfactory experiments, the transverse stress is always a pressure. The radial strain is a compression at the inner surface if $\quad 3m-n > 2(m_1-n),$

which will be the case unless the layer be much less compressible than the adjacent media. The radial strain is algebraically greatest at the outer surface where it is always an extension. The radial stress is always a pressure at the inner surface, a tension at the outer, and varies continuously throughout the thickness.

The maxima values $\bar{s}$ and $\bar{S}$ of the greatest strain* and the stress-difference*—*i.e.*, the difference between the algebraically greatest and least stresses at a point—occur at the outer surface, and we have

$$\left.\begin{aligned}
\bar{s} &= \tfrac{4}{3}\pi\rho^2h^2/(m+n),\\
\bar{S} &= 4\pi\rho^2h^2n/(m+n)
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (82).$$

It is easy to prove that the dilatation vanishes over the outer surface and elsewhere is negative, so that the volume occupied by the layer is reduced.

As first approximations for the increments in the radius e and thickness h of the layer we have

$$\left.\begin{aligned}
\delta e/e &= -\tfrac{2}{3}\pi\rho^2h^2/(m+n),\\
\delta h/h &= \tfrac{4}{3}\pi\rho^2\frac{h^2}{m+n}\,\frac{m_1-m}{m_1+n}
\end{aligned}\right\} \quad \ldots \quad (83).\dagger$$

* The magnitude of one or other of these quantities is frequently regarded as measuring the "tendency to rupture" in the material. See *Philosophical Magazine*, September 1891, pp. 239-242.

† The letter δ denotes the increment of the quantity denoted by the following letter.

Thus the radius of the shell is always reduced. The thickness is increased or diminished according as the shell is less or more compressible than the adjacent media.

Gravitational Medium, Two Layers.

§ 14. The case of two gravitating layers in which the elastic constants have the values m_1, n_1, while in the surrounding media they have the values m, n, may be deduced from the treatment of the two electrostatic layers by changing the signs of ρ_1^2, ρ_2^2 but not of $\rho_1\rho_2$, where ρ_1, ρ_2 denote the densities of the inner and outer layers, the radii of whose surfaces in ascending order of magnitude are e, c, b, a. We shall only glance at the case when the thicknesses

$$c - e = h_1, \quad a - b = h_2$$

are very small. Putting

$$\rho_1 h_1 = \sigma_1, \quad \rho_2 h_2 = \sigma_2,$$

we find from (56), (57), (58) as first approximations, employing k as before for the bulk modulus outside the layers,

in the core

$$\frac{u}{r} = \frac{\widehat{rr}}{3k} = -\tfrac{2}{3}\frac{\pi}{m+n}\{\sigma_1^2 + 2\sigma_1\sigma_2(c/b)^2 + \sigma_2^2\} \qquad \dots (84),$$

between the layers

$$u_2 = -\tfrac{2}{3}\frac{\pi}{m+n}[r\{\sigma_2^2 + 2\sigma_1\sigma_2(c/b)^2\} + c^3r^{-2}\sigma_1^2] \qquad \dots (85),$$

$$\widehat{rr}_2 = -\tfrac{2}{3}\frac{\pi}{m+n}[3k\{\sigma_2^2 + 2\sigma_1\sigma_2(c/b)^2\} - 4n(c/r)^3\sigma_1^2] \dots (86),$$

outside the outer layer

$$\frac{-u_4}{r} = \frac{\widehat{rr}_4}{4n} = \tfrac{2}{3}\frac{\pi r^{-3}}{m+n}\{\sigma_1^2c^3 + 2\sigma_1\sigma_2c^2b + \sigma_2^2b^3\} \qquad \dots (87).$$

To a first approximation we have

$$u/r = (u_2/r)_c, \text{ and } (u_2/r)_b = (u_4/r)_a.$$

Thus, since the radial displacement is continuous, we deduce that the transverse strain in each layer has a nearly constant value, that value being given for the inner layer by (84) and for the outer layer by the value of u_4/r in (87) when $r = a$. We also know the values of the radial stress over the surfaces of the layers from the fact that there is no discontinuity in that stress. All the strains and stresses in the present case vary as the squares or products of the thicknesses of the layers.

In the previous gravitational problems the stresses we have found are those which maintain equilibrium when forces at a distance act. If we suppose the stresses reversed we get the stress system required to propagate gravitational forces in a hypothetical medium. These reversed stresses are to be regarded as residing in the medium and it is this medium and not any sensible substance to which the elastic constants of the solution belong. How such stresses may be excited, or what connection there may exist between the medium and sensible matter does not come within the scope of the present enquiry.

Single Gravitating Shell.

§ 15. The problems previously considered are of a speculative nature referring to the action of some medium to be classed under the general title "ether". To prevent misconception we add the solution of the corresponding problems in their relation to the actual visible matter of which the spherical layers are composed.

The first problem then is that of a spherical shell of ordinary isotropic material, say of density ρ and elastic constants m, n, existing alone in space, acted on by no surface forces and no bodily forces other than its own gravitation. The potential of the bodily forces in the shell, supposing its radii e and c, is given by

$$V = -\tfrac{2}{3}\pi\rho(r^2 + 2e^3r^{-1}).$$

The solution in arbitrary constants is analogous to (27), but the constants must now be determined from the conditions that the radial stress vanishes over both surfaces. It is unnecessary to record the values found for the constants. The displacement is

$$u = \tfrac{2}{3}\frac{\pi\rho^2}{m+n}\Big[\tfrac{1}{5}r^3 + e^3 - \frac{r}{(c^3-e^3)(3m-n)}\{\tfrac{1}{5}(c^5-e^5)(5m+n)$$
$$+ 2e^3(c^2-e^2)(m-n)\} - \frac{r^{-2}}{2(c^3-e^3)n}\{\tfrac{1}{10}e^3c^3(c^2-e^2)(5m+n)$$
$$- e^5c^2(c-e)(m-n)\}\Big] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (88).$$

The strains and stresses may easily be deduced. The radial stress it will be found vanishes, as it ought, over both surfaces.

§ 16. The most interesting case for comparison with the previous problems is that of a very thin shell. For it we find, with our previous notation,

$$\left.\begin{aligned} u &= -\frac{\pi\rho^2 h e^2(m+n)}{2n(3m-n)}\left\{1+\tfrac{4}{3}\frac{h}{e}\frac{m}{m+n}-2\frac{\xi}{e}\frac{m-n}{m+n}\right\}, \\ \widehat{rr} &= -2\pi\rho^2\xi(h-\xi)\left\{1+\tfrac{4}{3}\frac{h}{e}\frac{m}{m+n}-\tfrac{2}{3}\frac{\xi}{e}\frac{m+3n}{m+n}\right\} \end{aligned}\right\} \quad \ldots \quad (89).$$

Comparing (89) with (81) we see that in both cases the radial stress is of order h^2, but the radial displacement in (81) is also of order h^2 whereas in (89) it is of order h and so enormously greater. In the case of the "ether" media the contraction of the layer was opposed by the contiguous media, and it was the action of the latter media that sustained the gravitational forces. In the present case the forces opposing the contraction of the shell are derived solely from the elastic stresses in itself, and to produce transverse stresses of sufficient intensity for this end requires a contraction of very much greater magnitude than in the previous case. For purposes of comparison it will suffice to confine our attention to the first approximation in the present case. We shall employ the same notation as before, and in addition shall put

$$\begin{aligned} n(3m-n)/m &= \mathrm{E}, \\ (m-n)/2m &= \eta, \\ 4\pi\rho h &= g, \end{aligned}$$

so that E is Young's modulus and η Poisson's ratio for the shell, while g is the acceleration at its outer surface due to the gravitational forces. We easily find

$$\left.\begin{aligned} u/r = \delta e/e &= -\tfrac{1}{4}g\rho e\frac{1-\eta}{\mathrm{E}}, \\ \frac{du}{dr} = \delta h/h &= \tfrac{1}{2}g\rho e\frac{\eta}{\mathrm{E}}, \\ \widehat{rr} &= -\tfrac{1}{2}g\rho e\frac{\xi(h-\xi)}{eh}, \\ \overline{\mathrm{S}} &= -\widehat{\theta\theta} = \tfrac{1}{4}g\rho e \end{aligned}\right\} \quad \ldots \quad \ldots \quad (90).$$

Strictly $\overline{\mathrm{S}}$ is the difference between the radial and the transverse stresses, but the former is negligible compared to the latter. The largeness of the transverse stress is perhaps the most striking feature of the solution. To show that to the present degree of approximation it balances the gravitational forces, we employ the known result that when a spherical membrane of radius e contains a gas at pressure p the requisite tension T in the membrane is given by

$$p = 2\mathrm{T}/e.$$

In other words the resultant of the tensions is a normal force $2\mathrm{T}/e$ directed inwards. Thus if instead of tensions in a membrane we have transverse pressures in an elastic shell whose intensity is $-\widehat{\theta\theta}$, or T′, over unit area, and so for the entire thickness $\mathrm{T}'h$ per unit arc of surface, their resultant is an outwardly directed force whose value is $2\mathrm{T}'h/e$ or by (90) is $2\pi(\rho h)^2$. But this is numerically equal and oppositely directed to the gravitational action of the shell on itself.

From (90) we see that the strains and the transverse stress are all to a first approximation constant throughout the thickness. Assuming Poisson's ratio positive, the radial strain is everywhere an extension and the thickness of the shell is increased. The transverse strain is always a compression and the radius of the shell is diminished. The radial and transverse stresses are both pressures. The former is to a first approximation a maximum at the mid thickness and diminishes numerically as we approach either surface.

§ 17. As an idea of the actual magnitude of the stresses may be of use, let us consider the approximate value of $\overline{\mathrm{S}}$ in a shell of the radius of the earth's outer surface, due to its own gravitation only. Let g' be the value of "gravity" at the surface of a solid sphere of the same radius e and density ρ as the shell, then

$$g' = \tfrac{4}{3}\pi\rho e,$$
$$\overline{\mathrm{S}} = \tfrac{3}{4}g'\rho h.$$

At the earth's surface a cubic foot of water weighs about $62\frac{1}{2}$ lbs., and thus if the specific gravity of the shell be equal to the mean value of the earth, say 5·5, we deduce for the approximate value of the maximum stress-difference in a shell i miles thick, and 4000 miles radius,

$$\overline{\mathrm{S}} = (4{\cdot}2)i \text{ tons weight per square inch.}$$

The conditions are of course totally different from those existing in the outer layer of a gravitating *solid* sphere.

Two Gravitating Shells.

§ 18. The case of two concentric shells of ordinary matter acted on solely by their gravitational forces may be solved in a similar way. In the inner shell, which is assumed not to touch the outer, the solution is exactly the same as in the previous case because the forces exerted by the outer shell are nil. The forces on the outer shell arise partly from its mutual gravitation, partly from the attraction of the inner shell. The consequences of the former set of forces we know already, and so, as strains are superposable when kept within the limits to which the mathematical theory applies, we need now investigate only the action of the inner shell on the outer.

The forces exerted by a shell of radii e, c and density ρ_1 on an external shell are derived from the potential

$$\tfrac{4}{3}\pi\rho_1(c^3 - e^3)/r = \mathrm{M}/r \text{ say.}$$

The solution thus obtained for the action solely of the inner shell on the outer is

$$u = -\tfrac{1}{2}\frac{\mathrm{M}\rho_2}{m+n}\left\{1 - 2r\frac{a^2-b^2}{a^3-b^3}\,\frac{m-n}{3m-n} + \frac{a^2b^2}{r^2}\,\frac{a-b}{a^3-b^3}\,\frac{m-n}{2n}\right\} \quad \ldots \ (91).$$

From this we find for any thickness of shell

$$\frac{du}{dr} = \mathrm{M}\rho_2\frac{1}{a^2+ab+b^2}\,\frac{m-n}{m+n}\left(\frac{a+b}{3m-n} + \frac{a^2b^2}{2nr^3}\right) \quad \ldots \quad (92),$$

$$\frac{u_a - u_b}{a-b} = \frac{\delta h}{h} = \tfrac{3}{4}\mathrm{M}\rho_2\frac{a+b}{a^2+ab+b^2}\,\frac{m-n}{n(3m-n)} \quad \ldots \quad \ldots \quad (93),$$

$$\widehat{rr} = -\mathrm{M}\rho_2\frac{(a-r)(r-b)(ab+ar+br)}{r^3(a^2+ab+b^2)}\,\frac{m-n}{m+n} \quad \ldots \quad (94).$$

The radial stress is thus everywhere a pressure; it vanishes of course over both surfaces. The radial strain is everywhere an extension.

The displacement, and thence the strains and stresses in the outer shell, due to its mutual gravitation, may be deduced from (88) by replacing ρ, e, c, by ρ_2, b, a respectively.

When the shell is very thin let

$$a - b = h, \ r - b = \xi,$$

and let

$$\tfrac{4}{3}\pi\rho_1\frac{(c^3-e^3)}{a^2} = g_1, \ 4\pi\rho_2 h_2 = g_2,$$

so that g_1 and g_2 are respectively the accelerations at the outer surface of the outer shell due to the attraction of the inner shell and its own gravitation. Then to a first approximation the complete values for the strains and stresses in the outer shell are as follows:—

$$\left.\begin{aligned}
u/r &= \delta a/a = -\tfrac{1}{4}(g_2+2g_1)\rho_2 a(1-\eta)/\mathrm{E},\\
\frac{du}{dr} &= \delta h/h = \tfrac{1}{2}(g_2+2g_1)\rho_2 a\eta/\mathrm{E},\\
\widehat{rr} &= -\tfrac{1}{2}g_2\rho_2 a\frac{\xi(h-\xi)}{ah},\\
\overline{\mathrm{S}} &= -\widehat{\theta\theta} = \tfrac{1}{4}(g_2+2g_1)\rho_2 a
\end{aligned}\right\} \quad \cdots \quad (95).$$

The intensity of the actual bodily force in the outer shell varies regularly from g_1 at the inner to g_1+g_2 at the outer surface. Thus the above results show that to a first approximation the strains and the transverse stress in the shell are the same as if the bodily forces had at every point of the thickness a constant value equal to the mean of the actual values. The value of the radial stress depends even to a first approximation on the law of distribution of the bodily forces, but this stress is negligible compared to the transverse stress. So far as concerns the results (95) the inner shell may be a solid core or a shell of any thickness. The only limitation is that the two shells must not be in contact.

The Elements of Quaternions (*Second Paper*).

Discussion of the Proofs of the Laws of the Quaternionic Algebra.

[*Abstract.*]

By Dr William Peddie.

Three main laws regulate the treatment of ordinary algebraic quantities. These are the Associative Law, the Distributive Law, and the Commutative Law. If a, b, c, ..., represent quantities dealt with in the algebra, the associative law of multiplication asserts that $a(bc)=(ab)c$, where the brackets have the usual meaning that the quantity within them is to be regarded as a single quantity: the distributive law of multiplication asserts that $(a+b)(c+d)=ac+bc+ad+bd$: and the commutative law gives $ab=ba$. With regard to addition, the associative law asserts that $(a+b)+c=a+(b+c)$: and the commutative law gives $a+b=b+a$.

In ordinary algebra, all the quantities are scalars. In a vector algebra, the further idea of direction is ntroduced, and so we cannot assert *a priori* that the quantities dealt with in that algebra will satisfy the laws of ordinary algebra. The matter is one for investigation. With certain fundamental assumptions, some of the ordinary laws may hold and others may not; and the particular set which holds, and that which does not, will depend on the nature of these assumptions.

In framing a new algebra, the first care should be to make its laws agree as far as possible with the laws of the old; former assumptions are only to be discarded when they stand in the way of farther development. The adoption of certain assumptions may make the algebra more readily applicable in some directions than in others; in which case the maximum of general applicability, consistent with ease of application in the most important directions, is to be aimed at. And, in dealing solely with the new quantities introduced in the new algebra, we may assume characteristics totally different from those which typify the old quantities if such assumptions enable us to follow out the above rules, while all others prevent us from doing so. Indeed, such a choice of characteristics might be most advantageous even in circumstances in which the old characteristics would also enable us to observe these rules.

When one vector α is changed into another β, the change may be represented as due to the addition of a third vector γ to the former: so that $\alpha+\gamma=\beta$. And, since the notion of a vector quantity involves only the ideas of magnitude and direction—*not* the idea of position—we see that a geometrical interpretation of this equation is that the relative position of two points in a plane is fully given either by means of the straight line joining the two or by means of the two sides of any triangle described with that line as base: and, similarly, it may be given by the remaining sides of any polygon of which the line joining the two points forms the other side. From this we at once see that the associative and commutative laws must apply to the addition (and subtraction) of vectors.

Again when α is changed into β, we may represent the result as due, not to some addition to α but, to some operation performed upon α. This is represented by the equation $q\alpha=\beta$, where q is the required operator. Such a method is as natural, as important, and, in many cases, more appropriate than the former. The Calculus of

Quaternions, regarded as a vector algebra, recognises and employs both methods.

The operator q turns α, in a plane parallel to the directions of α and β, through an angle equal to that contained between two lines drawn in these directions respectively through a fixed point; and it also changes its length if necessary until it becomes equal to that of β. Now, to determine the plane, two numbers (such as (1) the azimuth, in a fixed plane, of the line of intersection of the fixed plane with the required one, and (2) the obliquity of the planes) are needed. Then another number is needed to determine the amount of rotation in the plane; after which yet another is needed to determine the amount of lengthening (or shortening). In all *four* numbers are required; and hence q is conveniently called a *quaternion.*

An algebra which deals with such operators is, *ipso facto*, an algebra of vectors *plus* quaternions, and so *may be* more complex than another in which the subject is not regarded from this operational point of view. On the other hand, since we have $q\alpha = \alpha + \gamma$, it is evident that we can, by means of suitable definitions, express a quaternion in terms of vectors; and it may be possible to do this so simply that special symbols for quaternions need never be introduced, while, on the one hand, the greater complexity spoken of becomes vanishingly small, and, on the other, greater freedom of treatment is attained. In accordance with the usages of ordinary algebra, we may regard $q\alpha$ as the *product* of q into α. That is to say, $q\alpha$ is the product of two vector quantities; or, more strictly, of a vector and a function of vectors. Now, in physical enquiries, we have constantly to deal with products of vector quantities—which products may be either scalar or vector. Hence a vector algebra, which recognises the quaternion, may be made to deal *naturally* (and, it may be, very simply) with such physical investigations. On the contrary, the algebra which does not recognise the quaternion must have introduced into it new fundamental definitions, totally unconnected with anything else, if it is to deal with scalar or vector products of directed quantities. And the introduction of these new definitions into the algebra will make possible the quaternionic treatment of vectors by its means; so that it would be quite correct to call it a calculus of quaternions whether developed or not. Indeed the cry that vectors should be treated vectorially is merely a play upon words. A vector calculus deals with vectors and functions of vectors; and, as we have seen, in any quaternion calculus, a quaternion can be represented as a function of vectors; so that the quaternion calculus is, in a sense, *purely* a calculus of vectors. This is preeminently the case with Hamilton's system.

We know of only two fundamental classes of vectors—vectors having reference to translation along a line, and vectors having reference to rotation around an axis. Hamilton's system takes account of both ideas without introducing separate symbols: the same vector acts translationally, or rotationally, according as it is added to another, or is multiplied into another: and there is no possible confusion of meaning. And, further, provision is made, simply, for the treatment of scalar products of vectors. But, before considering the assumptions by means of which these advantages are attained, it is necessary to consider the laws of multiplication of quaternions: and, in doing so, it is not necessary to consider the stretching part (or Tensor) of the quaternion—for that part is a mere number and so obeys all the laws of ordinary algebra.

We may represent quaternions by plane angles or by arcs of great circles on a unit sphere. Thus, if PQR be a spherical triangle whose sides p, q, r are portions of great circles on the unit sphere, the quantities p, q, r may represent the corresponding quaternions. Let α be the vector from the origin to the point Q. Then $p\alpha$ is the vector to the point R, and $q\cdot p\alpha$ is the vector to P. But this is also $r\alpha$, if r is measured from Q to P while p and q are measured from Q to R, and from R to P, respectively. And we are at liberty to define $r = qp$, so that $q\cdot p\alpha = qp\alpha$. This makes the associative law hold when α, $p\alpha$, and $qp\alpha$ are vectors—a fact which is pointed out by Hamilton, *Lectures*, § 310, and by Tait, *Elements*, § 54. It defines quaternion multiplication.

Various proofs that the associative law holds in the multiplication of quaternions have been given. Of these, Hamilton's proof (*Lectures*, § 296; *Elements*, § 270; and Tait's *Elements*, §§ 57-60) by spherical arcs and elementary properties of spherical conics involves, by definition, the particular assumption of association just alluded to. His alternative proof, by more elementary geometry (*Lectures*, §§ 298-301), makes use of the same definition; and the same remark applies to the proof given in §§ 358, 359 of the *Lectures*. On the other hand, the geometrical proof given in Hamilton's *Elements*, §§ 266, 267, 272, is based upon the definition of the reciprocal of a quaternion, which makes the product of a quaternion and its reciprocal unity, and leads to the result that the versor of a product of quaternions is equal to the product of their versors. It involves the definition, above alluded to, of a quaternion in terms of vectors: which, in turn, partially assumes the associative law for vectors ($\beta\alpha^{-1}.\alpha = \beta.\alpha^{-1}\alpha$).

The complete proof of the law, by this method, is given in § 272 of the *Elements*. Other possible proofs are indicated in the *Elements*. In the proof, by spherical conics, given in the *Lectures*, § 302, and the *Elements*, §§ 265, 271, a quaternion is represented by a spherical angle.

Hamilton also gives proofs (*Lectures*, § 489; *Elements*, § 223) of the associative law for quaternions when the distributive law for vector multiplication is granted or proved. This proof is also given by Tait, *Elements*, § 85. It involves the representation of a quaternion as the sum of a scalar and a vector. The proof that this representation is possible and definite (*Lectures*, § 406; *Elements*, §§ 201, 202; Tait's *Quaternions*, § 77) necessitates the association of vectors, as above, to the extent $\beta\alpha^{-1}.\alpha = \beta.\alpha^{-1}\alpha$, (and the distributive law to the extent $(\alpha + \gamma\alpha)\alpha^{-1} = \alpha\alpha^{-1} + \gamma\alpha\alpha^{-1}$). Indeed all the laws of combination of rectangular vectors are taken for granted in this proof of the associative law.

The addition of quaternions is defined by the equation $(q + r)\alpha = q\alpha + r\alpha$ where α is a vector. From this (*Lectures*, § 449) it at once follows that the associative law holds in such addition. This definition of course is virtually an assumption of the distributive law in the particular case when α, $q\alpha$, $r\alpha$, are vectors.

Hamilton's proof (*Lectures*, §§ 451–455; *Elements*, §§ 210–212; Tait's *Quaternions*, § 81) of the distributive law in the multiplication of quaternions employs this definition of the addition of quaternions together with the partial assumption of the associative law of vectors involved in the definition of a quaternion in terms of vectors. It also assumes the possibility of representing a quaternion as the sum of a scalar and a vector. The law may be proved, as Tait indicates (§ 62), by means of this definition and the assumption of the laws of combination of vectors. Tait's other proof (§ 62) by means of the properties of spherical conics involves, in its complete generality, the proof of the commutativeness of quaternion addition.

When the partial assumption of association of vectors, used in Hamilton's fundamental expression for a quaternion in terms of vectors, is made along with the partial assumption of distribution used in the definition of quaternion addition, the commutativeness of quaternion addition follows at once (*Lectures*, §§ 448, 449; *Elements*, §§ 195–207; Tait's *Elements*, § 61) from the obvious commutativeness of vector addition.

The results obtained up to this point are the following: (1) The addition of vectors is commutative and associative. (2) A quaternion may be represented as a function of vectors. In Hamilton's system the quaternion q in the equation $q\alpha = \beta$ is defined to be $\beta\alpha^{-1}$; and so $q\alpha = \beta\alpha^{-1}.\alpha = \beta.\alpha^{-1}\alpha = \beta$, for $\alpha^{-1}\alpha$ is defined to be unity; and the steps of the process are consistent with association in vector multiplication. (3) A definition of quaternion addition, which does not conflict with the distributive law of multiplication, and which subjects the process to the associative law, is given. (4) With no further definitions, it is found that the associative and distributive laws hold in the multiplication of quaternions. Thus all the results yet obtained are consistent with the rules which must be observed in the formation of a new calculus.

The graphical representation of quaternion (or versor) multiplication shows at once (*Elements*, § 168, Tait's *Quaternions*, § 54) that quaternion multiplication is not in general commutative. And another peculiarity is that, if q be a versor which turns any vector in a given plane through a right angle, the double application of the operation q reverses any vector in that plane. If α be such a vector, we get $q.q\alpha = q^2\alpha = -\alpha$; so that we may put $q^2 = -1$ in the case of any quadrantal versor. And if p, q, r be rectangular quadrantal versors we get

$$p^2 = q^2 = r^2 = -1;\ pq = -qp = r,\ qr = -rq = p,\ rp = -pr = q.$$

Now consider three rectangular unit vectors i, j, k; and let them be perpendicular respectively to the planes of rotation of p, q, r, so that we may say that i is parallel to the axis of p, etc. We get at once $pj = k$, $pq = r$; $qi = -k$, $qp = -r$; $r^2 = -1$. Whence if we write $p = i$, $q = j$, $r = k$, we shall have the immense simplification that no special symbols are needed for versors—a vector acting translationally in addition (or subtraction), rotationally in multiplication (or division).

With this assumption, vector multiplication is associative, and distributive; but is not commutative; and the square of a unit vector is negative unity; the laws for unit rectangular vectors being $ij = -ji$, and $i^2 = -i$, etc.

Now the idea of a vector is one entirely foreign to ordinary algebra, in which the square of any unit is positive unity. Hence the fact that the square of a unit vector is negative unity has no disadvantage. It makes the scalar part of the product of β into α equal to the product of the lengths of these vectors into the cosine of the supplement of the angle between their positive directions; and it makes the reciprocal of a vector have a direction opposite to that of the vector itself; all of which conditions are as natural and simple as their opposites.

Finally, it is shown by Hamilton, by strict reasoning (*Lectures*, §§ 49–56), that these laws for the multiplication of unit rectangular vectors *must* hold if no one direction in space is to be regarded as eminent above another and if the ordinary rules of algebra are to apply in so far that, (1) to multiply either factor by any number positive or negative, multiplies the product by the same, (2) the product of two determined factors is itself determined, (3) the distributive and associative principles hold. We see then that Hamilton's system is one which preeminently satisfies the conditions of correspondence to ordinary algebra as far as possible.

Note on a Problem in Analytical Geometry.

By A. J. PRESSLAND, M.A.

[*Abstract.*]

The theorem, "If upon the sides of a triangle as diagonals parallelograms be described, whose sides are parallel to two given lines, then the other three diagonals will intersect in the same point," occurs in Hutton's *Course of Mathematics*, 12th ed., vol. II., p. 191.

For a proof, see Smith's *Conic Sections*, p. 40.

If we are given the point of intersection of the diagonals, and wish to find the directions of the sides of the parallelograms, the discussion resolves itself into describing a conic through three points to have its centre at a given point. The asymptotes of this conic are the directions required. For a solution, see Eagles' *Constructive Geometry of Plane Curves*, pp. 124, 173, and notice Taylor, *Ancient and Modern Geometry of Conics*, p. 164, Ex. 454.

If A, B, C be the three points, D, E, F the mid points of BC, CA, AB then if the centre lies inside DEF the asymptotes are imaginary, but they are real if the centre lies inside AEF, etc.

From the theorem, the following proof of the nine-point circle is obtainable (Fig. 27).

Take any line CK, and draw rectangles as in the figure, we have

$$\angle MPK = \angle PMC - \angle DKC = \angle ECM - \angle DCK = \angle EFD,$$

therefore F, P, D, E are concyclic. If KD is perpendicular to AC, PM is perpendicular to BC, and their intersection is the mid point of CO, where O is the orthocentre. If CK and CB coincide, P is the foot of the perpendicular from A on BC.

Seventh Meeting, May 12, 1893.

JOHN ALISON, Esq, M.A., F.R.S.E., President, in the Chair.

On the History of the Fourier Series.

By GEORGE A. GIBSON, M.A.

§ 1. The treatment of the Fourier Series, that is, of the series which proceeds according to sines and cosines of multiples of the variable, is in most English text-books very unsatisfactory; in many cases it shows almost no advance upon that of Poisson and, even where a more or less accurate reproduction of Dirichlet's investigations is given, there is no attempt at indicating the advantages it possesses over the so-called proof of Poisson. Nor is the *uniformity* of the convergence of the series so much as mentioned, not to say discussed. I have therefore thought it might be useful to give a fairly complete outline of the historical development of the series so far as the materials at my disposal allow. I do not think that any important contribution to the theory is omitted, but, as I indicate at one or two places, there are some memoirs to which I have not had access and which I only know at second hand.

Again it is to be understood that only series of the form

$$A_0 + \sum_{n=1}^{n=\infty}(A_n \cos nx + B_n \sin nx),$$

n being *an integer*, are dealt with, those cases in which n is not integral being omitted in the meantime.

In many of the memoirs referred to in what follows historical notes of the work of predecessors will be found, but there are two writers to whose work I am deeply indebted. In fact these two have done their work so thoroughly as to leave practically nothing for later investigation. The first of these is Riemann, who devotes the introductory pages of his *Habilitationsschrift*, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (*Werke*, pp. 213–253) to a summary of the views of preceding mathematicians, that is, those prior to 1854. This summary is masterly though it is very curious when we consider the influence Poisson has had in this connection on English writers to note that nowhere does Riemann allude to his proof. The other writer referred to is Arnold Sachse who, in his *Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variabele durch trigonometrische Reihen* (Göttingen, 1879), has in a manner completed the summary of Riemann; this dissertation is also of very great value and contains some important additions to the theory due to Schwarz and derived from his lectures. Unfortunately the German text is out of print, but a translation appears in Darboux's *Bulletin* for 1880. It is this translation which I quote when referring to Sachse's Essay. I may also refer to Reiff's *Geschichte der unendlichen Reihen* (Tübingen, 1889) where the connection of the trigonometric series with the theory of infinite series in general is carefully discussed.

It may be useful to remark at the outset :—

That up till the appearance of Fourier's memoir on the "Analytical Theory of Heat" the possibility of the expansion of an *arbitrary* function in a trigonometric series was not admitted by any mathematician.

That Fourier had a thorough grasp of the nature of such expansions and gave in broad outline, though not in such detail as its importance demanded, a sound proof of the expansion, so that from the time his memoir became known the validity of the expansion has never been questioned.

That Dirichlet was the first to give a proof in which the restrictions on the function to be expanded, in other words the limits of its arbitrariness, are carefully stated.

That the work of subsequent writers has consisted largely in extending the limits given by Dirichlet, while following in the main his methods, though new ground was broken by Riemann.

And finally, that in comparatively recent times the series has been shown to be in general uniformly convergent. We have thus to keep before us these three points: first, the possibility of the expansion of an arbitrary function; second, the limits to the arbitrariness of the function in order that the series which represents it may converge to the value of the function; and third, the nature of the convergence, whether uniform or not.

§ 2. The controversy as to the possibility of expanding an arbitrary function of one variable in a series of sines and cosines of multiples of the variable arose about the middle of last century in connection with the problem of vibrating chords. To appreciate properly the difficulty which the expansion presented to the mathematicians of that day we must bear in mind that their conception of a function was much more limited than ours. In the *Introductio in Analysin Infinitorum*, vol. II., cap. I., § 9, Euler says that curves may be divided into *continuous* and *discontinuous* or *mixed*; a curve is continuous when its nature can be expressed by one definite function (*i.e.*, analytical expression) of the variable; if on the other hand different portions of the curve require different functions to express them the curves are called discontinuous or mixed or irregular as not following the same law through their whole course but being composed of portions of continuous curves. Curves which are discontinuous in this sense seem to have been considered to be beyond the scope of analysis; on this point reference may be made to Lagrange, *Oeuvres*, I., p. 68, and to D'Alembert, *Opuscules*, I., p. 7. As a consequence or accompaniment of this view it was supposed that if two functions of a variable were equal for any definite range of values of the variable they must be so for all values so that if the curves which represent them coincide for any interval they must do so entirely. Thus the objection was constantly urged that an algebraic function could not be represented by a trigonometric series for the latter gives a periodic curve while the former does not. Fourier was the first to see and state that when a function is defined for a given range of values of the argument its course outside that range is in no way determined. One obvious consequence of these views is that no one before Fourier could have properly understood the representation of an arbitrary function by a trigonometric series.

§ 3. D'Alembert in the *Mémoires de l'Académie de Berlin* for 1747, vol. III., page 214, discusses the problem of the vibrating chord. The origin of co-ordinates being at one end of the chord whose length is l, the axis of x in the direction of the chord and y the displacement at time t, he shows that y must satisfy the equation $\frac{\delta^2 y}{\delta t^2} = a^2 \frac{\delta^2 y}{\delta x^2}$ (In the memoir $a = 1$, but I keep the usual form). He obtains the solution $y = f(at + x) + \phi(at - x)$, and since $y = 0$ for $x = 0$ and $x = l$ he finds $y = f(at + x) - f(at - x)$ and shows that f represents such a function that $f(z) = f(z + 2l)$. In a memoir immediately following this one in the same volume (p. 220) he seeks to find functions which satisfy this relation of periodicity.

In the *Mémoires* for the following year (1748) vol. IV., p. 69, Euler discusses the same problem. He observes that the motion of the string will be completely determined if its form and the velocity of each point of it be known for any one position. He deduces the equation $y = \phi(x + at) + \phi(x - at)$ where ϕ is such that $\phi(at) + \phi(-at) = 0$ and $\phi(l + at) + \phi(l - at) = 0$ for every t; and from these equations which ϕ must satisfy he concludes that every curve *whether regular or irregular* which consists of repetitions alternately below and above the axis of any given curve which the string may be supposed to take (each point where the curve crosses the axis being a centre of the curve) is suitable for representing ϕ. He then shows how the ordinate of any point at any given time may be determined by a simple geometrical construction. He gives on p. 84 as a particular solution for $\phi(x)$ the equation

$$\phi(x) = \alpha \sin\frac{\pi x}{l} + \beta \sin\frac{2\pi x}{l} + \gamma \sin\frac{3\pi x}{l} + \text{etc.}$$

Euler's solution is clearly more general than that of D'Alembert who always supposes the curve taken by the chord to be regular; but in the *Mémoires* for 1750, vol. VI., p. 355, the latter objects that Euler's solution is not more general than his own because the extension to *irregular* curves is illegitimate. He does not attack any special point in Euler's investigation, but seems rather to rest his objection on the illegitimacy of concluding from regular to irregular curves since the latter, not being expressible by one definite function through their whole course, cannot form the subject of analysis. Euler replies in the *Mémoires* for 1753, vol. IX., p. 156, by presenting his solution in great detail and asking where in his proof the law of continuity is assumed. D'Alembert does not seem to have answered Euler's challenge directly although repeating his previous objection (*Opuscules*, vol. I.). Lagrange while agreeing that Euler's solution is more general than that of D'Alembert still holds his proof to be unsatisfactory on what I suppose to be the same general grounds as D'Alembert. (Lagrange, *Oeuvres*, I., p. 68). If, as Lagrange seems to hold, and as Euler himself in the *Introd. in Anal. Inf.* leads us to think, an irregular curve cannot form the subject of mathematical investigation, there can be no question, I think, of the soundness of the objection to Euler's proof, and it was precisely because of his doubts that Lagrange undertook his investigation of the problem. Euler, however, seems always to have held to the accuracy of his solution and the other two to their objections, the one of these two to the generalisation and the mode of reaching it, the other not to the generalisation but only to the mode of reaching it; the difficulty was only explained by a better insight into the nature of functions and their mathematical treatment.

§ 4. The bearing of these memoirs and of the discussions as to the generality of the solution on the subject of this paper is fully seen when we consider an article by Daniel Bernoulli on the same subject which appeared in the Berlin *Mémoires* for 1753, vol. IX., p. 173. In that article Bernoulli approaches the consideration of the problem of the vibrating chord from the physical rather than from the mathematical side and proposes a synthetical solution of it. Basing his arguments on the expression given by Brook Taylor in his treatise *De Methodo Incrementorum* for a particular integral of the differential equation, namely, $y = \mathrm{A}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$, and on the principle of the Coexistence of Small Motions, he maintains that any position of the string may be given by the equation

$$y = \alpha\sin\frac{\pi x}{l} + \beta\sin\frac{2\pi x}{l} + \gamma\sin\frac{3\pi x}{l} + \text{etc.}$$

His arguments are not mathematical and he nowhere attempts to find the values of the coefficients α, β, γ, etc. A proof of the same nature as Bernoulli's in the mode of approaching the question but much more efficiently developed may be found in Lord Rayleigh's *Theory of Sound*, vol. I., cap. VI. Bernoulli observes in § XIII. that Euler had given the same equation as he does (in the memoir of 1748 referred to above), but he holds against Euler that this gives a perfectly general solution.

Euler combats Bernoulli's position in the memoir of 1753 already noticed in connection with D'Alembert. The earlier part of it deals with Bernoulli's solution. Euler admits that if it be general it is much better than his own; but he does not admit its generality, for that would be equivalent to admitting that every curve could be represented by a trigonometric series and this proposition he considers to be certainly false, seeing that a curve given by a trigonometric series is periodic—a property not possessed by all curves. In seeking to establish his position he remarks (p. 200) that it might be argued that since there is an infinite number of disposable constants, α, β, γ, etc., at disposal, it must be possible to make the proposed curve coincide with any given curve, but he states explicitly that Bernoulli himself has not used this argument. Bernoulli indeed does not seem in his memoir of 1753 to have quite grasped the mathematical consequences of his solution; his results seemed so satisfactory in their explanation of the facts of observation that he was prepared to maintain the generality of his solution on that ground alone. In a letter addressed to Clairaut and published in the *Journal des Scavans* for March 1759, pp. 59-80, he states very clearly the substance of his memoirs of 1753 and the line of reasoning that had led him to his treatment of the problem. In criticising Euler's views of his memoirs he (p. 77) explicitly accepts the argument from the infinite number of disposable constants, though in so doing he really detracts from the merit of his work. On p. 78 he indicates a proceeding that would appear to be that subsequently developed by Lagrange. He takes seven points on a curve and says he succeeded in determining α, β, γ, etc., so as to make the trigonometric curve pass through these points, and he adds that the process might be continued. He gives, however, no proof of his statements.

§ 5. When the controversy was at this stage a memoir by Lagrange on the *Nature and Propagation of Sound* appeared in the first volume (1759) of the *Miscellanea Taurinensia* (Lagrange;

Oeuv., vol. I.). In the introduction he gives a lucid statement of the methods of the three writers we have named, accepts Euler's solution as the most general, but objects to his mode of demonstration, and proposes to obtain a satisfactory solution by first considering the case of a finite number of vibrating particles and then seeking the limit for an infinite number—that is for a chord. The theory deduced in his fourth chapter for a finite number of particles is the same as that of Bernoulli on whose synthetical solution he bestows high praise (§ 32); but for our purposes the thirty-seventh article is the most important, in which he seeks the limit for an infinite number of particles. The length of the string being a and the initial co-ordinates of a point on it being (X, Y) the first part of the equation for the ordinate of the point (x, y) at time t is given by

$$y = \frac{2}{a}\int dx.\mathrm{Y}\left(\sin\frac{\pi\mathrm{X}}{a}\sin\frac{\pi x}{a}\cos\frac{\pi\mathrm{H}}{\mathrm{T}}t + \sin\frac{2\pi\mathrm{X}}{a}\sin\frac{2\pi x}{a}\cos\frac{2\pi\mathrm{H}}{\mathrm{T}}t \right.$$
$$\left. + \sin\frac{3\pi\mathrm{X}}{a}\sin\frac{3\pi x}{a}\cos\frac{3\pi\mathrm{H}}{\mathrm{T}}t + \text{etc.}\right)$$

"where the integral sign $\int$ is used to express the sum of all these series and the integrations are to be made on the supposition that X, Y are the variables and t, x constants." This seems undoubtedly to be a Fourier series in the proper sense of the term; yet it appears to me doubtful if Lagrange actually supposed it to be such. It could hardly have escaped his notice that for a definite value of t this is simply Bernoulli's solution. It was doubtless no part of Lagrange's purpose, as Reiff remarks (p. 134), to determine the co-efficients in Bernoulli's series, but rather to obtain the functional solution given by D'Alembert as he actually does by summing the series by trigonometric methods. At the same time if Lagrange had really meant the summation to be what we now call an integration his subsequent evaluation of the series would not have possessed that generality he contended for, as it starts from a result that implies the continuity of Y. Exactly the same objections he urges (§ 15) against Euler could have been brought against himself. Many parts of the investigation of § 38, where he sums the series, are according to modern notions very loose; yet leaving this aside the investigation shows great analytical skill, and in some respects anticipates the procedure of Fourier as will be pointed out later. All the same I do not think that Lagrange himself nor any of his contemporaries can have understood the above series as anything else than a *finite* series, and I believe that the m used by Lagrange is not made really infinite until he has summed the series and passes to the functional solution. Further Lagrange was quite alive to the merits of Bernoulli's solution and even proposes a proof (*Oeuvres*, I., pp. 514–516) of the proposition that the initial figure of the chord, *when it has one*, is contained in the equation

$$y = \alpha \sin\frac{\pi x}{a} + \beta \sin\frac{2\pi x}{a} + \gamma \sin\frac{3\pi x}{a} + \text{etc.}$$

With this result before him it is almost beyond belief that Lagrange would fail to see its identity with his own formula quoted above, had he supposed m to be really infinite. With m infinite his solution would have been complete and the subsequent investigations mere transformations of it without adding anything to it.

Another investigation by Lagrange belonging to the same series of memoirs on Sound and printed on pages 552–554 of the first volume of his collected works is that repeatedly quoted by Poisson and others as the first investigation of the representation of a function by a trigonometric series. I think, however, that this investigation stands on the same footing as that just discussed and I hold that Riemann's view of it is correct. It is no doubt hard for us to understand how near Lagrange came to the conception of expanding an arbitrary function in an infinite series without ever actually attaining to it, especially when we see him in this memoir adopting the method of passing a trigonometric curve through a finite number of points on a given curve and succeeding in solving the necessary equations in the manner used later by Dirichlet (Dove's Repertorium). That he did not really solve the problem of expansion in trigonometric series is I think best understood from the circumstance that neither he nor any of his contemporaries (unless perhaps Bernoulli) believed such expansion to be possible. It would be interesting to have documentary evidence of the truth of Riemann's statement (*Werke*, p. 219) that Lagrange strongly objected to Fourier's conclusions in regard to such expansions.

§ 6. For the next forty years there seems to have been almost no progress made towards a solution of the difficulties raised in these discussions; but before passing to Fourier mention must be made of certain results given by Euler. In his memoir *Subsidium Calculi Sinuum* (*Novi. Comm. Petrop.*, tom V., ad annos 1754–55) he obtains (p. 204) the following equations:—

$$\sin\phi - \tfrac{1}{2}\sin 2\phi + \tfrac{1}{3}\sin 3\phi - \text{etc.} = \tfrac{1}{2}\phi$$

$$\cos\phi - \tfrac{1}{4}\cos 2\phi + \tfrac{1}{9}\cos 3\phi - \text{etc.} = \frac{\pi^2}{12} - \frac{\phi^2}{4}.$$

Reiff remarks (p. 128) that these series are the first in which rational functions are expressed by series of sines and cosines of multiples of the variable. It is somewhat remarkable that Euler should have accepted these results, but his views on the validity of results derived from the use of the series were extremely loose.

A more important result for the general theory is contained in a memoir presented by him to the St Petersburg Academy in 1777 but not published till 1798, long after his death. In the *Nova Acta*, vol. XI., p. 114, this memoir appears, and in it he says that if Φ can be expanded in a series of the form $A + B\cos\phi + C\cos 2\phi + \text{etc.}$ then

$A = \dfrac{1}{\pi}\displaystyle\int \Phi d\phi$, $B = \dfrac{2}{\pi}\displaystyle\int \Phi d\phi \cos\phi$, etc., where the limits of the integrals are 0 and π. Fourier's method of determining the coefficients is thus explicitly given by Euler as Jacobi remarked (Crelle's *Journal*, vol. II., p. 2); but the use that Euler makes of the series and the words in which he introduces his memoir seem to me to render it doubtful if Euler, as Sachse appears to think (p. 47), drew the hint that led to his method from Lagrange's memoirs. Except for the mode of determining the coefficients the memoir goes but a very little way towards settling the possibility of representing an arbitrary function by a trigonometric series.

§ 7. Glancing for a moment over the work of Euler, Bernoulli, and Lagrange, it is easy for us to see where the difficulties of the subject lay; they lay in the inadequacy of the notion of a function. Both Euler and Lagrange seem at times as if they had in part transcended the limits of their original conception, Euler in giving his geometrical constructions for the solution of the equation for the vibrating chord and Lagrange in his method of constructing the equation to a curve by first finding the equation of a curve passing through the vertices of an inscribed polygon. Yet I do not think either of them got beyond the old notion of continuity and its consequences in any of their writings on the subject of trigonometric series. But great part of their work could be and was of immense service to Fourier, as he himself indicates (*Théorie Anal. de la Chaleur*, § 428), when he approached the consideration of the subject with his conception of a function as given graphically.

§ 8. Fourier's first investigations on the Theory of Heat were communicated to the Academy of Sciences on the 21st December 1807. The memoir of 1807 has never been printed though it has now been recovered, and it is to the memoir sent to the Academy in 1811 and crowned on the 6th January of the following year that we must look for Fourier's exposition of the representation of arbitrary functions by trigonometric series. In all essential points the treatise *Théorie Analytique de la Chaleur*, published in 1822, is a reproduction of the memoir of 1811, and I shall therefore refer always to the treatise, in the recent edition of it by Darboux.

The third and the ninth chapters of the treatise are those in which the trigonometric expansions are most fully considered, and even a casual reading of these is sufficient to show how thoroughly Fourier cleared away the difficulties which had puzzled his predecessors. Even before the publication of Dirichlet's proof in 1829, which has generally been considered to be the first satisfactory exposition from the mathematical standpoint, Fourier's results had been universally accepted; no doubt some of Fourier's series were criticised but in many cases the errors were those of the printer and not of Fourier himself. At the same time there can be no question of the general acceptance of his main theorem on the subject of the expansion of an arbitrary function.

The treatise is so well known that I need not spend much time in analysing it; but I may call attention to one or two points. In article 428, 12° and 13° Fourier sums up his views on the nature of a function which admits of expansion; it is not necessarily *continuous* in the old sense of that word but may be composed of *separate functions* or *parts of functions*. By these phrases he means a function $f(x)$ which has values while x lies between given limits but is zero for all other values of x. The function may even become infinite between the limits (§ 417) and in general the function need only be given graphically. Again, Fourier has accurate conceptions of the convergency of series (§§ 177, 185, 228, 235, etc.) though he occasionally makes slips (for example in § 218 where he puts

$1 - 1 + 1 - 1 + \text{etc.} = \frac{1}{2}$, see also § 420, p. 506); in this respect both Euler and Lagrange leave much to be desired. A more important question remains, namely, how far did Fourier succeed in his mathematical demonstration that the series which represents the function actually converges to the value of the function? In special cases which he gives the convergency of the series and its equivalence with the function are, as he says, easily demonstrated; but it is usually maintained (*e.g.*, by Riemann, *Werke*, p. 220) that he gave no mathematical proof of the general theorem. I think, however, that in this respect Fourier has received less than justice. No doubt, the investigations of chapter III. can hardly be accepted as doing more than suggesting the truth of the general theorem, but it is different with those of chapter IX. In these as in nearly all the special series of chapter III. he adopts the method, followed afterwards by Dirichlet, of taking n terms of the series and seeking the limit for n infinite. This method indeed seems to me to be that of Lagrange in § 38 of his first memoir referred to above, and it is unquestionably the most satisfactory. Fourier's treatise being in everybody's hands I need hardly do more than refer to § 423 and suggest that it should be compared with Dirichlet's proof. At bottom Fourier's reasoning is, I believe, quite sound and it seems to me to contain the kernel of Dirichlet's proof. No doubt Fourier did not develope his proof with the extreme precision that the importance of the theorem demanded and that Dirichlet afterwards gave to it; still the substantial accuracy of his reasoning is beyond dispute. Darboux in a note pp. 511, 512 of his edition of the treatise calls attention to the matter, and his contention on behalf of Fourier seems to me quite justified by the facts. Before seeing this note I had formed the opinion I have expressed and I was glad to find it confirmed by so able an authority.

I content myself with this meagre reference to Fourier, because his treatise is so universally read even yet by all beginners in the study of mathematical physics that it would be waste of time to delay over it. I cannot pass from it however without remarking that it seems to me peculiarly unfortunate that instead of studying Fourier's own mode of presenting the proof of his series-theorem, or, what would have been even better, taking Dirichlet's memoirs on the same subject as guide, English writers have usually drawn their exposition from Poisson who studiously denied to Fourier his just claims in this field. As a matter of fact Poisson's proof is invalid and seems to have been recognised as such almost from the first by the great continental writers. At any rate, Dirichlet does not, I think, allude to it and Cauchy lays his finger on the weak point, as I shall indicate shortly. Nor does Riemann in his historical notice refer to Poisson except to call in question his estimate of Lagrange's position. No doubt the integral that Poisson makes use of is of great importance, and has played a fundamental part in many modern developments; but its value appears *after* the Fourier series has been established and not in the proof of the series itself.

§ 9. Poisson has treated the trigonometric series now dealt with in several places and always in practically the same way. I may refer to the *Journal de l'École Polytechnique* vol. XI. (1820), vol. XII. (1823), and to the treatise *Théorie de la Chaleur* (1835). His process is as follows:—

When $p<1$, $\dfrac{1-p^2}{1-2p\cos(x-a)+p^2}=1+2\sum_{n=1}^{n=\infty} p^n\cos n(x-a)$

Multiplying by $f(a)$ and integrating between $-\pi$ and π, he gets

$$\int_{-\pi}^{\pi}\frac{(1-p^2)f(a)da}{1-2p\cos(x-a)+p^2}=\int_{-\pi}^{\pi}f(a)\{1+2\Sigma p^n\cos n(x-a)\}da$$

When $p=1$, the integral on the left has all its elements zero except when $a=x$. Putting then $p=1-g$, where g is small, and $x-a=z$ he gets for the value of the integral $2f(x)\int_{-\epsilon}^{\epsilon'}\dfrac{gdz}{g^2+z^2}$ where ϵ, ϵ' are small; but no error will be introduced by making the limits infinite, so that when $p=1$, the integral is equal to $2\pi f(x)$. Making $p=1$ on the right side he deduces

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(a)\{1+2\Sigma\cos n(x-a)\}da$$

The proof is usually extended so as to include the cases in which $f(x)$ presents discontinuities.

On this proof there are two remarks to be made. In the first place, if the series $\int_{-\pi}^{\pi}f(a)\{1+2\Sigma p^n\cos n(x-a)\}da$ be denoted by $\Sigma A_n p^n$, and if we write $F(p)=\Sigma A_n p^n$, then we are only justified in assuming $F(1)=\Sigma A_n$ when the series ΣA_n is convergent. This theorem is generally quoted as Abel's Theorem (see Chrystal's *Algebra* Pt. II. p. 133). But in the present case this procedure amounts to assuming that the trigonometric series is convergent, and the convergency of the series is not proved by Poisson. In other words one of the greatest difficulties of the subject is tacitly passed over. It may be added that unless the function $f(x)$ be very greatly restricted it does not seem possible to prove the convergence of the series from a consideration of the integrals which give the coefficients. In the second place, the quantity p has no natural connection with the series and is a source of ambiguity that is not inherent in the series itself. This is seen when the integral $\int_{-\pi}^{\pi}\frac{(1-p^2)f(\alpha)d\alpha}{1-2p\cos(x-\alpha)+p^2}$ is more carefully studied, as in the writings of Schwarz (see his two memoirs on the *Integration of the Equation* $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0$ in his Collected Works, vol. II). If we describe a circle with radius unity and take a point in it having (p, x) for its polar co-ordinates, then the limit of the integral for $p=1$ depends, except when $f(x)$ is continuous and periodic, on the path by which the point approaches the circumference. Thus if $f(\pi-0)\neq f(-\pi+0)$, the limit when $x=\pi$ is $\pi\{f(\pi-0)+f(-\pi+0)\}+\theta\{f(\pi-0)-f(-\pi+0)\}$ where θ may have any value between π and $-\pi$. But the limit of the *series* when $x=\pi$ is perfectly definite, namely the value of the above expression when $\theta=0$. However valuable, then, Poisson's integral may be in other respects it does not seem to furnish a satisfactory starting point for the investigation of the series in question.

§ 10. After Poisson, Cauchy attacked the problem in his *Mémoire sur les dévelopements des fonctions en séries périodiques* (*Mem. de l'Inst.* vol. VI.; read 27th Feb. 1826). He starts with the series

$$\int_0^a f(\mu)d\mu+2\int_0^a\sum_{n=1}^{n=\infty}\cos\frac{2n\pi}{a}(x-\mu)f(\mu)d\mu.$$

To prove that this has for sum $af(x)$ he replaces it by another series

$$\int_0^a f(\mu)d\mu+\sum_{n=1}^{n=\infty}\theta^{n-1}\int_0^a e^{\frac{2n\pi i}{a}(x-\mu)}f(\mu)d\mu+\sum_{n=1}^{n=\infty}\theta^{n-1}\int_0^a e^{-\frac{2n\pi i}{a}(x-\mu)}f(\mu)d\mu$$

where $\theta=1-\epsilon$ and ϵ is a small quantity. The series, when summed, gives

$$\int_0^a \left\{1 + \frac{1}{e^{\frac{-2\pi i}{a}(x-\mu)} - \theta} + \frac{1}{e^{\frac{2\pi i}{a}(x-\mu)} - \theta}\right\} f(\mu)d\mu$$

and this integral being evaluated in Poisson's manner is equal to $af(x)$. But Cauchy recognises one of the faults of Poisson's proof and tries to prove the convergence of the series when $\theta = 1$. To do this he throws it into the form

$$f(x) = \frac{1}{a}\int_0^a f(\mu)d\mu + \frac{1}{ai}\int_0^\infty \left\{\frac{f(a+\nu i) - f(\nu i)}{e^{\frac{2\pi xi}{a}} e^{\frac{2\pi\nu}{a}} - 1} - \frac{f(a-\nu i) - f(-\nu i)}{e^{\frac{-2\pi xi}{a}} e^{\frac{2\pi\nu}{a}} - 1}\right\} d\nu$$

This equation, as Cauchy remarks later, may be deduced by integration of the functions $f(z)/\left\{e^{\pm\frac{2\pi}{a}(z-x)i} - 1\right\}$ round a properly selected boundary. As to the function $f(z)$ it must remain finite for all real or imaginary values of z. He now, instead of examining the integral in its closed form, throws it again into a series of which the general term is, if $z = 2n\pi\nu/a$

$$\frac{1}{2n\pi i} e^{\frac{-2n\pi xi}{a}} \int_0^\infty e^{-z} \left\{f\left(a + \frac{ai}{2n\pi}z\right) - f\left(\frac{ai}{2n\pi}z\right)\right\} dz$$

$$- \frac{1}{2n\pi i} e^{\frac{2n\pi xi}{a}} \int_0^\infty e^{-z} \left\{f\left(a - \frac{ai}{2n\pi}z\right) - f\left(-\frac{ai}{2n\pi}z\right)\right\} dz$$

so that when n is very large the general term approximates to

$$\left(f(0) - f(a)\right) . \frac{1}{n\pi} \sin \frac{2n\pi x}{a}$$

The series of which this is the general term is convergent, and he therefore concludes the trigonometric series to be convergent.

Now in regard to this proof two points in particular require notice. First, as Dirichlet noticed, there may be two series whose terms differ infinitely little from each other when $n = \infty$, and yet the one series diverges while the other converges; for example $\Sigma\frac{(-1)^n}{\sqrt{n}}$ converges while $\Sigma\frac{(-1)^n}{\sqrt{n}}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$ diverges. Cauchy's proof of the convergence thus fails. But, as Reiff remarks (p. 189), it is easy to see that the integral in closed form is finite if $0 < x < a$, so that this objection might be overcome. But, secondly, the conditions imposed on $f(z)$ reduce that function to a constant. Riemann, who pointed this out, states that Cauchy's conditions are not really necessary for his proof; it is sufficient that the function $f(x+iy)$ be determinable such that for all values of y it remains finite and for $y=0$ becomes $f(x)$. That such a function is determinable Riemann holds to be established, and therefore apparently that Cauchy's proof is valid. This remark of Riemann's is pretty fully considered by Sachse, pp. 48–52, and I content myself with referring to him, only adding that Riemann's proof of the possibility of determining a function by means of its values along a boundary is not now accepted, and that the necessity of using other methods of establishing the proposition in question carries with it the invalidity of Cauchy's proof.

For another and more general investigation by Cauchy I would refer to his *Oeuvres complétes*, vol. VII. (2nd ser.), p. 393.

§ 11. I come now to the classical investigations of Dirichlet. Of his two memoirs dealing with the subject of the Fourier Series the first appeared in Crelle's *Journal*, 1829, vol. IV., pp. 157–169, the second in Dove's *Repertorium der Physik*, vol. I., pp. 152–174. This second memoir is so clear and simple that it has become a model of nearly every discussion on the series in question contained in continental text-books, and probably there is no memoir in the whole range of mathematical journalism that has been so completely and so literally transferred to the text-books. Dirichlet saw that the convergence of the series does not depend solely on the decrease of the terms, but is due also to the presence of negative terms. (See the introduction to his paper on expansion in Spherical Harmonics in Crelle's *Journal*, vol. XVII.). Hence he adopts the method, which Fourier had employed, of summing to n terms and finding the limit for $n=\infty$. It will be convenient to follow the second rather than the first memoir.

The first $2n+1$ terms of the series for $\phi(x)$ may be written

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha.\phi(\alpha)\frac{\sin(2n+1)\frac{\alpha-x}{2}}{2\sin\frac{\alpha-x}{2}}$$

and this integral may be divided into

$$\frac{1}{\pi}\int_0^{\frac{\pi+x}{2}} d\beta.\phi(x-2\beta)\frac{\sin(2n+1)\beta}{\sin\beta}+\frac{1}{\pi}\int_0^{\frac{\pi-x}{2}} d\beta.\phi(x+2\beta)\frac{\sin(2n+1)\beta}{\sin\beta}$$

and the limit for $n=\infty$ has to be found. The investigation hinges upon the limit for $k=\infty$ of $\int_0^h \frac{\sin k\beta}{\sin\beta} f(\beta)d\beta$ where $k=2n+1$ and $0<h<$ or $=\pi/2$. The function $f(\beta)$ is supposed in the first place to be continuous, positive, and not increasing, while β goes from 0 to h. The integral is decomposed into a series of partial integrals with limits $0, \pi/k$; $\pi/k, 2\pi/k$; etc.; $r\pi/k, h$ where $r\pi/k$ is the greatest multiple of π/k contained in h. Each of these integrals is less in absolute value than its predecessor and the signs of them are alternately positive and negative. The integral is thus found to lie between limits which for $n=\infty$ coincide in the value $\frac{1}{2}\pi f(0)$. The restrictions on $f(\beta)$ are then partly removed; it may either be constant or negative or a not decreasing function as β goes from 0 to h. It follows immediately that $\underset{k=\infty}{\mathrm{L}} \int_g^h \frac{\sin k\beta}{\sin\beta} f(\beta)d\beta = 0$ if $0<g<h<$ or $=\frac{\pi}{2}$. By this last result it is possible to extend the first theorem to all continuous functions which have a finite number of maxima and minima, while if $f(\beta)$ be discontinuous for $\beta=0$ the limit is $\frac{1}{2}\pi f(+0)$ if h be positive but $-\frac{1}{2}\pi f(-0)$ if h be negative. The limit for $n=\infty$ of the sum of the first $2n+1$ terms of the trigonometric series is thus $\frac{1}{2}\{\phi(x+0)+\phi(x-0)\}$ if $x \neq \pm\pi$ but $\frac{1}{2}\{\phi(\pi-0)+\phi(-\pi+0)\}$ if $x=\pm\pi$.

The results may therefore be summed up as follows:—The limit for $n=\infty$ of the series $\frac{1}{2}a_0 + \sum_{m=1}^{m=n} (a_m \cos mx + b_m \sin mx)$ where

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(a)\cos ma\,da,\quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(a)\sin ma\,da$$

is $\quad \frac{1}{2}\{\phi(x-0)+\phi(x+0)\} \quad$ if $x \neq \pm\pi$

but $\quad \frac{1}{2}\{\phi(\pi-0)+\phi(-\pi+0)\} \quad$ if $x=\pm\pi$

provided that while $-\pi =$ or $< x =$ or $<\pi$, $\phi(x)$ has a finite number of maxima and minima, a finite number of discontinuities, and does not become infinite. Of course if $\phi(x)$ is continuous near x, the value is simply $\phi(x)$. These conditions (with another regarding infinite values of $\phi(x)$ to be given presently) are usually called *Dirichlet's conditions.*

It is perhaps worth observing that the mode of conducting the investigation prescribes the order in which the terms are to be taken, and the order is of course essential when the series is semi-convergent.

§ 12. The definite form which Dirichlet gives to the sum of the trigonometric series suggests that the phrases "the function $\phi(x)$ can be expanded in a series" or "the series represents the function $\phi(x)$" should be precisely defined, for where there is a breach of continuity in the function the series has a definite value while the function has not. The natural definition seems to be that adopted by Sachse (p. 55), namely, a series represents a function in a given interval if its values coincide with those of the function for all points in the interval with the exception of a limited number of known points. A Fourier series therefore represents a function which satisfies Dirichlet's conditions.

There is one point in Dirichlet's demonstration which has been subjected to criticism in some quarters. According to Dirichlet the value of the series at a point of discontinuity in the function is the arithmetic mean of the values of the function at that point. It has been contended on the other hand by Schläfli and Du Bois-Reymond that the value is really indeterminate (compare also Thomson and Tait, *Nat. Phil.*, vol. I., pt. I., p. 59) and that the sum may have all values between the two values of the function at the point. Sachse (pp. 56–58) discusses the point and as I have not had access to Schläfli's pamphlet (*Einige Zweifel an der allg. Darst. . . . durch trig. Reihe*, Berne, 1874) nor to Du Bois-Reymond's memoir (*Sprungweise Werthveränderungen, Math. Ann.*, vol. VII.) I must simply refer to Sachse for a fuller notice and also to Heine's *Kugelfunctionen*, vol. II., p. 347. At the same time I may say that these objections, so far as I understand them, do not seem to me to be sound as they rest upon the evaluation of a *double limit* while in the case of the series there is but one variable to be considered. I have already referred to the ambiguity of a similar character introduced by Poisson's proof.

§ 13. Had Dirichlet not written his first memoir, the paper which follows his in the same volume of Crelle (vol. IV., p. 170) by Dircksen would have been a notable contribution to the theory of trigonometric series. It proceeds on the same general lines as Dirichlet's though obviously it is quite independent; but neither in elegance nor in generality is it comparable with his, and it has practically fallen into oblivion.

Bessel in Schumacher's *Astronomische Nachrichten*, vol. XVI., p. 229, sought to simplify Dirichlet's proof, but he can hardly be said to have succeeded, and he certainly added nothing to the general theory.

§ 14. The conditions given by Dirichlet in his first memoir, as those which a function must satisfy if it is to be represented by a trigonometric series, are certainly very general, and in an addition to his memoir on the representation of an arbitrary function by a series of Spherical Harmonics (Crelle's *Journal*, 1837, vol. XVII., p. 54) he shows that the function $\phi(\beta)$ may become infinite at a finite number of points provided that $\int\phi(\beta)d\beta$ remain finite and continuous. This condition will be included among *Dirichlet's conditions* when these are referred to. But Dirichlet believed that a function, with fewer restrictions than those implied in his conditions, could be represented by a trigonometric series, and at the end of his first memoir promises a paper on the subject. Nothing, however, except the note in the seventeenth volume of Crelle, just mentioned, has appeared from his pen in the way of carrying out the promise. In particular it should be noticed that Dirichlet's conditions do not include all continuous functions, since they exclude every function with an infinite number of maxima and minima; but if a function have an infinite number of oscillations in the neighbourhood of a point it may be continuous when the amplitude of the oscillations is infinitely small. Thus the function $x\cos(1/x)$ is continuous between $-\pi$ and π on the understanding that it is zero for $x=0$, yet this would be excluded from Dirichlet's conditions. One of the main objects of later investigations has been to extend the limits of the arbitrariness allowable to a function which may still be represented by a trigonometric series, but it is a somewhat striking fact that the conditions do not yet include all continuous functions, and Du Bois-Reymond has even proved that there are continuous functions such that the trigonometric series which represent them become infinite at certain points, that is, cease to represent them at these points. The belief then that every continuous function can be represented by a trigonometric series is unwarranted.

§ 15. The first published attempt to show that a function having an infinite number of maxima and minima may be represented by a trigonometric series is that of Lipschitz in his memoir *De explicatione per series trigonometricas*, etc. (Crelle's *Journal*, 1864, vol. LXIII., p. 296). His proof depends on the evaluation of the two integrals noticed as fundamental in Dirichlet's method, and he shows that these still maintain their validity if in the neighbourhood of those points β for which $f(\beta)$ oscillates $f(\beta+\delta)-f(\beta)$ is less in absolute value than $B\delta^{a}$ where a is positive and B a constant. As an extension of Dirichlet's conditions the result is important, but it is to be observed that there may be continuous functions not satisfying this condition. $f(\beta)$ will be continuous near β if, given an arbitrarily small quantity ϵ, a value h can be found such that for all values of δ less numerically than h, mod. $\{f(\beta+\delta)-f(\beta)\}$ is less than ϵ. Lipschitz's condition implies that $\epsilon = \text{or} < Bh^{a}$ or $h = \text{or} > \sqrt[a]{(\epsilon/B)}$, a relation not necessary for continuity. Again, Lipschitz's results would hold if $\underset{\delta=0}{L} \log\delta\{f(\beta+\delta)-f(\beta)\}=0$ and this is a form which Dini uses in his treatise *Sopra la Serie di Fourier*, and is less restrictive than the other.

§ 16. I now come to Riemann's investigations as contained in his great memoir *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe.* Though prepared for his *Habilitationschrift* in 1854, it was not published till after his death, appearing in vol. XIII, of the *Göttingen Abhandlungen*, 1867; it is reprinted in his collected works, pp. 213–253, with notes by Weber.

The memoir is divided into three main sections. The first section, arts. 1–3, is historical and has been several times referred to in the earlier part of this paper. The second, arts. 4–6, contains a thorough investigation of the fundamental principles of definite integrals, and in particular determines in what cases a function has an integral. We see here the great extension of meaning which the word *function* has gained in modern times, chiefly under the guidance of Fourier, Dirichlet, and Riemann himself, and which is essential to the modern function theory. The third section completes the memoir and is devoted to the representation of a function by a trigonometric series without special suppositions as to the nature of the function. The problem proposed for solution is the following:—If a function can be represented by a trigonometric series, what follows respecting the march of the function, respecting the change in its value for a continuous change in the argument? The preceding investigations argued from the function to the series; here the series is supposed given and the conclusion is to the nature of the function.

Riemann denotes the series $A_0 + A_1 + A_2 + \text{etc.} + A_n + \text{etc.}$ where $A_0 = \frac{1}{2}b_0$, $A_n = a_n \sin nx + b_n \cos nx$ by Ω, and when it is convergent its value is denoted by $f(x)$, so that $f(x)$ only exists for those values of x for which the series is convergent. He first supposes Ω to be such that for every value of x A_n becomes infinitely small when n becomes infinitely great. If the series Ω be integrated twice and the series thus formed be denoted by $F(x)$ so that

$$F(x) = C + C'x + \tfrac{1}{2}A_0x^2 - A_1 - \text{etc.} - \frac{1}{n^2}A_n - \text{etc.}$$

he shows that $F(x)$ is convergent for every value of x, is continuous, and is integrable. He then proves—

(I.) That when the series Ω converges, the expression

$$\{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)\}/4\alpha\beta$$

converges to the value $f(x)$ when α and β become infinitely small, but such that their ratio remains finite;

(II.) That $\{F(x+2\alpha) + F(x-2\alpha) - 2F(x)\}/2\alpha$ becomes infinitely small with α; and

(III.) That the integral $\mu^2\int_b^c F(x)\cos\mu(x-a)\lambda(x)dx$ becomes infinitely small with $1/\mu$, where b, c denote two arbitrary constants $(c>b)$, $\lambda(x)$ a function which with its first derivative is continuous between b and c and vanishes at the limits and whose second derivative has not an infinite number of maxima and minima.

By means of these theorems he proves that if a periodic function $f(x)$, of period 2π, can be represented by a trigonometric series whose terms become ultimately indefinitely small there must exist a continuous function $F(x)$ such that

$$\{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)\}/4\alpha\beta$$

converges to the value $f(x)$ when α, β converge to zero, their ratio remaining finite. Further, the integral of (III), subject to the conditions there given, must become infinitely small with $1/\mu$.

Conversely, when these conditions are satisfied, there exists a trigonometric series whose terms become infinitely small and which is such that, where it converges, it represents the function. For, determining C', A_0 so that $F(x) - C'x - \frac{1}{2}A_0x^2$ has the period 2π, and then developing this function by the Fourier method the term A_n where

$$A_n = -\frac{n^2}{\pi}\int_{-\pi}^{\pi}\{F(t) - C't - \tfrac{1}{2}A_0t^2\}\cos n(x-t)dt$$

will become infinitely small with $1/n$ and therefore the series $A_0 + A_1 + A_2 + \text{etc.}$ will, whenever it converges, converge to $f(x)$. In Weber's note (Riemann's *Works*, p. 252) the proof for this assertion about A_n is fully given.

Riemann then shows that the convergence of the series for a definite value of x depends only on the behaviour of the function in the neighbourhood of that value. A proof of this important theorem, independent of Riemann's general theorems and due to Schwarz, is given by Sachse, pp. 89 *et seq.*

It will have been observed that as yet Riemann has given no criterion for determining when the coefficients of the series Ω will in fact become infinitely small. In art. 10 he comes to this point, and he there states that in many cases this question can not be settled by consideration of their expression as definite integrals, but must be determined in other ways. For the very important case in which $f(x)$ is integrable, finite throughout the range of the variable, and (he should have added) has only a finite number of maxima and minima, he proves that the coefficients do become infinitely small and therefore that the series represents $f(x)$ whenever it is convergent.

In art. 11 he takes up the case in which the terms of Ω do not become ultimately indefinitely small for every value of x, and shows that the series can converge only for those values of x which are symmetrically placed with respect to those for which the integral

$$\mu^2\int_b^c F(x)\cos\mu(x-a)\lambda(x)dx$$

does not become infinitely small with $1/\mu$.

In art. 12 he considers the possibility of the function becoming infinite, and gives as necessary and sufficient conditions that when $f(x)$ is infinite for $x=a$, $tf(a+t)$ and $tf(a-t)$ become infinitely small for $t=0$ and $f(a+t)+f(a-t)$ be integrable up to $t=0$, it being understood that $f(x)$ has not an infinite number of maxima and minima.

In the last article, art. 13, he deals with functions having an infinite number of maxima and minima. In this connection he first shows by an example that there may be integrable functions having an infinite number of maxima and minima which are yet not capable of representation by a Fourier series. He here takes $f(x) = \frac{d}{dx}\left(x^{\nu}\cos\frac{1}{x}\right)$ where $0 < \nu < \frac{1}{2}$. He shows in the second place by examples that there may be functions having a finite number of maxima and minima and not integrable which nevertheless may be represented by a trigonometric series (on these examples, see a paper by Genocchi, *Atti della R. Acc. di Torino*, vol. X., 1875, *Intorno ad alcune serie*).

Riemann has thus given a very general solution of the problem of representation of functions by trigonometric series and his theorems (I), (II), (III), are of fundamental importance in the subsequent investigations of Heine, Cantor, and Du Bois-Reymond. But other methods than those he gives must in many cases be resorted to to determine when the series is convergent, and as a matter of fact, Dirichlet's integrals seem indispensable for this purpose.

§ 17. Hitherto I have said nothing of the contributions of English writers to the theory of expansion in trigonometric series, and I am sorry to add that the main reason for this is that their contributions are so few. It is, I think, very unfortunate that Poisson's treatment of the Fourier series has become the basis of nearly every investigation in our text-books, because, as has been pointed out, that method is radically faulty. Dirichlet's proof seems to have been long unknown, for except in a note to a paper of Stokes's, to be mentioned presently, I do not remember to have seen it even mentioned till the publication of Todhunter's treatise on *Laplace's Functions*. In his *Integral Calculus*, Todhunter makes no mention whatever of it, except in the reference to his treatise on *Laplace's Functions*, and even there it is only given as an alternative to the other. The reference he makes to Abel's theorem on p. 170 of the treatise is curious as it tacitly assumes the whole thing to be proved, for it assumes that $\Sigma(2n+1)u_n$ is convergent.

De Morgan's *Calculus* is often referred to for the demonstration of the Fourier series, but while it is quite true that De Morgan gives many helpful illustrations and examples like other English writers (Donkin, in particular, in his *Acoustics*), it cannot be said that he has advanced beyond Poisson. I do not understand how such a careful writer as De Morgan could have allowed some of the statements he makes to pass. Thus (p, 607) he says $1+\cos\theta+\cos 2\theta+\ldots\ldots=\frac{1}{2}$ in every case unless $\theta=2\pi m$ when it is infinite, and he thinks (p. 640) there is no reason to doubt that the infinite series $1-1+1-$ etc. (namely, the value of that series for $\theta=\pi$) represents half a unit. This example might have been sufficient to show the uncertainty of reasoning from the convergence of $\Sigma a^n u_n$ to that of Σu_n.

§ 18. Hamilton in his great memoir *On Fluctuating Functions* (*Trans. R.I.A.*, vol. XIX., 1842) has much that bears on the subject of periodic series but no set proof of the Fourier series itself. His integrals, however, include the integrals of Dirichlet as particular cases, and the paper deserves more careful study than it usually receives. A good restatement of Hamilton's principal results in regard to these integrals will be found in Neumann's treatise, *Über die nach Kreis-Kugel-und Cylinder-Functionen fortsch. Entwickelungen*, which contains a good statement of the Fourier series and integrals for the case of *vernünftige Functionen.*

Stokes's memoir *On the Critical Values of the Sums of Periodic Series* (*Camb. Phil. Trans.*, 1847, vol. VIII., p. 533, reprinted in *Collected Works*, vol. I., p. 236) is important in the history of series, for he there (section III.) draws attention to what has since been called the uniform convergence of series, though this honour is usually attributed to Seidel, whose paper (evidently quite independent of Stokes's) did not appear till 1848. In the first section Stokes discusses the expansion of a function in a series of sines and also in a series of cosines, and adopts the method of Poisson as that which he employed when he first began the investigations and which best harmonised with the rest of the paper. He points out, however, in a note to page 251 (*Coll. Works*) that had he been aware of Dirichlet's memoir in Crelle and of Hamilton's paper at an earlier stage of his work he would probably have adopted the method of summing the first n terms of the series and then considering the limit of n infinite. The investigation as carried out is a little tedious but it forms a great advance on the way in which Poisson's mode of treating the subject is usually conducted. There are many things in the paper which make it still valuable, but as it is so easily accessible I need do no more than refer to it.

The investigation given by Thomson and Tait in their *Treatise* *on Natural Philosophy*, vol. I., pt. I., pp. 55–60 has much in common with Poisson's proof and also with Cauchy's. It will be noticed that in passing from their equation (11) to equation (12) the continuity of the series up to and including the value $e=1$ is assumed. But as has been repeatedly stated this assumption is only legitimate if the convergence of the series for $e=1$ is otherwise known, so that the same objection applies here as in Poisson's own proof. In general the convergence of the series is only comparable with that whose general term is A/n, and this result does not carry us far in determining the convergence. The remark on p. 59 that "if exactly the critical value is assigned to the independent variable, the series cannot converge to any definite value" is an *ex cathedra* statement which Dirichlet's proof shows to be quite incorrect.

§ 19. The course of the history of the Fourier series now takes a new departure. In the preceding work it has been seen that under certain circumstances the series will converge to the value of the function, but in more recent times it has been recognised that mere convergence is not sufficient for most of the applications for which the series is needed; the convergence must be uniform. Suppose, for instance, that we have for $f(x)$ the series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{n=\infty}(a_n\cos nx + b_n\sin nx)$$

and we wish to evaluate $\int_\alpha^\beta f(x)\phi(x)dx$ by means of the series; then we can only safely assert the equation

$$\int_\alpha^\beta f(x)\phi(x)dx = \tfrac{1}{2}a_0\int_\alpha^\beta \phi(x)dx + \Sigma\int_\alpha^\beta (a_n\cos nx + b_n\sin nx)\phi(x)dx$$

if the series be uniformly convergent. Unless then the series is to be shorn of much of its value its uniform convergence must be established.

Another difficulty that this conception of uniform convergency raises is that the old proof for the uniqueness of the expansion becomes invalid, as resting upon an integration the legitimacy of which is not proved.

§ 20. The first to call attention to the points just mentioned was Heine in a paper contributed to Crelle's *Journal* vol. LXXI. (1870)

p. 353, *Über trigonometrische Reihen.* In § 2 he gives the definition of uniform convergence, and it is interesting to note as illustrative of the immense influence of Weierstrass, in spite of his comparatively few published papers, that Heine's attention seems to have been first directed to the matter of uniform convergence by Weierstrass or one of his pupils rather than by the writings of Seidel or Stokes. (As regards Stokes, Reiff in his *Geschichte* (p. 207) seems to have been the first to give him due credit in this connection.) He shows that the Fourier series can not converge uniformly in the neighbourhood of a point at which the function is discontinuous, and establishes the following theorems :—

(1) The Fourier series for a finite function $f(x)$ with a finite number of maxima and minima converges uniformly if $f(x)$ be continuous for $-\pi =$ or $< x =$ or $< \pi$ and $f(-\pi) = f(\pi)$; in all other cases it is only uniformly convergent *in general*, that is, it converges uniformly for every interval which does not include a point of discontinuity, these points being snpposed finite in number. The points $\pm\pi$ are to be considered points of discontinuity if $f(-\pi) \neq f(\pi)$.

(2) If a trigonometric series is in general uniformly convergent, and is in general equal to zero ($-\pi =$ or $< x =$ or $< \pi$) then will every co-efficient be zero. For the proof of this theorem he falls back on Riemann's proposition regarding $\underset{a=0}{\mathrm{L}}\{F(x+a)+F(x-a)-2F(x)\}/a = 0$. The proof of theorem (1) follows the lines of Dirichlet's proof, and is reproduced in greater detail in his *Kugelfunctionen*, vol. I., pp. 53 *et seq.*

§ 21. Heine's second theorem shows that there cannot be two different expansions of a function if these are to be (in general) uniformly convergent. Cantor has proved the more general theorem that even if uniform convergence be not demanded there can be but one convergent expansion in a trigonometric series and it is that of Fourier. Cantor's memoirs appear in Crelle's *Journal*, vol. LXXII., p. 130, *Über einen . . . Lehrsatz* and p. 139 *Beweis dass eine für jeden reelen Werth, etc.*, vol. LXXIII, p. 294, *Notiz zu dem Aufsatze: Beweis, etc.* In the first of these he proves that if two infinite series, a_1, a_2, etc., b_1, b_2, etc., are such that $\underset{n=\infty}{\mathrm{L}}(a_n \sin nx + b_n \cos nx) = 0$ where x is real and lies in a given interval a, b, then $\mathrm{L}a_n = 0$, $\mathrm{L}b_n = 0$ for

$n=\infty$. In the second memoir he takes the function $F(x)$ of Riemann, the conditions imposed on it being shown, by the proposition just stated, to be satisfied and forms the quotient $\{F(x+a)-2F(x)+F(x-a)\}/a^2$. This quotient is zero for $a=0$ and $F(x)$ is continuous; and it now follows by a theorem due to Schwarz that $F(x)$ must be a linear function of x. (Of course, if $F(x)$ be supposed to have continuous first and second derivatives, this theorem is evident). Giving to $F(x)$ a linear value and adopting the notation of Riemann, we have

$$\tfrac{1}{2}A_0x^2+C_1x+C_2=A_1+\tfrac{1}{4}A_2+\ldots\ldots+\frac{1}{n^2}A_n+\ldots\ldots$$

The right hand member being periodic, it follows that $A_0=0=C_1$ and then by multiplying by $\sin nx$ or $\cos nx$ and integrating between $-\pi$ and π (a process now allowable) it is seen that $a_n=0=b_n$ for every value of n. Hence a convergent trigonometric series can represent zero only if every coefficient is zero, from which the uniqueness of the trigonometric expansion at once follows.

In the third memoir quoted above he gives a simplified form of the proof, due to Kronecker, which dispenses with the necessity of the investigation of the first memoir. In an article, *Über die Ausdehnung eines Satzes, etc.*, *Math. Ann.*, vol. V., Cantor extended his theorem to functions having an infinite number of discontinuities, provided these be distributed in a particular way, but, unfortunately, I have not had access to this article.

§ 22. Du Bois-Reymond's name has occurred more than once incidentally in this paper, and one memoir of his has now to be briefly noticed. His contributions to the theory of series in general and of the Fourier series in particular have been both numerous and important, but I can hardly do more than give a brief notice of one memoir and a statement of some interesting results of a second. These two memoirs are very important, and they contain notices of the work of predecessors and full references to his own papers bearing on the subject; but a detailed analysis would carry me far beyond the limits of this paper.

The first of these two memoirs appears in the *Abhandlungen der Bayerischen Academie*, vol. XII. (1875) p. 117, *Beweis dass die Coeff. der trig. Reihe, etc.* He there proves that the coefficients of the series $f(x)=\sum_{p=0}^{p=\infty}(a_p\cos px+b_p\sin px)$ have the values

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha),\quad a_p=\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha)\cos p\alpha,\quad b_p=\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha)\sin p\alpha,$$

whenever these integrals are finite and determinate. This proposition includes of course the theorem that $f(x)$ can be expanded in only one way in a Fourier series. In the proof Riemann's theorems (I.) and (II.) and Schwarz's theorem, quoted above, play an important part Putting

$$F(x)=\tfrac{1}{2}a_0x^2-\sum_{n=1}^{n=\infty}\frac{1}{p^2}(a_p\cos px+b_p\sin px)\ \ldots\ \ldots\ (1),$$

and supposing, in the first place, $f(x)$ to be continuous, he seeks to express $F(x)$ by $f(x)$. For every value of x between $-\pi$ and π

$$\frac{d^2}{dx^2}\int_{-\pi}^{x} d\alpha\int_{-\pi}^{\alpha} d\beta . f(\beta)=f(x).$$

If $\Phi(x)=F(x)-\int_{-\pi}^{x} d\alpha\int_{-\pi}^{\alpha} d\beta f(\beta)$ it follows that $\underset{\epsilon=0}{L}\ \Delta^2\Phi/\epsilon^2=0$

where $\Delta^2\Phi=\Phi(x+\epsilon)-2\Phi(x)+\Phi(x-\epsilon)$, and therefore $\Phi(x)=c_0+c_1x$

and $F(x)=\int_{-\pi}^{x} d\alpha\int_{-\pi}^{\alpha} d\beta f(\beta)+c_0+c_1x=F_1(x)+c_0+c_1x$, suppose, (2).

Multiplying (1) by $\cos nx$, $\sin nx$ respectively and integrating between $-\pi$ and π we get

$$\int_{-\pi}^{\pi} F(\alpha)\cos n\alpha d\alpha=(-1)^n\frac{2\pi}{n^2}a_0-\frac{\pi}{n^2}a_n;\quad \int_{-\pi}^{\pi} F(\alpha)d\alpha=\frac{\pi^3}{3}a_0;$$

$$\int_{-\pi}^{\pi} F(\alpha)\sin n\alpha d\alpha=-\frac{\pi}{n^2}b_n.$$

Replacing $F(x)$ by its value given by (2), integrating by parts and noticing that a_n, b_n, $\int_{-\pi}^{\pi} d\alpha f(\alpha)\cos n\alpha$, $\int_{-\pi}^{\pi} d\alpha f(\alpha)\sin n\alpha$ vanish with $1/n$ he finds

$$c_0=\frac{1}{4\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha)\left\{\frac{\pi^2}{3}-(\pi-\alpha)^2\right\},\quad c_1=-\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha)(\pi-\alpha),$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha),\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha)\cos n\alpha,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha f(\alpha)\sin n\alpha$$

Du Bois-Reymond now, instead of supposing $f(x)$ to have discontinuities, proceeds to consider the case where $f(x)$ is supposed only to be integrable. In this case $\underset{\epsilon=0}{\mathrm{L}}\, \Delta^2\Phi/\epsilon^2$ is not, or at least is not provable to be, generally zero. He proves, however, that it follows from the integrability of $f(x)$ that $\Phi(x)$ is a linear function of x, but the proof is too long and complicated to be reproduced here. When once this point is established the reasoning is as before. He then considers the possibility of $f(x)$ having infinite values.

In the same volume of the *Bayerischen Abhandlungen, Zweite Abtheilung*, pp. 1–102, Du Bois-Reymond has a long article entitled *Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungs-Formeln.* The memoir forms rather laborious reading, but is, nevertheless, a very important contribution to the theory of the Fourier series. It is specially valuable on account of the thorough discussion of the Dirichlet integral $\underset{h=\infty}{\mathrm{L}} \int_0^b d\alpha f(\alpha) \frac{\sin h\alpha}{\alpha}$. By considering special forms of $f(\alpha)$ he succeeds in showing that there do exist continuous functions of x such that for special values of x the Fourier series does not converge. In the last chapter of his essay Sachse gives an example, due to Schwarz, of such a function; the example is included in Du Bois-Reymond's more general ones, but is simpler both in definition and in proof. In the *Comptes Rendus*, vol. XCII., p. 915 and p. 962, will be found a short statement by Du Bois-Reymond himself of his investigations on integrals of the form $\underset{h=\infty}{\mathrm{L}} \int_a^b f(x)\phi(x, h)dx$.

§ 23. The memoirs of Du Bois-Reymond may be said in a sense to include all the results of previous writers and to push the inquiry as to the nature of the functions which can be represented by a Fourier series when the co-efficients are determined as definite integrals very near its utmost limits. In what follows I will therefore refer chiefly to certain investigations on the Dirichlet integral, and to some articles which bear on the integrals and which tend to simplify proofs and to clear up one or two doubtful points. But before doing so I would specially recommend to any one who wishes to have in a compact form a thorough and rigorous treatment of the Fourier series in all its bearings the treatise by Ulisse

Dini, entitled *Serie di Fourier e altre Rappresentazioni analitiche delle Funzioni di una Variabile Reale* (Pisa, 1880). As the title indicates, the book contains much more than the Fourier series proper, and the whole treatment is carried through on a uniform plan and with scrupulous accuracy of statement. A careful reading of it is quite an education in some of the most delicate points of the integral calculus and of the theory of functions.

In the appendix to the second volume of his *Kugelfunctionen* (Berlin, 1881) Heine returns to the discussion of the Fourier series, and shows how, by a certain procedure, great simplification may be introduced into the mode of presenting Dirichlet's proof, which is apt to become rather tedious from the great number of different cases that have to be considered. In particular, the simplification affects the consideration of the uniform convergence of the series, and throws light on certain difficulties raised by Schläfli.

In some respects Heine's treatment in this appendix resembles that suggested by Jordan in a paper *Sur la Série de Fourier* (Comptes Rendus, 1881, vol. XCII., p. 228); for the decomposition of the function, as proposed by Heine, into the sum of functions which are either not increasing or not decreasing, secures the same end as Jordan obtains by his conception of *fonctions à oscillation limitée.* In his *Cours d' Analyse*, vol. II. (first edition) Jordan systematically uses the *fonction à oscillation (variation) limitée* in discussing the integrals of Dirichlet and Du Bois-Reymond, and thus simplifies the treatment considerably. In the paper just mentioned he gives a new condition for $F(x)$, such that

$$\underset{p=\infty}{L}\int_0^b F(x)\frac{\sin px}{x}dx=\frac{\pi}{2}F(0)$$

is still true.

Conditions including that of Jordan are developed in an article by O. Hölder, *über eine neue Bedingung, etc.* (Berliner Berichte, 1885, p. 419)

In the *Berliner Berichte* for the same year (p. 641) Kronecker has a memoir *Über die Dirichletsche Integral*, which is particularly noteworthy because of the variety of forms to which he reduces the conditions for the validity of the equation $\underset{n=\infty}{L}\int_0^h f(x)\frac{\sin nx}{x}dx=\frac{\pi}{2}f(0)$.

These conditions include those of Dirichlet, Lipschitz, Jordan, and Hölder. Kronecker thinks the variety of the results is due to his method of putting $f(x) = f_0(x) + f(0)$ so that $f_0(x)$ vanishes with x and using $f_0(x)$ in the integral.

I cannot conclude without calling attention to a remarkable memoir by Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen* (Berliner Ber., 1885, p. 633 and p. 789). He there proves the remarkable theorem that if $f(x)$ be a single-valued, continuous, and periodic function (x real), then, given an arbitrarily small positive magnitude g, a finite Fourier series can be formed in a variety of ways which is such that the difference between it and the function $f(x)$ does not exceed g for any value of x. Further, every such function $f(x)$ (period $= 2c$) may be represented as a sum whose terms are finite Fourier series with the period $2c$. This series converges absolutely for every value of x and uniformly in each finite interval.

In the *Comptes Rendus* for 1891, (vol. CXII., p. 183) Picard has proved the first theorem by using Poisson's integral. (*Sur la représentation approchée des fonctions*).

In the foregoing paper there are some points in connection with the Fourier series which I have not touched upon, and in particular the differentiability of the series. I have also avoided all reference to series other than the Fourier series strictly so called. To have taken up these points would have added considerably to the length of the paper, already perhaps too long. I would fain hope that no important contribution to the theory of the Fourier series has been altogether passed over, and that the paper may prove useful in directing attention to a most interesting side of mathematical theory.

Eighth Meeting, June 9, 1893.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

Japanese Arithmetic.

By Prof. C. G. KNOTT, D.Sc., F.R.S.E.

The paper was mainly an account of the abacus, as used in China and Japan. The instrument was shown, and the various operations of addition, subtraction, multiplication, division, and extraction of square and cube roots, were illustrated. The multiplication and division tables were fully described, the latter being especially interesting. The historic development of the abacus in the East was also touched upon. A full account of the Chinese and Japanese abacus will be found in a paper by the author, entitled, "The Abacus, in its Historic and Scientific Aspects," published in the *Transactions of the Asiatic Society of Japan* (vol. XIV., 1886). A copy of this paper is in the library of the Edinburgh Mathematical Society.

Revue Semestrielle des Publications Mathématiques:

Rédigée sous les Auspices de la Société Mathématique d' Amsterdam.

Tome I. (Première Partie): Amsterdam, W. Versluys, 1893, pp. 104.

The appearance of this *Review* is a significant indication of the enormous development of mathematical studies in recent years. Nearly every one who has attempted to keep himself abreast of mathematical research has been obliged sooner or later to recognise the practical impossibility of mastering the literature of every branch, and has resigned himself to a comparatively elementary study of the general subject while devoting his main energies to special departments. Even then, so numerous are the Societies that publish *Proceedings* and *Transactions*, and so varied are the Journals that are chiefly mathematical in their content, that it is no easy matter for the mathematician to get a knowledge of what is being done in any special field by workers outside (sometimes in) his own country. The need for a publication that will, without undue delay, furnish a conspectus of the literature of the subject is thus a very real one. That the need has been felt is sufficiently shown by the synopses of the contents of other Journals, given in such publications as Darboux's *Bulletin*, and more especially by the excellent *Jahrbuch über die Fortschritte der Mathematik*. The last completed issue of the *Jahrbuch*, that for 1889, extends to upwards of 1300 pages, while that for 1890, two parts of which have appeared, will evidently be as large. It is almost inevitable that a work of this magnitude should be a little late in appearing, and it is possibly in view of this fact that the Mathematical Society of Amsterdam lay stress on their intention to issue their notices of the various Journals "without any delay of importance" The first part of the first volume, a copy of which has been sent to our Society, contains a statement of the principles on which the *Review* is to be conducted, and it may be of interest to the members of the Society to have these presented in outline.

The object of the *Review* is to facilitate the study of the mathematical sciences by making known, without any delay of importance, the title and the principal contents of the mathematical memoirs published in the principal scientific Journals, and it is issued under the editorship of Messrs Schoute, Korteweg, Kapteyn, Kluyver, and Zeeman, who are assisted by a pretty large staff of contributors. In general, notices are to be given of memoirs on pure mathematics and mechanics, including hydrodynamics and the theory of elasticity, but excluding applied mechanics, mathematical physics, and astronomy. This selection seems to me a somewhat arbitrary one, but it may perhaps prove better in practice than the statement of it would suggest, since it is proposed to give the titles and the classification of memoirs on excluded subjects when these appear in Journals which are almost exclusively mathematical. All the same, it is not easy to understand why the line should be drawn at the theory of elasticity, which certainly offers no problems of greater mathematical interest than, for example, does the mathematical theory of electricity or even that of the conduction of heat.

As regards the notices of the memoirs the guiding principle is the following:—The title of the memoir will be preceded by one or more letters (notations) in accordance with the system of classification adopted by the International Congress on the Bibliography of the Mathematical Sciences; this will be followed by a very short abstract of the contents of the memoir, but in cases where the title and the letters of classification indicate sufficiently the contents of the paper no abstract will be added. The use of symbols of classification has many advantages and in large numbers of instances gives about as much information as a short abstract would do. Any one who has had experience in summarising mathematical papers knows the extreme difficulty, one might say the impossibility, of compressing into a few lines the substance of a really good article, and it is quite clear that in several instances the writers of the abstracts have felt themselves cramped in dealing with the important memoirs. The value of the *Review*, it seems to me, will be chiefly in the exhaustive list of titles of papers and in the system of classification adopted; the abstracts are in many instances admirably done, but even then do not furnish very much more information than the title and classification convey.

If the Society succeed in their intention of publishing the two parts which make up the annual volume at the times they propose, the *Review* should have a useful career before it; these are to appear on the first of January and the first of July respectively. The first part will contain an analysis of all works published between first March and first October of the preceding year, while the second will deal with those published between the latter date and the first March of the current year. It should be possible, I think, to keep to the dates proposed; any postponement would probably prove fatal to the success of the venture.*

It would seem that notices of text-books and independent works do not come within the scope of the *Review*. It would, I think, be

* I note, with pleasure, that the Second Part of Vol. I. has appeared at the promised date.

a valuable addition to have even the titles of the more important mathematical books that are annually published; they are often a considerable time in making their way outside the country that produces them.

The *Review* should prove very useful to all engaged in mathematical work. To every mathematician—and does any one deserve the name of mathematician who does not spend some portion of his leisure in reading original memoirs—this *Review* makes its appeal.

GEORGE A. GIBSON.

Edinburgh Mathematical Society.

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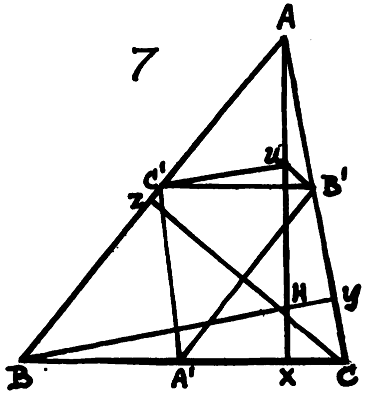

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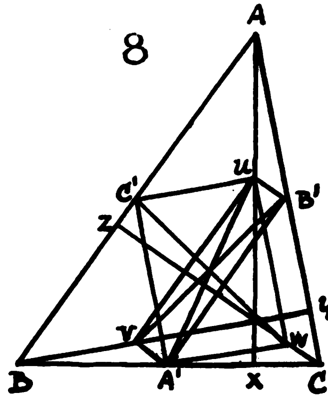

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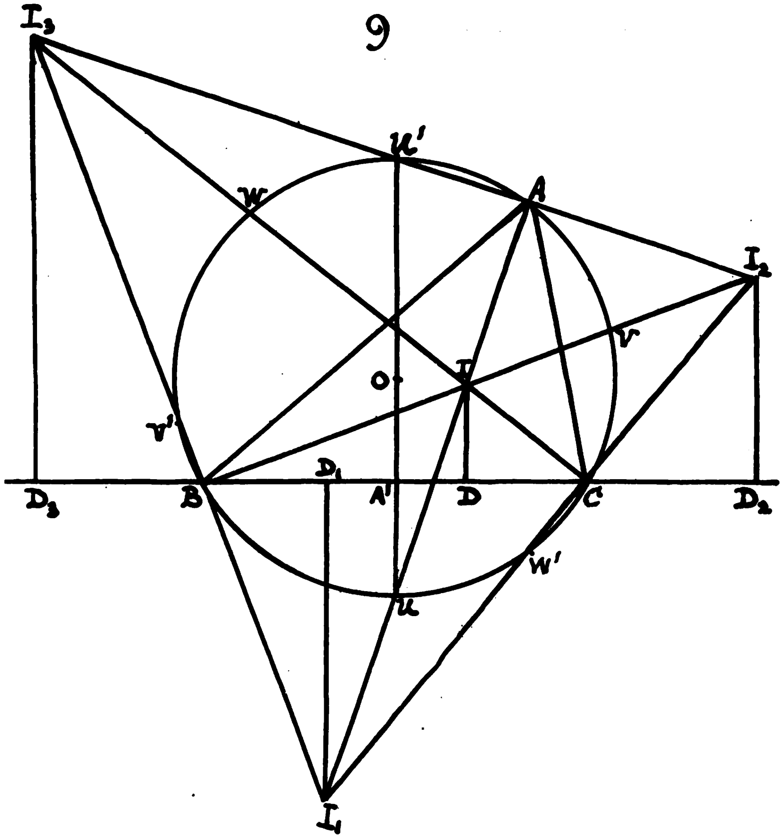

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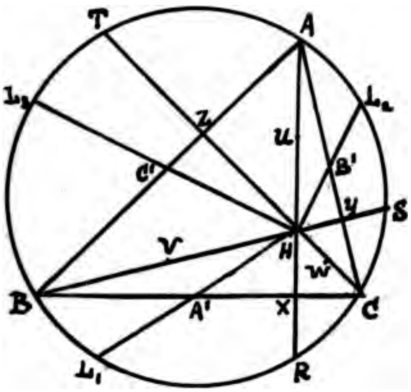
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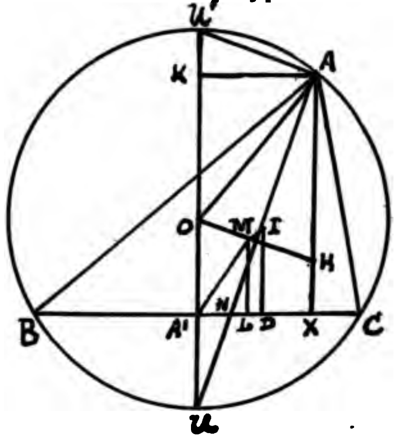
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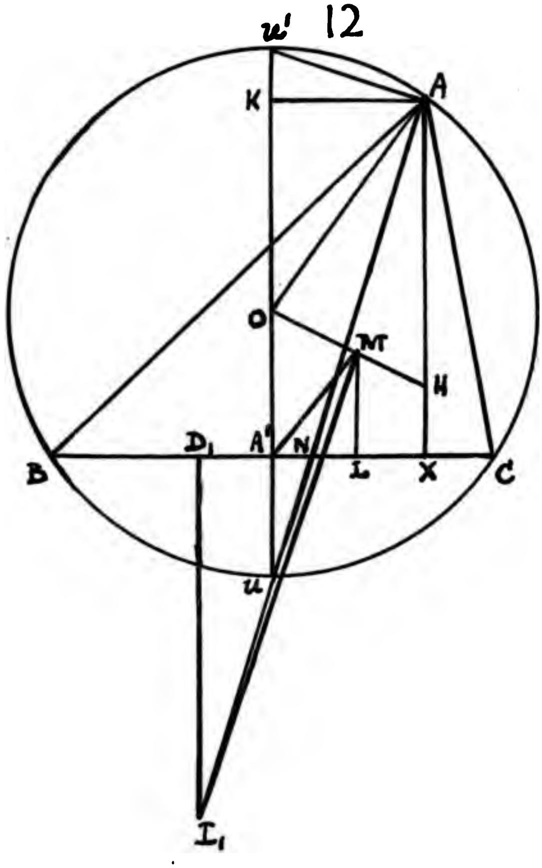
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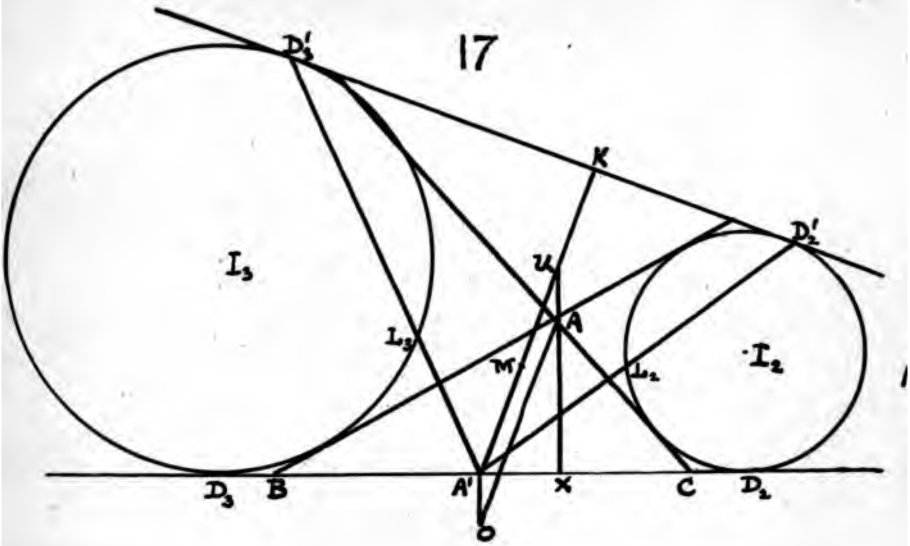

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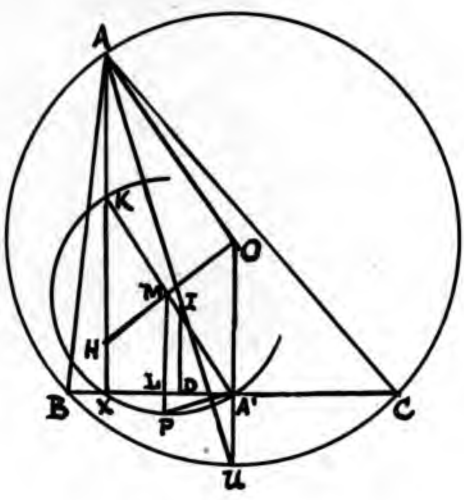

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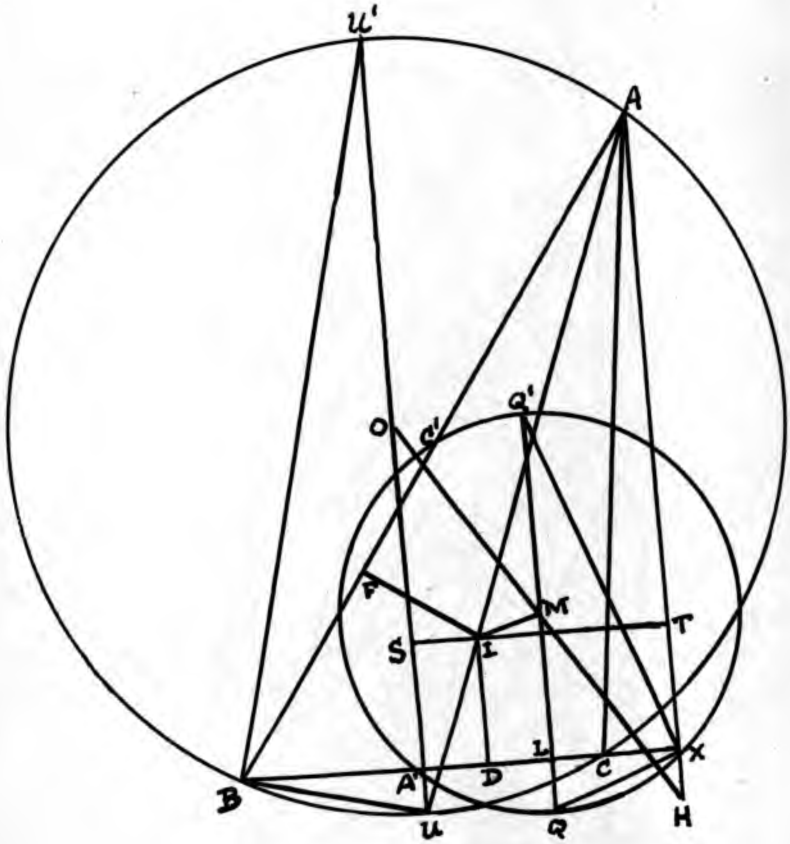

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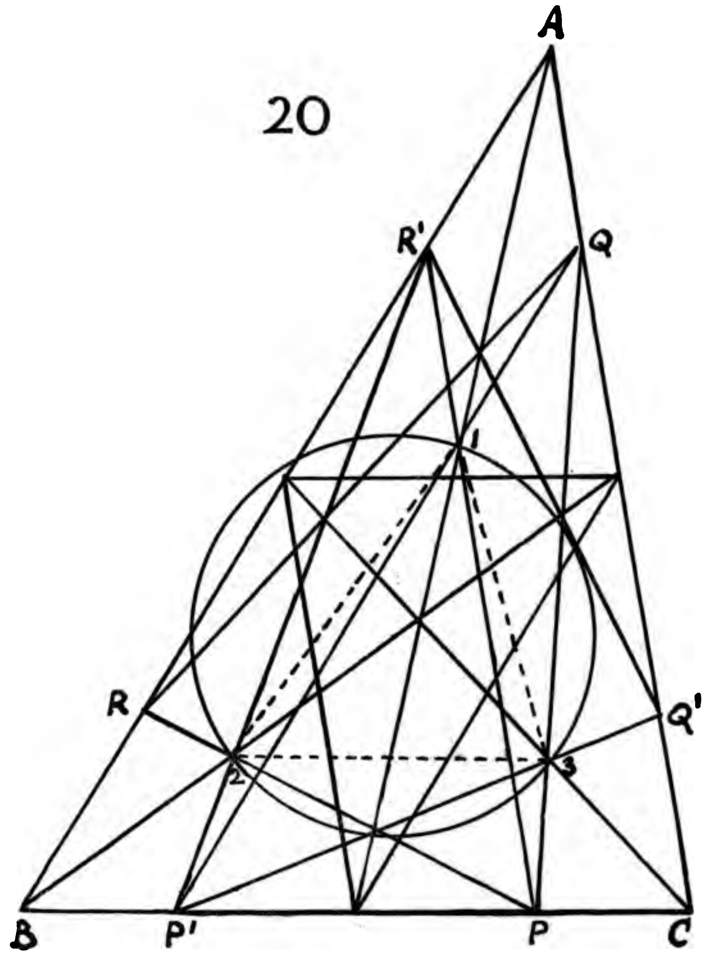

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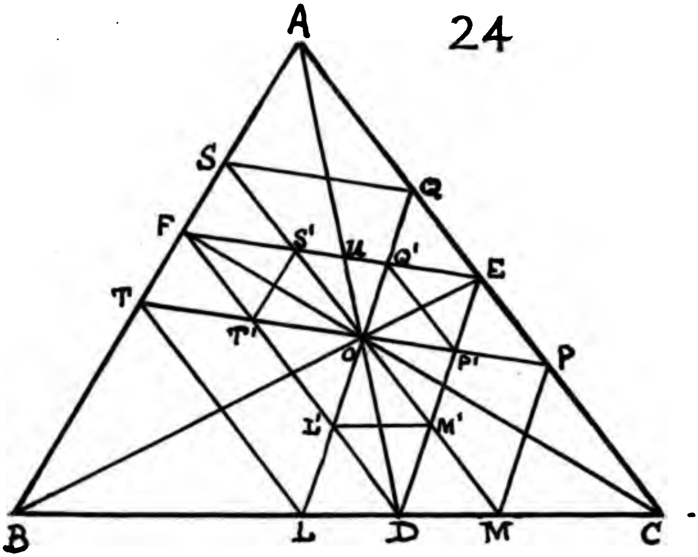

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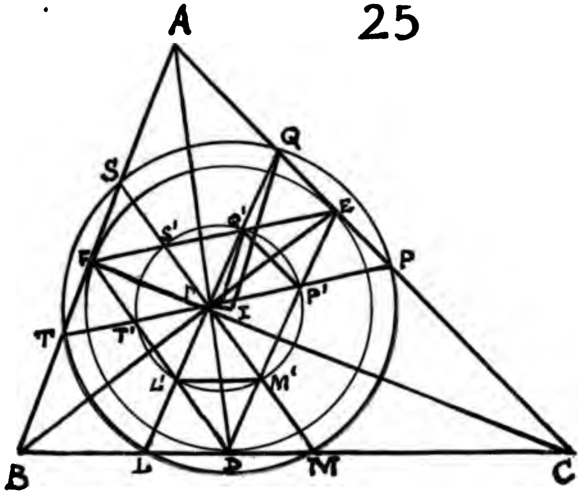

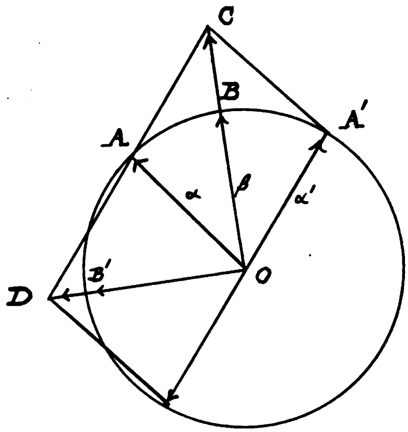

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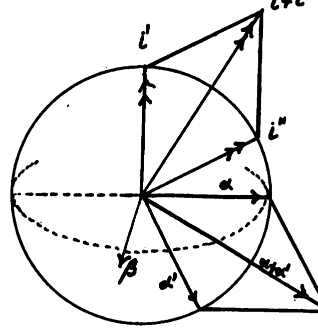

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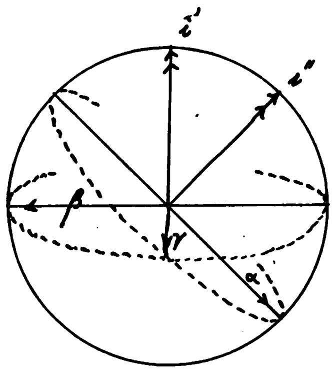

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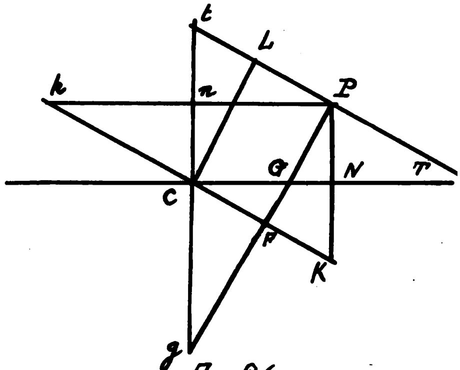

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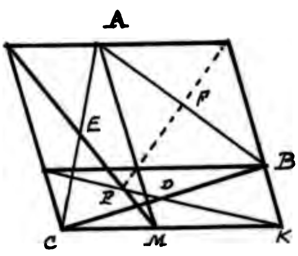

Fig. 27.

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OF THE

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VOLUME XII.

SESSION 1893-94.

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

TWELFTH SESSION, 1893-94.

First Meeting, November 10th, 1893.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

For this Session the following Office-bearers were elected:—

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The Geometrography of Euclid's Problems.

By J. S. MACKAY, M.A., LL.D.

The term Geometrography is new to mathematical science, and it may be defined, in the words of its inventor, as "the art of geometrical constructions."

Certain constructions are, it is well known, simpler than certain others, but in many cases the simplicity of a construction does not consist in the practical execution, but in the brevity of the statement, of what has to be done. Can then any criterion be laid down by which an estimate may be formed of the relative simplicity of several different constructions for attaining the same end?

This is the question which Mr Émile Lemoine put to himself some years ago, and which he very ingeniously answered in a memoir read at the Oran meeting (1888) of the French Association for the Advancement of the Sciences. Mr Lemoine has since returned to the subject, and his maturer views will be found in another memoir read at the Pau meeting (1892) of the same Association. The object of the present paper is to give an account of Mr Lemoine's method of estimation, to suggest a slight modification of it, and to apply it to the problems contained in the first six books of Euclid's *Elements*.

In the first place Mr Lemoine restricts himself, as Euclid does, to constructions executed with the ruler and the compasses, and these he divides into the following elementary operations:

To place the edge of the ruler in coincidence with a point	R_1
To draw a straight line	R_2
To put a point of the compasses on a determinate point	C_1
To put a point of the compasses on an indeterminate point of a line	C_2
To describe a circle	C_3

No account is taken of the length of the lines that are described; if any portion of a straight line be drawn the operation is R_2, if a small arc only or the whole circumference be described, the operation is C_3.

It ought also to be added that to place the edge of the ruler in coincidence with two points is $2R_1$; to put one point of the compasses on a determinate point and the other point of the compasses on another determinate point is $2C_1$.

Every construction therefore is finally represented by

$$l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2 + m_3C_3$$

where l_1, m_1, etc., are coefficients denoting the number of times any particular operation is performed.

The number $(l_1 + l_2 + m_1 + m_2 + m_3)$ is called the *coefficient of simplicity*, or more shortly, *the simplicity of the construction;* it denotes the total number of operations. The number $l_1 + m_1 + m_2$ is called the *coefficient of exactitude*, or more shortly, the *exactitude of the construction**; it denotes the number of preparatory operations, on which and not on the tracing operations, the exactitude of the construction depends. The number of straight lines drawn is l_2; the number of circles m_3.

An objection at once presents itself to the reader, as it did to Mr Lemoine. Is it legitimate to suppose the operations R_1, R_2, C_1, C_2, C_3 identical in value, in order to make up the coefficient of simplicity or exactitude? They are evidently not identical in execution, and hence Geometrography does not furnish us with an *absolute* measure of simplicity or exactitude in the sense in which measure is usually employed, the comparison of one magnitude with a unit of the same kind. The various operations however are assimilated because they are incapable of decomposition into others more simple, and because, speculatively, any one is neither more simple nor less simple than another.

In one case it may be said that Geometrography does furnish an absolute measure, the case namely when all the coefficients in one construction are smaller than the respective coefficients in the other. This case occurs pretty frequently.

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* Mr Lemoine remarks that the simplicity and the exactitude of an operation vary inversely as the numbers he sums; but since no confusion is possible, he prefers names recalling the object aimed at to the more logical terms *coefficient of complication* and *coefficient of inexactitude.*

Such is Mr Lemoine's scheme of comparison, which he applies to more than sixty of the principal problems of elementary geometry, with some very unexpected results.

To justify his procedure in denoting by $2R_1$ and $2C_1$ the operations of placing the edge of the ruler and the two points of the compasses in coincidence with two given points, Mr Lemoine says in a note on the problem

To take with the compasses a given length AB :

"It is clear that the operation of putting the first point of the compasses on A is not the same as that of keeping the first point on A and placing the second on B; and yet we denote them both by C_1. We believe that there is no inconvenience in that, because we are only making an *ideal* theory of operations. Thus we suppose that all the lines of the figure intersect within the limits of the drawing, that it is indifferent whether these lines intersect at a very acute angle, and so on; so that it appears to us quite sufficient to denote by the symbol C_1 the general operation which consists in putting *one* of the points of the compasses on *one* point. The reader however who, after reflection, does not share our opinion has only to denote by C_1' the operation which consists in putting on a given point the movable point of the compasses while the other is kept fixed.

"In like manner, since we call R_1 the operation which consists in putting the edge of the ruler in contact with a point, it is evident from the manner in which it is performed that the operation which consists in putting the edge of the ruler in coincidence with two given points is not exactly twice the operation R_1. One might also denote by $R_1 + R_1'$ the operation which consists in placing the edge of the ruler in contact with two points; but if one practises Geometrography a little I believe he will come to recognise that this distinction is a useless complication.

"We might also have assimilated the operations C_1 and C_2 and have kept for the two only one symbol C_1; but we have not done so, because if *theoretically* R_1 and R_1' come to the same thing, C_1 and C_2 are *theoretically* different. C_2 however occurs much more rarely than the other symbols, and in general with a very small coefficient."

I am not sure that I understand in what respect the practical operations C_1 and C_2 are theoretically different, unless it be that in performing C_2 there is one degree more of freedom than in performing C_1. But this is so also in the case of the two operations denoted by $2R_1$; for the ruler can be placed in coincidence with one point by a motion either of translation or of rotation or of both, while it can be placed in coincidence with the other point, the coincidence with the first being maintained, only by rotation.

In the case of the two operations denoted by $2C_1$ it is clear also that there is less freedom in placing the second point of the compasses than there is in placing the first, and hence if, for the sake of convenience, two operations which are not precisely identical may be denoted by the reduplication of the same symbol, there does not seem to be any imperative reason why the operations C_1 and C_2 should not be regarded as equivalent. The fact also that in estimating the simplicity and exactitude of constructions the symbol C_2 rarely occurs, and the manifest advantage of having only four units instead of five have induced me to propose the following modification of Mr Lemoine's scheme:

To place the edge of the ruler in coincidence with one point	R_1
To place the edge of the ruler in coincidence with two points	$2R_1$
To draw a straight line	R_2
To put one point of the compasses on a determinate point	C_1
To put the points of the compasses on two determinate points	$2C_1$
To describe a circle	C_2

On another matter (of small importance) I have ventured to differ from Mr Lemoine.

Given A, B, C, the three vertices of a triangle, to construct the triangle.

Mr Lemoine estimates this operation as $6R_1 + 3R_2$. I have estimated it as $4R_1 + 3R_2$. To put the ruler in contact with A, B is $2R_1$; to draw AB is R_2. Now as the ruler is in contact with B, I estimate the putting of it in contact with B and C as only an additional R_1; to draw BC is R_2. Again as the ruler is in contact with C, I estimate the putting of it in contact with C and A as another R_1; to draw CA is R_2—in all, $4R_1+3R_2$.

The following remarks are extracted from one of Mr Lemoine's letters:

Geometrography may be divided into several branches.

(1) That of the canonical geometry of the straight line and the circle, the only instruments being the ruler and the compasses.

(2) Add the carpenter's square, with two new symbols. This branch may be applied especially to descriptive geometry.

(3) Add graduated rulers, for application to graphical statics.

(4) The geometrography of the ruler alone.

(5) The geometrography of the compasses alone.*

A sub-section may be made of the geometrography of the ruler and one single opening of the compasses.†

In what follows, the notation I. 1, etc., denotes Euclid's *Elements*, Book *First*, Proposition *First*, etc. It will be seen that, except in the fourth book, Euclid does not group his problems together.

I. 1.

To describe an equilateral triangle on a given finite straight line.

$$3R_1+2R_2+3C_1+2C_2$$

Simplicity 10; exactitude 6; lines 2; circles 2.

I. 2.

From a given point to draw a straight line equal to a given straight line.

$$5R_1+3R_2+7C_1+4C_2$$

The problem may be solved with much less complication, namely,

$$R_1+R_2+3C_1+C_2$$

I. 3.

From the greater of two given straight lines to cut off a part equal to the less.

$$5R_1+3R_2+9C_1+5C_2$$

* See Mascheroni's *Geometria del compasso* (1795).

† See *Proceedings of the Edinburgh Mathematical Society*, V. 2-22 (1887).

The problem may be solved with much less complication, namely,

$$3C_1 + C_2$$

I. 9.

To bisect a given rectilineal angle.

$$2R_1 + R_2 + 4C_1 + 3C_2$$

In this estimate the operations for drawing the sides of the equilateral triangle which occurs in Euclid's construction are omitted. The construction may be effected by

$$2R_1 + R_2 + 3C_1 + 3C_2$$

I. 10.

To bisect a given finite straight line

$$2R_1 + R_2 + 3C_1 + 2C_2.$$

Some of Euclid's operations are not counted, as they are needed only for the demonstration. The construction may be effected by

$$2R_1 + R_2 + 2C_1 + 2C_2$$

I. 11.

To draw a straight line perpendicular to a given straight line from a given point in the same.

$$2R_1 + R_2 + 4C_1 + 3C_2$$

The construction may be effected by

$$2R_1 + R_2 + 3C_1 + 3C_2$$

Or thus:

FIGURE 1.

Let AB be the straight line, C the point in it.

Take any point D outside AB; with D as centre and DC as radius describe a circle cutting AB again at E.

Join ED, and produce it to meet the circle at F; join FC.

$$3R_1 + 2R_2 + C_1 + C_2$$

I. 12.

To draw a straight line perpendicular to a given straight line from a given point outside it.

$$2R_1 + R_2 + 5C_1 + 3C_2$$

From the way in which Euclid describes his construction, the formula for it would be

$$4R_1 + 2R_2 + 5C_1 + 3C_2$$

But if the construction be fully carried out it will be seen that the drawing of the final straight line is unnecessary. Hence the formula is as first stated.

The construction may be effected by

$$2R_1 + R_2 + 3C_1 + 3C_2$$

Or thus:

FIGURE 2.

Let AB be the straight line, C the point outside it.

Take any point D in AB; with D as centre and DC as radius describe a circle cutting AB at E.
With E as centre and EC as radius describe a circle cutting the previous one again at F; join FC.

$$2R_1 + R_2 + 4C_1 + 2C_2$$

I. 22.

To make a triangle the sides of which shall be equal to three given straight lines.

Euclid does not use any of the given straight lines as a side of the triangle.

$$3R_1 + 3R_2 + 9C_1 + 4C_2$$

I. 23.

At a given point in a given straight line to make an angle equal to a given angle.

$$2R_1 + R_2 + 9C_1 + 3C_2$$

The construction may be effected by

$$2R_1 + R_2 + 5C_1 + 3C_2$$

I. 31.

Through a given point to draw a straight line parallel to a given straight line.

$$3R_1 + 2R_2 + 9C_1 + 3C_2$$

The construction is frequently effected by

$$2R_1 + R_2 + 5C_1 + 3C_2$$

The following method is due to Mr Gaston Tarry.

FIGURE 3.

Let A be the given point, BC the given straight line.

Draw any circle passing through A and cutting BC at D and E. With E as centre and radius AD describe a circle to cut the previous one at F. Join AF.

$$2R_1 + R_2 + 4C_1 + 2C_2$$

I. 42.

To describe a parallelogram that shall be equal to a given triangle and have one of its angles equal to a given angle.

Euclid constructs his parallelogram on the half of one of the sides of the triangle.

$$10R_1 + 6R_2 + 30C_1 + 11C_2$$

The construction may be effected by

$$8R_1 + 4R_2 + 15C_1 + 9C_2$$

I. 44.

To a given straight line to apply a parallelogram which shall be equal to a given triangle and have one of its angles equal to a given angle.

$$28R_1 + 17R_2 + 81C_1 + 28C_2$$

I. 45.

To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given angle.

Euclid takes a quadrilateral for the given rectilineal figure.

$$40R_1 + 24R_2 + 111C_1 + 39C_2$$

I. 46.

To describe a square on a given straight line.

$$10R_1 + 6R_2 + 24C_1 + 10C_2$$

The construction may be effected by

$$6R_1 + 3R_2 + 7C_1 + 5C_2$$

II. 11.

To divide a given straight line in medial section.

$$6R_1 + 3R_2 + 12C_1 + 7C_2$$

I have left out several of Euclid's operations, as they are necessary only for the demonstration.

If the given straight line AB be denoted by 2, the greater segment of it will be denoted by $\sqrt{5} - 1$. Hence to obtain the required section of AB, a geometrical construction for $\sqrt{5}$ must be found. This geometrical construction can be found from a right-angled triangle whose sides containing the right angle are 2 and 1 (Euclid's method). It may also be found from a right-angled triangle whose hypotenuse is 3 and one of its sides 2.

The following method (which in substance has been long known) depends upon the second construction for $\sqrt{5}$, and was communicated to me by Mr Lemoine, to whom it had been sent by Mr Bernès. Mr Bernès remarked that he would probably not have discovered it without the aid of Geometrography, or that if he had, he would have attached no special importance to it. And yet it is the simplest of all the solutions yet discovered.

FIGURE 4.

Produce BA, the given straight line.
With centre A and radius AB describe a circle cutting BA produced at C. With centre C and the same radius describe a circle cutting the previous circle in D, D′.
Join DD′, cutting AC in E.
With centre E and radius AB cut DD′ in F. With centre F and radius EB describe a circle cutting BA in G, and BA produced in G′. These are the required points of internal and external section.

$$4R_1 + 2R_2 + 7C_1 + 4C_2$$

For other solutions see the *Proceedings of the Edinburgh Mathematical Society*, IV. 60 (1886), and Mr Lemoine's memoir of 1892, already cited.

II. 14.

To describe a square equal to a given rectilineal figure.

Euclid describes a rectangle equal to the given rectilineal figure, which can be done by the extremely complicated construction of I. 45; and then finds the side of a square equal to the rectangle. This latter process he performs by

$$6R_1 + 3R_2 + 7C_1 + 4C_2$$

III. 1.

To find the centre of a given circle.

$$4R_1 + 3R_2 + 6C_1 + 4C_2.$$

The following solution, due to J. H. Swale of Liverpool (1830), is probably the simplest yet discovered.

FIGURE 5.

Take any point P on the given circumference, and with P as centre describe a circle ABC cutting the given circle at A and B. In this circle place the chord BC equal to BP; and join AC cutting the given circumference in D. Then BD or CD is the radius of the given circle.

$$2R_1 + R_2 + 5C_1 + 4C_2$$

III. 17.

To draw a tangent to a circle from an external point.

Euclid begins by finding the centre of the circle. I shall suppose the centre to be given.

$$8R_1 + 4R_2 + 6C_1 + 4C_2$$

To draw the two tangents, there would be required

$$10R_1 + 6R_2 + 6C_1 + 4C_2$$

A common solution is to join the external point to the centre of the circle, and on this line as diameter to describe a circle. This, giving the two tangents, is effected by

$$7R_1 + 4R_2 + 4C_1 + 3C_2$$

If the ruler alone is used, the two tangents can be obtained by

$$14R_1 + 10R_2$$

III. 25.

A segment of a circle being given, to describe the circle of which it is the segment.

$$6R_1 + 3R_2 + 14C_1 + 6C_2$$

The construction can be effected by

$$5R_1 + 3R_2 + 6C_1 + 5C_2$$

III. 30.

To bisect a given arc of a circle.

$$4R_1 + 2R_2 + 3C_1 + 2C_2$$

The construction may be effected by

$$2R_1 + R_2 + 2C_1 + 2C_2$$

III. 33.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.

$$6R_1 + 3R_2 + 18C_1 + 9C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 11C_1 + 6C_2$$

III. 34.

From a given circle to cut off a segment containing an angle equal to a given angle.

I shall suppose the centre of the given circle to be known.

$$6R_1 + 3R_2 + 13C_1 + 6C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 8C_1 + 4C_2$$

IV. 1.

In a given circle to place a chord of given length.

$$3R_1 + 2R_2 + 3C_1 + C_2$$

The construction may be effected by

$$2R_1 + R_2 + 3C_1 + C_2$$

IV. 2.

In a given circle to inscribe a triangle equiangular to a given triangle.

I shall suppose the centre of the circle to be known.

$$10R_1 + 4R_2 + 22C_1 + 9C_2$$

The construction may be effected by

$$9R_1 + 5R_2 + 10C_1 + 6C_2$$

IV. 3.

About a given circle to circumscribe a triangle equiangular to a given triangle.

$$14R_1 + 7R_2 + 30C_1 + 15C_2$$

The construction may be effected by

$$10R_1 + 7R_2 + 12C_1 + 8C_2$$

IV. 4.

To inscribe a circle in a given triangle.

$$6R_1 + 3R_2 + 14C_1 + 10C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 11C_1 + 6C_2$$

IV. 5.

To circumscribe a circle about a given triangle.

$$4R_1 + 2R_2 + 8C_1 + 5C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 5C_1 + 4C_2$$

IV. 6.

To inscribe a square in a given circle.

$$8R_1 + 6R_2 + 3C_1 + 2C_2$$

IV. 7.

To circumscribe a square about a given circle.

$$11R_1 + 6R_2 + 19C_1 + 14C_2$$

IV. 8.

To inscribe a circle in a given square.

$$10R_1 + 6R_2 + 26C_1 + 11C_2$$

IV. 9.

To circumscribe a circle about a given square.

$$4R_1 + 2R_2 + 2C_1 + C_2$$

IV. 10.

To describe an isosceles triangle having each of the base angles double of the vertical angle.

$$9R_1 + 6R_2 + 17C_1 + 9C_2$$

IV. 11.

To inscribe a regular pentagon in a given circle.

$$28R_1 + 16R_2 + 47C_1 + 24C_2$$

The following construction, given in the first book of Ptolemy's *Almagest*, is much simpler than Euclid's.

FIGURE 6.

Draw AB any diameter of the given circle. From the centre C draw CD perpendicular to AB and meeting the circumference at D. Bisect AC at E; and from EB cut off EF equal to ED. Then DF is a side of the inscribed regular pentagon.

$$11R_1 + 8R_2 + 11C_1 + 9C_2$$

IV. 12.

To circumscribe a regular pentagon about a given circle.

$$43R_1 + 22R_2 + 67C_1 + 39C_2$$

IV. 13.

To inscribe a circle in a regular pentagon.

$$6R_1 + 3R_2 + 12C_1 + 8C_2$$

IV. 14.

To circumscribe a circle about a regular pentagon.

$$4R_1 + 2R_2 + 7C_1 + 5C_2$$

IV. 15.

To inscribe a regular hexagon in a given circle.

Euclid states as a corollary to this problem that the side of the regular hexagon is equal to the radius of the circle. Hence his construction, if he were not concerned with demonstration, would be as simple as possible.

IV. 16.

To inscribe in a circle a regular figure of fifteen sides.

It does not seem worth while to evaluate the simplicity of Euclid's solution of this problem. The solution depends on the inscription of a regular pentagon in the circle, and Euclid's construction for this is more than twice as complicated as it need be.

V.

The propositions in Euclid's fifth book are all theorems.

VI. 9.

From a given straight line to cut off any aliquot (n^{th}) part.

On the supposition that all the points of division are to be marked on the auxiliary line, the compasses being lifted from the paper each time, the result is

$$6R_1 + 4R_2 + (n+9)C_1 + (n+3)C_2$$

VI. 10.

To divide a given straight line similarly to a given divided straight line.

Euclid's given divided straight line consists of three consecutive segments.

$$9R_1 + 6R_2 + 27C_1 + 9C_2$$

VI. 11.

To find a third proportional to two given straight lines.

Euclid's two given straight lines are drawn from the same point, and the third proportional is found on one of them.

$$8R_1 + 5R_2 + 12C_1 + 4C_2$$

VI. 12.

To find a fourth proportional to three given straight lines.

$$5R_1 + 5R_2 + 18C_1 + 6C_2$$

VI. 13.

To find a mean proportional between two given straight lines.

Euclid's two given straight lines are placed contiguous to each other and in the same straight line.

$$4R_1 + 2R_2 + 9C_1 + 6C_2$$

If the lengths of the two given straight lines had to be measured off on another straight line, the result would be

$$4R_1 + 3R_2 + 15C_1 + 8C_2$$

The following is the simplest solution yet obtained.

FIGURE 7.

Let M, N be the two given straight lines, M being greater than N.

Draw any straight line AB, and with A as centre and M as radius describe a circle cutting AB in B. With B as centre and N as radius describe a circle cutting BA, between A and B, at C; with C as centre and N as radius describe a circle cutting the second circle in D and E. Join DE and let it cut the first circle at F. BF is the mean proportional.

$$2R_1 + 2R_2 + 7C_1 + 3C_2$$

If BF be drawn, the result is

$$4R_1 + 3R_2 + 7C_1 + 3C_2$$

This solution is practically identical with that communicated in 1684 by Thomas Strode to Dr John Wallis of Oxford. See Wallis's *Treatise of Algebra, Additions and Emendations*, p. 164 (1685), or his *Opera Mathematica*, I. 301 (1695).

It does not seem worth while to consider the remaining problems of the sixth book. Euclid's construction of VI. 18,

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure,

is as simple as possible; his construction of VI. 25,

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure,

depends on I. 45, and is therefore unnecessarily complicated. The problems VI. 28, 29 have now no practical but only a historical interest, and VI. 30 is merely II. 11 over again.

Two Circular Notes.

By R. Tucker, M.A.

I.

1. In Fig. 8, take

$$\angle \mathrm{BFD} = \theta = \angle \mathrm{CDE} = \angle \mathrm{AEF},$$

and denote DE, EF, FD by x, y, z respectively; then

$$\begin{array}{llll} x\sin \mathrm{B}\sin(\mathrm{C}+\theta) & & +z\sin \mathrm{C}\sin\theta = a\sin \mathrm{B}\sin \mathrm{C} & = \mathrm{P}, \\ x\sin \mathrm{A}\sin\theta & +y\sin \mathrm{C}\sin(\mathrm{A}+\theta) & & = \mathrm{P}, \\ & y\sin \mathrm{B}\sin\theta & + z\sin \mathrm{A}\sin(\mathrm{B}+\theta) & = \mathrm{P}. \end{array}$$

From these equations

$$x[\sin^3\theta + \sin(\mathrm{A}+\theta)\sin(\mathrm{B}+\theta)\sin(\mathrm{C}+\theta)]$$
$$= x\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}\sin(\theta+\omega)/\sin\omega = \mathrm{P}\sin \mathrm{A},$$

i.e., $$x = a\sin\omega/\sin(\theta+\omega).$$

Hence $\triangle$DEF is similar to ABC, and the modulus of similarity is $\sin\omega/\sin(\theta+\omega)$.

2. The preceding result is got, in an elegant geometrical manner, in Milne's *Companion* (Simmons, Cap. V., the Tucker Circles, pp. 127–137); *c.f.* also H. M. Taylor, On the relations of the intersections of a circle with a triangle, L.Math. Soc. *Proceedings*, vol. XV., pp. 122–139.

3. We readily deduce the equation to the circle DEF to be

$$(\lambda^2 \equiv a^2b^2 + b^2c^2 + c^2a^2)$$

$$\lambda^2\sin^2(\theta+\omega)\,.\,\Sigma(a\beta\gamma) = \Sigma(a\alpha)\,.\,\Sigma\{b^2c^2\sin(\mathrm{A}+\theta)\,.\,\alpha\}\,.\,\sin\theta. \quad \text{(i.)}$$

This circle cuts the sides BC, CA, AB, in points D′, E′, F′ such that $\angle \mathrm{AF'E'} = \theta = \angle \mathrm{BD'F'} = \angle \mathrm{CE'D'}$.

4. The equation to this group of circles is given in the form

$$(\alpha - \mathrm{K}a)(\beta - \mathrm{K}b)(\gamma - \mathrm{K}c) - \alpha\beta\gamma = 0.$$

(Simmons l.c. p. 136, and Casey, *Conics* 2nd edition, p. 421. In the 1st edition the form was utterly wrong.)

5. For the "T.R." circle, $\theta = \omega$, and (i.) becomes, since

$$\sin(A+\omega) = (b^3 + c^2)\sin A/\lambda,$$
$$K^2\Sigma(\alpha\beta\gamma) = \Sigma(a\alpha) \,.\, \Sigma\{bc(b^2+c^2)\alpha\} ;$$

which is a neater form than either of the forms given by me in Quar. Jour. of Math. (The "Triplicate-Ratio" Circle, vol. XIX., p. 346).

6. For the "Cosine"-Circle, $\theta = \frac{\pi}{2}$, and (i.) becomes

$$K^2\Sigma(\alpha\beta\gamma) = 4\Sigma(a\alpha) \,.\, \Sigma\{b^2c^2\cos A \,.\, \alpha\} .$$

II.

1. In Fig. 9 I connect any point O with the vertices A, B, C of the triangle and then bisect the angles BOC, COA, AOB by lines meeting BC, CA, AB in D, E, F.

I propose to consider some properties of the group of triangles DEF.

2. For AO, BO, CO write l, m, n; and denote the triangles BOC, COA, AOB by δ_a, δ_b, δ_c, then the trilinear co-ordinates of O are

$$2\delta_a/a, \quad 2\delta_b/b, \quad 2\delta_c/c .$$

3. From Euc. vi. 3

AD, BE, CF cointersect in a point, P, and the points D, E, F are given by

$$\left.\begin{array}{llll} D, & o & cn & bm \\ E, & cn & o & al \\ F, & bm & al & o \end{array}\right\} .$$

Hence the equations to

$$\left.\begin{array}{l} AD, \\ BE, \\ CP, \end{array}\right\} \text{ are } \begin{array}{l} bm\beta = cn\gamma \,, \\ cn\gamma = al\alpha \,, \\ al\alpha = bm\beta \,, \end{array} \left\{ \begin{array}{c} \text{and the point P is} \\ \\ al\alpha = bm\beta = cn\gamma . \end{array}\right. \qquad \text{(i.)}$$

4. Now
$$\frac{\text{BD}}{m}=\frac{\text{CD}}{n}=\frac{a}{m+n},$$
$$\frac{\text{CE}}{n}=\frac{\text{AE}}{l}=\frac{b}{n+l},$$
$$\frac{\text{AF}}{l}=\frac{\text{BF}}{m}=\frac{c}{l+m};$$

and therefore
$$2\triangle\text{BDF}=\text{BD}\,.\,\text{BF}\,.\,\sin\text{B}$$
$$=m^2\,.\,2\triangle/(m+n\,.\,l+m);$$
$$\therefore\ \triangle\text{DEF}=\triangle\,.\,2lmn/(l+m\,.\,m+n\,.\,n+l). \qquad \text{(ii.)}$$

5. The equation to the circle DEF is

$$2abc\,.\,l+m\,.\,m+n\,.\,n+l\,.\,\Sigma(a\beta\gamma)= \qquad \text{(iii.)}$$
$$\Sigma(a\alpha)\,.\,\Sigma[al\,.\,\alpha\{b^2\,.\,l+m\,.\,m+n+c^2\,.\,m+n\,.\,n+l-a^2\,.\,n+l\,.\,l+m\}].$$

6. The equations to DE, EF, FD are

$$\left.\begin{array}{r} al\alpha+bm\beta-cn\gamma=0, \\ -al\alpha+bm\beta+cn\gamma=0, \\ al\alpha-bm\beta+cn\gamma=0. \end{array}\right\} \qquad \text{(iv.)}$$

7. If D′, E′, F′ are the Harmonic Conjugates to D, E, F respectively, we have

$$\frac{\text{BD}'}{m}=\frac{\text{CD}'}{n}=\frac{a}{m-n},\ \text{etc.},$$

and the points

$$\left.\begin{array}{l}\text{D}'\\ \text{E}'\\ \text{F}'\end{array}\right\}\ \text{are given by}\ \begin{vmatrix} o & -cn & bm \\ cn & o & -al \\ -bm & al & o \end{vmatrix}.$$

Hence the line D′E′F′, the axis of Perspective of DEF and ABC, has for its equation

$$al\alpha+bm\beta+cn\gamma=0.$$

8. If the ratio $l:m:n$ is given, O is, of course, determined by the intersections of the circles on DD′, EE′, FF′ as diameters.

9. The equations to the circles on DD′, FF′, referred to BC, BA as axes are

$$(m^2-n^2)(x^2+y^2+2xy\cos B)-2m^2ax-2m^2ay\cos B+m^2a^2=0\,,$$

$$(m^2-l^2)(x^2+y^2+2xy\cos B)-2m^2cx\cos B-2m^2cy+m^2c^2=0\,.\quad\text{(v.)}$$

10. The circle (iii.) cuts BC again in d, so that

$$\left.\begin{aligned}&\frac{Bd}{c^2(m+n)(n+l)+a^2(n+l)(l+m)-b^2(l+m)(m+n)}\\&=\frac{Cd}{a^2(n+l)(l+m)+b^2(l+m)(m+n)-c^2(m+n)(n+l)}\end{aligned}\right\}.$$

11. If $l=m=n$, O is the circumcentre, P the centroid, and (iii.) the nine-point circle.

If O is Ω, $\qquad l:m:n=cb^2:ac^2:ba^2$,
and P is $\qquad b\alpha=c\beta=a\gamma$;
and if O is Ω' $\qquad l:m:n=c^2b:a^2c:b^2a$,
and P is $\qquad c\alpha=a\beta=b\gamma$.

12. If P is Ω, then $l:m:n=ca:ab:bc$,
and if P is Ω', then $l:m:n=ab:bc:ca$.

13. If P is the circumcentre, then

$$l:m:n=\operatorname{cosec}2A:\operatorname{cosec}2B:\operatorname{cosec}2C\,;$$

and if P is the orthocentre, then

$$l:m:n=\cot A:\cot B:\cot C\,,$$

and (iii.) is, of course, the nine-point circle.

14. If O is the orthocentre, then

$$l:m:n=\cos A:\cos B:\cos C\,,$$

and P is $\qquad \alpha\sin 2A=\beta\sin 2B=\gamma\sin 2C$.

15. If $l:m:n=s-a:s-b:s-c$, then P is the Gergonne point, and (iii.) is the In-circle.

16. If P is the Symmedian-point, then (iii.) is

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2)\Sigma(a\beta\gamma)=\Sigma(a\alpha)\,.\,\Sigma[bca(b^4+c^4-a^4+\lambda^2)],$$

where $\lambda^2 \equiv a^2b^2+b^2c^2+c^2a^3$ as before.

17. If $n=0$, $m=a$, $l=b$, (iii.) becomes

$$(a+b)\Sigma(a\beta\gamma)=\Sigma(a\alpha)\,.\,[b\cos A\alpha+a\cos B\beta]$$

which cuts AB in the points where the bisector of $\angle$C and the perpendicular from C meet it.

18. If the locus of P is the line

$$p\alpha+q\beta+r\gamma=0\,,$$

then the envelope of DE is found from the equations

$$\left.\begin{aligned}\frac{p}{al}+\frac{q}{bm}+\frac{r}{cn}&=0\,,\\ \text{and}\qquad al\alpha+bm\beta-cn\gamma&=0\,.\end{aligned}\right\}$$

It is readily seen to be the in-conic

$$p^2\alpha^2+q^2\beta^2+r^2\gamma^2-2pq\alpha\beta+2qr\beta\gamma+2rp\gamma\alpha=0\,.$$

The chords of contact are

$$\left.\begin{aligned}p\alpha-q\beta+r\gamma&=0\,,\\ -p\alpha+q\beta+r\gamma&=0\,,\\ p\alpha+q\beta+r\gamma&=0\,.\end{aligned}\right\}$$

In like manner, with the same condition, the envelope of D′E′F′ is

$$p^2\alpha^2+q^2\beta^2+r^2\gamma^2-2qr\beta\gamma-2rp\gamma\alpha-2pq\alpha\beta=0\,.$$

19. If P, P′ are inverse points, and O, O′ are given by (l, m, n), (l', m', n'), then we have

$$a^2ll'=b^2mm'=c^2nn'. \qquad \text{(vi.)}$$

The equation to PP′, in the general case, is

$$all'\alpha[m'n-mn']+\ldots+\ldots=0$$

hence, if (vi.) is satisfied, and the line passes through the symmedian point, we must have

$$\Sigma[m'n-mn']=0\,. \qquad \text{(vii.)}$$

20. If O, O′, O″ are points such that P, P′, P″ are collinear, then we have

$$\Sigma l'l''mn[m''n' - m'n''] = 0 .$$

21. If P, P′ are inverse points, the equation (vii.) is satisfied by the cubic

$$\Sigma\left(\frac{m-n}{a^2 l}\right) = 0 \quad :$$

a value is

$$\frac{1}{l} : \frac{1}{m} : \frac{1}{n} = \lambda^2 - 2b^2c^2 : \lambda^2 - 2c^2a^2 : \lambda^2 - 2a^2b^2 .$$

An Arithmetical Problem.

By Dr Wm. Peddie.

Second Meeting, December 8th, 1893.

JOHN ALISON, Esq., ex-President, in the Chair.

Note on the number of numbers less than a given number and prime to it.

By Professor STEGGALL.

The following proof of the well-known result, n being any number, a, b, c the different prime factors that singly, or multiply, compose it

$$\phi(n) = n\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\ldots$$

seems worthy of notice.

Consider any number N: let p be any smaller number; consider also the number aN where a is a prime factor of N: then the numbers

$$p, \; p + \mathrm{N}, \; p + 2\mathrm{N}, \; .. \; p + (a-1)\mathrm{N}$$

are all less than aN.

They are also all prime to N (and to aN) if p is, and not otherwise; for p has no prime factor of N (and of aN) and N, 2N .. etc., have every prime factor of N (and of aN).

Hence if a is a prime factor of N the number of numbers less than aN and prime to it (sometimes called the totient of aN) is a times the totient of N,

or $$\phi(a\mathrm{N}) = a\phi(\mathrm{N}). \qquad (1)$$

Again let b be a prime number not a factor of N; then of the numbers

$$p, \; p + \mathrm{N}, \; p + 2\mathrm{N}, \; p + (b-1)\mathrm{N}$$

one, and one only, is divisible by b.

Hence as before if p is prime to N, $b-1$ of the above numbers are prime to bN. Thus if b is a prime non-factor of N the totient of bN is $(b-1)$ times the totient of N, or

$$\phi(bN) = (b-1)\phi(N) \qquad (2)$$

Now $\phi(a) = a - 1$ and we note that the totient of any prime a is $(a-1)$ times that of unity, if we call that of unity one.

We see then that in multiplying the prime factors a p times, b q times, etc., in any order, the totient of the numbers resulting is once (viz., at the first introduction of a new factor a) increased in the ratio $a-1$, and at every other introduction of a in the ratio a; similarly for b, c, etc. Hence

$$\phi(a^p b^q c^r \ldots) = a^{p-1} b^{q-1} c^{r-1} (a-1)(b-1)(c-1)\ldots \times \phi(1)$$
$$= a^{p-1} b^{q-1} c^{r-1} \ldots (a-1)(b-1)(c-1)\ldots$$

or

$$\phi(n) = n\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right)\ldots$$

Professor STEGGALL also exhibited two geographical models, and said :—

I had intended to bring before you a simple definition of a ridge line on a surface, and to shortly discuss the equation deduced; but since offering my paper I learnt that Dr M'Cowan had developed very fully the consequences of an equivalent definition; and therefore, lest I should accidentally impair the interest of his paper, I shall leave with him the treatment of the whole subject, a treatment that I believe includes all I had to say.

The models I exhibit were originally made with a view to the presentation in a concrete form of the mathematical conceptions of contour lines, lines of slope, saddle points, indicatrices, and ridges in surfaces. As a minor and secondary object, the educational value of the representation of an actual hill seemed to justify the construction of the model of a real mountain rather than that of any surface derived from merely arbitrary design, or from fixed equation. Besides this, there is a probability that such a model as I show may result in a more lasting impression in the mind of the student; and there seems more than a probability that the application of mathematical terms to a surface that has not been constructed to suit the requirements of the analysis will serve to fix the general theories and their wide application securely in the learner's mind.

For my first model, Ben Cruachan was selected, because of the richness of feature it presents; for the other model, which was left in a state to show the method of construction, Glencoe was selected, because of its steepness, the deep indentations in its hills, and its general interest.

The scale of Ben Cruachan is three inches to the mile horizontal, and four vertical. That of Glencoe is $2\frac{1}{2}$ inches to the mile in both directions.

On Ridge Lines and Lines connected with them.*

By J. M'Cowan, D.Sc.

* Printed in *Philosophical Magazine*, February, 1894.

Third Meeting, January 12th, 1894.

Dr C. G. KNOTT, President, in the Chair.

On certain Maxima and Minima.

By G. DUTHIE, M.A.

1. If x, y, z are the distances of the point P from the sides BC, CA, AB of a triangle, to find P so that the product xyz may be a maximum.

If the product xyz has a value very near the critical value, suppose yz to remain constant then x also remains constant, that is, at the required point the curves $yz = \text{constant}$ and $x = \text{constant}$ touch, or the tangent to the hyperbola $yz = \text{constant}$ is parallel to BC. But the intercept on this tangent by the sides AB, AC is bisected at the point of contact, therefore AP produced bisects BC. So also BP bisects AC, and CP bisects AB and therefore P is the centroid of the triangle ABC.

2. The usual method of solution is:—

If xyz is a maximum

$ax \,.\, by \,.\, cz$ is a maximum,

which is true when $ax = by = cz$, that is when

$$\triangle PAB = \triangle PBC = \triangle PCA.$$

3. This latter method may be applied to a figure of any number of sides, for if x, y, z, w, etc., are the perpendiculars then the product $xyzw$, etc., is a maximum when $ax \,.\, by \,.\, cz \,.\, dw$, etc., is a maximum. But ax, by, cz are connected by the symmetrical relation $ax + by + cz + \text{etc.} = 2\triangle$ where $\triangle$ is the area of the polygon and therefore for the critical position $ax = by = cz = \text{etc.}$, that is, all

the triangles with vertex P and sides of the polygon as bases must be equal.

In general no such solution is possible. For example, in the case of a quadrilateral the point P exists only when one diagonal bisects the quadrilateral and the point P is then the middle point of that diagonal.

4. Treating the case of the quadrilateral by the method of § 1 an interesting result follows.

At the required point P the hyperbolas

$$xy = \text{constant and } zw = \text{constant}$$

must touch. Also at the same point the pairs of curves

$$xz = \text{constant and } yw = \text{constant}$$

and

$$yz = \text{constant and } xw = \text{constant}$$

must touch.

5. PROBLEM:—To find the locus of the point of contact of hyperbolas whose asymptotes are the pairs of sides of a quadrilateral drawn respectively from opposite angular points.

Let $x=0$, $y=0$ be two sides drawn from one angular point, and $\frac{x}{a_1}+\frac{y}{b_1}=1$ and $\frac{x}{a_2}+\frac{y}{b_2}=1$ the two drawn from the opposite angular point.

The two hyperbolas will be

$$xy = c_1^2 \quad \text{and} \quad \left(\frac{x}{a_1}+\frac{y}{b_1}-1\right)\left(\frac{x}{a_2}+\frac{y}{b_2}-1\right)=c_2^2.$$

The tangents to these curves at the point x_1, y_1 are

$$\frac{x}{x_1}+\frac{y}{y_1}=2$$

and

$$(x-x_1)\left\{\frac{1}{a_1}\left(\frac{x_1}{a_2}+\frac{y_1}{b_2}-1\right)+\frac{1}{a_2}\left(\frac{x_1}{a_1}+\frac{y_1}{b_1}-1\right)\right\}$$

$$+(y-y_1)\left\{\frac{1}{b_1}\left(\frac{x_1}{a_2}+\frac{y_1}{b_2}-1\right)+\frac{1}{b_2}\left(\frac{x_1}{a_1}+\frac{y_1}{b_1}-1\right)\right\}=0.$$

These tangents must be coincident and therefore

$$x_1\left\{\frac{1}{a_1}\left(\frac{x_1}{a_2}+\frac{y_1}{b_2}-1\right)+\frac{1}{a_2}\left(\frac{x_1}{a_1}+\frac{y_1}{b_1}-1\right)\right\}$$

$$=y_1\left\{\frac{1}{b_1}\left(\frac{x_1}{a_2}+\frac{y_1}{b_2}-1\right)+\frac{1}{b_2}\left(\frac{x_1}{a_1}+\frac{y_1}{b_1}-1\right)\right\}$$

$$=\{\text{sum of absolute terms}\}.$$

These two equations are equivalent to one only and therefore the required locus is

$$\left(\frac{x}{a_1}-\frac{y}{b_1}\right)\left(\frac{x}{a_2}+\frac{y}{b_2}-1\right)+\left(\frac{x}{a_2}-\frac{y}{b_2}\right)\left(\frac{x}{a_1}+\frac{y}{b_1}-1\right)=0$$

which is the equation of the centre locus of all conics passing through the four points of intersection of the four sides of the quadrilateral, other than the two through which the asymptotes are drawn.

6. Thus interpreting § 4 by means of this result the point P in a quadrilateral for which the product $xyzw$ is a maximum is *the intersection of the three centre-loci of all conics passing through the three sets of four of the angular points of a complete quadrilateral.*

This point will be real when the quadrilateral is bisected by one of its diagonals, and is the middle-point of that diagonal.

7. The method of § 1 may be extended by finding the curve, the tangent at any point of which is divided by two given lines and by the point of contact in a given ratio.

Let Ox, Oy (Fig. 10) be the given lines, and XPP′Y a line passing through two consecutive points on the curve.

$$\frac{\delta y}{\delta x}=\frac{P'N}{PN}=\pm\frac{y}{OX-x}=\pm\frac{y}{x}\cdot\frac{x}{OX-x}=\pm\frac{y}{x}\cdot\frac{YP}{PX}$$

$$=\pm\frac{y}{x}\cdot\frac{l}{m}$$ where $l:m$ is the given ratio.

Taking the negative sign $l\frac{\delta x}{x}+m\frac{\delta y}{y}=0$ or $x^l y^m=\kappa$.

Whence, by § 1, the maximum value of $x^l y^m z^n$ in a triangle is at the point P where AP divides BC in the ratio of $n:m$, BP divides CA in the ratio $l:n$ and CP divides AB in ratio $m:l$. (Fig. 11.)

By taking the positive sign in the above equation we find $x^{-l}y^m z^n$ a maximum for a point similarly found, but BP, CP divide AC, AB externally.

Examples :—For incentre $x^a y^b z^c$ is a maximum.

For the Gergonne point $x^{\frac{1}{s-a}} y^{\frac{1}{s-b}} z^{\frac{1}{s-c}}$ is a maximum.

8. It is not difficult to show that any point in the plane of a triangle is the maximum or minimum position for an infinite number of functions of the distances of the point from the sides of a triangle.

For example, to find the critical position for

$$\phi(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta.$$

Applying the ordinary criterion we find

$$\frac{\partial\phi}{\partial\alpha_0}\Big/a = \frac{\partial\phi}{\partial\beta_0}\Big/b = \frac{\partial\phi}{\partial\gamma_0}\Big/c$$

which are also the equations for finding the centre $\alpha_0, \beta_0, \gamma_0$ of the conic $\phi(\alpha, \beta, \gamma) = 0$.

Applying still further the criterion for a true maximum or minimum we find the same condition as that $\phi(\alpha, \beta, \gamma) = 0$ should be an ellipse. So that to find various functions for which a given point is the critical position it is necessary only to find the equations to the various ellipses with centre at the given point. For this we have two equations among five unknown quantities and therefore there is a triply infinite number of functions.

To take a few particular cases, let $u' = v' = w' = 0$ then u, v, w are found from the equation

$$\frac{u\alpha_0}{a} = \frac{v\beta_0}{b} = \frac{w\gamma_0}{c}$$

$\alpha_0, \beta_0, \gamma_0$ being a given point.

For the centre of gravity we have $a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$ a minimum.

For the positive Brocard point $b^2a^2\alpha^2+c^2b^2\beta^2+a^2c^2\gamma^2$ is a minimum, and for the orthocentre and the Gergonne point we have

$a\alpha^2\cos A+b\beta^2\cos B+c\gamma^2\cos C$ and $(s-a)a^2\alpha^2+(s-b)b^2\beta^2+(s-c)c^2\gamma^2$

respectively the minimum values.

In the first example given we have a function of the second degree of $a\alpha$, $b\beta$, $c\gamma$ as a critical value at the centroid. This point would be the critical point equally well for any symmetrical function whatever of $a\alpha$, $b\beta$, $c\gamma$ since we have the symmetrical relation $a\alpha+b\beta+c\gamma=2\Delta$.

9. The following are two theorems on the critical value of $r_1^m+r_2^m+r_3^m$ where r_1, r_2, r_3 are the distances of a point from the vertices of a triangle.

(1) If near the point $r_2^m+r_3^m$ remains constant r_1^m also remains constant and therefore the direction of r_1 is normal to the curve $r_2^m+r_3^m=\text{constant}$.

In this curve by differentiating we have (Fig. 12)

$$\frac{dr_2}{ds}:-\frac{dr_3}{ds}=r_3^{m-1}:r_2^{m-1}$$

but $$\frac{dr_2}{ds}=\sin AOB \quad\text{and}\quad -\frac{dr_3}{ds}=\sin AOC$$

therefore $\sin BOC:\sin AOC:\sin AOB=r_1^{m-1}:r_2^{m-1}:r_3^{m-1}$.

Example :—If $m=1$, $\sin AOC=\sin AOB=\sin BOC$ that is, $r_1+r_2+r_3$ is a minimum when the angles AOB, AOC, BOC are equal, which is Fermat's theorem.

(2) Again, if AO meet BC in L

$$\begin{aligned} BL:LC &= r_2\sin AOB : r_3\sin AOC \\ &= 1/r_2^{m-2} : 1/r_3^{m-2}. \end{aligned}$$

If $m=2$ then BL = LC and we have the known result that $r_1^2+r_2^2+r_3^2$ is a minimum for the centre of gravity.

Notes on Factoring.

By J. W. Butters.

§ 1. When it is necessary to find the factors of such an expression as $x^2 - 7x - 120$ the difficulty for beginners lies in finding the pair of factors of 120 which satisfy the middle term: in this case so that their difference is 7. They may write down the pairs in order (1×120; 2×60; etc.) and then choose the suitable pair, but this method is often long; or they may guess until they chance to find the pair required. This is often done in so haphazard a fashion that much time is wasted, especially if the given expression have no *rational* factors.

§ 2. A combination of these methods may be used as follows: Take any pair of factors as a first trial—say 10×12; here the difference is too small; hence the required factors must *differ more* than 10 and 12, and therefore the smaller factor must be made still smaller. Try now *in order the natural numbers* less than 10. We find that 9 is not a factor of 120, but that 8 is (for $120 = 8 \times 15$) and that this gives the required difference. Hence

$$x^2 - 7x - 120 = (x + 8)(x - 15).$$

§ 3. It sometimes happens that the first trial gives a sum or difference which *differs greatly* from the one required, and then the above method is rather long. For example, if for $x^2 + 59x + 168$ we take 8×21 as a first trial, we get in succession 7×24; 6×28; 4×42; and 3×56 which is the pair required. In such a case we may begin by removing a factor from one of the pair and multiplying it into the other. We should then have (say) 8×21; 4×42; 3×56. It ought to be noticed, however, that by this method we often *pass* the required pair. Thus we might have (by transferring 4 instead of 2) 8×21; 2×84 where the first pair has too small a sum and the second too great. In most cases the method of § 2 is sufficient, and if in using it we find two consecutive pairs with their (algebraic) sums too great and too small respectively, we see that the given expression cannot have *rational* factors.

§ 4. Just as from the identity

$$(x + a)(x + b) \equiv x^2 + (a + b)x + ab$$

we know that an expression of the form x^2+px+q may be resolved into factors provided we can find two factors of q whose sum is p, so we may extend this method* to the form Ax^2+Bx+C by means of the identity

$$(ax+m)(bx+n) \equiv abx^2+(an+bm)x+mn.$$

Here we see that it is necessary to find two factors of AC whose sum is B (viz., an and bm); also that the required factors are obtained by removing a and b from the expressions $(abx+an)$ and $(abx+bm)$, viz., $(bx+n)(ax+m)$.

§5. One or two numerical examples will show how this works out in practice.

Find the factors of $18x^2-111x+80$.

Begin by writing down the product 18×80. [It is convenient to keep throughout the smaller number first.] This gives a sum of 98 instead of 111. Hence 18 must be reduced. Applying now the method of *natural numbers* as in §2 we find 16×90 (sum $=106$, too small) and 15×96 (sum 111, as required). Affix the proper signs : -15×-96. Now divide $(18x-15)$, $(18x-96)$ by 3 and 6 (the G C.M.'s of 18 and 15 and 18 and 96 respectively) and we get $(6x-5)(3x-16)$ as the required factors.

§6. In the following the necessary work alone is given:

$12x^2-5x-72$	12×72 (A)
$\equiv(4x+9)(3x-8)$	24×36
	$+27\times -32$

[Explanation: (A) The difference between 12 and 72 being *very much* greater than 5, the factor 2 is transferred from 72 to 12 before applying the method of the *natural numbers* as in §2.]

§7. This example may be used to present the theory in another form.

$$12x^2-5x-72 \equiv \frac{1}{12}\{(12x)^2-5(12x)-12\times 72\}$$

$$\equiv \frac{1}{12}(12x+27)(12x-32) \equiv \frac{1}{3}(12x+27)\cdot\frac{1}{4}(12x-32)$$

$$\equiv (4x+9)(3x-8)$$

* The extension is due to Mr Jas. M'Kean, F E.I.S., Lecturer on Mathematics in the Heriot-Watt College, Edinburgh.

Since "factoring" is eminently a *practical* part of Algebra it is perhaps better not to burden the working of each case with the exemplification of the theory but to treat all as in § 6.

§ 8. This form of the theory suggests that instead of transforming quadratic equations from $ax^2 + bx + c = 0$ to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, it is better to change to $(ax)^2 + b(ax) + ac = 0$. We thus avoid fractions and introduce early the important notion of "change of variable."

Against a Current Pseudo-Definition of Varying Velocity.

By Mr R. F. Muirhead

Fourth Meeting, February 9th, 1894.

Dr J. M'Cowan, Vice-President, in the Chair.

On the Geometrical Interpretation of i^i.

By T. B. Sprague, M.A., LL.D., F.R.S.E., etc.

If we put $\theta = \pi/2$ in the well known equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we get $e^{i\pi/2} = i$; and raising both sides to the power i,

$$i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2}.$$

Before proceeding further it will be useful to consider the former of these results. Putting for $e^{i\pi/2}$ the equivalent series, we have

$$i = 1 - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 - \dots$$
$$+ i\left\{\frac{\pi}{2} - \frac{1}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}\right)^5 - \dots\right\}$$

It is obvious that each of the infinite series here involved is convergent, and a very little numerical calculation is sufficient to show that their limits are 0 and 1 respectively. In fact, taking $\pi = 3{\cdot}1416$, or $\pi/2 = 1{\cdot}5708$, the two series become

$$1 - 1{\cdot}2337 + {\cdot}2537 - {\cdot}0208 + {\cdot}0009 = {\cdot}0001,$$

and $$1{\cdot}5708 - {\cdot}6459 + {\cdot}0797 - {\cdot}0047 + {\cdot}0001 = 1{\cdot}0000.$$

Returning now to the equation $i^i = e^{-\pi/2}$, and substituting for e and π their numerical values, we get

$$i^i = {\cdot}20788;$$

and the question I propose to consider is how this result is to be understood or interpreted. It will be convenient first to consider the more general expression a^i, where a is a complex number. We first observe that

$$(a^i)^i = a^{i^2} = a^{-1};$$

so that the effect of the operation $(\ )^i$, if performed twice upon a, is to give us the reciprocal of a; or the operation is one which goes half way towards the reciprocal. Next, writing $re^{i\theta}$ for a, we have

$$(re^{i\theta})^i = r^i e^{-\theta} = e^{-\theta} e^{i\log r} = e^{-\theta}(\cos\log r + i\sin\log r).$$

This shows us that the operation $(\ )^i$, when performed on a complex with modulus r and amplitude θ, gives us a new complex, of which the modulus is $e^{-\theta}$, and the amplitude $\log r$. Performing the same operation on this new complex, we have

$$(r^i e^{-\theta})^i = r^{-1} e^{-i\theta};$$

the result being the reciprocal of our original complex. Proceeding in the same way, we get

$$(r^{-1}e^{-i\theta})^i = r^{-i}e^{-i^2\theta} = r^{-i}e^{\theta};$$

$$(r^{-i}e^{\theta})^i = r^{-i^2}e^{i\theta} = re^{i\theta}.$$

We have thus produced the original complex, and we see that the

moduli of the four complexes are ... ... r, $e^{-\theta}$, r^{-1}, e^{θ};

and the amplitudes ... ... θ, $\log r$, $-\theta$, $-\log r$.

If OP_1 in Figure 13 represents $re^{i\theta}$, so that $OP_1 = r$, $P_1OA = \theta$, then the other three complexes will be represented by OP_2, OP_3, OP_4; where $AOP_1 = AOP_3$, $AOP_2 = AOP_4$, and $OP_1 . OP_3 = OP_2 . OP_4 = 1$. The figure is drawn for the case where $r = \frac{3}{4}$, $e^{-\theta} = \frac{1}{2}$; so that

the moduli are ... ... $\frac{3}{4}$, $\frac{1}{2}$, $\frac{4}{3}$, 2;

and the amplitudes ... ... $\begin{cases} \cdot693, & -\cdot288, & -\cdot693, & \cdot288, \\ \text{or } 39^\circ\cdot7, & -16^\circ\cdot6, & -39^\circ\cdot7, & 16^\circ\cdot6. \end{cases}$

If the complex $(re^{i\theta})^i$ is a real number, the point P_2 lies in OA or in AO produced; and the condition for this is that the amplitude shall be 0 or π. If now we suppose r to approach 1, the angles AOP_2 and AOP_4 gradually diminish, and ultimately vanish when $r=1$. In this case

the moduli become	...	...	1,	$e^{-\theta}$,	1,	e^{θ};
and the amplitudes	...	...	θ,	0,	$-\theta$,	0;

so that the points, P_2, P_4, lie in OA, and the equation $(re^{i\theta})^i = r^i e^{-\theta}$ becomes $(e^{i\theta})^i = e^{-\theta}$. If we now suppose θ to approach $\pi/2$, or the line OP_1 to approach the perpendicular OB,

the moduli become ...	...		1,	$e^{-\pi/2}$,	1,	$e^{\pi/2}$;
and the amplitudes ...	...		$\pi/2$,	0,	$-\pi/2$,	0.

In this case $e^{i\theta}$, which $=\cos\theta + i\sin\theta$, becomes $=i$, and we have $i^i = e^{-\pi/2} = \cdot 20788$. We thus see that this apparently anomalous result admits of a simple geometrical interpretation.

Another special case deserving of notice is when $AOP_1 = AOP_4$. This will happen when $\theta = -\log r$ or $r = e^{-\theta}$; and then

the moduli are	...	...	... $e^{-\theta}$,	$e^{-\theta}$,	e^{θ},	e^{θ};
and the amplitudes	...	...	θ,	$-\theta$,	$-\theta$,	θ.

(See Figure 14.)

In the foregoing investigation I have not taken into account the possibility that a^i may have a multiplicity of values, and I will now consider that point. If l is any integer, we have

$$e^{2il\pi} = \cos 2l\pi + i\sin 2l\pi = 1.$$

Hence $$re^{i\theta} = re^{i(\theta + 2l\pi)}$$

and $$(re^{i\theta})^i = \{re^{i(\theta+2l\pi)}\}^i = r^i e^{-\theta-2l\pi} = e^{-\theta-2l\pi} e^{i\log r}.$$

We thus see that, instead of the first member having a single value, as we have hitherto assumed, it has an infinite number of values, all of which have the same amplitude, $\log r$; while the moduli are $e^{-\theta}$, $e^{-\theta\pm 2\pi}$, $e^{-\theta\pm 4\pi}$, etc., and form a geometric series of which the ratio is $e^{\pm 2\pi}$.

Repeating the operation $(\,)^i$, since $e^{2im\pi}=1$, where m is any integer,

$$(r^i e^{-\theta-2l\pi})^i = (r^i e^{2im\pi}.e^{-\theta-2l\pi})^i$$

$$= r^{-1}e^{-2m\pi}.e^{-i\theta}.e^{-2il\pi}$$

$$= r^{-1}e^{-2m\pi}.e^{-i\theta}$$

so that, instead of the reciprocal $r^{-1}e^{-i\theta}$, we have an infinite number of complexes, all of which have the same amplitude, $-\theta$; while the moduli are r^{-1}, $r^{-1}e^{\pm 2\pi}$, $r^{-1}e^{\pm 4\pi}$, etc., and form a geometric series, of which the ratio is $e^{\pm 2\pi}$. Since l and m each denote any integer, and they do not occur in the same formula, we may say that by successive repetitions of $(\,)^i$ we get a series of complexes, of which

the moduli are ... ... r, $e^{-\theta+2l\pi}$, $r^{-1}e^{2l\pi}$, $e^{\theta+2l\pi}$, $re^{2l\pi}$,...

and the amplitudes ... θ, $\log r$, $-\theta$, $-\log r$, θ,...

It thus appears that we are not entitled to reason, as we did above, that $(a^i)^i = a^{ii} = a^{i^2} = a^{-1}$. This is analogous to what occurs with fractional indices; for instance, $(a^{\frac{1}{2}})^2 = a$; while $(a^2)^{\frac{1}{2}}$ is not a, but $\pm a$.

We have seen that $(\,)^i$ is a periodic operation with a period 4, subject to the remark that the original complex is only one of a series that are produced by performing the operation four times. Subject to a similar remark, we may say that $(\,)^z$ is a periodic operation, the period of which is n, if z is a primary n^{th} root of unity. Suppose that $z = x + iy = \cos 2\pi/n + i\sin 2\pi/n$; then

$$(re^{i\theta})^z = (re^{i\theta}.e^{2if\pi})^z, \text{ where } f \text{ is any integer}$$

$$= r^z e^{iz\theta}.e^{2ifz\pi}$$

$$= r^z e^{iz\theta}.e^{2ifz\pi}.e^{2ig\pi}, \text{ where } g \text{ is any integer.}$$

Performing the operation $(\ )^z$ again,

$$(re^{i\theta})^{z^2} = r^{z^2} e^{iz^2\theta} . e^{2ifz^2\pi} . e^{2igz\pi} . e^{2ih\pi}, \quad \text{where } h \text{ is any integer.}$$

Proceeding in this way, we get

$$(re^{i\theta})^{z^n} = r^{z^n} e^{iz^n\theta} . e^{2ifz^n\pi} . e^{2igz^{n-1}\pi} \dots e^{2isz\pi}$$

$$= re^{i\theta} . e^{2i\pi(gz^{n-1} + hz^{n-2} + \dots + sz)}.$$

The index of e in the last factor becomes

$$2i\pi\{g\cos 2(n-1)\pi/n + h\cos 2(n-2)\pi/n + \dots + s\cos 2\pi/n\}$$
$$- 2\pi\{g\sin 2(n-1)\pi/n + h\sin 2(n-2)\pi/n + \dots + s\sin 2\pi/n\}$$

where $g\, h \dots s$ are any integers.

The preceding investigation was suggested to me by a perusal of Hayward's *Vector Algebra and Trigonometry.* In chap. 5 Mr Hayward gives the result (p. 115), $(4{\cdot}810475\dots)^i = i$; and this at once leads to $i^i = (4{\cdot}810475\dots)^{-1} = {\cdot}20788$. He then discusses the interpretation of A^B, where A and B are complex numbers; and shows that it has an infinite number of values, forming a series with a constant ratio; and he explains how these may be geometrically represented as derived from a "fundamental vector". He also considers several "particular cases"; but not specially the case where $B = i$, which is the one I have mostly had in view.

A Proof of the Uniform Convergence of the Fourier Series, with Notes on the Differentiation of the Series.

By George A. Gibson, M.A.

1. My only justification for presenting this paper to the Society lies in the fact that, so far as I am aware, the uniform convergence of the Fourier Series is nowhere alluded to, and far less discussed, in any English textbook; while the precautions that are necessary in differentiating the series are hardly ever mentioned even in treatises which give a very thorough treatment of its convergence. I have confined myself almost exclusively to what may be called ordinary functions, as a complete discussion of what has been done in recent years for functions that lie outside the category of "ordinary" would make the paper much too long. For information as to the original authorities, I would refer to the paper which I communicated to the Society last session *On the History of the Fourier Series*. It is sufficient to say here that the proof I now give is simply an adaptation of that of Heine (*Kugelfunctionen*, Bd. I. 57–64, Bd. II. 346–353) and of that of Neumann (Über die nach Kreis . . . Functionen fortsch. Entwickelungen, 26–52).

2. The chief instrument employed in the investigation is *The Second Theorem of Mean Value*. A sketch of a proof of this theorem will be found in a paper by me in Vol. VI., pp. 40–42 of the Society's *Proceedings;* but for a satisfactory treatment of this and other theorems of the Integral Calculus I would refer to Cathcart's translation of Harnack's *Introduction to the Calculus*. The theorem, so far as we are now concerned with it, may be stated thus:—If, for all values of x between a and b, $f(x)$ and $\phi(x)$ are finite integrable functions of x and if $f(x)$ is either not increasing or not decreasing (*i.e.*, has no turning points) in the interval (a, b), then

$$\left.\begin{aligned}\int_a^b f(x)\phi(x)dx &= f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx \qquad a \leqq \xi \leqq b \\ &= f(a)\int_a^b \phi(x)dx + \{f(b)-f(a)\}\int_\xi^b \phi(x)dx.\end{aligned}\right\} \quad (1)$$

It is to be understood that $f(a)=f(a+0) = \underset{\epsilon=0}{\mathrm{L}} f(a+\epsilon)$ and $f(b)=f(b-0)=\underset{\epsilon=0}{\mathrm{L}} f(b-\epsilon)$ where $a<b$ and ϵ is positive.

3. When a single-valued function $f(x)$ is said to be given arbitrarily in the interval (a, b), then to each value of x corresponds one value of $f(x)$ in such a manner that the value of the function for one value of the argument in no way conditions its value for another value of the argument. An arbitrary function of this kind is, however, too general for representation by a Fourier (or any other) series, and for the present paper we subject the function to the following restrictions, though some of them may be removed and a Fourier representation of the function still exist. The argument x is, merely for convenience, restricted to the interval $(-\pi, \pi)$.

(i.) $f(x)$ is to be finite, having an upper limit G to its numerical values; (ii.) it is to be continuous except for a finite number of values of x for which it may be discontinuous, but only so that $f(x+0)$ and $f(x-0)$ are each definite, though unequal; (iii.) it is to have a finite number of maxima and minima—*i.e.*, a finite number of turning points. Such a function is necessarily integrable.

4. Let $$S_{2n+1}=\tfrac{1}{2}A_0+\sum_{r=1}^{n}(A_r\cos rx+B_r\sin rx)$$

where $$A_r=\frac{1}{\pi}\int_{-\pi}^{\pi}f(a)\cos ra\,da,\quad B_r=\frac{1}{\pi}\int_{-\pi}^{\pi}f(a)\sin ra\,da \quad . \qquad (2)$$

Hence $$S_{2n+1}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(a)\{\tfrac{1}{2}+\sum_{r=1}^{n}\cos r(a-x)\}da$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(a)\frac{\sin(2n+1)\frac{a-x}{2}}{2\sin\frac{a-x}{2}}da$$

$$\left.\begin{aligned}&=\frac{1}{\pi}\int_{0}^{\frac{\pi-x}{2}}f(x+2a)\frac{\sin(2n+1)a}{\sin a}da\\&+\frac{1}{\pi}\int_{0}^{\frac{\pi+x}{2}}f(x-2a)\frac{\sin(2n+1)a}{\sin a}da\end{aligned}\right\} \qquad (3)$$

It has to be shown that, *excluding values of x in the neighbourhood* of the points of discontinuity, though not these points themselves*, a finite value of n can be found, which is the same for every x, such that the difference between S_{2n+1} and $\frac{1}{2}\{f(x+0)+f(x-0)\}$ shall be as small as we please, except that when $x=\pm\pi$ the value $\frac{1}{2}\{f(-\pi+0)+f(\pi-0)\}$ is to be taken instead of $\frac{1}{2}\{f(x+0)+f(x-0)\}$. If $f(\pi-0)\neq f(-\pi+0)$, the points $x=\pm\pi$ are to be reckoned points of discontinuity. In other words, the convergence of the series has to be shown to be in general uniform.

The demonstration is divided into two parts, in the first of which it is shown, *inter alia*, that the coefficients A_n, B_n have zero for their limit.

5. Consider the integral $\int_a^b \phi(a)\sin ma da$ where m is any positive number, integral or fractional, and a, b and $\phi(a)$ may contain a parameter x (*e.g.*, $\phi(a)$ might be $\phi(x+2a)$) and $\phi(a)$ is subject to the same restrictions as $f(x)$ in § 3.

Let $a_1, a_2, \ldots a_p$ be the turning points or points of discontinuity of $\phi(a)$, p in number and let G be an upper limit to the numerical values of $\phi(a)$. Divide the interval (a, b) into the $p+1$ partial intervals (a, a_1) . . . . (a_r, a_{r+1}) . . . . (a_p, b) and take the integral for the interval (a_r, a_{r+1}) to which theorem (1) is applicable.

$$\int_{a_r}^{a_{r+1}} \phi(a)\sin ma da = \phi(a_r)\int_{a_r}^{\xi} \sin ma da + \phi(a_{r+1})\int_{\xi}^{a_{r+1}} \sin ma da\,;\quad a_r \leqq \xi \leqq a_{r+1}$$

$$= \phi(a_r)\frac{\cos ma_r - \cos m\xi}{m} + \phi(a_{r+1})\frac{\cos m\xi - \cos ma_{r+1}}{m}$$

$$< \frac{2}{m}\left\{\phi(a_r) + \phi(a_{r+1})\right\} < \frac{4G}{m}\quad .$$

$$\therefore \int_a^b \phi(a)\sin ma da < \frac{4(p+1)G}{m} \quad \text{in absolute value.} \tag{4}$$

Now m can be chosen so large that $4(p+1)G/m$ shall be as small as we please, and this independently of the parameter x; in other words, the integral converges uniformly to zero with $1/m$.

* See §6 for a definition of "neighbourhood of a point."

In the same way it may be shown that $\int_a^b \phi(\alpha)\cos m\alpha d\alpha$ converges uniformly to zero with $1/m$.

These results show that the coefficients of the series, given by (2), are ultimately zero. It should be observed, however, that A_n, B_n are in general only of the order $1/n$; it is only in exceptional cases, as will be seen later, that they are of the order $1/n^2$. Hence the series is usually semi-convergent, while the series obtained by differentiating a Fourier series term by term will usually not be convergent.

The presence of the factor $p+1$ in (4) should be noted as an indication of the necessity for the restrictions (ii.) and (iii.) of § 3.

We may also deduce another result that will be useful immediately. If we suppose $Q > P > 0$, then

$$\int_0^Q \frac{\sin\alpha}{\alpha}d\alpha - \int_0^P \frac{\sin\alpha}{\alpha}d\alpha = \int_P^Q \frac{\sin\alpha}{\alpha}d\alpha < \frac{2}{P} + \frac{2}{Q} \text{ numerically.}$$

But we know that the value of the first integral for $Q = \infty$ is $\pi/2$. Hence if $P > 0$

$$\int_0^P \frac{\sin\alpha}{\alpha}d\alpha < \frac{\pi}{2} + \frac{2}{P} \quad \text{in absolute value.} \tag{5}$$

6. By means of equations (4) and (5) we can find limits for each of the integrals in (3).

Let an arbitrarily small but finite distance δ be measured off on each side of the points for which the function is discontinuous, thus forming what we may call the *neighbourhood* of the points; the points $\pm\pi$ are also to be dealt with in the same way. A point is thus in the neighbourhood of c if its abscissa lies between $c - \delta$ and $c + \delta$. Then suppose in the first place, $-\pi + \delta \lesseqgtr x \lesseqgtr \pi - \delta$ and let x have any value in this range *except those in the neighbourhood of the points of discontinuity*, though it may have exactly a value for which the function is discontinuous.

We shall now show that, given an arbitrarily small quantity 2ϵ, we can choose n so large, but finite, that S_{2n+1} shall differ from $\frac{1}{2}\{f(x+0) + f(x-0)\}$ by a quantity that is less than $2\epsilon/\pi$ and that such n is the same for every x.

Consider $\int_0^{\frac{\pi-x}{2}} f(x+2a)\frac{\sin ma}{\sin a}da$ where $m=2n+1$,

although the particular form of m is of no moment; it may be either integral or fractional. Put $\psi(x, a)=(a/\sin a)f(x+2a)$, so that $\psi(x, 0)$ is equal to $f(x+0)$. Take η an arbitrarily small but finite quantity, such that $0<\eta\leqq(\pi-x)/2$. Then

$$\int_0^{\frac{\pi-x}{2}} f(x+2a)\frac{\sin(2n+1)a}{\sin a}da=\int_0^{\frac{\pi-x}{2}}\psi(x, a)\frac{\sin ma}{a}da$$

$$=\int_0^{\eta}\psi(x, a)\frac{\sin ma}{a}da+\int_{\eta}^{\frac{\pi-x}{2}}\psi(x, a)\frac{\sin ma}{a}da \qquad (6)$$

To the last of these integrals equation (4) is applicable if we denote by G the greatest value of $\psi(x, a)$ and then write G/η in place of G of (4), since G/η will be the greatest value of $\psi(x, a)/a$. Hence

$$\int_{\eta}^{\frac{\pi-x}{2}}\psi(x, a)\frac{\sin ma}{a}da<\frac{G'}{m\eta} \quad \text{numerically} \qquad (7)$$

where G' is a finite constant independent of x, though depending on p.

Again, no matter what the value of x, we can choose η so small but finite that $\psi(x, a)$ shall have no turning point for any value of a between 0 and η; hence we can apply equation (1) to the first integral on the right of (6). Thus

$$\int_0^{\eta}\psi(x, a)\frac{\sin ma}{a}da=\psi(x, 0)\int_0^{\eta}\frac{\sin ma}{a}da$$

$$+\{\psi(x, \eta)-\psi(x, 0)\}\int_{\xi}^{\eta}\frac{\sin ma}{a}da$$

and therefore

$$\int_0^{\eta}\psi(x, a)\frac{\sin ma}{a}da=f(x+0)\int_0^{m\eta}\frac{\sin\beta}{\beta}d\beta+\{\psi(x, \eta)-\psi(x, 0)\}\int_{m\xi}^{m\eta}\frac{\sin\beta}{\beta}d\beta \quad (8)$$

where $0\leqq\xi\leqq\eta$.

Further, we can choose η so small, but finite, that $\psi(x, \eta) - \psi(x, 0)$ shall for every x be less than any arbitrarily small quantity ϵ',

while $\int_{m\xi}^{m\eta} \frac{\sin\beta}{\beta} d\beta \ngtr \int_0^{\infty} \frac{\sin\beta}{\beta} d\beta \ngtr \frac{\pi}{2}$. Hence by (5) we may

now write (8)

$$\int_0^{\eta} \psi(x, a)\frac{\sin ma}{a} da < f(x+0)\left(\frac{\pi}{2} + \frac{2}{m\eta}\right) + \frac{\pi}{2}\epsilon'$$

$$< \frac{\pi}{2} f(x+0) + \frac{2G}{m\eta} + \frac{\pi}{2}\epsilon' \qquad (9)$$

since $$f(x+0) < G.$$

By means of (7) and (9) we can now write (6) thus

$$\int_0^{\frac{\pi - x}{2}} f(x+2a)\frac{\sin(2n+1)a}{\sin a} da < \frac{\pi}{2} f(x+0) + \frac{G'}{m\eta} + \frac{2G}{m\eta} + \frac{\pi}{2}\epsilon'$$

$$< \frac{\pi}{2} f(x+0) + \frac{G''}{(2n+1)\eta} + \frac{\pi}{2}\epsilon' \qquad (10)$$

where G'' is a finite constant independent of x.

The theorem to be proved is now obvious. Suppose ϵ given, we first choose ϵ' so that $\pi\epsilon'/2$ is less than $\epsilon/2$; we then choose η to satisfy this value of ϵ'; n is then chosen so large that $G''/(2n+1)\eta$ shall also be less than $\epsilon/2$. The right-hand side of (10) is then less than $\frac{\pi}{2} f(x+0) + \epsilon$.

The same considerations apply to the second integral in (3) so that we have $S_{2n+1} < \frac{1}{2}\{f(x+0) + f(x-0)\} + \frac{2\epsilon}{\pi}$ and the larger of the two values of n thus determined is the one required. In other words, $\underset{n=\infty}{L} S_{2n+1} = \frac{1}{2}\{f(x+0) + f(x-0)\}$ and the convergence is uniform.

7. Suppose now, in the second place, that x is in the neighbourhood of a point of discontinuity. From equations (6) and (7) it is evident that the integrals whose limits are 0, $(\pi - x)/2$ and 0, η

respectively converge to the same value, as m becomes infinite, *even if* η *at the same time converge to zero*, provided only that $m\eta$ at the same time becomes infinite; this would be the case if, *e.g.*, $\eta = 1/\sqrt{m}$. This result is at first sight a little startling, though it is by no means without parallel in the theory of limiting values. In the case now under consideration it will contribute greatly to clearness to draw the curve $y = f(x+2a)$ with a for abscissa, giving to x values that lie within the neighbourhood of a point of discontinuity. Let c be a value for which $f(x)$ is discontinuous, and consider values of x lying between $c-\delta$ and $c-0$. Suppose $x = c-\delta$; then when $a=0$, $y = f(c-\delta)$, As a increases from 0 to $\delta/2$, y changes from $f(c-\delta)$ to $f(c-0)$. In forming the integrals (6) for the values $c-\delta$ of x, the greatest value one would give to η would be $\delta/2$, in order that the interval $(0, \eta)$ should not contain the point of discontinuity, though of course η might require to be smaller in order to satisfy the conditions respecting ϵ' and ϵ, in the equations corresponding to (8), (9), (10). Again, if $c-\delta < x < c$, then η would not be so large as $\delta/2$, and as x approaches indefinitely near to c, η becomes indefinitely small, and therefore m must be taken indefinitely great in order that $m\eta$ may also be indefinitely great. The convergence thus becomes infinitely slow for the part of the series depending on $f(x+2a)$ and we can assign no finite value for m such that for *every* x in the neighbourhood of c the value of $1/m\eta$ shall become arbitrarily small. In other words, while by equations (6) and (7) in virtue of the observation made at the beginning of this paragraph we can for any *given* x find a value of m which will make the series converge, we must when we take a new value of x usually take a new value of m, and m becomes ultimately indefinitely great; that is, the series does not converge uniformly in the neighbourhood of the point of discontinuity. These considerations do not, however, apply if $x = c$; the convergence of each of the integrals is simply of the nature of the cases of § 6. Even in the case of non-uniform convergence, it is only one of the integrals in (3) that falls under the class considered in this paragraph. It is of course known from general principles that when a series represents a discontinuous function the convergence is non-uniform in the neighbourhood of the points of discontinuity; but I have thought it better to deduce the result in this case from the analysis used to establish the general theorem.

8. The values $\pm\pi$ have still to be considered. For these the reasoning may be gone over afresh; but it is simpler to suppose $f(x)$ continued for values of x greater than π and less than $-\pi$, so that $f(x+2\pi)=f(x)$ and therefore $f(\pi+0)=f(-\pi+0)$ and $f(-\pi-0)=f(\pi-0)$. If the origin from which x is measured be then shifted, these points become ordinary points or points of discontinuity like those previously discussed; hence in the neighbourhood of $\pm\pi$, if $f(\pi)\neq f(-\pi)$ the convergence is not uniform, and if $x=\pm\pi$ the value to which S_{2n+1} converges is $\frac{1}{2}\{f(-\pi+0)+f(\pi-0)\}$ which of course reduces to $f(\pi)$ if $f(\pi)=f(-\pi)$.

9. There is one point in the proof just given that perhaps deserves notice. Is it quite clear that the neighbourhood of points at which the function is a maximum or a minimum is not of the same character as that of a point of discontinuity? Must not η for such points diminish and m increase beyond all limit as x approaches the value for which the function has a turning point? Whatever difficulty may seem to exist is at once removed by the observation of Heine that if $f(c)$ be a maximum value, then, between the two adjacent minima that include $f(c)$, $f(x)$ can be put in the form $\phi(x)+\psi(x)$

where $\phi(x)=f(x)$, $\psi(x)=0$ for $x\leqq c$

and $\phi(x)=f(c)$, $\psi(x)=f(x)-f(c)$ for $x\geqq c$.

Evidently $\phi/(x)$ is not decreasing and $\psi(x)$ not increasing, so that to each the Mean Value Theorem is applicable. Turning points are thus of the same nature as ordinary points. Heine's treatment of the Fourier series based on the representation of a function as the sum of functions which are either not increasing or not decreasing is very instructive.

10. Equations (6) and (7) establish Riemann's Theorem for functions of the kind we are considering, namely, that the value of the series at any point depends only on the behaviour of the function in the neighbourhood of the point. In establishing this result and the convergence generally, the integral $\int_0^h \phi(a) \,.\, \sin ma \,.\, da/a$, first thoroughly discussed by Dirichlet, plays a leading part and has been appropriately called *Dirichlet's Integral* by Kronecker.

11. The restrictions imposed on $f(x)$, generally called *Dirichlet's conditions*,* still leave a great amount of arbitrariness to it; when we seek to remove restrictions the investigation begins to get rather complicated. Some of them however can be removed, in part at least, by considerations founded simply on the nature of an integral. Suppose, for instance, that for $x=c$, $f(x)$ is infinite but $(x-c)^r f(x)$ finite and definite when $x=c$, r being positive and less than unity. $f(x)$ would still be integrable over the whole range, and excluding the neighbourhood of c the series would still converge for all other values as before. Functions which oscillate indefinitely often in the neighbourhood of a finite number of points can be dealt with in the same way. But cases like these are of comparatively little practical importance. Instances of trigonometric series in which the function is infinite for particular values of the argument are common enough in analysis. (See, *e.g.*, Chrystal's *Algebra* II., p. 310, *ex.* 13, 14.)

12. It is now obvious that the Fourier series can be integrated and therefore that the expansion is unique, when the function is supposed to possess the properties laid down in § 3 for the whole interval. The resulting series will not however, in general, be a Fourier series since the term $\frac{1}{2}A_0$ will introduce the term $\frac{1}{2}A_0 x$. The case is different when we consider the differentiation of the series. We suppose the differential coefficient $f'(x)$ to have the properties required by § 3.

The first theorem to be given is this:—$f'(x)$ will only be obtained by differentiating the series for $f(x)$ term by term if $f(x)$ be throughout continuous and besides $f(\pi)=f(-\pi)$.

Suppose

$$f(x)=\tfrac{1}{2}A_0+\Sigma A_n \cos nx+\Sigma B_n \sin nx \qquad (11)$$

$$f'(x)=\tfrac{1}{2}C_0+\Sigma C_n \cos nx+\Sigma D_n \sin nx. \qquad (12)$$

If we calculate the values of C_n, D_n we find

$$C_0=0 \quad C_n=nB_n, \quad D_n=-nA_n$$

so that $f'(x)$ is got by differentiating the series (11).

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* It is to be understood that the addition "that $f(x)$ may become infinite" is not included, so that these are Dirichlet's conditions in the narrower acceptation of the phrase.

Suppose however that $f(x)$ is continuous but $f(\pi) \neq f(-\pi)$; we then get

$$C_0 = \frac{f(\pi) - f(-\pi)}{\pi}, \quad C_n = \frac{f(\pi) - f(-\pi)}{\pi}\cos n\pi + nB_n, \quad D_n = -nA_n$$

so that $f'(x)$ even in this simple case is not got by differentiating (11) as it stands.

Denoting $\{f(\pi) - f(-\pi)\}$ by C and expanding $\frac{1}{2}Cx$ into the Fourier series $-C\Sigma\frac{\cos n\pi}{n}\sin nx$ we can write (11) in the form

$$f(x) = \tfrac{1}{2}A_0 + \frac{1}{2\pi}Cx + \Sigma A_n \cos nx + \Sigma\left(B_n + \frac{C\cos n\pi}{n\pi}\right)\sin nx. \quad (13)$$

The differentiation of this series gives the proper value for $f'(x)$. In (11) the value of B_n is $-C\cos n\pi/n\pi + C_n/n$, so that the series obtained by differentiating (11) will usually be oscillating, or divergent. The procedure adopted in using the form (13) may be compared with that of representing the product series for $\sin x$ in the form

$$\sin \pi x = \pi x \Pi\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}}$$

13. Suppose now that $f(x)$ is discontinuous for $x = c$ and put $f(c+0) - f(c-0) = D$. Take a function $\phi(x) = f(x) + D\theta(x)$ where

$$\theta(x) = 1 \text{ from } x = -\pi \text{ to } x = c - 0$$

and

$$\theta(x) = 0 \text{ from } x = c + 0 \text{ to } x = \pi.$$

The function $\phi(x)$ is continuous in the neighbourhood of c, since $\phi(c-0) = f(c+0) = \phi(c+0)$. Moreover $\phi(x)$ has the same differential coefficient as $f(x)$. The Fourier expansion for $\phi(x)$ is, using A_n, B_n to denote the cofficients in (11),

$$\phi(x) = \tfrac{1}{2}\left\{A_0 + \frac{\pi + c}{\pi}D\right\} + \Sigma\left\{A_n + \frac{D\sin nc}{n\pi}\right\}\cos nx$$

$$+ \Sigma\left\{B_n + \frac{D(\cos n\pi - \cos nc)}{n\pi}\right\}\sin nx. \quad (14)$$

Before we can differentiate we must take account of the values for $\pm\pi$. Denoting $f(\pi) - f(-\pi)$ by D′ we have

$$\phi(\pi) - \phi(-\pi) = D' - D$$

and the corresponding equation to (11) will be

$$\phi(x)=\tfrac{1}{2}\left\{A_0+\frac{\pi+c}{\pi}D\right\}+\frac{1}{2\pi}(D'-D)x+\Sigma\left\{A_n+\frac{D\sin nc}{n\pi}\right\}\cos nx$$
$$+\Sigma\left\{B_n-\frac{D\cos nc}{n\pi}+\frac{D'\cos n\pi}{n\pi}\right\}\sin nx. \quad (15)$$

If we differentiate (15) term by term we get the value of the series (12), as may be readily proved by going through the necessary differentiations and integrations.

In the same way if $f(x)$ were also discontinuous for $x=c_1$, we should put $\phi(x)=f(x)+D\theta(x)+D_1\theta_1(x)$ where D and $\theta(x)$ are as before, $D_1=f(c_1+0)-f(c_1-0)$ and

$$\theta_1(x)=1 \text{ from } x=-\pi \text{ to } x=c_1-0$$
$$\theta_1(x)=0 \text{ from } x=c_1+0 \text{ to } x=\pi.$$

The equation corresponding to (15) would then be formed and so on for any number of points of discontinuity.

The procedure of this paragraph was suggested by the method of Heine referred to in § 9, although Heine does not discuss the differentiation of the series except in regard to the cases mentioned in § 12.

14. It may perhaps be of interest to give some examples. In dealing with series which represent a function that is given only for a portion of the interval $(-\pi, \pi)$, it is important to bear in mind that we may have an infinite number of series which for that portion coincide with the function, each series being determined by the way in which we suppose the function continued beyond the given portion so as to be defined for the complete interval $(-\pi, \pi)$. The most frequently occurring cases are those for which (i.) $f(-x)=f(x)$ and (ii.) $f(-x)=-f(x)$, giving rise to cosine and to sine series respectively.

Take the two series

$$e^{ax}=\frac{2}{\pi}\sum_1^\infty\frac{n}{a^2+n^2}(1-e^{a\pi}\cos n\pi)\sin nx \quad (16)$$

$$e^{a}=\frac{e^{a\pi}-1}{a\pi}+\frac{2a}{\pi}\sum_1^\infty\frac{e^{a\pi}\cos n\pi-1}{a^2+n^2}\cos nx. \quad (17)$$

If we differentiate (17) term by term we get (16), but we do not get (17) by differentiating (16); on the other hand, we should get (17) by integrating (16) term by term, but we should not, immediately at least, get (16) by integrating (17) term by term. The reason for the difference in behaviour is obvious after what has been said. Equation (16) is deduced on the understanding that in the interval $(0, \pi)$ $f(x)=e^{ax}$ and in the interval $(-\pi, 0)$ $f(x)=-e^{-ax}$; we have therefore $f(\pi)-f(-\pi)$ equal to $2e^{a\pi}=D'$ while $f(+0)=1$ $f(-0)=-1$ and $\therefore$ $f(+0)-f(-0)=2=D$. Equation (16) must therefore be put in the form (15), namely

$$\phi(x)=1+\frac{e^{a\pi}-1}{\pi}x+\frac{2}{\pi}\Sigma\frac{e^{a\pi}\cos n\pi-1}{n}\cdot\frac{a^2}{a^2+n^2}\sin nx$$

and this equation when differentiated yields (17).

Equation (17), on the other hand, is obtained on the understanding that in the interval $(0, \pi)$ $f(x)=e^{ax}$ and in the interval $(-\pi, 0)$ $f(x)=e^{-ax}$, so that $f(\pi)=f(-\pi)$ and $f(+0)=f(-0)$. By § 12 it follows that $f'(x)$ is obtained by differentiating (17) as it stands.

Suppose again we have

$$\cos x=\frac{2}{\pi}\sum_1^\infty\frac{1+(-1)^n}{n^2-1}n\sin nx. \tag{18}$$

In this case $f(x)=\cos x$ in the interval $(0, \pi)$ but $f(x)=-\cos x$ in interval $(-\pi, 0)$. Hence

$$f(\pi)-f(-\pi)=-2=D',\ f(+0)-f(-0)=2=D.$$

Hence the form corresponding to (15) is

$$\phi(x)=1-\frac{2x}{\pi}+\frac{4}{\pi}\sum_1^\infty\frac{\sin 2nx}{2n(4n^2-1)}$$

from which we obtain

$$\sin x=\frac{2}{\pi}-\frac{4}{\pi}\sum_1^\infty\frac{\cos 2nx}{4n^2-1}.$$

These examples are sufficient to show the care necessary in differentiating infinite series.

It is perhaps not out of place to add that the most natural method of passing to Fourier's Integral Formulae is to start from the equations of §§ 5 and 6, as is done by Neumann.

Geometrical Note, II.

By R. Tucker, M.A.

Figure 15.

In continuation of the results given in Vol. XI. of the *Proceedings* I note that the equations to AP, BQ, CR are respectively

$$\beta/nc = \gamma/mb : \gamma/na = a/mc : a/nb = \beta/ma :$$

and to AP′, BQ′, CR′ are

$$\beta/mc = \gamma/nb : \gamma/ma = a/nc : a/mb = \beta/na.$$

Now BQ, CR cut in α_1; CR, AP β_1; AP, BQ γ_1 and BQ′, CR′ cut in α_2; CR′, AP′ β_2; AP′, BQ′ γ_2;

where the points are given by the following

$$\left.\begin{array}{llll}(\alpha_1), & mnbc, & m^2ca, & n^2ab, \\ (\beta_1), & n^2bc, & mnca, & m^2ab, \\ (\gamma_1), & m^2bc, & n^2ca, & mnab,\end{array}\right\} \quad \left.\begin{array}{llll}(\alpha_2), & mnbc, & n^2ca, & m^2ab, \\ (\beta_2), & m^2bc, & mnca, & n^2ab, \\ (\gamma_2), & n^2bc, & m^2ca, & mnab,\end{array}\right\}$$

where the common modulus is $2\Delta/(m^2+mn+n^2)abc$. Hence the triangles $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, are concentroidal with ABC.

The straight lines $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$ are parallel to the sides BC, CA, AB respectively, and since the equation to $\alpha_1\alpha_2$ is

$$-a\alpha(m^2+n^2)+mn(b\beta+c\gamma)=0,$$

it is seen that the above lines intersect in points α_3, β_3, γ_3 where α_3 is given by $a\alpha mn/(m^2-mn+n^2)=b\beta=c\gamma$, *i.e.*, the points are on the respective medians. From these coordinates we at once obtain that the modulus of similarity for the triangle $\alpha_3\beta_3\gamma_3$ is

$$(m-n)^2/(m^2+mn+n^2).$$

The equations to the circles $\alpha_1\beta_1\gamma_1$; $\alpha_2, \beta_2, \gamma_2$; are

$$abc(m^2+mn+n^2)^2\Sigma\alpha\beta\gamma = \Sigma a\alpha \,.\, \Sigma\{a\alpha\,.\,(m^2b^2+n^2c^2+2mnbc\cos A)\}$$
$$abc(m^2+mn+n^2)^2\Sigma\alpha\beta\gamma = \Sigma a\alpha \,.\, \Sigma\{a\alpha\,.\,(n^2b^2+m^2c^2+2mnbc\cos A)\}$$

and their radical is $\Sigma[a\alpha(b^2-c^2)]=0$, *c.f.* (1).

The equation to the conic through the last-named six points is

$$mn(a^2\alpha^2+b^2\beta^2+c^2\gamma^2)=(m^2-mn+n^2)(bc\beta\gamma+ca\gamma\alpha+ab\alpha\beta);$$

this is an in-ellipse if $m^2-3mn+n^2=0$, *i.e.*, if

$$m/n=(3\pm\sqrt{5})/2.$$

It is similar and similarly situated to (4). The triangles $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, are equal to one another and

$$=(m-n)^2\mathrm{ABC}/(m^2+mn+n^2).$$

The perimeter of the hexagon $\alpha_1\beta_2\gamma_1\,\alpha_2\beta_1\gamma_2$

$$=(m-n)(m^2+n^2)/(m^2+mn+n^2)\times \text{the perimeter of ABC}.$$

The lines $\alpha_1\beta_2$, $\alpha_1\gamma_2$; are parallel respectively to AB, AC; with like results for the analogous lines.

The equation to $\alpha_1\alpha_3$ is

$$-mn(m+n)a\alpha+n^3b\beta+m^3c\gamma=0,$$

whence we see that if this line cuts CB in W, then

$$\mathrm{CW}:\mathrm{BW}=m^3:n^3.$$

If BQ′, CR; CR′, AP; AP′, BQ; intersect in α_4, β_4, γ_4, these points are given by

$$\left.\begin{array}{llll}(\alpha_4) & nbc, & mca, & mab,\\ (\beta_4) & mbc, & nca, & mab,\\ (\gamma_4) & mbc, & mca, & nab,\end{array}\right\}$$

and the modulus of similarity for $\alpha_4\beta_4\gamma_4$ is $(m-n)/(2m+n)$.

The equation to the circle $\alpha_4\beta_4\gamma_4$ is

$$(2m+n)^2abc\Sigma\alpha\beta\gamma = \Sigma a\alpha\,.\,\Sigma[ma\alpha\{-ma^2+(m+n)(b^2+c^2)\}].$$

Again, if CR′, BQ; AP′, CR; BQ′, AP; intersect in α_5, β_5, γ_5, these points are given by

$$\left.\begin{array}{llll} (\alpha_5) & mbc, & nca, & nab, \\ (\beta_5) & nbc, & mca, & nab, \\ (\gamma_5) & nbc, & nca, & mab, \end{array}\right\}$$

and the modulus of similarity for $\alpha_5\beta_5\gamma_5$ is

$$(m-n)/(m+2n).$$

The equation to the circle $\alpha_5\beta_5\gamma_5$ is

$$(2n+m)^2abc\Sigma\alpha\beta\gamma = \Sigma a\alpha\,.\,\Sigma[na\alpha\{-ma^2+(m+n)(b^2+c^2)\}].$$

All the triangles are concentroidal with ABC.

If AP′$(\beta/mc = \gamma/nb)$ cuts QR$(-mna\alpha + n^2b\beta + m^2c\gamma = 0)$ in p', this point is given by

$$a\alpha/(m+n) = b\beta/m = c\gamma/n = \Delta,$$

i.e., p' is on the mid-parallel to BC, (say ZY), and is the mid point of QR. It is also readily seen that

$$Zp' = n\,.\,ZY.$$

Hence p', q', r' (analogous points to p') form the medial triangle of PQR.

In like manner if AP cuts Q′R′ in p, then pqr is the medial triangle of P′Q′R′, and $Zp = m\,.\,ZY$.

We see then that $pqrp'q'r'$ are the exact analogues on

YZ, ZX, XY of PQRP′Q′R′ on BC, CA, AB.

I consider two envelopes, viz., of $\beta_1\gamma_4$, $\beta_4\gamma_3$.

The line $\beta_1\gamma_4$ is given by

$$ma\alpha(m+n) - b\beta(m^2+mn+n^2) + c\gamma mn = 0,$$

or, as it may be written,

$$m^2(b\beta + c\gamma) - m(a\alpha + b\beta + c\gamma) + b\beta = 0$$

The envelope, therefore, is

$$(a\alpha + b\beta + c\gamma)^2 = -4b\beta(b\beta + c\gamma),$$

an hyperbola of which the asymptotes are

$$\beta = 0, \ \beta b + c\gamma = 0$$

i.e., CA, and the parallel to BC through A.

The line $\beta_4\gamma_3$ is given by

$$n^2 a\alpha + m(m-n)b\beta - mnc\gamma = 0,$$

or by

$$n^2(a\alpha + 2b\beta + c\gamma) - n(3b\beta + c\gamma) + b\beta = 0.$$

The envelope of which is

$$(b\beta + c\gamma)^2 = 4aba\beta,$$

which is also an hyperbola, the asymptotes being given by

$$a = 0, \ 3b\beta + c\gamma = 0.$$

Fifth Meeting, March 9th, 1894.

Dr C. G. KNOTT, President, in the Chair.

Coordonnées Tangentielles.

By M. PAUL AUBERT.

Cette note a pour objet de familiariser les élèves avec l'emploi des coordonnées tangentielles, en appliquant ces coordonnées, concurremment avec les coordonnées ponctuelles, à la résolution d'un certain nombre de questions, d'ordre très général, concernant les surfaces du second ordre.

I. FOYERS ET FOCALES.

DÉFINITION :—Soit $f(x, y, z) = 0$

l'équation d'une surface du second ordre. On dit qu'un point α, β, γ est foyer de cette surface si l'on peut trouver deux expressions L et M linéaires en x, y, z satisfaisant à l'identité

$$(1) \quad (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 + eL^2 + e'M^2 = \lambda f(x, y, z)$$

où l'on a $e = \pm 1$, $e' = \pm 1$, λ étant une constante. La droite qui a pour équations $L = 0$, $M = 0$, est la directrice correspondante.

L'identité (1) conduit à considérer un foyer comme le centre d'une sphère de rayon nul doublement tangente à la surface.

Nous ramènerons l'étude des foyers d'une surface à un autre problème en établissant les deux propositions suivantes :

I°. *Tout sommet d'un cône de révolution circonscrit à la surface est un foyer de celle-ci.*

En effet l'équation d'un cône de révolution ayant pour sommet le point (α, β, γ) est de la forme

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - P^2 = 0,$$

$P = 0$ étant l'équation d'un plan passant par le point α, β, γ. Si ce cône est circonscrit à la surface considérée, on a l'identité

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - P^2 = \lambda f(x, y, z) + Q^2$$

où Q est linéaire en x, y, z. On peut donc écrire

$$\lambda f(x, y, z) = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - P^2 - Q^2$$

et le point (α, β, γ) est foyer de la surface.

II°. *Tout foyer de la surface est le sommet d'un cône de révolution circonscrit à cette surface.*

Prenons en effet le point (α, β, γ) donné comme foyer de la surface, pour origine de coordonnées ; l'équation de la surface deviendra de la forme

$$f(x, y, z) = x^2 + y^2 + z^2 + eL^2 + e'M^2 = 0.$$

Or le cône circonscrit ayant pour sommet l'origine est

$$4f(x, y, z)f(x_1, y_1, z_1) - (xf'_{x_1} + yf'_{y_1} + zf'_{z_1} + tf'_{t_1})^2 = 0$$

où l'on fait $x_1 = y_1 = z_1 = 0$ et $t = t_1 = 1$.

Si donc nous posons

$$\sqrt{e}\,L = ax + by + cz + dt = L_1$$

$$\sqrt{e'}M = a'x + b'y + c'z + d't = M_1$$

l'équation du cône sera

$$(d^2 + d'^2)(x^2 + y^2 + z^2 + L_1^2 + M_1^2) - (L_1 d + M_1 d')^2 = 0$$

ou $(d^2 + d'^2)(x^2 + y^2 + z^2) - (L_1 d + M_1 d')^2 + L_1^2(d^2 + d'^2) + M_1^2(d^2 + d'^2) = 0$

c'est à dire $\quad (d^2 + d'^2)(x^2 + y^2 + z^2) + (L_1 d' - M_1 d)^2 = 0$

équation d'une surface de révolution; le cône est donc de révolution.

Cela posé, nous allons chercher les directions principales d'un cône circonscrit à la surface donnée par le point (α, β, γ), puis nous écrirons que ce cône est de révolution.

PROBLÈME :—*Déterminer les axes d'un cône de sommet donné circonscrit à une quadrique donnée.*

Supposons la quadrique définie par son équation tangentielle

$$Au^2 + Bv^2 + Cw^2 - h^2 = 0,$$

l'équation du point donné, (α, β, γ) étant

$$u\alpha + v\beta + w\gamma + h = 0 \qquad (2)$$

nous avons immédiatement l'équation tangentielle du cône circonscrit par ce point à la surface

$$Au^2 + Bv^2 + Cw^2 - (u\alpha + v\beta + w\gamma)^2 = 0. \qquad (3)$$

Or le cône réciproque a mêmes directions d'axes que celui-ci, et, son équation en coordonnées ponctuelles étant

$$\phi(x, y, z) = Ax^2 + By^2 + Cz^2 - (\alpha x + \beta y + \gamma z)^2 = 0$$

il nous suffit de déterminer les directions principales de ce cône.

Posons $$\rho = \alpha x + \beta y + \gamma z.$$

Nous savons que les coordonnées du point directeur d'une direction principale sont les solutions communes aux équations

$$\begin{aligned} (A - s)x - \alpha\rho &= 0 \\ (B - s)y - \beta\rho &= 0 \\ (C - s)z - \gamma\rho &= 0 \\ \alpha x + \beta y + \gamma z - \rho &= 0 \end{aligned}$$

s étant une inconnue auxiliaire assujettie à vérifier par cela même la condition

$$\begin{vmatrix} A - s & 0 & 0 & \alpha \\ 0 & B - s & 0 & \beta \\ 0 & 0 & C - s & \gamma \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = 0$$

ou bien

$$(A - s)(B - s)(C - s)\left[\frac{\alpha^2}{A - s} + \frac{\beta^2}{B - s} + \frac{\gamma^2}{C - s} - 1\right] = 0 \qquad (4)$$

Toute racine de cette équation substituée à s dans les équations qui précédent donnera une direction principale.

Condition pour que le cône soit de révolution :—Il faut et il suffit que le cône réciproque soit de révolution, et pour cela, que l'équation (4) ait une racine double.

Je dis qu' une racine double de cette équation ne peut être que l'une des valeurs A, B, ou C, c'est à dire, une racine de l'équation en s relative à la forme

$$Ax^2 + By^2 + Cz^2.$$

En effet, si s est une racine double de l'équation (4) la forme $\phi(x, y, z) - s(x^2 + y^2 + z^2)$ est carré parfait.
On a donc

$$Ax^2 + By^2 + Cz^2 - s(x^2 + y^2 + z^2) - (\alpha x + \beta y + \gamma z)^2 = Q^2$$

ou

$$Ax^2 + By^2 + Cz^2 - s(x^2 + y^2 + z^2) = P^2 + Q^2$$

P et Q étant linéaires et homogènes par rapport à x, y, z. Donc s est une racine de l'équation en s relative à la forme

$$Ax^2 + By^2 + Cz^2.$$

Cherchons alors la condition pour que l'une de ces racines, C par exemple soit racine de l'équation (4). Le terme ne contenant pas en facteur $(s - C)$ dans cette équation est

$$\gamma^2(A - C)(B - C)$$

Si donc on suppose $(A - C)(B - C) \gtrless 0$ on doit avoir $\gamma = 0$ et il reste

$$(A - s)(B - s)\left[\frac{\alpha^2}{A - s} + \frac{\beta^2}{B - s} - 1\right] = 0.$$

Cette équation doit encore admettre la racine $s = C$ qui était racine double de la première. On doit donc avoir

$$\frac{\alpha^2}{A - C} + \frac{\beta^2}{B - C} - 1 = 0.$$

Ainsi le point (α, β, γ) est déterminé par les conditions

$$\left.\begin{aligned} &\gamma = 0 \\ &\frac{\alpha^2}{A - C} + \frac{\beta^2}{B - C} - 1 = 0 \end{aligned}\right\}$$

Si l'on avait pris $s = A$ on aurait trouvé de même

$$\left.\begin{aligned} &\alpha = 0 \\ &\frac{\beta^2}{B - A} + \frac{\gamma^2}{C - A} - 1 = 0 \end{aligned}\right\}$$

et pour $s = B$ on aurait eu

$$\left.\begin{aligned} &\beta = 0 \\ &\frac{\gamma^2}{C - B} + \frac{\alpha^2}{A - B} - 1 = 0 \end{aligned}\right\}$$

On obtient ainsi trois courbes situées dans les plans principaux de la surface donnée, et lieux des sommets de cônes de révolution circonscrits à cette surface. Tous ces points sont, nous l'avons vu, des foyers de la surface. Aussi a-t-on donné á ces trois courbes le nom de *focales* de la surface considérée.

Remarque.—Le calcul précédent s'applique au problème suivant. Trouver les directions principales d'un cône de sommet donné et passant par une conique donnée,

$$\frac{x^2}{A}+\frac{y^2}{B}-1=0.$$

Il suffit de faire $C=0$ dans les résultats obtenus.

Ainsi considérons la conique

$$Au^2+Bv^2-h^2=0.$$

La trace d'un plan tangent au cône de sommet (α, β, γ) et dont la directrice est cette conique, sur le plan de cette courbe sera tangente à la conique. On aura donc

$$u\alpha+v\beta+w\gamma+h=0$$
$$Au^2+Bv^2-h^2=0$$

L'élimination de h donnera l'equation du cône, et on achèvera comme précédemment, en introduisant le cône réciproque.

II. Surfaces Homofocales.

On dit que deux quadriques sont homofocales si elles admettent les mêmes focales.

Soit $$f(u, v, w, h)=0$$

l'équation tangentielle d'une surface, et

$$u\alpha+v\beta+w\gamma+h=0$$

l'équation du point α, β, γ. Le cône circonscrit par ce point s'obtient en éliminant h entre ces deux equations, ce qui donne $\psi(u, v, w)=0$, et le point (α, β, γ) est foyer si ce cône est de révolution. Or les calculs nous ont montré que dans ce cas la forme

$$\psi(u, v, w) - s(u^2 + v^2 + w^2)$$

est carré parfait. *

Il est clair que si on considère une surface dont l'équation est

$$f(u, v, w, h) + \lambda(u^2 + v^2 + w^2) = 0$$

le cône circonscrit correspondant sera

$$\psi(u, v, w) + \lambda(u^2 + v^2 + w^2) = 0$$

et il sera de révolution en même temps que le cône $\psi(u, v, w)$. On peut le voir en prenant le cône réciproque.

Ainsi, toutes les surfaces de la famille représentée par l'équation

$$f(u, v, w, h) + \lambda(u^2 + v^2 + w^2) = 0$$

où λ désigne un paramètre arbitraire ont les mêmes focales.

Pour interpréter ce résultat, remarquons que toutes ces surfaces sont inscrites dans une même surface développable, circonscrite à la

fois à $$f(u, v, w, h) = 0$$

et à $$u^2 + v^2 + w^2 = 0$$

puisque les coordonnées (u, v, w, h) de tout plan tangent à la fois à ces deux surfaces vérifient l'équation précédente.

Cherchons si parmi toutes les surfaces qui constituent la famille considérée il peut y avoir des coniques.

Pour cela rapportons la quadrique $f(u, v, w, h)$ à ses plans principaux, et soit

$$Au^2 + Bv^2 + Cw^2 - h^2 = 0$$

son équation. L'équation

$$Au^2 + Bv^2 + Cw^2 - h^2 + \lambda(u^2 + v^2 + w^2) = 0$$

représentera une conique si le discriminant de la forme premier membre est nul. On retrouve ainsi les trois focales. L'équation

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* On doit exprimer en effet que la trace de la surface sur le plan de l'infini est bitangente au cercle de l'infini. Le calcul est le même qu'en coordonnées ponctuelles, ce qui s'explique en remarquant que par la transformation réciproque, à deux coniques bitangentes correspondent deux coniques bitangentes.

donne, il est vrai, pour λ une quatrième racine, de valeur infinie à laquelle correspond le cercle de l'infini.

Ces trois coniques sont entièrement situées sur la développable dont nous avons parlé, dont elles constituent des lignes de points doubles. Chacun de leurs points en effet, étant sommet d'un cône isotrope bitangent à la quadrique, est l'intersection de deux génératrices de la développable tangentes à la quadrique.

Ainsi : *Pour obtenir les trois focales d'une quadrique, on égale à zéro le discriminant de la forme* $f(u, v, w, h) + \lambda(u^2 + v^2 + w^2)$. *L'équation ainsi obtenue donne trois valeurs finies de* λ *pour chacune desquelles on obtient une conique focale d'équation*

$$f(u, v, w, h) + \lambda(u^2 + v^2 + w^2) = 0.$$

Reprenons l'équation d'une quadrique à centre rapportée à ses plans principaux

$$Au^2 + Bv^2 + Cw^2 - h^2 = 0 \quad \dots \quad \dots \quad \dots \quad (1)$$

L'équation d'une surface homofocale pourra s'écrire

$$(A - s)u^2 + (B - s)v^2 + (C - s)w^2 - h^2 = 0. \quad \dots \quad (2)$$

La condition pour qu'elle passe par un point α, β, γ est

$$\frac{\alpha^2}{A - s} + \frac{\beta^2}{B - s} + \frac{\gamma^2}{C - s} - h^2 = 0 \quad \dots \quad \dots \quad (3)$$

d'où l'on déduit trois valeurs réelles de s donnant trois surfaces homofocales à la première et passant par le point.

Nous avons vu que les directions principales du cône circonscrit à la quadrique (1) par le point (α, β, γ) sont données par

$$(A - s)x + \rho\alpha = 0$$
$$(B - s)y + \rho\beta = 0$$
$$(C - s)z + \rho\gamma = 0$$

où s est racine précisément de l'équation (3).

Les cosinus directeurs qui définissent les axes du cône sont proportionnels à

$$\frac{\alpha}{A - s}, \ \frac{\beta}{B - s}, \ \frac{\gamma}{C - s}.$$

Or ces quantités sont aussi proportionnelles aux cosinus directeurs de la normale à la quadrique (2) au point α, β, γ. On reconnait ainsi que

Les axes du cône circonscrit à une quadrique sont les normales aux trois surfaces homofocales à cette quadrique qui passent par le sommet du cône. On en déduit que ces trois surfaces *se coupent orthogonalement.*

Quand une surface est définie par son équation ponctuelle

$$F(x, y, z, t) = 0$$

on dit que deux points (x_1, y_1, z_1, t_1) et (x_2, y_2, z_2, t_2) sont conjugués par rapport à cette surface si on a

$$x_1 F'_{x_2} + y_1 F'_{y_2} + z_1 F'_{z_2} + t_1 F'_{t_2} = 0.$$

Pareillement nous dirons que deux plans (u_1, v_1, w_1, h_1) et (u_2, v_2, w_2, h_2) sont conjugués par rapport à la surface dont l'équation tangentielle est

$$F(u, v, w, h) = 0$$

si l'on a

$$u_1 F'_{u_2} + v_1 F'_{v_2} + w_1 F'_{w_2} + h_1 F'_{h_2} = 0.$$

THÉORÈME :—*Si deux plans sont conjugués par rapport à deux surfaces homofocales ils sont rectangulaires.*

En effet soient
$$f(u, v, w, h) + \lambda(u^2 + v^2 + w^2) = 0$$
$$f(u, v, w, h) + \mu(u^2 + v^2 + w^2) = 0$$

les deux surfaces considérées. On a par hypothèse

$$u_1 f'_{u_2} + v_1 f'_{v_2} + w_1 f'_{w_2} + h_1 f'_{h_2} + \lambda(u_1 u_2 + v_1 v_2 + w_1 w_2) = 0$$
$$u_1 f'_{u_2} + v_1 f'_{v_2} + w_1 f'_{w_2} + h_1 f'_{h_2} + \mu(u_1 u_2 + v_1 v_2 + w_1 w_2) = 0.$$

On en déduit immédiatement

$$u_1 u_2 + v_1 v_2 + w_1 w_2 = 0$$

ce qui établit la proposition.

THÉORÈME : *Les pôles d'un plan fixe par rapport à toutes les surfaces homofocales d'une même famille sont situés sur la normale à celle de ces surfaces qui touche le plan donné, au point de contact.*

En effet, le pôle d'un plan u_1, v_1, w_1, h_1 a pour coordonnées

$$x = \frac{f'_{u_1} + 2\lambda u_1}{f'_{h_1}}, \quad y = \frac{f'_{v_1} + 2\lambda v_1}{f'_{h_1}}, \quad z = \frac{f'_{w_1} + 2\lambda w_1}{f'_{h_1}}$$

Or en remarquant que le point de contact du plan (u_1, v_1, w_1, h_1) avec la surface λ_1 est

$$x_1 = \frac{f'_{u_1} + 2\lambda_1 u_1}{f_{h_1}}, \quad y_1 = \frac{f'_{v_1} + 2\lambda_1 v_1}{f_{h_1}}, \quad z = \frac{f'_{w_1} + 2\lambda_1 w_1}{f'_{h_1}}$$

et en posant $$2(\lambda - \lambda_1) = \kappa \times f_{h_1}$$

on peut écrire

$$x = x_1 + \kappa u_1, \quad y = y_1 + \kappa v_1, \quad z = z_1 + \kappa w_1.$$

Il en résulte bien que le point x, y, z est sur la normale au plan donné au point x_1, y_1, z_1 puisque les cosinus directeurs de cette normale sont, on le sait, proportionnels aux coordonnées u_1, v_1, w_1 de ce plan.

Remarque.—A une famille de surfaces homofocales correspond en vertu du principe du dualité, une famille de surfaces

$$f(x, y, z, t) + \lambda(x^2 + y^2 + z^2) = 0 \quad \ldots \quad \ldots \quad (4)$$

passant par la courbe d'intersection des deux surfaces

$$f(x, y, z, t) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$x^2 + y^2 + z^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Aux propriétés établies plus haut correspondent les propriétés corrélatives suivantes.

I. *Il existe trois surfaces de la famille* (4) *tangentes à un plan donné, et les droites qui joignent l'origine, centre du cône isotrope, aux trois points de contact sont rectangulaires.*

En effet, soit (u, v, w, h) un plan tangent à (4) on aura

$$\begin{aligned} \tfrac{1}{2}f'_x + \lambda x &= \rho u \\ \tfrac{1}{2}f'_y + \lambda y &= \rho v \\ \tfrac{1}{2}f'_z + \lambda z &= \rho w \\ \tfrac{1}{2}f'_t \qquad &= \rho h \end{aligned}$$

$$ux + vy + wz = ht = 0.$$

Les points de contact s'obtiendront en remplacant λ par les trois racines de l'équation qui exprime que ce système a une solution en x, y, z, t, ρ.

Désignons par (x_1, y_1, z_1, t_1) une des solutions du système, correspondant à la racine λ_1, le point (x, y, z, t) correspondant à la valeur λ. On peut écrire

$$\begin{aligned} \tfrac{1}{2}f'_{x_1} + \lambda_1 x_1 &= \rho_1 u \\ \tfrac{1}{2}f'_{y_1} + \lambda_1 y_1 &= \rho_1 v \\ \tfrac{1}{2}f'_{z_1} + \lambda_1 z_1 &= \rho_1 w \\ \tfrac{1}{2}f'_{t_1} \qquad &= \rho_1 h \end{aligned}$$

$$ux_1 + vy_1 + wz_1 + ht_1 = 0.$$

Multiplions respectivement par (x_1, y_1, z_1, t_1) les quatre premières relations du premier système, puis ajoutons, en tenant compte de la dernière relation du second système, il vient

$$\tfrac{1}{2}(x_1 f'_x + y_1 f'_y + z_1 f'_z + t_1 f'_t) + \lambda\,(xx_1 + yy_1 + zz_1) = 0.$$

En opérant de même sur le second système on a

$$\tfrac{1}{2}(x\, f'_{x_1} + y\, f'_{y_1} + z\, f'_{z_1} + t\, f'_{t_1}) + \lambda_1(x_1 x + y_1 y + z_1 z) = 0.$$

En retranchant ces deux relations membre à membre, il vient puisque $(\lambda - \lambda_1) \gtrless 0$

$$x\,x_1 + y\,y_1 + z\,z_1 = 0.$$

On démontrerait de même,

$$\begin{aligned} x\,x_2 + y\,y_2 + z\,z_2 &= 0 \\ x_1 x_2 + y_1 y_2 + z_1 z_2 &= 0 \end{aligned}$$

ce qui établit la proposition.

Interprétation géométrique.—Une surface (4) tangente au plan (u, v, w, h) est coupée par ce plan suivant deux droites formant l'un des couples de sécantes communes aux coniques sections du plan donné par les deux surfaces (5) et (6). Le point de contact est donc un sommet du triangle conjugué commun à ces deux coniques. Le tétraèdre formé par les droites joignant les trois points de contact du même plan et des trois surfaces (4), à l'origine, est donc conjugué par rapport au cône isotrope. Par suite les trois arêtes issues du sommet du cône sont rectangulaires.

On établirait aussi simplement la seconde propriété corrélative à savoir: *Les deux droites qui joignent l'origine à deux points conjugués par rapport à deux surfaces de la famille* (4) *sont rectangulaires.*

La troisième propriété peut s'énoncer ainsi. *Les plans polaires d'un point fixe par rapport à toutes les surfaces de la famille* (4) *passent par une droite fixe, intersection du plan tangent à la surface particulière qui passe au point donné, en ce point par le plan mené par l'origine perpendiculairement à la droite joignant le point donnè à l'origine.*

On montre en effet que les coordonnées de l'un quelconque de ces plans ont la forme

$$\begin{aligned} x &= u_1 + kx_1 \\ y &= v_1 + ky_1 \\ z &= w_1 + kz_1 \\ h &= h_1 \end{aligned}$$

où u_1, v_1, w_1, h_1 sont les coordonnées du plan tangent à la surface λ_1 qui passe par le point donné (x_1, y_1, z_1, t_1).

II. Sections Circulaires; sommets d'une quadrique en coordonnées tangentielles.

Problème:—*Une surface du second ordre étant définie par son équation tangentielle, exprimer qu'un plan u, v, w, h la coupe suivant un cercle.*

Il suffit évidemment de considérer ici le cône asymptote de la surface donnée. Supposons l'origine des coordonnées au sommet de ce cône, et désignons par p, q, r les coordonnées d'un plan tangent quelconque à ce cône, en sorte que son équation tangentielle soit

$$f(p, q, r) = 0.$$

Le point de contact du plan (p, q, r) situé dans le plan de l'infini a pour coordonnées ponctuelles

$$t=0, \quad x=f'_p, \quad y=f'_q, \quad z=f'_r.$$

Ce point sera dans le plan (u, v, w, h) si on a

$$uf'_p+vf'_q+wf'_r=0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

En y joignant

$$f(p, q, r)=0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

on obtient les valeurs de (p, q, r) définissant une génératrice passant par le point de contact considéré. La section plane (u, v, w, h) sera un cercle si le point de contact en question est sur la surface

$$x^2+y^2+z^2=0$$

c'est à dire, si l'on a

$$(f'_p)^2+(f'_q)^2+(f'_r)^2=0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Il faudra donc écrire que les équations (1), (2), et (3) ont un système de solutions communes en p, q, r.

On peut interpréter cette condition en remarquant que si (p, q, r) représentaient les coordonnées d'un point, ce serait exprimer que le plan (1) contient les génératrices communes aux deux cônes (2) et (3).

Le calcul peut être dirigé ainsi : on a

$$pf'_p+qf'_q+rf'_r=0$$

et aussi

$$uf'_p+vf'_q+wf'_r=0$$

d'où

$$\frac{f'_p}{wq-vr}=\frac{f'_q}{ur-wp}=\frac{f'_r}{vp-uq}$$

On est donc ramené à écrire que les équations

$$f(p, q, r)=0$$
$$uf'_p+vf'_q+wf'_r=0$$
$$(wq-vr)^2+(ur-wp)^2+(vp-uq)^2=0,$$

ont un système de solutions communes en p, q, r.

Dans certains cas il est plus simple, pour exprimer que le plan u, v, w, h est un plan cyclique du cône, de considérer le cône réciproque. Le plan donné coupant le premier cône suivant deux droites isotropes, les plans tangents au second cône par la droite perpendiculaire au plan donné passeront par une des focales de ce cône. Chacun de ces plans touche en effet le premier cône le long d'une des génératrices isotropes. On aura donc à exprimer que la droite perpendiculaire au plan (u, v, w, h) est une focale du cône réciproque du cône donné ; ce qui peut être plus simple que le calcul précédent.

PROBLÈME :—*Une surface du second ordre étant définie par son équation tangentielle, exprimer qu'un plan (u, v, w, h) la coupe suivant une hyperbole équilatère.*

On prendra l'équation ponctuelle du cône réciproque du cône asymptote de la surface donnée, soit

$$f(x, y, z) = 0$$

et il suffira d'écrire que les plans tangents menés à ce cône par la droite dont les cosinus directeurs sont proportionnels à u, v, w, sont rectangulaires. On est ramené à une question bien connue.

PROBLÈME :—*Une surface du second ordre étant définie par son équation tangentielle, trouver les sommets de cette surface.*

Ecrivons l'équation donnée sous la forme

$$f(u, v, w, h) = \phi(u, v, w) + 2Ph + h^2 = 0$$

où l'on a posé

$$\phi(u, v, w) = au^2 + a'v^2 + a''w^2 + 2bvw + 2b'wu + 2b''uv$$
$$P = cu + c'v + c''w.$$

Le centre de la surface est le pôle du plan de l'infini. Ses coordonnées ponctuelles sont donc c, c', et c''.

Le point de contact du plan u, v, w, h a pour coordonnées ponctuelles

$$\frac{\phi'_u + 2ch}{2P + 2h}, \quad \frac{\phi'_v + 2c'h}{2P + 2h}, \quad \frac{\phi'_w + 2c''h}{2P + 2h}.$$

Les paramètres directeurs de la droite qui joint le centre au point de contact, c'est à dire, du diamètre conjugué du plan u, v, w sont donc

$$\frac{\phi'_u + 2ch}{2P + 2h} - c, \quad \frac{\phi'_v + 2c'h}{2P + 2h} - c', \quad \frac{\phi'_w + 2c''h}{2P + 2h} - c''$$

quantités proportionnelles à

$$\phi'_u - 2cP, \quad \phi'_v - 2c'P, \quad \phi'_w - 2c''P.$$

Ce sera un axe de la surface si on a

$$\frac{\phi'_u - 2c\mathrm{P}}{u} = \frac{\phi'_v - 2c'\mathrm{P}}{v} = \frac{\phi'_w - 2c''\mathrm{P}}{w}$$

Désignons par $2s$ la valeur de ces rapports, et développons les expressions ϕ'_u, ϕ'_v, ϕ'_w, nous obtenons les équations

$$(a - c^2 - s)u + (b'' - cc')v + (b' - cc'')w = 0$$
$$(b'' - cc')u + (a' - c'^2 - s)v + (b - c'c'')w = 0$$
$$(b' - cc'')u + (b - c'c'')v + (a'' - c''^2 - s)w = 0.$$

La forme de ces équations conduit à considérer la fonction homogène

$$\mathrm{F}(u, v, w) = (a - c^2)u^2 + (a' - c'^2)v^2 + (a'' - c''^2)w^2 + 2(b - c'c'')wv + 2(b' - c''c)uw + 2(b'' - cc')uv.$$

Les premiers membres des équations précédentes sont les trois derivées partielles de la forme

$$\mathrm{F}(u, v, w) - s(u^2 + v^2 + w^2).$$

La question est donc identique à celle qui concerne l'équation en s d'une quadrique en coordonnées ponctuelles. On remarquera toutefois qu'ici l'équation contient les coefficients c, c', c'' des termes qui correspondent aux termes du premier degré dans l'équation ponctuelle d'une quadrique.

Ayant déterminé les racines de l'équation

$$\begin{vmatrix} a - c^2 - s & b'' - cc' & b' - cc'' \\ b'' - cc' & a' - c'^2 - s & b - c'c'' \\ b' - cc'' & b - c'c'' & a'' - c''^2 - s \end{vmatrix} = 0$$

on les portera dans le système des équations précédentes, qui fournira les valeurs de u, v, w correspondant aux plans principaux de la surface. L'équation

$$\phi(u, v, w) + 2\mathrm{P}h + h^2 = 0$$

où l'on portera ces valeurs de u, v, w donnera deux valeurs de h qui feront connaître les deux plans tangents parallèles au plan principal et par suite détermineront les points de contact ou sommets correspondants de la surface.

Two Triplets of Circum-Hyperbolas.

By R. TUCKER, M.A.

PART I.

1. Let one of the series of circles, which can be drawn so as to touch the sides AB, AC of a triangle, touch those sides in K, L; and let AK = AL = δ.

Then the points K, L are

$$b(c-\delta),\ a\delta,\ 0;\quad c(b-\delta),\ 0,\ a\delta;$$

and the lines BL, CK are given by

$$a\alpha\delta = c(b-\delta)\gamma,\ \ a\alpha\delta = b(c-\delta)\beta.$$

Hence eliminating δ we get for the locus of P the hyperbola

$$(b-c)\beta\gamma + a\gamma\alpha - a\alpha\beta = 0. \qquad (A_1) \qquad \text{(i.)}$$

The allied hyperbolas are

$$-b\beta\gamma + (c-a)\gamma\alpha + b\alpha\beta = 0, \qquad (B_1)$$
$$c\beta\gamma - c\gamma\alpha + (a-b)\alpha\beta = 0. \qquad (C_1).$$

2. We shall in the main confine our attention to the consideration of the conic A_1.

3. Its centre is evidently the mid point of BC and the four fixed points through which the triple system passes are A, B, C and the Gergonne-point of ABC (or $a\alpha(s-a) = b\beta(s-b) = c\gamma(s-c)$).

4. The polar of any point $(\alpha', \beta', \gamma')$ with respect to (i.) is

$$a\alpha(\gamma' - \beta') + \beta[(b-c)\gamma' - a\alpha'] + \gamma[(b-c)\beta' + a\alpha'] = 0. \quad \dots \quad \text{(ii.)}$$

The tangent at A therefore is $\beta = \gamma$, *i.e.*, the bisector of the $\angle$ A touches the curve at A.

The tangents at B, C are

$$a\alpha - (b-c)\gamma = 0,\ \ a\alpha + (b-c)\beta = 0,$$

hence they intersect on the external bisector of $\angle$ A.

The tangent at the Gergonne-point is

$$a(b-c)(s-a)^2\alpha+b^2(s-b)^2\beta-c^2(s-c)^2\gamma=0,$$

if this line, and the allied ones, cut the sides in $p, q, r,$ then Ap, Bq, Cr cointersect in the point

$$a^2(s-a)^2\alpha=b^2(s-b)^2\beta=c^2(s-c)^2\gamma\,^* \quad \ldots \quad \text{(iii.)}$$

The polar of the incentre is $\beta(s-b)=\gamma(s-c)$, hence Ap_1, Bq_1, Cr_1 cointersect in $\alpha(s-a)=\beta(s-b)=\gamma(s-c)$.† ... ... (iv.)

The polar of the orthocentre is

$$a\alpha\cos A(\cos B-\cos C)-b\cos B\beta(1-\cos A)+c\cos C\gamma(1-\cos A)=0,$$

and Ap_2, Bq_2, Cr_2 meet in $a\cos A\alpha=b\cos B\beta=c\cos C\gamma$. ... (v.)

The polar of the circumcentre is

$$a\alpha\cos\frac{A}{2}\sin\frac{B-C}{2}-b\beta\sin\left(C-\frac{A}{2}\right)\sin\frac{A}{2}+c\gamma\sin\left(B-\frac{A}{2}\right)\sin\frac{A}{2}=0,$$

hence Ap_3, Bq_3, Cr_3 intersect in

$$a\alpha\operatorname{cosec}\left(A-\frac{B}{2}\right)=b\beta\operatorname{cosec}\left(B-\frac{C}{2}\right)=c\gamma\operatorname{cosec}\left(C-\frac{A}{2}\right). \quad \ldots \text{(vi.)}$$

The polars of the mid points of CA, AB are

$$a\alpha+(b-2c)\beta+c\gamma=0, \ (\therefore \text{ parallel to CA}),$$

and

$$a\alpha+b\beta-(2b-c)\gamma=0, \ (\therefore \text{ parallel to AB}),$$

and these meet in $\dfrac{\beta}{b}=\dfrac{\gamma}{c}=\dfrac{-a\alpha}{(b-c)^2}$,

i.e., on the symmedian line through A.

The polar of the centroid is

$$a\alpha(b-c)+b\beta(b-2c)+c\gamma(2b-c)=0,$$

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* The mode of procedure adopted in the following paragraphs is the same, viz., p, q, r points are on BC, CA, AB respectively.

† The polars of $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ are respectively

$$\beta(s-c)=\gamma(s-b)$$

($\therefore$ this is the isogonal conjugate of the polar of the incentre),

$$a\alpha-(s-b)\beta+(s-b)\gamma=0, \text{ and } a\alpha+(s-c)\beta-(s-c)\gamma=0.$$

hence Ap_4, Bq_4, Cr_4 cointersect in

$$a\alpha/(b+c-3a) = b\beta/(c+a-3b) = c\gamma/(a+b-3c). \quad \ldots \quad \text{(vii.)}$$

5. The circumcircle cuts (i.) in the additional point

$$-a\alpha\cos\frac{A}{2}\cos\frac{B-C}{2} = b\beta\sin\frac{A}{2}\cos\left(C-\frac{A}{2}\right) = c\gamma\sin\frac{A}{2}\cos\left(B-\frac{A}{2}\right),$$

hence Ap', Bq', Cr' meet in

$$a\alpha\sec\left(A-\frac{B}{2}\right) = b\beta\sec\left(B-\frac{C}{2}\right) = c\gamma\sec\left(C-\frac{A}{2}\right). \quad \ldots \quad \text{(viii.)}$$

6. If $$\sigma \equiv a\alpha + b\beta + c\gamma,$$

then the asymptotes are given by the equations

$$4abc\alpha(\beta+\gamma) = (b-c)(\sigma^2 - 4bc\beta\gamma). \quad \ldots \quad \text{(ix.)}$$

These cut CA in m_1, m_2 determined by

$$4abc\gamma\alpha = (b-c)(a\alpha + c\gamma)^2, \quad \ldots \quad \ldots \quad \text{(x.)}$$

whence $$\frac{a\alpha}{c\gamma} = \frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}+\sqrt{c}}, \text{ or } = \frac{\sqrt{b}+\sqrt{c}}{\sqrt{b}-\sqrt{c}},$$

i.e., Bm_1, Bm_2 are isotomic conjugate lines with respect to CA.

7. The foci are determined from

$$\frac{a^2bc}{4\Delta^2}\alpha_1^2 + \frac{(b-c)^2}{4} = \frac{a^2bc}{4\Delta^2}\beta_1^2 - \frac{a^2c}{2\Delta}\beta_1 + \frac{a^2}{4}$$

$$= \frac{a^2bc}{4\Delta^2}\gamma_1^2 - \frac{a^2b}{2\Delta}\gamma_1 + \frac{a^2}{4}. \quad \ldots \quad \text{(xi.)}$$

8. Reverting to § 1, and calling Q, R, the points corresponding to P, we see that, for the same value δ, AP, BQ, CR meet in O, given by $$a\alpha/(a-\delta) = b\beta/(b-\delta) = c\gamma/(c-\delta);$$
hence the locus of O is the incentroidal axis

$$a\alpha(b-c) + b\beta(c-a) + c\gamma(a-b) = 0.$$

If $\delta = s$, *i.e.*, if the circles are the excircles, then O is the point

$$a\alpha/(s-a) = b\beta/(s-b) = c\gamma/(s-c).$$

9. The isogonal transformation of (i.) is

$$(b-c)a+a\beta-a\gamma=0,$$

if this, and the allied lines for B_1, C_1, meet the sides in l, m, n, then Al, Bm, Cn meet in the incentre.

10. If we take the polars of any point $(\alpha', \beta', \gamma')$ with respect to the triple system, we find that they meet in a point (π), viz.,

$$\begin{aligned}&[-a\alpha'(s-a)+b\beta'(s-b)+c\gamma'(s-c)],\\ &[\ \ a\alpha'(s-a)-b\beta'(s-b)+c\gamma'(s-c)],\\ &[\ \ a\alpha'(s-a)+b\beta'(s-b)-c\gamma'(s-c)].\end{aligned}$$

If the given point be the centroid, then π is the point (vii.): if it be the in-centre, then π is the point $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$.

11. If $(\alpha', \beta', \gamma')$ be taken on the line

$$p\alpha+q\beta+r\gamma=0,$$

then the polar of any point on it with reference to A_1 passes through the point $a\alpha/(b-c)=\beta=\gamma$, and therefore for the triple system the polars pass through the in-centre.

12. If $(\alpha', \beta', \gamma')$ be situated on the circumcircle then the envelope of its polar with regard to A_1 is the conic

$$\begin{aligned}&a^2\alpha^2(b+c)^2+(a^2-bc+c^2)^2\beta^2+(a^2-b^2+bc)^2\gamma^2\\ &-2a\alpha\beta(b-c)(a^2+bc+c^2)+2a\alpha\gamma(b-c)(a^2-b^2-bc)\\ &+2\beta\gamma[bc(b-c)^2+a^2(b^2-c^2-a^2)]=0.\end{aligned}$$

13. We may note that the equation to A_1, referred to the mid-point of BC as centre and axes parallel to AB, AC, is

$$bx^2-cy^2=bc(c-b)/4.$$

PART II.

14. In the second system KL is an antiparallel to angle A, such that $AK=\lambda$, $AL=\mu$, hence

$$c\lambda=b\mu.$$

The equations to BL, CK are

$$(b-\mu)c\gamma=a\alpha\mu,\quad a\alpha\lambda=(c-\lambda)b\beta$$

whence P is $\quad a\alpha\lambda\mu=(c-\lambda)b\mu\beta=(b-\mu)c\lambda\gamma.$

The locus of P is the rectangular hyperbola

$$(b^2-c^2)\beta\gamma+ab\gamma\alpha-ca\alpha\beta=0.\qquad (A_2)\quad\ldots\quad\ldots\quad \text{(i.)}$$

15. The centre is at the mid point of BC.

16. The polar of any point $(\alpha', \beta', \gamma')$ with respect to A_2 is

$$a\alpha(b\gamma'-c\beta')+\beta[(b^2-c^2)\gamma'-ca\alpha']+\gamma[(b^2-c^2)\beta'+ab\alpha']=0.\qquad \text{(ii.)}$$

The tangent at A is the symmedian line through A; at B and C the tangents are

$$ca\alpha=(b^2-c^2)\gamma,\quad -ab\alpha=(b^2-c^2)\beta,$$

these intersect in $\quad \dfrac{a\alpha}{b^2-c^2}=\dfrac{\beta}{-b}=\dfrac{\gamma}{c},$

i.e., they are parallel.

The polar of the symmedian point is $\quad \dfrac{\beta\cos B}{b^2}=\dfrac{\gamma\cos C}{c^2},$

hence Ap_5, Bq_5, Cr_5 meet in $\quad \dfrac{\alpha\cos A}{a^2}=\dfrac{\beta\cos B}{b^2}=\dfrac{\gamma\cos C}{c^2}.\quad \ldots\quad \text{(iii.)}$

The polar of the centroid is

$$a\alpha(b^2-c^2)+b\beta(b^2-2c^2)+c\gamma(2b^2-c^2)=0.\quad\ldots\quad \text{(iv.)}$$

17. Let A′, B′, C′ be the extremities of the diameters through A, B, C, then the tangent at A′ is

$$a\alpha(b^2-c^2)+b^3\beta-c^3\gamma=0,\quad\ldots\quad\ldots\quad \text{(v.)}$$

and Ap_6, Bq_6, Cr_6 meet in $\quad a^3\alpha=b^3\beta=c^3\gamma.\quad\ldots\quad\ldots\quad \text{(vi.)}$

The line (v.) has the point $-(b^2+c^2)/a,\ b,\ c$ on it.

The tangents themselves meet in

$$a^2\alpha/\cos A = b^2\beta/\cos B = c^2\gamma/\cos C, \quad \ldots \quad \ldots \quad \text{(vii.)}$$

which is the inverse of point (iii.) above.

The tangent at the orthocentre is given by

$$2R\cos^2 A\sin(B-C)\alpha + b\beta\cos^2 B - c\gamma\cos^2 C = 0, \quad \ldots \quad \text{(viii.)}$$

hence Ap_7, Bq_7, Cr_7 meet in

$$a\alpha\cos^2 A = b\beta\cos^2 B = c\gamma\cos^2 C. \quad \ldots \quad \ldots \quad \text{(ix.)}$$

18. The curve A_2 obviously cuts the circumcircle in a fourth point determined by producing the join of the orthocentre and the mid point of BC to meet the circle.

The polar of this point is

$$\alpha(b^2-c^2) + ab\beta\cos^2 C - ca\gamma\cos^2 B = 0, \quad \ldots \quad \text{(x.)}$$

hence Ap_8, Bq_8, Cr_8 meet in

$$a\alpha\sec^2 A = b\beta\sec^2 B = c\gamma\sec^2 C. \quad \ldots \quad \ldots \quad \text{(xi.)}$$

19. The asymptotes are

$$a\alpha(b-c) + b\beta(b+c) - c\gamma(b+c) = 0, \quad (a)$$

$$a\alpha(b+c) + b\beta(b-c) - c\gamma(b-c) = 0. \quad (b)$$

The (a) set pass through the point

$$a\alpha/(b+c) = b\beta/(c+a) = c\gamma/(a+b)\,;$$

hence they are readily constructed.

20. The polar of $(-a,\ b,\ c)$ is $\beta\cos C = \gamma\cos B$, *i.e.*, the diameter of the circumcircle through A produced.

The polars of $(a,\ -b,\ c)$, $(a,\ b,\ -c)$ are

$$b\alpha - c\cos B\beta + b\cos B\gamma = 0,$$

$$c\alpha + c\cos C\beta - b\cos C\gamma = 0,$$

hence they meet BC where the symmedian line through A meets it.

21. The condition that the polars of a point $(\alpha', \beta', \gamma')$ with respect to A_1 and A_2 should be parallel is that the point should lie on the median parallel to BC.

22. The co-ordinates of the point of intersection of the polars of $(\alpha', \beta', \gamma')$ with regard to A_1, A_2 are

$$-\alpha'(a\alpha' + b\beta' + c\gamma'),\ \beta'(a\alpha' + b\beta' - c\gamma'),\ \gamma'(a\alpha' - b\beta' + c\gamma').$$

Hence for the centroid, the point is $(-3bc, ca, ab)$.

23. The polar of $(\alpha', \beta', \gamma')$ with regard to A_1 meets the polars with regard to B_2, C_2, in

$$a\alpha(-a^2 + bc + c^2) = b\beta(a^2 - bc + c^2) = c\gamma(a^2 + bc - c^2)\ ;$$
$$a\alpha(-a^2 + b^2 + bc) = b\beta(a^2 - b^2 + bc) = c\gamma(a^2 + b^2 - bc).$$

24. The equation to A_2 with reference to axes parallel to AB, AC through the mid point of BC is

$$x^2 - y^2 = cx - by.$$

Sixth Meeting, April 13th, 1894.

JOHN ALISON, Esq., Ex-President, in the Chair.

Note on a Third Mode of Section of the Straight Line.

By W. WALLACE, M.A.

The rational treatment of Geometry has this important disadvantage, that for want of suitable demonstrations it seems impossible to preserve the natural grouping of the facts developed. The study of Rational Geometry, in fact, should always be supplemented by a systematic attempt to array the facts demonstrated according to their subject-matter; for it will hardly be denied that a direct and systematic knowledge of the Properties of Geometrical Figures has an intrinsic value apart from the knowledge of their demonstrations. In pursuing such a retrospective scheme as this in connection with the Second Book of Euclid, I have found that a very comprehensive view of the subject-matter is obtained by adding a Third Mode of Section of a straight line to the two which are already recognised. This third mode of section, for which I have not been able to find a more suitable name than "Circuitous Section," along with the other two known as Internal and External Section respectively, exhausts the possible modes of section of a line—for three-dimensional space at any rate. From this point of view, the elementary treatment of the subject may be arranged as follows. It will be observed that several important properties of triangles and polygons acquire a new interpretation as cases of circuitous section.

1. A straight line is divided into two parts—

 internally, when the point of section lies on the line, between its extremities;

 externally, when the point of section lies on the line, beyond its extremities;

 circuitously, when the point of section lies outside the line.

2. Instead of the axiom,

A line is equal to the sum of its parts;

substitute the lemma,

A line is equal to the sum of the projections of its parts upon it.

Then the successive segments are completely represented

by their respective lengths $\lambda_1 \lambda_2$, etc.;

and their inclinations $\theta_1 \theta_2$, etc.,

reckoned counter-clockwise from the right-hand parallels through the points of section;

and the lemma becomes expressible in the form

$$L = \Sigma(\lambda \cos\theta).$$

3. Proposition I.—The square on a two-part line

is equal to the sum of the rectangles contained by the line and the projections upon it of its two parts.

[sq. AB = rect. AB, AK$\cos\theta_1$ + rect. AB, KB$\cos\theta_2$]

(Particular cases):

$\theta_1 = 0 \;\; \theta_2 = 2\pi$ (internal section) Euc. II. 2.

$\theta_1 = 0 \;\; \theta_2 = \pi$ (external section) Euc. II. 3.

The general theorem is true for an n-part line.

[sq. AB = Σ(rect. AB, $\lambda\cos\theta$)]

4. Proposition II.—The square on a two-part line

is equal to the sum of the squares on the two segments diminished by twice the rectangle contained by either segment and the projection of the other upon it.

[sq. AB = (sq. AK + sq. KB)—2(rect. AK, KB$\cos\phi$)].

(Particular cases):

$\phi = \pi$	Internal section	Euc. II. 4
$\phi = 0$	External section	Euc. II. 7
$\phi = \frac{\pi}{2}$	Circuitous section	Euc. I. 47
$\phi > \frac{\pi}{2}$	Circuitous section	Euc. II. 12
$\phi < \frac{\pi}{2}$	Circuitous section	Euc. II. 13

5. Proposition III.—The sum of the squares on the segments of a two-part line

is double the sum of the squares on half the line and on the line between the middle and the point of section.

[sq. AK + sq. KB = 2(sq. AM + sq. MK)].

(Particular cases):

	Internal section	Euc. II. 9
	External section	Euc. II. 10
X	Circuitous section	Apollonius' (?) Theorem.

6. Proposition IV.—The rectangle contained by either segment of a two-part line and the projection of the other upon it

is equal to the square on the line between the middle and the point of section, diminished by the square on half the line.

[rect. AK, KB$\cos\phi$ = (sq. MK − sq. AM)].

(Particular cases):

$\phi = \pi$	Internal section	Euc. II. 5
$\phi = 0$	External section	Euc. II. 6
ϕ bet. 0 and π	Circuitous section.	

Note.—By combining Propositions III. and IV. a verification of Proposition I. is obtained.

7. Proposition V.—The locus of points which divide a straight line in a given ratio is a circle.

8. Proposition VI.—The joins of the corresponding points of section of two similarly divided parallel straight lines are concurrent.

The point of concurrence is the "centre of similarity" of the two lines.

(Particular cases):

	Internal and external section	Employed in the well known method of dividing a line proportionally to a given divided line.
X	Circuitous section	Well known theorem of centre of similarity of similar polygons.

Three Parabolas connected with a Plane Triangle.

By R. Tucker, M.A.

1. The parabolas considered in the present Note are obtained in the following manner :

From any point in one side of a triangle perpendiculars are let fall on the other two sides. The join of the feet of these perpendiculars envelopes a parabola.

Let AD, BE, CF be the perpendiculars from the angles on the opposite sides and let DK, DL be the perpendiculars on AC, AB. Then the envelope of KL is the parabola P_a.

It is evident that BE, CF are particular positions of KL, as also are AC, AB : hence P_a is also the envelope of the analogue of KL for the triangle BHC.

Now since the circumcircles of AEB, AFC intersect in D, D is the focus of P_a; and as E, F are the orthocentres of the above triangles, EF is the directrix of the curve.

2. Draw the perpendicular DX to EF cutting KL in V, then since KL bisects DX, V is the vertex of P_a.

Now $\qquad DV = DK\cos B = b\sin C\cos C\cos B,$

hence the Latus-Rectum $(L_a) = 2b\sin 2C\cos B,$

$$= 2R\sin 2B\sin 2C.$$

If L_b, L_c are the corresponding lines for P_b, P_c, we have

$$a'.L_a = 8\Delta\cos A\cos B\cos C = b'.L_b = c'.L_c,$$

and
$$L_a.L_b.L_c = 16\Delta'^2/R',$$

where a', Δ', R' refer to the Pedal triangle DEF.

3. If we draw Db, Dc' $\perp$ to DF, DE respectively to meet AB, AC produced in b, c', then these are the points where P_a touches AB, AC.

Now $\quad Ac' = b\sin A\sin C/\cos B, \; Ab = c\sin A\sin B/\cos C,$

$$\therefore \qquad \frac{Ac'}{Ab} = \frac{\cos C}{\cos B} = \frac{DK}{DL},$$

and $\quad Bb = c\cos A\cos B/\cos C, \quad Cc' = b\cos C\cos A/\cos B,$

$$\therefore \quad Bb \,.\, Cc' = bc\cos^2 A\,;$$

also $\quad Db\,.\,Dc' = DA^2, \quad \angle DbA = 90° - C, \quad \angle Dc'A = 90° - B.$

4. If the productions of $c'D$, bD, meet HB, HC in b_1, c_2 then these are the points where P_a touches HB, HC.

5. The diameter through A, being parallel to DX, passes through the circumcentre of ABC.

Since $\quad LK = AL\sin A/\sin B = 2R\sin A\sin B\sin C,$

$\therefore$ the intercepts on the vertical tangents of P_a, P_b, P_c by pairs of sides of ABC are equal, *i.e.*, they are equally distant from the centre of the "Taylor" circle of ABC.

6. Since $\quad \angle KLA = C = \angle DFL,$

$\therefore$ the portions intercepted on KL between the sides of the Pedal triangle and the sides of ABC are equal half of the adjacent sides of DEF.

7. If c_1, a_2; a_1, b_2; are points obtained as in §4, we have the relations

$$Aa_1 = Ha_2 = b\cos C\cos A/\sin A,$$
$$Aa_2 = Ha_1 = c\cos A\cos B/\sin A,$$

hence $\quad Ha_1\,.\,Hb_1\,.\,Hc_1 = Ha_2\,.\,Hb_2\,.\,Hc_2 = 8R^3\cos^2 A\cos^2 B\cos^2 C$

$$= HD\,.\,HE\,.\,HF\,.$$

Also the triangles DEF, $a_1b_1c_1$, $a_2b_2c_2$, are in Perspective with ABC, having the orthocentre for Perspective centre.

Plainly the Nine-point Circle bisects a_1a_2, b_1b_2, c_1c_2.

If δ be the area of $a_1b_1c_1$, or $a_2b_2c_2$, then

$$8\triangle\,.\,\delta = \cos A\cos B\cos C(a^4 + b^4 + c^4).$$

8. Let x_1, x_1' be the projections of b_1 on HF, HD, and x_2, x_2' be the projections of b_2 on the same lines, then

$$x_1 + x_1' = a\cos B\cos C,$$
$$x_2 + x_2' = c\cos A\cos B,$$

hence their sum $\quad = b\cos B = DF.$

9. If Cb_2, Bc_1, meet in X and HX cut BC in R, then

$$Hb_2 . Hc_2 . BR = Hb_1 . Hc_1 . CR,$$

whence BR = CD and HR is the isotomic of HD and X lies on AO, where O is the circumcentre.

The lines Cb_1, Bc_2 intersect on HD.

Numerous other properties of these points can be readily obtained, but I have not succeeded in getting simple equations to the circles $a_1b_1c_1$, $a_2b_2c_2$, or to the parabolas through $b_1b_2c_1c_2$, etc.

10. Parallels through D to AC, AB, being perpendicular to BE, CF, meet those lines where KL meets them, *i.e.*, in y_1, y_2 suppose.

The triangles Hy_1y_2, HFE are inversely similar. Again, the triangle Dy_1y_2 is similar to AFE and to ABC and is in Perspective with them.

Since $y_1y_2 = a\cos B\cos C$, the modulus of similarity of Dy_1y_2, with regard to ABC, is $\cos B\cos C$, and

$$y_1y_2 + EF = KL.$$

11. Let y_1', y_1'' ; y_2', y_2'' be the analogous points for the angles B, C, then

$$\begin{aligned} Hy_1 &= 2R\cos B\cos^2 C, & Hy_2 &= 2R\cos^2 B\cos C, \\ Hy_1' &= 2R\cos C\cos^2 A, & Hy_2' &= 2R\cos^2 C\cos A, \\ Hy_1'' &= 2R\cos A\cos^2 B, & Hy_2'' &= 2R\cos^2 A\cos B. \end{aligned}$$

Hence $\Pi . Hy_1 = 8R^3\cos^3 A\cos^3 B\cos^3 C = \Pi . Hy_2$.

Again
$$\begin{aligned} Hy_1 . Hy_2'' &= 4R^2\cos^2 A\cos^2 B\cos^2 C \\ &= \lambda^2 (\text{suppose}) = Hy_1' . Hy_2 ; \end{aligned}$$

hence H is the radical centre of the three circles which pass through y_1, y_2'', y_1', y_2, etc. ; and their orthotomic circle is the incircle of the Pedal triangle.

12. Taking the lines HB, HC as axes, the equation to the circle $y_1y_2''y_1'y_2$ is

$$x^2 + y^2 - 2xy\cos A - 2gx - 2fy + \lambda^2 = 0,$$

and to the parabolas through them is

$$(x \pm y)^2 - 2gx - 2fg + \lambda^2 = 0,$$

where $$g \equiv R\cos B(\cos^2 C + \cos^2 A),$$

and $$f \equiv R\cos C(\cos^2 A + \cos^2 B);$$

hence axes of the parabolas are known as to direction.

If ρ_a, ρ_b, ρ_c are the radii of the circles, then, since

$$y_2 y_2'' = 2\rho_a \cos B,$$

$$\rho_a^2 = R^2[\sin^2 A(\cos_2 B - \sin^2 C) + \sin^2 B \sin^2 C],$$

and $$2\Sigma(\rho^2) = 3 - [\Sigma \sin^2 A \sin^2 B + \Sigma \cos^2 A \cos^2 B].$$

13. If we suppose P to be the point in BC from which perpendiculars Pq, Pr are drawn to AC, AB, and if we take $BP = ma$, $CP = na$, $m + n \equiv 1$, then the equation of qr, in trilinear co-ordinates is

$$-(mb + nc\cos A)(mb\cos A + nc)\alpha + mb\cos B(mb + nc\cos A)\beta$$
$$+ nc\cos C(mb\cos A + nc)\gamma = 0,$$

i.e., $$am^2\cos B\cos C \,.\, \Sigma(a\alpha)$$
$$+ cm[a\alpha(\cos A\cos B - \cos C) + b\beta\cos A\cos B - \gamma\cos C(c + a\cos B)]$$
$$+ c^2(\gamma\cos C - \alpha\cos A) = 0,$$

the envelope of which is

$$[a\alpha(\cos A\cos B - \cos C) + b\beta\cos A\cos B - \gamma\cos C(c + a\cos B)]^2$$
$$= 4a\cos B\cos C\Sigma(a\alpha) \,.\, (\gamma\cos C - \alpha\cos A),$$

i.e., a parabola.

This equation can be put into the two forms

$$(\alpha\sin^2 A + \beta\cos A\cos B + \gamma\cos C\cos A)^2 = 4\beta\gamma\cos B\cos C, \qquad \text{(A)}$$

$$[-\alpha(1 + \cos^2 A) + \beta\cos A\cos B + \gamma\cos C\cos A]^2$$
$$= 4(\alpha\cos A - \beta\cos B)(\alpha\cos A - \gamma\cos C). \qquad \text{(B)}$$

Hence we see that the equation to bc' (§ 3) is

$$\alpha\sin^2 A + \beta\cos A\cos B + \gamma\cos C\cos A = 0, \qquad \text{(C)}$$

and to $b_1 c_2$ (§ 4) is

$$-\alpha(1 + \cos^2 A) + \beta\cos A\cos B + \gamma\cos C\cos A = 0. \qquad \text{(D)}$$

From (A) we see that the curve cuts BC in

$$\frac{\beta\cos B}{\gamma\cos C} = \tan^2\left(\frac{\pi}{4} \pm \frac{A}{2}\right).$$

If we take P′ the isotomic of P on BC, *i.e.*, write

m for n and n for m in the equation to qr.

we see that the two lines intersect on

$$a\alpha(b\cos B - c\cos C) + b^2\beta\cos B - c^2\cos C\gamma = 0,$$

which is the diameter through H, (⊥ to EF).

14. If V′, V″ are the vertices of P_b, P_c, then AV, BV′, CV″ meet in

$$\alpha/a^2\sec A = \beta/b^2\sec B = \gamma/c^2\sec C.$$

15. The equation to the conic, through b, c', and the four analogous points, is

$$\cos A\cos B\cos C\Sigma(\alpha^2\sin^2 A) + \Sigma[\beta\gamma\cos A(\cos^2 B\cos^2 C + \sin^2 B\sin^2 C)] = 0.$$

16. If the lines BK, CL; etc.; intersect in t_1, t_2, t_3, then At_1, Bt_2, Ct_3 cointersect in the point

$$\alpha\operatorname{cosec}A\cos^2 A = \beta\operatorname{cosec}B\cos^2 B = \gamma\operatorname{cosec}C\cos^2 C.$$

17. The equation to bc' (C), on the supposition of ∠A constant, and writing θ for B, is

$$\gamma\cos A\sin A\sin\theta + (\beta - \gamma\cos A)\cos A\cos\theta + \alpha\sin^2 A = 0,$$

hence the envelope of bc' is

$$\cos^2 A(\beta^2 + \gamma^2 - 2\beta\gamma\cos A) = \alpha^2\sin^4 A;$$

which is a hyperbola, if A is acute,
and an ellipse, if A is obtuse.

18. The equation to the circle Abc is

$$\Sigma(\alpha\beta\gamma) + (\Sigma a\alpha).\cot A(\gamma\tan B\cos C + \beta\tan C\cos B) = 0.$$

19. From (C) and (D) we see that bc' and b_1c_2 cut BC where the harmonic conjugate to AD with regard to AB_1AC_1 cuts it, as they should do by the geometry of the figure.

20. If we take a point P″ on BC such that $BP''.BP = BC^2$, *i.e.*, in the equation in § 13 write $1/m$ for m, we get for the equation to the tangent

$$c^2(\gamma\cos C - a\cos A)m^2 + cm[\ldots] + a\cos B\cos C\Sigma(a\alpha) = 0\ ;$$

hence the locus of the intersection of the two lines is

$$a\cos B\cos C\Sigma(a\alpha) = c^2(\gamma\cos C - a\cos A),$$

a straight line parallel to BE and cutting BC in a point D′ given by $BD' = a^2\cos B/c$.

21. The polars of B and C are

$$\begin{aligned}\alpha\sin^2 A\cos A + \beta\cos^2 A\cos B - \gamma(1+\cos^2 A) &= 0,\\ \alpha\sin^2 A\cos A - \beta\cos B(1+\cos^2 A) + \gamma\cos^2 A\cos C &= 0\ ;\end{aligned}$$

hence they intersect on AD; in fact, in the point where AD cuts EF.

The polar of H is

$$-\alpha(1+\cos^2 A) + (\beta+\gamma)\cos A = 0.$$

The Pedal Triangle.

By Professor J. E. A. STEGGALL.

The area of the pedal triangle of a given triangle is easily shown by trilinear co-ordinates to bear to that of the original triangle the ratio $R^2 - S^2 : 4R^2$ where S is the distance of the point from the circumcentre of the triangle. A proof, by purely geometrical methods, of this theorem was read before the Society (*Proceedings*, Vol. III., pp. 78–79) by Mr Alison.

The following geometrical proof proceeds on somewhat different lines.

FIGURE 16.

Let ABC be the triangle, P′ the point, P′L′, P′M′, P′N′ the perpendiculars; P the point where AP′ meets the circumcircle, PL, PM, PN its perpendiculars, AK perpendicular to BC, OY perpendicular to AP.

Let AL cut M′N′ in L″; then from similarity, P′L″ is parallel to PL, and therefore collinear with P′L′.

$$\text{Area } L'M'N' : \text{area } P'M'N' = L'L'' : L''P'$$

$$\text{area } P'M'N' : \text{area } PMN = L''P'^2 : LP^2$$

$$= AP' \,.\, L''P' : AP \,.\, LP$$

$$\text{area } PMN : \text{area } PBC = PM^2 : PC^2$$

$$= PY^2 : PO^2$$

$$= PA^2 : 4PO^2$$

because $\angle PCM = PBA = POY.$

$$\text{Area } PBC : \text{area } ABC = PL : AK$$

$\therefore$ $$\text{area } L'M'N' : \text{area } ABC = L'L'' . AP' . AP : 4OP^2 . AK$$

But $$L'L'' : AK = LL'' : AL = PP' : AP$$

$\therefore$ $$L'L'' . AP = AK . PP'$$

$\therefore$ $$\text{area } L'M'N' : \text{area } ABC = AK . AP' \cdot PP' : 4OP^2 . AK$$

$$= PP' . AP' : 4OP^2$$

$$= OP^2 - OP'^2 : 4OP^2$$

which is the theorem required.

Formulae connected with the Radii of the Incircle and the Excircles of a Triangle.

By J. S. MACKAY, M.A., LL.D.

The notation employed in the following pages is that recommended in a paper of mine on "The Triangle and its Six Scribed Circles"* printed in the first volume of the *Proceedings of the Edinburgh Mathematical Society.* It may be convenient to repeat all that is necessary for the present purpose.

a, b, c	= the sides BC, CA, AB of triangle ABC.
h_1, h_2, h_3	= the perpendiculars from A, B, C on BC, CA, AB.
m_1, m_2, m_3	= the medians from A, B, C.
r	= the radius of the incircle.
r_1, r_2, r_3	= the radii of the 1st, 2nd, 3rd excircles. These are frequently denoted by r_a, r_b, r_c.
s	= semiperimeter † of ABC.
s_1, s_2, s_3	$= s - a$, $s - b$, $s - c$. ‡
$\triangle$	= area of ABC.

When an equation expresses a property of a triangle relating to one of the excircles it is easily enough transformed into the corresponding equation for either of the other excircles. It is not however so easy at first sight to transform an equation relating to the incircle into the corresponding one relating to an excircle. The following table gives the substitutions that are necessary to effect the required transformation.

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* The title is somewhat of a misnomer. Five only of these circles are treated of. The sixth (the nine-point circle) is discussed in the eleventh volume of the *Proceedings.*

† On the Continent of Europe p is generally employed instead of s.

‡ This notation was suggested by Thomas Weddle in 1842. See *Lady's and Gentleman's Diary* for 1843, p. 78. Professor Neuberg proposes p_1, p_2, p_3 instead of $p-a$, $p-b$, $p-c$ in *Mathesis,* III. 167 (1883).

When r is changed into r_1

a	b	c	s	s_1	s_2	s_3	become
a	$-b$	$-c$	$-s_1$	$-s$	s_3	s_2	
r_1	r_2	r_3	h_1	h_2	h_3		become
r	$-r_3$	$-r_2$	$-h_1$	h_2	h_3		

A	B	C	R	Δ	become
$-$A	$180^\circ-$B	$180^\circ-$C	$-$R	$-\Delta$	

The greater part of this table is given in the *Lady's and Gentleman's Diary* for 1871, p. 93, and it is due either to the editor of the *Diary*, W. S. B. Woolhouse, or to one of his correspondents, W. B. G. (William Bywater Grove?). No demonstration however is offered of the law of transformation thus enunciated.

A discussion of this law by Mr E. Lemoine will be found in the *Bulletin de la Société Mathématique de France*, XIX. 133–141 (1891), in Mr De Longchamps' *Journal de Mathématiques Élémentaires*, 4th series, I. 62–69, 91–93, 103–106 (1892), and in *Mathesis*, 2nd series, II. 58–64, 81–92 (1892). Two articles on the same subject by Edouard Lucas will be found in *Nouvelle Correspondance Mathématique*, II. 384–391 (1876), III. 1–5 (1877).

The following algebraical identities will be found useful.

$$\left.\begin{aligned} s-s_1&=s_2+s_3=a\\ s-s_2&=s_3+s_1=b\\ s-s_3&=s_1+s_2=c \end{aligned}\right\}\qquad \text{I.}$$

$$\left.\begin{aligned} s+s_1&=b+c \qquad & s_2-s_3&=c-b\\ s+s_2&=c+a & s_3-s_1&=a-c\\ s+s_3&=a+b & s_1-s_2&=b-a \end{aligned}\right\}\qquad \text{II.}$$

$$\left.\begin{aligned} s+s_1+s_2+s_3&=2s\\ s-s_1+s_2+s_3&=2a\\ s+s_1-s_2+s_3&=2b\\ s+s_1+s_2-s_3&=2c \end{aligned}\right\}\qquad \text{III.}$$

$$\left.\begin{aligned} s_1 + s_2 + s_3 &= s \\ s - s_3 - s_2 &= s_1 \\ s - s_1 - s_3 &= s_2 \\ s - s_2 - s_1 &= s_3 \end{aligned}\right\} \qquad \text{IV.}$$

$$s^2 + s_1^2 + s_2^2 + s_3^2 = a^2 + b^2 + c^2 \qquad \text{V.}$$

$$\left.\begin{aligned} ss_2 - s_1 s_3 - s_2 s_1 + s_3 s &= a^2 \\ ss_3 - s_2 s_1 - s_3 s_2 + s_1 s &= b^2 \\ ss_1 - s_3 s_2 - s_1 s_3 + s_2 s &= c^2 \end{aligned}\right\} \qquad \text{VI.}$$

$$\left.\begin{aligned} s\,s_1 + s\,s_2 + s\,s_3 &= s^2 \\ s_1 s - s_1 s_3 - s_1 s_2 &= s_1^2 \\ s_2 s - s_2 s_1 - s_2 s_3 &= s_2^2 \\ s_3 s - s_3 s_2 - s_3 s_1 &= s_3^2 \end{aligned}\right\} \qquad \text{VII.}$$

$$ss_1 + s_2 s_3 = bc, \quad ss_2 + s_3 s_1 = ca, \quad ss_3 + s_1 s_2 = ab \qquad \text{VIII.}$$

$$\left.\begin{aligned} 2(ss_1 - s_2 s_3) &= -a^2 + b^2 + c^2 \\ 2(ss_2 - s_3 s_1) &= a^2 - b^2 + c^2 \\ 2(ss_3 - s_1 s_2) &= a^2 + b^2 - c^2 \end{aligned}\right\} \qquad \text{IX.}$$

$$\left.\begin{aligned} 4(s_2 s_3 + s_3 s_1 + s_1 s_2) &= 2(bc + ca + ab) - (a^2 + b^2 + c^2) \\ 4(s_3 s_2 - s_2 s - s\,s_3) &= 2(bc - ca - ab) - (a^2 + b^2 + c^2) \\ 4(s_1 s_3 - s_3 s - s\,s_1) &= 2(ca - ab - bc) - (a^2 + b^2 + c^2) \\ 4(s_2 s_1 - s_1 s - s\,s_2) &= 2(ab - bc - ca) - (a^2 + b^2 + c^2) \end{aligned}\right\} \qquad \text{X.}$$

$$\left.\begin{aligned} &2(as_1 + b\,s_2 + c\,s_3) = 2(bc + ca + ab) - (a^2 + b^2 + c^2) \\ &2(as_2 + bs_3 + c\,s_1) = 2(as_3 + b\,s_1 + c\,s_2) = a^2 + b^2 + c^2 \\ &-as_1 + bs_2 + c\,s_3 = 2s_2 s_3 \qquad -as_3 + bs + cs_1 = 2ss_1 \\ &as_1 - bs_2 + c\,s_3 = 2s_3 s_1 \qquad as_2 - bs_1 + cs = 2ss_2 \\ &as_1 + b\,s_2 - c\,s_3 = 2s_1 s_2 \qquad as + bs_3 - cs_2 = 2ss_3 \\ &s_1(b - c) + s_2(c - a) + s_3(a - b) = 0 \end{aligned}\right\} \qquad \text{XI.}$$

$$\left.\begin{aligned}
&s_1^3 + s_2^3 + s_3^3 + 3abc = s^3 \\
&ss_1s_2s_3\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} - \frac{1}{s}\right) = abc \\
&as_2s_3 + bs_3s_1 + cs_1s_2 + 2s_1s_2s_3 = abc \\
&as_1^2 + bs_2^2 + cs_3^2 + 2s_1s_2s_3 = abc \\
&4(as_1^2 + bs_2^2 + cs_3^2) \\
&= a^3 + b^3 + c^3 + 6abc - b^2c - bc^2 - c^2a - ca^2 - a^2b - ab^2 \\
&a(b-c)s_1^2 + b(c-a)s_2^2 + c(a-b)s_3^2 = 0
\end{aligned}\right\} \quad \text{XII.}$$

$$r = \frac{\Delta}{s} \quad r_1 = \frac{\Delta}{s_1} \quad r_2 = \frac{\Delta}{s_2} \quad r_3 = \frac{\Delta}{s_3} \qquad (1)$$

where $\Delta = \sqrt{ss_1s_2s_3} = \frac{1}{2}ah_1 = \frac{1}{2}bh_2 = \frac{1}{2}ch_3$

These results may be put into a variety of other forms, such as

$$\left.\begin{aligned}
&sr = s_1r_1 = s_2r_2 = s_3r_3 \\
&\frac{r}{r_1} = \frac{s_1}{s} \quad \frac{r}{r_2} = \frac{s_2}{s} \quad \frac{r}{r_3} = \frac{s_3}{s} \\
&\frac{r_2}{r_3} = \frac{s_3}{s_2} \quad \frac{r_3}{r_1} = \frac{s_1}{s_3} \quad \frac{r_1}{r_2} = \frac{s_2}{s_1}
\end{aligned}\right\} \qquad (2)$$

$$rr_1r_2r_3 = \Delta^2 = ss_1s_2s_3 \qquad (3)$$

$$\left.\begin{aligned}
&\frac{r_1r_2r_3}{s} = \Delta = \frac{s_1s_2s_3}{r} \\
&\frac{r\,r_3r_2}{s_1} = \Delta = \frac{s\,s_3s_2}{r_1} \\
&\frac{r\,r_1r_3}{s_2} = \Delta = \frac{s\,s_1s_3}{r_2} \\
&\frac{r\,r_2r_1}{s_3} = \Delta = \frac{s\,s_2s_1}{r_3}
\end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned}
&r_2r_3 : s^2 = r^2 : s_2s_3 \\
&r_3r_1 : s^2 = r^2 : s_3s_1 \\
&r_1r_2 : s^2 = r^2 : s_1s_3
\end{aligned}\right\} \qquad (5)$$

Other proportions may be obtained by putting for the extremes rr_1, ss_1, etc., and for the means r_1^2, s_1^2, etc.

$$\left.\begin{array}{l} r_1r_2r_3 : s^3 = r^3 : s_1s_2s_3 \\ r\,r_3r_2 : s_1^3 = r_1^3 : s\,s_3s_2 \\ r\,r_1r_3 : s_2^3 = r_2^3 : s\,s_1s_3 \\ r\,r_2r_1 : s_3^3 = r_3^3 : s\,s_2s_1 \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{llll} r^2 = \dfrac{s_1s_2s_3}{s}, & r_1^2 = \dfrac{ss_3s_2}{s_1}, & r_2^2 = \dfrac{ss_1s_3}{s_2}, & r_3^2 = \dfrac{ss_2s_1}{s_3} \\ s^2 = \dfrac{r_1r_2r_3}{r}, & s_1^2 = \dfrac{rr_3r_2}{r_1}, & s_2^2 = \dfrac{rr_1r_3}{r_2}, & s_3^2 = \dfrac{rr_2r_1}{r_3} \end{array}\right\} \quad (7)$$

By means of (7) and I., II., III., IV., a large number of expressions may be obtained. The following is given as a specimen:

$$\sqrt{\frac{r_1r_2r_3}{r}} = \sqrt{\frac{r\,r_3r_2}{r_1}} + \sqrt{\frac{r\,r_1r_3}{r_2}} + \sqrt{\frac{r\,r_2r_1}{r_3}} \quad (8)$$

$$\left.\begin{array}{lll} rr_1 = s_2s_3 & rr_2 = s_3s_1 & rr_3 = s_1s_2 \\ ss_1 = r_2r_3 & ss_2 = r_3r_1 & ss_3 = r_1r_2 \end{array}\right\} \quad (9)$$

$$rr_1 + r_2r_3 = bc \qquad rr_2 + r_3r_1 = ca \qquad rr_3 + r_1r_2 = ab \quad (10)$$

See VIII.

$$2(r_2r_3 + r_3r_1 + r_1r_2 + rr_1 + rr_2 + rr_3) = 2(bc + ca + ab) \quad (11)$$

$$2(r_2r_3 + r_3r_1 + r_1r_2 - rr_1 - rr_2 - rr_3) = a^2 + b^2 + c^2 \quad (12)$$

See IX.

$$\left.\begin{array}{ll} a^2 + 4r_2r_3 = (b+c)^2 & a^2 - 4rr_1 = (b-c)^2 \\ b^2 + 4r_3r_1 = (c+a)^2 & b^2 - 4rr_2 = (c-a)^2 \\ c^2 + 4r_1r_2 = (a+b)^2 & c^2 - 4rr_3 = (a-b)^2 \end{array}\right\} \quad (13)$$

$$\left.\begin{aligned}
\frac{bc - r_2r_3}{r_1} = \frac{ca - r_3r_1}{r_2} = \frac{ab - r_1r_2}{r_3} = r \\
\frac{bc - s_2s_3}{s_1} = \frac{ca - s_3s_1}{s_2} = \frac{ab - s_1s_2}{s_3} = s
\end{aligned}\right\} \quad (14)$$

Similar expressions may be obtained for r_1, r_2, r_3 and for s_1, s_2, s_3.

$$\left.\begin{aligned}
r_2r_3 + r_3r_1 + r_1r_2 &= s^2 \\
r_3r_2 - r_2r \ - r\,r_3 &= s_1^2 \\
r_1r_3 - r_3r \ - r\,r_1 &= s_2^2 \\
r_2r_1 - r_1r \ - r\,r_2 &= s_3^2
\end{aligned}\right\} \quad (15)$$

See VII.

$$\left.\begin{aligned}
- r_2r_3 + r_3r_1 + r_1r_2 &= s\,(2a - s) \\
r_2r_3 + r_2r \ + r\,r_3 &= s_1(2a + s_1) \\
r_3r_1 + r_3r \ + r\,r_1 &= s_2(2b + s_2) \\
r_1r_2 + r_1r \ + r\,r_2 &= s_3(2c + s_3)
\end{aligned}\right\} \quad (16)$$

See III.

$$\left.\begin{aligned}
4r\,(r_1 + r_2 + r_3) &= 2(bc + ca + ab) - (a^2 + b^2 + c^2) \\
4r_1(r \ - r_3 - r_2) &= 2(bc - ca - ab) - (a^2 + b^2 + c^2) \\
4r_2(r \ - r_1 - r_3) &= 2(ca - ab - bc) - (a^2 + b^2 + c^2) \\
4r_3(r \ - r_2 - r_1) &= 2(ab - bc - ca) - (a^2 + b^2 + c^2)
\end{aligned}\right\} \quad (17)$$

See X.

$$\left.\begin{aligned}
r\,(r_1 + r_2 + r_3) &= bc - s_1^2 = \quad ca - s_2^2 = \quad ab - s_3^2 \\
r_1(r \ - r_3 - r_2) &= bc - s^2 = -ca - s_3^2 = -ab - s_2^2 \\
r_2(r \ - r_1 - r_3) &= ca - s^2 = -ab - s_1^2 = -bc - s_3^2 \\
r_3(r \ - r_2 - r_1) &= ab - s^2 = -bc - s_2^2 = -ca - s_1^2
\end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned}
4r_2r_3 + r_3r_1 + r_1r_2 - 4rr_1 - rr_2 - rr_3 &= 4m_1^2 \\
4r_3r_1 + r_1r_2 + r_2r_3 - 4rr_2 - rr_3 - rr_1 &= 4m_2^2 \\
4r_1r_2 + r_2r_3 + r_3r_1 - 4rr_3 - rr_1 - rr_2 &= 4m_3^2
\end{aligned}\right\} \quad (19)$$

$$\left.\begin{array}{llll} \frac{h_1}{2r}=\frac{s}{a} & \frac{h_1}{2r_1}=\frac{s_1}{a} & \frac{h_1}{2r_2}=\frac{s_2}{a} & \frac{h_1}{2r_3}=\frac{s_3}{a} \\ \frac{h_2}{2r}=\frac{s}{b} & \frac{h_2}{2r_1}=\frac{s_1}{b} & \frac{h_2}{2r_2}=\frac{s_2}{b} & \frac{h_2}{2r_3}=\frac{s_3}{b} \\ \frac{h_3}{2r}=\frac{s}{c} & \frac{h_3}{2r_1}=\frac{s_1}{c} & \frac{h_3}{2r_2}=\frac{s_2}{c} & \frac{h_3}{2r_3}=\frac{s_3}{c} \end{array}\right\} \quad (20)$$

$$\frac{h_2h_3}{4r_2r_3}=\frac{rr_1}{bc} \qquad \frac{h_3h_1}{4r_3r_1}=\frac{rr_2}{ca} \qquad \frac{h_1h_2}{4r_1r_2}=\frac{rr_3}{ab} \quad (21)$$

$$\frac{8s^3}{abc}r^3=\frac{8s_1^{\,3}}{abc}r_1^{\,3}=\ldots \quad =h_1h_2h_3 \quad (22)$$

$$\triangle = sr_1\left(1-\frac{2r}{h_1}\right)=s_1r\left(1+\frac{2r_1}{h_1}\right)=\ldots \quad (23)$$

$$\left.\begin{array}{l} \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}= \quad \frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3} \\ \frac{1}{r_1}=\frac{1}{r}-\frac{1}{r_3}-\frac{1}{r_2}=-\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3} \\ \frac{1}{r_2}=\frac{1}{r}-\frac{1}{r_1}-\frac{1}{r_3}=-\frac{1}{h_2}+\frac{1}{h_3}+\frac{1}{h_1} \\ \frac{1}{r_3}=\frac{1}{r}-\frac{1}{r_2}-\frac{1}{r_1}=-\frac{1}{h_3}+\frac{1}{h_1}+\frac{1}{h_2} \end{array}\right\} \quad (24)$$

These may be put into many other forms; for example

$$\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$$

$$\left.\begin{array}{ll} \frac{1}{r}-\frac{1}{r_1}=\frac{1}{r_2}+\frac{1}{r_3}=\frac{2}{h_1} & \frac{2rr_1}{r_1-r}=\frac{2r_2r_3}{r_2+r_3}=h_1 \\ \frac{1}{r}-\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_1}=\frac{2}{h_2} & \frac{2rr_2}{r_2-r}=\frac{2r_3r_1}{r_3+r_1}=h_2 \\ \frac{1}{r}-\frac{1}{r_3}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{h_3} & \frac{2rr_3}{r_3-r}=\frac{2r_1r_2}{r_1+r_2}=h_3 \end{array}\right\} \quad (25)$$

$$\left.\begin{array}{ll}
\dfrac{1}{r}+\dfrac{1}{r_1}=\dfrac{2}{h_2}+\dfrac{2}{h_3} & \dfrac{2rr_1}{r+r_1}=\dfrac{h_2h_3}{h_2+h_3} \\[2ex]
\dfrac{1}{r}+\dfrac{1}{r_2}=\dfrac{2}{h_3}+\dfrac{2}{h_1} & \dfrac{2rr_2}{r+r_2}=\dfrac{h_3h_1}{h_3+h_1} \\[2ex]
\dfrac{1}{r}+\dfrac{1}{r_3}=\dfrac{2}{h_1}+\dfrac{2}{h_2} & \dfrac{2rr_3}{r+r_3}=\dfrac{h_1h_2}{h_1+h_2}
\end{array}\right\} \quad (26)$$

$$\left.\begin{array}{ll}
\dfrac{1}{r_3}-\dfrac{1}{r_2}=\dfrac{2}{h_2}-\dfrac{2}{h_3} & \dfrac{2r_2r_3}{r_3-r_2}=\dfrac{h_2h_3}{h_2-h_3} \\[2ex]
\dfrac{1}{r_1}-\dfrac{1}{r_3}=\dfrac{2}{h_3}-\dfrac{2}{h_1} & \dfrac{2r_3r_1}{r_1-r_3}=\dfrac{h_3h_1}{h_3-h_1} \\[2ex]
\dfrac{1}{r_2}-\dfrac{1}{r_1}=\dfrac{2}{h_1}-\dfrac{2}{h_2} & \dfrac{2r_1r_2}{r_2-r_1}=\dfrac{h_1h_2}{h_1-h_2}
\end{array}\right\} \quad (27)$$

$$\left.\begin{array}{l}
r=\dfrac{r_1r_2r_3}{r_2r_3+r_3r_1+r_1r_2}=\dfrac{h_1h_2h_3}{h_2h_3+h_3h_1+h_1h_2} \\[2ex]
r_1=\dfrac{r\,r_3r_2}{r_3r_2-r_2r-r\,r_3}=\dfrac{h_1h_2h_3}{-h_2h_3+h_3h_1+h_1h_2} \\[2ex]
r_2=\dfrac{r\,r_1r_3}{r_1r_3-r_3r-r\,r_1}=\dfrac{h_1h_2h_3}{-h_3h_1+h_1h_2+h_2h_3} \\[2ex]
r_3=\dfrac{r\,r_2r_1}{r_2r_1-r_1r-r\,r_2}=\dfrac{h_1h_2h_3}{-h_1h_2+h_2h_3+h_3h_1}
\end{array}\right\} \quad (28)$$

$$\left.\begin{array}{l}
\dfrac{r_1^2r_2^2r_3^2}{r_2r_3+r_3r_1+r_1r_2}=\dfrac{r^2r_3^2r_2^2}{r_3r_2-r_2r-r\,r_3} \\[2ex]
=\dfrac{r^2r_1^2r_3^2}{r_1r_3-r_3r-r\,r_1}=\dfrac{r^2r_2^2r_1^2}{r_2r_1-r_1r-r\,r_2} \\[2ex]
=\Delta^2 \\[1ex]
=\text{the reciprocal of} \\[1ex]
\left(\dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3}\right)\left(-\dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3}\right)\left(-\dfrac{1}{h_2}+\dfrac{1}{h_3}+\dfrac{1}{h_1}\right)\left(-\dfrac{1}{h_3}+\dfrac{1}{h_1}+\dfrac{1}{h_2}\right)
\end{array}\right\} \quad (29)$$

$$\left.\begin{aligned}
\frac{1}{rr_1}+\frac{1}{r_2r_3}&=\frac{1}{s_2s_3}+\frac{1}{ss_1}=\frac{4}{h_2h_3}=\frac{bc}{\Delta^2}\\
\frac{1}{rr_2}+\frac{1}{r_3r_1}&=\frac{1}{s_3s_1}+\frac{1}{ss_2}=\frac{4}{h_3h_1}=\frac{ca}{\Delta^2}\\
\frac{1}{rr_3}+\frac{1}{r_1r_2}&=\frac{1}{s_1s_2}+\frac{1}{ss_3}=\frac{4}{h_1h_2}=\frac{ab}{\Delta^2}
\end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned}
\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)&=\left(\frac{1}{s_1}-\frac{1}{s}\right)\left(\frac{1}{s_2}+\frac{1}{s_3}\right)=\frac{4}{h_1^2}=\frac{a^2}{\Delta^2}\\
\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)&=\left(\frac{1}{s_2}-\frac{1}{s}\right)\left(\frac{1}{s_3}+\frac{1}{s_1}\right)=\frac{4}{h_2^2}=\frac{b^2}{\Delta^2}\\
\left(\frac{1}{r}-\frac{1}{r_3}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)&=\left(\frac{1}{s_3}-\frac{1}{s}\right)\left(\frac{1}{s_1}+\frac{1}{s_2}\right)=\frac{4}{h_3^2}=\frac{c^2}{\Delta^2}
\end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned}
\frac{1}{rr_1}+\frac{1}{rr_2}+\frac{1}{rr_3}+\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}&=\frac{4}{h_2h_3}+\frac{4}{h_3h_1}+\frac{4}{h_1h_2}\\
&=\frac{bc+ca+ab}{\Delta^2}
\end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned}
\frac{1}{rr_1}+\frac{1}{rr_2}+\frac{1}{rr_3}-\frac{1}{r_2r_3}-\frac{1}{r_3r_1}-\frac{1}{r_1r_2}&=\frac{2}{h_1^2}+\frac{2}{h_2^2}+\frac{2}{h^2}\\
&=\frac{a^2+b^2+c^2}{2\Delta^2}
\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned}
\frac{1}{r^2}&=\frac{1}{s_2s_3}+\frac{1}{s_3s_1}+\frac{1}{s_1s_2}\\
\frac{1}{r_1^2}&=\frac{1}{s_3s_2}-\frac{1}{s_2s}-\frac{1}{ss_3}
\end{aligned}\right\} \tag{34}$$

and so on.

$$\left.\begin{aligned}
\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}&=\frac{4}{h_1^2}+\frac{4}{h_2^2}+\frac{4}{h_3^2}\\
&=\frac{a^2+b^2+c^2}{\Delta^2}
\end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned}\frac{1}{r^2}+\frac{1}{r_1^{\,2}}-\frac{1}{r_2^{\,2}}-\frac{1}{r_3^{\,2}}&=\frac{8}{h_2h_3}\\ \frac{1}{r^2}-\frac{1}{r_1^{\,2}}+\frac{1}{r_2^{\,2}}-\frac{1}{r_3^{\,2}}&=\frac{8}{h_3h_1}\\ \frac{1}{r^2}-\frac{1}{r_1^{\,2}}-\frac{1}{r_2^{\,2}}+\frac{1}{r_3^{\,2}}&=\frac{8}{h_1h_2}\end{aligned}\right\}\quad(36)$$

$$\left.\begin{aligned}\frac{a}{h_1}+\frac{b}{h_2}+\frac{c}{h_3}+\tfrac{1}{2}\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)&=\frac{s}{r}\\ \frac{a}{h_1}+\frac{b}{h_2}+\frac{c}{h_3}-\tfrac{1}{2}\left(\frac{a}{r}+\frac{b}{r_3}+\frac{c}{r_2}\right)&=\frac{s_1}{r_1}\end{aligned}\right\}\quad(37)$$

and so on.

$$\left.\begin{aligned}\frac{h_1+h_2+h_3-(r_1+r_2+r_3)}{s}-3\left(\frac{r}{a}+\frac{r}{b}+\frac{r}{c}\right)+\frac{r_1}{a}+\frac{r_2}{b}+\frac{r_3}{c}&=0\\ \frac{h_1-h_2-h_3+(r-r_3-r_2)}{s_1}-3\left(\frac{r_1}{a}-\frac{r_1}{b}-\frac{r_1}{c}\right)+\frac{r}{a}+\frac{r_3}{b}+\frac{r_2}{c}&=0\end{aligned}\right\}\quad(38)$$

and so on.

$$\left.\begin{aligned}\frac{h_2+h_3}{r_1}+\frac{h_3+h_1}{r_2}+\frac{h_1+h_2}{r_3}&=6\\ \frac{h_2+h_3}{r}-\frac{h_3-h_1}{r_3}+\frac{h_1-h_2}{r_2}&=6\end{aligned}\right\}\quad(39)$$

and so on.

$$\left.\begin{aligned}\frac{h_1+h_2+h_3}{r}-\left(\frac{h_1}{r_1}+\frac{h_2}{r_2}+\frac{h_3}{r_3}\right)&=6\\ \frac{-h_1+h_2+h_3}{r_1}+\left(\frac{h_1}{r}+\frac{h_2}{r_3}+\frac{h_3}{r_2}\right)&=6\end{aligned}\right\}\quad(40)$$

and so on

$$\left.\begin{array}{ll} r a = s_1(r_1 - r) & r_1 a = s(r_1 - r) \\ r b = s_2(r_2 - r) & r_1 b = s_3(r_1 + r_3) \\ r c = s_3(r_3 - r) & r_1 c = s_2(r_2 + r_1) \\ & \\ r_2 a = s_3(r_2 + r_3) & r_3 a = s_2(r_2 + r_3) \\ r_2 b = s(r_2 - r) & r_3 b = s_1(r_1 + r_3) \\ r_2 c = s_1(r_1 + r_2) & r_3 c = s(r_3 - r) \end{array}\right\} \quad (41)$$

$$\left.\begin{array}{ll} s a = r_1(r_2 + r_3) & s_1 a = r(r_3 + r_2) \\ s b = r_2(r_3 + r_1) & s_1 b = r_3(r_2 - r) \\ s c = r_3(r_1 + r_2) & s_1 c = r_2(r_3 - r) \\ & \\ s_2 a = r_3(r_1 - r) & s_3 a = r_2(r_1 - r) \\ s_2 b = r(r_1 + r_3) & s_3 b = r_1(r_2 - r) \\ s_2 c = r_1(r_3 - r) & s_3 c = r(r_2 + r_1) \end{array}\right\} \quad (42)$$

Many formulae may be obtained from (41) and (42) by appropriate grouping.

$$\left.\begin{array}{c} a(br_3 - cr_2) = r(r_3^2 - r_2^2) \qquad b(cr_1 - ar_3) = r(r_1^2 - r_3^2) \\ c(ar_2 - br_1) = r(r_2^2 - r_1^2) \\ a(br_2 - cr_3) = r_1(r_2^2 - r_3^2) \qquad b(ar_2 - cr) = r_1(r_2^2 - r^2) \\ c(ar_3 - br) = r_1(r_3^2 - r^2) \\ a(br_1 - cr) = r_2(r_1^2 - r^2) \qquad b(cr_3 - ar_1) = r_2(r_3^2 - r_1^2) \\ c(br_3 - ar) = r_2(r_3^2 - r^2) \\ a(cr_1 - br) = r_3(r_1^2 - r^2) \qquad b(cr_2 - ar) = r_3(r_2^2 - r^2) \\ c(ar_1 - br_2) = r_3(r_1^2 - r_2^2) \end{array}\right\} \quad (43)$$

$$\left.\begin{array}{c} \dfrac{a}{r_1} + \dfrac{b}{r_2} + \dfrac{c}{r_3} = \dfrac{2(r_1 + r_2 + r_3)}{s} \\[2ex] \dfrac{a}{r} + \dfrac{b}{r_3} + \dfrac{c}{r_2} = \dfrac{2(-r + r_3 + r_2)}{s_1} \end{array}\right\} \quad (44)$$

and so on.

$$\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)\frac{a+b+c}{r_1+r_2+r_3}=4$$

$$\left(\frac{a}{r}+\frac{b}{r_3}+\frac{c}{r_2}\right)\frac{-a+b+c}{-r+r_3+r_2}=4 \qquad (45)$$

and so on.

$$r_1-r=\frac{arr_1}{\Delta}=\frac{a\Delta}{r_2r_3} \qquad r_2+r_3=\frac{ar_2r_3}{\Delta}=\frac{a\Delta}{rr_1}$$

$$r_2-r=\frac{brr_2}{\Delta}=\frac{b\Delta}{r_3r_1} \qquad r_3+r_1=\frac{br_3r_1}{\Delta}=\frac{b\Delta}{rr_2} \qquad (46)$$

$$r_3-r=\frac{crr_3}{\Delta}=\frac{c\Delta}{r_1r_2} \qquad r_1+r_2=\frac{cr_1r_2}{\Delta}=\frac{c\Delta}{rr_3}$$

$$\frac{s^2-r_2r_3}{s}=a \qquad \frac{s^2-r_3r_1}{s}=b \qquad \frac{s^2-r_1r_2}{s}=c \qquad (47)$$

$$r_1(r_2+r)=s_3(c+a) \qquad r_1(r_3+r)=s_2(a+b)$$

$$r_2(r_3+r)=s_1(a+b) \qquad r_2(r_1+r)=s_3(b+c) \qquad (48)$$

$$r_3(r_1+r)=s_2(b+c) \qquad r_3(r_2+r)=s_1(c+a)$$

See II.

$$r(r_2-r_3)=s_1(b-c) \qquad r_1(r_2-r_3)=s(b-c)$$

$$r(r_3-r_1)=s_2(c-a) \qquad r_2(r_3-r_1)=s(c-a) \qquad (49)$$

$$r(r_1-r_2)=s_3(a-b) \qquad r_3(r_1-r_2)=s(a-b)$$

See II.

Many formulae may be obtained from (48) and (49) by appropriate grouping.

$$ar_3-br=\frac{rr_3}{r_2}(b+a)$$

$$ar_2-br_1=\frac{r_1r_2}{r_3}(b-a) \qquad (50)$$

and so on.

$$\left.\begin{aligned} r_1+r &: r_1-r = b+c : a \\ r_2+r &: r_2-r = c+a : b \\ r_3+r &: r_3-r = a+b : c \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} r_2-r_3 &: r_2+r_3 = b-c : a \\ r_3-r_1 &: r_3+r_1 = c-a : b \\ r_1-r_2 &: r_1+r_2 = a-b : c \end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned} (r_1+r)(r_2-r_3) &= (b+c)(b-c) \\ (r_2+r)(r_3-r_1) &= (c+a)(c-a) \\ (r_3+r)(r_1-r_2) &= (a+b)(a-b) \end{aligned}\right\} \quad (53)$$

$$\left.\begin{aligned} & s^2 : r_1^2 = r_2+r_3 : r_1-r = s_1^2 : r^2 \\ & s^2 : r_2^2 = r_3+r_1 : r_2-r = s_2^2 : r^2 \\ & s^2 : r_3^2 = r_1+r_2 : r_3-r = s_3^2 : r^2 \\[1ex] & s_1^2 : r_2^2 = r_3-r : r_1+r_2 \qquad s_1^2 : r_3^2 = r_2-r : r_3+r_1 \\ & s_2^2 : r_3^2 = r_1-r : r_2+r_3 \qquad s_2^2 : r_1^2 = r_3-r : r_1+r_2 \\ & s_3^2 : r_1^2 = r_2-r : r_3+r_1 \qquad s_3^2 : r_2^2 = r_1-r : r_2+r_3 \end{aligned}\right\} \quad (54)$$

$$\left.\begin{aligned} (r_2+r_3)(r_1-r) &= a^2 \\ (r_3+r_1)(r_2-r) &= b^2 \\ (r_1+r_2)(r_3-r) &= c^2 \end{aligned}\right\} \quad (55)$$

See I.

$$\left.\begin{aligned} \frac{r_1^2(r_2+r_3)^2}{r_2r_3+r_3r_1+r_1r_2} &= a^2 \\[1ex] \frac{r_2^2(r_3+r_1)^2}{r_2r_3+r_3r_1+r_1r_2} &= b^2 \\[1ex] \frac{r_3^2(r_1+r_2)^2}{r_2r_3+r_3r_1+r_1r_2} &= c^2 \end{aligned}\right\} \quad (56)$$

$$\left.\begin{aligned}
\frac{r_1(r_2+r_3)}{a}&=\frac{r_2(r_3+r_1)}{b}=\frac{r_3(r_1+r_2)}{c}\\
\frac{r\,(r_3+r_2)}{a}&=\frac{r_3(r_2-r\,)}{b}=\frac{r_2(r_3-r\,)}{c}\\
\frac{r_3(r_1-r\,)}{a}&=\frac{r\,(r_1+r_3)}{b}=\frac{r_1(r_3-r\,)}{c}\\
\frac{r_2(r_1-r\,)}{a}&=\frac{r_1(r_2-r\,)}{b}=\frac{r\,(r_2+r_1)}{c}
\end{aligned}\right\}\quad(57)$$

$$\left.\begin{aligned}
\frac{r_2r_3(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}&=bc\\
\frac{r_3r_1(r_1+r_2)(r_2+r_3)}{r_2r_3+r_3r_1+r_1r_2}&=ca\\
\frac{r_1r_2(r_2+r_3)(r_3+r_1)}{r_2r_3+r_3r_1+r_1r_2}&=ab
\end{aligned}\right\}\quad(58)$$

$$\left.\begin{aligned}
(r_2+r_3)(r_3+r_1)(r_1+r_2):abc&=s:r\\
(r_1-r\,)(r_2-r\,)(r_3-r\,):abc&=r:s
\end{aligned}\right\}\quad(59)$$

$$\left.\begin{aligned}
ss_1:s_2s_3&=a^2:(r_1-r)^2=(r_2+r_3)^2:a^2\\
ss_2:s_3s_1&=b^2:(r_2-r)^2=(r_3+r_1)^2:b^2\\
ss_3:s_1s_2&=c^2:(r_3-r)^2=(r_1+r_2)^2:c^2
\end{aligned}\right\}\quad(60)$$

$$\left.\begin{aligned}
2(r_2r_3-rr_1)&=-a^2+b^2+c^2\\
2(r_3r_1-rr_2)&=a^2-b^2+c^2\\
2(r_1r_2-rr_3)&=a^2+b^2-c^2
\end{aligned}\right\}\quad(61)$$

$$\left.\begin{aligned}
\frac{r_2r_3-rr_1}{r_2r_3+rr_1}&=\frac{-a^2+b^2+c^2}{2bc}\\
\frac{r_3r_1-rr_2}{r_3r_1+rr_2}&=\frac{a^2-b^2+c^2}{2ca}\\
\frac{r_1r_2-rr_3}{r_1r_2+rr_3}&=\frac{a^2+b^2-c^2}{2ab}
\end{aligned}\right\}\quad(62)$$

$$\left.\begin{aligned}
&\frac{r_1 - r}{rr_1}\Delta = (r_1 - r)\sqrt{\frac{r_2 r_3}{rr_1}} \\
&= \frac{r_2 + r_3}{r_2 r_3}\Delta = (r_2 + r_3)\sqrt{\frac{rr_1}{r_2 r_3}} = a \\
&\frac{r_2 - r}{rr_2}\Delta = (r_2 - r)\sqrt{\frac{r_3 r_1}{rr_2}} \\
&= \frac{r_3 + r_1}{r_3 r_1}\Delta = (r_3 + r_1)\sqrt{\frac{rr_2}{r_3 r_1}} = b \\
&\frac{r_3 - r}{rr_3}\Delta = (r_3 - r)\sqrt{\frac{r_1 r_2}{rr_3}} \\
&= \frac{r_1 + r_2}{r_1 r_2}\Delta = (r_1 + r_2)\sqrt{\frac{rr_3}{r_1 r_2}} = c
\end{aligned}\right\} \quad (63)$$

$$\left.\begin{aligned}
2r\,r_1 &= -as_1 + bs_2 + cs_3 \\
2r_2 r_3 &= -as_3 + bs + cs_1
\end{aligned}\right\} \quad (64)$$

and so on.

See XI.

$$\left.\begin{aligned}
\frac{r_1 + r}{rr_1}\Delta &= (r_1 + r)\sqrt{\frac{r_2 r_3}{rr_1}} = b + c \\
\frac{r_2 + r}{rr_2}\Delta &= (r_2 + r)\sqrt{\frac{r_3 r_1}{rr_2}} = c + a \\
\frac{r_3 + r}{rr_3}\Delta &= (r_3 + r)\sqrt{\frac{r_1 r_2}{rr_3}} = a + b
\end{aligned}\right\} \quad (65)$$

$$\left.\begin{aligned}
\frac{s}{r}(r_1 + r)(r_2 + r)(r_3 + r) &= (b + c)(c + a)(a + b) \\
\frac{s_1}{r_1}(r_1 + r)(r_3 - r_1)(r_2 - r_1) &= (b + c)(a - c)(a - b)
\end{aligned}\right\} \quad (66)$$

and so on.

$$\left.\begin{aligned}\frac{r_2-r_3}{r_2r_3}\Delta&=(r_2-r_3)\sqrt{\frac{rr_1}{r_2r_3}}=b-c\\ \frac{r_3-r_1}{r_3r_1}\Delta&=(r_3-r_1)\sqrt{\frac{rr_2}{r_3r_1}}=c-a\\ \frac{r_1-r_2}{r_1r_2}\Delta&=(r_1-r_2)\sqrt{\frac{rr_3}{r_1r_2}}=a-b\end{aligned}\right\}\quad(67)$$

$$\left.\begin{aligned}\frac{r}{s}(r_2-r_3)(r_3-r_1)(r_1-r_2)&=(b-c)(c-a)(a-b)\\ \frac{r_1}{s_1}(r_2-r_3)(r_2+r)(r+r_3)&=(b-c)(c+a)(a+b)\end{aligned}\right\}\quad(68)$$

and so on.

$$\left.\begin{aligned}\Delta^2\left(\frac{1}{r^2}-\frac{1}{r_1^2}\right)&=(r_2+r_3)(r+r_1)=a(b+c)\\ \Delta^2\left(\frac{1}{r^2}-\frac{1}{r_2^2}\right)&=(r_3+r_1)(r+r_2)=b(c+a)\\ \Delta^2\left(\frac{1}{r^2}-\frac{1}{r_3^2}\right)&=(r_1+r_2)(r+r_3)=c(a+b)\end{aligned}\right\}\quad(69)$$

$$\left.\begin{aligned}\Delta^2\left(\frac{1}{r_3^2}-\frac{1}{r_2^2}\right)&=(r_2-r_3)(r_1-r)=a(b-c)\\ \Delta^2\left(\frac{1}{r_1^2}-\frac{1}{r_3^2}\right)&=(r_3-r_1)(r_2-r)=b(c-a)\\ \Delta^2\left(\frac{1}{r_2^2}-\frac{1}{r_1^2}\right)&=(r_1-r_2)(r_3-r)=c(a-b)\end{aligned}\right\}\quad(70)$$

$$\left.\begin{aligned}(r_1+r)(r_2+r)(r_3+r):(b+c)(c+a)(a+b)&=r:s\\ (r_2-r_3)(r_3-r_1)(r_1-r_2):(b-c)(c-a)(a-b)&=s:r\end{aligned}\right\}\quad(71)$$

$$\begin{aligned}
\Delta & \\
&= \frac{r_2 r_3 (r_1 + r)}{b + c} = \frac{r r_1 (b + c)}{r_1 + r} = \frac{r r_1 (r_2 - r_3)}{b - c} = \frac{r_2 r_3 (b - c)}{r_2 - r_3} \\
&= \frac{r_3 r_1 (r_2 + r)}{c + a} = \frac{r r_2 (c + a)}{r_2 + r} = \frac{r r_2 (r_3 - r_1)}{c - a} = \frac{r_3 r_1 (c - a)}{r_3 - r_1} \\
&= \frac{r_1 r_2 (r_3 + r)}{a + b} = \frac{r r_3 (a + b)}{r_3 + r} = \frac{r r_3 (r_1 - r_2)}{a - b} = \frac{r_1 r_2 (a - b)}{r_1 - r_2}
\end{aligned} \quad (72)$$

$$\left.\begin{aligned}
\frac{(r_1 - r)(r_2 - r)(r_3 - r)\Delta}{r^2} &= abc \\
\frac{(r_2 + r_3)(r_3 + r_1)(r_1 + r_2) r^2}{\Delta} &= abc
\end{aligned}\right\} \quad (73)$$

$$(r_1 + r)(r_2 + r)(r_3 + r)(r_2 - r_3)(r_3 - r_1)(r_1 - r_2) = (b^2 - c^2)(c^2 - a^2)(a^2 - b^2) \quad (74)$$

$$(r_1 - r)(r_2 - r)(r_3 - r)(r_2 + r_3)(r_3 + r_1)(r_1 + r_2) = a^2 b^2 c^2 \quad (75)$$

$$(r_1 + r_2 + r_3 - r)\Delta = abc \quad (76)$$

$$\Delta^3 \left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = abc \quad (77)$$

$$\Delta^3 \left(\frac{1}{r r_3 r_2} + \frac{1}{r r_1 r_3} + \frac{1}{r r_2 r_1} - \frac{1}{r_1 r_2 r_3}\right) = abc \quad (78)$$

$$\frac{1}{r_3} - \frac{1}{r_1^3} - \frac{1}{r_2^3} - \frac{1}{r_3^3} = \frac{24}{h_1 h_2 h_3} \quad (79)$$

$$\left.\begin{aligned}
\frac{r_2 r_3}{r_1^3} + \frac{r_3 r_1}{r_2^3} + \frac{r_1 r_2}{r_3^3} &= s^2 r \left(\frac{1}{r_1^4} + \frac{1}{r_2^4} + \frac{1}{r_3^4}\right) \\
\frac{r_2 r_3}{r^3} + \frac{r_2 r}{r_3^3} + \frac{r r_3}{r_2^3} &= s_1^2 r_1 \left(\frac{1}{r^4} + \frac{1}{r_3^4} + \frac{1}{r_2^4}\right)
\end{aligned}\right\} \quad (80)$$

and so on.

In the following notes an endeavour has been made to assign the various formulae to the authors who first published them more or less explicitly. But it would be presumptuous to suppose that this endeavour has met with more than a partial success. I shall be grateful to any one who will inform me of earlier sources than those I have been able to find.

(1) Both of the expressions $sr = \Delta$ and $ss_1s_2s_3 = \Delta^2$ are given by Heron of Alexandria in his treatise "On the Dioptra." See Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, *pp.* 235-7 (1864).

(2) Weddle in the *Lady's and Gentleman's Diary* for 1845, p. 82.

(3) It is stated in a note in Gergonne's *Annales*, I. 150 (1810-11), that Mahieu, professor of mathematics at the College of Alais, discovered the theorem $\Delta^2 = rr_1r_2r_3$ about 1807, and that Lhuilier gives it in his *Élémens d'Analyse*, p. 224 (1809).

(4) The first half of the first expression is given by Feuerbach, *Eigenschaften... des...Dreiecks*, § 4 (1822). All the expressions which have *r's* in the numerator are given by L.P.F.R. in Gergonne's *Annales*, XIX. 214 (1829).

(5) T. S. Davies in the *Philosophical Magazine*, II. 28 (1827). In the same place Davies gives also the first proportion of (6).

(7) The first expression of the first line is given by Euler in *Novi Commentarii Academiae...Petropolitanae*, for the years 1747-8, I. 54 (1750); the second by T. S. Davies in the *Ladies' Diary* for 1835, p. 56. The first expression of the second line is given, implicitly, by Feuerbach, § 4 (1822); all the expressions in the second line are given in Gergonne's *Annales*, XIX. 214 (1829).

(9) The first three expressions are given by T. S. Davies in the *Philosophical Magazine*, II. 28 (1827); the second three by L.P.F.R. in Gergonne's *Annales*, XIX. 214 (1829).

(10) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 86.

(11), (12) Feuerbach, § 6 (1822).

(15) The first expression occurs in the *Ladies' Diary* for 1759; all four occur in Gergonne's *Annales*, XIX. 214 (1829).

(16) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 83 (1892).

(17) The first expression is given by C. J. Matthes in his *Commentatio de Proprietatibus Quinque Circulorum*, p. 9 (1831).

(18) The first of these expressions is given by Mr R. Knowles in *Mathematical Questions from the Educational Times*, XLI. 93 (1884).

(19), (21) Weddle in the *Lady's and Gentleman's Diary* for 1848, p. 76, and for 1845, p. 78.

(23) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 81 (1892).

(24) The relation between r and r_1, r_2, r_3 was given by Steiner and Bobillier in 1828. See Steiner's *Gesammelte Werke*, I. 214. The relation between r and h_1, h_2, h_3 was given by L.P.F.R. in Gergonne's ***Annales***, XIX. 212 (1829).

(25) Half of these expressions were given by Lowry and Rutherford in the *Ladies' Diary* for 1836, p. 54; the other half and also the whole of (26) and (27) by Weddle in the *Lady's and Gentleman's Diary* for 1843, pp. 90-1.

(28) The expression for r in terms of r_1, r_2, r_3 occurs in the *Ladies' Diary* for 1759. The expressions for r, r_1, r_2, r_3 in terms of the h's are given by Lowry and Rutherford in the *Ladies' Diary* for 1836, pp. 53, 55; the other expressions, by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 81.

(29) The expressions for Δ^2 in terms of the r's were given by Steiner in 1828. See his *Gesammelte Werke*, I. 214. The expression in terms of the h's was given by J. A. Grunert in *Supplemente zu Klügels Wörterbuche der reinen Mathematik*, I. 703 (1833).

(30)—(33) Weddle in the *Lady's and Gentleman's Diary* for 1843, pp. 91-2.

(34) First expression given by T. S. Davies in the *Philosophical Magazine*, II. 29 (1827).

(35) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 91.

(36) Mr David Trowbridge in Runkle's *Mathematical Monthly*, III. 188 (1861).

(37), (38), (40) The first expression in each of these is given by Thomas Dobson in the *Lady's and Gentleman's Diary* for 1862, pp. 95-6; the first expression in (39) is given by Dobson in *Mathematical Questions from the Educational Times*, III. 104 (1865).

(41) The values of r_3b, r_2c are given in the *Ladies' Diary* for 1759; those of ra, rb, rc, and of r_1a, r_2b, r_3c, in a slightly different form, were given by Steiner in 1828. See his *Gesammelte Werke*, I. 215.

(42) The values of sa, sb, sc are given in the *Ladies' Diary* for 1759. All the expressions in (41) and (42) were given by Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 84.

(43) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 81 (1892).

(44), (45) Thomas Dobson in the *Lady's and Gentleman's Diary* for 1865, p. 53, and for 1864, p. 83.

(46) Mr Bernhard Möllmann in Grunert's *Archiv*, XVII. 380-1 (1851).

(47) L.P.F.R. in Gergonne's *Annales*, XIX. 214 (1829).

(48), (49) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 85.

(50) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 81 (1892). The first proportion in (51) and the last in (52) are given by C. J. Matthes in his *Commentatio*, p. 52 (1831).

(53), (54), (55) Weddle in the *Lady's and Gentleman's Diary* for 1843, pp. 85, 87, 80.

(56) and the first line of (57) are given by L.P.F.R. in Gergonne's *Annales*, XIX. 214-5 (1829).

(58) T. S. Davies in the *Ladies' Diary* for 1836, p. 51.

(61) Mr C. Hellwig in Grunert's *Archiv*, XIX. 50 (1852).

(63) The expressions in which $-$ occurs were given by Steiner in 1828. See his *Gesammelte Werke*, I. 215. C. J. Matthes in his *Commentatio*, p. 52 (1831), gives one of the others.

The first value of $b+c$ in (65) and the first of $a-b$ in (67) are given by C. J. Matthes in his *Commentatio*, p. 52 (1831).

(66), (68) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 82 (1892).

(69), (70) The expressions where Δ^2 occurs are given by Mr Lemoine in *Mathesis*, 2nd series, II. 91 (1892); the others by Mr C. Hellwig in Grunert's *Archiv*, XIX. 50 (1852).

(71), (72) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 86.

(76)—(78) T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 90.

(80) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 84 (1892).

Seventh Meeting, May 11th, 1894.

Dr C. G. KNOTT, President, in the Chair.

On the Division of a Parallelepiped into Tetrahedra without making new corners.

By Professor CRUM BROWN.

The method employed in this paper is first to ascertain in how many ways a cube can be cut into tetrahedra without making new corners, and then, taking each of these divisions of the cube as the type of a genus of divisions of the general parallelepiped, determining the number of species in each genus.

We first fix a system of notation of the corners of the cube. Call any corner A and let the three corners at the other ends of face-diagonals from A, be called B, C, and D, noting them in the positive sense as seen from A. Then let $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, respectively be the corners at the opposite ends of body-diagonals from A, B, C and D. So that passing along an edge we change both letter and sign, passing along a face-diagonal we change letter but not sign, passing along a body diagonal we change sign but not letter.

The number of distinct forms of tetrahedra which can be cut out of a cube without making new corners is five. We shall give each of them a symbol and indicate one position of each in the cube by noting its four corners. 1st. Ω, ABCD, this is the regular tetrahedron, its volume is $\frac{1}{3}$ of that of the cube. 2nd. Δ, $A\bar{B}\bar{C}\bar{D}$, this tetrahedron has as one of its corners an undivided corner of the cube, its volume is $\frac{1}{6}$ of the volume of the cube. 3rd. I, $A\bar{A}BC$, its volume is $\frac{1}{6}$ of the volume of the cube. 4th and 5th. Two enantiomorph tetrahedra L, $A\bar{A}B\bar{C}$, and Γ, $A\bar{A}\bar{B}C$, each of these has $\frac{1}{6}$ of the volume of the cube. Forms such as $AB\bar{C}\bar{D}$, and $A\bar{A}B\bar{B}$, do not correspond to tetrahedra, because their corners are all in one plane.

There is only one combination containing Ω, which makes up a cube, in this a Δ is applied to each face of the Ω. This is the only quinquepartite division of the cube into tetrahedra. The other divisions of the cube into tetrahedra without making new corners are necessarily sexpartite, because all our tetrahedra except Ω have $\frac{1}{6}$ of the volume of the cube.

Let i, δ, l, γ represent the number of the tetrahedra, I, Δ, L and Γ respectively in a combination making up a cube. Noting that I has one face which is $\frac{1}{12}$ of the surface of the cube, Δ three such faces, and L and Γ each two such faces, we have, on account of the volume, (1), $i+\delta+l+\gamma=6$, and on account of the surface, (2), $i+3\delta+2l+2\gamma=12$. From these we have at once $i=\delta$, and $l+\gamma$ an even number. Not only is $i=\delta$ but each I has a particular Δ attached to it, the equilateral triangular face of the one fitting to the equilateral triangular face of the other to form a figure which we may call IΔ. This IΔ is $\frac{1}{3}$ of the cube and is an oblique square pyramid with a face of the cube for base and for apex a corner of the cube. It can be cut into an I and a Δ by a plane containing the apex and the two corners of the cube which have the same sign as the apex. It can be cut into two tetrahedra in another way, namely, by a plane through the apex and the two corners of the base which differ in sign from the apex, the two tetrahedra into which it is thus cut are an L and a Γ. So that an IΔ can always be replaced by an LΓ pair.

We shall call those combinations which contain only L's and Γ's *central* combinations because all the tetrahedra in them have an edge bisected by the centre of the cube. Half a cube can be made up either by two L's with a Γ between them or by two Γ's with an L between them, there are therefore three central combinations, namely—3L,3Γ; 4L,2Γ; and 4Γ,2L. The first is uniaxial because all the six tetrahedra meet in one body-diagonal, the other two are biaxial because the three tetrahedra forming one half of the cube meet in one body-diagonal and the three forming the other half of the cube meet in another body-diagonal. From these three central combinations all the others can be derived by replacing LΓ pairs by IΔ's. It is to be noted that when we have two IΔ's we have always two distinct combinations, in one of which the plane between the one I and its Δ is parallel ($\|$) to the plane between the other I and its Δ, while in the other these two planes are inclined ($\nparallel$) to one another.

We have therefore the following sexpartite combinations:— Uniaxial—3L, 3Γ ; IΔ, 2L, 2Γ ; ‖ 2I, 2Δ, L, Γ ; ⫲ 2I, 2Δ, L, Γ ; and 3I, 3Δ. Biaxial—4L, 2Γ ; 4Γ, 2L ; I, Δ, 3L, Γ ; I, Δ, 3Γ, L ; ‖ 2I, 2Δ, 2L ; ‖ 2I, 2Δ, 2Γ ; ⫲ 2I, 2Δ, 2L ; and ⫲ 2I, 2Δ, 2Γ. In all five uniaxial combinations and eight biaxial in four enantiomorph pairs.

As in the general parallelepiped none of the eight corners are interchangeable, differences of position in the divisions of the cube become differences of form in the corresponding divisions of the general parallelepiped. We can therefore obtain the number of species in each genus by counting the number of ways in which each cubic combination can be placed in a cubic box, the corners of which are distinctively noted.

Of Ω,4Δ	there are	2	species	
„ 3L,3Γ	„	4	„	Uniaxial.
„ I,Δ,2L,2Γ,	„	24	„	
„ ‖ 2I,2Δ,L,Γ,	„	12	„	
„ ⫲ 2I,2Δ,L,Γ	„	24	„	
„ 3I,3Δ	„	8	„	
„ 4L,2Γ	„	6	„	Biaxial.
„ 4Γ,2L	„	6	„	
„ I,Δ,3L,Γ	„	24	„	
„ I,Δ,3Γ,L	„	24	„	
„ ‖ 2I,2Δ,2L	„	12	„	
„ ‖ 2I,2Δ,2Γ	„	12	„	
„ ⫲ 2I,2Δ,2L	„	12	„	
„ ⫲ 2I,2Δ,2Γ	„	12	„	

In all 72 species of the five uniaxial combinations and 108 species of the eight biaxial combinations.

This investigation is published in greater detail, with figures of the tetrahedra and of the combinations, in the *Transactions* of the Royal Society of Edinburgh

The Solutions of the Differential Equations

$$\left\{\cos\left(\lambda \cdot \frac{d}{dx}\right)\right\} \cdot y = f(x) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\left\{sin\left(\lambda \cdot \frac{d}{dx}\right)\right\} \cdot y = \phi(x) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

By F. H. Jackson, M.A.

These equations are solved by the ordinary methods applicable to linear equations with constant coefficients.

The complementary function of equation (1) is the primitive of

$$\left\{\cos\left(\lambda \cdot \frac{d}{dx}\right)\right\} \cdot \mathrm{Y} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Now $\mathrm{Y} = e^{m \cdot x}$ will be a particular solution of (3) if m be such as to make $\cos(\lambda \cdot m) = 0$.

The required values of m are found by giving to r all positive integral values in succession, from zero to infinity in $\left(r + \frac{1}{2}\right)\frac{\pi}{\lambda}$ and $-\left(r + \frac{1}{2}\right)\frac{\pi}{\lambda}$. Thus $e^{(r+\frac{1}{2})\frac{\pi}{\lambda}}$ and $e^{-(r+\frac{1}{2})\frac{\pi}{\lambda}}$ are each particular integrals of equation (3) r having any integral value from 0 to $+\infty$.

Hence the solution of equation (3) is

$$\mathrm{Y} = \sum_{r=0}^{r=\infty}\left\{\mathrm{A}_r e^{(r+\frac{1}{2})\frac{\pi x}{\lambda}} + \mathrm{B}_r e^{-(r+\frac{1}{2})\frac{\pi x}{\lambda}}\right\} \quad \ldots \quad \ldots \quad (4)$$

The expression on the right side of the above equation is the complementary function of equation (1).

In order to find a particular integral of the equation we write it in the form

$$y = \left\{\frac{1}{\cos(\lambda \cdot \mathrm{D})}\right\} \cdot f(x) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Decomposing the operating function into partial fractions we obtain

$$\frac{1}{\cos\lambda . \mathrm{D}} = \left(\frac{1}{\lambda . \mathrm{D} + \frac{\pi}{2}} - \frac{1}{\lambda . \mathrm{D} - \frac{\pi}{2}}\right) - \left(\frac{1}{\lambda . \mathrm{D} + \frac{3\pi}{2}} - \frac{1}{\lambda . \mathrm{D} - \frac{3\pi}{2}}\right) + \dots \text{ to infin.}$$

$$= \sum_{r=0}^{r=\infty} \frac{(-1)^r}{\lambda} . \left\{\frac{1}{\mathrm{D} + (r+\frac{1}{2})\frac{\pi}{\lambda}} - \frac{1}{\mathrm{D} - (r+\frac{1}{2})\frac{\pi}{\lambda}}\right\}$$

Equation (5) may now be written

$$y = \sum_{r=0}^{r=\infty} \frac{(-1)^r}{\lambda} . \left\{\frac{1}{\mathrm{D} + (r+\frac{1}{2}) . \frac{\pi}{\lambda}} - \frac{1}{\mathrm{D} - (r+\frac{1}{2}) . \frac{\pi}{\lambda}}\right\} . f(x) \qquad \dots \quad (6)$$

Using the general theorem $\mathrm{D}^{-1}\{e^{ax}\mathrm{X}\} = e^{ax} . \frac{1}{\mathrm{D}+a} . \mathrm{X}$
equation (6) becomes

$$y = \sum_{r=0}^{r=\infty} \frac{(-1)^r}{\lambda} . \left\{e^{-(r+\frac{1}{2}) . \frac{\pi . x}{\lambda}} \int e^{(r+\frac{1}{2}) . \frac{\pi . x}{\lambda}} . f(x)dx - e^{(r+\frac{1}{2})\frac{\pi . x}{\lambda}} \right.$$

$$\left. \int e^{-(r+\frac{1}{2})\frac{\pi . x}{\lambda}} . f(x)dx\right\}$$

This is the particular solution of equation (1), and the complete solution is found by adding the complementary function to the expression on the right side of the above equation.

In the equation $\left\{\sin\left(\lambda . \frac{d}{dx}\right)\right\}y = \phi(x)$, the solution takes a simpler form than in the first equation.

The complementary function is the primitive of

$$\left\{\sin\left(\lambda . \frac{d}{dx}\right)\right\}\mathrm{Y} = 0 \, ;$$

$y = e^{m . x}$ will be a particular solution of this if m be such as to make $\sin(\lambda . m) = 0$. The required values of m are found by giving to r all integral values (including zero) from $+\infty$ to $-\infty$ in $\frac{r . \pi .}{\lambda}$

Hence $Y = \sum_{r=-\infty}^{r=+\infty} \left\{ A_r e^{\frac{r\pi x}{\lambda}} \right\}$ the expression on the right side of this equation is the complementary function in the solution of the second equation.

To find the particular integral we write equation (2) in the form $y = \left\{ \frac{1}{\sin(\lambda . D)} \right\} \phi(x)$ and decompose the operator into partial fractions.

$$\frac{1}{\sin\lambda . D} = \frac{1}{\lambda . D} - \frac{1}{\lambda . D - \pi} - \frac{1}{\lambda . D + \pi} + \frac{1}{\lambda . D - 2\pi} + \frac{1}{\lambda . D + 2\pi} - \text{etc. to infinity}$$

$$= \sum_{r=-\infty}^{r=+\infty} \frac{(-1)^r}{\lambda} \frac{1}{D + \frac{r\pi}{\lambda}}$$

$$\therefore \quad y = \sum_{r=-\infty}^{r=+\infty} \left\{ \frac{(-1)^r}{\lambda} \cdot \frac{1}{D + \frac{r\pi}{\lambda}} \right\} \cdot \phi(x)$$

$$= \sum_{r=-\infty}^{r=+\infty} \frac{(-1)^r}{\lambda} e^{\frac{-r\pi x}{\lambda}} \int e^{\frac{r\pi x}{\lambda}} \phi(x) dx$$

The complete solution is found by adding the complementary function to the expression on the right of the above equation.

The solution of the equation (2) is therefore

$$y = \sum_{r=-\infty}^{r=+\infty} \left\{ A_r e^{\frac{r\pi x}{\lambda}} + \frac{(-1)^r}{\lambda} e^{\frac{-r\pi x}{\lambda}} \int e^{\frac{r\pi x}{\lambda}} \phi(x) dx \right\} \quad \ldots \quad \ldots \quad (7)$$

and the complete solution of equation (1) is

$$y = \sum_{r=0}^{r=\infty} \left\{ A_r e^{(r+\frac{1}{2})\frac{\pi . x}{\lambda}} + B_r e^{-(r+\frac{1}{2})\frac{\pi . x}{\lambda}} + \frac{(-1)^r}{\lambda} e^{-(r+\frac{1}{2})\frac{\pi . x}{\lambda}} \cdot \right.$$
$$\left. \int e^{(r+\frac{1}{2}) . \frac{\pi x}{\lambda}} \phi(x) dx - \frac{(-1)^r}{\lambda} e^{(r+\frac{1}{2}) . \frac{\pi . x}{\lambda}} \int e^{-(r+\frac{1}{2}) . \frac{\pi . x}{\lambda}} \phi(x) dx \right\} \quad \ldots \quad (8)$$

Eighth Meeting, June 8th, 1894.

Dr C. G. KNOTT, President, in the Chair.

On the Highest Wave of Permanent Type.

By JOHN M'COWAN, D.Sc.

On a Problem in Tangency.

By G. E. CRAWFORD, M.A.

FIGURE 17.

Let AOB, DOE be any two intersecting chords of a circle. Required to inscribe a circle in one of the compartments, as BOE.

Draw OF bisecting the angle BOE and let it cut the circle in F. Draw FM perpendicular to AB.

Join C' (the centre of the circle) to O and produce to cut the circle in G. [Draw GH in any direction equal to FM, and join C'H. Draw OK parallel to GH cutting C'H in K.] Cut off OL = LQ = OK. Join LF and with Q as centre and radius FM or GH describe a circle cutting LF in S. Draw QSR and draw a parallel to it through C' cutting OF in F' and the circle in P'. Then F' shall be the centre of the required circle.

PROOF.—Draw F′M′ perpendicular to OB, draw through F a parallel CP to C′P′, and join OP′, producing it to cut CF in P.

Then $LQ : QS :: OL : FM :: OK : GH :: C'O : C'G$
$:: C'O : C'P'$

$\therefore$ the $\triangle$'s LQS and OC′P′ have an angle common and the sides about that angle proportional.

$\therefore$ $\angle SLQ = \angle P'OC'$

$\therefore$ LF is parallel to OP.

Hence $OL : FP :: CL : CF$
$:: QL : QS$, since the $\triangle$'s QSL, CFL are similar
$:: OL : FM$

$\therefore$ $FP = FM$.

But $F'P' : FP :: OF' : OF$
$:: F'M' : FM$

But $FP = FM$

$\therefore$ $F'P' = F'M'$

and C′ is the centre of circle AGB

$\therefore$ a circle with centre F′ and radius F′M′ will touch both OE and OB and will touch the circle AGB at P′.

The Algebraic Solution of the Cubic and Quartic in x by means of the Substitution

$$\frac{\lambda x_1 + \mu}{1 + x}$$

By CHARLES TWEEDIE, M.A., B.Sc.

E. Carpenter's Proof of Taylor's Theorem.

By R. F. MUIRHEAD.

The idea of the following proof was communicated to me some years ago by Mr Edward Carpenter of Millthorpe, Derbyshire, formerly Fellow of Trinity Hall, Cambridge; who remarked that it seemed to afford a demonstration of Taylor's Theorem which came very naturally and directly from the definition of a differential coefficient. The chief difficulty seemed to arise in dealing with the negligible small quantities which are produced in great numbers. However, I found it not difficult to complete the proof for the case when *all* the successive differential coefficients of $f(x)$ are finite and continuous.

It occurred to me lately that this proof might interest the Society: and it is here given with the addition of a modified proof leading to an expansion in m terms with a remainder.

1. If $f(x)$ possesses a differential coefficient $f'(x)$, then

$$\operatorname*{Lim}_{h=0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

that is

$$f(x+h) = f(x) + hf'(x) + h\alpha_1 \quad \ldots \quad \ldots \quad (1)$$

where α_1 is a quantity which vanishes with h.

Similarly

$$f'(x+h) = f'(x) + hf''(x) + h\alpha_2 \quad \ldots \quad \ldots \quad (2)$$

$$f''(x+h) = f''(x) + hf'''(x) + h\alpha_3 \quad \ldots \quad \ldots \quad (3)$$

and so on, α_2 α_3 etc. being quantities which vanish with h.

Writing $x+h$ for x in (1) we have

$$f(x+2h) = f(x+h) + hf'(x+h) + h\alpha_1'$$

where α_1' also vanishes with h; and hence, by (2)

$$f(x+2h) = f(x) + 2hf'(x) + h^2 f''(x) + h(\alpha_1 + \alpha_1') + h^2\alpha_2$$

Repeating this process n times, we get

$$f(x+nh) = f(x) + {}^nC_1 h f'(x) + {}^nC_2 h^2 f''(x) + \dots + {}^nC_n h^n f^n(x)$$
$$+ h(a_1 + a_1' + a_1'' + \dots) + h^2(a_2 + a_2' + \dots) + \dots + h^n a_n \qquad (4)$$

where ${}^nC_r$ is the number of r-combinations of n things, and the number of the quantities $a_r, a_r', a_r'' \dots$ is ${}^nC_r$.

Now if η be the numerical value of the numerically greatest of the quantities a, the sum of the terms in the 2nd line of (4) is

$$< \eta({}^nC_1 h + {}^nC_2 h^2 + \dots + {}^nC_n h^n)$$
$$< \eta(\overline{1+h}^n - 1) \qquad < \eta(e^{nh} - 1).$$

Now write y for nh and (4) becomes

$$f(x+y) = f(x) + \frac{y}{1} f'(x) + \frac{y.(y-h)}{1 \cdot 2} f''(x) + \dots$$
$$+ \frac{y(y-h)\dots(y-\overline{n-1}h)}{n!} f^n(x) + \epsilon \qquad (5)$$

where $\epsilon < \eta(e^y - 1)$.

It is clear that if we take any finite number of the terms of the series (5), say $m+1$, these terms will differ from the first $m+1$ terms of Taylor's Expansion of $f(x+y)$ by a finite number of quantities which vanish with h; the sum of which we may denote by ϵ', a quantity which likewise vanishes with h.

If then Taylor's Expansion is absolutely convergent, it is clear that the above investigation affords a complete proof of his Theorem. In fact if R be the remainder after $m+1$ terms of Taylor's Expansion, and L the corresponding remainder (excluding ϵ) of the expansion in (5), then each term of L is numerically less than the corresponding term of R and L can be made less than $\frac{\delta}{3}$ when δ is an arbitrary small finite quantity, by taking m great enough, but still finite.

Thus from (5) we have

$$f(x+y) = f(x) + y.f'(x) + \frac{y^2}{2!} f''(x) + \dots + \frac{y^m}{m!} f^m(x) + L + \epsilon + \epsilon'.$$

Now take m so that $L < \frac{\delta}{3}$.

Then take n so that $\epsilon < \frac{\delta}{3}$ and $\epsilon' < \frac{\delta}{3}$.

Thus $$f(x+y) = f(x) + yf'(x) + \ldots + \frac{y^m}{m\,!} f^m(x)$$
$$+ \text{ a quantity less than } \delta.$$

This investigation will apply only when *all* the differential coefficients of $f(x)$ are finite and continuous, and Taylor's Expansion an absolutely convergent series for all values of the independent variable from x up to $x+y$.

2. In order to get a formula with a remainder after m terms, m being *any* integer, without assuming anything about the differential coefficients after the m^{th} we may modify our procedure thus :—

Proceed as before until we get

$$f(x+mh) = f(x) + {}^mC_1 hf'(x) + \ldots + {}^mC_m h^m f^m(x)$$
$$+ h(a_1 + a_1' + \ldots) + h^2(a_2 + a_2' + \ldots) + \ldots + h^m a_m.$$

In taking the succeeding steps up to the n^{th}, proceed as before except that the m^{th} differential coefficient is left unaltered each time, when we are substituting for $f(x+h), f'(x+h) \ldots$ the values given by (1), (2) ...

After the n^{th} step we have

$$f(x+nh) = f(x) + {}^nC_1 hf'(x) + \ldots + {}^nC_{m-1} h^{m-1} f^{m-1}(x)$$
$$+ {}^{n-1}C_{m-1} h^m f^m(x) + {}^{n-2}C_{m-1} h^m f^m(x+h) + \ldots$$
$$+ {}^mC_{m-1} h^m f^m(x + \overline{n-m-1}h) + {}^mC_m h^m f^m(x + \overline{n-m}h)$$
$$+ h(a_1 + a_1' + \ldots) + h^2(a_2 + a_2' + \ldots) + \ldots + h^m(a_m + \ldots) \qquad (6)$$

Here the number of the quantities a_1 is ${}^nC_1$ and that of the quantities a_r is ${}^nC_r$. Hence if η is the numerical value of the numerically greatest a, the last line of (6) is

$$< \eta\{{}^nC_1 h + {}^nC_2 h^2 + \ldots + {}^nC_m h^m\} < \eta\{\overline{1+h}^n - 1\} < \eta(e^y - 1)$$

where $y = nh$. Denoting the sum of these small terms by ϵ, we see that ϵ is less than a quantity which vanishes with h.

Again the sum of the coefficients of the terms which contain differential coefficients of the m^{th} order is

$$ {}^{n-1}C_{m-1} + {}^{n-2}C_{m-1} + \dots + {}^{m}C_{m-1} + {}^{m-1}C_{m-1}, \quad \text{since} \quad {}^{m}C_{m} = {}^{m-1}C_{m-1} $$

and this is $= {}^{n}C_{m}$. Multiplying it by h^m we get

$$ y(y-h)(y-2h)\dots(y-\overline{m-1}h)/m! $$

which differs from $y^m/m!$ by a quantity ζ which vanishes with h. Hence the terms in question are

$$ = (y^m/m! - \zeta) \times \text{a mean of the values } f^m(x), f^m(x+h)\dots f^m(x+y-mh). $$

This mean may obviously be written $f^m(x+\theta y)$, where θ is a proper fraction, so that we have $y^m f^m(x+\theta y) + \epsilon'$ where $\epsilon' = -\zeta f^m(x+\theta y)$, a quantity which vanishes with h.

Again the first line of (6) differs from

$$ f(x) + yf'(x) + \frac{y^2}{2!} f''(x) + \dots + \frac{y^{m-1}}{(m-1)!} f^{m-1}(x) $$

by a quantity ϵ'' which also vanishes with h.

Now take n so large that $\epsilon < \frac{\delta}{3}$, $\epsilon' < \frac{\delta}{3}$, and $\epsilon'' < \frac{\delta}{3}$ where δ is a finite quantity but may be as small as we please.

Thus $$ f(x+y) = f(x) + yf'(x) + \dots + \frac{y^{m-1}}{(m-1)!} f^{m-1}(x) $$
$$ + \frac{y^m}{m!} f^m(x+\theta h) + \text{a quantity less than } \delta. $$

Notes on an Orthocentric Triangle.

By R. TUCKER, M.A.

FIGURE 18.

1. In the accompanying figure DEF is the pedal triangle of ABC and P, Q, R are the orthocentres of AFE, BDF, CED.

AP, BQ, CR evidently meet in the circumcentre, O, of ABC, which is the orthocentre of PQR,

2. Now the circumradius of $AFE(\rho_a) = R\cos A$,

$$\therefore \; EP = 2\rho_a \cos B = 2R\cos A \cos B$$
$$= FH = DQ:$$

hence the sides of PQR are equal and parallel to the sides of DEF, *i.e.*, the triangles are congruent, and their centre of perspective, L, bisects DP, EQ, FR.

The hexagon DQFPER = twice the pedal triangle.

3. The coordinates of P, D are respectively

$$R[\cos A - \cos 2A\cos(B-C)], \quad 2R\cos^2 A\cos B, \quad 2R\cos^2 A\cos C;$$
$$0 \quad , \quad R\sin 2C\sin B \; , \quad R\sin 2B\sin C \; ;$$

hence the coordinates of L are as

$$\cos A - \cos 2A\cos(B-C), \quad \cos B - \cos 2B\cos(C-A),$$
$$\cos C - \cos 2C\cos(A-B), \quad \text{or} \quad \sin A.\Pi\sin A - \cos A.\Pi\cos A, \; \ldots, \; \ldots,^*$$

4. The perpendicular from P on $EF = 2R\cos A\cos B\cos C =$ the perpendicular from D on QR; and so for the other vertices and sides.

5. The equation to the circle PQR is

$$\sin A\sin B\sin C.\Sigma\alpha\beta\gamma = \Sigma a\alpha.\Sigma\alpha\sin A\cos^2 A(1 + 2\cos 2B\cos 2C)$$

and the α - coordinate of the centre is

$$R[\cos C\sin(2A - C) + \cos B\sin(2A - B)]/2\sin A.$$

The equation to PQ is

$$\alpha\sin 2A\cos 2B\cos A + \beta\cos 2A\sin 2B\cos B + \gamma\sin C(1 + \cos 2A\cos 2B) = 0.$$

* See *Proceedings*, London Mathematical Society, Vol. xv., Appendix.)

6. If v is the incentre of PQR then

$$Pv = DH = 2R\cos B\cos C,$$

and the a – ordinate of v is $2R\cos A(\cos^2 B + \cos^2 C)$:
hence vLH is a straight line, as is also readily seen from the symmetry of the figure, L being the mid point of vH.

7. The a – ordinate of H′, the orthocentre of DEF, is (*l.c.*),

$$-R\cos 2A\cos(B - C),$$

hence OLH′ is a straight line, in fact it is the circum-Brocard axis of ABC.

8. If OP, OQ, OR cut EF, FD, DE in X, Y, Z,

then since $$\frac{EX}{XF} = \frac{\tan C}{\tan B},$$

it is seen that DX, EY, FZ cointersect in a point, viz., in v.

9. If a side, as PR, is cut by EF, ED in x, y, then

$$Px : xy : yR = \sin 2A : \sin 2B : \sin 2C.$$

[Since the above Note was written, vol. i. of the Society's *Proceedings* has been published. I must refer readers to Figs. 58, 59, from which it will be seen that some of my points were noted in that admirable piece of Geometry by Dr Mackay.]

Edinburgh Mathematical Society.

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2. American Journal of Mathematics.
   (Johns Hopkins University.)
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3. Bulletin of the New York Mathematical Society.
   Vol. II.—Nos. 7-10.
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4. The Mathematical Magazine.
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6. Bulletin de la Société Mathématique de France.
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7. Journal de Mathématiques Élémentaires.
   Publié par M. H. VUIBERT.
   Nos. 1-18. 1893-4.

8. Journal de Mathématiques Élémentaires.
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11. Mathematical Papers (3 Miscellaneous).
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12. Jornal de Sciencias Mathematicas e Astronomicas.
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13. Revue Semestrielle des Publications Mathématiques.
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14. Nieuw Archief voor Wiskunde. Amsterdam.
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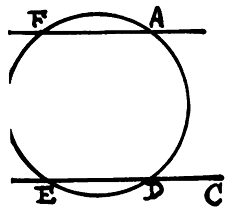

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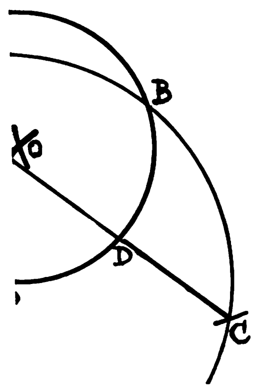

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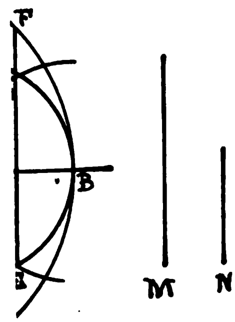

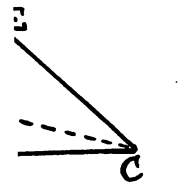
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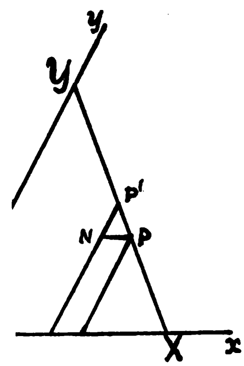
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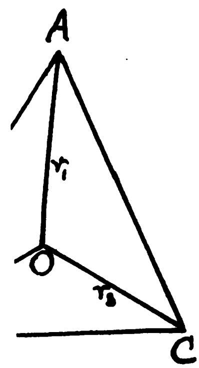
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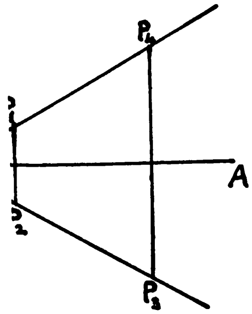

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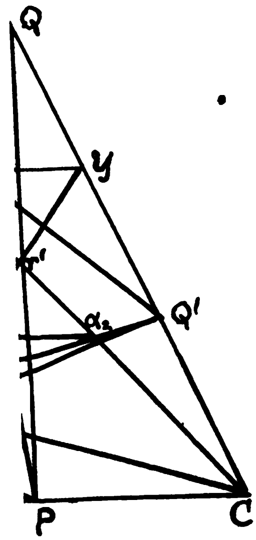

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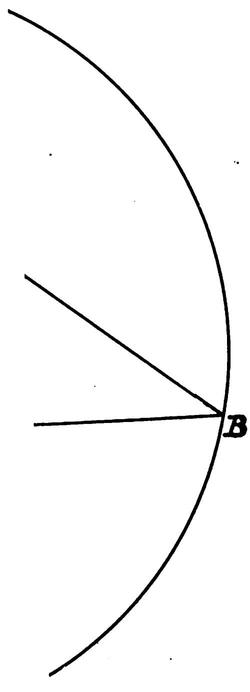
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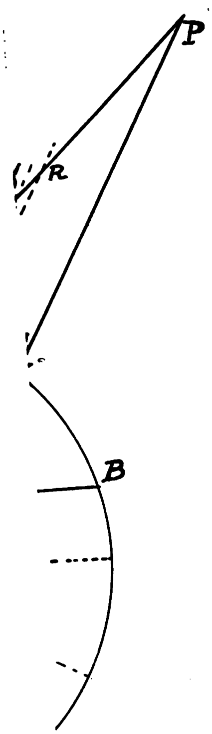
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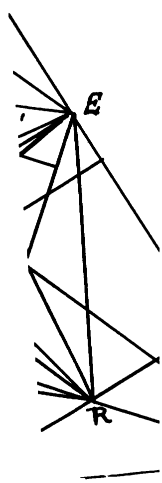
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